Date: 2020-09-20
Announcements:

0.1 Limits and Continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

The limit either exists, or does not. But if it exists, the limit must be the same value from every direction in the Domain.

Example 1 (Limit - hard DNE)**.**

$$\lim_{(x,y)\to(0,0)} \left(\frac{2x^2y}{x^4y^2}\right)$$

At $(0,0)$ *the function evaluates to* $\frac{0}{0}$

- x *axis* $\to y = 0$

$$\lim_{x\to 0} \frac{2x^2 \cdot 0}{x^4 + 0}$$
$$= \lim_{x\to 0} \frac{0}{x^4} = 0$$

- y *axis* $\to x = 0$

$$\lim_{y\to 0} \frac{2 \cdot 0^2 \cdot y}{0^4 + y^2}$$
$$= \lim_{y\to 0} \frac{0}{y^2} = 0$$

- $y = kx$, k *constant* $\neq 0$

$$\lim_{x\to 0} \frac{2x^2(kx)}{x^4 + (kx)^2}$$
$$= \lim_{x\to 0} \frac{2kx^3}{x^4 + k^2x^2}$$
$$= \lim_{x\to 0} \frac{2kx}{x^2 + k^2}$$
$$= \lim_{x\to 0} \frac{0}{k^2} = 0$$

- $y = kx^2$

$$\lim_{x\to 0} \frac{2x^2(kx^2)}{x^4 + (kx^2)^2}$$
$$\lim_{x\to 0} \frac{2kx^4}{x^4 + k^2x^4}$$
$$\lim_{x\to 0} \frac{2k}{1 + k^2} = \frac{2k}{1 + k^2}$$

This breaks it. The z values approach differnt values, $\boxed{DNE}$ •

Example 2 (Limit - easy DNE).

$$\lim_{(x,y)\to(0,0)}\left(\frac{2x^2+y^2}{x^2+y^2}\right)$$

- *x- axis* $\to y = o$

$$\lim_{(x)\to 0}\left(\frac{2x^2+0^2}{x^2+0^2}\right)$$
$$= \lim_{(x)\to 0}\left(\frac{2x^2}{x^2}\right) = 2$$

- *y axis* $\to x = 0$

$$\lim_{y\to 0}\left(\frac{2(0)^2+y^2}{(0)^2+y^2}\right)$$
$$\lim_{y\to 0}\left(\frac{y^2}{y^2}\right)$$
$$\lim_{y\to 0} 1 = 1$$

This breaks it. The z values approach differnt values, $\boxed{DNE}$ •

Example 3 (Limit Exists - solution requires algebraic technique).

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2-xy}{\sqrt{x}-\sqrt{y}}\right)$$

Conjugate

$$= \lim_{(x,y)\to(0,0)}\left(\frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}\right)$$
$$= \lim_{(x,y)\to(0,0)}\left(\frac{x(x-y))(\sqrt{x}+\sqrt{y})}{x-y}\right)$$
$$= \lim_{(x,y)\to(0,0)}\left(x(\sqrt{x}+\sqrt{y})\right) = \boxed{0}$$

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Example 4 (Limit (Does Exist)).

$$\lim_{(x,y)\to(0,0)}\left(\frac{4xy^2}{x^2+y^2}\right)$$

- *y-axis* $\to x = 0$

$$= \lim_{y\to 0}\left(\frac{0}{y^2}\right) = 0$$

- *x-axis* $\to y = 0$

$$= \lim_{y\to 0}\left(\frac{0}{x^2}\right) = 0$$

- *lines* $y = kx, k \neq 0$

$$= \lim_{x\to 0}\left(\frac{4xk^2x^2}{x^2+k^2x^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{4xk^2}{1+k^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{0}{1+k^2}\right) = 0$$

- $y = kx^2$, $k \neq 0$

$$= \lim_{x\to 0}\left(\frac{4xk^2x^4}{x^2+k^2x^4}\right)$$
$$= \lim_{x\to 0}\left(\frac{4k^2x^5}{x^2(1+k^2x^2)}\right)$$
$$= \lim_{x\to 0}\left(\frac{4k^2x^3}{1+k^2x^2}\right) = 0$$

In the interest of time, this limit does go towards zero, but can't be proven with this method.

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0.1.1 Going Polar

Example 5 (Limit (Same problem)).

$$\lim_{(x,y)\to(0,0)}\left(\frac{4xy^2}{x^2+y^2}\right)$$

You can try to change the problem to Polar Coordinates
This helps, because you only need r to go to 0

$$(x,y) \to (r,\theta)$$
$$x = r\cos\theta a$$
$$y = r\sin\theta a$$

$$\lim_{r\to 0}\left(\frac{4r\cos\theta(r\sin\theta)^2}{r^2\cos^2\theta + r^2\sin^2\theta}\right)$$
$$\lim_{r\to 0}\left(\frac{4r^3\cos\theta\sin^2\theta}{r^2(\cos^2\theta\sin^2\theta)}\right)$$
$$\lim_{r\to 0}\left(\frac{4r^3\cos\theta\sin^2\theta}{r^2}\right)$$
$$\lim_{r\to 0} 4r\cos\theta\sin^2\theta = 4*0*\textit{ finite }*\textit{ finite } = \boxed{0}$$

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Definition 1 (Squeeze Theorem)**.** *if $g(x) \leq f(x) \leq h(x)$ everywhere close to a, except possibly at a.*
and if $\lim_{x\to ag(x)}$ and $\lim_{x\to a} h(x)$ both exist and are equal, then $\lim_{x\to a} f(x)$ must exist and equal the same value. •

0.1.2 Squeezing

Example 6 (Limit (same problem)).

$$\lim_{(x,y)\to(0,0)}\left(\frac{4xy^2}{x^2+y^2}\right)$$

We suspect from previous attempts that the limit goes to zero.
If the absolute value goes to zero, $|\ldots| \to 0$, then $\ldots \to 0$

$$0 \leq \left|\frac{4xy^2}{x^2+y^2}\right| = \left|4x\cdot\frac{y^2}{x^2+y^2}\right|$$

$\frac{y^2}{x^2+y^2}$ *is always going to be less than or equal to one.*

$$0 \leq \left|\frac{4xy^2}{x^2+y^2}\right| = \left|4x \cdot \frac{y^2}{x^2+y^2}\right| \leq |4x|$$

$$0 \leq \left|\frac{4xy^2}{x^2+y^2}\right| \leq |4x|$$

$$\lim_{(x,y)\to(0,0)} 0 = 0$$

$$\lim_{(x,y)\to(0,0)} |4x| = 0$$

$$\implies_{(\text{ squeeze theorem })} \lim_{(x,y)\to(0,0)} \left|\frac{4xy^2}{x^2+y^2}\right| = 0$$

therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \boxed{0}$$

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0.1.3 Continuous

Definition 2 (Continuous)**.** *A function of two variables is called continuous at* (a, b) *if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say that f *is continuous on* D *if it is continous at every point* (a, b) *in* D*. "If you know it is continuous, then you can interpret it this way"* •

The followin are continuous on their domain

1. Polynomial
2. Rational
3. Root
4. Trig
5. Inverse Trig
6. Exponential
7. Log