Lecture Notes

# Linear Algebra

Fall 2020

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Instructed by:
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# Contents

# Chapter 1

# Systems of Linear Equations and Matrices

DATE: 2020-08-17

ANNOUNCEMENTS:

Instructor - Kathleen Kane

Office Hours - MWF 11:30am - 12:30pm TR 8:00am - 9:00 am

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Book - Elementary Linear Algebra: Applications Version by Howard Anton and Chris Rorries, 11th edition 9781118434413

**Assignment(Aug 19 08:50): practice uploading 3 scanned pages in a single pdf**

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## 1.1 Policies and Procedures

### 1.1.1 Learning Outcomes

1. Perform basic operations with vectors in n-dimensional space.
2. Perform basic operations with matrices.
3. Solve a system of m linear equations in n unknowns.
4. Prove basic theorems in a vector space.
5. Perform basic operations with vectors in the standard matrix spaces and function space.
6. Find the matrix repersentation of a linear transformation between two vector spaces.
7. Find eigenvalues and eigenvectors for a given matrix.
8. Perform basic operations in an inner product space
9. Prove basic theorems in an inner product space.

### 1.1.2 Evaluation

1. Assignments (10%)
2. Testes (70%)
3. Final (weighted) (20%)

### 1.1.3 Testing

1. Required to scan test and submit via pdf
2. 50 minutes each test and 10 minutes to submit test
3. No make up tests
4. One test may be substituted with final exam grade
5. Missing final is automatic F.

## 1.2 Introduction to Systems of Linear Equations and Matrices

**Example 1** (One solution). *Solve:*

$$\begin{aligned} 3x + y &= 6 \\ 5x - 3y &= 10 \end{aligned}$$

**Solution 1.**

$$\begin{aligned} 3x + y = 6 &\implies \\ 9x + 3y &= 18 \\ +[5x - 3y &= 10] \\ \hline 14x = 28 &\implies \\ \boxed{x = 2} \end{aligned}$$

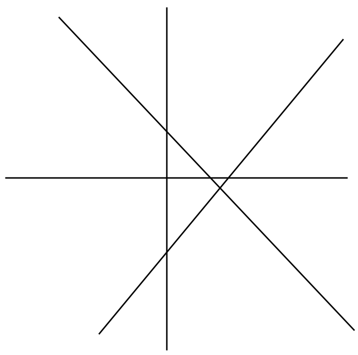

Figure 1.1: one solution

**Example 2** (Infinite solutions). *Solve:*

$$\begin{aligned} 2x - y &= 7 \\ 4x - 2y &= 14 \end{aligned}$$

**Solution 2.**

$$\begin{aligned} 4x - 2y = 14 &\implies \\ 2x - y = 7 \\ -[2x - y = 7] \\ \hline \boxed{0 = 0} &\textit{ (no solution)} \end{aligned}$$

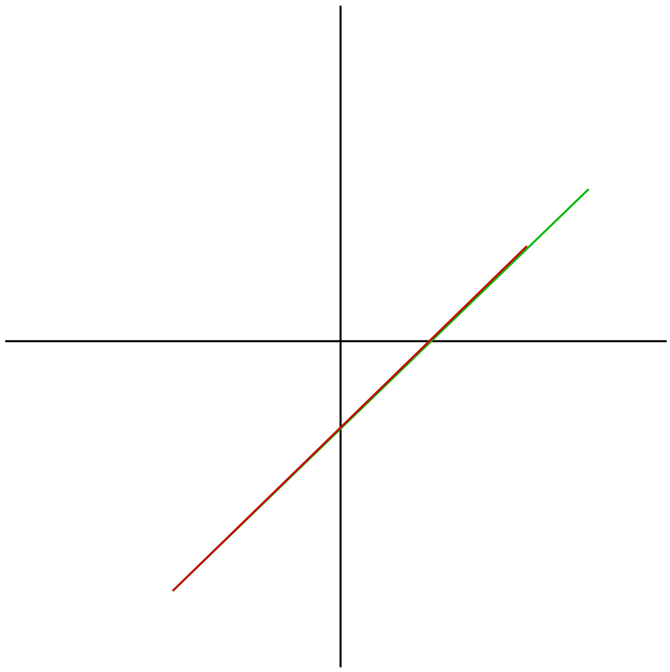

Figure 1.2: infinite solutions

**Example 3** (No solutions)**.**

$$\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 6 \end{aligned}$$

**Solution 3.**

$$\begin{aligned} [4x - 2y = 6] &\implies \\ 2x - y = 3 \\ -[2x - y = 6] \\ \hline \boxed{0 = -3} &\textit{ (false equation)} \end{aligned}$$

# Chapter 2

# Systems of Linear Equations and Matrices (Cont'd)

## Date: 2020-08-19

## Announcements:

20 minutes missed in lecture time will be posted as a video later today on blackboard.
**Assignment: Section 1.2 Numbers 5, 6, 7, 8**

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## 2.1 Introduction

Working towards solving systems of equations in a systematic way, vs. the algebra days when you attack it at any angle with substitution and elimination. This systematic way is setting up for the theoretical

**Example 4** (1)**.** 1. *take a system of equation like this*

$$3x + 3y - 2z = 13$$
$$6x + 2y - 5z = 13$$
$$7x + 5y - 3z = 26$$

2. *Use the algebraic operations:*
   (a) *Multiply an equation by a nonzero constant*
   (b) *Interchange any equation*
   (c) *Add a constant times one equation to another equation*

   $-1E2 + E3 \rightarrow E3$

$$\begin{aligned} -6x - 2y + 5z &= 13 \\ 7x + 5y - 3z &= 26 \end{aligned}$$

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$$x + 3y + 2z = 13$$

*to get a variable in the first position with a coeficient of 1 (The equation you add to [ E3 ] is always the equation you replace.)*

$$6x + 2y - 5z = 13$$

$$3x + 3y - 2z = 13$$

$$x + 3y + 2z = 13$$

*Interchange to get the equation with coeficient 1 on top, denoted like this:*

$E1 \leftrightarrow E3$

$$x + 3y + 2z = 13$$

$$6x + 2y - 5z = 13$$

$$3x + 3y - 2z = 13$$

3. *Use the $x$ to eliminate the $x$'s from the other equations below.*

   $-6E1 + E2 \rightarrow E3$

   $-3E1 + E3 \rightarrow E3$

$$\begin{aligned} x + 3y + 2z &= 13 \\ -16y - 17z &= -65 \\ -6y - 8z &= -26 \end{aligned}$$

4. *Get the next variable in column 2 with coeficient 1.*

   $\frac{-1}{16}E2 \rightarrow E2$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ -6y - 8z &= -26 \end{aligned}$$

5. *Use the $y$ to elimintate the $y$'s from below.*

$$6E1 + E3 \rightarrow E3$$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ -\frac{13}{8}z &= -\frac{13}{8} \end{aligned}$$

6. *Get the last variable in column 3 with coeficient 1. (Continue until diagonal of coeficient 1)*

   $\frac{-8}{13}E3 \rightarrow E3$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ z &= 1 \end{aligned}$$

7. *Work to eliminate variables from the bottom up.*
$-\frac{17}{16}E3 + E2 \to E2$
$-2E3 + E1 \to E1$

$$\begin{aligned} x + 3y &= 11 \\ y &= 3 \\ z &= 1 \end{aligned}$$

$-3E2 + E1 \to E1$

$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= 1 \end{aligned}$$

8. *Write solution as ordered tripple:* $(2, 3, 1)$

Because this system of equations could be written in this way, it has **only one** solution. There are other cases where there are problems in solving the system this way, a situation where there is no soution or infinite solutions.

**Example 5** (2)**.** *Solve:*

$$\begin{aligned} x + y + z &= 1 \\ -2x + y + z &= -2 \\ 3x + 6y + 6z &= 5 \end{aligned}$$

$2E_1 + E_2 \to E_2$
$-3E_1 + E_3 \to E_3$

$$\begin{aligned} x + y + z &= 1 \\ 3y + 3z &= 0 \\ 3y + 3z &= 2 \end{aligned}$$

$\frac{1}{3}E_2 \to E_2$

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 0 \\ 3y + 3z &= 2 \end{aligned}$$

$-3E_2 + E_3 \to E_3$

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 0 \\ 0 &= 2 \leftarrow \textit{false, no solution} \end{aligned}$$

*You could make the determination of no solution eariler, as soon as you catch the false statement.*

**Example 6** (3)**.**

$$\begin{aligned} 2x - y_z &= -1 \\ x + 3y - 2z &= 2 \\ -5x + 6y - 5z &= 5 \end{aligned}$$

$E_1 \leftrightarrow E_2$

$$\begin{aligned} x + 3y - 2z &= 2 \\ 2x - y + z &= -1 \\ -5x + 6y - 5z &= 5 \end{aligned}$$

$-2E_1 + E_2 \to E_2$
$5E_1 + E_3 \to E_3$

$$\begin{aligned} x + 3y - 2z &= 2 \\ -7y + 5z &= -5 \\ 21y - 5z &= 15 \end{aligned}$$

$\frac{-1}{7}E_2 \to E_2$

$$\begin{aligned} x + 3y - 2z &= 2 \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 21y - 5z &= 15 \end{aligned}$$

$-21E_2 + E_3 \to E_3$

$$\begin{aligned} x + 3y - 2z &= 2 \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 0 &= 0 \end{aligned}$$

*Where the last equation is a* ***true*** *statement, but there is no coeficient of 1, is a situation where there are* ***infinite solutions****. You must continue to elimintate up with what you got.*
$-3E_2 + E_1 \to E_1$

$$\begin{aligned} x + \frac{1}{7}z &= -\frac{1}{7} \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 0 &= 0 \end{aligned}$$

*This is now in as few variables as possible, and can give us a final solution*

$$\begin{aligned} x + \frac{1}{7}z = -\frac{1}{7} &\to \boxed{x = -\frac{1}{7}z - \frac{1}{7}} \\ y - \frac{5}{7}z = \frac{5}{7} &\to \boxed{y = \frac{5}{7}z + \frac{5}{7}} \\ 0 &= 0 \end{aligned}$$

$$\boxed{(x, y, z) = \left(-\frac{1}{7}z - \frac{1}{7}, \frac{5}{7}z + \frac{5}{7}, z\right)}$$

Thinking about this solution in terms of vectors, $z(-\frac{1}{7}, \frac{5}{7}, 1) + (-\frac{1}{7}, \frac{5}{5}, 0)$, this describes a line.

**Example 7** (4)**.**

$$\begin{aligned} x + 3y + 4z &= 1 \\ 2x + 6y + 8z &= 2 \\ 3x + 9y + 12z &= 3 \end{aligned}$$

$-2E_1 + E_2 \to E_2$
$-E_1 + E_3 \to E_3$

$$\begin{aligned} x + 3y + 4z &= 1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

*both equations with no variables are ture equations, therefore* ***infinite solutions****, and we have one equations to solve for x.*

$x + 3y + 4z = 1 \to x = -3y - 4z + 1$

$\boxed{(x, y, z) = (-3y + 4z + 1, y, z)}$

You can look at this as the sum of three vectors,
$(-3y, y, 0) + (-4z, 0, z) + (1, 0, 0)$ or $y(-3, 1, 0) + z(-4, 0, 1) + (1, 0, 0)$
This is just a geometric interpretation, whatever it means, but it is not a line

# Chapter 3

# Systems of Linear Equations and Matrices (Cont'd)

## Date: 2020-08-21

## Announcements:

**Assignment: 1.1 (5-10) 1.2 (1-22)** Solve them using Gaus-Jordan Elimination, as was shown, not gausian elimination.

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**Example 8** (solve)**.**

$$\begin{aligned} 2y + 3z &= 8 \\ 2x + 3y + z &= 5 \\ x - y - 2z &= -5 \end{aligned}$$

**Definition 1** (Matrix)**.** *A rectangular array of numbers.*

coeficient matrix

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

Constan matrix

$$\begin{bmatrix} 8 \\ 5 \\ -5 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix}$$

## 3.1 Reduced Row Echelon Form

1. If a row does not consist entirely of zeros, then the first non zero entry is a 1. (Leading one).
2. If any rows that are all zero, they appear at the bottom of the matrix.
3. In any two successive rows that are not all zeros, the leading one in lower row is further to the right than the 1 in the higher row. (This qualifies as Row Eschelon Form)
4. Each column that contains a leading 1 has zeros everywhere else in that column.

**Solution 4** (1)**.**

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$\frac{1}{5}R_2 \to R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} (Row Exchelon Form)$$

$-R_3 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$2R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} (Reduced)$$

$$x = 0$$
$$y = 1$$
$$z = 2$$

$\boxed{(0, 1, 2)}$

**Example 9** (Is this Row-eschelon or Reduced?). $\begin{bmatrix} 1 & 1 & 3 & 4 & 9 \\ 0 & 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Solution 5.** *Answer: Just row eschelon, there must be a zero above and below every leading one.*
$-2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -10 & -5 \\ 0 & 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ReducedRowEschelonForm$$

$(R_1)$ $x_1 + x_2 - 10x_3 = -5$
$(R_2)$ $x_3 + 7x_4 = 7$
$(R_1)$ $x_1 = -5 - x_2 + 10x_3$
$(R_2)$ $x_3 = 7 - 7x_4$

**Definition 2** (n-tuple). *Ordered pair with n entries in it.*
*E.g 4-tuple of solution above.*

$$(x_1, x_2, x_3, x_4) = (-5 - x_2 + 10x_4, x_2, 7 - 7x_4, x_4)$$

The tuple is always going to be parameterized, replacing $x_1$ with s's and $x_2$ with t's, etc.

$$(-5 - t + 10s, t, 7 - 7s, s)$$

# Chapter 4

# Matrices and Matric Operations

## Date: 2020-08-26

## Announcements:

**Assignment: Set 1.3 (1-6, 11-16, 23, 24)**

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### 4.1 Matrices

**Definition 3** (Matrix). *A rectangular array of numbers.*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Size: # Rows x # Columns
A : 2x3
B : 3x2

**Example 10** (A General matrix). *A is $m \times n$ elements*
*m-rows*
*n-columns*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_1n \\ a_{21} & a_{22} & a_{23} & \dots & a_2n \\ \dots & \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & a_{m_3} & \dots & a_{mn} \end{bmatrix}$$

### 4.1.1 Square matrix

**Definition 4** (Square Matrix)**.** *a matrix where #rows = #columns*

$$\begin{bmatrix} a_{11} & a_{22} & \dots & a1n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & \dots & a_{mn} \end{bmatrix}$$

*where $a_{11} \to a_{mn}$ is the main diagonal*

## 4.2 Matrix operations

**Theorem 1** (Matrix equality)**.** *Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.*

### 4.2.1 addition and subtraction

**Theorem 2.** *The sum of matrices A and B is written $A + B$ and it is the matrix obtained by adding corresponding entries of two matrices of the same size.*

**Example 11** (matrix addition)**.**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & -7 & 8 \\ -1 & 0 & 5 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 6 & -4 & 12 \\ 4 & 6 & 12 & 17 \end{bmatrix}$$

#### Notation

**A -** entire matrix

$a_{ij}$ **-** individual entries

$(A + B)_{ij}$ **-** notation of entry addition $(A)_{ij} + (B)_{ij}$

$(A - B)_{ij}$ **-** $(A)_{ij-(B)_{ij}}$

**Example 12** (Matrix addition)**.**

$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -7 & 9 \end{bmatrix} = \begin{bmatrix} -4 & -4 & 13 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -5 & -7 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 10 & -5 \end{bmatrix}$$

### 4.2.2 Product of a scalar, c, and a Matrix, A

The product of a scalar and a matrix, cA, is produced by multiplying each entry of A by c.

$$(cA)_{ij} = c(A)_{ij}$$

**Example 13.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$
$$B = \begin{bmatrix} -1 & 5 \\ -3 & 0 \\ 9 & 7 \end{bmatrix}$$
$$3A - B$$
$$\begin{bmatrix} 3 & 6 \\ 9 & 12 \\ 15 & 18 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ -3 & 0 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 12 & 12 \\ 6 & 11 \end{bmatrix}$$

### 4.2.3 Product of matrices

**Definition 5.** *The producto of two matrices A and B, written AB, is only defined when the number of columns of matrix A is equal to the number of rows of matrix B.*

$$A_{m\times r}B_{r\times n}$$

*The size of the product will be the rows of A by the columns of B.*

$$C_{m\times n}$$

**Example 14.**

$$A_{3\times 5} \; and \, B_{5\times 3}$$

*5 and 5: this can be done.*
*3 and 3: the size of the result*

#### Getting the entries

To find the entries in Row i and Column j of AB, single out the ith row of A and the jth column of B, multiply their corresponding entries and add the results.

**Example 15.**

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & 5 & -2 & 2 \end{bmatrix}_{2\times 4}$$

**Solution 6.**

$$\begin{bmatrix} 3\cdot -1 + 4\cdot 1 & 3\cdot 1 + 4\cdot 5 & 3\cdot 2 + 4\cdot 2 & 9+8 \\ -2+1 & 2+5 & 4+2 & 6+2 \\ -3+2 & 3+10 & 6+4 & 9+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 23 & 14 & 17 \\ -1 & 7 & 6 & 8 \\ -1 & 13 & 10 & 13 \end{bmatrix}$$

**Example 16.**

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 7 \\ 2 & 1 & 9 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Ax$$

**Solution 7.**

$$Ax = \begin{bmatrix} 2x_1 + x_2 + 3x_3 \\ 4x_1 - x_2 + 7x_3 \\ 2x_1 + x_2 + 9x_3 \end{bmatrix}$$

**Definition 6** (Transpose of a matrix)**.** *$A^T$ is the matrix obtained when the rows and columns of A are interchanged.*

**Example 17.**

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 9 & 3 & 1 \end{bmatrix}$$

**Solution 8.** $A^T = \begin{bmatrix} 1 & 9 \\ 2 & 3 \\ 5 & 1 \end{bmatrix}$