2.152 Nonlinear Control Spring 2020 HW5

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Problem 1

Suppose we chose $u = v + x_2^2 x_1$. The second equation would simplify to

$$\dot{x}_2 + x_2^2 x_1 = u \Leftrightarrow \dot{x}_2 + x_2^2 x_1 = v + x_2^2 x_1 \Leftrightarrow \dot{x}_2 = v$$

Treating x_2 as the input to the first equation, if we could set x_2 equal to $x_2^d = -2x_1 \cos(x_1) - x_1$, we would make x_1 globally asymptotically stable:

$$\ddot{x}_1 + 2\dot{x}_1 - 2x_1 \cos(x_1) = x_2 \Leftrightarrow \ddot{x}_1 + 2\dot{x}_1 + x_1 = 0$$

In other words, if we could choose u to meet our desired x_2^d, then x_2^d would drive x_1 to 0 and x_2 would similarly follow to 0. To find u, consider the Lyapunov function candidate:

$$V(x_1, x_2, x_2^d) = \frac{1}{2}x_1^2 + \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}(x_2 - x_2^d)^2$$

Its time derivative is:

$$\begin{aligned}
\dot{V}(x_1, x_2, x_2^d) &= x_1\dot{x}_1 + \dot{x}_1\ddot{x}_2 + (x_2 - x_2^d)(\dot{x}_2 - \dot{x}_2^d) \\
&= x_1\dot{x}_1 + \dot{x}_1(x_2 - x_2^d - x_1 - 2\dot{x}_1) + (x_2 - x_2^d)(v - \dot{x}_2^d) \\
&= -2\dot{x}_1^2 + \dot{x}_1(x_2 - x_2^d) + (x_2 - x_2^d)(v - \dot{x}_2^d)
\end{aligned}$$

Choosing $v = \dot{x}_2^d - \dot{x}_1$ will thus ensure x_1 is stabilized, and choosing $u = v + x_2^2 x_1 = \dot{x}_2^d - \dot{x}_1 + x_2^2 x_1$ will ensure x_2 and x_1 are both stabilized to the origin.

Problem 2

(a) Consider $\{\dot{x} = f_i(x,t)\}_{i=1}^n$, where each system $\dot{x} = f_i(x,t)$ is contracting for the same $\theta(x)$, and further consider $\{\alpha_i\}_{i=1}^n$ such that $\sum_i \alpha_i > 0$ and $\alpha_i \geq 0$ for all i, t. We wish to show that the superposition

$$\dot{x} = \underbrace{\sum_i \alpha_i(t) f_i(x,t)}_{f(x,t)}$$

is also contracting. To show this, we need to show two facts. First, $\theta^T\theta > 0$. This is trivially met by considering any of the systems individually. Second, the generalized Jacobian is uniformly negative definite. To show this, we use the definition above and the fact that each generalized Jacobian for f_i is uniformly negative definite.

$$\begin{aligned}
F &\stackrel{\text{def}}{=} \theta(\partial_x f)\theta^{-1} + \dot{\theta}\theta^{-1} \\
&= \theta(\partial_x \sum_i \alpha_i(t) f_i(x,t))\theta^{-1} + \dot{\theta}\theta^{-1} \\
&= \sum_i \alpha_i(t) \underbrace{\theta(\partial_x f_i(x,t))\theta^{-1} + \dot{\theta}\theta^{-1}}_{<0} \\
&< 0
\end{aligned}$$

where the last line follows because α_i are non-negative and because at least one must be positive.

(b) Assume that $\dot{z_1} = F_{11} z_1$ is a contracting system with respect to θ_1 and $\dot{z_2} = F_{22} z_2$ is a contracting system with respect to z_2. Our goal is to show that $\begin{bmatrix} \dot{\delta z_1} \\ \dot{\delta z_2} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \delta z_1 \\ \delta z_2 \end{bmatrix}$ is also a contracting system if F_{21} is bounded. This requires picking a θ and showing that (a) $\theta^T\theta > 0$ and (b) $\theta\partial_x f\theta^{-1} + \partial_t\theta\theta^{-1} < 0$. Choose

$$\theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$$

which satisfies the first criteria and yields the following block matrix for the generalized Jacobian of the combined system:

$$F \stackrel{\text{def}}{=} \theta \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \theta^{-1} + \dot{\theta}\theta^{-1} = \begin{bmatrix} \theta_1 F_{11}\theta_1^{-1} + \dot{\theta_1}\theta_1^{-1} & 0 \\ \theta_2 F_{21}\theta_1 & \theta_2 F_{22}\theta_2^{-1} + \dot{\theta_2}\theta_2^{-1} \end{bmatrix}$$

To see that F is uniformly negative definite, consider

$$\partial_t(\delta z^T F \delta z) = \partial_t \delta z_1^T(\theta_1 F_{11}\theta_1^{-1} + \dot{\theta_1}\theta_1^{-1})\delta z_1 + \delta z_2^T(\theta_2 F_{22}\theta_2^{-1} + \dot{\theta_2}\theta_2^{-1})\delta z_2 + z_2^T\theta_2 F_{21}\theta_1 z_1$$

We know that the first two terms on the right-hand side will tend to 0 exponentially fast, which implies that $\delta z_1 \to 0$ and $\delta z_2 \to 0$ exponentially fast, meaning the third term also tends to 0 exponentially quickly.

(c) We want to show that the system

$$\begin{bmatrix} \dot{\delta z_1} \\ \dot{\delta z_2} \end{bmatrix} = \begin{bmatrix} F_{11} & kG \\ -G^T & F_{22} \end{bmatrix} \begin{bmatrix} \delta z_1 \\ \delta z_2 \end{bmatrix}$$

is contracting if the individual systems $\delta z_1, \delta z_2$ are separately contracting. Define the contraction metric:

$$\theta = \begin{bmatrix} I & 0 \\ 0 & \sqrt{k}I \end{bmatrix}$$

The generalized Jacobian is thus:

$$F \stackrel{\text{def}}{=} = \theta \begin{bmatrix} F_{11} & kG \\ -G^T & F_{22} \end{bmatrix} \theta^{-1} = \begin{bmatrix} F_{11} & \sqrt{k}G \\ -\sqrt{k}G^T & F_{22} \end{bmatrix}$$

The generalized Jacobian is:

$$F_{sym} = \frac{1}{2} \begin{bmatrix} F_{11} + F_{11}^T & 0 \\ 0 & F_{22} + F_{22}^T \end{bmatrix}$$

Since the sum of two uniformly negative definite matrices is uniformly negative definite, and since the block diagonal composition of uniformly negative definite matrices is still uniformly negative definite, we conclude that the overall system is contracting.

Problem 3

(a) Let $I(0)$. Then $\partial_t I(t) = aSI - \kappa I - \gamma I = 0 \Rightarrow I(t) = 0 \Rightarrow \partial_t S = \alpha - \gamma S \Rightarrow S(t) = \frac{\alpha}{\gamma} + S(0)e^{-\gamma t} \Rightarrow \lim_{t\to\infty} S(t) = \frac{\alpha}{\gamma}$. We interpret this to mean that under this model (which assumes that population growth is independent of population size), the susceptible population converges to α/γ.

(b) We want to show that $\dot{S}(t)|_{S=0,I\geq 0,R\geq 0}, \dot{I}|_{S\geq 0,I=0,R\geq 0}, \dot{R}|_{S\geq 0,I\geq 0,R=0} \geq 0$. We consider the time derivatives of each evaluated

$$
\begin{aligned}
\partial_t S\big|_{S=0,I\geq 0,R\geq 0} &= \alpha - 0 - 0 \\
&\geq 0 \\
\partial_t I\big|_{S\geq 0,I=0,R\geq 0} &= 0 - 0 - 0 \\
&\geq 0 \\
\partial_t I\big|_{S\geq 0,I\geq 0,R=0} &= \kappa I - 0 \\
&\geq 0
\end{aligned}
$$

(c) We consider the time derivative of $N(t) \stackrel{\text{def}}{=} S(t) + I(t) + R(t)$:

$$
\begin{aligned}
\partial_t N &= \partial_t S + \partial_t I + \partial_t R \\
&= \alpha - \gamma S - aSI + aSI - \kappa I - \gamma I + \kappa I - \gamma R \\
&= \alpha - \gamma(S + I + R) \\
&= \alpha - \gamma N \\
N(t) &= \frac{\alpha}{\gamma} + N(0)e^{-\gamma t}
\end{aligned}
$$

From this we see that $N(t)$ is contracting to α/γ. We see that $N_\infty = S_0$. I don't think that $S(t), I(t), R(t) \leq N_\infty$ unless $N(0) \leq N_\infty$. With this additional assumption, we know that $N(t)$ converges exponentially to its limit, and since $S(t), I(t), R(t)$ are bounded between 0 and $N(t)$, where $N(t) < N_\infty$ by assumption, we reach the appropriate conclusion.

(d) Let $\theta(t) = I(t)$. Then

$$\partial_t \theta = aSI - \kappa I - \gamma I = \partial_t I$$

To calculate the Jacobian J of the virtual system, take the derivative w.r.t. θ:

$$J \stackrel{\text{def}}{=} \partial_\theta \Big(aS\theta - \kappa\theta - \gamma\theta \Big) = aS - \kappa - \gamma$$

In order for the virtual system to be contracting, we need the Jacobian to be negative. We know from the previous two parts that $S(t) \in [0, \alpha/\gamma]$. For the lower bound, the inequality is trivially satisfied:

$$-\kappa - \gamma < 0$$

For the upper bound, the inequality is satisfied if

$$a\frac{\alpha}{\gamma} - \kappa - \gamma < 0$$

Assuming this condition is met, we know that $\theta(t)$ is contracting and therefore $I(t)$ is contracting. Treating $I(t)$ as the input to the system, $S(t)$ and $R(t)$ are also contracting. There is one equilibrium point in the positive orthant.

(e) One equilibrium point is $S = S_0, I = 0, R = 0$. We can verify this by immediately seeing that $\partial_t I = 0, \partial_t R = 0$ and that $\partial_t S = \alpha - \gamma\frac{\alpha}{\gamma} = 0$. To find the other, I use the hint that $S^* = \frac{\kappa+\gamma}{a}$. This yields the following system of linear equations with two unknowns (I, R):

$$\begin{aligned}
0 &= \alpha - \gamma\frac{\kappa+\gamma}{a} - a\frac{\kappa+\gamma}{a}I \\
0 &= a\frac{\kappa+\gamma}{a}I - \kappa I - \gamma I \\
0 &= KI - \gamma R
\end{aligned}$$