Problem Set 7

Due: Th, April 9, 6.00 PM

Topics: L^2 spaces, compatibility check, weak derivatives, separability

1. **Reading.** Read sections 4.1 to 4.3 in Holland's book. Optional reading: sections 4.4 and 4.5, appendix 4.A (Sobolev functions can be identified with the set of absolutely continuous functions, in one dimension; the appendix talks about these functions).

2. **L^2 practice.** Define $f(x) = 1/x^r$ for $x \in (0;1)$. Prove that $f \in L^2((0;1);dx)$ if $0 < r < 1/2$ and $f \notin L^2((0;1);dx)$ if $r > 1/2$. What about $r = 1/2$?

3. **L^2 practice.** Define $f(x) = 1/x^r$ for $x \in [1;\infty)$. Prove that $f \in L^2((1;\infty);dx)$ if $r > 1/2$ and $f \notin L^2((1;\infty);dx)$ if $0 < r < 1/2$. What about $r = 1/2$?

4. **L^2 practice.** Let $f(x) = 1/x^r$ for $x \in (0;\infty)$. For which values $r > 0$ does f belong to $L^2((0;1);dx)$, $L^2((1;\infty);dx)$ and $L^2((0;\infty);dx)$?

5. **Compatibility.** Consider the function g defined by $g(x) = \frac{1}{2}\log\big((1+x)/(1-x)\big)$ and define the linear operator ℓ through
$$\ell(y) = -\frac{d}{dx}\Big((1-x^2)\frac{dy}{dx}\Big).$$
Show that $\ell(g) \in L^2((-1;1);dx)$, that is, g is compatible with ℓ.

6. **Distributional derivatives.** This exercise provides some complementary background on the Sobolev space $H^1(\mathbb{R}, dx)$ and distributional derivatives. Consider the real Hilbert space $L^2(\mathbb{R};dx)$. We say that $\psi \in L^2(\mathbb{R};dx)$ *has weak or distributional derivative* $\zeta \in L^2(\mathbb{R};dx)$ if for every[1] $\varphi \in C_c^\infty(\mathbb{R})$ it holds true that
$$\int_{\mathbb{R}} dx\ \psi(x)\varphi'(x) = -\int_{\mathbb{R}} dx\ \zeta(x)\varphi(x).$$
If such a distributional derivative ζ exists, we denote it symbolically by $\zeta \equiv \psi'$. Show that the set of $L^2(\mathbb{R}, dx)$ functions that have a distributional derivative in $L^2(\mathbb{R};dx)$ forms a vector space. Denote this space by $H^1(\mathbb{R};dx)$, as in class[2]. Prove that if $\psi \in C^1(\mathbb{R}) \cap L^2(\mathbb{R};dx)$ is continuously differentiable, then its usual derivative is equal to its distributional derivative. Explain why
$$\langle \phi, \psi\rangle_{H^1} = \int_{\mathbb{R}} dx\ \Big[\phi(x)\psi(x) + \phi'(x)\psi'(x)\Big]$$
defines an inner product in $H^1(\mathbb{R};dx)$. In class it was mentioned that $H^1(\mathbb{R};dx)$ is a Hilbert space. Assuming, as in class, that $L^2(\mathbb{R};dx)$ is a Hilbert space, give a proof of this fact that $\big(H^1(\mathbb{R};dx); \langle\cdot,\cdot\rangle_{H^1}\big)$ is a Hilbert space.

[1] A function ϕ lies in $\phi \in C_c^\infty(\mathbb{R})$ if and only if ϕ is smooth and if there exists a bounded interval $J = [a;b]$ such that outside of J, we have that $\phi_{|\mathbb{R}\setminus J} \equiv 0$. We say that ϕ has *compact support*.

[2] The definition given in class actually turns out to be equivalent to the one given in this problem - this is taught in, for instance, Math 212: graduate analysis.

Definition: Let $(V, \|\cdot\|)$ be a normed space. A set $S \subset V$ is called *dense in V* iff for all $x \in V$ and $\varepsilon > 0$, it holds true that $B_\varepsilon(x) \cap S \neq \emptyset$. Here, $B_\varepsilon(x)$ denotes the open ball $B_\varepsilon(x) = \{y \in V : \|x - y\| < \varepsilon\}$ of radius ε around $x \in V$.

*7. **Separable Hilbert spaces.** *This problem is optional and does not give credit.* Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with its standard norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

a) Suppose H has a complete orthonormal basis $(\phi_j)_{j\in\mathbb{N}}$. Show that

$$S = \Big\{ \sum_{j=1}^{N} a_j \phi_j : a_j \in \mathbb{Q} \; \forall \, j = 1, \ldots, N \;\text{ and }\; N \in \mathbb{N} \Big\}$$

is a countable, dense subset of H. A space that contains a countable, dense subset is called *separable.*

b) Conversely, suppose H is separable, as defined in the previous part, that is, H contains a countable, dense subset. Prove that in this case, H has a complete orthonormal basis.

Hint: Recall the Gram-Schmidt procedure to construct orthonormal bases.