MATH110 Spring 2020 Exam 1

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1. (5 points) Solve the initial value problem $xy' + y = 12x^5$ with $y(1) = 7$.

I first solve the first order linear differential equation with one free parameter, then solve for the initial value.

$$\begin{aligned} xy' + y &= 12x^5 \\ (xy)' &= 12x^5 \\ xy &= 2x^6 - c \\ y &= 2x^5 - \frac{c}{x} \end{aligned}$$

Solving for the initial value:

$$\begin{aligned} 7 &= y(1) \\ &= 2(1) - c \\ c &= -5 \end{aligned}$$

Thus, our solution is $y(x) = 2x^5 + \frac{5}{x}$. We check that this is correct by plugging back in:

$$\begin{aligned} y &= 2x^5 + \frac{5}{x} \\ y' &= 10x^4 - \frac{5}{x^2} \\ xy' &= 10x^5 - \frac{5}{x} \\ xy' + y &= 10x^5 - \frac{5}{x} + 2x^5 + \frac{5}{x} \\ &= 12x^5 \end{aligned}$$

Looks good!

2. (2+3 points) Let $\ell(y) = \frac{1}{2}y'' - \frac{3}{2}y' + \frac{9}{8}y$.

(a) Solve the initial value problem $\ell(y) = 0$ with $y(0) = y'(0) = 1$ using the method of the characteristic polynomial.

We start with the ansatz that the solution has the form $y(x) = e^{rx}$. Then

$$\begin{aligned}
y' &= re^{rx} \\
y'' &= r^2e^{rx} \\
0 = l(y) &= \frac{1}{2}y'' - \frac{3}{2}y' + \frac{9}{8} \\
0 &= [\frac{1}{2}r^2 - \frac{3}{2}r + \frac{9}{8}]e^{irx} \\
&= r^2 - 3r + \frac{9}{4} \\
&= (r - \frac{3}{2})^2
\end{aligned}$$

The root is $r = \frac{3}{2}$, with multiplicity 2. We know from lecture (February 20th, 2020) that the general solution to a constant coefficient ODE with only one real root will have the form

$$y(x) = c_1e^{rx} + c_2xe^{rx}$$

How do we know this? We know that the kernel of a second order linear ODE is two dimensional, so we know that a second function must exist. We then can use the Wronskian to find a second, linearly independent function by solving

$$f_2(x) = f_1(x)\int_{x_0}^{x} \frac{W(t)}{f_1(t)}dt$$

Returning to the problem, we solve for initial conditions i.e. $y(0) = y'(0) = 1$:

$$\begin{aligned}
y(0) = 1 &= c_1e^{r(0)} + c_2(0)e^{r(0)} \\
1 &= c_1(1) + 0 \\
1 &= c_1 \\
y'(0) = 1 &= re^{r(0)} + c_2e^{r(0)} + c_2(0)e^{r(0)} \\
1 &= \frac{3}{2} + c_2 \\
-\frac{1}{2} &= c_2
\end{aligned}$$

Thus, the solution to the initial value problem is

$$y(x) = e^{3x/2} - \frac{1}{2}xe^{3x/2}$$

(b) Let V denote the space of polynomials with real coefficients of degree less than or equal to 2. Show that $\ell : V \to V$, i.e., that ℓ maps V to itself. Given $p \in V$, does there always exist some $q \in V$ such that $\ell(q) = p$? **Justify your answer.**

Consider a polynomial $p(x) = ax^2 + bx + c$, with $a, b, c \in \mathbb{R}$. Note that $p(x)$ could describe any element in V. We show that that $l(p) \in V$:

$$
\begin{aligned}
l(p) &= \frac{1}{2}p'' - \frac{3}{2}p' + \frac{9}{8}p \\
&= \frac{1}{2}(2a) - \frac{3}{2}(2ax+b) + \frac{9}{8}(ax^2+bx+c) \\
&= (\frac{9}{8}a)x^2 + (-3a + \frac{9}{8}b)x + (a - \frac{3}{2}b + \frac{9}{8}c)
\end{aligned}
$$

Note that $l(p)$ is at most a second-degree polynomial. Since each of the terms inside the parentheses are real, the coefficients are all real. Thus, $l(p) \in V$ and we conclude that l maps V to itself.

Regarding the second part of the question, suppose $p \in V$. Does there always exist $q \in V$ such that $l(q) = p$? **Yes**. Suppose $p = a'x^2 + b'x + c'$. We first solve $a' = \frac{9}{8}a$ for a, then solve $b' = -3a + \frac{9}{8}b$ for b, and finally solve $c' = a - \frac{3}{2}b + \frac{9}{8}c$ for c. For each step, we have one equation and one unknown, and since we never divide by a zero coefficient, a solution must exists.

3. (2.5+2.5 points) Consider the first order linear differential equation

$$ay' + by = h$$

where $a(x) = (x^2 + 1), b(x) = 3x$ and $h(x) = e^{-x} + 2x + 1$ for all $x \in \mathbb{R}$.

(a) Show that the functions a, b and h are analytic at the point $x_0 = 0$.

Recall that a real function $f : x \to \mathbb{R}$ is analytic on the interval $I \overset{\text{def}}{=} [a, b]$ if $\forall x \in I$, (i) the Taylor Series of f converges, and (ii) the Taylor Series of f is equal to f.

First, recall that all real finite polynomials are convergent (specifically, the infinite series converges to the polynomial itself), which allows us to immediately conclude that a, b are convergent. The Taylor Series of a at $x_0 = 0$ is equal to itself:

$$T_{a,x_0=0} = \sum_{k=0}^{\infty} \frac{a^{(n)}}{k!}(x-0)^k = 1 + x^2 = a(x)$$

And the Taylor Series of b at $x_0 = 0$ is also equal to itself:

$$T_{b,x_0=0} = \sum_{k=0}^{\infty} \frac{b^{(n)}}{k!}(x-0)^k = 3x = b(x)$$

From this, we conclude that a, b are analytic. By a similar argument, $2x + 1$ is also analytic at $x_0 = 0$. I next show that e^{-x} is analytic at $x_0 = 0$. Adapting my answer from Homework 2, let $x_0 \in \mathbb{R}$ and $f : x \to e^{-x}$. I show the series converges at $x_0 = 0$:

$$T_{f,x_0=0} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

Using a_n to refer to the nth term in the sequence, we use a ratio test to show that $T_{f,x_0=0}$ converges via the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)!} \frac{n!}{(-1)^n x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1$$

One way to define $f : x \to e^{-x}$ is as the function for which $f(x) = -\frac{d}{dx}f(x)$ and $f(0) = 1$. We use this to show that the Taylor Series converges to the function:

$$\frac{d}{dx}T_{f,x_0=0} = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = -\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = -\sum_{k=0}^{\infty} \frac{x^k}{k!} = -T_{f,x_0=0}$$

I also need to show that the Taylor series centered at 0 and evaluated at x=0 is 1:

$$T_{f,0}(0) = \sum_{k=0}^{\infty} \frac{(0-0)^k}{k!} = 1 + 0 + 0 + ... = 1$$

Thus, $f : x \to e^{-x}$ is analytic for $x_0 = 0$ (as an aside, if we modify some details, this same approach also shows that f is analytic for all $x \in \mathbb{R}$). Next, recall that the sum of analytic functions is itself an analytic function; this is true because the sum of convergent sequences converges to the sum of the sequences' limits, and the Taylor series terms merely add, so if each of the constituent functions $f_i(x)$ have Taylor series equal to the function itself, then the sum of the functions will be equal to the sum of the Taylor series. Thus, because h is the sum of two analytic functions,e^{-x} and $1 + 2x$, h is also analytic.

(b) Use a power series expansion (the method of undetermined coefficients) to find the first four non-vanishing coefficients of the solution y to $l(y) = h$ with $y(0) = 4$.

Consider our first order linear ODE $a(x)y' + b(x)y = h(x)$, where $a(x) = x^2 + 1$, $b(x) = 3x$ and $h(x) = e^{-x} + 2x + 1$. We first Taylor Series expand all functions:

$$\begin{aligned}
a(x) &= 1 + 0x + x^2 \\
b(x) &= 0 + 3x \\
e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\
h(x) &= 1 + 2x + e^{-x} \\
&= 2 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\
y(x) &= y_0 + y_1 x + y_2 x^2 + y_3 x^3 + y_4 x^4 + ... \\
b(x)y(x) &= (3x)(y_0 + y_1 x + y_2 x^2 + y_3 x^3 + y_4 x^4 + ...) \\
y'(x) &= y_1 + 2y_2 x + 3y_3 x^2 + 4y_4 x^3 + 5y_5 x^4 + ... \\
a(x)y'(x) &= (1 + x^2)(y_1 + 2y_2 x + 3y_3 x^2 + 4y_4 x^3 + 5y_5 x^4 + ...)
\end{aligned}$$

We solve for the first-four non-zero coefficients, considering terms in order of increasing power of x:

$$\begin{aligned}
y(0) = 4 &\Rightarrow y_0 = 4 \\
y_1 + 0 = 2 &\Rightarrow y_1 = 2 \\
2y_2 + 3y_0 = 1 &\Rightarrow y_2 = -\frac{11}{2} \\
y_1 + 3y_3 + 3y_1 = \frac{1}{2} &\Rightarrow y_3 = -\frac{5}{2}
\end{aligned}$$

4. (1+1+1+2 points) **True or False?** Indicate whether the following claims are true or false. **Justify in each case your answer.**

(a) The function $x \mapsto \sin\left(x - \frac{\pi}{2}\right)$ is analytic at $x_0 = 0$.
Hint: Recall Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ *for all* $x \in \mathbb{R}$.

True. As shown on the homework and earlier on this exam, we know that e^{ix} and e^{-ix} are analytic over all $x \in \mathbb{R}$, and as previously argued, the sum of analytic functions is analytic. I derive a formula for sin() as the sum of the two exponentials. Euler's formula states that $e^{ix} = \cos(x) + i\sin(x) \Rightarrow e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$, where the last equality follows becaues cosine is an even function and sine is an odd function. Thus,

$$\begin{aligned} e^{ix} - e^{-ix} &= \cos(x) + i\sin(x) - \cos(x) + i\sin(x) \\ \frac{e^{ix} - e^{-ix}}{2i} &= \sin(x) \end{aligned}$$

However, we're interested in whether $\sin(x - \frac{\pi}{2})$ is analytic at $x_0 = 0$. This is equivalent of asking whether $\sin(x)$ is analytic at $\frac{\pi}{2}$, which is true since we know that $\sin(x)$ is analytic over all $x \in \mathbb{R}$. conclude that $\sin(x - \frac{\pi}{2})$ is analytic at $x_0 = 0$.

(b) Let $a, b, c \in \mathbb{R}$ and let $l(y) = ay'' + by' + cy$ for y in the space V of smooth functions. Then, $\ker(l)$ has dimension 2.

False. Suppose $a = 0, b \neq 0, c \neq 0$. Then the second order linear operator $l(y) = ay'' + by' + cy$ becomes a first order linear operator $l(y) = by' + cy$, and we know that the kernel of a first order linear operator is 1-dimensional; to see why, recall that for the first order linear operator,

$$l(y) = 0 \Leftrightarrow ay_n' + by_n = 0 \Leftrightarrow \frac{y_n'}{y_n} = \frac{b}{a} \Leftrightarrow \text{span}(\exp\Big(-\int_{x_0}^{x} \frac{b(t)}{a(t)}dt\Big))$$

If $a \neq 0$, then yes, the kernel of the second order linear differential operator would have dimension 2.

(c) If the Wronskian $W(f, g) = fg' - f'g$ of two smooth functions f and g in $\mathbb{R}$ vanishes at each point in $\mathbb{R}$, then f and g are linearly dependent.

False. We saw this on Worksheet 3, Problem 4 c. The point was that $W(f, g) \neq 0 \Rightarrow f, g$ are linearly independent, but the inverse is not true. The example we saw was

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad g(x) = \begin{cases} 0 & x \geq 0 \\ e^{-1/x^2} & x < 0 \end{cases}$$

These functions are linearly independent, but the Wronskian $W = f'g - fg'$ is zero regardless of whether $x < 0, x = 0, x > 0$:

$$\begin{aligned} x < 0 &: W = (0)g' - (0)g = 0 \\ x = 0 &: W = 0(0) - 0(0) = 0 \\ x > 0 &: W = f'(0) - f(0) = 0 \end{aligned}$$

But I'm not sure whether these functions are smooth since the derivative near 0 is wonky. I propose two other functions that are clearly smooth:

$$f(x) = x^2 \qquad g(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

Note that the two functions are linearly independent since no constant can make $f = g$ for all x. The Wronksian vanishes:

$$x \geq 0 : W = 2xx^2 - x^2 2x = 0$$
$$x < 0 : W = 2x(-x^2) - x^2(-2x) = 0$$

(d) Let $C^\infty(I, \mathbb{R})$ be the vector space of smooth functions mapping $I = [0; 1]$ to $\mathbb{R}$. Then $d : C^\infty(I; \mathbb{R}) \times C^\infty(I; \mathbb{R}) \to \mathbb{R}$ defined by

$$d(\psi, \phi) = \int_0^1 dt \, \left(\psi'(t) - \phi'(t)\right)^2$$

defines a metric on $C^\infty(I, \mathbb{R})$. Here, f' denotes the derivative $\frac{d}{dx} f$ of $f \in C^\infty(I; \mathbb{R})$.

True. Recall that a metric needs to meet 3 properties: (1) $d(\psi, \phi) \geq 0$ and $d(\psi, \phi) = 0 \Leftrightarrow \psi = \phi$, (2) symmetry i.e. $d(\psi, \phi) = d(\phi, \psi)$ and (3) triangle inequality i.e. $d(\psi, \phi) \leq d(\psi, \chi) + d(\chi, \phi)$. I prove each in turn:

i. Non-negativity: Since both ψ, ϕ map to the reals, their derivatives are also real. Because their derivatives are real, $\forall t \in [0, 1], (\psi'(t) - \phi'(t))^2 \geq 0$. Since each posssible value of the integrand is non-negative, the total integral must also be non-negative. In order for the integral to be zero, $\psi'(t) - \phi'(t)$ must hold for all $t \in [0, 1]$, meaning $d(\psi, \phi) = 0 \Leftrightarrow \psi = \phi$.

ii. Symmetry: $d(\psi, \phi) = \int_0^1 dt \, (\psi'(t) - \phi'(t))^2 = \int_0^1 dt \, (\phi'(t) - \psi'(t))^2 = d(\phi, \psi)$

iii. Triangle Inequality: Rather than arduously proving that the triangle inequality holds as we did in the homework, I'll point out that we can view the integral as the limit of a Riemann sum i.e.

$$d(\psi, \phi) = \int_0^1 dt \, (\phi'(t) - \psi'(t))^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\psi'(\frac{n}{N}) - \phi'(\frac{n}{N}))^2$$

Here, we see that the metric is the integral equivalent of the d_2 metric, and since we know that the d_2 metric obeys the triangle inequality, this $d(\cdot, \cdot)$ also obeys the triangle inequality.