Lecture Notes

Calculus 3

Fall 2020

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Contents

Chapter 1

Vectors and the Geometry of Space

DATE: 2020-09-01

ANNOUNCEMENTS:
Review chapter 12.1 to 12.5 and complete assignments

1.1 Three-Diensional Coordinate Systems

$$\boxed{|P_1P_2| = \sqrt{(x_1 - x_1)^2 + (y_2 - y_2)^2 + (z_2 - z_1)^2}} \tag{1.1}$$

Dinstance Formula in Three Dimensions

$$\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2} \tag{1.2}$$

Equation of a sphere with a center $C(h, k, l)$ and a radius r

$$\boxed{|\overline{a}| = \sqrt{a_1^2 + a_2^2}} \tag{1.3}$$

$$\boxed{|\overline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}} \tag{1.4}$$

Length of a Vector

$$\boxed{\overline{a} + \overline{b} = \langle a_1 + b_1, a_2 + b_2 \rangle} \tag{1.5}$$

Vector Addition

$$\boxed{\overline{a} - \overline{b} = \langle a_1 - b_1, a_2 - b_2 \rangle} \tag{1.6}$$

Vector Subtraction

$$\boxed{c\overline{a} = \langle ca_1, ca_2 \rangle} \tag{1.7}$$

Scalar Multiplication

$$\boxed{\overline{a} \cdot \overline{b} = a_1b_1 + a_2b_2 + a_3b_3} \tag{1.8}$$

Dot Product

$$\begin{aligned}
\overline{a} \cdot \overline{a} &= |\overline{a}|^2 \\
\overline{a} \cdot \overline{b} &= \overline{b} \cdot \overline{a} \\
\overline{a} \cdot (\overline{b} + \overline{c}) &= \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c} \\
(c\overline{a}) \cdot \overline{b} &= c(\overline{a} \cdot \overline{b} = \overline{a} \cdot (c\overline{b}) \\
0 \cdot \overline{a} &= 0
\end{aligned}$$

Theorem 1 (Dot product)**.** $\overline{a} \cdot \overline{b} = |a|\,|b| \cos\theta$

Corollary 1. $\cos\theta = \frac{\overline{a} \cdot \overline{b}}{|\overline{a}||\overline{b}|}$

Direction Angles

$$\begin{aligned}
\cos\alpha &= \frac{a_1}{|\overline{a}|} \\
\cos\beta &= \frac{a_2}{|\overline{a}|} \\
\cos\gamma &= \frac{a_3}{|\overline{a}|}
\end{aligned}$$

Projections

$$\boxed{comp_a\overline{b} = \frac{\overline{a} \cdot \overline{b}}{|\overline{a}|}} \tag{1.9}$$

Scalar projection of b onto a

$$\boxed{proj_a\overline{b} = \left(\frac{\overline{a} \cdot \overline{b}}{|\overline{a}|}\right) \frac{\overline{a}}{|\overline{a}|} = \frac{\overline{a} \cdot \overline{b}}{|\overline{a}|}\overline{a}} \tag{1.10}$$

Vector projection of b onto a

$$\boxed{\bar{a} \times \bar{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle} \tag{1.11}$$

Cross product

Theorem 2. $\left|\bar{a} \times \bar{b}\right| = |\bar{a}| \left|\bar{b}\right| \sin\theta$

Theorem 3 (parallel)**.** $\bar{a} \times \bar{b} = 0 \implies$ *a and b are parallel*

$$\begin{aligned}
\bar{a} \times \bar{b} &= -\bar{b} \times \bar{a} \\
(c\bar{a}) \times \bar{b} &= c(\bar{a} \times \bar{b}) = \bar{a} \times (c\bar{b}) \\
\bar{a} \times (\bar{b} + \bar{c}) &= \bar{a} \times \bar{b} + \bar{a} \times \bar{c}
\end{aligned}$$

1.2 Equations of Lines and Planes

1.2.1 Lines

$$\boxed{\bar{r} = \bar{r}_0 + t\bar{v}} \tag{1.12}$$

Vector equation of a line

r vector that traces the line

$\bar{r}_0$ given point on the line ($\bar{r}_0 = \langle x_0, y_0, z_0 \rangle$)

$\bar{v}$ vector in the direction of $\bar{r}$

t parameter

$$\implies \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Parametric equations for a line through a point (x_0, y_0, z_0) parallel to the vector $\bar{v} = \langle a, b, c \rangle$

$$\begin{aligned}
x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{aligned}$$

Chapter 2

Cylinders and Quatric Surfaces

DATE: 2020-09-03

ANNOUNCEMENTS:

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Definition 1 (Cylinder)**.** *A cylinder is a surface that consists of al lines (called rulings) that are parallel to a given line and pass through a given plane curve.*

Definition 2 (plane curve)**.** *A plane curve is any graph in a 2-d space.*

2.1 Quadric Surfaces

Definition 3 (Quadric surface)**.** *The most general form is* $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ *where A-J are constants.*

Using rotation and translation will be brought into one of the two standard forms:

Definition 4 (Standard quadric forms)**.**

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

Example 1 (What we do with this)**.** $z = \frac{x^2}{9} + \frac{y^2}{4}$

Solution 1. *You should know this is an Elliptic Paraboloid and what that looks like.*

What if you dont recognize the quadric surface form?

Looking at the equation, figure out **TRACES**, either **horizontal** (z = constant) or **vertical**

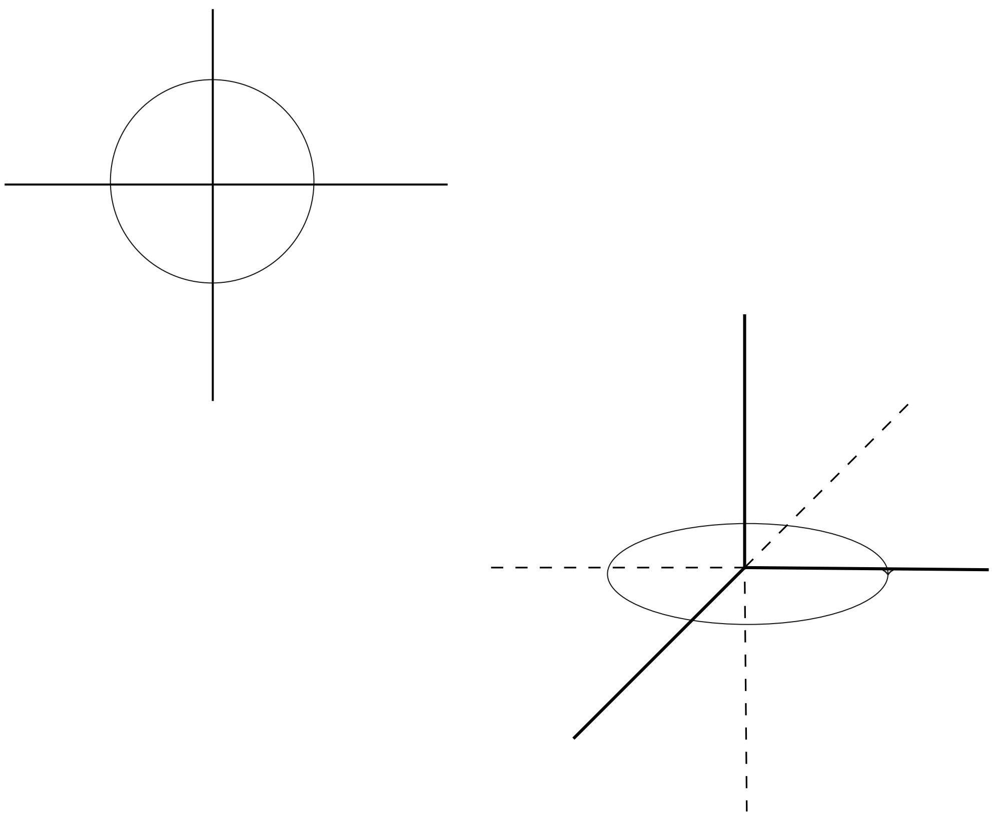

Figure 2.1: How to draw a circle in Calc 3

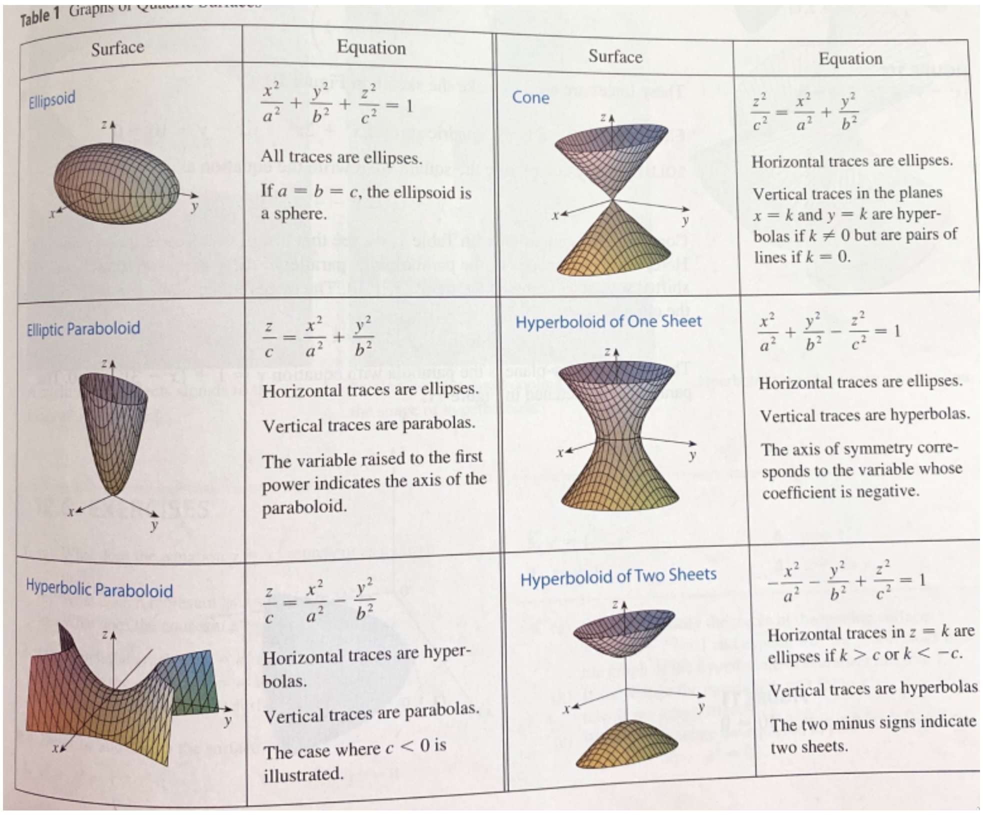

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone	$\dfrac{z^2}{c^2}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid	$\dfrac{z}{c}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\dfrac{z}{c}=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets	$-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas The two minus signs indicate two sheets.

Figure 2.2: Graphs of Quadric Surfaces

Chapter 3

Vector Functions

DATE: 2020-09-06

ANNOUNCEMENTS:

3.1 Vector Functions and Space Curves

Vectors

$$\begin{aligned}
\overline{r(t)} &= \langle f(t), g(t), h(t)\rangle \\
&= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \\
\overline{r(t)} &= \left\langle t^2+1, \frac{1}{t-1}, \sqrt{t+3}\right\rangle \\
\overline{r(2)} &= \left\langle 5, 1, \sqrt{5}\right\rangle
\end{aligned}$$

Domain

$$\begin{aligned}
t^2+1 &\rightarrow \text{ polonomial, Domain: } (-\infty, \infty) \\
\frac{1}{t+1} &\rightarrow \text{ Rational, Domain: } (-\infty, 1) \cup (1, \infty) \\
\sqrt{t+3} &\rightarrow \text{ Radical, Domain: } [-3, \infty) \\
\text{Domain of } \overline{r(t)} &= [-3, 1) \cup (1, \infty)
\end{aligned}$$

Example 2 (Domain)**.**

$$\overline{r(t)} = \left\langle \cos t, \ln(t+r), \frac{t}{t^2-23}\right\rangle$$

$$\begin{aligned}\cos t &\to (-\infty, \infty) \\ \ln(t+4) &\to (-4, \infty) \\ \frac{t}{t^2-25} &\to (-\infty,-5)\cup(-5,5)\cup(5,\infty) \\ \text{Domain } &= (-4,5)\cup(5,\infty)\end{aligned}$$

Graphs

2-D plane curve

3-D space curve

Plane curve

$$\overline{r(t)} = \langle t-3, t^2 \rangle$$

...	
$t=-2$	$\langle -5, 4\rangle$
$t=-1$	$\langle -4, 1\rangle$
$t=0$	$\langle -3, 0\rangle$
$t=1$	$\langle -2, 1\rangle$
$t=2$	$\langle -1, 4\rangle$
...	

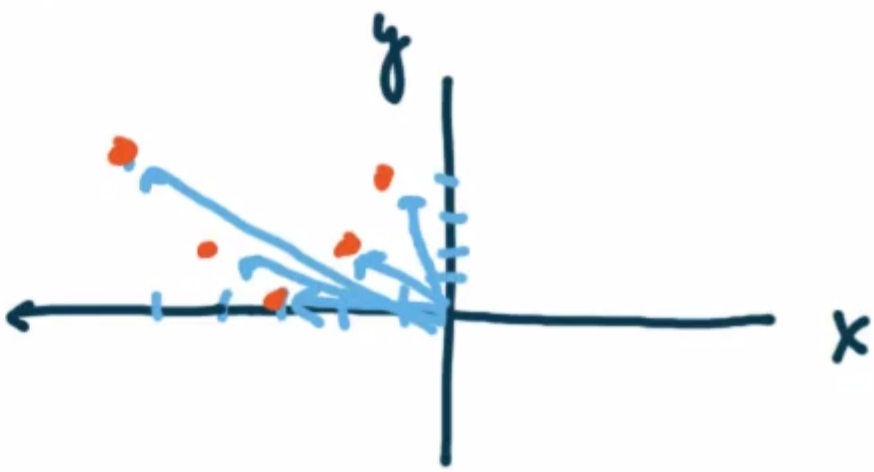

Figure 3.1: 2-D vector graph

Space Curve

$$\overline{r(t)} = \langle 3+t, 4-2t, t-5 \rangle$$

$$x = 3 + t$$
$$y = 4 - 2t$$
$$z = t - 5$$

(Parametric form)

$$\overline{r(t)} = \langle 3, 4t, -5 \rangle + \langle t, -2t, t \rangle$$
$$\left(\overline{r(t)} = \langle 3, 4t, -5 \rangle + t \langle 1, -2, 1 \rangle \right)$$

(Equation of line in 3-D)

$$x - 3 = \frac{y - 4}{-2} = z + 5$$

(Symetric form)

3.1.1 Limit

Definition 5 (Limit of a Vector)**.** *If* $\overline{r(t)} = \langle f(t), g(t), h(t), \rangle$, *then*

$$\lim_{t \to a} \overline{r(t)} = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example 3 (Limit of a Vector)**.**

$$\overline{r(t)} = \left\langle \frac{\sin^2 t}{t^2}, te^{-t}, t^2 + 1 \right\rangle$$

$$\lim_{t \to 0} \overline{r(t)} = \lim_{t \to 0} \left\langle \frac{\sin^2 t}{t^2}, te^{-t}, t^2 + 1 \right\rangle$$
$$= \left\langle \lim_{t \to 0} \frac{\sin^2 t}{t^2}, \lim_{t \to 0} te^{-t}, \lim_{t \to 0} t^2 + 1 \right\rangle$$
$$= \langle a, b, c \rangle$$

$$a = \frac{0}{0} (\text{ \textit{L'H Rule} })$$
$$a = \lim_{t \to 0} \frac{2 \sin t \cos t}{2t}$$
$$a = \lim_{t \to 0} \frac{\sin t \cos t}{t}$$
$$a = \frac{0}{0} (\text{ \textit{L'H Rule} })$$
$$a = \lim_{t \to 0 \frac{-\sin^2 t + \cos^2 t}{1}}$$
$$a = 1$$
$$b = 0$$
$$c = 1 \implies$$

$$= \boxed{\langle 1, 0, 1 \rangle}$$

DATE: 2020-09-06
ANNOUNCEMENTS:

3.2 Derivatives and Integrals of Vector Functions

Definition of the Derivative

Definition 6 (Derivative)**.**

$$\frac{d\overline{r}}{dt} = \overline{r}'(t) = \lim_{h\to 0} \frac{\overline{r}(t+h) - \overline{r}(t)}{h}$$

$$\overline{r}(t) = \langle f(t), g(t), m(t) \rangle$$

1. $\overline{r}(t+h) = \langle r(t+h), g(t+h), m(t+h) \rangle$
2. $\overline{r}(t+h) - \overline{r}(t)$
$$= \langle f(t+h) - f(t), g(t+h) - g(t), m(t+h) - m(t) \rangle$$
3. multiply $\frac{1}{h}$
$$= \frac{1}{h} \langle f(t+h) - f(t), g(t+h) - g(t), m(t+h) - m(t) \rangle$$
$$= \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{m(t+h) - m(t)}{h} \right\rangle$$
4. $\overline{r}'(t) =$
$$\lim_{h\to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{m(t+h) - m(t)}{h} \right\rangle$$
$$= \left\langle \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h\to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h\to 0} \frac{m(t+h) - m(t)}{h} \right\rangle$$
$$\boxed{\overline{r}'(t) = \langle f'(t), g'(t), m'(t) \rangle}$$

Example 4 (Tangent line)**.** *find the parametric equation for the line tangent to*

$$\overline{r}(t) = \left\langle t^2 + 1, 4\sqrt{t}, e^{t^2 - t} \right\rangle$$

at the point $(2, 4, 1)$

Solution 2 (Tangent line)**.** *Need:*

1. *point* $(2, 4, 1)$
2. *direction vector*
$$\overline{r}'(t) = \left\langle 2t, 2t^{-\frac{1}{2}}, e^{t^2 - t}(2t - 1) \right\rangle$$

3. *parameter value*

$$\overline{r}(t) = (2,4,1) \implies t = 1 \implies$$
$$\overline{r}'(1) = \langle 2,2,1\rangle$$
$$\boxed{\langle 2,4,1\rangle + t\langle 2,2,1\rangle}$$
$$\boxed{\langle 2+2t, 4+2t, 1+t\rangle}\ \textit{(vector)}$$

$$x = 2 + 2t$$
$$y = 4 + 2t$$
$$z = 1 + t$$

3.2.1 The Unit Tangent Vector

Definition 7 (Unit Tangent Vector)**.**

$$\overline{T}(t) = \frac{1}{|\overline{r}'(t)|}\overline{r}'(t)$$

Example 5 (Unit Tangent Vector)**.**

$$\overline{r}(t) = \left\langle t^2 + 3t, 3t - 8, t^3 - 1\right\rangle$$

Find $\overline{T}(2)$

Solution 3 (Unit Tangent Vector)**.**

$$\overline{r}'(t) = \left\langle 2t + 3, 3, 3t^2\right\rangle$$
$$\overline{r}'(2) = \langle 7, 3, 12\rangle$$
$$|\overline{r}'(2)| = \sqrt{49 + 9 + 144}$$
$$|\overline{r}'(2)| = \sqrt{202}$$
$$\overline{T}(2) = \frac{1}{\sqrt{202}}\langle 7, 3, 12\rangle$$
$$\overline{T}(2) = \boxed{\left\langle \frac{7}{\sqrt{202}}, \frac{3}{\sqrt{202}}, \frac{12}{\sqrt{202}}\right\rangle}$$

Theorem 4. *Suppose that* $\overline{u}$ *and* $\overline{v}$ *are differentiable vector functions, c is a scalar, and f is a real valued function*

1. $\frac{d}{dt}[\overline{u}(t) + \overline{v}(t)] = \overline{u}'(t) + \overline{v}'(t)$
2. $\frac{d}{dt}[c\overline{u}(t)] = c\overline{u}'(t)$
3. $$\frac{d}{dt}[f(t)\overline{u}(t)] = \langle f(t)u_1'(t) + f'(t)u_1(t), f(t)u_2'(t) + f'(t)u_2(t), f(t)u_3'(t) + f'(t)u_3(t)\rangle$$ $$= f(t)\overline{u}'(t) + f'(t)\overline{u}(t)$$
4. $$\frac{d}{dt}\left[\overline{u}(t)\cdot\overline{v}(t)\right] = \frac{d}{dt}\left[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)\right]$$ $$= \overline{u}(t)\cdot\overline{v}'(t) + \overline{u}'(t)\cdot\overline{v}(t)$$
5. $\frac{d}{dx}\left[\overline{u}(t)\times\overline{v}(t)\right] = \overline{u}(t)\times\overline{v}'(t) + \overline{u}'(t)\times\overline{v}(t)$
6. $\frac{d}{dt}\left[\overline{u}(f(t))\right] = f'(t)\overline{u}'(f(t))$

Example 6 (Prove the vector differentiation theorems)**.** *Exercise for the reader.*

Date: 2020-09-06
Announcements:

3.2.2 Integrals

Definition 8 (Integral)**.**

$$\int_a^b \overline{r}(t)dt = \lim_{n\to\infty}\sum_{i=1}^{n}\overline{r}(t_i^*)\Delta t$$

$$= \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

Example 7 (Integral)**.**

$$\int_1^4 \left\langle 4t^{\frac{3}{2}}, t^2, \cos t \right\rangle dt$$

$$= \left\langle \int_1^4 4t^{\frac{3}{2}}dt, \int_1^4 t^2 dt, \int_1^4 \cos t dt \right\rangle$$

$$= \frac{8}{5}t^{\frac{5}{2}}\bigg|_1^4, \frac{1}{3}t^3\bigg|_1^4, \sin t\bigg|_1^4$$

$$= \left\langle \frac{256}{5} - \frac{8}{5}, \frac{64}{3} - \frac{1}{3}, \sin 4 - \sin 1 \right\rangle$$

$$= \boxed{\left\langle \frac{248}{5}, 21, \sin 4 - \sin 1 \right\rangle}$$

Example 8 (Integral)**.**

$$\int \left\langle te^{t^2}, te^t, \cos(5t) \right\rangle dt$$

$$= \left\langle \int te^{t^2}dt, \int te^t dt, \int \cos(5t)dt \right\rangle$$

$$= \left\langle \frac{1}{2}e^{t^2} + C_1, te^t - \int e^t dt, \frac{1}{5}\sin(5t) + C_3 \right\rangle$$

$$= \left\langle \frac{1}{2}e^{t^2} + C_1, te^t - -e^t + C_2, \frac{1}{5}\sin(5t) + C_3 \right\rangle$$

Sometimes, you want to pull out the constants:

$$= \left\langle \frac{1}{2}e^{t^2}, te^t - e^t, \frac{1}{5}\sin(5t) \right\rangle + \langle C_1, C_2, C_3 \rangle$$

$$= \left\langle \frac{1}{2}e^{t^2}, te^t - e^t, \frac{1}{5}\sin(5t) \right\rangle + \overline{C}$$

Example 9 (Integral)**.**

$$r'(t) = \left\langle t^2, e^{3t}, \sqrt{t} \right\rangle$$
$$\overline{r}(0) = \langle 4, 1, 5 \rangle$$

Find $\overline{r}(t)$

$$\begin{aligned}
\overline{r}(t) &= \int \left\langle t^2, e^{3t}, \sqrt{t} \right\rangle dt \\
&= \left\langle \int t^2 dt, \int e^{3t} dt, \int \sqrt{t} dt \right\rangle \\
&= \left\langle \frac{1}{3}t^3 + C_1, \frac{1}{3}e^{3t} + C_2, \frac{2}{3}t^{\frac{3}{2}} + C_3 \right\rangle
\end{aligned}$$

Use initial condition:

$$\begin{aligned}
&= \left\langle C_1, \frac{1}{3} + C_2, C_3 \right\rangle = \langle 4, 1, 5 \rangle \implies \\
C_1 &= 4 \\
C_2 &= \frac{2}{3} \\
C_3 &= 5 \implies \\
\overline{r}(t) &= \boxed{\left\langle \frac{1}{3}t^3 + 4, \frac{1}{3e^{3t} + \frac{2}{3}, \frac{2}{3}t^{\frac{3}{2}} + 5} \right\rangle}
\end{aligned}$$

Date: 2020-09-11
Announcements:

3.3 13.3 Arc Length and Curvature

3.3.1 Review

dx form

$$s = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

dy form

$$s = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} dy$$

dt form

$$s = \int_{t_0}^{t_1} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

3.3.2 Calc 3 Version

Definition 9 (Arclength of a Space Curve)**.** .

Larange form

$$s = \int_{t_0}^{t_1} \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$$

Leibnitz form

$$s = \int_{t_0}^{t_1} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt$$

Vector form

$$\begin{aligned} \overline{r}(t) &= \langle f(t), g(t), h(t) \rangle \implies \\ \overline{r}'(t) &= \langle f'(t), g'(t), h'(t) \rangle \implies \\ |\overline{r}'(t)| &= \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} \\ s &= \int_{t_0}^{t_1} |\overline{r}'| \, dt \end{aligned}$$

Example 10 (Find arclength$0 \leq t \leq 1$)**.**

$$\begin{aligned}
\overline{r}'(t) &= \langle 2, 2t, t^2 \rangle \\
|\overline{r}'(t)| &= \sqrt{4 + 4t^2 + t^4} \\
&= \sqrt{(t^2 + 2)^2} \\
&= t^2 + 2 \\
s &= \int_0^1 t^2 + 2dt \\
&= \frac{1}{3}t^3 + 2t \Big|_0^1 \\
&= \boxed{\frac{7}{3}}
\end{aligned}$$

Definition 10 (Unit Vector)**.**

$$\boxed{\overline{T}(t) = \frac{1}{|\overline{r}'(t)|}\overline{r}'(t)} \tag{3.1}$$

Unit (Tangent) Vector

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Definition 11 (Principle Unit Normal Vector)**.**

$$\boxed{\overline{N}(t) = \frac{1}{\left|\overline{T}'(t)\right|}\overline{T}'(t)} \tag{3.2}$$

Normal Vector

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Definition 12 (Binormal Vector)**.**

$$\boxed{\overline{B}(t) = \overline{T}(t) \times \overline{N}(t)} \tag{3.3}$$

Binormal Vector

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Example 11.

$$\begin{aligned}
\overline{r}(t) &= \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle \\
\overline{r}'(t) &= \langle 2, 2t, t^2 \rangle \\
|\overline{r}'(t)| &= t^2 + 2 \\
\overline{T}(t) &= \frac{1}{t^2 + 2} \langle 2, 2t, t^2 \rangle
\end{aligned}$$

$$\overline{T}'(t) = \frac{1}{t^2 + 2} \langle 0, 2, 2t \rangle - (t^2 + 2)^{-2}(2t) \langle 2, 2t, t^2 \rangle$$

$$\overline{T}'(t) = \left\langle 0, \frac{2}{t^2 + 2}, \frac{2t}{t^2 + 2} \right\rangle + \left\langle -\frac{4t}{(t^2 + 2)^2}, -\frac{4t^2}{(t^2 + 2)^2}, -\frac{2t^3}{(t^2 + 2)^2} \right\rangle$$

$$\overline{T}'(t) = \left\langle -\frac{4t}{(t^2+2)^2}, -\frac{-2t^2+4}{(t^2+2)^2}, \frac{4t}{(t^2+2)^2} \right\rangle$$
$$\overline{T}'(t) = \frac{1}{(t^2+2)^2} \left\langle -4t, -2t^2+4, 4t \right\rangle$$

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Date: 2020-09-12

Announcements:

3.3.3 Curvature

Definition 13 (Curvature)**.** .

- *A measure of how quickly the curve changes direction at that point*
- *The magnitude of the rate of change of the unit tangent vector with respect to arc length.*

$$\kappa = \left|\frac{d\overline{T}}{ds}\right|$$

•

Problem: don't want to use the arclength.

$$\overline{T}(t) \to \frac{d\overline{T}}{dt}$$

$$\frac{d\overline{T}}{dt} \cdot \frac{dt}{ds} \to$$

$$\left|\frac{d\overline{T}}{dt}\right|$$

$$\frac{\frac{d\overline{T}}{ds}}{\frac{ds}{dt}}$$

$$\implies \kappa = \frac{\frac{d\overline{T}}{dt}}{\frac{ds}{dt}}$$

Arclength so far function:

$$s = \int_a^t |\overline{r}'(u)|\, du$$

$$\frac{ds}{st} = |\overline{r}'(t)|$$

$$\left|\frac{ds}{st}\right| = |\overline{r}'(t)|$$

Definition 14 (Curvature)**.**

$$\kappa = \frac{\left|\overline{T}'(t)\right|}{|\overline{r}'(t)|}$$

•

Example 12 (Curvature).
$$\overline{r}(t) = \langle 3t+2, t, t-7\rangle$$
, find κ •

Solution 4 (Curvature).
$$r'(t) = \langle 3,1,1\rangle$$
$$|r'(t)| = \sqrt{9+1+1} = \sqrt{11}$$
$$\overline{T} = \frac{1}{\sqrt{11}}\langle 3,1,1\rangle$$
$$\overline{T}' = \frac{1}{\sqrt{11}}\langle 0,0,0\rangle$$
$$\overline{T}' = \langle 0,0,0\rangle$$
$$\left|\overline{T}'\right| = 0$$
$$\kappa = \frac{0}{\sqrt{11}} = 0$$

•

Example 13 (Curvature). *let* $\overline{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t}\right\rangle$*, Find* κ •

Solution 5 (Curvature).
$$r'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$$
$$|r'(t)| = \sqrt{2+e^{2t}+e^{-2t}}$$
$$|r'(t)| = \sqrt{e^{2t}+2+e^{-2t}}$$
$$|r'(t)| = \sqrt{(e^t+e^{-t})^2}$$
$$|r'(t)| = e^t+e^{-t}$$
$$\overline{T}(t) = \frac{1}{e^t+e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$$
$$\overline{T}(t) = \frac{e^t}{e^{2t}+1}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$$
$$\overline{T}(t) = \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle$$
$$\overline{T}'(t) = \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle + -1(e^{2t}+1)^{-2}2e^{2t}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle$$
$$\overline{T}'(t) = \left\langle \frac{\sqrt{2}e^t}{e^{2t}+1}, \frac{2e^{2t}}{e^{2t+1}}, 0\right\rangle + \left\langle \frac{-2\sqrt{2}e^{3t}}{(e^{2t}+1)^2}, \frac{-2e^{4t}}{(e^{2t}+1)^2}, \frac{2e^{2t}}{(t^{2t+1)^2}}\right\rangle$$
$$\overline{T}'(t) = \left\langle \frac{\sqrt{2}e^{3t}+\sqrt{2}e^t-2\sqrt{2}e^{3t}}{(e^{2t}+1)^2}, \dots\right\rangle$$
$$\overline{T}'(t) = \left\langle \frac{-\sqrt{2}e^{3t}+\sqrt{2}e^t}{(e^{2t}+1)^2}, \dots\right\rangle$$
$$\overline{T}'(t) = \left\langle \frac{-\sqrt{2}e^{3t}+\sqrt{2}e^t}{(e^{2t}+1)^2}, \frac{2e^{2t}}{(e^{2t}+1)^2}, \frac{2e^{2t}}{(e^{2t}+1)^2}\right\rangle$$
$$\overline{T}'(t) = \frac{\sqrt{2}e^t}{(e^{2t}+1)^2}\left\langle 1-e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t\right\rangle$$

$$\left|c\overline{V}\right| = |c|\,|\overline{v}|$$

$$\left|\overline{T}'(t)\right| = \frac{\sqrt{2}e^t}{(e^{2t}+1)^2}\sqrt{1-2e^{2t}+e^{4t}+2e^{2t}+2e^{2t}}$$

$$\left|\overline{T}'(t)\right| = \frac{\sqrt{2}e^t}{(e^{2t}+1)^2}\sqrt{e^{4t}+2e^{2t}+1}$$

$$\left|\overline{T}'(t)\right| = \frac{\sqrt{2}e^t}{(e^{2t}+1)^2}\sqrt{(e^{2t}+1)^2}$$

$$\left|\overline{T}'(t)\right| = \frac{\sqrt{2}e^t}{e^{2t}+1}$$

$$\kappa = \frac{\frac{\sqrt{2}e^t}{e^{2t}+1}}{e^t+e^{-t}}$$

$$\kappa = \frac{\frac{\sqrt{2}e^{2t}}{e^{2t}+1}}{e^{2t}+1}$$

$$\kappa = \frac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$$

•

This example worked, but is a contrived textbook version of the problem. A better formula is still needed..

Date: 2020-09-12
Announcements:

3.3.4 Another formula for curvature

$$\kappa = \frac{\left|\overline{T}'\right|}{|\overline{r}'|}$$

$$\overline{T} = \frac{1}{|\overline{r}'|}\overline{r}' \rightarrow \overline{r}' = |\overline{r}'|\,\overline{T}$$

$$\overline{r}''(t) = |\overline{r}'|\,\overline{T}' + |\overline{r}'|'\,\overline{T}$$

$$\overline{r}' \times \overline{r}'' = \overline{r}' \times \left(|\overline{r}'|\,\overline{T}' + |\overline{r}'|'\,\overline{T}\right)$$

$$= |\overline{r}'|\left(\overline{r}' \times \overline{T}'\right) + |\overline{r}'|'\left(\overline{r}' \times \overline{T}\right)$$

Second term is equivalent to a scallar times $(\overline{r}' \times \overline{r}')$, evaluates to zero vector.

$$\overline{r}' \times \overline{r}'' = |\overline{r}'|\,(\overline{r}' \times \overline{T}')$$

Multiply by one: $\frac{|\overline{r}'|}{|\overline{r}'|}$

$$\overline{r}' \times \overline{r}'' = |\overline{r}'|^2\,(\overline{T} \times \overline{T}')$$

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T} \times \overline{T}'\right|$$

$\left|\overline{a} \times \overline{b}\right| = |\overline{a}|\left|\overline{b}\right|\sin\theta$
$\overline{T}$ and $\overline{T}'$ are $\perp$

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T}\right|\left|\overline{T}'\right|\sin 90$$

$\overline{T}$ is a unit vector and $\sin 90$ is 1

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T}'\right|$$

$$\left|\overline{T}'\right| = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$

$$\kappa = \frac{\overline{T}'}{\overline{r}'} = \frac{|\overline{r}' \times \overline{r}''|}{\left|\overline{r}^3\right|}$$

$$\boxed{\kappa = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}}$$

Example 14 (Curvature)**.** *Given* $\overline{r} = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle$, *find* κ •

Solution 6 (Curvature)**.**

$$\overline{r}' = \left\langle 2, 2t, t^2 \right\rangle$$
$$|\overline{r}'| = \sqrt{4+4t^2+t^4}$$
$$= 2+t^2$$
$$\overline{r}'' = \langle 0, 2, 2t\rangle$$
$$\overline{r}' \times \overline{r}'' = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{bmatrix}$$
$$= \left\langle 2t^2, -4t, 4 \right\rangle$$
$$|\overline{r}' \times \overline{r}''| = \sqrt{4t^4+16t^2+4}$$
$$= 2(t^2+1)$$
$$\kappa = \frac{2(t^2+1)}{(2+t^2)^3}$$

•

3.3.5 Curvature of a plane curve (2-D)

Definition 15 (Curvature of a plane curve)**.**

$$\kappa(x) = \frac{|f'(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}$$

•

Curvature of a plane curve.

$$\boxed{\kappa = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}}$$

$$y = f(x)$$
$$\overline{r}(t) = \langle t, f(t), 0\rangle$$
$$\overline{r}'(t) = \langle 1, f'(t), 0\rangle$$
$$\overline{r}''(t) = \langle 0, f''(t), 0\rangle$$
$$|\overline{r}'| = \sqrt{1+(f'(t))^2}$$
$$\overline{r}' \times \overline{r}'' = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{bmatrix}$$
$$= \langle 0, 0, f''(t)\rangle$$
$$|\overline{r}' \times \overline{r}''| = |f''(t)|$$
$$\kappa(x) = \frac{|f'(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}$$

□

Example 15 (Finding plane curvature with derivation of fromula)**.** *Given* $y = 4 - x^2$*, find curvature.* •

Solution 7 (Finding plane curvature with derivation of fromula).

$$\begin{aligned}
\overline{r}(t) &= \left\langle t, 4-t^2, 0\right\rangle \\
\overline{r}'(t) &= \langle 1, -2t, 0\rangle \\
\overline{r}''(t) &= \langle 0, -2, 0\rangle \\
|\overline{r}'(t)| &= \sqrt{1+4ts^2} \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2t & 0 \\ 0 & -2 & 0 \end{bmatrix} \\
&= \langle 0, 0, -2\rangle \\
|\overline{r}' \times \overline{r}''| &= \sqrt{(-2)s^2} = 2 \\
k &= \frac{2}{(\sqrt{1+4t^2})^3}
\end{aligned}$$

$$k = \boxed{\frac{2}{(\sqrt{1+4t^2})^3}}$$

•

3.3.6 Review equation of a plane

1. need point on plane
$$(x_0, y_0, z_0)$$
2. vector normal to plane
$$\overline{n} = \langle a, b, c\rangle$$

1. turn point into generic vector in plane
$$\langle x - x_0, y - y_0, z - z_0\rangle$$
2. $\overline{n}$ is perpendicular to every vector in plane so
$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Date: 2020-09-12
Announcements:

3.4 Planes on a Space Curve

Definition 16 (Normal Plane). .
Determined by Normal and Binormal vectors, $\overline{N}$ and $\overline{B}$ where $\overline{T}$ is normal vector to normal plane. The normal plane contains all lines orthogonal to $\overline{T}$ •

Definition 17 (Oscilating Plane). .
*Determined by $\overline{T}$ and $\overline{N} \to \overline{B}$ is normal vector to oscilating plane.
The oscilating plane lies closest to the space curve $\overline{r}(x)$ in 3-D space.* •

Definition 18 (Rectifying Plane). .
Contains tangent $\overline{T}$ and binormal $\overline{B}$ vectors so that the normal vector $\overline{N}$ is normal to the rectifying plane.

•

Table 3.1: Vectors of 3 Planes

Plane	Vectors contained	Orthogonal to
Normal	$\overline{N}$ $\overline{B}$	$\overline{T}$
Oscilating	$\overline{T}$ $\overline{N}$	$\overline{B}$
Rectifying	$\overline{T}$ $\overline{B}$	$\overline{N}$

Example 16 (Find 3 planes at a point). .
let $\overline{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$, find the equation of the Normal plane, Oscilating plane, and Rectifying plane at $t = \frac{\pi}{2}$ •

Solution 8 (Find 3 planes at a point). .
A) Normal Plane: Find Unit Tangent Vector
point: $t = \frac{\pi}{2}$

$$\overline{r}(\frac{\pi}{2}) = \left\langle \cos(\frac{3\pi}{2}), \frac{\pi}{2}, \sin(\frac{3\pi}{2}) \right\rangle$$
$$= \left\langle 0, \frac{\pi}{2}, -1 \right\rangle$$

generic vector $\left\langle x, y - \frac{\pi}{2}, z + 1 \right\rangle$

$$\overline{r}'(t) = \langle -3\sin(3t), 1, 3\cos(3t) \rangle$$
$$|\overline{r}'(t)| = \sqrt{9\sin^2(3t) + 1 + 9\cos^2(3t)}$$
$$|\overline{r}'(t)| = \sqrt{9\left(\sin^2(3t) + \cos^2(3t)\right) + 1}$$
$$= \sqrt{10}$$

$$\overline{T} = \frac{1}{|r'(t)|} r'(t)$$

$$\overline{T}(t) = \frac{1}{\sqrt{10}} \langle -3\sin(3t), 1, 3\cos(3t) \rangle$$

$$\overline{T}(\frac{\pi}{2}) = \frac{1}{\sqrt{10}} \langle 0, 0, -3 \rangle$$

$$= \left\langle 0, 0, -\frac{3}{\sqrt{10}} \right\rangle$$

$$\overline{N}(\frac{3}{2}) : \left\langle 0, 0, -\frac{3}{\sqrt{10}} \right\rangle \cdot \left\langle x, y - \frac{\pi}{2}, z+1 \right\rangle = 0$$

$$-\frac{3}{\sqrt{10}} z - \frac{3}{\sqrt{10}} = 0$$

Normal Plane:

$$\boxed{z = -1}$$

B) Rectifying Plane: Find Normal Tangent Vector

$$\overline{T}'(t) = \frac{1}{\sqrt{10}} \langle -9\cos(3t), 0, -9\sin(3t) \rangle$$

$$\left| \overline{T}'(t) \right| = \frac{1}{\sqrt{10}} \sqrt{81\cos^2(3t) + 0 + 81\sin^2(3t)}$$

$$= \frac{1}{\sqrt{10}} \sqrt{91}$$

$$= \frac{9}{\sqrt{10}}$$

$$\overline{N}(t) = \frac{1}{\frac{9}{\sqrt{10}}} \frac{1}{\sqrt{10}} \langle -9\cos(3t), 0, -9\sin(3t) \rangle$$

$$\overline{N}(t) = \frac{1}{9} \langle -9\cos(3t), 0, -9\sin(3t) \rangle$$

$$= \langle -\cos(3t), 0, -\sin(3t) \rangle$$

$$\overline{N}(\frac{\pi}{2}) = \langle 0, 0, 1 \rangle$$

$$\textit{Rectifying plane } = \langle 0, 0, -1 \rangle \cdot \left\langle x, y - \frac{\pi}{2}, z+1 \right\rangle = 0$$

$$0x + 0(y - \frac{\pi}{2}) + 1(z+1) = 0$$

$$z + 1 = 0$$

$$\textit{Rectifying Plane: } \boxed{z = -1}$$

C) Oscilating Plane: Find Binormal Unit Vector

$$\overline{B}(\frac{\pi}{2}) = \overline{T}(\frac{\pi}{2}) \times \overline{N}(\frac{\pi}{2})$$

$$= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0 \right\rangle$$

Oscilating Plane $\left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0 \right\rangle \cdot \left\langle x, y - \frac{\pi}{2}, z + 1 \right\rangle = 0$

$$-\frac{1}{\sqrt{10}}x + \frac{3}{\sqrt{10}}y - \frac{3}{2\sqrt{10}} + 0 = 0$$

C)

$$\boxed{-2x + 6y - 3 = 0}$$

•

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3.4.1 Oscilating Circle

Definition 19 (Oscilating Circle)**.** *The circle that lies in the oscilating plane of the curve at a point and has the same tangent, lies on the concave side of the curve and has a radius of* $\frac{1}{\kappa}$ •

To find a circle, you would need a center and the Radius. We know the radius is $\frac{1}{\kappa}$. To find the center, we must move the radius distance to the center from the edge touching the curve. This is in the direction of the unit normal vector.

Scale the unit normal vector by the length of the radius:

$$\frac{1}{\kappa}\overline{N}(t)$$

So, the vector from the origin to the center is the vector from the origin to the point on the curve + $\frac{1}{\kappa}\overline{N}$

Example 17 (oscilating circle)**.** *Given* $y = 4 - x^2$, $\overline{r}(t)0\left\langle t, 4-t^2, 0\right\rangle$, *find the equation of the oscilating circle when t=0, at* $(0, 4)$ •

Solution 9 (oscilating circle)**.**

$$\overline{r}(t) = \left\langle t, 4-t^2, 0\right\rangle$$

$$\kappa = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}$$

$$\overline{r}' = \langle 1, -2t, 0\rangle$$

$$\overline{r}'' = \langle 0, -2, 0\rangle$$

$$\overline{r}' \times \overline{r}'' = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2t & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= \langle 0, 0, -2\rangle$$

$$|\overline{r}' \times \overline{r}''| = 2$$

$$|\overline{r}'(t)| = \sqrt{1+4t^2}$$

$$\kappa = \frac{2}{(\sqrt{1+4t^2})^3}$$

$$\kappa(0) = 2$$

$$radius = \frac{1}{2}$$

We know the normal vector is in the negative-y direction because this is a prabola centered on the axis.

$$\langle 0, -1, 0\rangle$$

$$\langle 0,4,0\rangle + \frac{1}{2}\langle 0,-1,0\rangle = \left\langle 0,\frac{7}{2},0\right\rangle$$

$$(0,\frac{7}{2})$$

$$(x-h)^2+(y-k)^2=r^2$$

$$\boxed{x^2+(y-\frac{7}{2})^2=\frac{1}{4}}$$

•

Example 18 (oscilating circle)**.** *Given same as above, find oscilating circle at* $(2,0)$ •

Solution 10 (oscilating circle)**.**

$$\kappa(2)=\frac{2}{(\sqrt{1+4(2)^2})^3}$$

$$\frac{17\sqrt{17}}{2}$$

Center: $\langle 2,0,0\rangle + \frac{17\sqrt{17}}{2}\overline{N}(t)$ *We need principal normal unit vector at t=2.*

$$\overline{T}(t)=\frac{1}{\sqrt{1+4t^2}}\langle 1,-2t,0\rangle$$

$$\overline{T}'(t)=\frac{1}{\sqrt{1+4t^2}}\langle 0,-2,0\rangle - \frac{1}{2}(1+4t^2)^{-\frac{3}{2}}(8t)\langle 1,-2t,0\rangle$$

$$\overline{T}'(t)=\left\langle 0,-\frac{2}{\sqrt{1+4t^2}},0\right\rangle + \frac{-4t}{\left(\sqrt{1+4t^2}\right)^3}\langle 1,-2t,0\rangle$$

$$\overline{T}'(2)=\left\langle 0,-\frac{2}{\sqrt{1+4(2)^2}},0\right\rangle + \frac{-4(2)}{\left(\sqrt{1+4(2)^2}\right)^3}\langle 1,-2(2),0\rangle$$

$$\overline{T}'(2)=\left\langle -\frac{8}{17\sqrt{17}},\frac{-2}{17\sqrt{17}},0\right\rangle$$

$$\left|\overline{T}'(2)\right| = \sqrt{\frac{64}{17^2*17}+\frac{4}{17^2*17}}=\frac{2}{17}$$

$$\overline{N}(2)=\frac{1}{\frac{2}{17}}\left\langle -\frac{8}{17\sqrt{17}},-\frac{2}{17\sqrt{17}},0\right\rangle$$

$$\left\langle -\frac{4}{\sqrt{17}},-\frac{1}{\sqrt{17}},0\right\rangle$$

Center: $\langle 2,0,0\rangle + \frac{17\sqrt{17}}{2}\left\langle -\frac{4}{\sqrt{17}},-\frac{1}{\sqrt{17}},0\right\rangle$

$$=\langle 2,0,0\rangle + \left\langle -34,-\frac{17}{2},0\right\rangle$$

$$=\left\langle -32,-\frac{17}{2},0\right\rangle$$

$$(x-h)^2+(y-k)^2=r^2$$

$$\boxed{(x+32)^2+(y+\frac{17}{2})^2=\frac{4913}{2}}$$

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DATE: 2020-09-12
ANNOUNCEMENTS:

3.5 13.4 Motion in Space

3.5.1 Velcocity and Acceleration

$\overline{r}(t)$ position

$\overline{r}'(t) = \overline{v}(t)$ velocity

$\overline{r}''(t) = \overline{v}'(t) = \overline{a}(t)$ acceleration

$|\overline{v}|$ speed

Example 19 (velocity)**.** *givem* $\overline{r}(t) = \left\langle t^2, 2t, \ln t \right\rangle$, *find* $\overline{v}(t)$ •

Solution 11 (velocity)**.**

$$\overline{v}(t) = \overline{r}'(t) = \left\langle 2t, 2, \frac{1}{t} \right\rangle$$

•

Example 20 (speed)**.** *Find speed* •

Solution 12 (speed)**.**

$$|\overline{v}(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}}$$
$$= \left| 2t + \frac{1}{t} \right|$$

Note: we need the absolute value unless we know the domain
Domain $\overline{r}(t)$ $t > 0$ $(0, \infty)$

$$\boxed{2t + \frac{1}{t}}$$

•

Example 21 (acceleration)**.** *Find acceleration.* •

Solution 13 (acceleration)**.**

$$\overline{a}(t) = \overline{v}'(t) = \left\langle 2, 0, -\frac{1}{t^2} \right\rangle$$

•

Example 22 (find position and velocity given acceleration)**.** *Given* $\overline{A}(t) = \langle t, e^t, e^{-t} \rangle$ *and* $\overline{v}(0) = \langle 0, 0, 1 \rangle$, *and* $\overline{r}(0) = \langle 0, 1, 1 \rangle$, *Find* $\overline{v}(t)$ *and* $\overline{r}(t)$ •

Solution 14 (find position and velocity given acceleration).

$$\overline{v}(t) = \int \overline{a}(t)dt$$

$$= \left\langle \frac{1}{2}t^2, e^t, -e^{-t} \right\rangle + \langle c_1, c_2, c_3 \rangle$$

Find constants:

$$\langle 0,0,1 \rangle = \langle 0,1,-1 \rangle + \langle c_1, c_2, c_3 \rangle$$

$$\langle 0,0,1 \rangle = \langle c_1, 1+c_2, -1+c_3 \rangle$$

$$\implies c_1 = 0, c_2 = -1, c_3 = 2$$

$$\overline{v}(t) = \boxed{\left\langle \frac{1}{2}t^2, e^t - 1, -e^{-t} + 2 \right\rangle}$$

$$\overline{r}(t) = \int \overline{v}(t)dt$$

$$= \left\langle \frac{1}{6}t^3, e^t - t, e^{-t+2t} \right\rangle + \langle c_1, c_2, c_3 \rangle$$

$$\langle 0,1,1 \rangle = \langle 0,1,1 \rangle + \langle c_1, c_2, c_3 \rangle$$

$$\implies c_1 = 0, c_2 = 0, c_3 = 0$$

$$\overline{r}(t) = \boxed{\left\langle \frac{1}{6}t^3, e^t - t, e^{-t+2t} \right\rangle}$$

•

3.5.2 Acceleration Vector

$$\overline{T} = \frac{1}{|\overline{r}'|}\overline{r}'$$

$$\overline{r}' = |\overline{r}'|\,\overline{T}$$

$$\overline{v} = |\overline{v}|\,\overline{T}$$

$$\overline{a} = \overline{v}'(t) = |\overline{v}|\,\overline{T}' + |\overline{v}|'\,\overline{T}$$

We are trying to manufacture the unit normal vector for $\overline{T}'$

$$|\overline{v}|\,\overline{T}' = |\overline{v}|\,\frac{\left|\overline{T}'\right|}{\left|\overline{T}'\right|}\overline{T}'$$

$$\overline{N} = \frac{\overline{T}'}{\left|\overline{T}'\right|}$$

$$\implies \overline{a} = |\overline{v}|\left|\overline{T}'\right|\overline{N} + |\overline{v}|'\,\overline{T}$$

We have a scalar times unit vector plus a scalar times a unit vector. Therefore, you can take the acceleration and break into two components, $a_{\overline{N}}$ which is the size of the acceleration that goes in the direction of the unit normal vector, and $a_{\overline{T}}$ which goes in the direction of the unit tangent vector.

$$\rightarrow \overline{a} = a_{\overline{N}}\overline{N} + a_{\overline{T}}\overline{T}$$

$a_{\overline{N}}$

$$a_{\overline{N}} = |\overline{v}| \left|\overline{T}'\right|$$

Going back to curvature:

$$\kappa = \frac{\left|\overline{T}'\right|}{|\overline{r}'|} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}$$
$$\left|\overline{T}'\right| = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$
$$a_{\overline{N}} = |\overline{r}'| \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$

$$\boxed{a_{\overline{N}} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|}} \tag{3.4}$$

Normal component of the acceleration

$$\implies \overline{a} = |\overline{v}| \left|\overline{T}'\right| \overline{N} + |\overline{v}|' \overline{T}$$

What if we dot the velocity with the acceleration?

$$\begin{aligned} \overline{v} \cdot \overline{a} &= \overline{v} \cdot \left(|\overline{v}| \left|\overline{T}'\right| \overline{N} + |\overline{v}|' \overline{T}\right) \\ &= |\overline{v}| \left|\overline{T}'\right| \left(\overline{v} \cdot \overline{N}\right) + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right) \end{aligned}$$

$\left(\overline{v} \cdot \overline{N}\right) \to \overline{r}' \cdot \overline{N} \to \frac{|\overline{r}'|}{|\overline{r}'|}\overline{r}' \cdot \overline{N} \to |\overline{r}'|\, \overline{T} \cdot \overline{N}$

$$\begin{aligned} &= |\overline{r}'|^2 \left|\overline{T}'\right| \left(\overline{T} \cdot \overline{N}\right) + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right) \\ &= 0 + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right) \text{ (Tan and normal vectors orthagonal)} \\ \overline{v} \cdot \overline{a} &= \left|\overline{V}\right|' \left(\overline{v} \cdot \overline{T}\right) \\ \overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' \left(\overline{v} \cdot \overline{T}\right) \\ \overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' \left(\overline{r}' \cdot \overline{T}\right) \end{aligned}$$

$\overline{r}' \cdot \overline{T} \to \frac{|\overline{r}'|}{|\overline{r}'|}\overline{r} \cdot \overline{T} \to |\overline{r}'|\, \overline{T} \cdot \overline{T}$

$$\begin{aligned} \overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' |\overline{r}'|\, \overline{T} \cdot \overline{T} \\ \overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' |\overline{r}'| \left|\overline{T}\right|^2 \end{aligned}$$

$\left|\overline{T}\right|^2 = 1$

$$\begin{aligned} \overline{r}' \cdot \overline{r}'' &= |\overline{v}|' |\overline{r}'| \\ \frac{\overline{r}' \cdot \overline{r}''}{|\overline{r}'|} &= |\overline{v}|' \end{aligned}$$

$$\boxed{a_{\overline{T}} = \frac{\overline{r}' \cdot \overline{r}''}{|\overline{r}'|}} \tag{3.5}$$

Tangential component of acceleration

Example 23 (Find components of acceleration)**.** *Given* $\overline{r}(t) = \left\langle t, 2e^{2t}, e^{-2t} \right\rangle$, *find tangential and normal components of acceleration* •

Solution 15 (Find components of acceleration)**.**

$$\begin{aligned}
\overline{r}'(t) &= \left\langle 1, 4e^{2t}, -2e^{-25} \right\rangle \\
\overline{r}''(t) &= \left\langle 0, 8e^{2t}, 4e^{-2t} \right\rangle \\
\overline{r}' \cdot \overline{r}'' &= 0 + 32e^{4t} - 8e^{-4t} \\
&= 32e^{4t} - 8e^{-4t} \\
&= 8e^{-4t}\left(4e^{8t} - 1\right) \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4e^{2t} & -2e^{-2t} \\ 0 & 8e^{2t} & 4e^{-2t} \end{bmatrix} \\
&= \left\langle 16 + 16, -(4e^{-2t}), 8e^{2t} - 0 \right\rangle \\
&= \left\langle 32, -4e^{-2t}, 8e^{2t} \right\rangle \\
|\overline{r}' \times \overline{r}''| &= \sqrt{32^2 + 16e^{-4t} + 64e^{4t}} \\
&= \sqrt{16(64 + e^{-4t} + 4e^{4t}} \\
&= 4\sqrt{4e^{4t} + 64 + e^{-4t}} \\
|\overline{r}'| &= \sqrt{1^2 + (4e^{2t})^2 + (-2e^{-2t})^2} \\
&= \sqrt{1 + 16e^{4t} + 4e^{-4t}}
\end{aligned}$$

$$\begin{aligned}
\implies a_{\overline{T}} &= \frac{8e^{-4t}(4e^{8t} - 1)}{\sqrt{1 + 16e^{4t} + 4e^{-4t}}} \\
\implies a_{\overline{N}} &= \frac{4\sqrt{4e^{4t} + 64 + e^{-4t}}}{\sqrt{1 + 16e^{4t} + 4e^{-4t}}}
\end{aligned}$$

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Chapter 4

(14) Partial Derivatives

DATE: 2020-09-20
ANNOUNCEMENTS:

4.1 Functions of Several Variables

A function of two variables is a rule that assigns to each ordered pair (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the Domain of f and its Range is the set of values that f takes on, that is $\{f(x, y)|(x, y) \in D\}$.

Single Variables

$$y = f(x)$$
$$(x, y)$$

$$f(x) = x^2 - 2x + 3$$
$$f(-3) = (-3)^2 - 2(-3) + 3 = 18$$
$$(-3, 18)$$

Two Variables

$$z = f(x, y)$$
$$(x, y, z)$$

$$f(x, y) = x^2y + xy$$
$$f(-3, 2) = (-3)^2(2) + (-3)(2) = 12$$

$$(-3, 2, 12)$$

These are both polynomial functions.

A. Domain (1 variable): $(-\infty, \infty)$, All real numbers.
B. Domain (2 variables): $\mathbb{R}^2$

$$f(x) = \frac{\sqrt{x-3}}{x-7}$$

need $x \geq 3$, $x \neq 7$

$$[3, 7) \cup (7, \infty)$$

$$f(x, y) = \frac{\sqrt{x+y-1}}{x-y}$$

need $y \geq -x+1$, $x \neq y$

You could graph this domain in the x,y-plane

$$f(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$$

need $x^2 + y^2 + z^2 < 1$

Recognize special quadric surface, an elipsoid that is a sphere, and graph it with dotted surface to represent the domain.

4.1.1 Level Curve

The level curves of a function of two variables are the curves with equations $f(x, y) = k$ where k is a constant in the Range of f.

Definition 20 (Contour Map)**.** *A 2-d representation of a surface drawn with level curves.* •

Example 24 (Unfamiliar surface)**.**

$$f(x, y) = e^{x-y}$$

This will never output a value of 0 or negative.

$$1 = e^{x-y}$$
$$\ln 1 = x - y$$
$$y = x$$

$$2 = e^{x-y}$$
$$\ln 2 = x - y$$
$$y = x - \ln 2$$

The larger the value k gets, the more negative the y-intercept gets.
This resembles a slide that slopes towards the x-y plane. •

Example 25.

$$f(x, y, z) = \sqrt{x^2 + y^2 - z}$$

Level surfaces

$$0 = \sqrt{x^2 + y^2 - z}$$
$$z = x^2 + y^2$$

(paraboloid that maps to zero in 4-D) $(x, y, z, 0)$

$$1 = \sqrt{x^2 + y^2 - z}$$
$$z = x^2 + y^2 - 1$$

(paraboloid that maps to one in 4-D) $(x, y, z, 1)$ *This paraboloid would shift down from input 0, and this could continue...*

$$(x, y, z, k)$$

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4.2 Limits and Continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

The limit either exists, or does not. But if it exists, the limit must be the same value from every direction in the Domain.

Example 26 (Limit - hard DNE)**.**

$$\lim_{(x,y)\to(0,0)} \left(\frac{2x^2y}{x^4y^2}\right)$$

At $(0,0)$ *the function evaluates to* $\frac{0}{0}$

- x *axis* $\to y = 0$

$$\lim_{x\to 0} \frac{2x^2 \cdot 0}{x^4 + 0}$$
$$= \lim_{x\to 0} \frac{0}{x^4} = 0$$

- y *axis* $\to x = 0$

$$\lim_{y\to 0} \frac{2 \cdot 0^2 \cdot y}{0^4 + y^2}$$
$$= \lim_{y\to 0} \frac{0}{y^2} = 0$$

- $y = kx$, k *constant* $\neq 0$

$$\lim_{x\to 0} \frac{2x^2(kx)}{x^4 + (kx)^2}$$
$$= \lim_{x\to 0} \frac{2kx^3}{x^4 + k^2x^2}$$
$$= \lim_{x\to 0} \frac{2kx}{x^2 + k^2}$$
$$= \lim_{x\to 0} \frac{0}{k^2} = 0$$

- $y = kx^2$

$$\lim_{x\to 0} \frac{2x^2(kx^2)}{x^4 + (kx^2)^2}$$
$$\lim_{x\to 0} \frac{2kx^4}{x^4 + k^2x^4}$$
$$\lim_{x\to 0} \frac{2k}{1 + k^2} = \frac{2k}{1 + k^2}$$

This breaks it. The z values approach differnt values, $\boxed{DNE}$ •

Example 27 (Limit - easy DNE)**.**

$$\lim_{(x,y)\to(0,0)} \left(\frac{2x^2+y^2}{x^2+y^2}\right)$$

- *x- axis* $\to y = o$

$$\lim_{(x)\to 0} \left(\frac{2x^2+0^2}{x^2+0^2}\right)$$
$$= \lim_{(x)\to 0} \left(\frac{2x^2}{x^2}\right) = 2$$

- *y axis* $\to x = 0$

$$\lim_{y\to 0} \left(\frac{2(0)^2+y^2}{(0)^2+y^2}\right)$$
$$\lim_{y\to 0} \left(\frac{y^2}{y^2}\right)$$
$$\lim_{y\to 0} 1 = 1$$

This breaks it. The z values approach differnt values, $\boxed{DNE}$ •

Example 28 (Limit Exists - solution requires algebraic technique)**.**

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2-xy}{\sqrt{x}-\sqrt{y}}\right)$$

Conjugate

$$= \lim_{(x,y)\to(0,0)} \left(\frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}\right)$$
$$= \lim_{(x,y)\to(0,0)} \left(\frac{x(x-y))(\sqrt{x}+\sqrt{y})}{x-y}\right)$$
$$= \lim_{(x,y)\to(0,0)} \left(x(\sqrt{x}+\sqrt{y})\right) = \boxed{0}$$

•

Example 29 (Limit (Does Exist))**.**

$$\lim_{(x,y)\to(0,0)} \left(\frac{4xy^2}{x^2+y^2}\right)$$

- *y-axis* $\to x = 0$

$$= \lim_{y\to 0} \left(\frac{0}{y^2}\right) = 0$$

- *x-axis* $\to y = 0$

$$= \lim_{y\to 0} \left(\frac{0}{x^2}\right) = 0$$

- *lines* $y = kx, k \neq 0$

$$= \lim_{x\to 0} \left(\frac{4xk^2x^2}{x^2+k^2x^2}\right)$$
$$= \lim_{x\to 0} \left(\frac{4xk^2}{1+k^2}\right)$$
$$= \lim_{x\to 0} \left(\frac{0}{1+k^2}\right) = 0$$

- $y = kx^2$, $k \neq 0$

$$= \lim_{x \to 0} \left(\frac{4xk^2x^4}{x^2 + k^2x^4} \right)$$
$$= \lim_{x \to 0} \left(\frac{4k^2x^5}{x^2(1 + k^2x^2)} \right)$$
$$= \lim_{x \to 0} \left(\frac{4k^2x^3}{1 + k^2x^2} \right) = 0$$

In the interest of time, this limit does go towards zero, but can't be proven with this method.

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4.2.1 Going Polar

Example 30 (Limit (Same problem)).

$$\lim_{(x,y) \to (0,0)} \left(\frac{4xy^2}{x^2 + y^2} \right)$$

You can try to change the problem to Polar Coordinates
This helps, because you only need r to go to 0

$$(x, y) \to (r, \theta)$$
$$x = r \cos \theta a$$
$$y = r \sin \theta a$$

$$\lim_{r \to 0} \left(\frac{4r \cos \theta (r \sin \theta)^2}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \right)$$
$$\lim_{r \to 0} \left(\frac{4r^3 \cos \theta \sin^2 \theta}{r^2 (\cos^2 \theta \sin^2 \theta)} \right)$$
$$\lim_{r \to 0} \left(\frac{4r^3 \cos \theta \sin^2 \theta}{r^2} \right)$$
$$\lim_{r \to 0} 4r \cos \theta \sin^2 \theta = 4 * 0 * \textit{finite} * \textit{finite} = \boxed{0}$$

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Definition 21 (Squeeze Theorem). *if $g(x) \leq f(x) \leq h(x)$ everywhere close to a, except possibly at a.*
and if $\lim_{x \to ag(x)}$ and $\lim_{x \to a} h(x)$ both exist and are equal, then $\lim_{x \to a} f(x)$ must exist and equal the same value. •

4.2.2 Squeezing

Example 31 (Limit (same problem)).

$$\lim_{(x,y) \to (0,0)} \left(\frac{4xy^2}{x^2 + y^2} \right)$$

We suspect from previous attempts that the limit goes to zero.
If the absolute value goes to zero, $|\ldots| \to 0$, then $\ldots \to 0$

$$0 \leq \left| \frac{4xy^2}{x^2 + y^2} \right| = \left| 4x \cdot \frac{y^2}{x^2 + y^2} \right|$$

$\frac{y^2}{x^2+y^2}$ *is always going to be less than or equal to one.*

$$0 \leq \left| \frac{4xy^2}{x^2+y^2} \right| = \left| 4x \cdot \frac{y^2}{x^2+y^2} \right| \leq |4x|$$

$$0 \leq \left| \frac{4xy^2}{x^2+y^2} \right| \leq |4x|$$

$$\lim_{(x,y)\to(0,0)} 0 = 0$$

$$\lim_{(x,y)\to(0,0)} |4x| = 0$$

$$\implies_{(\textit{ squeeze theorem })} \lim_{(x,y)\to(0,0)} \left| \frac{4xy^2}{x^2+y^2} \right| = 0$$

therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \boxed{0}$$

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4.2.3 Continuous

Definition 22 (Continuous)**.** *A function of two variables is called continuous at (a, b) if*

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is continuous on D if it is continous at every point (a, b) in D. "If you know it is continuous, then you can interpret it this way" •

The followin are continuous on their domain

1. Polynomial
2. Rational
3. Root
4. Trig
5. Inverse Trig
6. Exponential
7. Log

Date: 2020-09-20
Announcements:

4.3 Partial Derivatives

Definition 23 (Partial Derivative)**.**

$$z = f(x, y)$$

The partial derivative of f with respect to x

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z_x$$

How the output z changes with respect to x - the y stays constant.

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

•

Example 32 (Partial Derivative)**.**

$$\begin{aligned} z = f(x, y) &= x^2 y^3 \\ \frac{\partial z}{\partial x} &= \lim_{h \to 0} \frac{(x+h)^2 y^3 - x^2 y^3}{h} \\ &= \lim_{h \to 0} \frac{y^3 \left((x+h)^2 - x^2\right)}{h} \\ &= y^3 \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= y^3 \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= y^3 \lim_{h \to 0} 2x + h = \boxed{2xy^3} \end{aligned}$$

•

4.3.1 Higher Order Derivatives

Theorem 5 (Clairaut's Theorem)**.** *Suppose that f is defined on a disk D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D, then the*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

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Corollary 2 (Clairaut's Corollary)**.**

$$f_{xxy} = f_{xyx} = f_{yxx}$$
$$f_{yyx} = f_{yxy} = f_{xyy}$$

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DATE: 2020-09-22
ANNOUNCEMENTS:

4.4 Tangent Planes and Linear Approximations

Example 33 (Tangent Plane)**.**

$$f(x, y) = x^2\sqrt{y}$$

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Finding $f(2, 9)$ would be easy.
Finding $f(1.98, 9.03)$ by hand would be hard.
Approximate $f(2, 9)$. Do better than (12).
At the point on the surface at $(2, 9)$, there is a tangent plane that touches at $(2, 9, 12)$.
The "old" z value is 12 and there will be a change in the x and y to afect z.

$$\text{old z-value } + \frac{\text{change in z}}{\text{change in x}}(\text{ change in x }) + \frac{\text{change in z}}{\text{change in y}}(\text{ change in y })$$

$$12 + \frac{\partial z}{\partial x}(x - 2) + \frac{\partial y}{\partial x}(y - 9)$$

Theorem 6. *Suppose f has continuous partial derivatives. An equation of the tangetn plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is*

$$\boxed{z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)} \tag{4.1}$$

Tangent Plane

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Example 34 (Tangent Plane Cont'd)**.** *Approximate $f(1.98, 9.03)$ using the equation of the tangent plane at $(2, 9, 12)$.*

$$f(x, y) = x^2\sqrt{y}$$

1. $f(2, 9) = 12$
2. $f_x(x, y) = 2x\sqrt{y} \to f_x(2, 9) = 2(2)\sqrt{9} = 12$
3. $f_y(x, y) = \frac{1}{2}x^2y^{-\frac{1}{2}} \to f_y(2, 9) = \frac{2^2}{2\sqrt{9}} = \frac{2}{3}$
4. *Tangent Plane*

$$z = 12 + 12(x - 2) + \frac{4}{9}(y - 9)$$

5. *Solve for approximation*

$$\begin{aligned}
f(1.98, 9.03) &\approx 12 + 12(1.98 - 2) + \frac{2}{3}(9.03 - 9) \\
&\approx 12 + 12(\frac{-2}{100}) + \frac{2}{3}(\frac{3}{100}) \\
&\approx 12 - \frac{24}{100} + \frac{2}{100} \\
&\approx 12 - 0.22 \\
&\approx \boxed{11.78}
\end{aligned}$$

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