Date: 2020-09-09

Announcements:

Assignment: 1.6 1-19 odd

Turn in posted Friday, due Monday at 4 pm

Text chapter 1: 9/16

0.1 More on Linear Systems and Invertible Matrices

Example 1 (Solving system using the inverse). *Solve:*

$$\begin{aligned} 2x + 6y + 6z &= 1 \\ 2x + 7y + 6z &= 2 \\ 2x + 7y + 7z &= 3 \end{aligned}$$

$$Ax = b$$

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If A^{-1} exists,

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

$$\begin{gathered} \frac{1}{2}R_1 \to R_1 \\ -R_2 + R_3 \to R_3 \end{gathered}$$

$$\begin{aligned} &\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 2 & 7 & 6 & | & 0 & 1 & 1 \\ 2 & 7 & 7 & | & 0 & 0 & 1 \end{bmatrix} \\ \to &\begin{bmatrix} 1 & 3 & 3 & | & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & | & 0 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix} \\ \to &\begin{bmatrix} 1 & 3 & 0 & | & \frac{1}{2} & 3 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix} \\ \to &\begin{bmatrix} 1 & 0 & 0 & | & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\left(-\frac{11}{2}, 1, 1\right)}$$

Theorem 1 (1.6.1)**.** *A system of linear equations has zero, one or an infinite number of solutions.*

$$Ax = b$$

A *has Reduced Row Eschelon form that is I_n.*

or

A *has Reduced Row Eschelon form that is not.*

*Suppose that $Ax = b$ has **two** solutions x_1 and x_2 (proof by contradiction)*
We know

$$\begin{aligned} Ax_1 &= b \\ Ax_2 &= b \\ Ax_1 - Ax_2 &= A(x_1 - x_2) \\ b - b &= A(x_1 - x_2) \implies \\ A(x_1 - x_2) &= 0 \end{aligned}$$

By equivalence principal, the matrix $x_1 - x_2 = 0$ has only trivial solution $\implies \boxed{x_1 = x_2}$

If A is an $n \times n$ matrix, the following are equivalent

1. A is invertible
2. $Ax = 0$ has only the trivial solution
3. The reduced row eschelon form is I
4. A is expressible as a product of Elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix b. (1 solution or an infinite)
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b:

$$\begin{aligned} Ax &= b \\ x &= \boxed{A^{-1}b} \end{aligned}$$

0.1.1 A Fundamental Problem

Example 2 (A fundamental Problem)**.** *Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices b such that the system of equations $Ax = b$ is consistent.*

$$\begin{aligned} 2 + y &= b_1 \\ -2y + 4z &= b_2 \\ 3x - 2z &= b_3 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 0 & | & b_1 \\ 0 & -2 & 4 & | & b_2 \\ 3 & 0 & -2 & | & b_3 \end{bmatrix} \rightarrow$$

$$\frac{1}{2}R_1 \rightarrow R_1$$

$$\frac{-1}{2}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 3 & 0 & -2 & | & b_3 \end{bmatrix} \rightarrow$$

$$-3R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 0 & -\frac{3}{2} & -2 & | & b_3 - \frac{3}{2}b_1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 0 & 0 & -5 & | & b_3 - \frac{3}{2}b_1 - \frac{3}{4}b_2 \end{bmatrix} \rightarrow$$

Example 3 (Different End of Fundamental problem)**.**

$$\begin{bmatrix} 1 & 2 & 0 & b_1 \\ 9 & 1 & 0 & 2b_1 - 3b_2 \\ 0 & 0 & 0 & b_1 - 3b_2 + b_3 \end{bmatrix}$$

$$\rightarrow b_1 - 3b_2 + b_3 = 0$$

$$\rightarrow b_3 = 3b_2 - b_1$$

$$\boxed{\begin{bmatrix} b_1 \\ b_2 \\ 3b_2 - b_1 \end{bmatrix}}$$