Week 13

Topics: Hermite's operator, the wave equation

Review from Week 12:

- **Boundary value problems:** motivated by the vibrating wire, our goal is to set up a mathematical framework that deals with eigenvalue problems in infinite dimensions of the form:

$$\begin{cases} y'' &= \lambda y \\ y(0) &= y(L) = 0. \end{cases}$$

- **Eigenvalues and Spectrum:** If $T : D_T \to \mathcal{H}$ is a linear operator with domain $D_T \subset \mathcal{H}$ we say that $0 \neq \phi \in D_T$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{C}$ if and only if $T\phi = \lambda\phi$. Notice that there are two subtle details in this definition: $\phi \neq 0$ is not the zero vector and $\phi \in D_T$ must be in the domain of T. The spectrum $\sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of T } \} \subset \mathbb{C}.$$

- **Symmetric operators:** An operator $T : D_T \to \mathcal{H}$ is called symmetric if and only if

$$\langle \psi, T\phi \rangle_{\mathcal{H}} = \langle T\psi, \phi \rangle_{\mathcal{H}} \qquad \forall\ \psi, \phi \in D_T.$$

- **Self-adjoint operators:** An operator $T : D_T \to \mathcal{H}$ is called self-adjoint if and only if $T = T^*$ and $D_T = D_{T^*}$, where $T^* : D_{T^*} \to \mathcal{H}$ denotes the adjoint of T. It is defined through

$$\langle T\psi, \phi \rangle_{\mathcal{H}} = \langle \psi, T^*\phi \rangle_{\mathcal{H}} \qquad \forall\ \psi \in D_T, \phi \in D_{T^*}$$

and we always have that $D_T \subset D_{T^*}$. If T is symmetric, then T^* is an extension of T. The spectral theorem holds true for self-adjoint operators.

1. Hermite's operator 1. Consider Hermite's operator $\ell(y) = -e^{x^2/2}\partial_x\big(e^{-x^2/2}\partial_x y\big)$, defined on a suitable domain D_ℓ which forms a subspace of $L^2(\mathbb{R}, e^{-x^2/2}dx)$.

(a) Let $a < b \in \mathbb{R}$. Show that for all $\psi, \phi \in D_\ell$, it holds true that

$$\int_a^b dx\ \ell(\psi)(x)\overline{\phi}(x)e^{-x^2/2} = -\,e^{-b^2/2}(\partial_x\psi)(b)\overline{\phi}(b) + e^{-a^2/2}(\partial_x\psi)(a)\overline{\phi}(a) + \int_a^b dx\ (\partial_x\psi)(x)(\overline{\partial_x\phi})(x)e^{-x^2/2}.$$

(b) Using the facts from the lecture, prove that ℓ is a symmetric operator.

(c) Prove that ℓ is a positive semi–definite operator. Prove that this implies that all eigenvalues of ℓ are non-negative.

2. Hermite's operator 2. Consider Hermite's operator $\ell(y) = -e^{x^2/2}\partial_x\big(e^{-x^2/2}\partial_x y\big)$, defined on a suitable domain D_ℓ which forms a subspace of $L^2(\mathbb{R}, e^{-x^2/2}dx)$.

(a) By suitably rewriting Hermite's operator, prove that $\mathbb{N}_0 \subset \sigma(\ell)$.

(b) In fact, it turns out that $\mathbb{N}_0 = \sigma(\ell)$ and each eigenvalue is simple. Verify this for $\lambda_0 = 0$. The eigenvalue equation for λ_0 is a linear second order ODE which has, as we know from the first part of the course, **two** linearly independent solutions. Why is λ_0 nevertheless a simple eigenvalue?

(c) Prove that the Hermite polynomials H_n are orthogonal.

3. Hermite's operator 3. Consider Hermite's operator $\ell(y) = -e^{x^2/2}\partial_x\big(e^{-x^2/2}\partial_x y\big)$, defined on a suitable domain D_ℓ which forms a subspace of $L^2(\mathbb{R}, e^{-x^2/2}dx)$.

(a) Consider the standard monomials $x \mapsto 1, x \mapsto x, x \mapsto x^2, etc.$ as functions on $\mathbb{R}$. Show that the first $n \in \mathbb{N}$, $n \geq 2$ monomials are linearly independent.

(b) Prove that every polynomial of degree $n \in \mathbb{N}$ can be written as a linear combination of the first $n \in \mathbb{N}$ Hermite polynomials H_j, $j = 1, \ldots, n$.

(c) With the remarks from the lecture, we see that the Hermite polynomials form a complete orthonormal basis of $L^2(\mathbb{R}, e^{-x^2/2}dx)$. Use this fact to conclude that $\mathbb{N}_0 = \sigma(\ell)$ and that every eigenvalue is simple.

4. The 1d Wave Equation. Inspired by the physicist's derivation of the vibrating wire, we derived the wave equation

$$\frac{\partial^2}{\partial x^2} u = \frac{1}{\omega^2} \frac{\partial^2}{\partial t^2} u, \tag{1}$$

where $(x,t) \mapsto u(x,t)$ describes the vertical displacement of the wire and $\omega > 0$ is a constant related to the mass–density and the tension in the wire. We want to solve this equation under the boundary conditions that $u(0,t) = 0 = u(0,L)$, that is, we fix the wire of length $L > 0$ and $x = 0$ and $x = L$. In an actual physical experiment, like when we stroke the wire (we may think of a guitar, for instance), the wire has an initial position $x \mapsto u(x,0) \equiv u_0(x)$ and initial speed $x \mapsto (\partial_t u)(x,0) \equiv v_0(x)$.

Our goal is to derive the general solution of the above problem within our Hilbert space framework. To do so, we interpret the solution $(x,t) \mapsto u(x,t)$ as a time–dependent $L^2([0;L],dx)$ valued map $t \mapsto u(\cdot,t) \equiv u_t \in L^2([0;L],dx)$. That is, for every fixed time $t \geq 0$, we have that the displacement lies in $u_t \in L^2([0;L],dx)$. Since it solves (1), we actually assume that $[0;\infty) \ni t \mapsto u(\cdot,t) \equiv u_t \in D_\ell$, where $\ell(\phi) = \partial_x^2 \phi$ and

$$D_\ell = \{\phi \in H^2([0;L],dx) : \phi(0) = \phi(L) = 0.\}$$

(a) Find an orthonormal eigenbasis of the self-adjoint operator ℓ and determine $\sigma(\ell) \subset \mathbb{R}$.

(b) Given a fixed time $t \in [0;\infty)$, describe the general form of $u_t \in D_\ell$, the displacement at time $t \geq 0$.

(c) Using the wave equation (1), derive an equation for the time-dependent coefficients in the basis expansion of $u_t \in D_\ell$. Find the general form of the coefficients by solving a sequence of linear, second order ODE.

(d) Plug in the initial conditions $x \mapsto u(x,0) \equiv u_0(x)$ and $x \mapsto (\partial_t u)(x,0) \equiv v_0(x)$ to conclude the general form of the solution to the wave equation (1).

(e) As an explicit example, compute the solution for $u_0(x) = x\mathbf{1}_{[0;L/2]} + (L-x)\mathbf{1}_{[L/2;L]}$, $v_0 \equiv 0$, $L = 1$ and $\omega = 1$. A fun extra homework is to display the time evolution using your favorite computer program (MATLAB, mathematica, maple, etc.)!