2.152 Nonlinear Control System Design
Fall 2020

Solutions to the Midterm

Problem 1:

(a) (5 points)
If A is Hurtwitz, then for any positive definite $Q \in \mathbb{R}^{\ltimes \times \ltimes}$, there exists a unique positive definite $P \in \mathbb{R}^{\ltimes \times \ltimes}$ that satisfies Lyapunovs equation

$$PA + A^T P = -Q.$$

Thus, we can choose Q to be the identity matrix and use the resulting P in our quadratic Lyapunov function.

(b) (5 points)
$x_2(t) = x_2(0)e^{bt}$ and thus $\dot{x}_1(t) + x_1(t) = x_2(0)e^{(b+a)t}$. The system is unstable if $b > 0$ or if $b \leq 0$, then the system is unstable if $a > -b$. The system is marginally stable if $b = 0$ and $a \leq -b$ or if $b < 0$ and $a = -b$. The system is stable if $b < 0$ and $a < -b$.

c) (5 points)
Ignoring initial conditions, we have that:

$$|\tilde{x}(t)| \leq \frac{\phi}{\lambda^{n-1}}.$$

See review notes for proof.

D) (5 points)
In general this is not a good strategy. To satisfy the persistent excitation condition (page 366 in the textbook), the desired trajectory must be sufficiently rich.

Problem 2:

(a) (10 points)
The only equilibrium point is $x = 0$. With the radially unbounded positive definite Lyapunov function

$$V = 1/2x^2$$

we get

$$\dot{V} = x\dot{x} = -4x^{10} + 2x^4 \sin^6 x < 0 \qquad \text{if} \quad x \neq 0$$

Using Lyapunov's direct method, we therefore know that the origin is globally asymptotically stable.

(b) (10 points)
The system has a unique equilibrium point at $x = 0$. Let $a(\dot{x}) = \dot{x}^4 + 1$, $b(\dot{x}) = \dot{x}^5(\cos(3\dot{x})^4 + 1)$ and $c(x) = 3x(\sin^4 x + 1)e^{-3x}$. Notice that $a(\dot{x}) > 0, \forall \dot{x} \in \mathbb{R}$. We have that $\dot{x}b(\dot{x})$ and $xc(x)$ are positive definite. Consider the positive definite scalar

$$V = \int_0^{\dot{x}} za(z)dz + \int_0^{x} c(y)dy$$

with a negative semi-definite time derivative

$$\dot{V} = -\dot{x}b(\dot{x}) \leq 0$$

which is zero if $\dot{x} = 0$. The second derivate at $\dot{x} = 0$ is

$$\ddot{x} = -c(x)$$

and hence only vanishes when $x = 0$. Using the Invariant Set Theorem, we conclude $[x, \dot{x}]^T = [0, 0]^T$ is an asymptotical stable equilibrium point. Note that the term $\int_0^x c(y)dy$ does not approach ∞ as $x \to \infty$, thus we cannot conclude global asymptotic stability.

(c) (15 points)
The system is given as

$$\begin{aligned}
\dot{x}_1 &= (2\beta + 3)x_1 - 2.5x_2 \\
\dot{x}_2 &= 2.5x_1 + (-e^{\alpha} - \sin(t))x_2
\end{aligned}$$

Let's put this in matrix notation

$$\dot{\mathbf{x}} = \begin{bmatrix} 2\beta + 3 & -2.5 \\ 2.5 & -e^{\alpha} - \sin(t) \end{bmatrix} \mathbf{x}$$

So if we choose

$$\begin{aligned} V &= \mathbf{x}^T \mathbf{x} \\ \dot{V} &= \mathbf{x}^T \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \mathbf{x} \\ &= \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{x} \\ &= \mathbf{x}^T (\mathbf{A} + \mathbf{A}^T) \mathbf{x} \leq -\lambda_0 \mathbf{x}^T \mathbf{x} = -\lambda_0 V \end{aligned}$$

Therefore we know that

$$V = \mathbf{x}^T \mathbf{x} \leq e^{-\lambda_0 t} V(0)$$

and therefore

$$||\mathbf{x}|| \leq ||\mathbf{x}(\mathbf{0})||\; e^{\frac{-\lambda_0}{2} t}.$$

We see that the convergence of $\mathbf{x}$ to the origin is determined by $\frac{\lambda_0}{2}$. To guarantee the convergence of $\mathbf{x}$ with a convergence rate ≥ 2 we need the following constraint on the maximum eigenvalue of $\mathbf{A} + \mathbf{A}^T$

$$\frac{-\lambda_0}{2} \leq -2 \qquad \Rightarrow \qquad -\lambda_0 \leq -4$$

So, now looking at $\mathbf{A} + \mathbf{A}^T$

$$\mathbf{A} + \mathbf{A}^T = \begin{bmatrix} 2(2\beta + 3) & 0 \\ 0 & 2(-e^{\alpha} - \sin(t)) \end{bmatrix}$$

The eigenvalues are obviously $s_1 = 2(2\beta + 3), s_2 = 2(-e^{\alpha} - \sin(t))$ and

$$s_1 = 2(2\beta + 3) \leq -4$$

$$s_2 = 2(-e^{\alpha} - \sin(t)) \leq -4$$

Therefore we get

$$-e^{\alpha} - \sin(t) \leq -2 \qquad \Rightarrow \qquad \alpha \geq \ln 3$$

$$2\beta + 3 \leq -2 \qquad \Rightarrow \qquad \beta \leq 2.5$$

Thus a value for α and β will guarantee an exponential convergence of $\mathbf{x}$ to the origin with a convergence rate ≥ 2.

Problem 3: (20 points)

The system is given as

$$\dddot{x} + \ddot{x} + (1+b^3)\dot{x} + 2\cos(c)\cos(x)x^2 = u$$

Taking $\alpha_1 = (1+b^3)$ and $\alpha_2 = 2\cos(c)$ we have

$$\begin{array}{lll} -7 \le \alpha_1 \le 9 & \Rightarrow \quad \hat{\alpha_1} = 1 \quad , & |\tilde{\alpha_1}| \le A_1 = 8 \\ -2 \le \alpha_2 \le 2 & \Rightarrow \quad \hat{\alpha_2} = 0 \quad , & |\tilde{\alpha_2}| \le A_2 = 2 \end{array}$$

Define the sliding surface

$$s = \left(\frac{d}{dt} + \lambda\right)^2 \tilde{x} = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x} = \ddot{x} - \ddot{x}_r, \qquad \text{where } \ddot{x}_r = \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2\tilde{x}$$

with $\tilde{x} = x - x_d$. Its dynamics are

$$\dot{s} = \dddot{x} - \dddot{x}_r = -\ddot{x} - \alpha_1\dot{x} - \alpha_2\cos(x)x^2 + u - \dddot{x}_r$$

Determine the best estimate for the control input $\hat{u}$ by substituting the best estimates of the model and forcing $\dot{s} = 0$. This leads to

$$\hat{u} = \ddot{x} + \hat{\alpha_1}\dot{x} + \hat{\alpha_2}\cos(x)x^2 + \dddot{x}_r$$

To take care of the uncertainty add a switching term, resulting in the control law

$$u = \hat{u} - k\,\mathrm{sgn}(s)$$

Inserting this into the dynamics of s, we have

$$\begin{aligned} \dot{s} &= -\ddot{x} - \alpha_1\dot{x} - \alpha_2\cos(x)x^2 + \hat{u} - k\,\mathrm{sgn}(s) - \dddot{x}_r \\ &= \tilde{\alpha_1}\dot{x} + \tilde{\alpha_2}\cos(x)x^2 - k\,\mathrm{sgn}(s), \end{aligned}$$

where $\tilde{\alpha_1} = \hat{\alpha_1} - \alpha_1$ and $\tilde{\alpha_2} = \hat{\alpha_2} - \alpha_2$. To guarantee a rate of attraction η to the surface, the controller must satisfy

$$\frac{d}{dt}\tfrac{1}{2}s^2 = s\dot{s} \le -\eta\,|s|$$

and hence

$$s\dot{s} = s(\tilde{\alpha_1}\dot{x} + \tilde{\alpha_2}\cos(x)x^2) - k\,|s| \leq -\eta\,|s|$$

so that the choice of k is limited by

$$k \geq \eta + \left|\ \tilde{\alpha_1}\dot{x} + \tilde{\alpha_2}\cos(x)x^2\right|$$

This is satisfied if

$$k = \eta + A_1|\dot{x}| + A_2|\cos(x)|x^2 = \eta + 8|\dot{x}| + 2\,|\cos(x)|\,x^2$$

Problem 4: (25 points)

(a) (15 points)
Define the sliding surface as

$$s = \dot{\tilde{x}} + \lambda\tilde{x} = \dot{x} - \dot{x}_r, \qquad\qquad \dot{x}_r = \dot{x}_d - \lambda\tilde{x}$$

with $\tilde{x} = x - x_d$. Its dynamics are described by

$$\begin{aligned} m\dot{s} \;=\; m\ddot{x} - m\ddot{x}_r \;&=\; -b\cos x e^{-t}\dot{x}^9 + d + u - kx + kl - m\ddot{x}_r \\ &=\; u - \mathbf{Ya}, \end{aligned}$$

where

$$\mathbf{Y} = [\ \cos x e^{-t}\dot{x}^9 \quad -1 \quad x \quad \ddot{x}_r\ ] \qquad , \qquad \mathbf{a} = [\ b \quad d + kl \quad k \quad m\ ]^T$$

The best estimate for the control input $\hat{u}$ is determined by substituting the best estimates of the model and forcing $\dot{s} = 0$. This leads to

$$\hat{u} = \mathbf{Y\hat{a}}$$

The control law adds a negative feedback term for convergence, so that

$$u = \hat{u} - Ks = \mathbf{Y\hat{a}} - Ks$$

Substituting this control law into the dynamics of $\dot{s}$, we get

$$m\dot{s} = \mathbf{Y\tilde{a}} - Ks$$

where $\mathbf{\tilde{a}} = \mathbf{\hat{a}} - \mathbf{a}$.

Using the Lyapunov function

$$V = \frac{m}{2}s^2 + \tfrac{1}{2}\tilde{\mathbf{a}}^T\Gamma^{-1}\tilde{\mathbf{a}}$$

where Γ is a positive definite matrix, leads to

$$\dot{V} = ms\dot{s} + \dot{\tilde{\mathbf{a}}}^T\Gamma^{-1}\tilde{\mathbf{a}} = -Ks^2 + (s\mathbf{Y} + \dot{\hat{\mathbf{a}}}^T\Gamma^{-1})\tilde{\mathbf{a}},$$

where we have used the fact that $\dot{\hat{\mathbf{a}}} = \dot{\tilde{\mathbf{a}}}$. Using the adaptation law

$$\dot{\hat{\mathbf{a}}} = -\Gamma\mathbf{Y}^T s$$

results in

$$\dot{V} = -Ks^2 \leq 0$$

To verify convergence, we have to use Barbalat's Lemma. We need to show that $\dot{V}$ is uniformly continuous by showing that $\ddot{V} = -2Ks\dot{s}$ is bounded.

First we notice that the Lyapunov function V is positive definite and $\dot{V}$ is negative semi-definite. This implies that V is bounded and therefore s and $\tilde{\mathbf{a}}$ are also bounded. From the definition of s we can then conclude that $\tilde{x}$ and $\dot{\tilde{x}}$ are bounded. Assuming the the desired trajectory and its derivatives are bounded implies that $\mathbf{Y}$ is bounded. From the dynamics of s and the control law we finally get $\dot{s}+Ks = \mathbf{Y}\tilde{\mathbf{a}}$, which bounds $\dot{s}$. This shows that $\ddot{V}$ is indeed bounded. From Barbalat's Lemma we can conclude that $\dot{V}$ and thus s will converge to zero. From the definition of s as a stable filter, we also know that $\tilde{x}$ will converge to zero.

(b) (10 points)

If d is replaced by $d\sin(2t+\phi)$, $\mathbf{Y}$ and $\mathbf{a}$ must change accordingly.

Using the identity

$$\sin(x+\phi) = \sin(x)\,\cos(\phi) + \cos(x)\,\sin(\phi)$$

this can be rewritten as

$$d\sin(2t+\phi) = d\sin(2t)\,\cos(\phi) + d\cos(2t)\,\sin(\phi)$$

Using the same sliding surface as in part (a), we get

$$\begin{aligned} m\dot{s} = m\ddot{x} - m\ddot{x}_r &= -b\sin x e^{-t}\dot{x}^7 + u + kl - kx - m\ddot{x}_r + d\,\cos(\phi)\,\sin(2t) + d\,\sin(\phi)\,\cos(2t) \\ &= u - \mathbf{Ya}, \end{aligned}$$

where

$$\mathbf{Y} = [\,\sin x e^{-t}\dot{x}^7 \quad \ddot{x}_r \quad -\sin(2t) \quad -\cos(2t) \quad -1 \quad x\,], \qquad \mathbf{a} = [\,b \quad m \quad \alpha_1 \quad \alpha_2 \quad kl \quad k\,]^T$$

and $\alpha_1 = d\cos(\phi)$ and $\alpha_2 = d\sin(\phi)$.

The rest follows part (a) using the updated $\mathbf{Y}$ and $\mathbf{a}$.