# Week 3

*Topics: power series; Wronskian; Abel's formula; 2nd order ODE with constant coefficients*

***Review from Week 2:***

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads
$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right).$$

- **General solution to 1st order ODE:** Let $\ell(y) = ay' + by$, then the kernel $\ker(\ell)$ is one-dimensional. Suppose it is spanned by some function $f$. Then the general solution to $\ell(y) = h$ has the form $y = y_p + Cf$ for some particular solution $y_p$ and some constant $C \in \mathbb{R}$. We can find a particular solution using the **variation of parameters**. This yields a particular solution of the form $x \mapsto y_p(x) = f(x) \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt$

- **Analytic functions:** A function $x \mapsto f(x)$ is real-analytic at $x_0$ if its Taylor series $T_{f,x_0}$ converges in some interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, with $\varepsilon > 0$, and is equal to $f$ in that interval.

**1.** (a) Using the ratio test, find the radius of convergence $R$ about $x_0 = 0$ for $x \mapsto f(x) = e^x$ and $x \mapsto f(x) = \log(1 + x)$.

(b) Explain why the function $f$ with values $f(x) = \exp\left(-\frac{1}{x}\right)$ if $x > 0$ and $f(x) = 0$ otherwise is not analytic at $x_0 = 0$.

**2.** Use the method of undetermined coefficients to solve for the first four coefficients of the analytic solution of the following example from Holland's book:

$$(3x^2 + 1)y' - 2xy = x, \text{ with } y(0) = \frac{3}{2}.$$

**3.** Consider the space $V$ of solutions to the equation

$$y'' = y'.$$

(a) Show that the functions $x \mapsto y_1(x) = 1$ and $x \mapsto y_2(x) = e^x$ are in the space $V$. Does this mean that $x \mapsto y(x) = 3 + 2e^x$ is also in the space $V$?

(b) Are the vectors $y_1$ and $y_2$ linearly dependent or independent?

(c) Show that $V = \text{span}\{1, e^x\}$, i.e. that any solution to $y'' = y'$ is of the form

$$y = c_1 y_1 + c_2 y_2 = c_1 + c_2 e^x,$$

for some scalars $c_1, c_2$.

(d) What is the dimension of the space $V$?

**4.** (a) Assume that $y_1$ and $y_2$ are linearly dependent. Prove that the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$. This implies that if $W(y_1, y_2)$ is not equal to the zero function, then $y_1$ and $y_2$ are linearly independent.

(b) Try this for the functions $x \mapsto e^x$ and $x \mapsto e^{-x}$. Before you do the computation, take a moment and ask yourself what result you expect.

(c) The criterion from (a) for linear independence is not necessary[1]. That is, the Wronskian of two functions can be zero, but that does **not** mean they are linearly dependent. Indeed, check this for the two (smooth) functions

$$y_1(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}, \qquad y_2(x) = \begin{cases} e^{-1/x^2} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

[1]For a converse statement, you may have a look at Chapter 3.1.3 in *Mathematics for Physics* by Stone & Goldbart.

**Theorem 1** (Holland, Theorem 2.6)**.** *If $W(y_1, \ldots, y_n)$ is not the zero function on $[c, d]$, then the functions $y_1, \ldots, y_n$ are linearly independent on $[c, d]$.*

**5.** Prove it.

6. **Existence of solutions to second-order linear homogeneous equations.** Consider general second order linear operators

$$\ell(y) = ay'' + by' + cy.$$

- The initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$.
- We assume that $a, b$, and $c$ are analytic functions at $x_0$ and that $a(x_0) \neq 0$. For simplicity we also assume that $x_0 = 0$.
- **Theorem:** There exists a unique solution to the initial value problem!

(a) Using the power series method, solve $(2x + 1)y'' + y' + 2y = 0$ for arbitrary $y_0$ and $y_1$, keeping terms up through $x^4$.

**Note:** The proof consists in showing that the procedure cannot fail in general, provided you know $y_0$ and $y_1$ and that $a(0) \neq 0$.

For $\ell(y) = ay'' + by' + cy$, the initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$. From the previous theorem we deduce the following:

- **Theorem:** Solutions exist and every solution may be written in the form $y = y_0 f_1 + y_1 f_2$, where $f_1$ satisfies $f_1(0) = 1$, $f_1'(0) = 0$ and $f_2$ satisfies $f_2(0) = 0$, $f_2'(0) = 1$.

**7.** Use this result to solve the following problem.

(a) Prove that the kernel of $\ell$ is two-dimensional.

(b) Prove that every solution to $\ell(y) = h$ is the sum of a particular solution $y_p$ and a solution of the form $c_1 f_1 + c_2 f_2$.

**8.** Start with the second-order equation $\ell(y) = ay'' + by' + cy = 0$.

(a) Show that the Wronskian $W$ of any two independent solutions $f_1$ and $f_2$ of this equation satisfies the first order equation $aW' + bW = 0$. As a consequence, what is the formula for W in terms of $a$ and $b$? The result is called **Abel's Formula**.

(b) Knowing the Wronskian and one vector $f_1$ in the kernel of $\ell$, can you find a second vector $f_2$, independent of $f_1$, in the kernel of $\ell$?

**9.** Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$ and consider the equation $ay'' + by' + cy = 0$.

(a) If $x \mapsto y(x) = e^{rx}$ is a solution to the first order homogeneous ODE above, can you find an algebraic equation that is solved by $r$?

(b) Using this approach, find two independent solutions to the equation

$$y'' - y' - 2y = 0.$$

**10.** Now, we would also like to solve the second order *inhomogeneous* equation

$$\ell(y) = ay'' + by' + cy = h.$$

where still $a, b, c \in \mathbb{R}$ are constants, but $h$ is a function which in general is not just constant.

(a) Find a particular solution to the equation

$$l(y) = y'' + y' - 2y = x - 2x^3,$$

by guessing that $y_p$ is in the space $V = \operatorname{span}\{1, x, x^2, x^3\}$.

(b) Use the previous result to find the general solution to $\ell(y) = h$ where $h(x) = x - 2x^3$.

# Answers and Solutions.

**1.** (a) Recall that the Taylor series for $e^x$ about $x_0 = 0$ is

$$1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

and the ratio test says that for a series

$$S = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then the series $S$ converges (also if the limit if $> 1$ then the series diverges). Thus, consider

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{x}{n} \right| < 1,$$

and this holds for *any* value of $x$. In case we see that the radius of convergence is $R = \infty$.

For the function $\log(1 + x)$ write

$$\begin{aligned} \log(1+x) &= \int_0^x \frac{1}{1+t}\,dt, \\ &= \int_0^x \left[1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots\right] dt, \text{ geometric series,} \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n+1}\frac{1}{n}x^n \cdots \end{aligned}$$

and consider

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = \lim_{n\to\infty} |x| \left| \frac{n}{n+1} \right| = |x|.$$

Thus, the radius of convergence for $\log(x + 1)$ is $R = 1$.

(b) For $x > 0$, one proves by induction (recall the principle of induction and carry out the argument!) that for all $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \exp\left(-\frac{1}{x}\right)\left(\frac{c_1}{x^{n+1}} + \frac{c_2}{x^{n+2}} + \cdots + \frac{c_n}{x^{2n}}\right)$$

for some constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Given any $n \in \mathbb{N}$, we also see that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq \frac{x^{2n+2}}{(2n+2)!}$$

for all $x \geq 0$. This implies, that for all $n \in \mathbb{N}$, we have

$$0 \leq e^{-x} \leq \frac{(2n+2)!}{x^{2n+2}}$$

Proceeding as in class, we now see that for any $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \lim_{\substack{h \to 0, \\ h>0}} \frac{1}{h} \left( \frac{c_1}{h^{n+1}} \pm \frac{c_2}{h^{n+2}} \pm \cdots + \frac{c_n}{h^{2n}} \right) e^{-\frac{1}{h}} \\ &\leq \lim_{\substack{h \to 0, \\ h>0}} \frac{1}{h} \left( \frac{c_1}{h^{n+1}} \pm \frac{c_2}{h^{n+2}} \pm \cdots + \frac{c_n}{h^{2n}} \right) \frac{(2n+2)!}{(1/h)^{2n+2}} = 0 \end{aligned}$$

and that this implies that $f^{(n)}(0) = 0$ (can you explain how this follows?). Hence $f^{(n)}(0) = 0$ for *all* $n = 0, 1, 2, 3, \ldots$, which means that the Taylor series for $f$ is the zero function and this can not equal $f$ since $\exp\left[-1/x\right]$ is not zero for any small positive value $x > 0$!

**2.** Consider,

$$\begin{aligned}
y(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 \cdots \\
y'(x) &= \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 \cdots \\
3x^2 y'(x) &= \qquad\qquad 3\alpha_1 x^2 + 6\alpha_2 x^3 \cdots \\
-2xy(x) &= \quad -2\alpha_0 x - 2\alpha_1 x^2 - 2\alpha_2 x^3 + \cdots
\end{aligned}$$

and adding these up we find

$$\ell(y) = \alpha_1 + 2(\alpha_2 - \alpha_0)x + (\alpha_1 + 3\alpha_3)x^2 + 4(\alpha_2 + \alpha_4)x^3 + \cdots = x.$$

Since $y(0) = 3/2$ then $\alpha_0 = 3/2$ and equating coefficients in the last equation we find

$$\begin{aligned}
\alpha_1 &= 0, \\
2(\alpha_2 - \alpha_0) &= 1, \\
\alpha_1 + 3\alpha_3 &= 0, \\
4(\alpha_2 + \alpha_4) &= 0, \\
&\vdots
\end{aligned}$$

which gives

$$\begin{aligned}
\alpha_1 &= 0, \\
\alpha_2 &= 2, \\
\alpha_3 &= 0, \\
\alpha_4 &= -2, \\
&\vdots
\end{aligned}$$

and we could in principal continue finding coefficients to any order desired

$$y(x) = \frac{3}{2} + 2x^2 - 2x^4 + \frac{10}{3}x^6 + \cdots$$

**3.** (a) This is easy to confirm by direct calculation. For $y_1 = 1$ or $y = e^x$ we have,

$$\begin{aligned} y_1'' &= (1)'' = 0 = (1)' = y_1', \\ y_2'' &= (e^x)'' = e^x = (e^x)' = y_2'. \end{aligned}$$

Thus, the equation $y'' = y'$ is satisfied for both $y_1 = 1$ and $y_2 = e^x$, and so both of these functions are in $V$ (by definition!).

Note that since taking derivatives is a *linear* operation, and $y = 3 + 2e^x = 3y_1 + 2y_2$ is a *linear combination* of $y_1, y_2$, then $y = 3 + 2e^x$ must also be a solution. Check this directly yourself!

(b) Assume there exist constants $c_1, c_2$ such that

$$c_1 y_1 + c_2 y_2 = 0$$

This means that for all $x$

$$c_1 + c_2 e^x = 0$$

Rearranging, this means that

$$c_2 e^x = -c_1,$$

*for all* values of $x$! But this is impossible unless $c_1 = c_2 = 0$. Thus, $\{1, e^x\}$ is a linearly independent set.

(c) Let $y$ be any solution to the equation $y'' = y'$. In order to solve this for $y$, let's eliminate a derivative by introducing the function,

$$g = y'.$$

Now plugging this into the equation we want to solve we get a new equation for $g$,

$$g' = g.$$

Solving by separation we find,

$$g = c_2 e^x,$$

for some constant $c_2$. Using $g = y'$ we now want to solve,

$$y' = c_2 e^x.$$

By integration we find,

$$y = c_1 + c_2 e^x.$$

(d) Since $\{1, e^x\}$ is a linearly independent set that spans $V$, it must be a basis for $V$, and therefore $\dim V = 2$.

**4.** (a) Assuming that $y_1$ and $y_2$ are linearly dependent, then there exist constants $c_1, c_2$ (not both zero) such that,

$$c_1 y_1 + c_2 y_2 = 0.$$

Differentiating, we also have the relation

$$c_1 y_1' + c_2 y_2' = 0.$$

Thus, the matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

has a non-zero solution (i.e. a non-zero kernel). Recall that this means that,

$$\det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = 0.$$

This is equivalent to saying that the Wronskian vanishes for all values of $x$,

$$W(y_1, y_2) = y_1(x) y_2'(x) - y_1'(x) y_2(x) = 0.$$

(b) Compute,

$$\begin{aligned} W(e^x, e^{-x}) &= e^x \cdot \left(e^{-x}\right)' - (e^x)' \, e^{-x}, \\ &= -e^x \cdot e^{-x} - e^x \cdot e^{-x}, \\ &= -2 \neq 0. \end{aligned}$$

Thus, $e^x$ and $e^{-x}$ are (linearly) independent functions.

(c) To compute the Wronsikian, it is, by definition of $y_1$ and $y_2$ enough to know their derivatives at $x = 0$. Using exercise 1 ($b$) and the chain rule, we find that $y_1'(0) = y_2'(0) = 0$ and therefore that $W(y_1, y_2)(x) = 0$ for all $x$. But $y_1$ and $y_2$ are clearly not linearly dependent, because assuming that

$$c_1 y_1 + c_2 y_2 = 0$$

for *all* $x$ implies that necessarily $c_1 = 0$ and $c_2 = 0$ (because $y_1(x) = 0$ for $x < 0$, where $y_2(x) > 0$ and vice versa).

**5.** This is just a generalization of the argument we had for two functions. Assuming that $y_1, y_2, \ldots, y_n$ are linearly dependent, then there exist constants $c_1, c_2, \ldots, c_n$ (not all zero) such that,

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0.$$

Differentiating $n-1$ times, we also have the relations

$$\begin{aligned} c_1 y_1 + c_2 y_2 + \cdots + c_n y_n &= 0, \\ c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' &= 0, \\ &\vdots \\ c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} &= 0. \end{aligned}$$

Thus, the matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

has a non-zero solution (i.e. a non-zero kernel). Thus,

$$\det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} = 0.$$

Hence, linear dependence implies that $W(y_1, y_2, \ldots, y_n)(x) = 0$ for all values of $x$ in $[c, d]$. The contrapositive states that if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for some value of $x$ in $[c, d]$, then the functions $y_1, y_2, \ldots, y_n$ are linearly independent on $[c, d]$.

**6.** Consider,

$$\begin{aligned}
y(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \cdots \\
y'(x) &= \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + \cdots \\
y''(x) &= 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 + \cdots \\
2xy''(x) &= \qquad 4\alpha_2 x + 12\alpha_3 x^2 + \cdots \\
2y(x) &= 2\alpha_0 + 2\alpha_1 x + 2\alpha_2 x^2 + \cdots
\end{aligned}$$

We are given the constants $y(0) = y_0$ and $y'(0) = y_1$, and so $\alpha_0 = y_0$ and $\alpha_1 = y_1$. Adding the last four equations up we see that the constant term satisfies,

$$y_1 + 2\alpha_2 + 2y_0 = 0,$$

so that,

$$\alpha_2 = -y_0 - \frac{1}{2}y_1.$$

The coefficient of $x$ gives the relation,

$$2\alpha_2 + 6\alpha_3 + 4\alpha_2 + 2y_1 = 0,$$

so that,

$$\begin{aligned}
6\alpha_3 &= -6\alpha_2 - 2y_1, \\
6\alpha_3 &= -6\left(-y_0 - \frac{1}{2}y_1\right) - 2y_1, \text{ sub. } \alpha_2 \text{ from above,} \\
\alpha_3 &= y_0 + \frac{1}{6}y_1.
\end{aligned}$$

The coefficient of $x^2$ gives the relation,

$$3\alpha_3 + 12\alpha_4 + 12\alpha_3 + 2\alpha_2 = 0,$$

so that,

$$\begin{aligned}
12\alpha_4 &= -15\alpha_3 - 2\alpha_2, \\
12\alpha_4 &= -15\left(y_0 + \frac{1}{6}y_1\right) - 2\left(-y_0 - \frac{1}{2}y_1\right), \text{ sub. } \alpha_2, \alpha_3 \text{ from above,} \\
12\alpha_4 &= -13y_0 - \frac{3}{2}y_1, \\
\alpha_4 &= -\frac{13}{12}y_0 - \frac{1}{8}y_1.
\end{aligned}$$

The solution may then be written as:

$$y(x) = y_0\left(1 - x^2 + x^3 - \frac{13}{12}x^4 + \cdots\right) + y_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + \cdots\right).$$

**Note:**

- it is difficult to determine a closed form solution
- it turns out that we may always write solutions in the form

$$y = y_0 f_1 + y_1 f_2,$$

  where $f_1$ satisfies $f_1(0) = 1,\ f_1'(0) = 0$ and $f_2$ satisfies $f_2(0) = 0,\ f_2'(0) = 1$.

**7.** (a) To see that the kernel of $\ell$ is two-dimensional we will show that the pair of functions $\{f_1, f_2\}$ given in the theorem forms a *basis* for $\ker\ell$.

- To see that $f_1, f_2$ are linearly independent consider,

  $$W(x) = f_1(x)f_2'(x) - f_1'(x)f_2(x).$$

  Evaluating the Wronskian at $x = 0$ and using $f_1(0) = 1$, $f_1'(0) = 0$ and $f_2(0) = 0$, $f_2'(0) = 1$,

  $$W(0) = f_1(0)f_2'(0) - f_1'(0)f_2(0) = 1 \neq 0.$$

Thus, $\{f_1, f_2\}$ forms a basis for $\ker\ell$ and we have shown $\dim\ker\ell = 2$.

(b) Let $g = y - y_p$ and consider,

$$\begin{aligned} \ell(g) &= \ell(y - y_p), \\ &= \ell(y) - \ell(y_p), \text{ by linearity of } \ell, \\ &= h - h, \text{ by assumption}, \\ &= 0. \end{aligned}$$

Thus, $g \in \ker\ell = \operatorname{span}\{f_1, f_2\}$, and therefore

$$\begin{aligned} g &= c_1 f_1 + c_2 f_2, \\ y - y_p &= c_1 f_1 + c_2 f_2, \\ y &= y_p + c_1 f_1 + c_2 f_2. \end{aligned}$$

This proves the claim.