Week 7

Topics: Conclusion of local existence & uniqueness for ODE; introduction to inner product spaces

Review from Week 6:

- Our goal is to solve the general first order ODE initial value problem

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

- Our strategy is to find a continuous function $\mathbf{y} \in C((x_0 - \epsilon; x_0 + \epsilon); \mathbb{R}^n)$ such that

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \mathbf{y}(s))\, ds$$

- **Metric spaces.** The appropriate setting to generalize our fixed point strategy is that of *metric spaces*. A metric space (M, d) is a set M together with a function $d : M \times M \to [0; \infty)$ such that for all $x, y, z \in M$ we have that

$$\begin{aligned} &1)\ d(x,y) \geq 0 \ \text{ and } \ d(x,y) = 0 \leftrightarrow x = y, \\ &2)\ d(x,y) = d(y,x), \\ &3)\ d(x,y) \leq d(x,z) + d(z,y). \end{aligned}$$

- **Cauchy Sequences and Convergence**. We have the notions of Cauchy sequences and convergent sequences in metric spaces. $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M if for all $\varepsilon > 0$ there exists $N = N_\varepsilon > 0$ such that

$$d(x_n, x_m) \leq \varepsilon$$

for all $n, m \geq N$. The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent if there exists some $x \in M$ such that for all $\varepsilon > 0$ there exists $N = N_\varepsilon > 0$ such that

$$d(x_n, x) \leq \varepsilon$$

for all $n \geq N$.

- **Completeness.** A metric space (M, d) is called complete if every Cauchy sequence is also a convergent sequence. The space of real numbers $\mathbb{R}$ is complete (with the usual metric). Also, the spaces of continuous functions $(C[a; b]; \mathbb{R}^n), d_\infty)$ are complete.

1. **The convergence criterion: Banach's fixed point theorem.** Let (M, d) be a complete metric space with $M \neq \emptyset$. Suppose $T : M \to M$ is a map with the property that

$$(*) \qquad d(T(x), T(y)) \leq cd(x, y).$$

for all $x, y \in M$ and a constant $0 \leq c < 1$. We call such a map a *contraction*. Prove that T has a unique fixed point $x \in M$. This means that there exists a unique element $x \in M$ such that $T(x) = x$. To prove this fact, proceed as follows:

(a) Pick some fixed $x_1 \in M$ (it is here where we use that $M \neq \emptyset$). Define $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$. Prove that the resulting sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M, due to the contraction property of T.

We use the triangle inequality and iteratively $(*)$ so that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m-2}) + \dots + d(x_{n+1}, x_n) \\ &\leq c^{n+m-2} d(x_2, x_1) + c^{n+m-3} d(x_2, x_1) + \dots + c^{n-1} d(x_2, x_1) \\ &\leq c^{n-1} d(x_2, x_1) \underbrace{(1 + c + c^2 + \dots + c^{m-1})}_{\leq \sum_{j=0}^{\infty} c^j = \frac{1}{1-c}} \\ &\leq \frac{d(x_2, x_1)}{1-c} c^{n-1} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

(b) Conclude that the sequence from the previous part has a limit $x \in M$.

(M, d) is, by assumption, complete. This means every Cauchy sequence has a limit in M. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy by (a), there exists $x \in M$ such that $\lim_{n\to\infty} x_n = x$

(c) Prove that x is a fixed point of T. Prove that it is the unique fixed point of T.

We want to prove that $T(x) = x \iff d(T(x), x) = 0$, and

$$\begin{aligned} d(T(x), x) &\leq d(T(x), x_n) + d(x_n, x) = d(T(x), T(x_{n-1})) + d(x_n, x) \\ &\leq c\, d(x_{n-1}, x) + d(x_n, x) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

$\Rightarrow d(T(x), x) = 0$

For the uniqueness, assume $T(x) = x$, $T(y) = y$, then

$d(x, y) = d(T(x), T(y)) \leq c\, d(x, y)$ with $c < 1$

$\Rightarrow \quad d(x, y) = 0 \iff x = y$

2. More on metric spaces: closed sets. Let (M, d) be a metric space. A subset $C \subset M$ is called *closed* in M if for every sequence $(x_n)_{n\in\mathbb{N}}$ s.t. $x_n \in C$ for all $n \in \mathbb{N}$ and s.t. $\lim_{n\to\infty} x_n = x \in M$, we have that $x \in C$. In words: C contains all of its limit points (it is closed in the sense that limits can not escape the set).

(a) Show that the whole space M is a closed subset of M.

This follows directly from the definition: let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M with $\lim_{n\to\infty} x_n = x \in M$. Then the limit $x \in M$ lies in M, so M is closed.

(b) Suppose M is complete and that $C \subset M$ is closed. Show that the metric space (C, d) is complete.

Suppose $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (C, d). Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy in (M, d), because $C \subset M$. Since M is complete, there exists $x \in M$ with $x = \lim_{n\to\infty} x_n$. But C is closed so that $x \in C \subset M$. This shows that (C, d) is complete.

(c) As a concrete example, consider the metric space $(\mathbb{R}^n, d_2)$ from problem set **4**. Show that the set $B_\delta(\mathbf{y}_0) = \{\mathbf{y} \in \mathbb{R}^n : d_2(\mathbf{y}_0, \mathbf{y}) \leq \delta\}$ is a closed subset of $(\mathbb{R}^n, d_2)$. We call $B_\delta(\mathbf{y}_0)$ the *closed ball of radius δ around $\boldsymbol{y}_0 \in \mathbb{R}^n$*.

This is proved on problem set 5 (for the closed ball in a general metric space).

3. Existence argument for 1st order ODE. Let $f \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ be continuous, $n \in \mathbb{N}$, and let $(x_0, \mathbf{y}_0) \in \mathbb{R} \times \mathbb{R}^n$. We collect our previous results and argue that there exists a local solution to the initial value problem

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

In order to simplify the argument, we will assume that there exists some $L > 0$ such that

$$d_2(f(x, \mathbf{y}_1), f(x, \mathbf{y}_2)) \le L d_2(\mathbf{y}_1, \mathbf{y}_2).$$

In addition to that, we need to define some further objects (see also problem set **5**):

- let $\delta > 0$ and consider the closed and bounded set $K = [x_0 - \delta; x_0 + \delta] \times B_\delta(\mathbf{y}_0) \subset \mathbb{R} \times \mathbb{R}^n$. Denote by $M < \infty$ the maximum of f restricted to K.
- choose some $\varepsilon_0 > 0$ such that $\varepsilon_0 < \min(\delta, \frac{\delta}{2L}, \frac{\delta}{2M})$ and let $I = [x_0 - \varepsilon_0; x_0 + \varepsilon_0]$.
- on problem set **5**, you will prove that

$$A = \{g \in C(I; \mathbb{R}^n) : \sup_{x \in I} d_2(g(x), \mathbf{y}_0) \le \delta\}$$

is a closed subset of $C(I; \mathbb{R}^n)$. The pair (A, d_∞) is therefore a complete metric space.
- finally, define the map $T : A \to C(I; \mathbb{R}^n)$ by

$$\big(T(\phi)\big)(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \phi(s))\, ds$$

Prove that $T : A \to A$ and, moreover, that T is a contraction. Conclude from here that there exists a local solution $\mathbf{y} \in C(I; \mathbb{R}^n)$ to the above initial value problem.

1) $\underline{\underline{T : A \to A:}}$ fix $x \in I$ and let $\phi \in A$, then

$$d_2(T(\phi)(x), y_0) = d_2\Big(y_0 + \int_{x_0}^{x} ds\, f(s, \phi(s)), y_0\Big)$$

$$= \Big[\sum_{i=1}^{n} \Big(\int_{x_0}^{x} ds\, f_i(s, \phi(s))\Big)^2\Big]^{1/2} \quad \Big|\ \text{apply triangle inequality for integrals}$$

$$\le \int_{x_0}^{x} ds \sqrt{\sum_{i=1}^{n} f_i(\underset{\in I}{s}, \underset{\in B_\delta(y_0)}{\phi(s)})^2} \le |x - x_0| \cdot M \le \varepsilon \cdot M \le \frac{\delta}{2}$$

$$\Rightarrow \quad \sup_{x \in I} d_2(T(\phi)(x), y_0) \le \delta \quad \Rightarrow \quad T : A \to A$$

2) <u>T satisfies contraction property</u>: let $x \in I$, $\phi_1, \phi_2 \in A$, then

$$
\begin{aligned}
&d_2(T(\phi_1)(x), T(\phi_2)(x)) \\
&= \left\{ \sum_{i=1}^{n} \left(\int_{x_0}^{x} ds \left[f(s, \phi_1(s)) - f(s, \phi_2(s)) \right] \right)^2 \right\}^{1/2} \\
&\leq \int_{x_0}^{x} ds \underbrace{d_2(f(s, \phi_1(s)), f(s, \phi_2(s)))}_{\leq d_2(\phi_1(s), \phi_2(s)) \leq d_\infty(\phi_1, \phi_2)} \leq |x - x_0| \, d_\infty(\phi_1, \phi_2) \cdot L \\
&\leq \varepsilon \cdot L \, d_\infty(\phi_1, \phi_2) \leq \delta \, d_\infty(\phi_1, \phi_2) \quad \text{so that}
\end{aligned}
$$

$d_\infty(T(\phi_1), T(\phi_2)) \leq \delta \, d_\infty(\phi_1, \phi_2)$. If we choose $\delta < 1$, $T: A \to A$ fulfills assumptions of the Banach fixed point theorem and hence, we find $\phi \in A$ such that

$$\phi(x) = T(\phi)(x) = y_0 + \int_{x_0}^{x} ds \, f(s, \phi(s)) \quad \forall x \in I$$

$\Rightarrow \quad \phi \in C^1(I; \mathbb{R}^n)$ solves the IVP

$$\begin{cases} \phi'(x) = f(x, \phi(x)) \\ \phi(x_0) = y_0 \end{cases}$$

4. Uniqueness argument for 1st order ODE. In the previous problem, we proved the existence of a local solution $\mathbf{y} \in C(I; \mathbb{R}^n)$ to the initial value problem

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0 \end{cases}$$

for a suitable interval $I = [x_0 - \varepsilon_0; x_0 + \varepsilon_0]$.

(a) Recall the definition of the set $A \subset C(I; \mathbb{R}^n)$ from the previous problem. Why is the solution $\mathbf{y} \in C(I; \mathbb{R}^n)$ that we found the unique solution in the metric space (A, d_∞)?

Because this is a direct consequence of the Banach fixed point theorem!

(b) The set $A \subset C(I; \mathbb{R}^n)$ is quite a particular subset. Apart from our solution $\mathbf{y} \in C(I; \mathbb{R}^n)$, can there be any other solution $\mathbf{z} \in C(I; \mathbb{R}^n)$ with $\mathbf{z} \notin A$, that solves the above intial value problem?

Suppose there exists such a solution $z \in C(I; \mathbb{R}^n)$, $z \notin A$. Then $\{x \in I : d_2(z(x), y_0) > \delta\} \neq \emptyset$. Set $x_1 = \inf \{x \in I : d_2(z(x), y_0) > \delta\}$.

Then $d_2(z(x_1), y_0) = \delta$ (this follows by continuity; it is clear that $d_2(z(x_1) y_0) \geq \delta$ and if we assume $d_2(z(x_1), y_0) > \delta$, the continuity of z implies that there must exist $\tilde{x}_1 < x_1$ such that still $d_2(z(\tilde{x}_1), y_0) > \delta$, but this contradicts the definition of x_1).

Since z solves the ODE, we have

$$\delta = d_2(z(x_1), y_0) = d_2\left(y_0 + \int_{x_0}^{x} ds\, f(s, z(s)), y_0\right)$$

$$\underset{\substack{z(s) \in B_\delta(y_0) \\ \text{for } s \leq x_1}}{\leq} |x - x_0| \cdot M \leq \frac{\delta}{2} \quad ↯$$

$\Rightarrow$ every solution $z \in C(I; \mathbb{R}^n)$ is also in $z \in A$.