PROBLEM SET 3

Due: Tue, Feb 25, 6.00 PM

Topics: linear second order ODE, calculus with linear maps

1. Consider *Legendre's Equation*,

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

where $n = 0, 1, 2, 3\dots$ is a natural number.

(a) Using Abel's formula, show that the Wronskian for Legendre's equation is given by the following (with appropriate choice of constant):

$$W(x) = \frac{1}{1-x^2}.$$

(b) For a given value of n, let P_n, Q_n be two linearly independent solutions to Legendre's equation. If $P_0 = 1$ and $P_1 = x$, find Q_0, Q_1 by solving

$$P_n Q_n' - P_n' Q_n = W,$$

for Q_n when $n = 0, 1$.

2. Consider the n-th order initial value problem

$$\begin{cases} a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \\ \quad y(x_0) = y_0, \\ \quad y'(x_0) = y_1, \\ \quad\quad \vdots \\ y^{(n-1)}(x_0) = y_{n-1}. \end{cases}$$

for analytic functions $a_0, \dots, a_n$ on $\mathbb{R}$, $x_0 \in \mathbb{R}$ and $a_0(x_0) \neq 0$. Assume that, for any given initial data $y_0, y_1, \dots, y_{n-1} \in \mathbb{R}$, the above initial value problem has a unique, analytic solution y. Prove that the n-th order differential operator l, defined by

$$l(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y,$$

has a kernel of dimension n.

3. Invent an explicit example of a linear second order initial value problem $\ell(y) = ay'' + by' + cy = 0$ with initial conditions $y(0) = y_0 \in \mathbb{R}, y''(0) = y_2 \in \mathbb{R}$ that is **not** solvable, but such that the corresponding initial value problem $\ell(y) = ay'' + by' + cy = 0$ with the usual initial conditions $y(0) = w_0 \in \mathbb{R}, y'(0) = w_1 \in \mathbb{R}$, is always solvable.

4. By substituting $y(x) = x^\alpha$ for some constant α, find two functions that span the kernel of the operator ℓ, defined by $\ell(y) = x^2 y'' + xy' - y$.

5. Solve the following:

(a) Start with $\ell(y) = y'' + 4y' + 5y = 0$. Find a basis for the kernel by using the trial solution $x \mapsto e^{rx}$, then construct a solution that satisfies the initial conditions $y(0) = 1, y'(0) = 0$.

(b) Start with $\ell(y) = y'' - 4y' + 3y = 0$. Find a basis for the kernel by using the trial solution $x \mapsto e^{rx}$, then construct a solution that satisfies the initial conditions $y(0) = 3, y'(0) = 1$.

6. Recall the notation and results from problem set 2, problem 5.

(a) Show that $l_1 = D + M_x$ and $l_2 = D + M_{x+1}$ commute, that is, $l_1 l_2 = l_2 l_1$.

(b) Find a basis for the kernel of the operator l, defined by $l(y) = y'' + (2x+1)y' + (x^2 + x + 1)y$.