DATE: 2020-09-12

ANNOUNCEMENTS:

## 0.1 13.4 Motion in Space

### 0.1.1 Velcocity and Acceleration

$\overline{r}(t)$ position

$\overline{r}'(t) = \overline{v}(t)$ velocity

$\overline{r}''(t) = \overline{v}'(t) = \overline{a}(t)$ acceleration

$|\overline{v}|$ speed

**Example 1** (velocity)**.** *givem* $\overline{r}(t) = \left\langle t^2, 2t, \ln t \right\rangle$*, find* $\overline{v}(t)$ •

**Solution 1** (velocity)**.**

$$\overline{v}(t) = \overline{r}'(t) = \left\langle 2t, 2, \frac{1}{t} \right\rangle$$

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**Example 2** (speed)**.** *Find speed* •

**Solution 2** (speed)**.**

$$|\overline{v}(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}}$$

$$= \left| 2t + \frac{1}{t} \right|$$

*Note: we need the absolute value unless we know the domain*

*Domain* $\overline{r}(t)$ $t > 0$ $(0, \infty)$

$$\boxed{2t + \frac{1}{t}}$$

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**Example 3** (acceleration)**.** *Find acceleration.* •

**Solution 3** (acceleration)**.**

$$\overline{a}(t) = \overline{v}'(t) = \left\langle 2, 0, -\frac{1}{t^2} \right\rangle$$

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**Example 4** (find position and velocity given acceleration)**.** *Given* $\overline{A}(t) = \langle t, e^t, e^{-t} \rangle$ *and* $\overline{v}(0) = \langle 0, 0, 1 \rangle$*, and* $\overline{r}(0) = \langle 0, 1, 1 \rangle$*, Find* $\overline{v}(t)$ *and* $\overline{r}(t)$ •

**Solution 4** (find position and velocity given acceleration)**.**

$$\overline{v}(t) = \int \overline{a}(t)dt$$

$$= \left\langle \frac{1}{2}t^2, e^t, -e^{-t} \right\rangle + \langle c_1, c_2, c_3 \rangle$$

*Find constants:*

$$\langle 0, 0, 1 \rangle = \langle 0, 1, -1 \rangle + \langle c_1, c_2, c_3 \rangle$$

$$\langle 0, 0, 1 \rangle = \langle c_1, 1 + c_2, -1 + c_3 \rangle$$

$$\implies c_1 = 0, c_2 = -1, c_3 = 2$$

$$\overline{v}(t) = \boxed{\left\langle \frac{1}{2}t^2, e^t - 1, -e^{-t} + 2 \right\rangle}$$

$$\overline{r}(t) = \int \overline{v}(t)dt$$

$$= \left\langle \frac{1}{6}t^3, e^t - t, e^{-t+2t} \right\rangle + \langle c_1, c_2, c_3 \rangle$$

$$\langle 0, 1, 1 \rangle = \langle 0, 1, 1 \rangle + \langle c_1, c_2, c_3 \rangle$$

$$\implies c_1 = 0, c_2 = 0, c_3 = 0$$

$$\overline{r}(t) = \boxed{\left\langle \frac{1}{6}t^3, e^t - t, e^{-t+2t} \right\rangle}$$

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### 0.1.2 Acceleration Vector

$$\overline{T} = \frac{1}{|\overline{r}'|}\overline{r}'$$

$$\overline{r}' = |\overline{r}'|\,\overline{T}$$

$$\overline{v} = |\overline{v}|\,\overline{T}$$

$$\overline{a} = \overline{v}'(t) = |\overline{v}|\,\overline{T}' + |\overline{v}|'\,\overline{T}$$

We are trying to manufacture the unit normal vector for $\overline{T}'$

$$|\overline{v}|\,\overline{T}' = |\overline{v}|\,\frac{\left|\overline{T}'\right|}{\left|\overline{T}'\right|}\overline{T}'$$

$$\overline{N} = \frac{\overline{T}'}{\left|\overline{T}'\right|}$$

$$\implies \overline{a} = |\overline{v}|\left|\overline{T}'\right|\overline{N} + |\overline{v}|'\,\overline{T}$$

We have a scalar times unit vector plus a scalar times a unit vector. Therefore, you can take the acceleration and break into two components, $a_{\overline{N}}$ which is the size of the acceleration that goes in the direction of the unit normal vector, and $a_{\overline{T}}$ which goes in the direction of the unit tangent vector.

$$\rightarrow \overline{a} = a_{\overline{N}}\overline{N} + a_{\overline{T}}\overline{T}$$

$a_{\overline{N}}$

$$a_{\overline{N}} = |\overline{v}| \left|\overline{T}'\right|$$

Going back to curvature:

$$\kappa = \frac{\left|\overline{T}'\right|}{|\overline{r}'|} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^3}$$

$$\left|\overline{T}'\right| = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$

$$a_{\overline{N}} = |\overline{r}'| \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|^2}$$

$$\boxed{a_{\overline{N}} = \frac{|\overline{r}' \times \overline{r}''|}{|\overline{r}'|}} \tag{1}$$

*Normal component of the acceleration*

$$\implies \overline{a} = |\overline{v}| \left|\overline{T}'\right| \overline{N} + |\overline{v}|' \, \overline{T}$$

What if we dot the velocity with the acceleration?

$$\begin{aligned}
\overline{v} \cdot \overline{a} &= \overline{v} \cdot \left(|\overline{v}| \left|\overline{T}'\right| \overline{N} + |\overline{v}|' \, \overline{T}\right) \\
&= |\overline{v}| \left|\overline{T}'\right| \left(\overline{v} \cdot \overline{N}\right) + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right)
\end{aligned}$$

$\left(\overline{v} \cdot \overline{N}\right) \rightarrow \overline{r}' \cdot \overline{N} \rightarrow \frac{|\overline{r}'|}{|\overline{r}'|} \overline{r}' \cdot \overline{N} \rightarrow |\overline{r}'| \, \overline{T} \cdot \overline{N}$

$$\begin{aligned}
&= |\overline{r}'|^2 \left|\overline{T}'\right| \left(\overline{T} \cdot \overline{N}\right) + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right) \\
&= 0 + |\overline{v}|' \left(\overline{v} \cdot \overline{T}\right) \text{ (Tan and normal vectors orthagonal)} \\
\overline{v} \cdot \overline{a} &= \left|\overline{V}\right|' \left(\overline{v} \cdot \overline{T}\right) \\
\overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' \left(\overline{v} \cdot \overline{T}\right) \\
\overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' \left(\overline{r}' \cdot \overline{T}\right)
\end{aligned}$$

$\overline{r}' \cdot \overline{T} \rightarrow \frac{|\overline{r}'|}{|\overline{r}'|} \overline{r} \cdot \overline{T} \rightarrow |\overline{r}'| \, \overline{T} \cdot \overline{T}$

$$\begin{aligned}
\overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' |\overline{r}'| \, \overline{T} \cdot \overline{T} \\
\overline{r}' \cdot \overline{r}'' &= \left|\overline{V}\right|' |\overline{r}'| \left|\overline{T}\right|^2
\end{aligned}$$

$\left|\overline{T}\right|^2 = 1$

$$\begin{aligned}
\overline{r}' \cdot \overline{r}'' &= |\overline{v}|' \, |\overline{r}'| \\
\frac{\overline{r}' \cdot \overline{r}''}{|\overline{r}'|} &= |\overline{v}|'
\end{aligned}$$

$$\boxed{a_{\overline{T}} = \frac{\overline{r}' \cdot \overline{r}''}{|\overline{r}'|}} \tag{2}$$

*Tangential component of acceleration*

**Example 5** (Find components of acceleration). *Given $\overline{r}(t) = \left\langle t, 2e^{2t}, e^{-2t} \right\rangle$, find tangential and normal components of acceleration* •

**Solution 5** (Find components of acceleration).

$$\begin{aligned}
\overline{r}'(t) &= \left\langle 1, 4e^{2t}, -2e^{-25} \right\rangle \\
\overline{r}''(t) &= \left\langle 0, 8e^{2t}, 4e^{-2t} \right\rangle \\
\overline{r}' \cdot \overline{r}'' &= 0 + 32e^{4t} - 8e^{-4t} \\
&= 32e^{4t} - 8e^{-4t} \\
&= 8e^{-4t}\left(4e^{8t} - 1\right) \\
\overline{r}' \times \overline{r}'' &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4e^{2t} & -2e^{-2t} \\ 0 & 8e^{2t} & 4e^{-2t} \end{bmatrix} \\
&= \left\langle 16 + 16, -(4e^{-2t}), 8e^{2t} - 0 \right\rangle \\
&= \left\langle 32, -4e^{-2t}, 8e^{2t} \right\rangle \\
|\overline{r}' \times \overline{r}''| &= \sqrt{32^2 + 16e^{-4t} + 64e^{4t}} \\
&= \sqrt{16(64 + e^{-4t} + 4e^{4t}} \\
&= 4\sqrt{4e^{4t} + 64 + e^{-4t}} \\
|\overline{r}'| &= \sqrt{1^2 + (4e^{2t})^2 + (-2e^{-2t})^2} \\
&= \sqrt{1 + 16e^{4t} + 4e^{-4t}}
\end{aligned}$$

$$\begin{aligned}
\implies a_{\overline{T}} &= \frac{8e^{-4t}(4e^{8t} - 1)}{\sqrt{1 + 16e^{4t} + 4e^{-4t}}} \\
\implies a_{\overline{N}} &= \frac{4\sqrt{4e^{4t} + 64 + e^{-4t}}}{\sqrt{1 + 16e^{4t} + 4e^{-4t}}}
\end{aligned}$$

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