Problem Set 1

Topics: 1st order ODE; linear algebra

Due: Tuesday, February 11, 6:00 PM.

Assignments should be turned in to the course assistant's mailbox or through Canvas. You are encouraged to use LaTeX to typeset your problem set solutions. If you are new to LaTeX, the following may be helpful: https://www.overleaf.com/latex/templates/class-template-for-315-writeups/xwjydpcqvknp and https://www.overleaf.com/latex/learn/free-online-introduction-to-latex-part-1.

1. **Growing Interest Rates.** You accept a summer internship with a Big 3 auto firm, hoping to work on energy-efficient vehicles. Alas, you are assigned to the finance division. Your task is to study interest rates that start at 0 percent APR and grow linearly with time, while the rate of payment by the customer also grows linearly.

   So, if y is the loan balance, $y' = kxy - mx$ (here, k and m are positive real numbers). Solve this equation, and determine how $y(0), k$ and m must be related in order to make it impossible for the customer ever to pay off the loan balance.

2. **Linear Independence.** Let V be a real vector space. Recall from linear algebra that the vectors $v_1, v_2, v_3 \in V$ are called linearly independent if and only if

   $$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \quad \to \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

   Define $V = \text{span}\{1, x, x^2\}$ as the real subspace of polynomials on $\mathbb{R}$, spanned by the polynomials $e_1(x) = 1, e_2(x) = x, e_3(x) = x^2$. Prove that the vectors e_1, e_2, e_3 are linearly independent.

   Hint: Start with $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0$. Apply a suitable mathematical operation on both sides of the equation, possibly multiple times. Your goal is to deduce that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

3. **Infinite Dimensional Vector Spaces.** Argue why the real vector space $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ forms a real vector space that is not finite dimensional.

   Hint: Find a sequence of linearly independent vectors in $C(\mathbb{R})$. Then, assume that $C(\mathbb{R})$ is finite dimensional and find a contradiction.

4. **Finite-Dimensional Techniques.** Let V be the four-dimensional space spanned by the polynomials[1] $\{1, x, x^2, x^3\}$. Let $l : V \to V$ be the linear operator defined by $l(y) = (x+2)y' - 3y$. Write down the 4×4 matrix L that represents l relative to the given basis and use this to find the general solution to $l(y) = -x^2$. Also invent an $h(x) \in V$ for which $l(y) = h(x)$ has no solution.

   Remark: Recall the definition of the matrix representation of a linear map w.r.t. a given basis.

5. **Eigenvalue review.** Consider $l(y) = (x+1)y'$ as a linear operator on the space V spanned by $\{1, x, x^2\}$. Can you find a basis for V that consists of eigenvectors of l?

   Hint: Proceed as in Problem 3 and find first a matrix representation of l in terms of the given basis. Then recall from linear algebra how to find eigenvalues and eigenvectors of a matrix.

[1] As explained in the lecture, we use here and in the future some shorthand notation. By x^2, for instance, we actually mean the map $x \mapsto x^2$, etc. Contact us whenever the notation is not clear and/or confusing.

6. More on eigenvalues. Consider the sets $\ell^2(\mathbb{N})$ and $h^2(\mathbb{N})$ of sequences defined by

$$\ell^2(\mathbb{N}) = \Big\{(c_k)_{k\in\mathbb{N}} = (c_1, c_2, \dots) : c_k \in \mathbb{C}\ \forall k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |c_k|^2 < \infty\Big\},$$
$$h^2(\mathbb{N}) = \Big\{(c_k)_{k\in\mathbb{N}} = (c_1, c_2, \dots) : c_k \in \mathbb{C}\ \forall k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} k^4|c_k|^2 < \infty\Big\}$$

Verify that $\ell^2(\mathbb{N})$ and $h^2(\mathbb{N})$ form complex vector spaces if you equip them with a reasonable definition for addition and scalar multiplication of sequences. Define the operator $-\Delta : h^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $-\Delta(c_1, c_2, \dots, c_k, \dots) = (c_1, 4c_2, \dots, k^2c_k, \dots)$. Show that $-\Delta$ is a linear map. What are the eigenvalues of $-\Delta$?