DATE: 2020-09-06

ANNOUNCEMENTS:

0.1 Derivatives and Integrals of Vector Functions

Definition of the Derivative

Definition 1 (Derivative)**.**

$$\frac{d\overline{r}}{dt} = \overline{r}'(t) = \lim_{h \to 0} \frac{\overline{r}(t+h) - \overline{r}(t)}{h}$$

$$\overline{r}(t) = \langle f(t), g(t), m(t) \rangle$$

1. $\overline{r}(t+h) = \langle r(t+h), g(t+h), m(t+h) \rangle$
2. $\overline{r}(t+h) - \overline{r}(t)$
$$= \langle f(t+h) - f(t), g(t+h) - g(t), m(t+h) - m(t) \rangle$$
3. multiply $\frac{1}{h}$
$$= \frac{1}{h} \langle f(t+h) - f(t), g(t+h) - g(t), m(t+h) - m(t) \rangle$$
$$= \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{m(t+h) - m(t)}{h} \right\rangle$$
4. $\overline{r}'(t) =$
$$\lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{m(t+h) - m(t)}{h} \right\rangle$$
$$= \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \to 0} \frac{m(t+h) - m(t)}{h} \right\rangle$$
$$\boxed{\overline{r}'(t) = \langle f'(t), g'(t), m'(t) \rangle}$$

Example 1 (Tangent line)**.** *find the parametric equation for the line tangent to*

$$\overline{r}(t) = \left\langle t^2 + 1, 4\sqrt{t}, e^{t^2 - t} \right\rangle$$

at the point $(2, 4, 1)$

Solution 1 (Tangent line)**.** *Need:*

1. *point* $(2, 4, 1)$
2. *direction vector*
$$\overline{r}'(t) = \left\langle 2t, 2t^{-\frac{1}{2}}, e^{t^2 - t}(2t - 1) \right\rangle$$

3. *parameter value*

$$\overline{r}(t) = (2,4,1) \implies t = 1 \implies$$
$$\overline{r}'(1) = \langle 2,2,1 \rangle$$
$$\boxed{\langle 2,4,1 \rangle + t\,\langle 2,2,1 \rangle}$$
$$\boxed{\langle 2+2t, 4+2t, 1+t \rangle}\ \textit{(vector)}$$

$$x = 2+2t$$
$$y = 4+2t$$
$$z = 1+t$$

0.1.1 The Unit Tangent Vector

Definition 2 (Unit Tangent Vector).

$$\overline{T}(t) = \frac{1}{|\overline{r}'(t)|}\overline{r}'(t)$$

Example 2 (Unit Tangent Vector).

$$\overline{r}(t) = \left\langle t^2 + 3t, 3t - 8, t^3 - 1 \right\rangle$$

Find $\overline{T}(2)$

Solution 2 (Unit Tangent Vector).

$$\overline{r}'(t) = \left\langle 2t+3, 3, 3t^2 \right\rangle$$
$$\overline{r}'(2) = \langle 7,3,12 \rangle$$
$$|\overline{r}'(2)| = \sqrt{49+9+144}$$
$$|\overline{r}'(2)| = \sqrt{202}$$
$$\overline{T}(2) = \frac{1}{\sqrt{202}}\langle 7,3,12 \rangle$$
$$\overline{T}(2) = \boxed{\left\langle \frac{7}{\sqrt{202}}, \frac{3}{\sqrt{202}}, \frac{12}{\sqrt{202}} \right\rangle}$$

Theorem 1. *Suppose that* $\overline{u}$ *and* $\overline{v}$ *are differentiable vector functions, c is a scalar, and f is a real valued function*

1. $\frac{d}{dt}[\overline{u}(t) + \overline{v}(t)] = \overline{u}'(t) + \overline{v}'(t)$
2. $\frac{d}{dt}[c\overline{u}(t)] = c\overline{u}'(t)$
3. $\frac{d}{dt}[f(t)\overline{u}(t)] = \langle f(t)u_1'(t) + f'(t)u_1(t), f(t)u_2'(t) + f'(t)u_2(t), f(t)u_3'(t) + f'(t)u_3(t) \rangle$
$$= f(t)\overline{u}'(t) + f'(t)\overline{u}(t)$$
4. $\frac{d}{dt}\left[\overline{u}(t)\cdot\overline{v}(t)\right] = \frac{d}{dt}\left[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)\right]$
$$= \overline{u}(t)\cdot\overline{v}'(t) + \overline{u}'(t)\cdot\overline{v}(t)$$
5. $\frac{d}{dx}\left[\overline{u}(t)\times\overline{v}(t)\right] = \overline{u}(t)\times\overline{v}'(t) + \overline{u}'(t)\times\overline{v}(t)$
6. $\frac{d}{dt}\left[\overline{u}(f(t)\right] = f'(t)\overline{u}'(f(t))$

Example 3 (Prove the vector differentiation theorems). *Exercise for the reader.*