# MATH110 Spring 2020 HW7

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## Problem 2

Define $f(x) = x^{-r}$. Consider $r = \frac{1}{2}$.

$$\int_0^1 (x^{-1/2})^2 dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{x} dx = \lim_{\epsilon \to 0} \log x \Big|_\epsilon^1 = \lim_{\epsilon \to 0} \log(1) - \log(\epsilon) = \infty$$

Thus we conclude $f(x) = x^{-1/2} \notin L^2$. Next, we consider $r \in (0, 1/2) \Rightarrow 1 - 2r > 0$:

$$\int_0^1 (x^{-r})^2 dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-2r} dx = \lim_{\epsilon \to 0} \frac{1}{1-2r} x^{1-2r} \Big|_\epsilon^1 = \frac{1}{1-2r}(1 - \lim_{\epsilon \to 0} \epsilon^{>0}) < \infty$$

Thus we conclude $\forall 0 < r < 1/2, f(x) = x^{-r} \in L^2$. Finally, we consider $r > 1/2 \Rightarrow 1 - 2r < 0$:

$$\int_0^1 (x^{-r})^2 dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-2r} dx = \frac{1}{1-2r}(1 - \lim_{\epsilon \to 0} \frac{1}{\epsilon^{>0}}) = \infty$$

And we finally conclude that $\forall r > 1/2, f(x) = x^{-r} \notin L^2$.

## Problem 3

Define $f(x) = x^{-r}$. Let $r = 1/2$.

$$\int_1^\infty (x^{-1/2})^2 dx = \lim_{\epsilon \to \infty} \int_1^\epsilon \frac{1}{x} dx = \lim_{\epsilon \to \infty} (\log(\epsilon) - \log(1)) = \infty$$

Thus we conclude $f(x) = x^{-1/2} \notin L^2$. Next we consider $r > 1/2 \Rightarrow 1 - 2r < 0$:

$$\int_1^\infty (x^{-r})^2 dx = \lim_{\epsilon \to \infty} \int_1^\infty x^{-2r} dx = \frac{1}{1-2r}(\lim_{\epsilon \to \infty} \epsilon^{1-2r} - 1) < \infty$$

We conclude $\forall r > 1/2, f(x) = x^{-r} \in L$. Finally we consider $r \in (0, 1/2) \Rightarrow 1 - 2r > 0$:

$$\frac{1}{1-2r}(\lim_{\epsilon \to \infty} \epsilon^{1-2r} - 1) = \infty$$

And we finally conclude that $\forall r \in (0, 1/2), f(x) = x^{-r} \notin L^2$.

## Problem 4

We know from Problem 2 that $f \in L^2((0,1), dx)$ if $0 < r < 1/2$. We also know from Problem 3 that $f \in L^2((1, \infty), dx)$ if $r > 1/2$. We finally consider $L^2((0, \infty), dx)$.

$$\int_0^\infty x^{-2r} dx = \lim_{a \to 0} \lim_{b \to \infty} \frac{1}{1-2r}(b^{1-2r} - a^{1-2r})$$

In order for the first term to be finite, $1 - 2r < 0$, but in order for the second term to be finite, $1 - 2r > 0$. Since both cannot be true, we conclude that $\forall r > 0, f(x) = x^{-r} \notin L((0, \infty), dx)$.

# Problem 5

We showed in Problem Set 6 that $g(x) = \frac{1}{2}\log(\frac{1+x}{1-x}) \in L^2((-1,1),dx)$. We now consider $l(g)$, where $l(y) = -\frac{d}{dx}((1-x^2)\frac{dy}{dx})$

$$
\begin{aligned}
\frac{dg}{dx} &= \frac{d}{dx}\frac{1}{2}\log\frac{1+x}{1-x} \\
&= \frac{1}{2}\frac{d}{dx}\Big(\log(1+x) - \log(1-x)\Big) \\
&= \frac{1}{2}\Big(\frac{1-x}{1-x^2} + \frac{1+x}{1-x^2}\Big) \\
&= \frac{1}{1-x^2} \\
l(g) &= \frac{d}{dx}((1-x^2)\frac{dg}{dx}) \\
&= \frac{d}{dx}((1-x^2)\frac{1}{1-x^2}) \\
&= 0
\end{aligned}
$$

We then see that the integral of $l(g)^2$ must be finite i.e. $\int_{-1}^{1} l(g)^2 dx = \int_{-1}^{1} 0^2 dx = 0 < \infty$ and thus $l(g) \in L^2((-1,1),dx)$.

# Problem 6

## Show that $H^1$ is a function space.

To show that $H^1(\mathbb{R},dx)$ forms a function space, let $c_1, c_2 \in \mathbb{R}$ and let $f_1, f_2 \in H^1(\mathbb{R},dx)$ with distributional derivatives $\zeta_1, \zeta_2$ respectively.

$H^1(\mathbb{R},dx)$ is closed under scalar multiplication because $c_1 f_1 \in L^2$:

$$\int_{\mathbb{R}} (c_1 f_1(x))^2 dx = c_1^2 \int_{\mathbb{R}} f_1(x)^2 dx < \infty \Rightarrow c_1 f_1 \in L^2(\mathbb{R},dx)$$

and because $c_1 f_1 \in H^1$ since $c_1 f_1$ is guaranteed to have distributional derivative $c_1\zeta_1$:

$$\int_{\mathbb{R}} dx c_1 f_1(x)\psi'(x) = c_1 \int_{\mathbb{R}} dx f_1(x)\psi'(x) = -c_1 \int_{\mathbb{R}} dx \zeta_1 \psi'(x) = -\int_{\mathbb{R}} dx c_1 \zeta_1 \psi'(x)$$

$H^1(\mathbb{R},dx)$ is also closed under element-wise addition because $f_1 + f_2 \in L^2$:

$$\int_{\mathbb{R}} (f_1(x)+f_2(x))^2 dx = \int_{\mathbb{R}} f_1(x)^2 + 2f_1(x)f_2(x) + f_2(x)^2 dx \le \int_{\mathbb{R}} 2f_1(x)^2 + 2f_2(x)^2 dx < \infty \Rightarrow f_1 + f_2 \in L^2(\mathbb{R},dx)$$

and because $f_1 + f_2 \in H^1$ since $f_1 + f_2$ is guaranteed to have distributional derivative $\zeta_1 + \zeta_2$:

$$\int_{\mathbb{R}} dx (f_1 + f_2)\psi' = \int_{\mathbb{R}} dx f_1 \psi' + \int_{\mathbb{R}} dx f_2 \psi' = -\int_{\mathbb{R}} dx \zeta_1 \psi' + -\int_{\mathbb{R}} dx \zeta_2 \psi' = -\int_{\mathbb{R}} dx (\zeta_1 + \zeta_2)\psi'$$

Thus we conclude that $H^1(\mathbb{R},dx)$ forms a function space.

## Prove that if $\psi \in C^1 \cap L^2(\mathbb{R})$ and $\psi' \in L^2(\mathbb{R})$ , then $\psi' = \zeta$.

Because $\phi \in C^1$, we know that $\phi' \overset{\text{def}}{=} \frac{d}{dx}\phi$ exists. Consider $\psi \in C_c^\infty(\mathbb{R})$. Using integration by parts, I show that $\phi'$ meets the definition of a distributional derivative:

$$\int_{\mathbb{R}} dx\, \phi(x)\psi'(x) = \phi(x)\psi(x)|_\infty - \int_{\mathbb{R}} dx\, \phi'(x)\psi(x) = 0 - \int_{\mathbb{R}} dx\, \phi'(x)\psi(x) = -\int_{\mathbb{R}} dx\, \phi'(x)\psi(x)$$

## Explain why $\langle \psi, \phi \rangle_{H^1} = \int_{\mathbb{R}} dx \left[\phi(x)\psi(x) + \phi'(x)\psi'(x)\right]$ defines an inner product.

I show that $\langle \psi, \phi \rangle_{H^1}$ satisfies the three properties of being an inner product:

1. **Conjugate bilinear:**

$$\begin{aligned}
\langle c_1 f_1 + c_2 f_2, \psi \rangle_{H^1} &= \int_{\mathbb{R}} dx \left[(c_1 f_1 + c_2 f_2)\psi + (c_1 f_1 + c_2 f_2)'\psi'\right] \\
&= c_1 \int_{\mathbb{R}} dx \left[f_1 \psi + f_1' \psi'\right] + c_2 \int_{\mathbb{R}} dx \left[(f_2)\psi + f_2'\psi'\right] \\
&= c_1 \langle f_1, \psi \rangle_{H^1} + c_2 \langle f_2, \psi \rangle_{H^1} \\
\langle \phi, c_1 f_1 + c_2 f_2 \rangle_{H^1} &= \int_{\mathbb{R}} dx \left[\phi(c_1 f_1 + c_2 f_2) + \phi'(c_1 f_1 + c_2 f_2)'\right] \\
&= c_1 \int_{\mathbb{R}} dx \left[\phi f_1 + \phi' f_1'\right] + c_2 \int_{\mathbb{R}} dx \left[\phi f_2 + \phi' f_2'\right] \\
&= c_1 \langle \phi, f_1 \rangle_{H^1} + c_2 \langle \phi, f_2 \rangle_{H^1}
\end{aligned}$$

2. **Hermitian symmetric:**

$$\langle \phi, \psi \rangle_{H^1} = \int_{\mathbb{R}} dx \left[\phi(x)\psi(x) + \phi'(x)\psi'(x)\right] = \int_{\mathbb{R}} dx \left[\psi(x)\phi(x) + \psi'(x)\phi'(x)\right] = \langle \psi, \phi \rangle_{H^1}$$

3. **Positive definite:**

$$\langle \psi, \psi \rangle_{H^1} = \int_{\mathbb{R}} dx \left[\psi(x)^2 + \psi'(x)^2\right] \geq 0$$

   which is non-negative and equal to zero only if $\psi(x) = 0$ almost everywhere (implicitly handling the requirement that $\psi(x)' = 0$).

# Assuming that $L^2(\mathbb{R}, dx)$ is a Hilbert space, prove that $(H^1(\mathbb{R}, dx); \langle \cdot, \cdot \rangle_{H^1})$ is a Hilbert space.

Let $(\phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^1$ and let $(\phi_n')_{n\in\mathbb{N}}$ in $L^2$ be a sequence where $\phi_n'$ is the weak derivative of $\phi_n$. We first show that $(\phi_n)_n$ and $(\phi_n')_n$ are Cauchy in $L^2$. Since the sequence is Cauchy in $H^1$, we know that $\forall \epsilon > 0, \exists i \in \mathbb{N}$ such that $\forall j, k > i$

$$
\begin{aligned}
\epsilon &> ||\phi_j - \phi_k||^2_{H^1} \\
&> \langle \phi_j - \phi_k, \phi_j - \phi_k \rangle_{H^1} \\
&> \int_{\mathbb{R}} (\phi_j - \phi_k)^2 + (\phi_j' - \phi_k')^2 \\
&\Rightarrow \\
\epsilon &> \int_{\mathbb{R}} (\phi_j - \phi_k)^2 \\
\epsilon &> \langle \phi_j - \phi_k, \phi_j - \phi_k \rangle_{L^2} \\
\epsilon &> \int_{\mathbb{R}} (\phi_j' - \phi_k')^2 \\
\epsilon &> \langle \phi_j' - \phi_k', \phi_j' - \phi_k' \rangle_{L^2}
\end{aligned}
$$

and thus $(\phi_n)_n$ and $(\phi_n')_n$ are both Cauchy in $L^2$. Since Since $L^2$ is a Hilbert space, $L^2$ is complete, and thus both $(\phi_n)_n$ and $(\phi_n')_n$ converge in $L^2$. In the $L^2$ metric, define:

$$\phi \stackrel{\text{def}}{=} \lim_{n\to\infty} \phi_n \text{ and } \phi' \stackrel{\text{def}}{=} \lim_{n\to\infty} \phi_n'$$

We next show that $\phi'$ is the weak derivative of $\phi$. Choose $\psi \in C_c^\infty(\mathbb{R})$. Using integration by parts:

$$
\begin{aligned}
\int_{\mathbb{R}} dx \phi \psi' &= \int_{\mathbb{R}} dx \lim_{n\to\infty} \phi_n \psi' \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} dx \phi_n \psi' \\
&= \lim_{n\to\infty} \left[ \phi\psi|_\infty - \int_{\mathbb{R}} dx \phi_n' \psi \right] \\
&= \lim_{n\to\infty} - \int_{\mathbb{R}} dx \phi_n' \psi \\
&= \int_{\mathbb{R}} dx \lim_{n\to\infty} \phi_n' \psi \\
&= \int_{\mathbb{R}} dx \phi' \psi
\end{aligned}
$$

From $\phi' \in L^2$, we see that $\phi \in H^1$. Now we simply need to show that $\phi = \lim_{n\to\infty} \phi_n$ converges in the $H^1$ metric.

$$
\begin{aligned}
\lim_{n\to\infty} ||\phi - \phi_n||^2 &= \lim_{n\to\infty} \int_{\mathbb{R}} [dx(\phi - \phi_n)^2 + (\phi' - \phi_n')^2] \\
&= \int_{\mathbb{R}} [dx(\phi - \lim_{n\to\infty} \phi_n)^2 + (\phi' - \lim_{n\to\infty} \phi_n')^2] \\
&= \int_{\mathbb{R}} [dx(\phi - \phi)^2 + (\phi' - \phi')^2] \\
&\to 0
\end{aligned}
$$

We conclude that $H^1$ is complete and therefore a Hilbert space.

# Problem 7