Problem Set 8

Due: Th, April 16, 6.00 PM

Topics: Sobolev spaces, symmetric operators

1. **Reading.** Sections 4.4, 4.5, 2.8 and 2.9 (up to Theorem 2.14) in Holland's book.

2. **Distributional derivatives.** Consider the real Hilbert space $H^1(\mathbb{R}; dx)$, as defined in the lecture. We gave a different definition in problem **6** of problem set 7; let's denote the corresponding space by $\widetilde{H}^1(\mathbb{R}; dx)$. Show that $H^1(\mathbb{R}; dx) = \widetilde{H}^1(\mathbb{R}; dx)$.

   Hints:

   i) *You are allowed to use the product rule for $H^1(\mathbb{R}; dx)$, as discussed in class.*

   ii) *You are allowed to use the following analysis fact: if $f \in L^2(\mathbb{R}; dx)$ satisfies*

   $$\int_{\mathbb{R}} f(x)\varphi'(x) = 0$$

   for all $\varphi \in C_c^\infty(\mathbb{R})$, then $f(x) = C$ for some constant $C \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$.

3. **Symmetric Operators 1.** Consider the complex Hilbert space $\mathcal{H} = L^2([0;1]; dx)$ and the operator[1] $\ell = i\partial_x$ with domain $D_1 = \{\psi \in H^1([0;1]; dx) : \psi(0) = \psi(1) = 0\}$, introduced in the lecture. Prove that ℓ is a symmetric operator. Is this still true if we change the domain to $D_2 = \{\psi \in H^1([0;1]; dx) : \psi(0) = 0\}$?

4. **Symmetric Operators 2.** Consider the complex Hilbert space $\mathcal{H} = L^2([0;1]; dx)$ and the operator $\ell = -\partial_x^2$ with domain

   $$D_1 = \{\psi \in H^2([0;1]; dx) : \psi(0) = \psi(1) \text{ and } \psi'(0) = \psi'(1)\}.$$

   Prove that ℓ is a symmetric operator. Find all the eigenvalues and eigenfunctions of the operator ℓ. Is ℓ still symmetric if defined on the domain

   $$D_2 = \{\psi \in H^2([0;1]; dx) : \psi(0) = \psi(1) = 0\}$$

   and if so, what are its eigenvalues and eigenfunctions in this case?

5. **Symmetric Operators 3** Suppose $\mathcal{H}$ is a Hilbert space and $T : D_T \to \mathcal{H}$ is a symmetric operator. Suppose that T has a complete orthonormal eigenbasis. That is, T has eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ with corresponding eigenvectors $(\varphi_j)_{j\in\mathbb{N}}$ in D_T such that $T\varphi_j = \lambda_j\varphi_j$ and such that $(\varphi_j)_{j\in\mathbb{N}}$ forms a complete orthonormal basis of $\mathcal{H}$. Let $\psi \in D_T$. Prove that $T\psi$ is represented in terms of the $(\varphi_j)_{j\in\mathbb{N}}$ as

   $$T\psi = \sum_{j=1}^{\infty} \lambda_j \langle \psi, \varphi_j \rangle \varphi_j.$$

   This means, we can pull T inside the (infinite!) summation.

[1] There are several conventions for denoting (partial) derivatives in mathematics. My favorite notation is the ∂–notation. From now on, please identify $\psi' = \frac{d}{dx}\psi = \partial_x\psi$, $\psi'' = \frac{d^2}{dx^2} = \partial_x^2\psi$, etc..