Date: 2020-09-02

Announcements:

As an exercise for the reader, theorems 1.4.7 - 1.4.9 should be examined.
Assignment: 1.4 (1-22, 25-28, 29-31, 39, 40, 45, 49, 50)

0.1 Inverse

0.1.1 4×4 Example

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \,/= 0$, in which case the inverse is given by the formula

$$A^{-1} \neq \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- $ad - bc$ is called the determinant of A, $det(A)$.

Identity Proof

$$AA^{-1} = I$$
$$A^{-1}A = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix}$$

$$\frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 1. *Solve*

$$2x + 3y = 5$$
$$-x + 7y = 12$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}_{2\times 1}$$
$$[A][x] = b$$

If A is invertible:

$$A^{-1}Ax = A^{-1}b$$

The inverse must be a left side multiplication on both sides of equations.

$$x = A^{-1}b$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = []$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}_{2\times 1}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$\frac{1}{17}\begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}\begin{bmatrix} x \\ 7 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -1 \\ 29 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{17} \\ \frac{29}{17} \end{bmatrix}$$

Theorem 1. *If B and C are both inverses of A, then $B = C$*

Proof If B is an inverse of A, $AB = BA = I$
If C is an inverse of A, $AC = CA = I$

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

□

Theorem 2. *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof

$$(AB)(AB)^{-1} = (AB)B^{-1}A^{-1} = (AI)A^{-1} = AA^{-1} = I$$
$$(AB)^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = I$$

Corollary 1.

$$A_1, A_2, \ldots, A_n$$

are all invertible matrices of the same size, then

$$(A_1A_2 \ldots A_n)$$

is invertible and

$$(A_1A_2 \ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \ldots A_2^{-1}A_1^{-1}$$

Definition 1. *If A is a square matrix, then*

$$A^n = AAA \ldots A\{nfactors$$

$$A^0 = I$$

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \ldots A^{-1}$$

Theorem 3. *If A is invertible and n is a nonnegative integer, then:*
A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
kA *is invertible for any nonzero scalar* k*, and* $(kA)^{-1} = k^{-1}A^{-1}$

Theorem 4. *If the sizes of the matrices are such that the stated operations can be performed, then:*

Table 1: caption

(a)	$(A^T)^T = A$
(b)	$(A+B)^T = A^T + B^T$
(c)	$(A-B)^T = A^T - B^T$
(d)	$(kA)^T = kA^T$
(e)	$(AB)^T = A^T B^T$

Theorem 5. *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$