MATH110 Spring 2020 HW1

Rylan Schaeffer

February 6th, 2020

Problem 1

First we separate variables:

$$y' = kx(y - \frac{m}{k})$$

Then we integrate:

$$\log(y - \frac{m}{k}) - \log(y(x_0) - \frac{m}{k}) = \frac{1}{2}k(x^2 - x_0^2)$$

We reach the final solution:

$$y(x) = y(0)\exp\Big(\frac{1}{2}k(x^2 - x_0^2)\Big) + \frac{m}{k} - \frac{m}{k}\exp\Big(\frac{1}{2}k(x^2 - x_0)\Big)$$

We want to identify for which values of m, k permits the loan to be paid off. Assuming $x_0 = 0$, we consider the fixed point $y' = 0$:

$$\begin{aligned}
0 &= kx(y - \frac{m}{k}) && (1)\\
&= kx\Big[y(0)\exp\Big(\frac{1}{2}kx^2\Big) - \frac{m}{k}\exp\Big(\frac{1}{2}kx^2\Big)\Big] && (2)\\
&= kx[y(0) - \frac{m}{k}] && (3)
\end{aligned}$$

Since k, x are both positive, y' will be positive only when $y(0) - \frac{m}{k} > 0 \Leftrightarrow y(0) > \frac{m}{k}$. Therefore, to ensure the customer never pays off the loan, the principal must be greater than the ratio of m to k.

Problem 2

Assume that

$$\begin{aligned}
0 &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 && (4)\\
&= \lambda_1 + \lambda_2 x + \lambda_3 x^2 && (5)
\end{aligned}$$

Differentiating twice, we see that

$$\begin{aligned}
0 &= \lambda_2 + 2\lambda_3 x && (6)\\
0 &= \lambda_3 && (7)
\end{aligned}$$

Starting from the last equation and working backwards, $\lambda_3 = 0 \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_1 = 0$. Thus, $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$. In the opposite direction, assume that $\lambda_1 = \lambda_1 = \lambda_2 = \lambda_3 = 0$ immediately yields $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0$.

Problem 3

Let $C(\mathbb{R})$ be the real vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. We know from the previous problem that $1, x, x^2, ..., x^n$ are linearly independent. Let $\{1, x, x^2, ..., x^n\}$ be the generating set of $C(\mathbb{R})$. Assume for purposes of contradiction that n is finite and consider the function $f(x) = x^{n+1} \in C(\mathbb{R})$. Based on the same previous results, we know that $f(x) \notin \text{span}\{1, x, x^2, ..., x^n\}$, but $f(x) \in C(\mathbb{R})$, a contradiction! Thus n cannot be finite.

Problem 4

Let V be the four-dimensional space spanned by the polynomials $\{1, x, x^2, x^3\}$. Let $l : V \to V$ be the linear operator defined by $l(y) = (x + 2)y' - 3y$.

$$\begin{aligned}
l(1) &= (x+2)(1)' - 3(1) && (8)\\
&= -3 && (9)\\
l(x) &= (x+2)(x)' - 3(x) && (10)\\
&= 2 + (-2)x && (11)\\
l(x^2) &= (x+2)(x^2)' - 3(x^2) && (12)\\
&= 0 + 4x + (-1)x^2 && (13)\\
l(x^3) &= (x+2)(x^3)' - 3(x^3) && (14)\\
&= 6x^2 && (15)
\end{aligned}$$

The matrix L that represents l in the given basis is:

$$L = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using Gauss-Jordan elimination, I solve $l(y) = -x^2$ for y:

$$l(y) = Ly = -x^2 \Leftrightarrow \begin{bmatrix} -3 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 4/3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 8 \\ 12 \\ 6 \\ 1 \end{bmatrix}$$

where $d \in \mathbb{R}$.

Problem 5

$$\begin{aligned}
l(1) &= (x+1)(1)' = 0 && (16)\\
l(x) &= (x+1)(x)' = x + 1 && (17)\\
l(x^2) &= (x+1)(x^2)' = (x+1)(2x) = 2x^2 + 2x && (18)
\end{aligned}$$

The matrix L that represents l in the given monomial basis is $L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Diagonalizing L yields $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, meaning three unique eigenvalues exist: $2, 1, 0$. Since one eigenvalue is 0, the diagonal matrix has rank 2 and thus there is no basis of eigenvectors of L that span the rank 3 matrix V.

Problem 6

Define scalar multiplication of each sequence in the set as the scalar multiplying each element in the sequence i.e. $a(c_1, c_2, ...) = (ac_1, ac_2, ...)$ and define addition point-wise i.e. $(c_1, c2, ...) + (b_1, b_2, ...) = (c_1 + b_1, c_2 + b_2, ...)$. Then $l^2(\mathbb{N})$ and $h^2(\mathbb{N})$ each form a complex vector space.

To see this, let $a_1, a_2 \in \mathbb{R}$ and $s_1 = (c_1, c_2, ...), s_2 = (b_1, b_2, ...) \in l^2(\mathbb{N})$. Scalar multiplication keeps $as_1 \in l^2(\mathbb{N})$:

$$s_1 \in l^2(\mathbb{N}) \Rightarrow \sum_{k=1}^{\infty} |c_k|^2 < \infty \Rightarrow as_1 = \sum_{k=1}^{\infty} |ac_k|^2 = \sum_{k=1}^{\infty} |a|^2 |c_k|^2 = |a|^2 \sum_{k=1}^{\infty} |c_k|^2 \infty \Rightarrow as_1 \in l^2(\mathbb{N})$$

Addition keeps $s_1 + s_2 \in l^2(\mathbb{N})$:

$$s_1 + s_2 = (c_1 + b_1, c_2 + b_2, ...) \Rightarrow \sum_{k=1}^{\infty} |c_k + b_k|^2 \leq \sum_{k=1}^{\infty} |c_k|^2 + \sum_{k=1}^{\infty} |b_k|^2 + 2 \sum_{k=1}^{\infty} |c_k||b_k| < \infty$$

where the last inequality follows from the fact that $\sum_{k=1}^{\infty} |c_k||b_k| < \sum_{k=1}^{\infty} \max\{|c_k|, |b_k|\}^2$. An almost-identical argument follows for $h^2(\mathbb{N})$, the only difference being that there's an additional coefficient of k^4 in each term in the sum.

Define $-\Delta : h^2(\mathbb{N}) \to l^2(\mathbb{N})$. To show that $-\Delta$ is a linear map, as before let $a_1, a_2 \in \mathbb{R}$ and $h_1 = (c_1, c_2, c_3, ...), h_2 = (b_1, b_2, b_3, ...) \in h^2(\mathbb{N})$:

$$\begin{aligned}
-\Delta(a_1 h_1 + a_2 h_2) &= -\Delta(a_1 c_1 + a_2 b_1, a_1 c_2 + a_2 b_2, a_1 c_3 + a_2 b_3, ...) && (19)\\
&= (1^2(a_1 c_1 + a_1 b_1), 2^2(a_1 c_2 + a_2 b_2), 3^2(a_1 c_3 + a_2 b_3), ...) && (20)\\
&= (a_1 1^2(c_1) + a_1 1^2 b_1), a_1 2^2 c_2 + a_2 2^2 b_2, a_1 3^2 c_3 + a_2 3^2 b_3, ...) && (21)\\
&= a_1(1^2(c_1), 2^2(c_2), 3^2(c_3), ...) + a_2(1^2(b_1), 2^2(b_2), 3^2(b_3), ...) && (22)\\
&= a_1 - \Delta h_1 + a_2 - \Delta h_2 && (23)
\end{aligned}$$

Thus $-\Delta$ is linear. The eigenvalues of $-\Delta$ are $1^2, 2^2, 3^2, 4^2, ...$