Problem

Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

a. {1, 3, 5, 7, . . . }

b. { . . . ,−4,−2, 0, 2, 4, . . . }

c. {nl n = 2m for some m in N}

d. {nl n = 2m for some m in N, and n = 3k for some k in N}

e. {wl w is a string of 0s and 1s and w equals the reverse of w}

f. {nl n is an integer and n = n + 1}

Step-by-step solution

Step 1 of 5

a)

A set of all odd natural numbers.

b)

A set of all even integers.

Comment

Step 2 of 5

c)

A set of all even natural numbers.

(or) A set of all natural numbers divisible by 2.

Comment

Step 3 of 5

d)

A set of all natural numbers, divisible by both 2 and 3.

(or) A set of all natural multiples of 6.

Comment

Step 4 of 5

e)

The set of all strings comprising of 0's and 1's and every string is a palindrome.

Comment

Step 5 of 5

f)

The set of all integers that are equal to one added to that number.

Problem

Write formal descriptions of the following sets.

a. The set containing the numbers 1, 10, and 100

b. The set containing all integers that are greater than 5

c. The set containing all natural numbers that are less than 5

d. The set containing the string aba

e. The set containing the empty string

f. The set containing nothing at all

Step-by-step solution

Step 1 of 6

a.

Consider the given information:

The set containing the number of series of 10. It means n contain 10 and 10 to the power of m, contain the series of 0, 1, and 2, is as shown below:

$$\{n : n = 10^m \text{ for some } m \in \{0,1,2\}\}$$

Comment

Step 2 of 6

b.

Consider the given information:

The set contains the all integer number which is greater than 5.

Thus, the n which contains only those integer numbers which are greater than 5 is as shown below:

$$\{n : n \text{ is an integer and } n > 5\}$$

Comment

Step 3 of 6

c.

Consider the given information:

The set contains the all natural number which is less than 5.

Thus, the n which contains only those natural numbers which are less than 5 is as shown below:

$$\{n : n \text{is a natural number and } n < 5\}$$

Comment

Step 4 of 6

d.

Consider the given information:

The set contains the string which is aba.

Thus, the n which contains only the string ‘aba’, is as shown below:

$$\{\text{n : n contains } \{\text{aba}\}\}.$$

Comment

Step 5 of 6

e.

Consider the given information:

The set contains the empty string. Empty is denoted by ε.

Thus, the set which contains only the empty string, is as shown below:

$\{\varepsilon\}$

Comment

Step 6 of 6

f.

Consider the given information:

The set contains nothing, it means set contain only null. Null is denoted by ϕ.

Thus, the set which does not contains anything is as shown below:

$\varnothing$

Comment

Problem

Let A be the set {x, y, z} and B be the set {x, y}.

a. Is A a subset of B?

b. Is B a subset of A?

c. What is A ∪ B?

d. What is A ∩ B?

e. What is A × B?

f. What is the power set of B?

Step-by-step solution

Step 1 of 6

(a) No, A is not a subset of B. A contains an extra element z which is not there in B.

Comment

Step 2 of 6

(b) Yes, B is a subset of A, because every member of B also is a member of A. In fact B is a proper subset of A in this case.

Comment

Step 3 of 6

(c)

$$\begin{aligned} A \cup B &= \{x, y, z\} \cup \{x, y\} \\ &= \{x, y, z\} \\ &= A \end{aligned}$$

Comment

Step 4 of 6

(d)

$$\begin{aligned} A \cap B &= \{x, y, z\} \cap \{x, y\} \\ &= \{x, y\} \\ &= B \end{aligned}$$

Comment

Step 5 of 6

(e)

$$\begin{aligned} A \times B &= \{(a, b) : a \in A \text{ and } b \in B\} \\ &= \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\} \end{aligned}$$

Comment

Step 6 of 6

(f) Power set of B is $\{\varnothing, \{x\}, \{y\}, \{x, y\}\}$

Problem

If A has a elements and B has b elements, how many elements are in A × B? Explain your answer.

Step-by-step solution

Step 1 of 1

$A \times B$ will have $a \times b$ number of elements in it.

One of the method of constructing the Cartesian product is to select an element of $A(x_a)$ and pair it with each and every element of *B:* $B(y_1, y_2 ... y_b)$. This produces the pairings $\{(x_1, y_1), (x_1, y_2), ..., (x_1, y_b)\}$. On repeating this procedure for each remaining element of $A(x_2)$ through x_a.

First pairing will produce $\sum_{i=1}^{n}(x_1, y_1) = a$ pairs. As the iteration continues over the elements of *A*, *a* number of sets will get generated, each set having *b* pairs.

Thus, the number of elements in $A \times B$ is $a \times b$.

Comment

Problem

If C is a set with c elements, how many elements are in the power set of C? Explain your answer.

Step-by-step solution

Step 1 of 1

Consider the data, Number of elements in set $C = c$

Formula:

When the number of elements in the set S is *n*, then its power set consists of 2^n elements.

The Power set is the set of all subsets of the set *S*.

The set *C* contains *c* elements such as $\{c_1, c_2, c_3, \ldots c_c\}$. Substituting, '*c*' instead of '*n*'. The number of elements in the power set of C is '2^c' elements.

Example:

Assume $C = \{1,2,3,4\}$ .So, the number of elements in set *C* is 4.

The total subsets of the set *C* are:

$$\{\ \},\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},$$
$$\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}.$$

The set of all subsets is called a Power set. Thus, the power set is as follows:

$$P(C) = \left\{ \begin{array}{l} \{\ \},\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}, \\ \{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\} \end{array} \right\}$$

The number of elements in the power set is 16 $\left(2^4\right)$.

Thus, when the number of elements in a set *C* is *c*, then the number of elements in its power set consists of 2^c elements.

Comment

Problem

Let X be the set {1, 2, 3, 4, 5} and Y be the set {6, 7, 8, 9, 10}. The unary function f : X → Y and the binary function g : X ×Y →Y are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

a. What is the value of f(2)?

b. What are the range and domain of f?

c. What is the value of g(2, 10)?

d. What are the range and domain of g?

e. What is the value of g(4, f(4))?

Step-by-step solution

Step 1 of 5

a)

The value of the function *f* at n is 2 is 7.

Therefore, the value of $f(2)=7$

Comment

Step 2 of 5

b)

The set of all possible dependent value of the outputs of a function is called the Range.

The set of all possible inputs to the function is called its Domain.

From the given values n, and f(n), the Range and Domain are as follows:

• Range $R=\{6,7\}$

• Domain $D=\{1,2,3,4,5\}$

Comment

Step 3 of 5

c)

The value in the function $g(i,j)$, where *i* is the row label and *j* is the column label.

Then the value at the row labeled *2* and the column labeled *10* in the table is $\boxed{g(2,10)=6}$

Comment

Step 4 of 5

d)

The Range of the function *g* is $R = \{6\quad ,8,9,10\}$

The Domain of the function *g*, which is the input values in the table goes into the function $g(i, j)$

From the table $g(i, j)$

$$\begin{aligned} X &= \{1,2,3,4,5\} \\ Y &= \{6,7,8,9,10\} \end{aligned}$$

$$\begin{aligned} \text{Domain } D &= X \times Y \\ &= \{(1,6),(1,7),(1,8),(1,9),(1,10),(2,6),(2,7),(2,8), \\ &\quad (2,9),(2,10),(3,6),(3,7),(3,8),(3,9),(3,10)(4,6), \\ &\quad (4,7),(4,8),(4,9),(4,10)(5,6),(5,7),(5,8),(5,9),(5,10)\} \end{aligned}$$

Comment

Step 5 of 5

e)

- If f is a function of the form $f(a)=b$, *b* is the output value when the input value is *a*
- The entry at the row labeled *i* and the column labeled *j* in the table is the value of *g (i, j)*.
- The value of the function f at n=4 is 7, which is $f(4)=7$

$$\begin{aligned} \text{The value of function } g(4, f(4)) &= g(4,7) \\ &= 8 \end{aligned}$$

Comment

Problem

For each part, give a relation that satisfies the condition.

a. Reflexive and symmetric but not trai tive

b. Reflexive and transitive but not symmetric

c. Symmetric and transitive but not reflexive

Step-by-step solution

Step 1 of 4

A binary relation *R* is

Reflexive if for every *x*, xRx

Symmetric if for every *x* and *y*, xRy implies yRx

Transitive if for every *x, y,* and z, xRy and yRz implies *x*

Comments (1)

Step 2 of 4

(a)

Re◊exive and symmetric but not transitive

xRy : *x, y* are people and they share at least one biological parent.

• This relation is clearly re◊exive, since everyone has their own parents for parents.

• This relation is symmetric, if person *x* shares a parent with person *y*, then person *y* shares that same parent with person *x*.

• However, this relation is not transitive. Assume a person *m* who shares exactly one parent (a mother) with person *g* and exactly one parent with person(*s*) (a father). *g*'s father is not *s*'s father, and *g*'s mother is not *s*'s mother. Hence, while *gRm* and *mRs* are true, *gRs* is false.

Comment

Step 3 of 4

(b)

Re◊exive and transitive but not symmetric

xRy : $x, y \in$ N and $x - y \leq 0$.

• This relation is Re◊exive, because $x\text{-}x = 0$.

• This is transitive, because if xRy then $x \geq y$ and if yRz then $y \geq z$, thus xRx because $x \geq y \geq z$.

• However, this relation is not symmetric, because $5 - 3 \geq 0$, but $3 - 5 < 0$.

Comments (1)

Step 4 of 4

(c)

Symmetric and transitive but not re◊exive

xRy : $x, y \in$ Z and $i * j > 0$

• This relation is Symmetric, because multiplication is symmetric.

• It is transitive, as well, since if xRy then neither *x* nor *y* is zero and if yRz , then neither *y* nor *z* is zero. Thus xRz because neither *x* nor *z* is zero.

• However, this relation is not symmetric, because $0 * 0 = 0$

Problem

Consider the undirected graph G=(V,E) where V , the set of nodes, is {1, 2, 3, 4} and E, the set of edges, is {{1, 2}, {2, 3}, {1, 3}, {2, 4}, {1, 4}}. Draw the graph G. What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of G.

Step-by-step solution

Step 1 of 2

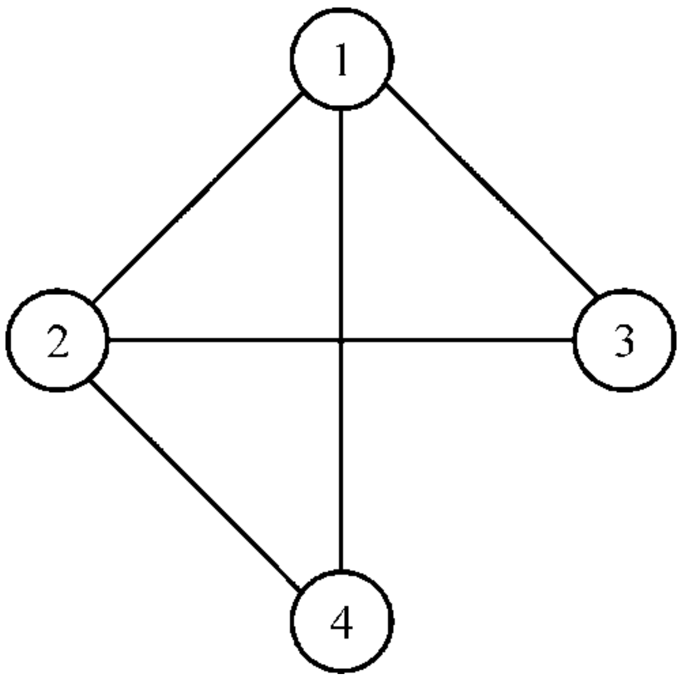
An undirected graph is a set of vertices (nodes) and edges where an edge connects a pair of vertices and it has no orientation.

Consider the undirected graph $G=(V,E)$

Where, V is the set of nodes, $\{1,2,3,4\}$

And, E is the set of edges, $\{\{1,2\},\{2,3\},\{1,3\},\{2,4\},\{1,4\}\}$.

The graph G is as shown below:



Degree of nodes:

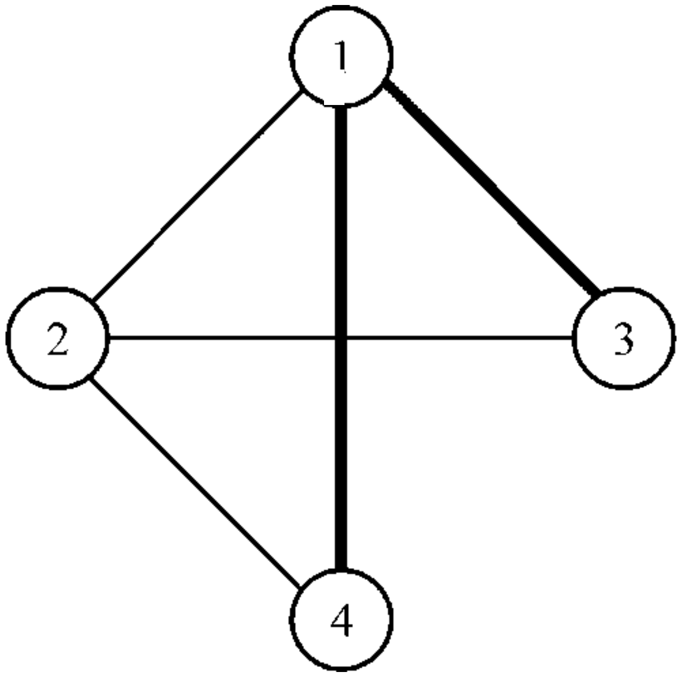
The degree of a node is the number of edges at that particular node. The degrees of each node of graph G are as shown in the table below:

Node	Degree
1	3
2	3
3	2
4	2

Comment

Path from node 3 to node 4:

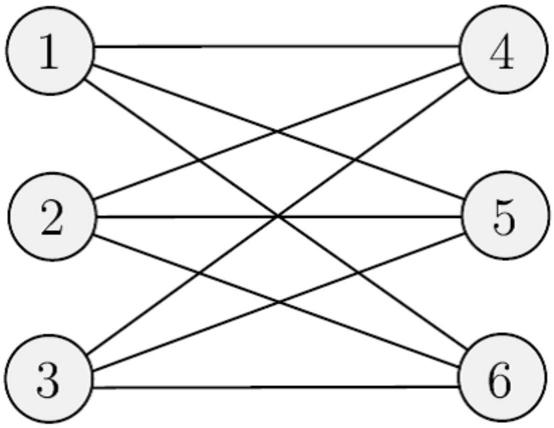
The path from node 3 to node 4 on the graph *G* is as shown below:



Comment

Problem

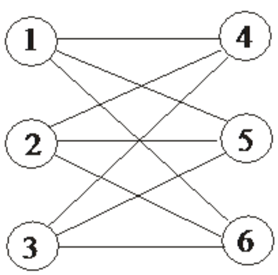
Write a formal description of the following graph.



Step-by-step solution

Step 1 of 3

Given graph:



Comment

Step 2 of 3

Formal description of the graph is

({1, 2, 3, 4, 5, 6}, {(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)})

It shows a set of nodes and a set of edges.

Comment

Step 3 of 3

In a graph G, we say G = (V, E) where V is the set of nodes and E is the set of edges.

$V = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

$E = \{\{1,\ 4\},\ \{1,\ 5\},\ \{1,\ 6\}, \{2,\ 4\},\ \{2,\ 5\},\ \{2,\ 6\}, \{3,\ 4\},\ \{3,\ 5\},\ \{3,\ 6\}\}$

Comment

Problem

Find the error in the following proof that 2 = 1. Consider the equation a = b. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, (a+b)(a–b) = b(a–b), and divide each side by (a–b) to get a+b = b. Finally, let a and b equal 1, which shows that 2 = 1.

Step-by-step solution

Step 1 of 2

To prove: 2 = 1

Given proof is:

• Consider the equation $a = b$

• Multiply both sides by a to obtain $a^2 = ab$

• Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$

• Now factor each side, $(a+b)(a-b) = b(a-b)$

• Divide each side by $(a-b)$, to get $a+b=b$

• Let *a* and *b* equal 1, which shows that 2 = 1.

Comment

Step 2 of 2

The fallacy lies in the step which involves division by $(a-b)$, which is zero. Since *a* is equal to *b* as stated in the problem statement, *a* – *b* gives zero. Since division by zero is undefined, the argument cannot be considered as valid.

Comment

Problem

Let S(n) = 1 + 2 + · · · + n be the sum of the first n natural numbers and let C(n) = $1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that C(n) = S^2(n) for every n.

a. $S(n) = \frac{1}{2}n(n+1)$.

b. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Step-by-step solution

Step 1 of 3

a) The sum of the first n natural numbers $(S_n = 1+2+3+...+n)$ is given by:

$$S_n = \frac{1}{2}n(n+1).$$

Here, induction method is used to prove the above equality. Let's write $S_n = 1+2+3+...+n$ in shorter form like:

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

• For n=1, it is true that

$$1 = \frac{1}{2}1(1+1).$$

• For n=2, it is true that

$$1+2 = \frac{1}{2}2(2+1).$$

• In the same way it is true for n

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

So, finally it has to prove that for n+1 or $S_{n+1} = \sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+2)$.

• If n+1 is added to each side of the next identity:

$$\begin{aligned} S_{n+1} &= 1+2+...+n+n+1 \\ &= S_n + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

• Which is equivalent to: $\sum_{i=1}^{n+1} i = \frac{1}{2}n(n+1) + \frac{2(n+1)}{2}$ or $S_{n+1} = \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

So, the above equality proves that the given equality for the sum of n natural or $S_n = \frac{1}{2}n(n+1)$ is also true for $(n+1)$. Hence the given equality is correct.

Comment

Step 2 of 3

b) The sum of the cube of the first n natural numbers or $C_n = 1^3 + 2^3 + ... + n^3$ is given by:

$$C_n = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{1}{4}n^2\left(n+1\right)^2$$

The above equality can be proved by induction method. Let's write $C_n = \frac{1}{4}n^2\left(n+1\right)^2$ in shorter form like:

$$\sum_{i=1}^{k=n} i^3 = 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2\left(n+1\right)^2$$

• For n=1, it is true that:

$1^3 = \frac{1}{4}1^2\left(1+1\right)^2$ or 1=1.

• In the same way it is true for n:

$$\sum_{i=1}^{k=n} i^3 = 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2\left(n+1\right)^2.$$

So, finally it has to prove that for n+1 or $C_n = 1^3 + 2^3 + ... + n^3 + \left(n+1\right)^3 = \frac{\left(n+1\right)^2\left(n+2\right)^2}{4}$

$$\begin{aligned}C_{n+1} &= 1^3 + 2^3 + 3^3 + ... + n^3 + (n+1)^3 \\ &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\ &= \frac{(n+1)^2(n+2)^2}{4}\end{aligned}$$

Therefore,

$$C_{n+1} = \frac{\left(n+1\right)^2\left(n+2\right)^2}{4}$$

So, the above equality proves that the given equality for the sum of the cube of n natural numbers is also true for n+1. Hence, the given equality is correct.

Comment

Step 3 of 3

From the above explanation:

$$C_n = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{1}{4}n^2\left(n+1\right)^2 = \left[\frac{n\left(n+1\right)}{2}\right]^2 = \left(S_n\right)^2$$

$C_n = \left(S_n\right)^2$.

It is concluded that "the sum of the cube of the first n natural number is equal to the square of the sum of first n natural number".

Comment

Problem

Find the error in the following proof that all horses are the same color.

CLAIM: In any set of h horses, all horses are the same color.

PROOF: By induction on h.

Basis: For h = 1. In any set containing just one horse, all horses clearly are the same color.

Induction step: For k ≥ 1, assume that the claim is true for h = k and prove that it is true for h = k+1. Take any set H of k+1 horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H^1 with just k horses. By the induction hypothesis, all the horses in H^1 are the same color. Now replace the removed horse and remove a different one to obtain the set H^2. By the same argument, all the horses in H^2 are the same color. Therefore, all the horses in H must be the same color, and the proof is complete.

Step-by-step solution

Step 1 of 2

The error has occurred in the last sentence. There are two statements "all horses in H_1 are the same or identical in color" And "all horses in H_2 are identical or the same color" for the choice of H_1 and H_2 which has taken.

Comment

Step 2 of 2

The induction is valid only for value greater than 2 but fails for 2.

• It indicates that all the horses present in $H = H_1 \cup H_2$ are identical (that is, the same color) if $H = H_1 \cap H_2$ is not empty which returns always true result except when exactly two horses $(h = 2)$ contained in H. Then exactly one horse out of the two horses will be consist in each H_1 and H_2.

• Certainly the single horse in H_1 and only the single horse in H_2 has its peculiar color. Therefore, it has no meaning to conclude that the horses have the same color.

• So the base case should be for value 2 not for 1. Since the base case is false, the above proof is wrong.

Comments (2)

Problem

Show that every graph with two or more nodes contains two nodes that have equal degrees.

Step-by-step solution

Step 1 of 1

Let us consider a graph G which has at least one edge and without having any loops or cycles in the graph. In graph G we would prove that there are at least two nodes with degree 1 i.e. with equal degree. This conclusion would be taken forward for a graph with 2 or more nodes.

In a graph G we must get a node, say V_1 at which only one edge is incident, i.e. $\text{degree}(V1)=1$. Let E_1 be this edge which is incident at V_1. Since *G* has no cycles other end of E_1 is not V_1. Let it be V_2. If there exists no other edges which is incident at V_2 then $\text{degree}(V2)=1$.

Otherwise let E_2 be the edge which is incident at V_2. Arguing similar way and proceeding in this way we get a node V_k having degree 1 and is equal to degree of V_1. Thus we get that graph G has at least two nodes of degree 1. Thus two nodes have the same degrees in a graph with two or more nodes.

Comment

Problem

Ramsey's theorem. Let G be a graph. A ***clique*** in G is a subgraph in which every two nodes are connected by an edge. An ***anti-clique***, also called an ***independent set***, is a subgraph in which every two nodes are not connected by an edge. Show that every graph with n nodes contains either a clique or an anti-clique with at least 1/2 $\log^2$ n nodes.

Step-by-step solution

Step 1 of 2

Consider the graph *G*. A clique is a subgraph of *G* in which every pair of vertices are connected. An anti-clique is a subgraph in which in which every pair of vertices are not connected.

Comment

Step 2 of 2

In order to show that every graph with *n* vertices contains either clique or anti-clique with at least $\frac{1}{2}\log_2 n$ vertices, create two piles *A* and *B* to store the vertices of a graph. Here, the pile *A* contains the vertices of a clique whereas the pile *B* contains the vertices of an anti-clique.

Procedure to identify a clique or an anti-clique is as follows:

• Take each vertex *v* of the graph *G*.

• If the degree of the vertex is greater than one half of the remaining vertices then add the vertex to pile *A*. Otherwise, add the vertex to the pile *B*.

• Discard all vertices to which *v* is not connected if it was added to the pile *A*.

• Discard all vertices to which *v* is connected if it was added to the pile *B*.

• Continue this procedure until no vertices left.

Consider the whole procedure as a step. For each step, at most half of the vertices are discarded. Thus, at least $\log_2 n$ steps occur before completion of the process. Each step adds a vertex to one of the piles. Thus, one of the piles contains at least $\frac{1}{2}\log_2 n$ vertices.

Therefore, it is proved that that every graph with *n* vertices contains either clique or anti-clique with at least $\frac{1}{2}\log_2 n$ vertices.

Comment

Problem

Use Theorem 0.25 to derive a formula for calculating the size of the monthly payment for amortgage in terms of the principal P, the interest rate I, and the number of payments t. Assume that after t payments have been made, the loan amount is reduced to 0. Use the formula to calculate the dollar amount of each monthly payment for a 30-year mortgage with 360 monthly payments on an initial loan amount of $100,000 with a 5% annual interest rate.

Step-by-step solution

Step 1 of 3

Given formula related to loan is

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

where,

P_t is the amount of loan outstanding after the t^{th} month.

P is the principal (original loan amount).

Y is the monthly payment.

t is the number of months in which loan is repaid.

I is the yearly interest rate.

M is the monthly multiplier $(M = 1 + I/12)$.

Now we have to derive the formula for calculating the size of the monthly payments for a mortgage in terms of the principal P, interest rate I, and the number of payments t.

Comment

Step 2 of 3

In order to derive the formula we have to get Y (monthly payment) on left hand side and remaining terms to right hand side.

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

$$Y\left(\frac{M^t - 1}{M - 1}\right) = PM^t - P_t$$

$$Y = \left(\frac{M - 1}{M^t - 1}\right)\left(PM^t - P_t\right)$$

The formula required for calculation is

$$Y = \left(\frac{M - 1}{M^t - 1}\right)\left(PM^t - P_t\right)$$

Comment

Step 3 of 3

Given data is

$P_t = \$0$

$P = \$100,000$

$t = 360$ months

$I = 5\% = \dfrac{5}{100} = 0.05$

$$M = 1 + I$$
$$= 1 + \frac{0.05}{12} \quad = 1.0042(\text{approx})$$

We get

$$Y = \left(\frac{M-1}{M^t-1}\right)\left(PM^t - P_t\right)$$
$$= \left(\frac{1.00417-1}{1.00417^{360}-1}\right)\left(100000 \times 1.00417^{360} - 0\right)$$
$$= \left(\frac{0.00417}{3.47309}\right)(100000 \times 4.47309)$$
$$= 0.0012 \times 447309$$
$$\approx 536.7708$$

Therefore, the monthly payment is $536.78.

Comment