MATH110 Spring 2020 HW1

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Problem 1

Our approach will be variation of parameters. Defining $l(y) = (x^2+1)y' - (1-x)^2 y$, we first find an element $f \in \ker(l)$:

$$\begin{aligned}
0 &= l(f)\\
\frac{f'}{f} &= \frac{(1-x)^2}{x^2+1}\\
&= \frac{x^2+1}{x^2+1} - \frac{2x}{x^2+1}\\
\log f(t)|_{t=x_0}^{t=x} &= \int_{t=x_0}^{t=x} dt - \int_{t=x_0}^{t=x} \frac{2t}{t^2+1} dt\\
\log f(x) - \log f(x_0) &= x - x_0 - \log(x^2+1) + \log(x_0^2+1)\\
f(x) &= f(x_0)\frac{x_0^2+1}{x^2+1} e^{x-x_0}
\end{aligned}$$

We know from Worksheet 2 that a general solution to the inhomogeneous equation $a(x)y'(x) + b(x)y(x) = h(x)$ is $y(x) = g(x)f(x)$, where $g(x) = \int_{t=x_0}^{t=x} \frac{h(t)}{a(t)f(t)} dt$. For our ODE, $g(x)$ is:

$$\begin{aligned}
g(x) &= \int_{t=x_0}^{t=x} \frac{h(t)}{t^2+1}\frac{t^2+1}{x_0^2+1}\frac{1}{f(x_0)} e^{x_0 - t} dt\\
&= \frac{e^{x_0}}{f(x_0)(x_0^2+1)} \int_{t=x_0}^{t=x} h(t)e^{-t} dt
\end{aligned}$$

Letting $h(x) = xe^{-x}$:

$$\begin{aligned}
g(x) &= \frac{e^{x_0}}{f(x_0)(x_0^2+1)} \int_{t=x_0}^{t=x} te^{-2t} dt\\
&= \frac{e^{x_0}}{f(x_0)(x_0^2+1)} \left(-\frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}\Big|_{t=x_0}^{t=x} \right)\\
&= \frac{e^{x_0}}{f(x_0)(x_0^2+1)} \left(-\frac{1}{4}e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x_0} + \frac{1}{2}x_0 e^{-2x_0} \right)
\end{aligned}$$

Letting $x_0 = 0$ and $y(x_0 = 0) = 1$:

$$
\begin{aligned}
y(x) &= f(x)g(x) \\
&= f(x_0)\frac{x_0^2+1}{x^2+1}(e^{x-x_0})\frac{e^{x_0}}{f(x_0)(x_0^2+1)}\bigg(-\frac{1}{4}e^{-2x}-\frac{1}{2}xe^{-2x}+\frac{1}{4}e^{-2x_0}+\frac{1}{2}x_0e^{-2x_0}\bigg) \\
&= \frac{e^x}{x^2+1}\bigg(-\frac{1}{4}e^{-2x}-\frac{1}{2}xe^{-2x}+\frac{1}{4}\bigg) \\
&= \frac{1}{x^2+1}\bigg(-\frac{1}{4}e^{-x}-\frac{1}{2}xe^{-x}+\frac{1}{4}e^{x}\bigg)
\end{aligned}
$$

Problem 2

Consider our first order linear ODE $a(x)y' + +b(x)y = h(x)$, where $a(x) = x^2+1$, $b(x) = -(1-x)^2$ and $h(x) = xe^{-x}$. We start by expanding each function into their Taylor series around $x_0 = 0$:

$$
\begin{aligned}
y(x) &= y(x_0) + \frac{y'(x_0)}{1}x + \frac{y''(x_0)}{2!}x^2 + \frac{y'''(x_0)}{3!}x^3 + ... \\
&= 1 + \frac{y'(x_0)}{1}x + \frac{y''(x_0)}{2!}x^2 + \frac{y'''(x_0)}{3!}x^3 + ... \\
y'(x) &= y'(x_0) + \frac{y''(x_0)}{1}x + \frac{y'''(x_0)}{2!}x^2 + \frac{y''''(x_0)}{3!}x^3 + ... \\
a(x) &= a(x_0) + \frac{a'(x_0)}{1}x + \frac{a''(x_0)}{2}x^2 + ... \\
&= \frac{0^2+1}{1} + \frac{2(0)}{1}x + \frac{2}{2}x^2 \\
&= 1 + x^2 \\
b(x) &= b(x_0) + \frac{b'(x_0)}{1}x + \frac{b''(x_0)}{2}x^2 + ... \\
&= -1 + 2x - 2x^2 \\
h(x) &= xe^{-x} \\
&= \frac{0e^0}{1} + \frac{e^0-(0)e^0}{1}x + \frac{-2e^0-(0)e^0}{2!}x^2 + ... \\
&= x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + ...
\end{aligned}
$$

For compactness, I drop x_0 as an argument to y and its derivatives. Our original equation is:

$$
(1+x^2)(y' + \frac{y''}{1}x + \frac{y'''}{2!}x^2 + ...) + (-1+2x-x^2)(y + \frac{y'}{1}x + \frac{y''}{2!}x^2 + \frac{y'''}{3!}x^3 + ...) = x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4
$$

I first group all terms with no x and use $y(x_0 = 0) = 1$ to solve for y':

$$
1y' + -1y(x_0) = 0 \Rightarrow y' = 1
$$

I then group all terms with x and solve for y'':

$$
y'' + 2y - 1y' = 1 \Rightarrow y'' + 2(1) - 1(1) = 1 \Rightarrow y'' = 0
$$

I next group all terms with x^2:

$$\frac{y'''}{2} + y' - \frac{y''}{2} + 2y' - y = -1 \Rightarrow y''' = -6$$

Repeating again for x^3:

$$\frac{1}{3!}y'''' + y'' - \frac{y'''}{3!} + y'' - y' = \frac{1}{2} \Rightarrow y'''' = 3$$

Thus, the solution up to the first four non-zero coefficients is:

$$y(x) = 1 + x + 0x^2 - x^3 + \frac{1}{8}x^4 + ...$$

I was too lazy to check my answer by hand, so I plugged the following query into Wolfram Alpha:

```
Series[(-0.25/E^x - (0.5 x)/E^x + 0.25 E^x)/(x^2 + 1), {x, 0, 8}]
```

Problem 3

Recall that a function is analytic if for some $\epsilon > 0$, the Taylor series of the function (a) converges in some interval $x \in [x_0 - \epsilon, x_0 + \epsilon]$, and (b) is equal to the function itself. Let $x_0 \in \mathbb{R}$ and $f : x \to e^x$. We start by showing the first property holds:

$$T_{f,x_0} = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!}$$

Using a_n to refer to the nth term in the sequence, we use a ratio test to show that T_{f,x_0} converges for all x_0 with a radius of ∞.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x - x_0)^{n+1}}{(n+1)!} \frac{n!}{(x - x_0)^n} \right| = \lim_{n\to\infty} \left| \frac{x - x_0}{n+1} \right| = 0$$

One way to define $f : x \to e^x$ is as the function for which $f(x) = \frac{d}{dx}f(x)$ and $f(0) = 1$. We use this to show that the second property holds:

$$\frac{d}{dx}T_{f,x_0} = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} = \sum_{k=1}^{\infty} \frac{(x - x_0)^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} = T_{f,x_0}$$

I also need to show that the Taylor series centered at 0 and evaluated at x=0 is 1:

$$T_{f,0}(0) = \sum_{k=0}^{\infty} \frac{(0 - 0)^k}{k!} = 1 + 0 + 0 + ... = 1$$

Thus, $f : x \to e^x$ is analytic for all $x_0 \in \mathbb{R}$.

Problem 4

In class, we saw that a function can be written as the first K terms of its Taylor series plus a remainder:

$$
\begin{aligned}
f(x) &= f(x_0) + \int_{t_1=x_0}^{t_1=x} dt_1\, f'(t_1) \\
&= f(x_0) + f'(x_0)(x-x_0) + \int_{t_1=x_0}^{t_1=x} dt_1 \int_{t_2=x_0}^{t_2=t_1} dt_2 f''(t_2) \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \int_{t_1=x_0}^{t_1=x} dt_1 \int_{t_2=x_0}^{t_2=t_1} dt_2 \int_{t_3=x_0}^{t_3=t_2} dt_3 f'''(t_3) \\
&= \sum_{j=1}^{k} \frac{f^{(j)}}{j!}(x-x_0)^j + \int_{x_0}^{x}\int_{x_0}^{t_1} ... \int_{x_0}^{t_k} dt_{k+1} f^{(k+1)}(t_{k+1})
\end{aligned}
$$

Starting with the LHS, we Taylor series expand $f(x)$ around x_0:

$$
\begin{aligned}
\text{LHS} &= \sup_{x\in[x_0-h,x_0+h]} \left| f(x) - \sum_{j=0}^{k} \frac{f^{(j)}}{j!}(x-x_0)^j \right| \\
&= \sup_{x\in[x_0-h,x_0+h]} \left| \sum_{j=1}^{k} \frac{f^{(j)}}{j!}(x-x_0)^j + \int_{x_0}^{x}\int_{x_0}^{t_1} ... \int_{x_0}^{t_k} dt_{k+1} f^{(k+1)}(t_{k+1}) - \sum_{j=0}^{k} \frac{f^{(j)}}{j!}(x-x_0)^j \right| \\
&= \sup_{x\in[x_0-h,x_0+h]} \left| \int_{x_0}^{x}\int_{x_0}^{t_1} ... \int_{x_0}^{t_k} dt_{k+1} f^{(k+1)}(t_{k+1}) \right| \\
&\leq \int_{x_0}^{x}\int_{x_0}^{t_1} ... \int_{x_0}^{t_{k-1}} \int_{x_0}^{t_k} dt_{k+1} |f^{(k+1)}(t_{k+1})| \\
&\leq \int_{x_0}^{x}\int_{x_0}^{t_1} ... \int_{x_0}^{t_{k-1}} dt_k |C_1(x-x_0)|
\end{aligned}
$$

where C_1 is the max value of $f^{(k+1)}$ over the interval $[x_0, x]$, which depends on f and k but not h. Continuing the same reasoning for the other integrals, and noting that $x_0 - h \leq x \leq x_0 + h \Rightarrow -h \leq x - x_0 \leq h \Rightarrow 0 < |x - x_0| \leq h$:

$$LHS \leq C|(x-x_0)^{k+1}| = Ch^{k+1}$$

This does not imply that f is analytic at x_0. Even if we consider the limit at $k \to \infty$, the function could be discontinuous such that the Taylor series fails to converge.

Problem 5

a

Consider $y \in V$ and define M_f and D as given. Then

$$DM_f(y) = D(fy) = f'y + fy'$$

and

$$(M_{f'} + M_f D)(y) = f'y + fy'$$

b

$$\begin{aligned}(M_f D + M_g)(D + M_x) &= M_f DD + M_f DM_x + M_g D + M_g M_x \\ &= M_f D^2 + M_f(M_{x'} + M_x D) + M_g D + M_{gx} \\ &= M_f D^2 + M_{fx'+gx} + M_{fx+g} D \\ &= M_f D^2 + M_{f+gx} + M_{fx+g} D\end{aligned}$$

Note that $(M_f D^2 + M_{f+gx} + M_{fx+g} D)(y) = 0$ if $(D + M_x)(y) = 0$. $(D + M_x)(y) = 0 \Leftrightarrow y' + xy = 0$ is a first order linear differential equation that we know has a non-zero solution; consequently, this solution will be an element of $\ker(l)$ and thus $\ker(l) \neq \{0\}$.

Problem 6

Consider the power series expansion of each term:

$$\begin{aligned}y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... \\ 2y(x) &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + ... \\ y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ... \\ -xy('x) &= 0 - a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 ... \\ y''(x) &= 2a_2 + 6a_3 x + 24a_4 x^2 + ...\end{aligned}$$

Our equation

$$y'' - xy' + 2y = 0$$

then becomes

$$(2a_2 + 6a_3 x + 24a_4 x^2 + ...) + (0 - a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 - ...) + (2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + ...) = 0$$

Recalling that $a_0 = y(0) = 1$ and that $a_1 = y'(0) = 0$, we first match terms with no x:

$$2a_2 + 2a_0 = 0 \Rightarrow a_2 + 1 = 0 \Rightarrow a_2 = -1$$

We next match terms with x:

$$6a_3 - a_1 + 2a_1 x = 0 \Rightarrow 6a_3 - 0 + 2(0) = 0 \Rightarrow a_3 = 0$$

Continuing along, we match terms with x^2:

$$24a_4 - 2a_2 + 2a_2 = 0 \Rightarrow 24a_4 - 2(-1) + 2(-1) = 0 \Rightarrow a_4 = 0$$

Since $a_3 = 0$ and $a_4 = 0$, then all subsequent coefficients should be zero for this 2nd order ODE. To check, we match terms with x^3:

$$120a_5 - 3a_3 + 2a_3 = 0 \Rightarrow a_5 = 0$$

Thus, our solution is $y(x) = 1 - x^2$. To confirm, we plug back in:

$$\begin{aligned}y'' - xy' + 2y &= 0 \\ -2 - x(-2x) + 2(1 - x^2) &= \\ -2 + 2x^2 + 2 - 2x^2 &= \\ 0 &=\end{aligned}$$

Nice!