Week 5

Topics: Local existence & uniqueness for 1st order ODE I: basics on metric spaces

Review from Week 4:

- **Second order constant coefficient equations:** Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$. We would like to solve the second order linear equation:

$$ay'' + by' + cy = 0$$

  Solutions are of form $x \mapsto y(x) = e^{rx}$, where r is a root of the equation $ar^2 + br + c = 0$

- **Case I: two real roots.** If $r = r_1, r_2$ are two distinct real roots then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- **Case II: double root.** If there is only one root r then

$$y(x) = c_1 e^{r\,x} + c_2 x e^{r\,x}$$

- **Case III: complex root.** If there is a complex root $r = \alpha + i\beta$ then

$$y(x) = e^{\alpha x} \left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

1. **General 1st Order ODE.** In the next few lectures, we want to start to discuss an important function space method to prove the existence of solutions to general first order ODE. More precisely, we consider the following: Let $f \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ be continuous, $n \in \mathbb{N}$, and let $(x_0, \mathbf{y}_0) \in \mathbb{R} \times \mathbb{R}^n$. The initial value problem related to an (not necessarily linear) *ordinary differential equation of first order* reads

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

   (a) One can recast our familiar 2nd linear order ODE $\ell(y) = ay'' + by' + cy = h$ as a (vector-valued) 1st order ODE. To this end, set $y_1 = y$ and $y_2 = y'$. Can you reformulate $\ell(y) = h$ as a 1st order ODE for the vector-valued function $\mathbf{y} = (y_1, y_2)$ with values in $\mathbb{R}^2$?

   (b) Show that a continuous function $\mathbf{y} \in C((x_0 - \epsilon; x_0 + \epsilon); \mathbb{R}^n)$ is a solution to the above initial value problem if and only if

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \mathbf{y}(s))\, ds$$

   for all $x \in (x_0 - \epsilon; x_0 + \epsilon)$.

2. An example of a fixed point argument. Suppose you want to prove the statement that there exists a real number $x \in \mathbb{R}$ such that $x^2 = 2$, i.e. that $\sqrt{2} \in \mathbb{R}$ exists. This is not a trivial question and below is an argument that uses the fact that every bounded, monotone sequence in $\mathbb{R}$ has a limit in $\mathbb{R}$.

(a) Verify that $x^2 = 2$ if and only if $x = f(x)$ for the function $f : (0; \infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{1}{2}x + \frac{1}{x}.$$

(b) Define the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = f(x_n)$. Use induction to prove that $2 \leq x_n^2$ and $x_n \leq 2$ for all $n \in \mathbb{N}$.

(c) Prove that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. Conclude the theorem.

3. First examples of metric spaces.

(a) Consider the set $\ell^1 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\right\}$ and define

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Prove that $d_1 : \ell^1 \times \ell^1 \to \mathbb{R}$ defines a metric.

(b) Consider the set $\ell^2 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$ and define

$$d_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}.$$

Prove that $d_2 : \ell^2 \times \ell^2 \to \mathbb{R}$ defines a metric.

4. Metric spaces for 1st order ODE.

Most important for the standard local existence and uniqueness theorem for 1st order ODE are spaces of continuous functions. To this end, fix $a < b$, $a, b \in \mathbb{R}$ and set

$$C([a;b]) = C([a;b];\mathbb{R}) = \{f : [a;b] \to \mathbb{R} : f \text{ continuous in } [a;b]\}$$

Recall that f is continuous at the point $x_0 \in [a;b]$ iff for all $\varepsilon > 0$ there exists a $\delta = \delta_\varepsilon > 0$ s.t. $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$. f is continuous in the interval $[a;b]$ if it is continuous at each $x_0 \in [a;b]$.

(a) Define $d_\infty : C([a;b]) \times C([a;b]) \to \mathbb{R}$ through

$$d_\infty(f, g) = \sup_{x \in [a;b]} |f(x) - g(x)|.$$

Prove that d_∞ defines a metric on $C([a;b])$.

(b) Set $C([a;b];\mathbb{R}^n) = \{f : [a;b] \to \mathbb{R}^n : f \text{ continuous in } [a;b]\}$ and define (with slight abuse of notation) $d_\infty : C([a;b];\mathbb{R}^n) \times C([a;b];\mathbb{R}^n) \to \mathbb{R}$ in this case through

$$d_\infty(f, g) = \sup_{x \in [a;b]} \Big(\sum_{i=1}^{n} |f(x) - g(x)|^2 \Big)^{1/2}.$$

Prove that d_∞ defines a metric on $C([a;b];\mathbb{R}^n)$.

5. Cauchy sequences in metric spaces.

In $(\mathbb{R}, |\cdot|)$, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty \in \mathbb{R}$ iff for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_\infty| \leq \varepsilon \quad \text{for all} \quad n \geq N.$$

A Cauchy sequence is a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m| \leq \varepsilon \quad \text{for all} \quad n, m \geq N.$$

(a) Suppose that (M, d) is a general metric space. Can you formulate an analogous definition for a convergent and for a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ ($x_n \in M$ for all $n \in \mathbb{N}$) in (M, d)?

(b) Show that every Cauchy sequence in a metric space (M, d) is bounded.

(c) Show that every convergent sequence in $\mathbb{R}$ is a Cauchy sequence. What about the same statement in a general metric space?

(d) Recall that every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$. Is this true in a general metric space as well?

6. Completeness of $C(a;b])$. Prove that $C([a;b])$ with the metric from problem **4** is complete: if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(C([a;b]), d_\infty\big)$, then there exists a limit $f_\infty \in C([a;b])$ such that $\lim_{n\to\infty} f_n = f_\infty$ in $C([a;b])$. To prove this theorem, proceed as follows:

(a) First, we have to find a good candidate f_∞ for our limit. To this end, fix $x \in [a;b]$ and prove that the sequence $(f_n(x))_{n\in\mathbb{N}}$ **of real numbers** is a Cauchy sequence in $\mathbb{R}$. Consequently, what function $f_\infty : [a;b] \to \mathbb{R}$ is a natural candidate for our limit?

(b) We have to prove that our candidate f_∞ is a continuous function in $[a;b]$. Prove that for any $x, y \in [a;b]$ and $n, m \in \mathbb{N}$, it holds true that

$$|f_\infty(x) - f_\infty(y)| \le |f_\infty(x) - f_m(x)| + |f_\infty(y) - f_m(y)| + |f_n(x) - f_n(y)| + 2d_\infty(f_n, f_m).$$

How does this help?

(c) Knowing that $f_\infty \in C([a;b])$, it remains to prove that $\lim_{n\to\infty} f_n = f_\infty$ in $\big(C([a;b]), d_\infty\big)$. It may be useful to observe that for all $n, m \in \mathbb{N}$, it holds true that

$$|f_\infty(x) - f_n(x)| \le |f_\infty(x) - f_m(x)| + d_\infty(f_n, f_m).$$