Week 2

Topics: recap linear algebra; variation of parameters; power series expansion; the Wronskian

Review from Week 1:

- **Integrating factor:** To solve $ay' + by = h$, multiply by

$$\mu(x) = \exp\left(\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right),$$

then LHS $= (\mu y)' = h =$ RHS, so solve by integrating and then dividing by μ.

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads

$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right).$$

1. Show that the function $\ell : V \to V$ given by the formula $\ell(y) = a(x)y' + b(x)y$ has the properties required for linearity, and the kernel of ℓ is a one-dimensional subspace of V. Find a vector (a function) $f \in V$ such that $\ker \ell = \text{span}\{f\}$.

2. Let V be the three-dimensional vector space of polynomials of degree no greater than 2, with basis $\mathfrak{B} = \{1, x, x^2\}$. Let ℓ be the linear differential operator

$$\ell(y) = (x+1)y' - 2y.$$

(a) Write down the matrix L that represents ℓ with respect to the basis $\mathfrak{B}$, and find a basis for the kernel of L.

(b) Then find, by algebraic methods, the general solution to $\ell(y) = -2x$.

(c) Is there any element $h(x) \in V$ for which $\ell(y) = h(x)$ cannot be solved?

3. Let V be the space of smooth functions and

$$\ell(y) = ay' + by.$$

for smooth functions a and b. The space of smooth functions is not finite-dimensional s.t. we cannot simply write down a matrix for ℓ, but from **Problem Set 1** we know how to find a vector (a function!) f that spans its kernel.

(a) Prove that if y_p is any **particular solution** to $\ell(y) = h$, then the general solution is $y = Cf + y_p$ for some constant C.

(b) Use this theorem and a bit of guesswork to find the general solution to

$$3xy' - y = \log x + 1.$$

4. Suppose that we want to solve $\ell(y) = h$ (where $\ell(y) = ay' + by$) and we have already found a function f that spans the kernel of ℓ. Use the variation of the parameters idea to find a particular solution to $\ell(y) = h$ (the expression that you get still depends on f, of course).

(a) Show that we can get a formula for the derivative g' and thereby (at least in principle) find a solution to the inhomogeneous equation $\ell(y) = h$.

(b) Apply this approach to the equation

$$xy' + 2y = x.$$

Notice that you could also solve this equation by using an integrating factor.

5. (a) Using the ratio test, find the radius of convergence R about $x_0 = 0$ for $x \mapsto f(x) = e^x$ and $x \mapsto f(x) = \log(1 + x)$.

(b) Explain why the function f with values $f(x) = \exp\left(-\frac{1}{x}\right)$ if $x \geq 0$ and $f(x) = 0$ otherwise is not analytic at $x_0 = 0$.

6. Use the method of undetermined coefficients to solve for the first four coefficients of the analytic solution of the following example from Holland's book:

$$(3x^2 + 1)y' - 2xy = x, \text{ with } y(0) = \frac{3}{2}.$$

7. Consider the space V of solutions to the equation

$$y'' = y'.$$

(a) Show that the functions $x \mapsto y_1(x) = 1$ and $x \mapsto y_2(x) = e^x$ are in the space V. Does this mean that $x \mapsto y(x) = 3 + 2e^x$ is also in the space V?

(b) Are the vectors y_1 and y_2 linearly dependent or independent?

(c) Show that $V = \text{span}\{1, e^x\}$, i.e. that any solution to $y'' = y'$ is of the form

$$y = c_1 y_1 + c_2 y_2 = c_1 + c_2 e^x,$$

for some scalars c_1, c_2.

(d) What is the dimension of the space V?

8. (a) Assume that y_1 and y_2 are linearly dependent. Prove that the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$. This implies that if $W(y_1, y_2)$ is not equal to the zero function, then y_1 and y_2 are linearly independent.

(b) Try this for the functions $x \mapsto e^x$ and $x \mapsto e^{-x}$. Before you do the computation, take a moment and ask yourself what result you expect.

(c) The criterion from (a) for linear independence is not necessary[1]. Indeed, check this for the two (smooth!) functions

$$y_1(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}, \qquad y_2(x) = \begin{cases} e^{-1/x^2} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

[1]For a converse statement, you may have a look at Chapter 3.1.3 in *Mathematics for Physics* by Stone & Goldbart.

Theorem 1 (Holland, Theorem 2.6)**.** *If* $W(y_1, \ldots, y_n)$ *is not the zero function on* $[c, d]$*, then the functions* $y_1, \ldots, y_n$ *are linearly independent on* $[c, d]$*.*

9. Prove it.