Week 1

Topics: first-order linear ODEs; integrating factor; recap of linear algebra; variation of parameters

1. Consider the differential equation $xy' + y = e^x$.

(a) Explain why it is a first-order linear inhomogeneous differential equation.

(b) By considering $(xy)'$, find its general solution (involving one arbitrary constant).

(c) Find a solution that satisfies the initial condition $y(1) = 0$.

(d) Then find the one and only value y_0 for which the initial value problem with $y(0) = y_0$ can be solved.

(e) What feature of the equation suggests that this equation cannot be solved for an arbitrary value of $y(0)$?

2. Suppose that your student loan balance is \$1K, that it accrues interest at a rate of 50% per decade, and that you pay it off at a steady rate of \$1K per decade. Denote by y the student loan balance as a function of x, the number of decades. What is the student loan balance for $x = 1$ decade? This amounts to solving the initial value problem

$$\begin{cases} y' &= 0.5y - 1 \\ y(0) &= 1 \end{cases}$$

Use the idea of the integrating factor to solve this problem.

3. Consider the general linear first-order equation,

$$a(x)y' + b(x)y = h(x).$$

Show that we may choose the integrating factor to be

$$\mu(x) = \exp\left(\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right),$$

and use this integrating factor to get a general solution to the equation for any region in which $x \mapsto b(x)/a(x)$ is integrable.

4. Show that the function $\ell : V \to V$ given by the formula $\ell(y) = a(x)y' + b(x)y$ has the properties required for linearity, and the kernel of ℓ is a one-dimensional subspace of V. Find a vector (function) $f \in V$ such that $\ker \ell = \operatorname{span}\{f\}$.

5. Let V be the three-dimensional vector space of polynomials of degree no greater than 2, with basis $\mathfrak{B} = \{1, x, x^2\}$. Let ℓ be the linear differential operator

$$\ell(y) = (x+1)y' - 2y.$$

(a) Write down the matrix L that represents ℓ with respect to the basis $\mathfrak{B}$, and find a basis for the kernel of L.

(b) Then find, by algebraic methods, the general solution to $\ell(y) = -2x$.

(c) Is there any element $h(x) \in V$ for which $\ell(y) = h(x)$ cannot be solved?

6. Let V be the infinite-dimensional space of differentiable functions (can you give a rigorous argument why it is infinite-dimensional?) and

$$\ell(y) = a(x)y' + b(x)y.$$

In this case we cannot write down a matrix for ℓ (since a basis for V has infinitely many vectors!), but we know how to find a vector (function) f that spans its kernel.

(a) Prove that if y_p is any **particular solution** to $\ell(y) = h$, then the general solution is $y = Cf + y_p$ for some constant C.

(b) Use this theorem and a bit of guesswork to find the general solution to

$$3xy' - y = \log x + 1.$$

7. Suppose that we want to solve $\ell(y) = h(x)$ (where $\ell(y) = ay' + by$) and we have already found a function $f(x)$ that spans the kernel of ℓ. Use the variation of the parameters idea to find a particular solution to $\ell(y) = h$.

(a) Show that we can get a formula for the derivative $g'(x)$ and thereby (at least in principle) find a solution to the inhomogeneous equation $\ell(y) = h$.

(b) Apply this approach to the equation

$$xy' + 2y = x,$$

which we could also solve by using an integrating factor.