Week 1

Topics: first-order linear ODEs; integrating factor; recap of linear algebra; variation of parameters

1. Consider the differential equation $xy' + y = e^x$.

(a) Explain why it is a first-order linear inhomogeneous differential equation.

(b) By considering $(xy)'$, find its general solution (involving one arbitrary constant).

(c) Find a solution that satisfies the initial condition $y(1) = 0$.

(d) Then find the one and only value y_0 for which the initial value problem with $y(0) = y_0$ can be solved.

(e) What feature of the equation suggests that this equation cannot be solved for an arbitrary value of $y(0)$?

2. Suppose that your student loan balance is \$1K, that it accrues interest at a rate of 50% per decade, and that you pay it off at a steady rate of \$1K per decade. Denote by y the student loan balance as a function of x, the number of decades. What is the student loan balance for $x = 1$ decade? This amounts to solving the initial value problem

$$\begin{cases} y' & = 0.5y - 1 \\ y(0) & = 1 \end{cases}$$

Use the idea of the integrating factor to solve this problem.

3. Consider the general linear first-order equation,

$$a(x)y' + b(x)y = h(x).$$

Show that we may choose the integrating factor to be

$$\mu(x) = \exp\left(\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right),$$

and use this integrating factor to get a general solution to the equation for any region in which $x \mapsto b(x)/a(x)$ is integrable.

4. Show that the function $\ell : V \to V$ given by the formula $\ell(y) = a(x)y' + b(x)y$ has the properties required for linearity, and the kernel of ℓ is a one-dimensional subspace of V. Find a vector (function) $f \in V$ such that $\ker\ell = \operatorname{span}\{f\}$.

Answers and Solutions.

1. (a) first-order: since the highest derivative term is y', which has only one derivative; linear: since the equation only involves y, y' (and not powers (y^3, $(y')^{-2}$, etc...), or other functions, $\sin(y)$, etc...); inhomogeneous: since terms not involving y, y' are non-zero.

(b) Observe that the LHS of the equation is $(xy)' = xy' + y$ and the equation becomes,

$$\begin{aligned}(xy)'(x) &= e^x,\\ \int (ty)'(t)\,dt &= \int e^t\,dt,\\ (xy)(x) &= e^x + C,\\ y(x) &= \frac{e^x}{x} + \frac{C}{x}\end{aligned}$$

(c) From the last computation we find $y(1) = e^1 + C = e + C = 0 \Rightarrow C = -e$. A solution that satisfies $y(1) = 0$ is then

$$y(x) = \frac{e^x}{x} - \frac{e}{x}.$$

(d) We need to evaluate the function $x \mapsto y(x) = \frac{e^x}{x} + \frac{C}{x}$ at $x = 0$. Clearly, we can not just plug this in and we need to consider the limit:

$$\lim_{x\to 0} y(x) = \lim_{x\to 0}\left[\frac{e^x + C}{x}\right].$$

In order for this limit to exist, we need the numerator $e^x + C \to 0$ as $x \to 0$ (at least as fast as $x \to 0$), which means that we need $C = -1$.

In this case,

$$\begin{aligned}\lim_{x\to 0} y(x) &= \lim_{x\to 0}\left[\frac{e^x - 1}{x}\right],\\ &= \lim_{x\to 0}\left[e^x\right], \text{ by L'Hôpital,}\\ &= 1.\end{aligned}$$

Thus, $y_0 = 1$ is the only value for which the initial value problem can be solved.

(e) The fact that the equation may be written as

$$y' + \frac{y}{x} = \frac{e^x}{x},$$

shows that we might expect some "singular" behavior as $x \to 0$ for an arbitrary value of $y(0)$.

2. First multiply both sides of the equation $y' - 0.5y = -1$, by an integrating factor μ

$$\mu y' - 0.5\mu y = -\mu.$$

We want the LHS of this last equation to have the form $(\mu y)' = \mu y' + \mu' y$, that is

$$\begin{aligned}
\mu y' - 0.5\mu y &\overset{!}{=} (\mu y)' = \mu y' + \mu' y, \\
-0.5\mu y &= \mu' y, \text{ cancelling } \mu y' \text{ from both sides,} \\
\mu' &= -0.5\mu, \text{ cancelling } y \text{ from both sides and rearranging.}
\end{aligned}$$

Thus, we want a solution (any solution will do!) to the equation

$$\begin{aligned}
\mu' &= -0.5\mu, \\
\int \frac{\mu'}{\mu}(x)\, dx &= -\int 0.5\, dx, \\
\log|\mu|(x) &= -0.5x + K, \\
|\mu|(x) &= \exp[-0.5x + K] = C\exp[-0.5x], \text{ where } C = e^K.
\end{aligned}$$

Since any solution will do, we are free to take $\mu = e^{-x/2}$. Going back to our original equation (multiplied by μ),

$$\begin{aligned}
\mu y' - 0.5\mu y &= -\mu, \\
(\mu y)' &= -\mu, \\
\int (e^{-x/2}y)'(x)\, dx &= -\int e^{-x/2}\, dx, \text{ plugging in } \mu = e^{-x/2}, \\
y(x) &= 2 + Ce^{x/2}.
\end{aligned}$$

Using the initial condition and our last result $y(0) = 2 + Ce^0 = 2 + C = 1 \Rightarrow C = -1$, and

$$y(x) = 2 - e^{x/2}.$$

Thus, there is $y(1) = 2 - \sqrt{e}$ kilobucks owing when $x = 1$ decade. Note that if $y(0) > 2$, then $C > 0$ and the loan keeps increasing!

3. Multiply both sides of the equation by an integrating factor μ and divide the equation by a. We are ignoring any problems that dividing by a might cause for now, and will worry about this for specific choices of a later. Also, recall that we assume that a, b and h are functions of x. We need to find the general solution to

$$\mu y' + \frac{b}{a}\mu y = \frac{h}{a}\mu.$$

Using the usual integrating factor method, we want

$$\begin{aligned} \mu'(x) &= \frac{b(x)}{a(x)}\mu(x), \\ \log|\mu|(x) &= \int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt + \log|\mu|(x_0), \\ |\mu|(x) &= |\mu|(x_0) \exp\left[\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right]. \end{aligned}$$

Notice that the x_0 is arbitrary. Since any integrating factor is sufficient for our purposes, set

$$\mu(x) = \exp\left[\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right].$$

The general solution is then given by

$$\begin{aligned} (\mu y)'(x) &= \frac{h(x)}{a(x)}\mu(x), \\ \mu(x)y(x) &= \int_{x_0}^{x} \frac{h(t)}{a(t)}\mu(t)\, dt \\ y(x) &= \frac{1}{\mu(x)} \int_{x_0}^{x} \frac{h(t)}{a(t)}\mu(t)\, dt + \frac{y(x_0)}{\mu(x)} \end{aligned}$$