Date: 2020-09-12
Announcements:

0.0.1 Another formula for curvature

$$\kappa = \frac{\left|\overline{T}'\right|}{|\overline{r}'|}$$

$$\overline{T} = \frac{1}{|\overline{r}'|}\overline{r}' \to \overline{r}' = |\overline{r}'|\,\overline{T}$$

$$\overline{r}''(t) = |\overline{r}'|\,\overline{T}' + |\overline{r}'|'\,\overline{T}$$

$$\overline{r}' \times \overline{r}'' = \overline{r}' \times \left(|\overline{r}'|\,\overline{T}' + |\overline{r}'|'\,\overline{T}\right)$$

$$= |\overline{r}'|\left(\overline{r}' \times \overline{T}'\right) + |\overline{r}'|'\left(\overline{r}' \times \overline{T}\right)$$

Second term is equivalent to a scallar times $(\overline{r}' \times \overline{r}')$, evaluates to zero vector.

$$\overline{r}' \times \overline{r}'' = |\overline{r}'|\,(\overline{r}' \times \overline{T}')$$

Multiply by one: $\frac{|\overline{r}'|}{|\overline{r}'|}$

$$\overline{r}' \times \overline{r}'' = |\overline{r}'|^2\,(\overline{T} \times \overline{T}')$$

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T} \times \overline{T}'\right|$$

$\left|\overline{a} \times \overline{b}\right| = |\overline{a}|\left|\overline{b}\right|\sin\theta$
$\overline{T}$ and $\overline{T}'$ are $\perp$

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T}\right|\left|\overline{T}'\right|\sin 90$$

$\overline{T}$ is a unit vector and $\sin 90$ is 1

$$|\overline{r}' \times \overline{r}''| = |\overline{r}'|^2\left|\overline{T}'\right|$$

$$\left|\overline{T}'\right| = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$

$$\kappa = \frac{\overline{T}'}{\overline{r}'} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}^3|}$$

$$\boxed{\kappa = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}}$$

Example 1 (Curvature)**.** *Given* $\overline{r} = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle$*, find* κ •

Solution 1 (Curvature)**.**

$$\begin{aligned}
\overline{r}' &= \left\langle 2, 2t, t^2 \right\rangle \\
|\overline{r}'| &= \sqrt{4 + 4t^2 + t^4} \\
&= 2 + t^2 \\
\overline{r}'' &= \langle 0, 2, 2t \rangle \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{bmatrix} \\
&= \left\langle 2t^2, -4t, 4 \right\rangle \\
|\overline{r}' \times \overline{r}''| &= \sqrt{4t^4 + 16t^2 + 4} \\
&= 2(t^2 + 1) \\
\kappa &= \frac{2(t^2 + 1)}{(2 + t^2)^3}
\end{aligned}$$

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0.0.2 Curvature of a plane curve (2-D)

Definition 1 (Curvature of a plane curve)**.**

$$\kappa(x) = \frac{|f'(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

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Curvature of a plane curve.

$$\boxed{\kappa = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}}$$

$$\begin{aligned}
y &= f(x) \\
\overline{r}(t) &= \langle t, f(t), 0 \rangle \\
\overline{r}'(t) &= \langle 1, f'(t), 0 \rangle \\
\overline{r}''(t) &= \langle 0, f''(t), 0 \rangle \\
|\overline{r}'| &= \sqrt{1 + (f'(t))^2} \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{bmatrix} \\
&= \langle 0, 0, f''(t) \rangle \\
|\overline{r}' \times \overline{r}''| &= |f''(t)| \\
\kappa(x) &= \frac{|f'(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}
\end{aligned}$$

□

Example 2 (Finding plane curvature with derivation of fromula)**.** *Given $y = 4 - x^2$, find curvature.* •

Solution 2 (Finding plane curvature with derivation of fromula).

$$
\begin{aligned}
\overline{r}(t) &= \left\langle t, 4-t^2, 0 \right\rangle \\
\overline{r}'(t) &= \langle 1, -2t, 0 \rangle \\
\overline{r}''(t) &= \langle 0, -2, 0 \rangle \\
|\overline{r}'(t)| &= \sqrt{1+4ts^2} \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2t & 0 \\ 0 & -2 & 0 \end{bmatrix} \\
&= \langle 0, 0, -2 \rangle \\
|\overline{r}' \times \overline{r}''| &= \sqrt{(-2)s^2} = 2 \\
k &= \frac{2}{(\sqrt{1+4t^2})^3}
\end{aligned}
$$

$$k = \boxed{\frac{2}{(\sqrt{1+4t^2})^3}}$$

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0.0.3 Review equation of a plane

1. need point on plane

$$(x_0, y_0, z_0)$$

2. vector normal to plane

$$\overline{n} = \langle a, b, c \rangle$$

1. turn point into generic vector in plane

$$\langle x - x_0, y - y_0, z - z_0 \rangle$$

2. $\overline{n}$ is perpendicular to every vector in plane so

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$