Fundamentals of Quantitative Finance

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Abstract— **This paper explores the fundamentals of quantitive finance and capital market theory, introducing the underlying quantitative methods driving capital asset allocation decision-making. Using these methods as building blocks, this paper will model financial market infrastructure to simulate optimal decision-making market participants.**

*Index terms—***finance, quantitative, capital, markets**

I. Introduction

II. Quantitative Methods

A. *Time-Value of Money*

Money is productive over time. Asset allocators forgo an opportunity cost when they choose to invest in one project over another. Determining where to allocate capital factors in other options that would yield a return. The `discount rate` is referred to as the `cost of capital`.

An obvious requirement of an investor is that the expected rate of return exceeds the discount rate or the cost of capital.

For example, if a corporation is borrowing money to finance a project, the discount rate or the cost of capital is at least equal to the cost of borrowing the money to finance the project.

In another example, an equity investor may decide to use a long-term average private equity return as the discount factor for investing in a private offering.

The discount rate is the rate of return an investor expects to receive in exchange for deferred payment.

1) *Discount Factor:*

The discount factor D_f is the periodic rate of return r discounted over n periods.

$$D_f = \frac{1}{(1+r)^n} \tag{1}$$

2) *Present Value:*

The present value PV is equal to the future cash flows CF over n periods discounted at a rate of r.

$$PV = CF * D_f = \frac{CF}{(1+r)^n} \tag{2}$$

3) *Future Value:*

The future value FV is equal to an initial investment cash flow CF_0 that is expected to return a rate of r over n periods.

$$FV = CF_0(1+r)^n \tag{3}$$

4) *Present Value of an Annuity (Fixed Cash Flows):*

The present value PV of an annuity is equal to a series of fixed cash flows CF received over n periods given a discount rate of return of r.

$$PV = \sum_{t=1}^{n} \frac{CF}{(1+r)^n} \tag{4}$$

5) *Present Value of Variable Cash Flows:*

The present value PV of a series of variable cash flows CF received over t time periods $(t_1, t_2, t_3, ..., t_n)$ given a discount rate of return r can be expressed as:

$$PV = \frac{CF_1}{1+r} + \frac{CF_2}{(1+r)^2} + ... + \frac{CF_n}{(1+r)^n} \tag{5}$$

Or,

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+r)^t} \tag{6}$$

6) *Net Present Value of an Investment:*

The net present value of NPV an investment is equal to the initial cash outlay CF_0, which is a negative cash flow, or the cost of the investment plus the future expected cash flows CF from the investment.

$$NPV = -CF_0 + \sum_{t=1}^{n} \frac{CF_t}{(1+r)^t} \tag{7}$$

B. *Bond Pricing Model*

1) *Coupon Payment:*

The coupon payment Coupon Payment is the `par` or `face` value of the bond Par_v multiplied by the interest rate r of the bond.

$$\text{Coupon Payment} = \text{Par}_v * r \tag{8}$$

2) Present Value of a Bond:

The present value of a bond B_v can be calculated by modifying the net present value formula, where instead of an initial cash outlay, the repayment of the `par value` (Par_v) of the bond (`face value`) is added to the future coupon payments (CP) (i.e. cash flows) over t time periods given a discount rate of r.

$$B_v = \text{Par}_v + \sum_{t=1}^{n} \frac{\text{CP}_t}{(1+r)^t} \tag{9}$$

3) Yield-to-Maturity (YTM):

The yield-to-maturity (YTM) for a bond can be calculated by taking the square root of the par value of the bond divided by the current price of the bond, minus one. The YTM represents the discount rate that makes the current market price of a bond equal to its calculated present value.

$$\text{YTM} = \sqrt[t]{\frac{\text{Par}_v}{\text{Par}_v + \sum_{t=1}^{n} \frac{\text{CP}_t}{(1+r)^t}}} - 1 \tag{10}$$

Or,

$$\text{YTM} = \sqrt[t]{\frac{\text{Par}_v}{B_v}} - 1 \tag{11}$$

Where B_v is the `Bond Current Value` using the `Bond Pricing Model` formula from above.

4) Macaulay's Duration:

Duration (D) is the weighted average of the time until each payment proportional to the present value of each payment, using the yield-to-maturity (y) as the discount rate, periodic coupon payment cash flows (CP_t), par value (Par_v), and the current bond value (B_v). Macaulay's Duration is a measure of a bond's sensitivity to changes in interest rates.

$$D = \left[\frac{\frac{1\text{CP}}{(1+y)^1} + \frac{2\text{CP}}{(1+y)^2} + \ldots + \frac{t(\text{CP} + \text{Par}_v)}{(1+y)^n}}{B_v} \right] \tag{12}$$

Or,

$$D = \frac{\sum_{t=1}^{n} \frac{t \times \text{CP}}{(1+y)^t} + \frac{n \times \text{Par}_v}{(1+y)^n}}{B_v} \tag{13}$$

where

t = Respective time period

CP = Periodic coupon payment

y = Periodic yield

n = Total number of periods

Par_v = Maturity (Par/Face) value

B_v = Current bond balue

5) Modified Duration:

Modified duration (MD) is an extension of Macaulay's Duration, and measures the average cash-weighted term to maturity of a bond.

$$\text{MD} = \frac{D}{1 + \left(\frac{y}{n}\right)} \tag{14}$$

C. Common Yield Measurements

1) Holding Period Yield (HPY):

$$\text{HPY} = \frac{P_1 - P_0 + D_1}{P_0} \tag{15}$$

2) Effective Annual Yield (EAY):

$$\text{EAY} = (1 + \text{HPY})^{\frac{365}{t}} - 1 \tag{16}$$

3) Bank Discount Yield (r_{BD}):

$$r_{BD} = \frac{\text{Par}_v - \text{Sale Price}}{\text{Par}_v} \times \frac{360}{t} \tag{17}$$

where

Par_v = Par value of the bond

t = Days to Maturity

4) Money Market Yield (CD Equivalent Yield) (r_{MM}):

$$r_{MM} = \text{HPY} \times \frac{360}{t} \tag{18}$$

Or,

$$r_{MM} = \frac{360 r_{BD}}{360 - (t)(r_{BD})} \tag{19}$$

5) Internal Rate of Return (IRR):

The Internal Rate of Return (IRR) is the discount rate that makes the net present value (NPV) of an investment equal to zero.

$$0 = \text{NPV} = \sum_{t=1}^{n} \frac{\text{CF}_t}{(1 + \text{IRR})^t} - \text{CF}_0 \tag{20}$$

The IRR must be calculated in reference to the periodic cash flow. An iterative solution is required to solve this equation, requiring adjusting the IRR until the NPV is equal to zero. This method is considered a *"guess-and-check"* algorithm.

D. *Statistical Methods*

1) *Population Mean:*

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N} \tag{21}$$

2) *Sample Mean:*

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{22}$$

where $n \leq N - 1$

3) *Geometric Mean:*

The geometric mean $(\overline{X}_G)$ is often used to average rates of return over time, particularly useful for time-series portfolio holding returns.

$$\overline{X}_G = \sqrt[t]{(1 + r_1)(1 + r_2)...(1 + r_t)} - 1 \tag{23}$$

Or,

$$\overline{X}_G = \left[\prod_{t=1}^{t} (1 + r_t) \right]^{\frac{1}{t}} - 1 \tag{24}$$

4) *Weighted Mean:*

$$\overline{X}_W = \sum_{i=1}^{n} w_i X_i \tag{25}$$

where $\sum_i w_i = 1$

5) *Harmonic Mean:*

The harmonic mean $(\overline{X}_H)$ is an inverse weighted-average of an observation proportional to its magnitude. One of it's limited applications is cost-averaging of investments.

$$\overline{X}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}} \tag{26}$$

Note: $\overline{X}_H$ should only be used at discrete time periods. Varying capital allocations over multiple time periods should use the weighted mean $\overline{X}_W$ formula.

6) *Quantiles:*

Quantiles are used to describe what fraction of a distribution is at or below a certain point. For example, quartiles divide the distribution into quarters to find the 25%, 50%, 75%, and 100% percentiles.

$$\text{Quartiles} = \frac{n}{4} \tag{27}$$

$$\text{Quintiles} = \frac{n}{5} \tag{28}$$

$$\text{Deciles} = \frac{n}{10} \tag{29}$$

$$\text{Percentiles} = \frac{n}{100} \tag{30}$$

where $X_i \leq X_{i+1}$ (Ascending order)

The index at which a percentile of the distribution is at or below (i_y) within an ascending distribution can be calculated as:

$$i_y = (n + 1)\left(\frac{y}{100} \right) \tag{31}$$

when $i_y \notin [\mathbb{W}]$(Whole Number Set), $i_y \geq 0$

Use *linear interpolation* between the ceiling and floor of i_y when the index value is not a whole number to find the value (P_y) at or below which y percent of the distribution lies.

$$P_y = X_{\lfloor i_y \rfloor} + \left(X_{i_y} - X_{\lfloor i_y \rfloor} \right)\left(X_{\lceil i_y \rceil} - X_{\lfloor i_y \rfloor} \right) \tag{32}$$

where

$\lceil i_y \rceil \in [\mathbb{W}]$

$\lfloor i_y \rfloor \in [\mathbb{W}]$

7) *Range:*

$$\text{Range} = X_{\max} - X_{\min} \tag{33}$$

8) *Deviation (Distance) from the Mean:*

$$X_i - \overline{X} \tag{34}$$

9) *Variance (Population & Sample):*

Population:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N} \tag{35}$$

Sample:

$$s^2 = \frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n - 1} \tag{36}$$

10) *Standard Deviation (Population & Sample):*

Population:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}} \tag{37}$$

Sample:

$$s = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n - 1}} \tag{38}$$

11) *Semideviation (Downside) Deviation:*

Remove all observations where $X_i > \overline{X}$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}} \qquad (39)$$

where $X_i \leq \overline{X}$

12) Coefficient of Variation (C_V):

The coefficient of variation measures the amount of risk (σ) per unit of mean return (μ), or the measure of *risk-to-return*.

$$C_V = \frac{\sigma}{\mu} \qquad (40)$$

Or, for <u>sample</u> observations

$$C_V = \frac{s}{\overline{X}} \qquad (41)$$

13) Sharpe Ratio:

The Sharpe Ratio is a common measurement for *return-to-risk* ($\frac{1}{C_V}$) with an adjusted mean return in excess of a risk-free rate. The larger the Sharpe ratio, the more attractive an asset's returns are to an investor, all else equal.

$$S_r = \frac{\mu_p - r_f}{\sigma_p} \qquad (42)$$

where

μ_p = Geometric Mean of Portfolio Returns

r_f = Risk-free Rate

σ_p = Standard Deviation of Portfolio Returns

14) Standard ("z") Scores:

The z-score represents the distance between the raw score x and the mean μ in units of standard deviation. The z-score is negative when the raw score is below the mean, and postive when above the mean.

$$z = \frac{x - \mu}{\sigma} \qquad (43)$$

15) Skewness:

$$S_K = \left[\frac{n}{(n-1)(n-2)}\right]\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^3}{s^3} \qquad (44)$$

Or,

$$S_K \approx \left(\frac{1}{n}\right)\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^3}{s^3} \qquad (45)$$

when $n \to \infty$

16) Sample Excess Kurtosis:

$$K_E = \frac{n(n+1)}{(n-1)(n-2)(n-3)}\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^4}{s^4} - \frac{3(n-1)^2}{(n-2)(n-3)}$$

Or,

$$K \approx \frac{1}{n}\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^4}{S^4} - 3 \qquad (46)$$

when $n \to \infty$

E. Probability Theory

1) Definition of a Probability:

- The probability of an event $P(E)$ is between 0 and 1

$$0 \leq P(E) \leq 1 \qquad (47)$$

- The sum of probabilities for any mutually exclusive and exhaustive set of events is equal to 1

$$\sum_{i=0}^{E} P(E_i) = 1 \qquad (48)$$

where

$P(E_i \cup E_{i+1}) = P(E_i) + P(E_{i+1}),$
$P(E_i \cap E_{i+1}) = 0$

2) Unconditional (Marginal) Probability:

An unconditional probability is the probability of an independent event occuring.

$$P(A) \qquad (49)$$

3) Conditional Probability:

A conditional probability is the probability of an event occuring given another event has occured.

$$P(A|B) \qquad (50)$$

where

$P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0$

4) Multiplicative Probability:

$P(AB) = P(A|B)P(B) = P(A)P(B)$

5) Additive Probability:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB) \qquad (51)$$

6) Independent Events:

Events A and B are independent, iff:

$$P(A|B) = P(A) \vee P(B|A) = P(B) \qquad (52)$$

7) Expected Value $E(X)$:

The expected value $E(X)$ of a random variable X is the probability weighted average of the observed outcomes of the random variable.

$$E(X) = \sum_{i=1}^{n} P(X_i)X_i \qquad (53)$$

8) *Variance of a Random Variable:*

The variance of a random variable is the expected value $E(X)$ of the squared deviations from the random variable's expected value.

$$\sigma^2(X) = \text{Var}(X) = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2 \qquad (54)$$

9) *Standard Deviation of a Random Variable:*

$$\sigma(X) = \sqrt{\sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2} \qquad (55)$$

10) *Conditional Expected Values:*

$$E(X|S) = \sum_{i=1}^{n} P(X_i \mid S)X_i \qquad (56)$$

11) *Total Probability Rule for Expected Value:*

Converts a conditional probability into an unconditional probability.

$$E(X) = \sum_{i=1}^{n} E(X_i \mid S_i)P(S_i) \qquad (57)$$

12) *Expected Portfolio Return:*

$$\text{ER}_p = E\left(\sum_{i=1}^{n} w_i R_i\right) \qquad (58)$$

13) *Covariance of Portfolio Holding Returns:*

$$\sigma(R_A, R_B) = \mathbb{E}[(R_A - \text{ER}_A)(R_B - \text{ER}_B)] \qquad (59)$$

Or,

$$\sigma(R_A, R_B) = \sum_{i=1}^{n} \sum_{j=1}^{n} P\left(R_{A_i}, R_{B_j}\right)\left(R_{A_i} - \text{ER}_A\right)\left(R_{B_j} - \text{ER}_B\right)$$

14) *Expected Portfolio Variance:*

$$\sigma^2(R_p) = \mathbb{E}\left\{\left[R_p - \text{ER}_p\right]^2\right\} \qquad (60)$$

Or,

$$\sigma^2(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma\left(R_i, R_j\right)$$

15) *Pearson's Correlation Coeffecient:*

For a population:

$$\rho(R_A, R_B) = \frac{\mathbb{E}[(R_A - \text{ER}_A)(R_B - \text{ER}_B)]}{(\sigma R_A)(\sigma R_B)} \qquad (61)$$

Or, for a sample:

$$r_{R_A R_B} = \frac{\sum_{i=1}^{n}(R_A - \text{ER}_A)(R_B - \text{ER}_B)}{\sqrt{\sum_{i=1}^{n}(R_A - \text{ER}_A)^2}\sqrt{\sum_{i=1}^{n}(R_B - \text{ER}_B)^2}}$$

where

$R_A \wedge R_B$ = Random Variables

$\text{ER}_A \wedge \text{ER}_B$ = Probability weighted average expected values of random variables

16) *Baye's Forumla:*

$P(E|I)$ can be expressed as the probability of new information given an event, divided by the unconditional probability of the new information, times the prior probability of the event. Baye's theorem uses the occurence of an event to infer the probability of the scenario generating it, and therefore is often described as an inverse probability.

$$P(E|I) = \frac{P(I|E)}{P(I)} \cdot P(E) \qquad (62)$$

where

E = Event

I = New Information

17) *Combinations (Binomial) and Permutations:*

A combinatorial is the number of ways to choose r options from a total of n options, when the order in which the r options are listed *does not* matter.

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} \qquad (63)$$

A permutation is the number of ways to choose r options from a total of n options, when the order in which the r options are listed *does* matter.

$$_nP_r = \frac{n!}{(n-r)!} \qquad (64)$$

18) *Binomial (Discrete Random) Distributions:*

Binomial Random Variable

$$X \sim B(n, p) \qquad (65)$$

Bernoulli Random Variable

$$Y \sim B(1, p) \qquad (66)$$

where

p = The probability of success (constant for all trials)

n = The number of trials (or occurences)

The probability of x successes in n trials is calculated as:

$$p(x) = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x} \qquad (67)$$

19) *Continuous Distributions:*