Lecture Notes

Linear Algebra

Fall 2020

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Instructed by:
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Contents

Chapter 1

Systems of Linear Equations and Matrices

Date: 2020-08-17

Announcements:

Instructor - Kathleen Kane

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Book - Elementary Linear Algebra: Applications Version by Howard Anton and Chris Rorries, 11th edition 9781118434413

Assignment(Aug 19 08:50): practice uploading 3 scanned pages in a single pdf

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1.1 Policies and Procedures

1.1.1 Learning Outcomes

1. Perform basic operations with vectors in n-dimensional space.
2. Perform basic operations with matrices.
3. Solve a system of m linear equations in n unknowns.
4. Prove basic theorems in a vector space.
5. Perform basic operations with vectors in the standard matrix spaces and function space.
6. Find the matrix repersentation of a linear transformation between two vector spaces.
7. Find eigenvalues and eigenvectors for a given matrix.
8. Perform basic operations in an inner product space
9. Prove basic theorems in an inner product space.

1.1.2 Evaluation

1. Assignments (10%)
2. Testes (70%)
3. Final (weighted) (20%)

1.1.3 Testing

1. Required to scan test and submit via pdf
2. 50 minutes each test and 10 minutes to submit test
3. No make up tests
4. One test may be substituted with final exam grade
5. Missing final is automatic F.

1.2 Introduction to Systems of Linear Equations and Matrices

Example 1 (One solution)**.** *Solve:*

$$\begin{aligned} 3x + y &= 6 \\ 5x - 3y &= 10 \end{aligned}$$

Solution 1.

$$\begin{aligned} 3x + y = 6 &\implies \\ 9x + 3y &= 18 \\ +[5x - 3y &= 10] \\ \hline 14x = 28 &\implies \\ \boxed{x = 2} \end{aligned}$$

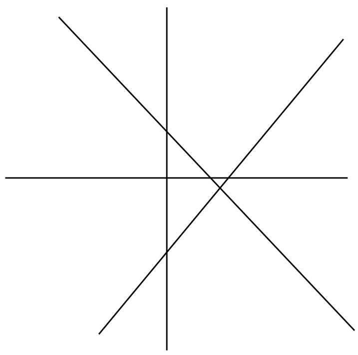

Figure 1.1: one solution

Example 2 (Infinite solutions)**.** *Solve:*

$$\begin{aligned} 2x - y &= 7 \\ 4x - 2y &= 14 \end{aligned}$$

Solution 2.

$$\begin{aligned} 4x - 2y = 14 &\implies \\ 2x - y = 7 & \\ -[2x - y = 7] & \\ \hline \boxed{0 = 0} &\ \textit{(no solution)} \end{aligned}$$

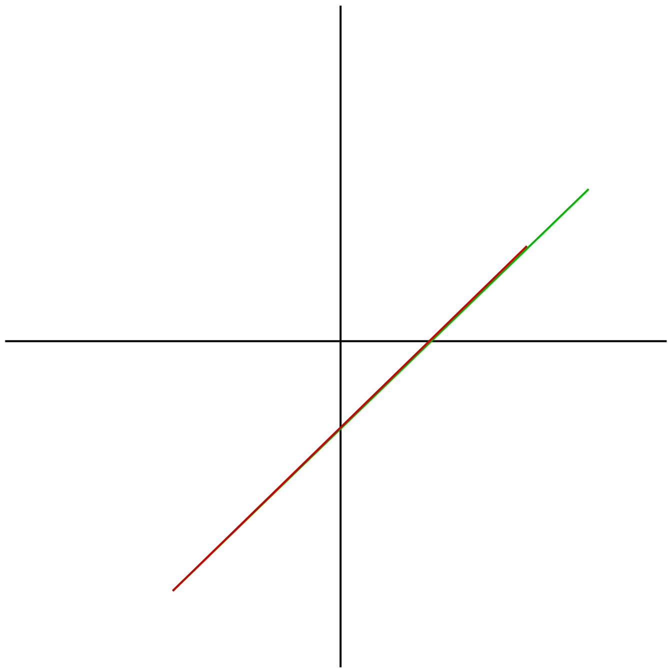

Figure 1.2: infinite solutions

Example 3 (No solutions)**.**

$$\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 6 \end{aligned}$$

Solution 3.

$$\begin{aligned} [4x - 2y = 6] &\implies \\ 2x - y = 3 & \\ -[2x - y = 6] & \\ \hline \boxed{0 = -3} &\ \textit{(false equation)} \end{aligned}$$

Date: 2020-08-19
Announcements:

20 minutes missed in lecture time will be posted as a video later today on blackboard.
Assignment: Section 1.2 Numbers 5, 6, 7, 8

1.2.1 1.1 Continued

Working towards solving systems of equations in a systematic way, vs. the algebra days when you attack it at any angle with substitution and elimination. This systematic way is setting up for the theoretical

Example 4 (1). *1. take a system of equation like this*

$$3x + 3y - 2z = 13$$

$$6x + 2y - 5z = 13$$

$$7x + 5y - 3z = 26$$

2. Use the algebraic operations:

(a) Multiply an equation by a nonzero constant

(b) Interchange any equation

(c) Add a constant times one equation to another equation

$-1E2 + E3 \to E3$

$$\begin{array}{r} -6x - 2y + 5z = 13 \\ 7x + 5y - 3z = 26 \\ \hline \\ x + 3y + 2z = 13 \end{array}$$

to get a variable in the first position with a coeficient of 1 (The equation you add to [E3] is always the equation you replace.)

$$6x + 2y - 5z = 13$$

$$3x + 3y - 2z = 13$$

$$x + 3y + 2z = 13$$

Interchange to get the equation with coeficient 1 on top, denoted like this:

$E1 \leftrightarrow E3$

$$x + 3y + 2z = 13$$

$$6x + 2y - 5z = 13$$

$$3x + 3y - 2z = 13$$

3. *Use the x to eliminate the x's from the other equations below.*
   $-6E1 + E2 \rightarrow E3$
   $-3E1 + E3 \rightarrow E3$

$$\begin{aligned} x + 3y + 2z &= 13 \\ -16y - 17z &= -65 \\ -6y - 8z &= -26 \end{aligned}$$

4. *Get the next variable in column 2 with coeficient 1.*
   $\frac{-1}{16}E2 \rightarrow E2$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ -6y - 8z &= -26 \end{aligned}$$

5. *Use the y to elimintate the y's from below.*

$$6E1 + E3 \rightarrow E3$$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ -\frac{13}{8}z &= -\frac{13}{8} \end{aligned}$$

6. *Get the last variable in column 3 with coeficient 1. (Continue until diagonal of coeficient 1)*
   $\frac{-8}{13}E3 \rightarrow E3$

$$\begin{aligned} x + 3y + 2z &= 13 \\ y + \frac{17}{16}z &= \frac{65}{16} \\ z &= 1 \end{aligned}$$

7. *Work to eliminate variables from the bottom up.*
   $-\frac{17}{16}E3 + E2 \rightarrow E2$
   $-2E3 + E1 \rightarrow E1$

$$\begin{aligned} x + 3y &= 11 \\ y &= 3 \\ z &= 1 \end{aligned}$$

   $-3E2 + E1 \rightarrow E1$

$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= 1 \end{aligned}$$

8. Write solution as ordered tripple: $(2,3,1)$

Because this system of equations could be written in this way, it has **only one** solution. There are other cases where there are problems in solving the system this way, a situation where there is no soution or infinite solutions.

Example 5 (2). *Solve:*

$$\begin{aligned} x+y+z&=1\\ -2x+y+z&=-2\\ 3x+6y+6z&=5 \end{aligned}$$

$2E_1+E_2\to E_2$
$-3E_1+E_3\to E_3$

$$\begin{aligned} x+y+z&=1\\ 3y+3z&=0\\ 3y+3z&=2 \end{aligned}$$

$\frac{1}{3}E_2\to E_2$

$$\begin{aligned} x+y+z&=1\\ y+z&=0\\ 3y+3z&=2 \end{aligned}$$

$-3E_2+E_3\to E_3$

$$\begin{aligned} x+y+z&=1\\ y+z&=0\\ 0&=2\leftarrow \textit{false, no solution} \end{aligned}$$

You could make the determination of no solution eariler, as soon as you catch the false statement.

Example 6 (3).

$$\begin{aligned} 2x-y_z&=-1\\ x+3y-2z&=2\\ -5x+6y-5z&=5 \end{aligned}$$

$E_1\leftrightarrow E_2$

$$\begin{aligned} x+3y-2z&=2\\ 2x-y+z&=-1\\ -5x+6y-5z&=5 \end{aligned}$$

$-2E_1+E_2\to E_2$
$5E_1+E_3\to E_3$

$$\begin{aligned} x + 3y - 2z &= 2 \\ -7y + 5z &= -5 \\ 21y - 5z &= 15 \end{aligned}$$

$\frac{-1}{7}E_2 \to E_2$

$$\begin{aligned} x + 3y - 2z &= 2 \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 21y - 5z &= 15 \end{aligned}$$

$-21E_2 + E_3 \to E_3$

$$\begin{aligned} x + 3y - 2z &= 2 \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 0 &= 0 \end{aligned}$$

*Where the last equation is a **true** statement, but there is no coeficient of 1, is a situation where there are **infinite solutions**. You must continue to elimintate up with what you got.*
$-3E_2 + E_1 \to E_1$

$$\begin{aligned} x + \frac{1}{7}z &= -\frac{1}{7} \\ y - \frac{5}{7}z &= \frac{5}{7} \\ 0 &= 0 \end{aligned}$$

This is now in as few variables as possible, and can give us a final solution

$$\begin{aligned} x + \frac{1}{7}z &= -\frac{1}{7} \to \boxed{x = -\frac{1}{7}z - \frac{1}{7}} \\ y - \frac{5}{7}z &= \frac{5}{7} \to \boxed{y = \frac{5}{7}z + \frac{5}{7}} \\ 0 &= 0 \end{aligned}$$

$$\boxed{(x, y, z) = \left(-\frac{1}{7}z - \frac{1}{7}, \frac{5}{7}z + \frac{5}{7}, z\right)}$$

Thinking about this solution in terms of vectors, $z(-\frac{1}{7}, \frac{5}{7}, 1) + (-\frac{1}{7}, \frac{5}{5}, 0)$, this describes a line.

Example 7 (4)**.**

$$\begin{aligned} x + 3y + 4z &= 1 \\ 2x + 6y + 8z &= 2 \\ 3x + 9y + 12z &= 3 \end{aligned}$$

$-2E_1 + E_2 \to E_2$
$-E_1 + E_3 \to E_3$

$$\begin{aligned} x + 3y + 4z &= 1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

both equations with no variables are ture equations, therefore ***infinite solutions****, and we have one equations to solve for x.*

$x + 3y + 4z = 1 \to x = -3y - 4z + 1$

$\boxed{(x, y, z) = (-3y + 4z + 1, y, z)}$

You can look at this as the sum of three vectors,
$(-3y, y, 0) + (-4z, 0, z) + (1, 0, 0)$ or $y(-3, 1, 0) + z(-4, 0, 1) + (1, 0, 0)$
This is just a geometric interpretation, whatever it means, but it is not a line

Date: 2020-08-21

Announcements:

Assignment: 1.1 (5-10) 1.2 (1-22) Solve them using Gaus-Jordan Elimination, as was shown, not gausian elimination.

1.3 Gaussian elimination

Example 8 (solve)**.**

$$\begin{aligned} 2y + 3z &= 8 \\ 2x + 3y + z &= 5 \\ x - y - 2z &= -5 \end{aligned}$$

Definition 1 (Matrix)**.** *A rectangular array of numbers.*

coeficient matrix

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

Constan matrix

$$\begin{bmatrix} 8 \\ 5 \\ -5 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix}$$

1.3.1 Reduced Row Eschelon Form

1. If a row does not consist entirely of zeros, then the first non zero entry is a 1. (Leading one).
2. If any rows that are all zero, they appear at the bottom of the matrix.
3. In any two successive rows that are not all zeros, the leading one in lower row is further to the right than the 1 in the higher row. (This qualifies as Row Eschelon Form)
4. Each column that contains a leading 1 has zeros everywhere else in that column.

Solution 4 (1)**.**

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$-2R_1 + R_2 \rightarrow R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$\frac{1}{5}R_2 \to R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} (RowExchelonForm)$$

$-R_3 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$2R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} (Reduced)$$

$$x = 0$$
$$y = 1$$
$$z = 2$$

$\boxed{(0, 1, 2)}$

Example 9 (Is this Row-eschelon or Reduced?). $\begin{bmatrix} 1 & 1 & 3 & 4 & 9 \\ 0 & 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution 5. *Answer: Just row eschelon, there must be a zero above and below every leading one.*
$-2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -10 & -5 \\ 0 & 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ReducedRowEschelonForm$$

(R_1) $x_1 + x_2 - 10x_3 = -5$
(R_2) $x_3 + 7x_4 = 7$
(R_1) $x_1 = -5 - x_2 + 10x_3$
(R_2) $x_3 = 7 - 7x_4$

Definition 2 (n-tuple)**.** *Ordered pair with n entries in it.*
E.g 4-tuple of solution above.

$$(x_1, x_2, x_3, x_4) = (-5 - x_2 + 10x_4, x_2, 7 - 7x_4, x_4)$$

The tuple is always going to be parameterized, replacing x_1 with s's and x_2 with t's, etc.

$$(-5 - t + 10s, t, 7 - 7s, s)$$

Date: 2020-08-26

Announcements:

Assignment: Set 1.3 (1-6, 11-16, 23, 24)

1.4 Matrices and Matrix opperations

Definition 3 (Matrix)**.** *A rectangular array of numbers.*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Size: # Rows x # Columns
A : 2x3
B : 3x2

Example 10 (A General matrix)**.** *A is $m \times n$ elements*
m-rows
n-columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_1n \\ a_{21} & a_{22} & a_{23} & \dots & a_2n \\ \dots & \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & a_{m_3} & \dots & a_{mn} \end{bmatrix}$$

1.4.1 Square matrix

Definition 4 (Square Matrix)**.** *a matrix where #rows = #columns*

$$\begin{bmatrix} a_{11} & a_{22} & \dots & a1n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & \dots & a_{mn} \end{bmatrix}$$

where $a_{11} \to a_{mn}$ is the main diagonal

1.4.2 Matrix Opperations

Theorem 1 (Matrix equality)**.** *Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.*

1.4.3 addition and subtraction

Theorem 2. *The sum of matrices A and B is written $A + B$ and it is the matrix obtained by adding corresponding entries of two matrices of the same size.*

Example 11 (matrix addition)**.**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & -7 & 8 \\ -1 & 0 & 5 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 6 & -4 & 12 \\ 4 & 6 & 12 & 17 \end{bmatrix}$$

Notation

A - entire matrix

a_{ij} **-** individual entries

$(A + B)_{ij}$ **-** notation of entry addition $(A)_{ij} + (B)_{ij}$

$(A - B)_{ij}$ **-** $(A)_{ij-(B)_{ij}}$

Example 12 (Matrix addition)**.**

$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -7 & 9 \end{bmatrix} = \begin{bmatrix} -4 & -4 & 13 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -5 & -7 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 10 & -5 \end{bmatrix}$$

1.4.4 Product of a scalar, c, and a Matrix, A

The product of a scalar and a matrix, cA, is produced by multiplying each entry of A by c.

$$(cA)_{ij} = c(A)_{ij}$$

Example 13.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$
$$B = \begin{bmatrix} -1 & 5 \\ -3 & 0 \\ 9 & 7 \end{bmatrix}$$
$$3A - B$$
$$\begin{bmatrix} 3 & 6 \\ 9 & 12 \\ 15 & 18 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ -3 & 0 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 12 & 12 \\ 6 & 11 \end{bmatrix}$$

1.4.5 Product of matrices

Definition 5. *The producto of two matrices A and B, written AB, is only defined when the number of columns of matrix A is equal to the number of rows of matrix B.*

$$A_{m\times r}B_{r\times n}$$

The size of the product will be the rows of A by the columns of B.

$$C_{m\times n}$$

Example 14.

$$A_{3\times 5}\ and\ B_{5\times 3}$$

5 and 5: this can be done.
3 and 3: the size of the result

Getting the entries

To find the entries in Row i and Column j of AB, single out the ith row of A and the jth column of B, multiply their corresponding entries and add the results.

Example 15.

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & 5 & -2 & 2 \end{bmatrix}_{2\times 4}$$

Solution 6.

$$\begin{bmatrix} 3\cdot -1 + 4\cdot 1 & 3\cdot 1 + 4\cdot 5 & 3\cdot 2 + 4\cdot 2 & 9+8 \\ -2+1 & 2+5 & 4+2 & 6+2 \\ -3+2 & 3+10 & 6+4 & 9+4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 23 & 14 & 17 \\ -1 & 7 & 6 & 8 \\ -1 & 13 & 10 & 13 \end{bmatrix}$$

Example 16.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 7 \\ 2 & 1 & 9 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Ax$$

Solution 7.

$$Ax = \begin{bmatrix} 2x_1 + x_2 + 3x_3 \\ 4x_1 - x_2 + 7x_3 \\ 2x_1 + x_2 + 9x_3 \end{bmatrix}$$

Definition 6 (Transpose of a matrix)**.** *A^T is the matrix obtained when the rows and columns of A are interchanged.*

Example 17.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 9 & 3 & 1 \end{bmatrix}$$

Solution 8. $A^T = \begin{bmatrix} 1 & 9 \\ 2 & 3 \\ 5 & 1 \end{bmatrix}$

Date: 2020-08-28

Announcements:

Exam date will be updeated on Blackboard today. **Assignmtent: first homework due Wednesday**

1.4.6 Trace of a Matrix

Definition 7 (Trace)**.** *If A is a square matrix, then the trace of the matrix, dentoed by $tr(A)$ is the sum of the entries on the main diagonal.*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$tr(A) = 15$$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

$$tr(A) \text{ not defined}$$

1.4.7 Linear Combinations

Definition 8 (Linear combination)**.** *If $A_1, A_2, \ldots, A_r$ are matrices of the same size and if $C_1, C_2, \ldots, C_r$ are scalars, then an expression of the form*

$$C_1A_1 + C_2A_2 + \ldots + C_rA_r$$

is called a linear combination of $A_1, A_2, \ldots, A_r$ with coefficients $C_1, C_2, \ldots, C_r$.

Theorem 3 (1.3.1)**.** *If A is an $m \times n$ matrix and if x is an $n \times 1$ column vector, then the product Ax can be expressed as a linear combination of the column vectors in A in which the coefficients are the entries of x.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 32 \\ 37 & 68 \end{bmatrix}$$

We begin thinking about this in different ways.

$$= \begin{bmatrix} 16 & 32 \\ 37 & 68 \end{bmatrix} \implies \begin{bmatrix} A_{b_1} & A_{b_2} \end{bmatrix}$$

In general:

$$AB = A[b_1b_2 \ldots b_n] = [Ab_1Ab_2 \ldots Ab_n]$$

Nothing says you can't partition differently.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 5 & 9 \end{bmatrix}$$
$$\implies \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} B$$
$$= \begin{bmatrix} a_1 B \\ a_2 B \end{bmatrix}$$

1.5 Inverse and Algebraic Properties of Matrices

The idea behind this section is we want to see these matrices like numbers in arithmetic.

Definition 9 (zero matrix). *a matrix of all zero's.*

Definition 10 (identity matrix). *An $n \times n$ identity matrix is a square matrix with 1's on the main diagonal and 0's everywhere else.*

Definition 11 (Additive inverse).

$$A + (-A) = 0$$

Definition 12 (multiplication inverse). A^{-1}

Date: 2020-08-31

Announcements:

Test: Friday, September 11th, or moved to Monday 14th if still catching up.
Assignment: Sec 1.4 (1-4, 51-58)

1.5.1 Matrix Arithmetic

Theorem 4 (1.4.1 Properties of matrix arithmetic)**.** *Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.*

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A(BC) = (AB)C$

$$A_{m\times n}B_{n\times r}C_{r\times w}$$

"Operations must be valid"

- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $A(B - C) = AB - AC$
- $(B - C)A = BA - CA$
- $a(A + B) = aA + aB$
- $a(A - B) = aA - aB$
- $(a + b)A = aA + bA$
- $(a - b)A = aA - bA$
- $a(bC) = (ab)C$
- $a(BC) = (aB)C = B(aC)$

How to Verify $a(A + B) = aA + aB$:

1. Must show that each side produces a matrix of the same size:
   let A be an $m \times n$ matrix.
   let B be an $m \times n$ matrix
   A+B is defined as an $m \times n$ matrix
   a(A+B) is defined and is an $m \times n$ matrix
   Let A be an $m \times n$ matrix. Let B be an $m \times n$ matrix
   aA is an $m \times n$ matrix.
   aB is an $m \times n$ matrix.
   $\boxed{aA + aB \text{ is defined and is an } m \times n \text{ matrix.}}$

2. Show that the corresponding entries of each side are equal:
   $a(A+B)$
   $(a(A+B))_{ij} = a(a_{ij} + b_{ij})$ {ALL SCALARS $\implies$
   $= aa_{ij} + ab_{ij}$
   $= a(A)_{ij} + a(B)_{ij}$
   $= aA + aB$

Example 18 (Theorem 1.4.1)**.** *Example to be done outside class. Prove 1.4.1 Theorems. 1.4.1c is the hardest.*

Same thing with 1.4.2

Inverse of a Matrix

Definition 13 (Inverse Matrix)**.** *If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be invertible (or nonsingular) and B is called the inverse of A. If no such B can be found, then A is said to be not invertible or singular.*

Date: 2020-09-02

Announcements:

As an exercise for the reader, theorems 1.4.7 - 1.4.9 should be examined.
Assignment: 1.4 (1-22, 25-28, 29-31, 39, 40, 45, 49, 50)

1.6 Inverse

1.6.1 4×4 Example

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \,/\!= 0$, in which case the inverse is given by the formula

$$A^{-1} \neq \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- $ad - bc$ is called the determinant of A, $det(A)$.

Identity Proof

$$AA^{-1} = I$$
$$A^{-1}A = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$\frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 19. *Solve*

$$2x + 3y = 5$$
$$-x + 7y = 12$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}_{2\times 1}$$
$$[A][x] = b$$

If A is invertible:

$$A^{-1}Ax = A^{-1}b$$

The inverse must be a left side multiplication on both sides of equations.

$$x = A^{-1}b$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = []$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}_{2\times 1}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$\frac{1}{17}\begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}\begin{bmatrix} x \\ 7 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -1 \\ 29 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{17} \\ \frac{29}{17} \end{bmatrix}$$

Theorem 5. *If B and C are both inverses of A, then $B = C$*

Proof If B is an inverse of A, $AB = BA = I$
If C is an inverse of A, $AC = CA = I$

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

□

Theorem 6. *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof

$$(AB)(AB)^{-1} = (AB)B^{-1}A^{-1} = (AI)A^{-1} = AA^{-1} = I$$
$$(AB)^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = I$$

Corollary 1.

$$A_1, A_2, \ldots, A_n$$

are all invertible matrices of the same size, then

$$(A_1A_2\ldots A_n)$$

is invertible and

$$(A_1A_2\ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\ldots A_2^{-1}A_1^{-1}$$

Definition 14. *If A is a square matrix, then*

$$A^n = AAA\ldots A\{n factors$$

$$A^0 = I$$

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\ldots A^{-1}$$

Theorem 7. *If A is invertible and n is a nonnegative integer, then:*
A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
kA *is invertible for any nonzero scalar k, and* $(kA)^{-1} = k^{-1}A^{-1}$

Theorem 8. *If the sizes of the matrices are such that the stated operations can be performed, then:*

Table 1.1: caption

(a)	$(A^T)^T = A$
(b)	$(A+B)^T = A^T + B^T$
(c)	$(A-B)^T = A^T - B^T$
(d)	$(kA)^T = kA^T$
(e)	$(AB)^T = A^T B^T$

Theorem 9. *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Date: 2020-09-04

Announcements:

Assignment: section 1.5 (1-20)

Watch weekend video.

1.7 Elementary Matrices and Methods for Finding A^{-1}

Definition 15 (Row Equivalent)**.** *Matrices A and B are called Row Equivalent if either is obtained from the other by a sequence of row operations.*

Definition 16 (Elementary Matrix)**.** *A matrix E is called an elementary matrix if it can be obtained from the identity matrix by performing one row operation.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product EA, the result is the same as performing the elementary row operation on a yourself.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$2R_2 \to R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix}$$

Is the same as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Theorem 10. *Every elementary matrix is invertible and its inverse is also an elementary matrix.*

Equivalent Statements Theorem

Theorem 11. *If A is an $n \times n$ matrix, then the following are equivalent, that is all are true or all are false,*

A *A is invertible.*

B *$Ax = 0$ has only the solution c* $\begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$ *(trivial solution).*

C *The reduced row echelon form of A is I_n.*

D A is expressible as a product of elementary matrices.

Proof: equivalent statements.

- Assume A true $\rightarrow$ B true $\rightarrow$ C true $\rightarrow$ D true $\rightarrow$ A true
- Assume A is invertible $\rightarrow A^{-1}$ exists and $AA^{-1} = A^{-1}A = I_n$
- $Ax = 0$
- $(A^{-1}Ax = A^{-1}0 \rightarrow Ix = \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \end{bmatrix}$
- $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vert & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & \vert & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vert & 0 \end{bmatrix} \begin{bmatrix} x_1 & & & \vert & 0 \\ & x_2 & & \vert & 0 \\ & & x_3 & \vert & 0 \end{bmatrix}$
- $[A|0]$, perform row operations and get $[I|0]$

$$E_n \dots E_2 E_1 A = I$$

$$E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1} E_n^{-1} E_n \dots E_2 E_1 A = E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1} E_n^{-1} I$$

- (theorem) If A and B are invertible then (AB) is invertible $\rightarrow$

$$(AB)^{-1} = B^{-1}A^{-1}$$

- $\rightarrow$ A is true.

$\square$

Date: 2020-09-09
Announcements:

1.7.1 Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of Elementary row operations that reduce A to the Identity and then perform that same sequence of operations on I_n to obtain A^{-1}

Example 20 (Inversion Algorithm)**.**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

$$R_1 + R_2 \to R_1$$

$$A = \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ -1 & 3 & 4 & | & 0 & 1 & 1 \\ 0 & 4 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 4 & 4 & | & 1 & 1 & 0 \\ 0 & 4 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_2 \to R_2$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$-4R_2 + R_3 \to R_3$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & | & -1 & -1 & 1 \end{bmatrix}$$

$$-R_3 \to R_3$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & | & 1 & 1 & -1 \end{bmatrix}$$

If a zero results in the main diagonal of the matrix you are inverting, it is not invertible.

$$-R_3 + R_2 \to R_2$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & \frac{-3}{4} & \frac{-3}{4} & 1 \\ 0 & 0 & 1 & | & 1 & 1 & -1 \end{bmatrix}$$

$$-R_2 + R_1 \to R_1$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{7}{4} & \frac{3}{4} & 1 \\ 0 & 1 & 0 & | & \frac{-3}{4} & \frac{-3}{4} & 1 \\ 0 & 0 & 1 & | & 1 & 1 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & 1 \\ \frac{-3}{4} & \frac{-3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}}$$

To prove it, multiply the origional by the inverse to get the identity:

$$AA^{-1} = I$$

$$A^{-1}A = I$$

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & 1 \\ \frac{-3}{4} & \frac{-3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 01 & 0 & \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & 1 \\ \frac{-3}{4} & \frac{-3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 01 & 0 & \\ 0 & 0 & 1 \end{bmatrix}$$

Date: 2020-09-09

Announcements:

Assignment: 1.6 1-19 odd

Turn in posted Friday, due Monday at 4 pm

Text chapter 1: 9/16

1.8 More on Linear Systems and Invertible Matrices

Example 21 (Solving system using the inverse)**.** *Solve:*

$$\begin{aligned} 2x + 6y + 6z &= 1 \\ 2x + 7y + 6z &= 2 \\ 2x + 7y + 7z &= 3 \end{aligned}$$

$$Ax = b$$

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If A^{-1} exists,

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

$$\begin{aligned} \frac{1}{2}R_1 &\to R_1 \\ -R_2 + R_3 &\to R_3 \end{aligned}$$

$$\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 2 & 7 & 6 & | & 0 & 1 & 1 \\ 2 & 7 & 7 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 3 & | & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & | & 0 & 1 & 1 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & | & \frac{1}{2} & 3 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\left(-\frac{11}{2}, 1, 1\right)}$$

Theorem 12 (1.6.1)**.** *A system of linear equations has zero, one or an infinite number of solutions.*

$$Ax = b$$

A *has Reduced Row Eschelon form that is I_n.*

or

A *has Reduced Row Eschelon form that is not.*

*Suppose that $Ax = b$ has **two** solutions x_1 and x_2 (proof by contradiction)*
We know

$$\begin{aligned} Ax_1 &= b \\ Ax_2 &= b \\ Ax_1 - Ax_2 &= A(x_1 - x_2) \\ b - b &= A(x_1 - x_2) \implies \\ A(x_1 - x_2) &= 0 \end{aligned}$$

By equivalence principal, the matrix $x_1 - x_2 = 0$ has only trivial solution $\implies \boxed{x_1 = x_2}$

If A is an $n \times n$ matrix, the following are equivalent

1. A is invertible
2. $Ax = 0$ has only the trivial solution
3. The reduced row eschelon form is I
4. A is expressible as a product of Elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix b. (1 solution or an infinite)
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b:

$$\begin{aligned} Ax &= b \\ x &= \boxed{A^{-1}b} \end{aligned}$$

1.8.1 A Fundamental Problem

Example 22 (A fundamental Problem)**.** *Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices b such that the system of equations $Ax = b$ is consistent.*

$$2 + y = b_1$$
$$-2y + 4z = b_2$$
$$3x - 2z = b_3$$
$$\begin{bmatrix} 2 & 1 & 0 & | & b_1 \\ 0 & -2 & 4 & | & b_2 \\ 3 & 0 & -2 & | & b_3 \end{bmatrix} \rightarrow$$
$$\frac{1}{2}R_1 \rightarrow R_1$$
$$\frac{-1}{2}R_2 \rightarrow R_2$$
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 3 & 0 & -2 & | & b_3 \end{bmatrix} \rightarrow$$
$$-3R_1 + R_3 \rightarrow R_3$$
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 0 & -\frac{3}{2} & -2 & | & b_3 - \frac{3}{2}b_1 \end{bmatrix} \rightarrow$$
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & \frac{1}{2}b_1 \\ 0 & 1 & -2 & | & -\frac{1}{2}b_2 \\ 0 & 0 & -5 & | & b_3 - \frac{3}{2}b_1 - \frac{3}{4}b_2 \end{bmatrix} \rightarrow$$

Example 23 (Different End of Fundamental problem)**.**

$$\begin{bmatrix} 1 & 2 & 0 & b_1 \\ 9 & 1 & 0 & 2b_1 - 3b_2 \\ 0 & 0 & 0 & b_1 - 3b_2 + b_3 \end{bmatrix}$$

$$\rightarrow b_1 - 3b_2 + b_3 = 0$$

$$\rightarrow b_3 = 3b_2 - b_1$$

$$\boxed{\begin{bmatrix} b_1 \\ b_2 \\ 3b_2 - b_1 \end{bmatrix}}$$

Date: 2020-09-11

Announcements:

Test on Wednesday, 8:50
Assignment: due Monday 4:00 PM

1.9 (1.7) Definitions of Matrices

Definition 17 (Diagonal Matrix)**.** *A square matrix where all the entries off of the main diagonal are zero.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Properties of diagonal matrices

- Multiplication is easy. Multiplying a matrix by a diagonal from the left gives the entry in the diagonal times the row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 8 \\ 15 & 18 \end{bmatrix}$$

- Multiplying from the right gives the diagonal entry times the column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 9 \\ -4 & 10 & 18 \end{bmatrix}$$

- Square a diagonal squares the entries.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

- Inverting the diagonal gives the reciprical of the entries

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 3 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 4 & | & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & | & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

- If it is invertible

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

D^{-1} (D is not invertible.)

Definition 18 (Triangular Matrices)**. Upper Triangular** *All non-zero entries are on the main diagonal or above.*

Lower Triangular *All non-zero entries are on the main diagonal or below.*

Example 24 (Upper triangular matrix)**.**

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Example 25 (Lower triangular matrix)**.**

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Theorem 13 (1.7.1)**.** *.*

1. *The **transpose** of a **lower** triangular is **upper** triangular and the transpose of an upper triangular matrix is lower triangular.*

2. *The **product** of **two lower** triangular matrices is **lower** triangular and the product of two upper triangular matrices is upper triangular.*

3. *A triangular matrix is **invertible if** and only if it's **diagonal** entries are all **non-zero**.*

4. *The **inverse** of an invertible **lower** triangular matrix **is lower** triangular and the inverse of an invertible **upper** triangular matrix **is upper** triangular.*

1.9.1 Symmetric Matrix

Definition 19 (Symmetric Matrix)**.** *A square matrix where* $A = A^T$*.*

$$(A)_{ij} = (A)_{ji}$$

Example 26 (Symmetric matrix)**.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

•

Theorem 14. *If A and B are symmetric matrices with the same size, and if k is any scalar, then*

1. *A^T is symmetric.*
2. *$A + B$ and $A - B$ are symmetric.*
3. *kA is symmetric.*

•

Theorem 15. *The product of tow symmetric matrices is symmetric if and only if the matrices commute.*

•