

Lecture 1 - Digital Signatures

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Note: This lecture is based on Princeton University's BTC-Tech: Bitcoin and Cryptocurrency Technologies Spring 2015 course.

Cryptography

- If you want to make information secret, hide its existence or make it intelligible
- Symmetric and Asymmetric Cryptography

Symmetric Cryptography

- An encryption scheme $SE = (K, E, D)$ in which the sender and receiver use the same key to encrypt and decrypt information.

Randomized *key generation* algorithm K that returns a string k . $Keys(SE)$ is the set of possible bitstrings that K can output (keys).

Encryption Algorithm E that takes a key $k \in K$.

decryption algorithm D that inputs a key $k \in Keys(SE)$ and *plaintext* $M \in \{0,1\}^*$.

$M \leftarrow D_K(C)$

- Message Space
 - The set of possible messages
- Key Space
- Ciphertext Space/Cipher Space
- Caesar Cipher
- One Time Pad
- Diffie-Hellman Key Exchange

A method of publically creating cryptographic keys.

Trapdoor Function: A function that is easy to compute in one direction, but not the other.

Perfect Forward Secrecy: The property in which compromised long-term keys do not invalidate the integrity of a session key.

1. Choose a modulus $p = 23$ and base $g = 5$.

g is a Primitive root modulo. A number g is a *Primitive root modulo* if every number a coprime to n is congruent to a power g of modulo n .

Restated: For every integer a coprime to n , there exists an integer k such that $g^k \equiv a \pmod{n}$.

Restated: g is a generator for the multiplicative group of integers modulo n .

2. Alice chooses a secret integer $a = 6$, and sends Bob $A = g^a \pmod{p}$.

$$A = 5^6 \pmod{23} = 8$$

3. Bob chooses secret integer $b = 15$, and sends Alice $B = g^b \bmod p$.
 $B = 5^{15} \bmod 23 = 19$
4. Alice computes $s = B^a \bmod p$.
 $s = 19^6 \bmod 23 = 2$
5. Bob computes $s = A^b \bmod p$.
 $s = 8^{15} \bmod 23 = 2$
6. They reached the same number because under mod p :
 $A^b \bmod p = g^{ab} \bmod p = g^{ba} \bmod p = B^a \bmod p$
 $= (g^a \bmod p)^b \bmod p = (g^b \bmod p)^a \bmod p$

Asymmetric Cryptography

- Properties

Only you can create a signature, but anyone can verify its validity

Tied to a document

Ex: Actual mail

- Public/Private keys

- Digital signature scheme

$(sk, pk) := generateKeys(keySize)$ Randomized key generation

$sig := sign(sk, message)$ Encryption Algorithm

$isValid := verify(pk, message, sig)$ Deterministic Decryption Algorithm

- Ensure only one person can decrypt your message

- Ensure a message was created by someone

- RSA

Ron Rivest, Adi Shamir, Leonard Adleman

- Basic Principle: $(m^e)^d \bmod n = m$

Even with m , e , and n , it is extremely difficult to find d .

- Key Distribution

Distribute public key (n, e) .

- Encryption

Change M to integer m . Make sure $0 \leq m < n$ and $\gcd(m, n) = 1$ through an agreed padding scheme.

Compute ciphertext c through $c \equiv (m^e)^d \equiv m \bmod n$

- Decryption

Use private key exponent d with $c^d \equiv (m^e)^d \equiv m \bmod n$

Then, reverse emphm to M using the padding scheme.

- Key Generation

Choose different prime numbers p and q . Find $n = pq$.

n is the key. $len(n) = keyLength$.

Find $\phi(n)$.

1. Choose two large, different primes p and q . Find $n = p \cdot q$

n is the key. $len(n) = keyLength$.

2. Find $\phi(n)$. This turns out to be $(p-1)(q-1)$
 ϕ is Euler's totient function. $\phi(n)$ is the number of positive integers less than n that are coprime to n . $\phi(1) = 1$.

Two numbers are *Coprime* when the only positive integer that divides them is 1.

Given n , a prime number, $\phi(n) = n - 1$. e.g. $n = 5$, so 1, 2, 3, 4 are coprime to 5.

- (a) However, for composite numbers, it works for some but not others.
- (b) e.g. $15 = 3 \cdot 5$ and $\phi(15) = \phi(3) \cdot \phi(5) = 2 \cdot 4 = 8$.
- (c) But doesn't hold for 4, 8, 9.

If m and n are coprime, $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$.

3. Find an integer e where $1 < e < \phi(n)$ and $\gcd(e, \phi(n)) = 1$. This means e and $\phi(n)$ are coprime.
4. Find d , the modular multiplicative inverse of $e \pmod{\phi(n)}$
This means: Solve for d where $d \cdot e \equiv 1 \pmod{\phi(n)}$
 e should have short length, and is usually $2^{16} + 1 = 65,537$.
 e is the public key, along with n .
 d is the private key.

– Example:

1. Find two distinct prime numbers
 $p = 11$ and $q = 7$
2. Find $n = pq$
 $n = 11 \cdot 7 = 77$
3. Compute the totient of n
 $\phi(77) = (11 - 1)(7 - 1) = 60$
4. Choose e such that $1 < e < 60$, where e is coprime to 60.
Let $e = 17$; check that 60 is not divisible by 17.
5. Compute d . Process is below.
 $d = 53$
 $d \cdot e \pmod{\phi(n)} = 1$
 $53 \cdot 17 \pmod{60} = 1$
6. The public key is $n = 77$ and $e = 17$.
 $c(m) = m^{17} \pmod{77}$
7. The private key is $d = 53$.
 $m(c) = c^{53} \pmod{77}$
8. To encrypt $m = 65$,
 $c = 65^{17} \pmod{77} = 32$
9. To decrypt $c = 32$,
 $m = 32^{53} \pmod{77} = 65$.

– Calculating d for above: use Extended Euclid's Algorithm

Basically finding \gcd with Euclid's Algorithm, but reversed.

$$\begin{aligned}
 \phi(77) &= 60 \\
 e \cdot d \pmod{60} &= 1 \\
 17 \cdot d \pmod{60} &= 1 \\
 60 &= 3(17) + 9 \\
 17 &= 1(9) + 8 \\
 9 &= 1(8) + 1
 \end{aligned}$$

Once we find 1, rewrite as:

$$\begin{aligned}1 &= 9 - 1(8) \\8 &= 2(9) - 17 \\9 &= 60 - 3(17)\end{aligned}$$

Therefore,

$$\begin{aligned}1 &= 9 - (17 - 1(9)) \\1 &= 2(9) - 17 \\1 &= 2(60 - 3(17)) - 17 \\1 &= 2(60) - 7(17) \\\phi(77) - 7 &= d \\d &= 53\end{aligned}$$

- Calculating d for above: use Extended Euclid's Algorithm
Basically finding gcd with Euclid's Algorithm, but reversed.
- Fermat's little theorem states $a^p = a \mod p$. This is equal to $a^{p-1} = 1 \mod p$.
- For RSA, this is insufficient. You need the Euler-Fermat generalisation:
 $a^{\phi(n)} = 1$

Group Theory: Multiplication for some sets of integers makes a group under modulo, if all the elements are coprime to the modulo used.

E is coprime to $\phi(pq)$. Group theory says there exists some integer that acts uniquely as an inverse and transforms under multiplication to the identity.

The identity element for multiplication is 1. The inverse is d .

Source: <http://crypto.stackexchange.com/questions/388/what-is-the-relation-between-rsa-fermats-little-theorem?lq=1>