MATH110 Spring 2020 HW9

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Problem 2

(a) Let $\phi \in D_l, \psi \in D_{l^*}$. We start by considering $\langle l\phi, \psi \rangle$:

$$\begin{aligned}
\langle l\phi, \psi \rangle &\stackrel{\text{def}}{=} \int_0^1 dx - \partial_x^2 \phi(x) \overline{\psi}(x) \\
&= -\partial_x \phi(x) \overline{\psi}(x)|_0^1 + \int_0^1 \partial_x \phi(x) \partial_x \overline{\psi}(x) \\
\partial_x \phi(0) \big(\overline{\psi}(1) - \overline{\psi}(0)\big) &= \int_0^1 \partial_x \phi(x) \overline{\partial_x \psi}(x)
\end{aligned}$$

Using this as a halfway point, we then consider $\langle \phi, l\psi \rangle$

$$\begin{aligned}
\langle \phi, l\psi \rangle &\stackrel{\text{def}}{=} \int_0^1 dx - \phi(x) \overline{\partial_x^2 \psi}(x) \\
&= -\phi(x) \overline{\partial_x \psi}(x)|_0^1 + \int_0^1 dx \partial_x \phi(x) \overline{\partial_x \psi}(x) \\
\phi(0)(\overline{\partial_x \psi}(1) - \overline{\partial_x \psi}(0)) &= \int_0^1 dx \partial_x \phi(x) \overline{\partial_x \psi}(x)
\end{aligned}$$

Setting the two sides equal and rearranging, we see that:

$$\partial_x \phi(0) \big(\overline{\psi}(1) - \overline{\psi}(0)\big) = \phi(0)(\overline{\partial_x \psi}(1) - \overline{\partial_x \psi}(0))$$

and thus

$$0 = \partial_x \phi(0) \big(\overline{\psi}(0) - \overline{\psi}(1)\big) + \phi(0)(\overline{\partial_x \psi}(1) - \overline{\partial_x \psi}(0))$$

(b) Fix $\psi \in D_{l^*}$ and choose $\phi(x) = 1 \in D_l$. Then $\partial_x \phi(x) = 0$ and the constraint simplifies to:

$$\begin{aligned}
0 &= \partial_x \phi(0) \big(\overline{\psi}(0) - \overline{\psi}(1)\big) + \phi(0)(\overline{\partial_x \psi}(1) - \overline{\partial_x \psi}(0)) \\
&= 0 + \overline{\partial_x \psi}(1) - \overline{\partial_x \psi}(0) \\
\partial_x \psi(1) &= \partial_x \psi(0)
\end{aligned}$$

For the same ψ, choose $\phi(x) = \frac{1}{2\pi} \sin(2\pi x)$. Then $\phi(0) = \phi(1) = 0$ and the constraint simplifies to:

$$
\begin{aligned}
0 &= \partial_x\phi(0)\big(\overline{\psi}(0)-\overline{\psi}(1)\big)+\phi(0)(\overline{\partial_x\psi}(1)-\overline{\partial_x\psi}(0)) \\
&= \big(\overline{\psi}(0)-\overline{\psi}(1)\big)+0 \\
\psi(1) &= \psi(0)
\end{aligned}
$$

Since ψ meets both boundary conditions, we conclude that $\psi \in D_l$ and l is therefore self-adjoint.

Problem 3

Let $\varphi_p, \varphi_q \in L^2([0,1],dx)$. This means that

$$
\int_0^1 dx|\varphi_p|^2 < \infty \quad \text{and} \quad \int_0^1 dx|\varphi_q|^2 < \infty
$$

Our first aim is to show that $\int_0^1 dx \int_0^1 dy|\varphi_p \otimes \varphi_q|^2 < \infty$

$$
\begin{aligned}
\int_0^1 dx \int_0^1 dy|\varphi_p \otimes \varphi_q|^2 &= \int_0^1 dx \int_0^1 dy|e^{2\pi ipx}e^{2\pi iqy}|^2 \\
&= \int_0^1 dx \int_0^1 dye^{2\pi i(px+qy)}e^{-2\pi i(px+qy)} \\
&= \int_0^1 dx \int_0^1 dy \\
&< \infty
\end{aligned}
$$

To show that $(\varphi_p \otimes \varphi_q)_{p,q\in\mathbb{Z}}$, consider the inner product:

$$
\langle \varphi_a \otimes \varphi_b, \varphi_c \otimes \varphi_d \rangle = \int_0^1 dxe^{2\pi(a-c)x} \int_0^1 dye^{2\pi(b-d)y}
$$

If $a = c$ and $b = d$, the above integral evaluates to:

$$
\langle \varphi_a \otimes \varphi_b, \varphi_a \otimes \varphi_b \rangle = \int_0^1 dx \int_0^1 dy = 1
$$

Consider the case with $a \neq c$; the left integral evaluates to:

$$
\begin{aligned}
\int_0^1 dxe^{2\pi(a-c)x} &= \frac{1}{2\pi(a-c)}\sin(2\pi(a-c)x)\Big|_0^1 - \frac{i}{2\pi(a-c)}\cos(2\pi(a-c)x)\Big|_0^1 \\
&= \frac{1}{2\pi(a-c)}(0-0) - \frac{i}{2\pi(a-c)}(1-1) \\
&= 0
\end{aligned}
$$

The same conclusion is reached if $b \neq d$ via the right integral. Thus we conclude the sequence is an orthonormal sequence.

To show that the sequence $(\varphi_p \otimes \varphi_q)_{p,q\in\mathbb{Z}}$ is a complete orthonormal sequence of $L([0,1]^2, dxdy)$, we use proof by contradiction. Assume that $(\varphi_p \otimes \varphi_q)_{p,q\in\mathbb{Z}}$ is not a complete orthonormal basis. This means there exists some $\psi \neq 0$ such that $\forall p, q \in \mathbb{Z}$, $\langle \psi, \varphi_p \otimes \varphi_q \rangle = 0$. I will show that $\psi = 0$ must necessarily hold, resulting a contradiction. To see this, we start with the given:

$$
\begin{aligned}
0 &= \langle \psi, \varphi_p \otimes \varphi_q \rangle \\
&= \int_0^1 dx e^{-2\pi i p x} \underbrace{\int_0^1 dy \psi e^{-2\pi i q y}}_{g(x)} \\
&= \int_0^1 dx e^{-2\pi i p x} g(x) \\
&= \langle g(x), \varphi_p \rangle
\end{aligned}
$$

Since $(\varphi_p)_p$ is a complete orthonormal basis in $L^2([0,1], dx)$, by the hint, $g(x) = 0 \in L^2([0,1], dx)$. We then have $0 = g(x) = \int_0^1 dy \psi e^{-2\pi i q y} = \langle \psi, \varphi_q \rangle$. Since $(\varphi_q)_q$ is also a complete orthonormal basis in $L^2([0,1], dx)$, by the hint again, $\psi = 0$. This contradicts our assumption that $\psi \neq 0$. Thus we conclude that $(\varphi_p \otimes \varphi_q)_{p,q \in \mathbb{Z}}$ is a complete orthonormal basis of $L^2([0,1]^2, dxdy)$.

Problem 4

Consider the power series expansion of $y \overset{\text{def}}{=} \sum_{i=0}^n a_i x^i$. It then follows that

$$
\begin{aligned}
ny &= \sum_{i=0}^n n a_i x^i \\
y' &= \sum_{i=0}^n i a_i x^{i-1} \\
xy' &= \sum_{i=0}^n i a_i x^i \\
y'' &= \sum_{i=0}^n i(i-1) a_i x^{i-2} \\
&= \sum_{i=-2}^{n-2} (i+2)(i+1) a_{i+2} x^i
\end{aligned}
$$

Using these power series and adding coefficients with equivalent monomials:

Polynomial	i	y''	xy	ny	$y'' - xy + ny$
1	0	$2a_2$	0	na_0	$2a_2 + na_0 = 0$
x	1	$3 * 2 * a_3$	a_1	na_1	$6a_3 - a_1 + na_1 = 0$
x^2	2	$4 * 3 * a_4$	$2a_2$	na_2	$12a_4 - 2a_2 + na_2 = 0$
x^3	3	$5 * 4 * a_5$	$3a_3$	na_3	$20a_4 - 3a_3 + na_3 = 0$
x^4	4	$6 * 5 * a_6$	$4a_4$	na_4	$30a_6 - 4a_4 + na_4 = 0$

For each n, every term with degree higher than n will be zero. For even n, choose $a_0 = 1, a_1 = 0$; by the second equation and subsequent even equations, all odd powers will have coefficient 0. For odd n, choose $a_0 = 0, a_1 = 1$; by the first equation and subsequent odd equations, all even powers will have coefficient 0.

To construct the first Hermite polynomial, choose $a_0 = 1, a_1 = 0$. By the first equation, $2a_2 = 0 \Rightarrow a_4, a_6, ... = 0$ and by the second equation $6a_3 + (0-1)(0) = 0 \Rightarrow a_3, a_5, ... = 0$.

$$H_0(x) = 1$$

To construct the second Hermite polynomial, choose $a_0 = 0, a_1 = 1$. By the first equation, $2a_2 = 0$, and thus all even coefficients are zero, and by the second equation, $6a_3 + (1-1)a_1 = 0$, and thus all subsequent odd coefficients are zero.

$$H_1(x) = x$$

To construct the third Hermite polynomial, choose $a_0 = -1, a_1 = 0$. By the first equation, $2a_2 + 2a_0 = 0 \Rightarrow a_2 = 1$. By the third equation, $12a_4 - 0 = 0$ and thus all subsequent even coefficients are zero. By the second equation, $6a_3 = 0$ and thus all subsequent odd coefficients are zero.

$$H_2(x) = x^2 - 1$$

Problem 5

Consider a n-degree polynomial $p_n(x) = \sum_{i=0}^{n} a_i x^i \in L(\mathbb{R}, e^{-x^2/2}dx)$. To show that this polynomial is compatible with the Hermite operator $l(y) = e^{-x^2/2}\partial_x(e^{-x^2/2}\partial_x y)$, we need to show that $l(p_n) \in L(\mathbb{R}, e^{-x^2/2}dx)$.

$$\begin{aligned}
l(p_n(x)) &= e^{-x^2/2}\partial_x(e^{-x^2/2}\partial_x \sum_{i=0}^{n} a_i x^i) \\
&= e^{-x^2/2}\partial_x(e^{-x^2/2} \sum_{i=0}^{n} i a_i x^{i-1}) \\
&= e^{-x^2/2}(e^{-x^2/2}(-x) \sum_{i=0}^{n} i a_i x^{i-1}) + e^{-x^2/2} \sum_{i=0}^{n} i(i-1) a_i x^{i-2}) \\
&= -x \sum_{i=0}^{n} i a_i x^{i-1} + \sum_{i=0}^{n} i(i-1) a_i x^{i-2} \\
&= -\sum_{i=0}^{n} i a_i x^{i} + \sum_{i=0}^{n} i(i-1) a_i x^{i-2}
\end{aligned}$$

By seeing that $l(p_n(x))$ is also a polynomial, we can conclude that $l(p_n(x)) \in L(\mathbb{R}, e^{-x^2/2}dx)$ by recalling from the book that

$$\int_{\mathbb{R}} dx e^{-x^2/2} |x^n|^2 = \frac{(2n)!}{n! 2^n} \sqrt{2n} < \infty$$

We consider the following integral, expand the square, group polynomials and then apply the above equality.

$$\int_{\mathbb{R}} dx e^{-x^2/2} \Big| -\sum_{i=0}^{n} i a_i x^i + \sum_{i=0}^{n} i(i-1) a_i x^{i-2} \Big|^2$$

Since each individual term is finite, and since there are finitely many term, the sum is also finite.