2.152 Nonlinear Control Spring 2020 HW2

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Problem 1

1. **No**. Consider two symmetric matrices $A = A^T, B = B^T$. In order for AB to be symmetric, $AB = (AB)^T = B^T A^T = BA$ must hold. But matrices are typically not commutative, so because $AB \neq BA$, we conclude no.
2. **No**. Consider $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, a positive definite matrix by Sylvester's criterion because all leading minors have positive determinant. But $AA = \begin{bmatrix} -1 & 2 \\ -4 & -1 \end{bmatrix}$ has a leading minor with negative determinant i.e. $\det(-1) = -1$, meaning that AA is not positive definite.
3. **No**. Define $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Note that both A, B are both symmetric and positive definite because all leading minors have positive determinants. But $AB = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}$, which has a leading minor with negative determinant. Thus we conclude that the product of symmetric positive definite matrices is not necessarily positive definite. I took this example from Robert Israel after posting on Math StackExchange for help in disproving my "proof" that the claim was true.
4. **No**. If the product of symmetric positive definite matrices isn't necessarily positive definite (part 3), the product certainly isn't necessarily symmetric positive definite.

Problem 2

To be a fixed point, all time derivatives must be zero. Since $R(\dot{q})$ has the same sign as its argument, $\dot{q} = 0 \Leftrightarrow R(\dot{q}) = 0$. This implies $L(\ddot{q})\ddot{q} + C(q) = 0 \Leftrightarrow \ddot{q} = -\frac{C(q)}{L(\ddot{q})}$. The only time $\ddot{q} = 0$ is either when $C(q) = 0 \Leftrightarrow x = 0$ or $L(\ddot{q}) \to \pm\infty$. Thus the fixed point is $(q = 0, \dot{q} = 0)$.

Next we construct our Lyapunov function. Define

$$V(q, \dot{q}) \overset{\text{def}}{=} \int_0^{\dot{q}} L(u)u \, du + \int_0^q C(v) \, dv$$

Taking the time derivative of our Lyapunov function, we see that:

$$\begin{aligned} \dot{V}(q, \dot{q}) &\overset{\text{def}}{=} L(\dot{q})\dot{q} + C(q)\dot{q} \\ &= -\dot{q}R(\dot{q}) \end{aligned}$$

where $V(q, \dot{q})$ is negative definite because $\dot{q}, R(\dot{q})$ have the same sign and are zero only when $\dot{q} = 0$. However, we do not know whether V is positive definite because although $L(\dot{p})$ is strictly positive, we don't know the behavior either integral. For instance, if $q \to -\infty$, $C(q)$ could possible $\to \int_0^q C(v)dv$. Consequently, by Theorem 3.4 (Local Invariant Set Theorem), we conclude that the origin is locally asymptotically stable. If we could guarantee that $V(q, \dot{q})$ is positive definite, then we could guarantee that the origin is globally asymptotically stable.

Problem 3

Let P, Q be symmetric positive definite matrices. I want to use λ for eigenvalues, so I'll replace λ in the problem with c. Assume that

$$A^T P + PA + 2cP = -Q$$

Our goal is to show that $\lambda_{max}(A) < -c$. Let v, λ be an eigenvector-eigenvalue pair of A. Left and right multiply by v:

$$\begin{aligned} v^T A^T P v + v^T P A v + 2c v^T P v &= -v^T Q v \\ 2(\lambda + c) v^T P v &= -v^T Q v \\ \lambda &= -c - \frac{v^T Q v}{2 v^T P v} \\ \lambda &< -c \end{aligned}$$

where the last line follows from P, Q being positive definite matrices i.e. $v^T Q v > 0$ and $v^T P v > 0$. This holds for all eigenvalues of A since nothing in our proof depended on which eigenvalue we chose.

Problem 4

We first show that $(y = 0, \dot{y} = 0)$ is a fixed point.

$$A_1 \ddot{y} + A_2(0) + A_3(0) = 0 \Rightarrow A_1 \ddot{y} = 0 \Rightarrow \ddot{y} = 0$$

where the last step follows from the fact that A_1 is symmetric positive definite. Define a Lyapunov function as:

$$V(y, \dot{y}) \stackrel{\text{def}}{=} \frac{1}{2} \dot{y}^T A_1 \dot{y} + \frac{1}{2} y^T A_3 y$$

Because both matrices are positive definite, $V(y, \dot{y})$ is positive definite. Since each term is quadratic in either y or $\dot{y}$, the function is also radially unbounded. We take the function's time derivative:

$$\begin{aligned} \dot{V} &= \dot{y}^T A_1 \ddot{y} + \dot{y} A_3 y \\ &= \dot{y}^T A_1 (-A_1^{-1} A_2 \dot{y} - A_1^{-1} A_3 y) + \dot{y}^T A_3 y \\ &= -\dot{y}^T A_2 \dot{y} - \dot{y}^T A_3 y + \dot{y}^T A_3 y \\ &= -\dot{y}^T A_2 \dot{y} \\ &\leq 0 \end{aligned}$$

Since A_2 is positive definite, $\dot{V}$ is negative definite. Thus, by Theorem 3.3 (Lyapunov's Direct Method for Global Stability), the equilibrium at the origin is globally asymptotically stable.

Problem 5

Consider a system $\dot{x} = Ax$ with $y = c^T x$. Suppose $\exists P = P^T$ such that $A^T P + PA = -cc^T$. Left multiply by x^T and right multiply by x:

$$
\begin{aligned}
A^T P + PA &= -cc^T \\
x^T A^T P x + x^T P A x &= -x^T c c^T x \\
\dot{x}^T P x + x^T P \dot{x} &= -y^T y \\
&\leq 0
\end{aligned}
$$

Define a scalar function $V(x) \stackrel{\text{def}}{=} x^T P x$. Note that because P is positive definite, the function V is positive definite and radially unbounded. Since P does not depend on time,

$$\dot{V} = \dot{x}^T P x + x^T \dot{P} x + x^T P \dot{x} = \dot{x}^T P x + x^T P \dot{x} \leq 0$$

Since $V(x)$ is radially unbounded and $\dot{V}(x)$ is negative semi-definite, by the Global Invariant Set Theorem (Theorem 3.5), the system is asymptotically stable at $x = (0, 0)$. Such a conclusion about the **global** stability could **not** be reached using the direct method because $\dot{V}$ fails to be negative definite.

Problem 6

To show that $\dot{x} = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix} x$ is globally asymptotically stable, define $x(t) = [x_1(t), x_2(t)]^T$. We first show that $x_2(t)$ is globally asymptotically stable:

$$\dot{x}_2 = -1x_2 \Rightarrow x_2(t) = x_2(0)e^{-t}$$

We then show that $x_1(t)$ is globally asymptotically stable. Consider:

$$\dot{x}_1 = -x_1 + e^{t/2}x_2 = -x_1 + x_2(0)e^{-t/2}$$

To find the solution to the inhomogeneous equation, I use an integrating factor e^t:

$$
\begin{aligned}
\dot{x_1} + x_1 &= x_2(0)e^{-t/2} \\
e^t(\dot{x_1} + x_1) &= x_2(0)e^{t/2} \\
\frac{d}{dt}(e^t x_1) &= x_2(0)e^{-t/2} \\
e^t x_1(t) - e^0 x_1(0) &= 2x_2(0)e^{t/2} \\
x_1(t) &= x_1(0)e^{-t} + 2x_2(0)e^{-t/2}
\end{aligned}
$$

We see that $x_1(t)$ is also globally asymptotically stable, meaning the system is globally asymptotically stable as both x_1, x_2 decay to 0.

Problem 7

For each sub-question, I use the notation $x(t) = [x_1(t), x_2(t)]^T$. We determine whether the following systems have stable equilibrium, and if so, whether the equilibria are asymptotically stable and whether the stability is local or global.

1. Consider $\dot{x} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} x$. We immediately see that

$$x_2(t) = x_2(0)e^{-2t}$$

Using the integrating factor e^{10t}, we see that

$$x_1(t) = x_1(0)e^{-10t} + \frac{1}{11}x_2(0)e^t$$

Thus we conclude the fixed point $x = [0, 0]^T$ is **unstable** because x_1 grows exponentially, proportional to $x_2(0)$.

2. Consider $\dot{x} = \begin{bmatrix} -1 & 2\sin(t) \\ 0 & -(t+1) \end{bmatrix} x$. We immediately see that

$$x_2(t) = x_2(0)e^{-\frac{1}{2}t^2 - t}$$

Using the integrating factor e^t, we see that

$$x_1(t) = x_1(0)e^{-t} + 2x_2(0)e^{-t} \int_{\tau=0}^{\tau=t} d\tau \, \sin(\tau)e^{-\frac{1}{2}\tau^2}$$

Wolfram Alpha tells me that the full solution is:

$$x_1(t) = x_1(0)e^{-t} + i\sqrt{\frac{\pi}{2}}x_2(0)e^{-t-1/2}\text{erf}(\frac{t+i}{\sqrt{2}}) - \sqrt{\frac{\pi}{2}}x_2(0)e^{-t-1/2}\text{erfi}(\frac{1+it}{\sqrt{2}})$$

This system **globally asymptotically stable** because $x(t) \to 0$.

3. Consider $\dot{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} x$. We immediately see that:

$$x_2(t) = x_2(0)e^{-2t}$$

Using the integrating factor e^t, we see that

$$x_1(t) = x_1(0)e^{-t} + x_2(0)$$

In the language of this course, we conclude the fixed point $x = [0, 0]^T$ is **unstable** because $x_2(0)$ is a constant offset that prevents approaching the fixed point. However, if $x_2(0) = 0$, then any point along

Problem 8

Because V is twice differentiable, $\ddot{V}$ exists, meaning I can take the limit of $\dot{V}$. We know that at any $\dot{V} = 0$, $\ddot{V} \leq 0$ must hold by contradiction: if $\ddot{V} > 0$, then by the fundamental theorem of calculus, $\dot{V}(t+\tau) = \dot{V}(t) + \int_t^{t+\tau} \ddot{V} > 0$, which violates the assumption that $\dot{V}(t+\tau) \leq 0$. Having established this fact, we consider the limit at any point t where $\dot{V}(t) = 0$:

$$\lim_{h \to 0} \frac{\dot{V}(t) - \dot{V}(t-h)}{h} \leq 0 \text{ and } \lim_{h \to 0} \frac{\dot{V}(t+h) - \dot{V}(t)}{h} \leq 0$$

Since $\dot{V}(t) = 0$, we see that:

$$\dot{V}(t-h) \geq 0 \text{ and } \dot{V}(t+h) \leq 0$$

Then by Rolle's Theorem, which informally states that if the points on the left side of t are non-negative and the points on the right side of t are non-positive and the function is continuous, then $\ddot{V}(t) = 0$.

Problem 9

1. Our goal is to design a switching controller for the following system:

$$\ddot{x} = -\frac{1}{m}(\alpha_1 + \alpha_2 \cos^2(x))|\dot{x}|\dot{x} + \frac{1}{m}d + \frac{1}{m}u$$

We set $m = 1$ and drop it from this part of the problem. Define $f(x, \dot{x}), \hat{f}(x, \dot{x})$ and F using the knowledge that $5 \leq \alpha_1 \leq 9, 2 \leq \alpha_2 \leq 4, -1 \leq d \leq 1$:

$$\begin{aligned} f(x, \dot{x}) &\stackrel{\text{def}}{=} -(\alpha_1 + \alpha_2 \cos^2 x)|\dot{x}|\dot{x} + d \\ \hat{f}(x, \dot{x}) &\stackrel{\text{def}}{=} -(7 + 3\cos^2 x)|\dot{x}|\dot{x} + 0 \\ F(x, \dot{x}) &\stackrel{\text{def}}{=} |(2 + \cos^2 x)|\dot{x}|\dot{x} + 1| \\ &\geq |f - \hat{f}| \end{aligned}$$

Next we define s:

$$s \stackrel{\text{def}}{=} \dot{\tilde{x}} + \lambda x \Rightarrow \dot{s} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

To construct our control law, we set $\dot{s} = 0 = \hat{f} + \hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}}$. To ensure our control law meets the sliding condition, we subtract $(F + \eta)\,\mathrm{sign}(s)$:

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} - (F + \eta)\,\mathrm{sign}(s)$$

This guarantees we meet the sliding condition:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}s^2 &= s\dot{s} \\ &= s(f + \hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}} \\ &= s(f - \hat{f}) - F|s| - \eta|s| \\ &\leq -\eta s \end{aligned}$$

2. Now we consider an unmodelled pendulum mode at 3 Hz and design a sliding controller for the system. For simplicity, we choose a fixed ϕ to simplify things since then $\dot{\phi} = 0$. Our approach is to take the previous controller and replace $(F + \eta)\,\text{sign}(s)$ with $(F + \eta)\text{sat}(s/\phi)$:

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} - (F + \eta)\text{sat}(s/\phi)$$

This satisfies the sliding window condition:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}s^2 &= s\dot{s} \\
&= s(f + \hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}}) \\
&= s(f + -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} - (F + \eta)\text{sat}(s/\phi) - \ddot{x}_d + \lambda\dot{\tilde{x}}) \\
&= s(f - \hat{f}) - sF\text{sat}(s/\phi) - s\,\eta\,\text{sat}(s/\phi) \\
&\leq -\eta|s|
\end{aligned}$$

3.

4. We now consider the same system with $1 \leq m \leq 3$:

$$\ddot{x} = -\frac{1}{m}(\alpha_1 + \alpha_2\cos^2(x))|\dot{x}|\dot{x} + \frac{1}{m}d + \frac{1}{m}u$$

We define the following variables to rewrite the equation:

$$\begin{aligned}
b &\overset{\text{def}}{=} \frac{1}{m} \Rightarrow \frac{1}{3} \leq b \leq 1 \\
\alpha_1' &\overset{\text{def}}{=} \frac{\alpha_1}{m} \Rightarrow \frac{5}{3} \leq \alpha_1' \leq 9 \\
\alpha_2' &\overset{\text{def}}{=} \frac{\alpha_2}{m} \Rightarrow \frac{2}{3} \leq \alpha_2' \leq 4 \\
d' &\overset{\text{def}}{=} \frac{d}{m} \Rightarrow -1 \leq d' \leq 1
\end{aligned}$$

We then rewrite our system:

$$\ddot{x} = -(\alpha_1' + \alpha_2'\cos^2 x)|\dot{x}|\dot{x} + d' + bu$$

We choose the following estimates for our unknown variables:

$$\begin{aligned}
\hat{b} &= \left(1\frac{1}{3}\right)^{1/2} = \frac{1}{\sqrt{3}} \\
\hat{\alpha}_1' &= \frac{1}{2}\left(\frac{5}{3} + 9\right) = \frac{16}{3} \\
\hat{\alpha}_2' &= \frac{1}{2}\left(\frac{2}{3} + 4\right) = \frac{7}{3} \\
\hat{d}' &= \frac{-1 + 1}{2} = 0
\end{aligned}$$

Let $f(x,\dot{x}) = -(\alpha_1' + \alpha_2' \cos^2 x)|\dot{x}|\dot{x} + d'$ and let $\hat{f}(x,\dot{x}) = -(\hat{\alpha}_1' + \hat{\alpha}_2' \cos^2 x)|\dot{x}|\dot{x}$. We again construct an upper bound on the absolute difference of the two:

$$|f - \hat{f}| \leq \left|(\frac{11}{3} + \frac{5}{3}\cos^2 x)|\dot{x}|\dot{x} + 1\right| \stackrel{\text{def}}{=} F$$

As before, define s as

$$s \stackrel{\text{def}}{=} \dot{\tilde{x}} + \lambda\tilde{x}$$

To meet the unmodeled pendulum mode at 3 Hz, we require that $\lambda \leq 3$.

We then calculate $\dot{s}$, set equal to 0 and solve for the control law:

$$\dot{s} = 0 = \ddot{x} - \ddot{x}_d + \lambda\tilde{x} = f + bu - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

To ensure the sliding condition is met, we need to subtract $(F+\eta)\text{sat}(s/\phi)$, yielding our control law:

$$\hat{u} = -b^{-1}\hat{f} + b^{-1}\ddot{x}_d - \lambda b^{-1}\dot{\tilde{x}} - b^{-1}(F+\eta)\text{sat}(s/\phi)$$

We check that our control law satisfies the sliding condition:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}s^2 &= s\dot{s} \\
&= s(f + b\hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}}) \\
&= s(f + b(-b^{-1}\hat{f} + b^{-1}\ddot{x}_d - \lambda b^{-1}\dot{\tilde{x}} - b^{-1}(F+\eta)\text{sat}(s/\phi)) - \ddot{x}_d + \lambda\dot{\tilde{x}}) \\
&= s(f - \hat{f}) - s\,F\text{sat}(s/\phi) - s\,\eta\,\text{sat}(s/\phi) \\
&\leq -\eta|s|
\end{aligned}$$