PROBLEM SET 10

THANKS FOR ALL YOUR FEEDBACK AND ACTIVE PARTICIPATION!
THIS IS THE LAST MATH 110 PROBLEM SET, ENJOY!

Due: ***Wednesday, April 29, 6.00 PM***

Topics: orthonormal bases, Hermite polynomials, time–dependent Schrödinger equation

1. **Reading.** Chapter 5 in Holland's book.

2. **Hilbert space bases.** Suppose $\mathcal{H}$ is a Hilbert space and $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$. Suppose that $(\varphi_k)_{k\in\mathbb{N}}$ has the following property: if $\zeta \in \mathcal{H}$, then

$$\langle \zeta, \varphi_k \rangle = 0 \ \ \forall\, k \in \mathbb{N} \quad \Rightarrow \quad \zeta = 0 \in \mathcal{H}.$$

Prove that this property implies that $(\varphi_k)_{k\in\mathbb{N}}$ is a complete orthonormal basis of $\mathcal{H}$.

Hint: if $\psi \in \mathcal{H}$, explain why $\varphi = \sum_{k=0}^{\infty} \langle \psi, \varphi_k \rangle \varphi_k \in \mathcal{H}$. Then analyze $\psi - \varphi \in \mathcal{H}$.

3. **Hermite polynomials.** Consider the Hermite polynomials $(H_n)_{n\in\mathbb{N}_0}$ as defined in class (with the help of problem set 9). Recall that the Hermite polynomials are normalized in the sense that $H_n - x^n$ is a polynomial of degree strictly smaller than n, for each $n \in \mathbb{N}$ (if $n = 0$, then $H_0 = 1$). In other words, H_n has leading coefficient equal to 1. Prove the following properties of the Hermite polynomials.

   a) $(-1)^n e^{x^2/2} \partial_x^n (e^{-x^2/2})$ is a polynomial of degree n with leading coefficient 1. With the lecture, this concludes that $H_n(x) = (-1)^n e^{x^2/2} \partial_x^n (e^{-x^2/2})$ for every $n \in \mathbb{N}_0$.

   b) By considering $\partial_x (e^{-x^2/2} H_n)$, prove that $\partial_x H_n = x H_n - H_{n+1}$.

   c) Prove that $\langle \partial_x H_n, H_m \rangle = \langle H_n, H_{m+1} \rangle$ and $\langle H_n, H_{m+1} \rangle = 0$ if $m \neq n-1$ and $\langle H_n, H_{m+1} \rangle = \sqrt{2\pi} n!$ if $m = n-1$.

   d) By writing $\partial_x H_n = \sum_{k=0}^{n-1} c_k H_k$ (why can we do that?), conclude that

$$H_{n+1} = x H_n - n H_{n-1}$$

   for all $n \in \mathbb{N}$. Determine H_0 up to H_4 explicitly.

Hint: You may use without proof that $\int_{\mathbb{R}} e^{-x^2/2}\, dx = \sqrt{2\pi}$.

4. 1d Wave–Equation. Consider $L^2([0;L],dx)$ and the 1d wave–equation

$$\begin{cases} \partial_t^2 u_t = \omega^2 \partial_x^2 u_t, \\ (u_t)_{|t=0} = u_0 \in H^2([0;1];dx), \quad (\partial_t u_t)_{|t=0} = v_0 \in H^2([0;1];dx). \end{cases} \tag{1}$$

Let $T(y) = \partial_x^2 y$ with domain $D_T = \{\psi \in H^2([0;1];dx) : \psi(0) = \psi(L) = 0\}$. Recall that (1) simply corresponds to the vector-valued second order, linear differential equation $u'' = \omega\, T(u)$ and we are looking for D_T–valued solutions $[0;\infty) \ni t \mapsto u_t$. Find the solution to Eq. (1) by proceeding as follows.

a) Find a complete orthonormal eigenbasis of the self-adjoint operator T and determine its spectrum $\sigma(T) \subset \mathbb{R}$.

b) Given a fixed time $t \in [0;\infty)$, describe the general form of $u_t \in D_T$, the displacement of the wire at time $t \geq 0$.

c) Using the wave equation (1), derive an equation for the time-dependent coefficients in the basis expansion of $u_t \in D_T$. Find the general form of each coefficient by solving a linear, second order ODE.

d) Plug in the initial conditions $x \mapsto u(x,0) = u_0(x)$ and $x \mapsto (\partial_t u)(x,0) = v_0(x)$ to conclude the general form of the solution of the wave equation (1).

e) As an explicit example, compute the solution for $L = 1$, $\omega = 1$, initial displacement $u_0(x) = 1/4 - (x - 1/2)^2$ and initial velocity $v_0(x) = 0$.

f^*) A fun extra exercise is to display the time evolution graphically using your favorite computer program (MATLAB, mathematica, maple, etc.)!