MATH 110, SPRING 2020

Week 3

Topics: power series; Wronskian; Abel's formula; 2nd order ODE with constant coefficients

Review from Week 2:

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads
$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right).$$

- **General solution to 1st order ODE:** Let $\ell(y) = ay' + by$, then the kernel $\ker(\ell)$ is one-dimensional. Suppose it is spanned by some function f. Then the general solution to $\ell(y) = h$ has the form $y = y_p + Cf$ for some particular solution y_p and some constant $C \in \mathbb{R}$. We can find a particular solution using the **variation of parameters**. This yields a particular solution of the form $x \mapsto y_p(x) = f(x) \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt$

- **Analytic functions:** A function $x \mapsto f(x)$ is real-analytic at x_0 if its Taylor series T_{f,x_0} converges in some interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, with $\varepsilon > 0$, and is equal to f in that interval.

1. (a) Using the ratio test, find the radius of convergence R about $x_0 = 0$ for $x \mapsto f(x) = e^x$ and $x \mapsto f(x) = \log(1 + x)$.

(b) Explain why the function f with values $f(x) = \exp\left(-\frac{1}{x}\right)$ if $x > 0$ and $f(x) = 0$ otherwise is not analytic at $x_0 = 0$.

2. Use the method of undetermined coefficients to solve for the first four coefficients of the analytic solution of the following example from Holland's book:

$$(3x^2 + 1)y' - 2xy = x, \text{ with } y(0) = \frac{3}{2}.$$

3. Consider the space V of solutions to the equation

$$y'' = y'.$$

(a) Show that the functions $x \mapsto y_1(x) = 1$ and $x \mapsto y_2(x) = e^x$ are in the space V. Does this mean that $x \mapsto y(x) = 3 + 2e^x$ is also in the space V?

(b) Are the vectors y_1 and y_2 linearly dependent or independent?

(c) Show that $V = \text{span}\{1, e^x\}$, i.e. that any solution to $y'' = y'$ is of the form

$$y = c_1 y_1 + c_2 y_2 = c_1 + c_2 e^x,$$

for some scalars c_1, c_2.

(d) What is the dimension of the space V?

4. (a) Assume that y_1 and y_2 are linearly dependent. Prove that the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$. This implies that if $W(y_1, y_2)$ is not equal to the zero function, then y_1 and y_2 are linearly independent.

(b) Try this for the functions $x \mapsto e^x$ and $x \mapsto e^{-x}$. Before you do the computation, take a moment and ask yourself what result you expect.

(c) The criterion from (a) for linear independence is not necessary[1]. That is, the Wronskian of two functions can be zero, but that does **not** mean they are linearly dependent. Indeed, check this for the two (smooth) functions

$$y_1(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}, \qquad y_2(x) = \begin{cases} e^{-1/x^2} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

[1]For a converse statement, you may have a look at Chapter 3.1.3 in *Mathematics for Physics* by Stone & Goldbart.

Theorem 1 (Holland, Theorem 2.6)**.** *If $W(y_1, \ldots, y_n)$ is not the zero function on $[c, d]$, then the functions $y_1, \ldots, y_n$ are linearly independent on $[c, d]$.*

5. Prove it.

6. Existence of solutions to second-order linear homogeneous equations. Consider general second order linear operators

$$\ell(y) = ay'' + by' + cy.$$

- The initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$.
- We assume that a, b, and c are analytic functions at x_0 and that $a(x_0) \neq 0$. For simplicity we also assume that $x_0 = 0$.
- **Theorem:** There exists a unique solution to the initial value problem!

(a) Using the power series method, solve $(2x + 1)y'' + y' + 2y = 0$ for arbitrary y_0 and y_1, keeping terms up through x^4.

Note: The proof consists in showing that the procedure cannot fail in general, provided you know y_0 and y_1 and that $a(0) \neq 0$.

For $\ell(y) = ay'' + by' + cy$, the initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$. From the previous theorem we deduce the following:

- **Theorem:** Solutions exist and every solution may be written in the form $y = y_0 f_1 + y_1 f_2$, where f_1 satisfies $f_1(0) = 1$, $f_1'(0) = 0$ and f_2 satisfies $f_2(0) = 0$, $f_2'(0) = 1$.

7. Use this result to solve the following problem.

(a) Prove that the kernel of ℓ is two-dimensional.

(b) Prove that every solution to $\ell(y) = h$ is the sum of a particular solution y_p and a solution of the form $c_1 f_1 + c_2 f_2$.

8. Start with the second-order equation $\ell(y) = ay'' + by' + cy = 0$.

(a) Show that the Wronskian W of any two independent solutions f_1 and f_2 of this equation satisfies the first order equation $aW' + bW = 0$. As a consequence, what is the formula for W in terms of a and b? The result is called **Abel's Formula**.

(b) Knowing the Wronskian and one vector f_1 in the kernel of ℓ, can you find a second vector f_2, independent of f_1, in the kernel of ℓ?

9. Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$ and consider the equation $ay'' + by' + cy = 0$.

(a) If $x \mapsto y(x) = e^{rx}$ is a solution to the first order homogeneous ODE above, can you find an algebraic equation that is solved by r?

(b) Using this approach, find two independent solutions to the equation

$$y'' - y' - 2y = 0.$$

10. Now, we would also like to solve the second order *inhomogeneous* equation

$$\ell(y) = ay'' + by' + cy = h.$$

where still $a, b, c \in \mathbb{R}$ are constants, but h is a function which in general is not just constant.

(a) Find a particular solution to the equation

$$l(y) = y'' + y' - 2y = x - 2x^3,$$

by guessing that y_p is in the space $V = \text{span}\{1, x, x^2, x^3\}$.

(b) Use the previous result to find the general solution to $\ell(y) = h$ where $h(x) = x - 2x^3$.