Math 110, spring 2020

Problem Set 2

Due: Tue, Feb 18, 6.00 PM

Topics: variation of the constant, power series expansion, calculus with linear maps

1. Find a formula for the general solution to $(x^2+1)y'-(1-x)^2y = h(x)$, then specialize to the case $h(x) = xe^{-x}, y(0) = 1$.

2. By using the method of undetermined coefficients, find the first four nonzero coefficients in the power series expansion of the solution to

$$(x^2+1)y' - (1-x)^2 y = xe^{-x}, y(0) = 1.$$

Check your answer by expanding the solution to the preceding problem in a power series.

3. We saw in class that the exponential function $x \mapsto e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is analytic at $x_0 = 0$. Prove that the exponential function is analytic at any $x_0 \in \mathbb{R}$.

4. Let $f \in C^\infty(\mathbb{R})$ be a smooth function, let $h > 0$ and let $x_0 \in \mathbb{R}$. Prove that

$$\sup_{x\in[x_0-h;x_0+h]} \left| f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!} x^j \right| \leq Ch^{k+1}$$

for some constant $C = C_{f,k} > 0$ that depends on the function f and the order $k \in \mathbb{N}$, but that is independent of $h > 0$. Does this imply that f is analytic at x_0?

5. Let V be the real vector space of real valued smooth functions on $\mathbb{R}$. Let f be a smooth function and let $M_f : V \to V$ be the linear map defined by $M_f(y) = fy$. Let $D : V \to V$ be the first order differential operator defined by $D(y) = y'$.

a) Show that, as a composition of linear maps, we have $DM_f = M_{f'} + M_f D$.

b) For two functions f and g, show that

$$(M_f D + M_g)(D + M_x) = M_f D^2 + M_{xf+g} D + M_{f+xg}.$$

Conclude that, no matter which f and g you choose, $\ker(l) \neq \{0\}$, where

$$l(y) = fy'' + (xf + g)y' + (f + xg)y.$$

6. Find the unique polynomial solution to the equation $y'' - xy' + 2y = 0$ with initial conditions $y(0) = -1$, $y'(0) = 0$.