2.152 Nonlinear Control Spring 2020 HW1

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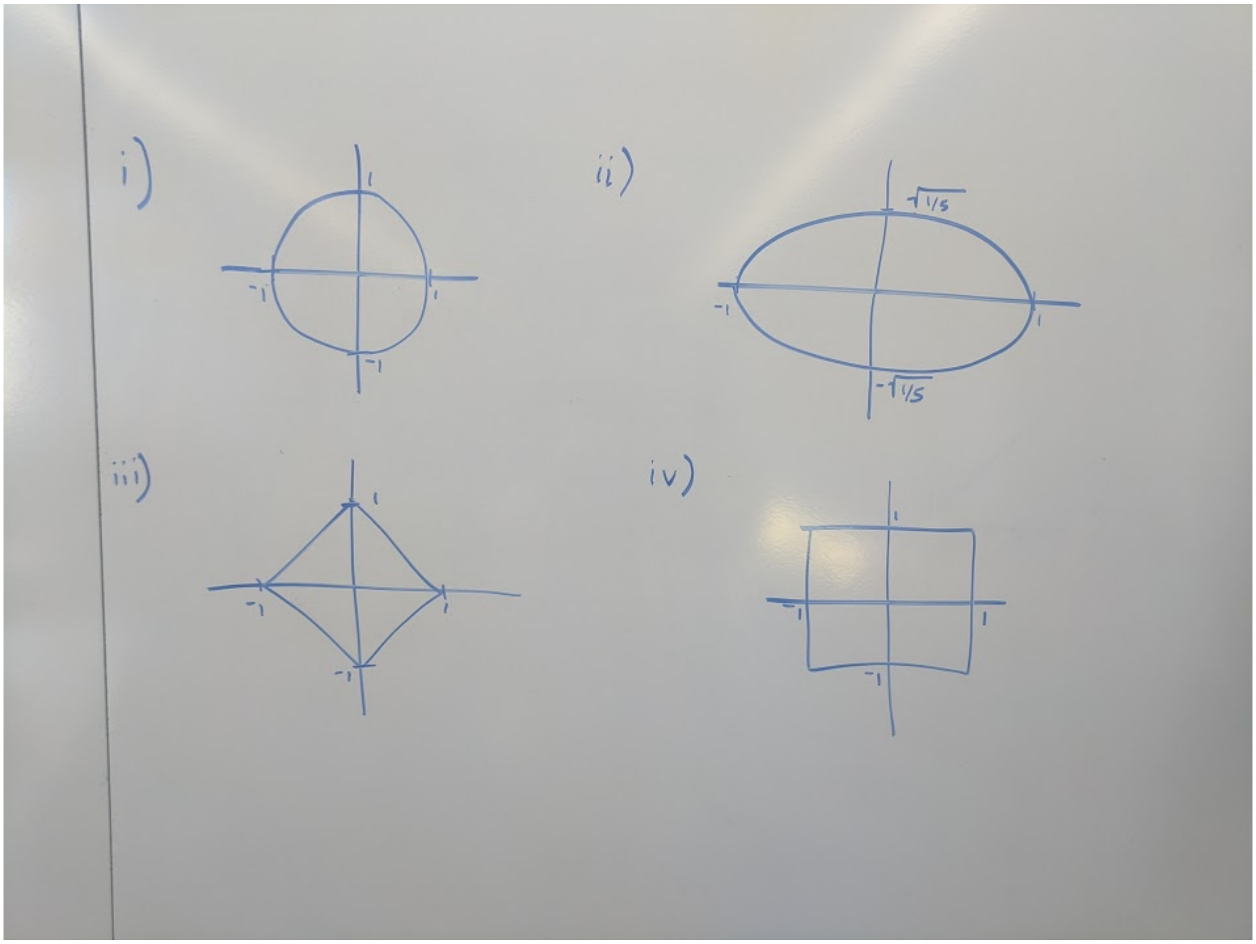
Problem 1



Problem 2

a

Fixed point(s):

$$\dot{x} = 0 = -x^3 + \sin^4 x \Rightarrow x^3 = \sin^4 x \Rightarrow x = 0$$

Define $V(x) = x^2$, a positive definite function. Then $\dot{V}(x) = 2x\dot{x} = 2x(-x^3 + \sin^4 x)$. Since $|x| \geq |\sin x|$, when $x > 0, -x^3 + \sin^4 x < 0$ and when $x < 0, -x^3 + \sin^4 x > 0$, leading us to conclude that $\dot{V}(x)$ is negative definite. Since $V(x)$ is also radially unbounded, by Theorem 3.3, the equilibrium point $x = 0$ is globally asymptotically stable.

b

Fixed point(s):

$$\dot{x} = 0 = (5 - x)^5 \Rightarrow 0 = 5 - x \Rightarrow x = 5$$

For clarity, perform a coordinate transform to bring the fixed point $x = 5$ to the origin by defining $u = 5 - x \Rightarrow \dot{u} = -\dot{x}$. Define $V(u) = u^2 \Rightarrow \dot{V}(u) = 2u\dot{u} = -2(5-x)(5-x)^5 = -2(5-x)^6 = -2u^6 \leq 0$. Since $V(u)$ is positive definite and radially unbounded, and $\dot{V}(u)$ is negative definite, by Theorem 3.3 again, the equilibrium point $u = 0 \Leftrightarrow x = 5$ is globally asymptotically stable.

c

Fixed point(s):

$$\ddot{x} + \dot{x} + x^7 = x^2 \sin^8(x) \cos^2(3x) \Rightarrow x^7 = x^2 \sin^8(x) \cos^2(3x) \Rightarrow (x, \dot{x}) = (0, 0)$$

Define $b(\dot{x}) \stackrel{\text{def}}{=} \dot{x}^5$ and $c(x) \stackrel{\text{def}}{=} x^7 - x^2 \sin^8(x) \cos^2(x)$. Note that

$$\begin{aligned}
\dot{x}b(\dot{x}) &= \dot{x}\dot{x}^5 = \dot{x}^6 \\
&= \geq 0 \\
xc(x) &= x(x^7 - x^2 \sin^8(x) \cos^2(x)) \\
&= x^3(x^5 - \sin^5(x) \sin^3(x) \cos^2(x)) \\
&= \geq 0
\end{aligned}$$

where the last line follows because when $x < 0$, $x^3 < 0$ and both $x^5 < 0$ and $-x^2 \sin^8(x) \cos^2(3x) \leq 0$ and when $x > 0$, $x^5 - \sin^5(x) \sin^3(x) \cos^2(x) \geq x^5 - x^5 \sin^3(x) \cos^2(x) > x^5 - x^5 > 0$. Define our Lyapunov function as

$$V(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2}\dot{x}^2 + \int_0^x c(u) du$$

This function is radially unbounded, and its time derivative is negative definite:

$$\begin{aligned}
\dot{V}(x, \dot{x}) &= \dot{x}\ddot{x} + c(x)\dot{x} \\
&= -\dot{x}b(\dot{x}) \\
&\leq 0
\end{aligned}$$

where $\dot{V}(x, \dot{x}) = 0$ only when $x = 0$. Then by Theorem 3.5, the equilibrium point $(0, 0)$ is globally asymptotically stable.

d

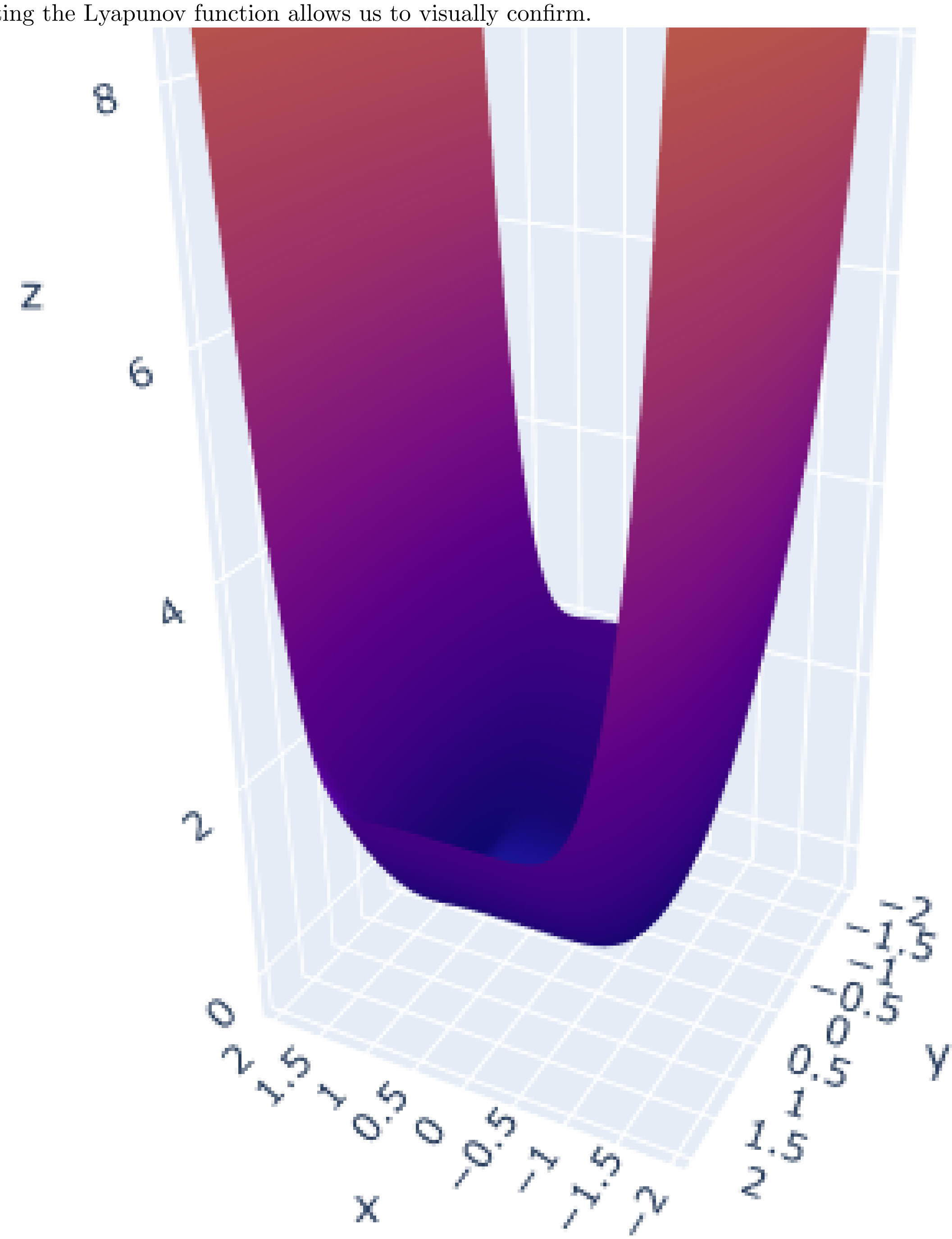
Fixed point(s):

$$\ddot{x} + (x - 1)^4 \dot{x}^7 + x^5 = x^3 \sin^3(x) \Rightarrow x^5 = x^3 \sin^3(x) \Rightarrow (x, \dot{x}) = (0, 0)$$

We take a similar approach as before. Define $b(\dot{x}) \stackrel{\text{def}}{=} (x-1)^4\dot{x}^7$ and $c(x) \stackrel{\text{def}}{=} x^5 - x^3\sin^3(x)$. Note that $\dot{x}b(\dot{x}) = (x-1)^4\dot{x}^8 > 0$ for $x \neq 0$ and that $xc(x) = x^4(x^2 - \sin(x)\sin^2(x)) > x^4(x^2 - \sin(x)x^2) > x^4(x^2 - x^2) \geq 0$ for $x \neq 0$.

Define $V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x c(u)du$. Then again by Theorem 3.5, the equilibrium point $(0, 0)$ is globally asymptotically stable.

Plotting the Lyapunov function allows us to visually confirm.



e

Fixed point(s):

$$\ddot{x} + (x-1)^2\dot{x}^7 + x = \sin(\pi x/2) \Rightarrow (x, \dot{x}) = (0,0), (1,0), (-1,0)$$

Define our Lyapunov function:

$$V(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x du(u - \sin(\frac{\pi}{2}u))$$

Then

$$\dot{V}(x,\dot{x}) = -\dot{x}^8(x-1)^2 \leq 0$$

which is zero at $\dot{x} = 0$ or $x = 1$. Recalling that

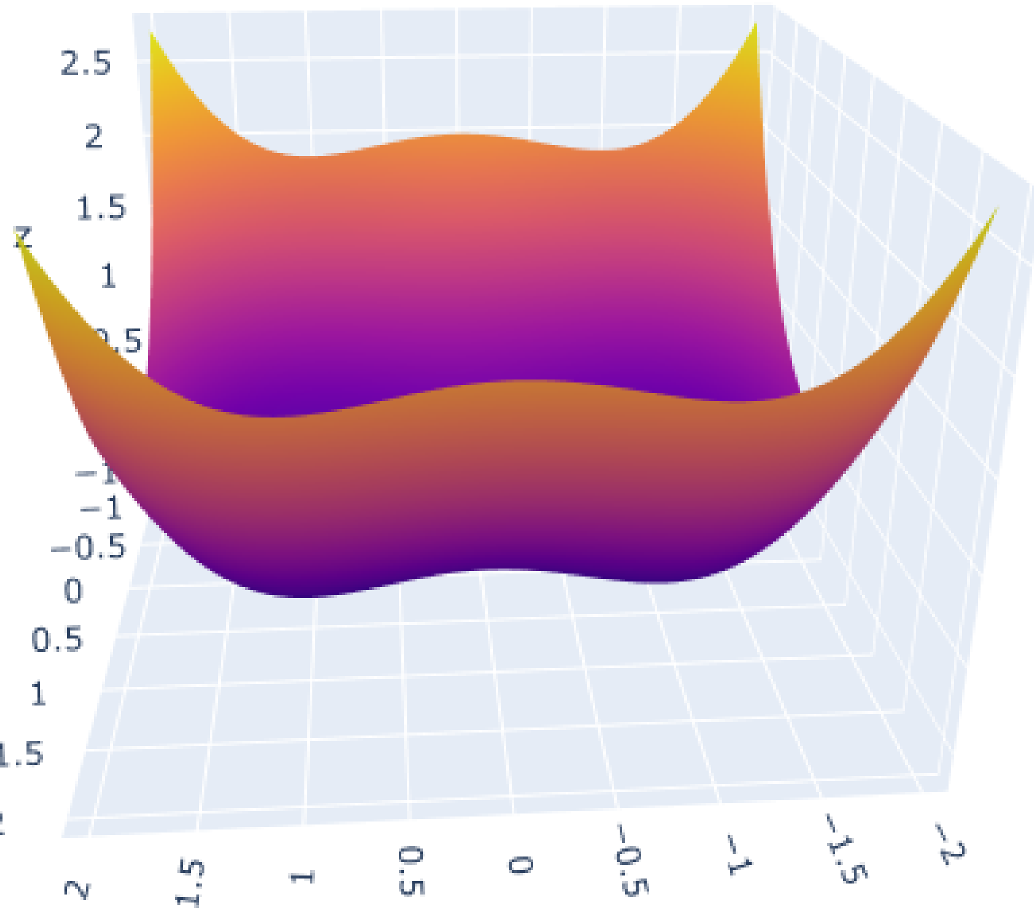
$$\ddot{x} = -(x-1)^2\dot{x}^7 - x + \sin(\pi x/2)$$

we evaluate at $\dot{x} = 0$ and $x = 1$.

$$\ddot{x}_{\dot{x}=0} = 0 - x + \sin(\pi x/2) \neq 0$$

$$\ddot{x}_{x=1} = 0 - 1 + 1 = 0$$

Plotting the Lyapunov landscape reveals that the fixed points (-1, 0) and (1, 0) are globally asymptotically stable, while (0, 0) is unstable.



Problem 3

For brevity, define $C(x, y) = x^2 + 2y^2 - 4$. We first note that $C(x, y)$ is an invariant set:

$$\begin{aligned}\frac{d}{dt}C(x, y) &= 2x\dot{x} + 4y\dot{y} \\ &= (-2xf_1(x) - 4yf_2(y))C(x, y)\end{aligned}$$

Note that $(0, 0)$ is is also a fixed point of the system since $\dot{x}|_{x=0} = 4f_1(x)|_{x=0} = 0$ and $\dot{y}|_{y=0} = 4f_2(y)|_{y=0}$ since $f_1(0) = 0$ and $f_2(0) = 0$. Define our Lyapunov function $V(x, y) = C^2$. We see that because V is a quartic function of x, y, V is radially unbounded. We then show that $\dot{V}(x, y)$ is negative semi-definite:

$$
\begin{aligned}
\dot{V}(x,y) &= 2C(2x\dot{x} + 4y\dot{y}) \\
&= 4Cx(4x^2y - f_1(x)C)) + 8Cy(-2x^3 - f_2(y)C) \\
&= 16Cx^3y - 4xf_1(x)C^2 - 16Cx^3y - 8yf_2(y)C^2 \\
&= -4xf_1(x)C^2 - 8yf_2(y)C^2 \\
&\leq 0
\end{aligned}
$$

where the last line follows because $xf_1(x) > 0 \forall x \neq 0$ and $yf_2(y) > 0 \forall y \neq 0$ and $C^2 \geq 0$. Then, by the Global Invariant Set Theorem (3.5), every solution must converge to either the origin (x, y) or the limit cycle. To show that all points other than the origin converge to the limit cycle, we use the Local Invariant Set Theorem (3.4). Define $l = 16 \Rightarrow \Omega_l = \{x | V(x) < 16\}$. This allows us to consider behavior of points on the "inside" of the limit cycle i.e. points within Ω_l while excluding the origin; by Theorem 3.4, all of these points must converge to the limit cycle. Thus we conclude that the system tends towards the limit cycle regardless of f_1, f_2 unless starting exactly at the origin.

Problem 4

Consider the system

$$
\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x
$$

Define our Lyapunov function $V(x) = x^T P x$ for $P = P^T = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$. We know from class that this matrix P must solve

$$
A^T P + PA = -I
$$

Starting with the bottom right element, then the upper right, then the top left:

$$
\begin{aligned}
-1 &= 0p_{12} - 3p_{22} + 0p_{12} - 3p_{22} \\
\frac{1}{6} &= p_{22} \\
0 &= 0p_{11} - 3p_{22} + 0p_{12} + 2p_{22} \\
\frac{1}{12} &= p_{12} \\
-1 &= -1p_{11} + 2p_{12} - 1p_{11} + 2p_{12} \\
\frac{2}{3} &= p_{11}
\end{aligned}
$$

Thus our matrix is $P = \frac{1}{12}\begin{bmatrix} 8 & 1 \\ 1 & 2 \end{bmatrix}$. This is a positive definite matrix because since the determinants $\det\left([8]\right) > 0, \det\left(\begin{bmatrix} 8 & 1 \\ 1 & 2 \end{bmatrix}\right) > 0$, a sufficient condition to prove that a matrix is positive definite.

Problem 5

All systems are at fixed points $\theta = 0$. I consider a rotation $d\theta$ of each shape around its center and ask how the system evolves if released.

1. Incorrect. Global asymptotic stability requires that the domain of attraction for this fixed point be defined for all possible $d\theta \in [0, 2\pi]$. However, for a range of $d\theta$, the square will stabilize to another face or a corner and not necessarily this initial face. I'd call this point locally asymptotically stable.

2. Correct. Every point on the circle's perimeter is a fixed point, meaning that after a $d\theta$ rotation, the system will not move but it will not return to $\theta = 0$ (unless $d\theta \mod 2\pi = 0$). The point is therefore stable but not asymptotically stable.

3. Correct. A small rotational perturbation will indeed move away from the fixed point, but to another fixed point

Problem 6

Let $V(x)$ be a Lynapunov function in a ball B_R and let ϕ be a scalar, differentiable, strictly monotonically increasing function of its scalar argument. To show that $V' \stackrel{\text{def}}{=} \phi(V(x)) - \phi(0)$ is also a Lyapunov function will require showing three facts:

1. $V'(x)$ is positive definite (PD). V is PD, which means that $V(x) = 0 \Leftrightarrow x = 0$ and thus $V'(0) = \phi(0) - \phi(0) = 0$. To show that $\forall x \neq 0$, $V'(x) > 0$, we know that because V is PD, $\forall x \neq 0$, $V > 0$ and because ϕ is a strictly monotonically increasing function, $\forall x \neq x$, $\phi(V(x)) - \phi(0) > 0$. Thus we conclude that V' is positive definite.

2. $V'(x)$ has continuous partial derivatives. We are told that ϕ is differentiable and by virtue of V being a Lyapunov function, it has continuous partial derivatives. Thus $\frac{\partial V'(x)}{\partial x_i} = \frac{d\phi(x)}{dV(x)} \frac{\partial V(x)}{\partial x_i}$ and we conclude that $V'(x)$ has continuous partial derivatives.

3. $\dot{V}'(x)$ is negative semi-definite (NSD). Because ϕ is a strictly monotonically increasing function, $\frac{d\phi(V(x))}{dV(x)} > 0$, and because V is a Lyapunov function, $\dot{V}(x) \leq 0$. By the chain rule, $\dot{V}'(x) = \frac{d}{dt} V'(x) = \frac{d\phi(V(x))}{dV(x)} \dot{V}(x)$. $\dot{V}'(x)$ is the product of a positive term and non-positive term, meaning $\dot{V}'(x) \leq 0$ is NSD.

Thus, we conclude that $V'(x)$ is also a Lyapunov function. Inituitively, we see that the effect of $V'(x)$ is to keep $V(x)$ pinned at the origin but permit $V(x)$ to be rescaled elsewhere, as long as the rescalings preserve the properties of V i.e. that it is PD and its time derivative is NSD.

Problem 7

Let $x^* = 0$ be exponentially stable. This means $\exists\, \alpha > 0, \lambda > 0, \delta > 0$ such that if $||x(0)|| < \delta$, then $\forall t \geq 0$, $||x(t)|| \leq \alpha ||x(0)|| e^{-\lambda t}$. To show that x^* is stable, we need to show that $\forall R > 0, \exists r > 0$ such that if $||x(0)|| < r$, then $||x(t)|| < R$ for all $t \geq 0$.

Choose $r = \min(\frac{R}{\alpha}, d)$. Assume that $||x(0)|| < r$. Then

$$
\begin{aligned}
||x(t)|| &\leq \alpha ||x(0)|| e^{-\lambda t} && \text{Defn. of exponential stability and } ||x(0|| < \delta \\
&\leq \alpha ||x(0)|| && e^{-\lambda t} \text{ is monotonically decreasing for increasing } t \geq 0 \\
&< \alpha r && \text{By assumption, } ||x(0)|| < r \\
&< \alpha \frac{R}{\alpha} && \text{Holds regardless of whether } r = \frac{R}{\alpha} \text{ or } r = d \\
&< R
\end{aligned}
$$