Problem Set 9

Due: Th, April 23, 6.00 PM

Topics: Self-adjoint operators, L^2 spaces, Hermite's operator

1. **Reading.** Sections 2.8, 2.9 (up to Theorem 2.14), 4.9, 4.10, 4.11 in Holland's book.

2. **Self-adjoint Operators.** Consider the complex Hilbert space $\mathcal{H} = L^2([0;1];dx)$ and the Laplace operator $\ell = -\partial_x^2$ with domain
$$D_\ell = \{\psi \in H^2([0;1];dx) : \psi(0) = \psi(1) \text{ and } \psi'(0) = \psi'(1)\}.$$
Prove that ℓ is self-adjoint by proceeding as follows:

   a) Using integration by parts, prove that for every $\phi \in D_\ell, \psi \in D_{\ell^*}$, we have that
   $$0 = \partial_x\phi(0)\Big(\overline{\psi}(0) - \overline{\psi}(1)\Big) + \phi(0)\Big(\overline{\partial_x\psi}(1) - \overline{\partial_x\psi}(0)\Big).$$

   b) Now, given $\psi \in D_{\ell^*}$, we need to conclude that $\psi \in D_\ell$. Notice that the identity from the previous part holds true for ***every*** $\phi \in D_\ell$. Make smart choices to conclude that $\psi \in D_\ell$. This proves $D_\ell = D_{\ell^*}$ and hence that ℓ is self-adjoint.

3. **L^2–Spaces.** Consider the space $L^2([0;1]^2, dxdy)$, defined by
$$L^2([0;1]^2, dxdy) = \Big\{\psi : [0;1]^2 \to \mathbb{C} : \int_0^1 dx \int_0^1 dy\, |\psi(x,y)|^2 < \infty\Big\}.$$
Denote by $(\varphi_p)_{p\in\mathbb{Z}}$ our standard orthonormal basis of $L^2([0;1], dx)$, $\varphi_p(x) = e^{2\pi ipx}$. Using the basis elements φ_p, define a new sequence $(\varphi_p \otimes \varphi_q)_{p,q\in\mathbb{Z}}$ by
$$\varphi_p \otimes \varphi_q(x,y) = \varphi_p(x)\varphi_q(y) = e^{2\pi ipx}e^{2\pi iqy}$$
for all $x, y \in [0;1]$. Prove that for every $p, q \in \mathbb{Z}$, we have $\varphi_p \otimes \varphi_q \in L^2([0;1]^2, dx)$ and that $(\varphi_p \otimes \varphi_q)_{p,q\in\mathbb{Z}}$ is an orthonormal sequence in $L^2([0;1]^2, dxdy)$. Using the hint below, provide a semi–rigorous argument[1] why this sequence forms a complete orthonormal basis of $L^2([0;1]^2, dxdy)$. It is called the *tensor product basis.*

   Hint: It is a useful fact that if $(\zeta_k)_{k\in\mathbb{N}}$ is a complete orthonormal basis in a Hilbert space $\mathcal{H}$, and $\langle f, \zeta_k\rangle_{\mathcal{H}} = 0$ for all $k \in \mathbb{N}$, then $f = 0 \in \mathcal{H}$. Why is that?

3. **Hermite's Equation.** Read carefully Section 2.8 and have a look at Exercise **1** of section 2.9 in Holland's book: For fixed $n \in \mathbb{N}_0$, consider the 2nd order linear ODE
$$y'' - xy' + ny = 0. \tag{1}$$
Use the power series expansion method to prove that for every $n \in \mathbb{N}_0$, the differential equation (1) admits a *polynomial solution H_n of degree $n \in \mathbb{N}_0$* with the property that if n is even, the polynomial $x \mapsto H_n(x)$ has only non-zero coefficients for even powers of x and if n is odd, $x \mapsto H_n(x)$ has only non-zero coefficients for odd powers of x. Compute H_0, H_1 and H_2.

4. **Hermite's Operator** Prove that every polynomial is compatible with the Hermite operator $\ell(y) = -e^{x^2/2}\partial_x(e^{-x^2/2}\partial_x y)$, as defined in class.

[1] Semi–rigorous means here that there are some details from Lebesgue integration that we, strictly speaking, do not know. Nevertheless, that's not going to keep us from providing a strong, intuitive argument!