2.152 Nonlinear Control System Design
Spring 2020

Problem Set #5

Issued : Tue 04/17/2020
Due : Tue 04/30/2020

Problem 1:

Consider the system

$$\begin{aligned} \ddot{x}_1 + 2\dot{x}_1 - 2x_1\cos(x_1) &= x_2 \\ \dot{x}_2 + x_2^2 x_1 &= u \end{aligned}$$

Design a controller to stabilize this system to the origin.

Problem 2:

Derive explicitly the parallel, feedback and hierarchical combination properties as stated in class (https://www.youtube.com/watch?v=e37T3OREB5g).

Problem 3:

In this problem we apply the tools we've learned in 2.152 to analyze a model that captures how disease spreads within a population. Let $S(t)$, $I(t)$, and $R(t)$ be the number of people are susceptible to the disease, infected by the disease, and those who have recovered from the disease, respectively. We assume that once you recover from the disease you can no longer be infected. Assuming a sufficiently large population count (to ignore stochastic effects) and that the population is in a well-mixed spatial environment (to ignore spatial effects), a simple model is given by

$$\begin{aligned} \frac{dS}{dt} &= \alpha - \gamma S - aSI, \\ \frac{dI}{dt} &= aSI - \kappa I - \gamma I, \\ \frac{dR}{dt} &= \kappa I - \gamma R, \end{aligned}$$

where α is the birth rate of the population, γ is the death rate (not due to the disease), a is the transmission rate of the disease, and κ is the rate at which infected individuals recover.

(a) Let $I(0) = 0$ (no one is initially infected) provide: a close from expression for $S(t)$, $S_0 = \lim_{t\to\infty} S(t)$, and a physical interpretation of this limit.

(b) For $x(t) \in \mathbb{R}^n$ with dynamics given by $\dot{x} = f(x)$. The set $\chi_+ = \{x \in \mathbb{R}^n : x_i \geq 0, \forall i\}$ (the positive orthant) is positively invariant if $f_i(x) \geq 0, \forall x \in \{x \in \chi_+ : x_i = 0\}, \forall i$ (essentially if the vector points inwards at the boundary). Show that for the disease model above, the positive orthant is a positively invariant set. Note that this consistent with our expectation that the states should be positive quantities.

(c) Let $N(t) = S(t) + I(t) + R(t)$ be the total population, what are the dynamics of $N(t)$? Show that these dynamics are contracting. What is $N_\infty = \lim_{t\to\infty} N(t)$ (express in terms of S_0)? Leveraging the result from part b, show that $S(t), I(t), R(t) \leq N_\infty$?

(d) Consider the virtual system

$$\frac{d\theta}{dt} = aS\theta - \kappa\theta - \gamma\theta,$$

show that $\theta(t) = I(t) = 0$ are solutions to the above system. Assuming $S(0), R(0) \geq 0$, and considering the bounds from part b and c, provide conditions to ensure the virtual system is contracting for all $\theta \geq 0$. Now use the hierarchical combination to show that the $S(t)$ and $R(t)$ dynamics are contracting. When contraction is guaranteed, what implications does this have on the number of equilibrium in the positive orthant?

(e) Determine all equilibria off the disease model and their stability in terms of S_0 and $S^* = \frac{\kappa+\gamma}{a}$ and their stability (linearization methods should suffice). How are these results consistent with the results of part d? Is there a stable equilibrium point with $I = 0$? If so, what conditions on a are required? One can show that that a is inversely proportional to the Mean Free Path (roughly the distance people most travel to find another person in our population model), so what do these results suggest about social distancing?