MATH110 Spring 2020 HW3

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Problem 1

a

Consider

$$
\begin{aligned}
0 &= (1-x^2)y'' - 2xy + n(n+1)y \\
&= y'' + \frac{2x}{(1-x^2)}y + \frac{n(n+1)}{(1-x^2)}
\end{aligned}
$$

Abel's Identity tells us that the Wronskian is

$$
\begin{aligned}
W &\propto \exp(-\int_{x_0}^{x} \frac{2x}{(1-x^2)} dx) \\
&\propto \exp(-\log(1-x^2) + \log(1-x_0^2)) \\
&\propto \exp(-\log(1-x^2)) \\
&\propto \frac{1}{1-x^2}
\end{aligned}
$$

where the proportionality is up to a multiplying constant.

b

Let $P_0 = 1$ and $P_1 = x$. We first find Q_0.

$$
\begin{aligned}
P_0 Q_0'(x) - P_0' Q_0(x) &= \frac{1}{1-x^2} \\
(1)Q_0'(x) - (0)Q_0(x) &= \\
Q_0(x) &= Q_0(x_0) + \int_{t=x_0}^{x} dt \, \frac{1}{1-t^2} \\
&= Q_0(x_0) + \int_{t=x_0}^{x} dt \, \frac{1-t+t}{1-t^2} \\
&= Q_0(x_0) + \int_{t=x_0}^{x} dt \, \frac{1+t}{1-t^2} - \int_{t=x_0}^{x} dt \, \frac{t}{1-t^2} \\
&= Q_0(x_0) - \log(1-x)|_{x_0}^{x} + \frac{1}{2}\log(1-x^2)|_{x_0}^{x} \\
&= c_1 - \frac{1}{2}\log(1-x) + \frac{1}{2}\log(1+x)
\end{aligned}
$$

where c_1 depends on initial conditions. Next we find Q_1:

$$
\begin{aligned}
P_1 Q_1'(x) - P_1' Q_1(x) &= \frac{1}{1-x^2} \\
x Q_1' - Q_1 &=
\end{aligned}
$$

I choose to solve this using an integrating factor. I want a function $f(x)$ such that:

$$
\begin{aligned}
\frac{1}{f}(Q_1 f)' = 0 &= Q_1' - \frac{1}{x} Q_1 \\
Q_1' f + Q_1 f' &= f Q_1' - \frac{f}{x} Q_1 \\
\frac{f'}{f} &= -\frac{1}{x} \\
f(x) &= f(x_0) + \frac{1}{x} - \frac{1}{x_0}
\end{aligned}
$$

Using the integrating factor:

$$
\begin{aligned}
x Q_1' - Q_1 &= \frac{1}{1-x^2} \\
(Q_1 f)' &= \frac{f}{x(1-x^2)} \\
Q_1 f &= Q_1(x_0) f(x_0) + \int_{t=x_0}^{t=x} dt\, \frac{f(t)}{t(1-t^2)} \\
Q_1 &= \frac{f(x_0)}{f(x)} Q_1(x_0) + \frac{1}{f(x)} \int_{t=x_0}^{t=x} dt\, \frac{f(t)}{t(1-t^2)} \\
&= c_1 x - \frac{1}{2} x \log(1-x) + \frac{1}{2} x \log(1+x) - 1
\end{aligned}
$$

Problem 2

I make a three part argument: (1) at least n unique functions are in $\ker(l)$, (2) these n functions are linearly independent, and (3) $\dim(\ker(l)) = n$. I'll use the term "original IVP" to refer to the given n-th order initial value problem.

1. At least n unique functions are in $\ker(l)$. To see this, consider functions $f_0, ..., f_{n-1}$ such that $f_i^{(i)}(x_0) = 1$ (i.e. the ith derivative at x_0 is 1) and $f_i^{(\neq i)}(x_0) = 0$ (i.e. all other derivatives at x_0 are 0). By assumption, for function $f_i(x)$ and associated initial values, there is a unique solution, and $f_i(x) \in \ker(l)$ because $l(f_i) = 0$. Thus we conclude that there are n functions in $\ker(l)$.

2. These n functions are linearly independent. We can see that the set of functions $\{f_i\}_{i=1}^n$ are linearly independent by considering the Wronskian W. If we evaluate the matrix of functions and their derivatives at x_0, the resulting matrix is the identity matrix (by construction of the initial conditions) and thus $W \neq 0 \Rightarrow \{f_i\}_{i=1}^n$ are linearly independent.

3. $\dim(\ker(l)) = n$. Consider a possible element $y \in \ker(l)$ with initial values $y(x_0), y'(x_0), ..., y^{(n-1)}(x_0)$. Is it possible that this function exists outside $\text{span}(\{f_i\}_{i=1}^n)$? No. Thanks to the well-chosen initial conditions of the set of functions, we can express y as a linear combination of the n functions: $y = y(x_0) f_0 + y'(x_0) f_1 + ... + y^{(n-1)}(x_0) f_{n-1}$. Thus any y can be written as a linear combination of the functions and thus $\dim(\ker(l)) = \text{span}(\{f_i\}_{i=1}^n) = n$.

Problem 3

Consider $l(y) = y'' + y = 0$. At $x = 0$, $y''(0) = -y(0)$. Thus, for given 0th and 2nd derivative initial conditions that are not negatives of one another e.g. $y(0) = 0$ and $y''(0) = 1$, the system is unsolvable. But for given 0th and 1st derivative initial conditions, we know that the system is always solvable..

Problem 4

Consider $y(x) = x^\alpha \Rightarrow y' = \alpha x^{\alpha-1} \Rightarrow \alpha(\alpha-1)x^{\alpha-2}$. Then

$$\begin{aligned} l(y) = 0 &= x^2\alpha(\alpha-1)x^{\alpha-2} + x\alpha x^{\alpha-1} - 1x^\alpha \\ &= [\alpha^2 - \alpha + \alpha - 1]x^\alpha \\ &= (\alpha+1)(\alpha-1)x^\alpha \end{aligned}$$

Since $\forall \alpha \in \mathbb{R}$, $x^\alpha \neq 0$, we have two solutions: $\alpha = \pm 1$. To check that the two functions $y_1(x) = x^1$ and $y_2(x) = x^{-1}$ span the kernel of l, we check that the two are linearly independent using the Wronskian:

$$W = y_1 y_2' - y_1' y_2 = (x^1)(-x^{-2}) - (1)(x^{-1}) = -x^{-1} - x^{-1} = -2x^{-1} \neq 0$$

Since the Wronskian is non-zero, we conclude that the two functions $y_1(x) = x^1$ and $y_2(x) = x^{-1}$ span the kernel of l.

Problem 5

a

We first solve for r:

$$0 = l(y) = (r^2 + 4r + 5)e^{rx} \Rightarrow r = -2 \pm i$$

The two functions $y_1(x) = e^{(-2+i)x}$ and $y_2(x) = e^{(-2-i)x}$ are linearly independent as shown by the non-zero Wronskian.

$$W = y_1 y_2' - y_1' y_2 = (-2 + i + 2 + i)e^{-4x} = 2ie^{-4x} \neq 0$$

The null solution is therefore $y_n(x) = c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x}$. Solving for initial conditions:

$$\begin{aligned} y(0) = 1 &= c_1 + c_2 \\ y'(0) = 0 &= c_1(-2+i) + c_2(-2-i) \\ &= (1 - c_2)(-2+i) - 2c_2 - ic_2 \\ 2ic_2 &= -2 + i \\ c_2 &= \frac{1}{2} + i \\ c_1 &= \frac{1}{2} - i \end{aligned}$$

Thus the null solution to this initial value problem is $y_n(x) = (\frac{1}{2} - i)e^{(-2+i)x} + (\frac{1}{2} + i)e^{(-2-i)x}$

b

$$0 = l(y) = (r^2 - 4r + 3)e^{rx} \Rightarrow r = 3, 1$$

The two functions $y_1(x) = e^{3x}$ and $y_2(x) = e^x$ are linearly independent as shown by the non-zero Wronskian:

$$W = y_1 y_2' - y_1' y_2 = 3e^{3x}e^x - e^{3x}e^x = 2e^{3x}e^x \neq 0$$

The null solution is therefore $y_n(x) = c_1 e^{3x} + c_2 e^x$. Solving for initial conditions:

$$\begin{aligned}
y(0) = 3 &= c_1 + c_2 \\
y'(0) = 1 &= 3c_1 + c_2 \\
&= 3(3 - c_2) + c_2 \\
&= 9 - 3c_2 + c_2 \\
-8 &= -2c_2 \\
4 &= c_2 \\
-1 &= c_1
\end{aligned}$$

The null solution to this initial value problem therefore $y_n(x) = -e^{3x} + 4e^x$.

Problem 6

a

Note that $l_1 l_2 = D^2 + DM_{x+1} + M_x D + M_{2x+1}$ and $l_2 l_1 = D^2 + DM_x + M_{x+1} D + M_{2x+1}$. We then show that $DM_{x+1} + M_x D = DM_x + M_{x+1} D$:

$$\begin{aligned}
(DM_{x+1} + M_x D)(y) &= D(xy + y) + M_x y' \\
&= y + 2xy' + y' \\
(DM_x + M_{x+1} D)(y) &= D(xy) + M_{x+1} y' \\
&= y + 2xy' + y'
\end{aligned}$$

We conclude that l_1 and l_2 commute.

b

We start by noting that the linear operator $l(y)$ can be written as $l_1 l_2(y)$:

$$\begin{aligned}
l_1 l_2(y) &= (D^2 + DM_{x+1} + M_x D + M_x M_{x+1})(y) \\
&= y'' + D(xy + y) + M_x y' + M_x(xy + y) \\
&= y'' + y + xy' + y' + xy' + x^2 y + xy \\
&= y'' + (2x + 1)y' + (x^2 + 2x + 1)y \\
&= l(y)
\end{aligned}$$

We can find a basis for the kernel of the operator l by using part a. We specifically look for functions in the kernels of l_1 and l_2:

$$\begin{aligned} 0 &= l_1(y) \\ &= (D + M_x)y \\ &= y' + xy \end{aligned}$$

The solution to this linear, first-order ODE is $y_1(x) = e^{-\frac{1}{2}x^2}$. We repeat for l_2:

$$\begin{aligned} 0 &= l_2(y) \\ &= (D + M_{x+1})y \\ &= y' + (x+1)y \end{aligned}$$

The solution to this linear, first order ODE is $y_2(x) = e^{-\frac{1}{2}x(x+2)}$. These two functions are linearly independent (exercise left to the grader) and thus form a basis of the kernel of l (although admittedly I haven't shown that the kernel of l is two dimensional):

$$l(c_1 e^{-\frac{1}{2}x^2} + c_2 e^{-\frac{1}{2}x(x+2)}) = 0$$