Week 9

Topics: inner products, norms, orthogonality, Cauchy-Schwarz inequality, L^2 spaces

Review from Week 7:

- motivated by the vibrating wire (Holland, section 3.1), we were lead to the study of the following *boundary value problem*: find all eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions y such that

$$\begin{cases} y'' &= \lambda y \\ y(0) &= y(L) = 0. \end{cases}$$

- the set $\{y : [0; L] \to \mathbb{R} : y(0) = y(L) = 0\}$ is a vector space and $\ell(y) = y''$ defines a linear operator on this vector space; the above problem can be interpreted as asking about all eigenvalues and eigenfunctions of the operator ℓ, the *one dimensional Laplace operator with Dirchlet boundary conditions*

- it turns out that there are countably infinite many eigenvalues $\lambda_n = \frac{\pi^2 n^2}{L^2}$, $n \in \mathbb{N}$, and corresponding eigenfunctions $y_n(x) = \sin(n\pi x/L)$; notice that all these functions are linearly independent

- the goal of the next part is to generalize the notions of inner products, norms, orthogonality, bases, symmetric matrices and more to infinite dimensional vector spaces that are complete with respect to a certain metric - these spaces are called *Hilbert spaces* and along with *self-adjoint linear operators*, they form the basic objects in quantum mechanics; they are also indispensable in view of solving partial differential equations

1. Inner products. Recall the notion of an inner product and the notion of orthogonality.

(a) **Conjugate linearity in the second factor:** Prove that $\langle v, \alpha w\rangle = \overline{\alpha}\langle v, w\rangle$.

$$\langle v, \alpha w\rangle = \overline{\langle \alpha w, v\rangle} = \overline{\alpha}\,\overline{\langle w, v\rangle} = \overline{\alpha}\langle v, w\rangle$$

(b) **Orthogonality and linear independence.** Prove that two orthogonal vectors are linearly independent. Does linear independence also imply orthogonality?

1) Let $v_1, v_2 \neq 0$ and $\lambda_1 v_1 + \lambda_2 v_2 = 0 \mid \langle \cdot, v_1\rangle$
$\rightsquigarrow \lambda_1\langle v_1, v_1\rangle = 0 \rightsquigarrow \lambda_1 = 0 \rightsquigarrow \lambda_2 = 0$

2) Consider $v_1 = \begin{pmatrix}1\\0\end{pmatrix}, v_2 = \begin{pmatrix}1\\1\end{pmatrix}$: lin. indep., but not orthog.

(c) Prove that $\mathbb{C}^n$ is an inner product space where for $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ we define the inner product by

$$\langle v, w\rangle = \sum_{j=1}^{n} v_j \overline{w_j}.$$

1) $\langle \alpha v + \beta w, z\rangle = \alpha \sum_j v_j \overline{z_j} + \beta \sum_j w_j \overline{z_j} = \alpha\langle v, z\rangle + \beta\langle w, z\rangle$

2) $\langle v, w\rangle = \overline{\left(\sum_j w_j \overline{v_j}\right)} = \overline{\langle w, v\rangle}$

3) $\langle v, v\rangle = \sum_j |v_j|^2 \geq 0$ and $= 0$ iff $v_j = 0 \ \forall j = 1, \ldots, n$

(d) Prove that the space $C([a;b], \mathbb{C})$ of complex valued continuous functions with domain $[a;b]$ is an inner product space with inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

- linearity in first slot and symmetry can be proved as above; also $\langle f, f\rangle \geq 0$
- let $\langle f, f\rangle = 0 \Rightarrow \int_a^b |f|^2 = 0$
- suppose $f(x_0) > 0$, then by continuity $f(x) \geq c > 0$ in $[x_0 - \delta, x_0 + \delta]$

$\rightsquigarrow \langle f, f\rangle \geq 2\delta \cdot c > 0$ ↯ $\rightsquigarrow f \equiv 0$!

2. The length of a vector. Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space (over $\mathbb{R}$ or $\mathbb{C}$). Using the inner product, we can define the *length or norm* of a vector through $\|v\| = \sqrt{\langle v, v\rangle}$. Prove that $\|\cdot\|$ has the following properties:

(a) $||\alpha f|| = |\alpha|||f||$ for $\alpha \in \mathbb{C}$.

$$\sqrt{\langle \alpha f, \alpha f\rangle} = \sqrt{\alpha \cdot \bar{\alpha} \langle f, f\rangle}$$
$$= \sqrt{|\alpha|^2 \langle f, f\rangle}$$
$$= |\alpha| \, \|f\|$$

(b) $||f|| \geq 0$.

$$\sqrt{\underbrace{\langle f, f\rangle}_{\geq 0}} \geq 0$$ (square root of a non-negative number!)

(c) $||f|| = 0$ if and only if $f = 0$.

$\|f\| = 0$ iff $\langle f, f\rangle = 0$

iff $f = 0$

3. Important inequalities. There are two fundamental inequalities in inner product spaces:

- **Cauchy-Schwarz:** $|\langle f, g\rangle| \leq ||f|| \cdot ||g||$.
- **Triangle:** $||f+g|| \leq ||f|| + ||g||$.

(a) Suppose $f, g \in V$ and assume V to be a real vector space, for simplicity. Show that, in order to prove Cauchy-Schwarz, you can assume w.l.o.g. that $\|f\| = \|g\| = 1$.

- $\|f\|=0$ or $\|g\|=0$ $\rightsquigarrow$ C.S. true!
- note that $\left\| f/\|f\| \right\| = 1$

(b) To prove the inequality in that case, remember the statement of the Pythagorean theorem and try to generalize this as follows: try to express the square of the length of f as the sum of the square of the length of f projected onto g and the square of the length of a vector orthogonal to g.

- $f = \langle f,g\rangle g + (f - \langle f,g\rangle g) \equiv \phi + \psi$
- note that $\langle \psi, g\rangle = \langle f,g\rangle - \langle f,g\rangle = 0$

$\rightsquigarrow 1 = \langle f,f\rangle = |\langle f,g\rangle|^2 + \underbrace{\langle \psi,\psi\rangle}_{\geq 0} = |\langle f,g\rangle|^2 \quad \square$

(c) Prove the **Triangle** inequality by expanding

$$||f+g||^2 = \langle f+g, f+g\rangle.$$

$$\begin{aligned}\langle f+g, f+g\rangle &= \|f\|^2 + \|g\|^2 + 2\mathrm{Re}\langle f,g\rangle \\ &\leq \|f\|^2 + \|g\|^2 + 2|\langle f,g\rangle| \\ &\leq (\|f\| + \|g\|)^2 \text{ by C.S.}\end{aligned}$$

(d) Show that $d(v,w) = \|v-w\|$, $v,w \in V$, defines a distance function (a metric). This means that, in an inner product space, *we can measure angles, lengths and distances.*

- $d(v,w) \geq 0$ and $= 0$ iff $v-w=0$ iff $v=w$
- $d(v,w) = d(w,v)$
- $d(v,w) \leq d(v,z) + d(z,w)$ by triangle

Note: inner product $\rightarrow$ normed $\rightarrow$ metric

4. Orthonormal Bases in Finite Dimensions.

(a) Suppose $v, w \in V$, $v, w \neq 0$, are two vectors in some inner product space $(V, \langle\cdot,\cdot\rangle)$. Remind yourself from the proof of Cauchy-Schwarz how to decompose v into a component proportional to w and one component orthogonal to w. Use this construction to find an orthonormal basis of $\text{span}\{v, w\} \subset V$.

• direction of w: $w/\|w\| = e_1$

$\rightsquigarrow v = \langle v, e_1\rangle e_1 + \underbrace{(v - \langle v, e_1\rangle e_1)}_{\phi \perp e_1}$

$e_2 = \phi/\|\phi\|$

(b) Apply the previous part to the case $V = C([0;1])$ with $\langle\psi, \phi\rangle = \int_0^1 dt\, \psi(t)\phi(t)$ and where v, w are defined by $v(x) = x$, $w(x) = x^2$ for all $x \in [0;1]$.

$e_1 = \frac{x}{\|x\|}$, $\|x\|^2 = \int_0^1 x^2 dx = 1/3$; $e_1 = \sqrt{3}x$

$\phi = x^2 - \langle e_1, x^2\rangle e_1$; $\langle x^2, e_1\rangle = \sqrt{3}\int_0^1 x^3 = \frac{\sqrt{3}}{4}$

$= x^2 - 3/4\, x$; $e_2 = \phi/\|\phi\| \ldots$

(c) Suppose now that V is a finite dimensional inner product space with basis $\{v_1, v_2, \ldots, v_n\}$. Generalize the case $n = 2$ to any $n \in \mathbb{N}$. More precisely, sketch an argument how to construct explicitly an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ from $\{v_1, v_2, \ldots, v_n\}$.

1) $e_1 = v_1/\|v_1\|$

2) $\phi_2 = v_2 - \langle v_2, e_1\rangle e_1 \perp e_1$

$\rightsquigarrow e_2 = \phi_2/\|\phi\|$

3) ϕ_3 should be orthogonal to e_1, e_2

$\rightsquigarrow \phi_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2$!

$\rightsquigarrow e_3 = \phi_3/\|\phi_3\|$

⋮

n) $\phi_n = v_n - \sum_{j=1}^{n-1} \langle v_n, e_j\rangle e_j$, $e_n = \frac{\phi_n}{\|\phi_n\|}$ ⋮

GRAM - SCHMIDT PROCEDURE

5. Orthonormal Bases in Hilbert Spaces. Suppose $(H, \langle\cdot,\cdot\rangle)$ is a (possibly infinite dimensional) Hilbert space. We say that the orthonormal sequence $(e_j)_{j\in\mathbb{N}}$ forms a *complete orthonormal basis or Hilbert space basis* of H if for every $x \in H$ there exist constants $(c_j)_{j\in\mathbb{N}}$ in $\mathbb{K}$ such that $x = \sum_{j=1}^{\infty} c_j e_j \in H$.

(a) Suppose $(e_j)_{j\in\mathbb{N}}$ is a Hilbert space basis of H and $x = \sum_{j=1}^{\infty} c_j e_j \in H$. Prove that then

$$c_k = \langle x, e_k\rangle$$

for all $k \in \mathbb{N}$. That is[1], $x = \sum_{j=1}^{\infty}\langle x, e_j\rangle e_j \in H$.

• compute $\langle x, e_k\rangle = \left\langle \sum_{j=1}^{\infty} c_j e_j, e_k \right\rangle = (*)$

• note by C.S: $\left|\left\langle \sum_{j=m}^{\infty} c_j e_j, e_k\right\rangle\right| \leq \left\| \sum_{j=m}^{\infty} c_j e_j \right\| \xrightarrow[m\to\infty]{} 0$

$\rightsquigarrow (*) = \sum_{j=1}^{\infty} c_j \langle e_j, e_k\rangle = \sum_{j=1}^{\infty} c_j \delta_{jk} = c_k$

(b) Consider the space $C([0;1];\mathbb{C})$ with inner product $\langle\psi,\phi\rangle = \int_0^1 dt\ \psi(t)\overline{\phi(t)}$. Show that the set $\mathcal{B} = \{x \mapsto \varphi_p = e^{ipx} : p \in \mathbb{Z}\setminus\{0\}\}$ forms an orthonormal set $C([0;1];\mathbb{C})$. Is it true that $f = \sum_{p\neq 0}\langle f, \varphi_p\rangle\varphi_p \in C([0;1];\mathbb{C})$ for any $f \in C([0;1];\mathbb{C})$?

Typo: $\varphi_p = e^{2\pi i p}$

1) $\int_0^1 dt\, e^{i2\pi(p-q)t} = \begin{cases} 1 & : p = q \\ \frac{1}{i(p-q)} e^{2\pi i(p-q)t}\Big|_0^1 = 0 & p \neq q \end{cases}$

2) No: $f \equiv 1$, then $\langle f, \varphi_p\rangle = 0\ \forall p \neq 0$, but $f \not\equiv 0$

(c) Consider the real vector space $\ell^2(\mathbb{R}) = \left\{(a_j)_{n\in\mathbb{N}} : a_j \in \mathbb{R}\ \forall\, j \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} a_j^2 < \infty\right\}$. Show that $\ell^2(\mathbb{R})$ can be turned into an inner product space. On the next problem set, you will prove that it is actually a Hilbert space. Once you've found the inner product, find an orthonormal basis of $\ell^2(\mathbb{R})$.

• inner product: $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$

• ONB: $e_j = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, 1 at j-th position

• indeed $x = \sum_{j=1}^{\infty}\langle x, e_j\rangle e_j \iff \left\| x - \sum_{j=1}^{m}\langle x, e_j\rangle e_j\right\|^2 \xrightarrow[m\to\infty]{} 0$

[1] In physics classes, one would say that $\sum_{j=1}^{\infty} |e_j\rangle\langle e_j|$ is the identity operator in H.

and $\left\| x - \sum_{j=1}^{m}\langle x, e_j\rangle e_j\right\|^2 = \sum_{j=m+1}^{\infty} |x_j|^2 \longrightarrow 0$

6. Parseval's identity. Suppose $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space H.

(a) Suppose $(c_j)_{j\in\mathbb{N}}$ is a sequence in $\mathbb{K}$. Show that

$$\sum_{j=1}^{\infty} c_j e_j \in H \qquad \leftrightarrow \qquad \sum_{j=1}^{\infty} |c_j|^2 < \infty.$$

$$\sum_{j=1}^{\infty} c_j e_j \in H \Leftrightarrow \Big(\sum_{j=1}^{m} c_j e_j\Big)_{m\in\mathbb{N}} \text{ Cauchy in } H$$

$$\Leftrightarrow \underbrace{\Big\| \sum_{j=1}^{m} c_j e_j - \sum_{j=1}^{n} c_j e_j \Big\|^2}_{= \sum_{j=n}^{m} |c_j|^2} \xrightarrow[m,n\to\infty]{} 0 \Leftrightarrow \Big(\sum_{j=1}^{m} |c_j|^2\Big)_{m\in\mathbb{N}} \text{ Cauchy in } \mathbb{R}$$

$$\Leftrightarrow \sum_{j=1}^{\infty} |c_j|^2 < \infty$$

(b) Suppose $x \in H$. Prove *Parseval's identity* which says that

$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2.$$

That means, norms of vectors can be computed as in finite dimensions!

$$\|x\|^2 = \langle x, x\rangle = \Big\langle x, \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j \Big\rangle$$

$$\underset{\text{Cont.}}{=} \sum_{j=1}^{\infty} \overline{\langle x, e_j\rangle} \langle x, e_j\rangle = \sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2$$

The L^2 spaces. L^2 spaces form *the fundamental examples* of Hilbert spaces. Modulo some measure theoretic complications[2], we define the spaces $L^2(I;\rho(x)dx)$ for some interval $I \subset \mathbb{R}$ and some non-negative function $\rho : I \to [0;\infty)$ through

$$L^2(I;\rho(x)dx) = \{\psi : I \to \mathbb{K} : \int_I dx\ |\psi(x)|^2 \rho(x)\cancel{dx} < \infty\}.$$

(a) Explain why $L^2(I;\rho(x)\ dx)$ is a vector space and show that $\langle\psi,\phi\rangle = \int_I dx\ \rho(x)\psi(x)\overline{\phi(x)}$ defines an inner product on it. A fundamental fact from analysis (measure and integration theory) is that the *L^2 spaces are Hilbert spaces.*

- $\psi, \phi \in L^2$: $\int dx\ \rho(x)\,|\alpha\psi(x) + \beta\varphi(x)|^2$
  $\leq 2|\alpha|^2 \int dx\ \rho(x)\,|\psi(x)|^2 + 2|\beta|^2 \int dx\ \rho(x)\,|\varphi(x)|^2$
  $< \infty \Rightarrow \alpha\psi + \beta\varphi \in L^2$
- Lebesgue fact: $\int dx\,\rho(x)\,|f(x)|^2 = 0 \Rightarrow f(x) = 0$ (a.s.)
  $\rightsquigarrow$ inner product check as in problem 1

(b) Show that any countable subset of $\mathbb{R}$ is a set of Lebesgue measure zero.

- let $S = \{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ be countable and $\varepsilon > 0$
- let $I_j = [a_j - \varepsilon/2^j,\ a_j + \varepsilon/2^j]$

$\rightsquigarrow \sum_{j=1}^{\infty} |I_j| = \sum_{j=1}^{\infty} \varepsilon/2^{j-1} = \varepsilon \cdot \sum_{j=0}^{\infty} 2^{-j} = 2\varepsilon$

$\rightsquigarrow$ S is of Lebesgue measure zero!

(c) Using the basic facts about Lebesgue integrals from the lecture, compute the $L^2([0;1]\ dx)$ norm of the functions $x \mapsto \underbrace{\mathbf{1}_{[0;1]\cap\mathbb{Q}}(x)}_{=:\psi(x)}$ and $x \mapsto \underbrace{\mathbf{1}_{[0;1]\setminus\mathbb{Q}}(x)}_{=:\varphi(x)}$.

- $\psi(x) = 0$ a.s., $\varphi(x) = 1$ a.s.

$\rightsquigarrow \|\psi\|_2 = \|0\|_2 = 0$ while $\|\varphi\|_2 = \|1\|_2 = 1$

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[2]See Math 114 in the fall ;).