2.152 Nonlinear Control Spring 2020 HW3

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Problem 1

Consider $\dot{x} = f(x)$ and let $f(0) = 0$ be a fixed point. **Approach 1: Mean Value Theorem**. By the Mean Value Theorem, we know $\exists c_x$ such that

$$f'(c_x) = \frac{f(x) - f(0)}{x - 0} \Leftrightarrow f(x) = f'(c_x)x$$

Define $A(x) \stackrel{\text{def}}{=} f'(c_x)$. Note that A depends on x implicitly because c_x depends on x. We immediately see that

$$\dot{x} = f(x) = f'(c_x)x = A(x)x$$

Approach 2: Jets. If we Taylor Series expand, the only term without an x vanishes i.e. $f(0)$. We factor x out from all other terms and define $A(x)$ as the sum of Taylor series terms:

$$\dot{x} = f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x-0)^k = f(0) + \underbrace{\Big(f'(0) + \frac{f''(0)}{2!}x + \frac{f'''(0)}{3!}x^2 + ...\Big)}_{A(x)} x = 0 + A(x)x = A(x)x$$

Note that each of the derivatives is a multi-dimensional array (maybe tensor is the correct word?) of increasing dimension e.g. f' is 2D, f'' is 3D, etc. However, the dimensions work out, because the exponentiated powers of x remove dimensions e.g. $f''(0)x$ is 2D,

$$f'''(0)x^2$$

is 2D, etc. I believe the correct mathematical term for this terms in the series is jet, and the correct way to write the kth jet of f at 0 is:

$$\frac{D^k j(0)}{k!} x^{\otimes k}$$

but I'm not familiar with this math, so my understanding of the terminology or notation may be off. Hopefully my idea is clear.

Problem 2

Define $s \stackrel{\text{def}}{=} \dot{\tilde{x}} + \lambda x$ with $\lambda > 0$. Consider the system:

$$\ddot{x} + \dot{x} + x = u + d \Leftrightarrow \ddot{x} = u + d - \dot{x} - x$$

We want to design an adaptive tracking controller. Suppose we consider the Lyapunov function candidate:

$$V = \frac{1}{2}s^2$$

Then its time derivative is

$$\dot{V} = s\dot{s} = s(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}}) = s(\ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}}) = s(u + d - \dot{x} - x - \ddot{x}_d + \lambda\dot{\tilde{x}})$$

If d was known, we could choose the control law that would guarantee stability canceling out the system's dynamics to meet the sliding window condition.

$$u \stackrel{\text{def}}{=} -ks - d + \dot{x} + x + \ddot{x}_d - \lambda\dot{\tilde{x}} \Rightarrow \dot{V} = -ks^2$$

However, d is unknown. We instead consider a time-varying estimate $\hat{d}$ with the time-varying estimate error $\tilde{d}(t) \stackrel{\text{def}}{=} \hat{d}(t) - d$. Consider the Lyapunov function candidate:

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{d}^T\tilde{d}$$

Its time derivative is:

$$\dot{V} = s\dot{s} + \dot{\hat{d}}\tilde{d} = s(u + d - \dot{x} - x - \ddot{x}_d + \lambda\dot{\tilde{x}}) + \dot{\hat{d}}^T\tilde{d}$$

We choose our adaptive control law:

$$u \stackrel{\text{def}}{=} -ks - \hat{d} + \dot{x} + x + \ddot{x}_d - \lambda\dot{\tilde{x}}$$

The time derivative of the Lyapunov function candidate then becomes:

$$\dot{V} = -ks^2 - s\tilde{d} + \dot{\hat{d}}^T\tilde{d}$$

In order to ensure the sliding condition is met, we choose $\dot{\hat{d}} = s$ and $k = \eta$. The adaptive control law in PID form is:

$$u = -\eta s + \dot{x} + x + \ddot{x}_d - \lambda\dot{\tilde{x}} - \int_{\tau=0}^{t} s(\tau)\, d\tau$$

Problem 3

We first consider the system with unknown constants $a_1 > 0, a_2, a_3$:

$$a_1\ddot{x} + a_2\dot{x}^2\sin(x) + a_3\cos(2x) = u$$

Define $\tilde{x} = x - x_d$, $s \stackrel{\text{def}}{=} \dot{\tilde{x}} + \lambda\tilde{x} \stackrel{\text{def}}{=} \dot{x} - \dot{x}_r$ and

$$a \stackrel{\text{def}}{=} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \hat{a} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \qquad \tilde{a} \stackrel{\text{def}}{=} \hat{a} - a \qquad Y \stackrel{\text{def}}{=} \begin{bmatrix} \ddot{x}_r \\ \dot{x}^2\sin x \\ \cos(2x) \end{bmatrix}$$

We consider the Lyapunov candidate function and its time derivative:

$$V = \frac{1}{2}a_1 s^2 + \frac{1}{2}\tilde{a}^T\tilde{a}$$
$$\dot{V} = s(u - Y^T a) + \tilde{a}^T\dot{\hat{a}}$$

We choose our control law, setting $k = \eta$:

$$u \stackrel{\text{def}}{=} -\eta s + Y^T\hat{a} \qquad \dot{\hat{a}} = -Ys$$

This ensures that the system will converge to the desired trajectory by ensuring the sliding condition is met:

$$
\begin{aligned}
\dot{V} &= s(u - Y^T a) + \tilde{a}^T P^{-1} \dot{\hat{a}} \\
&= -\eta s^2 + sY^T \tilde{a} + \dot{\hat{a}}^T \tilde{a} \\
&= -\eta s^2 + sY^T \tilde{a} - sY^T \tilde{a} \\
&= -\eta s^2
\end{aligned}
$$

We next consider the adaptive robust control problem for the following system:

$$
a_1 \ddot{x} + a_2 \dot{x}^2 \sin(x) + a_3 \cos(2x) + d(t) = u
$$

The new control law will largely be similar to the previous control law, but we need to now account for the time-varying $d(t)$. We define the new variable:

$$
s_\Delta \stackrel{\text{def}}{=} \begin{cases} 0 & |s| \leq \phi \\ s - \phi & s \geq \phi \\ s + \phi & s \leq \phi \end{cases}
$$

We choose the new control law:

$$
u \stackrel{\text{def}}{=} -(\eta + F)s_\Delta + \hat{d} + Y^T \hat{a} \qquad F \stackrel{\text{def}}{=} 1.5 \qquad \hat{d} \stackrel{\text{def}}{=} 0 \qquad \dot{\hat{a}} \stackrel{\text{def}}{=} -Y s_\Delta
$$

Using a slightly modified Lyapunov function from before, we see that this new control law ensures that the sliding condition is met:

$$
\begin{aligned}
V &= \frac{1}{2} a_1 s_\Delta^2 + \frac{1}{2} \tilde{a}^T \tilde{a} \\
\dot{V} &= s(u - Y^T a - d) + \tilde{a}^T \dot{\hat{a}} \\
&= s_\Delta(-(\eta + F)\operatorname{sign}(s_\Delta) + \hat{d} + Y^T \hat{a} - Y^T a - d) + \tilde{a}^T \dot{\hat{a}} \\
&= -\eta |s_\Delta| - F|s_\Delta| + s_\Delta(\hat{d} - d) \\
&\leq -\eta |s|
\end{aligned}
$$

Problem 4

I drop function arguments for brevity. We start with the proposed Lyapunov function:

$$
V = \frac{1}{2} sHs + \frac{1}{2} \tilde{a}^T \Gamma^{-1} \tilde{a} + \frac{1}{2} \tilde{q}^T (K_p + \lambda K_d) \tilde{q}
$$

I don't show this here, but the inverse of a positive definite (PD) matrix is PD, a positive constant times a PD matrix is PD, and the sum of two PD matrices is PD. Consequently, V is PD and therefore lower bounded. Its time derivative is

$$
\dot{V} = sH\dot{s} + \frac{1}{2} s\dot{H}s + \tilde{a}^T \Gamma^{-1} \dot{\hat{a}} + \tilde{q}^T (K_p + \lambda K_d) \dot{\tilde{q}}
$$

We can simplify terms:

$$
\begin{aligned}
\tilde{a}^T\Gamma^{-1}\dot{\tilde{a}} &= \tilde{a}^T\Gamma^{-1}(\dot{\hat{a}} - \dot{a}) \\
&= \tilde{a}^T\Gamma^{-1}\dot{\hat{a}} \\
&= -\tilde{a}\Gamma^{-1}\Gamma Y^T s \\
&= -\tilde{a}^T Y^T s \\
&= -sY\tilde{a} \\
&= -sY(\hat{a} - a) \\
&= -s(\hat{H}\ddot{q}_r + \hat{C}\dot{q}_r + \hat{g} - H\ddot{q}_r - C\dot{q}_r - g) \\
sH\dot{s} &= sH(\ddot{q} - \ddot{q}_r) \\
&= s(\tau - C\dot{q} - g) - sH\ddot{q}_r \\
&= s(\hat{H}\ddot{q}_r + \hat{C}\dot{q}_r + \hat{g} - K_d\dot{\tilde{q}} - K_p\tilde{q} - C\dot{q} - g) - sH\ddot{q}_r \\
sH\dot{s} + \tilde{a}^T\Gamma^{-1}\dot{\tilde{a}} &= -sK_d\dot{\tilde{q}} - sK_p\tilde{q} - sC\dot{q} + sC\dot{q}_r \\
-sK_D\dot{\tilde{q}} - sK_p\tilde{q} + \tilde{q}(K_p + \lambda K_d)\dot{\tilde{q}} &= -(\dot{\tilde{q}} + \lambda\tilde{q})^T K_D\dot{\tilde{q}} - (\dot{\tilde{q}} + \lambda\tilde{q})^T K_p\tilde{q} + \tilde{q}^T(K_p + \lambda K_d)\dot{\tilde{q}} \\
&= -\dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_D\dot{\tilde{q}} - \dot{\tilde{q}}^T K_p\tilde{q} - \lambda\tilde{q}^T K_p\tilde{q} + \tilde{q}^T(K_p + \lambda K_d)\dot{\tilde{q}} \\
&= -\dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q}
\end{aligned}
$$

The Lyapunov function time derivative is thus:

$$
\begin{aligned}
\dot{V} &= -sC\dot{q} + sC\dot{q}_r + \frac{1}{2}s\dot{H}s - \dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q} \\
&= -sCs + \frac{1}{2}s\dot{H}s - \dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q} \\
&= \frac{1}{2}s(\dot{H} - 2C)s - \dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q} \\
&= -\dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q}
\end{aligned}
$$

Because K_D, K_p are ND, $-\dot{\tilde{q}}^T K_D\dot{\tilde{q}} - \lambda\tilde{q}^T K_p\tilde{q}$ is ND, and because $\dot{H} - 2C$ is skew symmetric, $s(\dot{H} - 2C)s = 0$. Thus, $\dot{V}$ is ND. But what can we say about

$$\ddot{V}$$

?

$$\ddot{V} = -\dot{\tilde{q}}^T K_D\ddot{\tilde{q}} - \lambda\tilde{q}^T K_p\dot{\tilde{q}}$$

Since V is monotonically decreasing, V is upper bounded by its initial value, implying that s is bounded. Since s is bounded, so too are $\dot{q}, q$, and from the dynamics, if $\dot{q}, q$ are bounded, so too is $\ddot{q}$. Thus, we see that $\ddot{V}$ is a function of all bounded terms and thus $\ddot{V}$ is also bounded. Because V is bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is bounded, by Lemma 4.3, $\dot{V} \to 0 \to \tilde{q}, \dot{\tilde{q}} \to 0$. We conclude that the tracking error does fall to 0.

Problem 5

Perform a transformation of variables by defining $z_1 \stackrel{\text{def}}{=} x_1$:

$$
\begin{aligned}
z_1 &= x_1 \\
\dot{z}_1 &= \sin(x_2) \\
\ddot{z}_1 &= \cos(x_2)\dot{x}_2 \\
&= x_1^4\cos(x_2)^2 + \cos(x_2)u
\end{aligned}
$$

Define an intermediate variable:

$$v \stackrel{\text{def}}{=} x_1^4 \cos(x_2)^2 + \cos(x_2)u$$

and choose v as follows, with positive k_1, k_2:

$$v = \ddot{z}_d - k_1(\dot{z} - \dot{z}_d) - k_2(z - z_d)$$

Then

$$(\ddot{z} - \ddot{z}_d) + k_1(\dot{z} - \dot{z}_d) + k_2(z - z_d) = 0$$

represents an exponentially stable system meaning that z and its derivatives will converge to z_d and its derivatives, which in turn means x_1 and its derivatives will converge to x_{d1} and its derivatives. We then need to solve for u:

$$\begin{aligned} \ddot{z}_d - k_1(\dot{z}_1 - \dot{z}_d) - k_2(z_1 - z_d) &= v = x_1^4 \cos(x_2)^2 + \cos(x_2)u \\ \frac{\ddot{z}_d - k_1(\dot{z}_1 - \dot{z}_d) - k_2(z_1 - z_d) - x_1^4 \cos(x_2)^2}{\cos(x_2)} &= u \end{aligned}$$