MATH110 Spring 2020 HW6

Rylan Schaeffer

April 2nd, 2020

Problem 2

1. To save me typing, define $a \stackrel{\text{def}}{=} \sum^N \lambda_j \phi_j$ and $b \stackrel{\text{def}}{=} \sum^N \mu_j \phi_j$. First, note that

$$
\begin{aligned}
||\psi - \sum^N \mu_j \phi_j||^2 &= ||\psi - a + a - b||^2 \\
&= \langle \psi - a + a - b, \psi - a + a - b \rangle \\
&= \langle \psi - a, \psi - a \rangle + 2\langle \psi - a, a - b \rangle + \langle a - b, a - b \rangle \\
&= \langle \psi - a, \psi - a \rangle + \langle a - b, a - b \rangle
\end{aligned}
$$

The last line follows because $\langle \psi - a, a - b \rangle = 0$. To see why, note that inner products are projections of one vector onto another, but the left argument has had all its components in the $(\phi_j)_j$ basis subtracted out. We can be more specific by writing it out:

$$
\begin{aligned}
\langle \psi - a, a - b \rangle &= \langle \psi, a \rangle - \langle \psi, b \rangle - \langle a, a \rangle + \langle a, b \rangle \\
&= \sum^N \lambda_j^2 - \sum^N \mu_j \lambda_j - \sum^N \lambda_j^2 + \sum^N \mu_j \lambda_j \\
&= 0
\end{aligned}
$$

Returning to the first block, we replace a and b with their definitions, and continue along.

$$
\begin{aligned}
||\psi - \sum^N \mu_j \phi_j||^2 &= ||\psi - \sum^N \lambda_j \phi_j||^2 + ||\sum^N \lambda_j \phi_j - \sum^N \mu_j \phi_j||^2 \\
&= ||\psi - \sum^N \lambda_j \phi_j||^2 + ||\sum^N (\lambda_j - \mu_j) \phi_j||^2 \\
&= ||\psi - \sum^N \lambda_j \phi_j||^2 + \sum^N ||(\lambda_j - \mu_j) \phi_j||^2 \\
&= ||\psi - \sum^N \lambda_j \phi_j||^2 + \sum^N |\lambda_j - \mu_j|^2
\end{aligned}
$$

where the last line follows because $||(\lambda_j - \mu_j)\phi_j||^2 = |\lambda_j - \mu_j|^2 ||\phi||^2$ and $||\phi_j||^2 = 1$ by definition of orthonormality. We conclude that because only the second term on the RHS depends on μ_j and because the term is minimized by setting $\mu_j = \lambda_j$, we conclude that the minimum is attained at $(\mu_i, ..., \mu_N) = (\lambda_i, ..., \lambda_i)$

2. Using the previous subproblem, choose $\mu_j = 0$. Then

$$\begin{aligned}
||\psi||^2 &= ||\psi - \sum^N \mu_j \phi_j||^2 \\
&= ||\psi - \sum^N \lambda_j \phi_j||^2 + \sum^N |\lambda_j - \mu_j|^2 \\
&= ||\psi - \sum^N \lambda_j \phi_j||^2 + \sum^N |\lambda_j|^2 \\
&\geq \sum^N |\lambda_j|^2
\end{aligned}$$

where the last line follows because inner products are positive definite i.e. the inner product of an element with itself is non-negative.

3. Define $(S_N)_{N\in\mathbb{N}} = \left(\sum_{j=1}^N c_j \phi_j\right)_{N\in\mathbb{N}}$ as the sequence of truncated sums. Assume that the sequence converges in H to $S \overset{\text{def}}{=} \lim_{N\to\infty} S_N$. Assume that $S \in H \Rightarrow \langle S, S\rangle < \infty$. By Bessel's Inequality, we know that any $\forall N \in \mathbb{N}$:

$$\sum_j^N |\langle S, \phi_j|^2\rangle \leq \langle S, S\rangle < \infty$$

Since $\sum_j^N |\langle S, \phi_j\rangle|^2 = \sum_j^N |\langle \sum_i^N c_i\phi_i, \phi_j\rangle|^2 = \sum_j^N |c_j\langle \phi_j, \phi_j\rangle|^2 = \sum_j^N |c_j|^2$, we conclude that $\sum_j^N |c_j|^2 < \infty$. But this holds for all N, so we conclude that $\sum_j^\infty |c_j|^2 < \infty$.

In the other direction, assume that $\sum_{j=1}^\infty |c_j|^2 < \infty$. Note that $\sum_{j=1}^N |c_j|^2 = ||S_N||^2$. Because the sequence $(\sum_{j=1}^N |c_j|^2)_{N\in\mathbb{N}} < \infty$. is monotonically increasing with N but bounded from above, it must converge. This means that $(S_N)_{N\in\mathbb{N}}$ also converges. Because H is complete, the sequence converges in H to some S.

4. Because $(\phi_j)_j$ is now a complete orthonormal basis, we can write $\phi \in H$ as $\phi = \sum_{j=1}^\infty \langle H, \phi_j\rangle\phi_j$. Consequently,

$$\begin{aligned}
||\psi||^2 &= \langle \psi, \psi\rangle \\
&= \Big\langle \sum_{j=1}^\infty \langle \psi, \phi_j\rangle \phi_j, \sum_{k=1}^\infty \langle \psi, \phi_k\rangle \phi_k \Big\rangle \\
&= \sum_{j=1}^\infty \langle \psi, \phi_j\rangle \sum_{k=1}^\infty \langle \psi, \phi_k\rangle \underbrace{\langle \phi_j, \phi_k\rangle}_{\delta_{jk}} \\
&= \sum_{j=1}^\infty |\langle \psi, \phi_j\rangle|^2
\end{aligned}$$

Problem 3

1. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in L^2. We want to show that for the sequence comprised of the jth components is also a Cauchy sequence. Fix ϵ. Because $(x^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence, we know that for $\sqrt{\epsilon} > 0, \exists N$ such that $\forall a, b \in \mathbb{N}, a, b > N$,

$$||x^{(a)} - x^{(b)}|| < \sqrt{\epsilon}$$

. Using the metric induced by the inner product,

$$||x^{(a)} - x^{(b)}|| < \sqrt{\epsilon} \Leftrightarrow \langle x^{(a)} - x^{(b)}, x^{(a)} - x^{(b)} \rangle < \epsilon$$

Then, using the definition of the inner product:

$$\langle x^{(a)} - x^{(b)}, x^{(a)} - x^{(b)} \rangle = \sum_{j=1}^{\infty} (x_j^{(a)} - x_j^{(b)})^2 < \epsilon$$

Since each value is **real**, we know that each term in the sum is non-negative. Supposing all but the ith components are 0, we see that $\forall a, b > N$:

$$(x_i^{(a)} - x_i^{(b)})^2 < \epsilon$$

Thus we conclude that $\forall i$, $(x_i^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

2. To show that $x^{(\infty)} \in \ell^2$, we need to show that $\langle x^\infty, x^\infty \rangle < \infty$. Consider fixed n:

$$\begin{aligned}
\sqrt{\sum_j^N (x^\infty)^2} &= \sqrt{\sum_j^N (x_j^\infty - x_j^n + x_j^n)^2} \\
&\leq \sqrt{\sum_j^N (x_j^\infty - x_j^n)^2} + \sqrt{\sum_j^N (x_j^n)^2} \\
&\leq \sqrt{\sum_j^N (x_j^\infty - x_j^n)^2} + \sup_{n \in \mathbb{N}} ||x^n||
\end{aligned}$$

Define an upper bound $C \geq \sup_{n \in \mathbb{N}} ||x^n||$. Note that C is uniform in N. Taking the limit $n \to \infty$, the first term on the RHS vanishes and we see that:

$$\sum_j^N (x^\infty)^2 \leq C^2$$

Since C is finite, C^2 is finite and thus $x^\infty < \infty \Rightarrow x^\infty \in \ell^2$.

3. To show that $x^\infty = \lim_{n\to\infty} x^n$ in ℓ^2, we must show that the series $(x^n)_n$ converges to x^∞. This requires showing that $\forall \sqrt{\epsilon} > 0, \exists c \in \mathbb{N}$ s.t. $\forall n > c, ||x^n - x^\infty||^2 < \epsilon$.

$$||x^n - x^\infty||^2 = \langle x^n, x^n \rangle - 2\langle x^n, x^\infty \rangle + \langle x^\infty, x^\infty \rangle$$

Because each sequence $(x_j^n)_j$ is Cauchy in the reals, each sequence is convergent in the reals and converges to x_j^∞. This means that we can always find an integer c large enough that $\langle x^n, x^\infty \rangle$ will be arbitrarily close, and thus $|x^n - x^\infty||^2 < \epsilon$.

Problem 4

1. For $x \mapsto \log(x)$, define $u = \log(x), v = x \Rightarrow du = \frac{1}{x}dx, dv = dx$. Then

$$\int \log(x)dx = x\log(x) - \int x\frac{1}{x}dx = x\log(x) - x$$

For $x \mapsto (\log(x))^2$, define $u = (\log(x))^2, v = x \Rightarrow du = 2\log(x)\frac{1}{x}, dv = dx$. Then

$$\int (\log(x))^2 dx = x(\log(x))^2 - \int x2\log(x)\frac{1}{x}dx = x(\log(x))^2 - 2x\log x + 2x$$

2. To show that $f \in L((0,1);dx)$, we must show that the integral over the interval of the function squared is finite:

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &= x(\log(x))^2 - 2x\log x + 2x\Big|_0^1 \\ &= 0 - 0 + 2(1) - 0 + 0 - 0 \\ &= 2 \\ &< \infty\end{aligned}$$

3. To show that $f_1, f_2 \in L((-1,1);dx)$, we must show that for both functions, the integral over the interval of the function squared is finite:

$$\begin{aligned}\int_{-1}^1 |f_1(x)|^2 dx &= (x+1)((\log(x+1))^2 - 2\log(x+1) + 2)\Big|_{-1}^1 \\ &= 2((\log(2))^2 - 2\log(2) + 2) - 0 \\ &\leq \infty \\ \int_{-1}^1 |f_2(x)|^2 dx &= (1-x)((\log(1-x))^2 - 2\log(1-x) + 2)\Big|_{-1}^1 \\ &= 0 + 2((\log(2))^2 - 2\log(2) + 2) \\ &< \infty\end{aligned}$$

4. To show that $g \in L((-1,1);dx)$, we must show that the integral over the interval of the function squared is finite:

$$\begin{aligned}\int_{-1}^1 |g(x)|^2 dx &= \frac{1}{4}\int_{-1}^1 (\log(1+x) - \log(1-x))^2 dx \\ &= \frac{1}{4}\int_{-1}^1 (\log(1+x))^2 dx - \frac{2}{4}\int_{-1}^1 \log(1+x)\log(1-x)dx + \frac{1}{4}\int_{-1}^1 (\log(1-x))^2 dx \\ &= \frac{2}{4}((\log(2))^2 - 2\log(2) + 2) - \frac{4}{4}\int_0^1 \log(1+x)\log(1-x) + \frac{2}{4}((\log(2))^2 - 2\log(2) + 2) \\ &\leq \frac{2}{4}((\log(2))^2 - 2\log(2) + 2) - \int_0^1 (\log(1+x))^2 + \frac{2}{4}((\log(2))^2 - 2\log(2) + 2) \\ &< \infty\end{aligned}$$

Problem 5

I first show that $\phi_p \in L([0,1], dx)$:

$$\int_0^1 |\phi_p(x)|^2 dx = \int_0^1 \phi_p(x)\overline{\phi_p(x)} dx = \int_0^1 e^{2\pi ipx} e^{-2\pi ipx}\, dx = \int_0^1 dx = 1 < \infty$$

We know from lecture that $\langle \phi_j, \phi_k \rangle \stackrel{\text{def}}{=} \int_0^1 dx \phi_j(x)\overline{\phi_k(x)}$ defines an inner product for $L([0,1], dx)$ and we see from the above result that $\langle \phi_j, \phi_j \rangle = 1$; we next show that $\forall j, k \in \mathbb{N}, j \neq k$, the basis functions are orthogonal:

$$\begin{aligned}
\langle \phi_j, \phi_k \rangle &= \int_0^1 dx\, \phi_j(x)\overline{\phi_k(x)} \\
&= \int_0^1 dx\, e^{2\pi ijx} e^{-2\pi ikx} \\
&= \int_0^1 dx\, e^{2\pi i(j-k)x} \\
&= \int_0^1 dx\, \cos(2\pi(j-k)x) + i\sin(2\pi(j-k)x) \\
&= \sin(2\pi(j-k)x) - i\cos(2\pi(j-k)x)|_0^1 \\
&= (0-i) - (0-i) \\
&= 0
\end{aligned}$$

We find the Fourier coefficient for the pth basis function, using a u-substitution of $u = x, du = dx, v = -\frac{1}{2\pi ip} e^{-2\pi ipx}, dv = e^{-2\pi ipx} dx$:

$$\begin{aligned}
\langle \psi, \phi_p \rangle &= -\frac{1}{2\pi ip} x e^{-2\pi ip} - \Big(\frac{1}{2\pi ip}\Big)^2 e^{-2\pi ip}|_0^1 \\
&= -\frac{1}{(2\pi p)^2}\Big[-(2\pi ip)e^{-2\pi ip} + e^{2\pi ip} - 1\Big] \\
&= -\frac{1}{(2\pi p)^2}\Big[-(2\pi ip)(\cos(2\pi p) - i\sin(2\pi p)) + \cos(2\pi p) + i\sin(2\pi p) - 1\Big]
\end{aligned}$$

Because p is an integer, all sin terms vanish and cos terms are 1. This simplifies to:

$$\langle \psi, \phi_p \rangle = -\frac{1}{(2\pi p)^2}\Big[-(2\pi ip)(1-0) + 1 + i(0) - 1\Big] = -\frac{1}{2\pi ip} = \frac{i}{2\pi p}$$

The Fourier expansion is therefore:

$$\psi(x) = \sum_{p\in\mathbb{Z}} \langle \psi, \phi_p \rangle \phi_p = \sum_{p\in\mathbb{Z}} \frac{i}{2\pi p} \phi_p$$