Week 10

Topics: L^2 spaces, second order differential operators on L^2 spaces: compatibility and regularity

Review from Week 9:

- **Boundary value problems:** motivated by the vibrating wire, our goal is to set up a mathematical framework that deals with eigenvalue problems in infinite dimensions of the form:

$$\begin{cases} y'' &= \lambda y \\ y(0) &= y(L) = 0. \end{cases}$$

- **inner product spaces:** an inner product space is a vector space $(H, \langle \cdot, \cdot \rangle)$ with a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$, called the inner product, such that

$$\begin{aligned} i)\ \langle \alpha\psi + \beta\varphi, \zeta \rangle &= \alpha\langle \psi, \zeta \rangle + \beta\langle \varphi, \zeta \rangle, \\ ii)\qquad \langle \psi, \varphi \rangle &= \overline{\langle \varphi, \psi \rangle}, \\ iii)\qquad \langle \psi, \psi \rangle &\geq 0 \quad \text{and} \quad \langle \psi, \psi \rangle = 0 \ \leftrightarrow \ \psi = 0. \end{aligned}$$

- **orthogonal vectors:** $\psi, \varphi \in H$ are called orthogonal if $\langle \psi, \varphi \rangle = 0$.
- **normed spaces:** a normed space is a vector space $(V, \|\cdot\|)$ with a map $\|\cdot\| : V \to [0; \infty)$, called the norm, such that

$$\begin{aligned} i)\qquad \|\psi\| &= 0 \ \leftrightarrow \ \psi = 0, \\ ii)\qquad \|\alpha\psi\| &= |\alpha|\|\psi\|, \\ iii)\ \|\psi + \varphi\| &\leq \|\psi\| + \|\varphi\|. \end{aligned}$$

- **induced norms:** an inner product $\langle \cdot, \cdot \rangle$ defines a norm through $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$
- **Hilbert spaces:** a Hilbert space is an inner product space $(H, \langle \cdot, \cdot \rangle)$ that is complete with respect to the metric induced by the (inner product induced) norm
- **orthonormal bases:** we say that an orthonormal sequence $(\phi_j)_{j \in \mathbb{N}}$ forms a complete orthonormal basis of the Hilbert space H if and only if every $x \in H$ can be expanded as

$$x = \sum_{j=1}^{\infty} \langle x, \phi_j \rangle \varphi_j.$$

  The convergence is understood in the sense of the induced metric in H. The basis expansion is in close analogy to the basis expansion of vectors with respect to an orthonormal basis in finite dimensional spaces. For instance, we can compute norms through (Parseval's theorem)

$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, \phi_j \rangle|^2.$$

1. The L^2 spaces. L^2 spaces form *the fundamental examples* of Hilbert spaces. Modulo some measure theoretic complications[1], we define the spaces $L^2(I;\rho(x)dx)$ for some interval $I \subset \mathbb{R}$ and some non-negative function $\rho : I \to [0;\infty)$ through

$$L^2(I;\rho(x)dx) = \{\psi : I \to \mathbb{K} : \int_I dx\, |\psi(x)|^2 \rho(x) < \infty\}.$$

(a) Explain why $L^2(I;\rho(x)\,dx)$ is a vector space and show that $\langle \psi, \phi \rangle = \int_I dx\, \rho(x)\psi(x)\overline{\phi(x)}$ defines an inner product on it. A fundamental fact from analysis (measure and integration theory) is that the L^2 *spaces are Hilbert spaces.*

- there are two things we need to use

1) $\int dx\, \rho(x)\, |\psi(x)|^2 = 0 \Rightarrow \psi = 0$ a.s. by Lebesgue integral properties

2) $\psi, \varphi \in L^2$: $\int dx\, \rho(x)\, |\psi + \varphi|^2(x) \leq 2\|\psi\|_2^2 + 2\|\varphi\|_2^2$

(b) Show that any countable subset of $\mathbb{R}$ is a set of Lebesgue measure zero.

- let $S = \{x_j : j \in \mathbb{N}\} \subset \mathbb{R}$ be countable
- pick $\varepsilon > 0$ and intervals $I_j = [x_j - \varepsilon/2^j, x_j + \varepsilon/2^j]$ of length $\varepsilon \cdot 2^{-j+1} \Rightarrow S \subset \bigcup_{j \in \mathbb{N}} I_j$ and

$\sum_{j=1}^{\infty} |I_j| = \varepsilon \sum_{j=1}^{\infty} 2^{-j+1} = 2 \cdot \varepsilon$

(c) Using the basic facts about Lebesgue integrals from the lecture, compute the $L^2([0;1]\,dx)$ norm of the functions $x \mapsto \underbrace{\mathbf{1}_{[0;1]\cap\mathbb{Q}}(x)}_{f_1(x)}$ and $x \mapsto \underbrace{\mathbf{1}_{[0;1]\setminus\mathbb{Q}}(x)}_{f_2(x)}$.

- note: $f_1(x) = 0$ a.e. $\quad f_2(x) = 1$ a.e.

$\Rightarrow \|f_1\|_2 = 0, \quad \|f_2\|_2 = 1$

[1] See Math 114 in the fall ;).

2. Compatibility 1. We have learned that $L^2(I;\rho(x)\,dx)$ forms a Hilbert space, in particular a vector space. Motivated by the vibrating wire, our next goal is to make sense of linear, second order differential operators $\ell(y) = ay'' + by' + cy$ on $L^2(I;\rho(x)\,dx)$. There are a couple of issues we have to understand before we are able to define such an operator in $L^2(I;\rho(x)\,dx)$. The first one is the problem of *compatibility.*

Suppose the linear operator ℓ acts like $\ell(y) = ay'' + by' + cy$. Suppose you are given a function $y \in L^2(I;\rho(x)\,dx)$. Then, there are two possibilities that can happen:

$$a)\ \ell(y) \in L^2(I;\rho(x)\,dx),$$
$$b)\ \ell(y) \notin L^2(I;\rho(x)\,dx).$$

As an example, let's consider the space $L^2((0;1);\ dx)$ with $\ell(y) = -y''$ and the function $y(x) = x^{1/2}$. What possibility occurs?

- first of all, let's check that $y \in L^2$:

$$\int_0^1 (x^{1/2})^2\,dx = \int_0^1 x\,dx = \frac{1}{2} < \infty$$

- we have $y'(x) = \frac{1}{2}x^{-1/2}$ and

$$y''(x) = -1/4\,x^{-3/2}$$

- but $\int_0^1 dx \left(\frac{1}{x^{3/2}}\right)^2 = \int_0^1 dx\,\frac{1}{x^3} = \infty$!

3. Compatibility 2. As another example of the compatibility problem, consider the following operator, called *Bessel's operator*.

(a) Choose $L^2((0,1); x\,dx)$ as the Hilbert space and let $\ell(y) = -\frac{1}{x}(xy')'$. Which of the following monomials $x \mapsto 1, x \mapsto x, x \mapsto x^2$ is compatible with Bessel's operator?

- $x 1' = 0$, $x x' = x \rightsquigarrow -1/x \notin L^2$
- $x(x^2)' = 2x^2 \rightsquigarrow -1/x\,(x(x^2)')' = -4 \in L^2$

(b) What about a general polynomial of the form $x \mapsto y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$?

- as above: we need $a_1 = 0$, then in L^2

$x(a_0 + a_1 x + \dots)' = a_1 x + 2a_2 x^2 + \dots + n a_n x^n$

$$\frac{(\dots)'}{x} = \underbrace{\frac{a_1}{x}}_{\notin L^2} + \underbrace{\frac{4a_2 x}{x} + \dots + \frac{n^2 a_n x^{n-1}}{x}}_{\in L^2}$$

4. **Regularity 1.** The next problem we need to understand is that of *regularity*. In the previous examples, all functions were actually smooth, but $L^2(I;\rho(x)\,dx)$ spaces contain many more functions than that – which ones are we allowed to pick if we want to apply $\ell(y) = ay''+by'+cy$?

   (a) As an example, consider $L^2((-1;1);dx)$ and let $\ell(y)$ act like $\ell(y) = -y''$ if $y \in C^2((-1;1))$. Consider the function $x \mapsto \phi(x) = \frac{1}{2}x^2\mathbf{1}_{[0;1]}(x)$ and show that $\phi \notin C^2((-1;1))$, but $\phi, \phi' \in L^2((-1;1);dx)$. Discuss how you could still make sense of $\ell(\phi)$ – more precisely, how would you define it?

- $\phi'(x) = \begin{cases} 0 & x<0 \\ x & x \geq 0 \end{cases}$, $|\phi|, |\phi'| \leq 1 \Rightarrow \phi, \phi' \in L^2$
- for $x \neq 0$: $\phi''(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$

  $\Rightarrow$ ϕ' not differentiable at $x=0$
- since ϕ' is differentiable a.e. in $[-1,1]$, we just define $\phi''(x) = \begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$ (not continuous)

  $\rightsquigarrow \phi'' \in L^2$

   (b) Show that $\phi \in C^1((-1;1))$ and, moreover, that there exists a function $\zeta \in L^2((-1;1);dx)$ such that for all $x, x_0 \in (-1;1)$, it holds true that

$$\phi'(x) = \phi'(x_0) + \int_{x_0}^{x} ds\;\zeta(s).$$

   In other words, the fundamental theorem of calculus applies to ϕ' *although ϕ' is not continuously differentiable.*

- we computed ϕ' above and apparently it is continuous
- we let $\zeta := \phi'' \in L^2$ defined above, then

  $\phi'(x) = \phi'(x_0) + \int_{x_0}^{x} ds\,\zeta(s) = \max(x_0,0) + \int_{x_0}^{x} ds\,\zeta(s)$
- indeed

$$\int_{x_0}^{x} ds\,\zeta(s) = \int_{x_0}^{0} ds\,\zeta(s) + \int_{0}^{x} ds\,\zeta(s) = \max(x,0) - \max(0,x_0)$$

$$\rightsquigarrow \phi'(x) = \max(x,0) = \begin{cases} 0 & : x<0 \\ x & : x\geq 0 \end{cases}$$

5. Regularity 2. Let's continue our discussion with a similar example on $L^2((-\frac{1}{2};\frac{1}{2});dx)$ and with $\ell(y) = -y''$ for $y \in C^2((-\frac{1}{2};\frac{1}{2}))$. The previous problem suggests that we can make sense of $\ell(y)$ as long as y and y' have L^2 derivatives so that the fundamental theorem of calculus applies to them. To provide another example that supports this intuition, let's analyse the function $x \mapsto \phi(x) = x\mathbf{1}_{[0;\frac{1}{2}]}(x)$.

(a) Show that $\phi \notin C^1((-\frac{1}{2};\frac{1}{2}))$, but that ϕ' exists almost everywhere in $(-\frac{1}{2};\frac{1}{2})$ and that $\phi, \phi' \in L^2((-\frac{1}{2};\frac{1}{2});dx)$. Moreover, if you had to, what "value" would you assign to the second derivative $\phi''(0)$ at $x_0 = 0$? What "value" would you assign to $\phi''(x)$ for $x \neq 0$?

- $\phi'(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$, but ϕ not differentiable at $x=0$; note: $\phi''(x) = \begin{cases} 0 & x<0 \\ 0 & x>0 \end{cases}$
- it only makes sense to say formally $\phi''(0) = $ "∞" (look at the graph of ϕ'!)

(b) Suppose that, as in the previous problem, we can still make sense of $\ell(\phi) = \phi''$. More precisely, suppose there exists a function[2] $\delta \in L^2((-\frac{1}{2};\frac{1}{2});dx)$ such that

$$\phi'(x) = \phi'(x_0) + \int_{x_0}^{x} ds\ \delta(s).$$

With the discussion in class about absolutely continuous functions and the FTC, use integration by parts to show that under this assumption, it holds true that

$$\int_{-1/2}^{1/2} dx\ \psi(x)\delta(x) = \psi(0)$$

for all $\psi \in C^2((-\frac{1}{2};\frac{1}{2}))$. Use the Fourier series expansion from problem set 6 and Parseval's theorem to prove that then $\delta \notin L^2((-\frac{1}{2};\frac{1}{2});dx)$. Thus, it *does not* make sense to give meaning to $\ell(\phi)$ in the L^2 sense – functions that have discontinuity jumps in their first derivatives like ϕ' has are *not compatible* with ℓ!

- see class notes (April 7)

[2] In physics jargon, this would correspond to the *famous Dirac δ function!*