Week 4

Topics: Abel's formula; 2nd order ODE with constant coefficients; variation of parameters

Review from Weeks 2 & 3:

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads
$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\,dt\right).$$
- **General solution to 1st order ODE:** Let $\ell(y) = ay' + by$, then the kernel $\ker(\ell)$ is one-dimensional. Suppose it is spanned by some function f. Then the general solution to $\ell(y) = h$ has the form $y = y_p + Cf$ for some particular solution y_p and some constant $C \in \mathbb{R}$. We can find a particular solution using the **variation of parameters**. This yields a particular solution of the form $x \mapsto y_p(x) = f(x)\int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\,dt$
- **Analytic functions:** A function $x \mapsto f(x)$ is real-analytic at x_0 if its Taylor series T_{f,x_0} converges in some interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, with $\varepsilon > 0$, and is equal to f in that interval.
- **Wronskian:** A quick way to check linear independence of two functions y_1 and y_2 is to use the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. If $W(y_1, y_2)(x_0) \neq 0$ for some x_0, then y_1 and y_2 are linearly independent.
- **Existence of solutions:** For $\ell(y) = ay'' + by' + cy$, we consider the initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$. Under the usual assumptions on a, b, c, we have seen that solutions exist and every solution may be written in the form $y = y_0 f_1 + y_1 f_2$, where f_1 satisfies $f_1(0) = 1$, $f_1'(0) = 0$ and f_2 satisfies $f_2(0) = 0$, $f_2'(0) = 1$.

1. Start with the second-order equation $\ell(y) = ay'' + by' + cy = 0$.

(a) Show that the Wronskian W of any two independent solutions f_1 and f_2 of this equation satisfies the first order equation $aW' + bW = 0$. As a consequence, what is the formula for W in terms of a and b? The result is called **Abel's Formula**.

(b) Knowing the Wronskian and one vector f_1 in the kernel of ℓ, can you find a second vector f_2, independent of f_1, in the kernel of ℓ?

2. Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$ and consider the equation $ay'' + by' + cy = 0$.

(a) If $x \mapsto y(x) = e^{rx}$ is a solution to the first order homogeneous ODE above, can you find an algebraic equation that is solved by r?

(b) Using this approach, find two independent solutions to the equation

$$y'' - y' - 2y = 0.$$

3. Now, we would also like to solve the second order *inhomogeneous* equation

$$\ell(y) = ay'' + by' + cy = h.$$

where still $a, b, c \in \mathbb{R}$ are constants, but h is a function which in general is not just constant.

(a) Find a particular solution to the equation

$$l(y) = y'' + y' - 2y = x - 2x^3,$$

by guessing that y_p is in the space $V = \operatorname{span}\{1, x, x^2, x^3\}$.

(b) Use the previous result to find the general solution to $\ell(y) = h$ where $h(x) = x - 2x^3$.

4. Suppose our Ansatz $y(x) = e^{rx}$ leads only to one solution to the ODE. In this case, the only (real) solution to $ar^2 + br + c = 0$ is $r = -b/2a$ (why is this so?).

(a) We know that one solution is $x \mapsto f_1(x) = e^{rx}$. Use the Wronskian to find the second solution in the kernel of $\ell(y) = ay'' + by' + c$.

(b) As an application, solve the IVP $y'' + 4y' + 4y = 0$ with $y(0) = 1, y'(0) = -1$.

5. Suppose $r = \alpha + i\beta$ is a complex solution to $ar^2 + br + c = 0$.

(a) Using the fact that $y(x) = e^{rx}$ is a solution to the equation $ay'' + by' + cy = 0$, find two *real*, linearly independent solutions y_1, y_2 by using Euler's formula.

(b) As an example, find the general solution to $y'' + 2y' + 5y = 0$.

6. **Second Order Variation of Parameters.** Use the second order variation of parameters method discussed in class to find a particular solution of $ay'' + by' + cy = h$ as follows.

(a) Start to derive the relation $c_1' f_1' + c_2' f_2' = \frac{h}{a}$.

(b) We now have a pair of equations for c_1' and c_2', namely

$$\begin{cases} c_1' f_1 + c_2' f_2 &= 0, \\ c_1' f_1' + c_2' f_2' &= \frac{h}{a} \end{cases}$$

Solve this system for c_1', c_2' using the Wronskian $W(x) = \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right)$.

(c) Solve for c_1, c_2 and write down the final solution $x \mapsto y_p(x) = c_1(x) f_1(x) + c_2(x) f_2(x)$.

(d) As an example, use this method to find once more the general solution to $y'' + y' - 2y = x - 2x^3$. Compare your result with the solution to problem 3.

7. General 1st Order ODE. In the next few lectures, we want to start to discuss an important function space method to prove the existence of solutions to general first order ODE. More precisely, we consider the following: Let $f \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ be continuous, $n \in \mathbb{N}$, and let $(x_0, \mathbf{y}_0) \in \mathbb{R} \times \mathbb{R}^n$. The initial value problem related to an (not necessarily linear) *ordinary differential equation of first order* reads

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

(a) One can recast our familiar 2nd linear order ODE $\ell(y) = ay'' + by' + cy = h$ as a (vector-valued) 1st order ODE. To this end, set $y_1 = y$ and $y_2 = y'$. Can you reformulate $\ell(y) = h$ as a 1st order ODE for the vector-valued function $\mathbf{y} = (y_1, y_2)$ with values in $\mathbb{R}^2$?

(b) Show that a continuous function $\mathbf{y} \in C((x_0 - \epsilon; x_0 + \epsilon); \mathbb{R}^n)$ is a solution to the above initial value problem if and only if

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \mathbf{y}(s))\, ds$$

for all $x \in (x_0 - \epsilon; x_0 + \epsilon)$.

8. **An example of a fixed point argument.** Suppose you want to prove the statement that there exists a real number $x \in \mathbb{R}$ such that $x^2 = 2$, i.e. that $\sqrt{2} \in \mathbb{R}$ exists. This is not a trivial question and below is an argument that uses the fact that every bounded, monotone sequence in $\mathbb{R}$ has a limit in $\mathbb{R}$.

   (a) Verify that $x^2 = 2$ if and only if $x = f(x)$ for the function $f : (0; \infty) \to \mathbb{R}$, defined by
   $$f(x) = \frac{1}{2}x + \frac{1}{x}.$$

   (b) Define the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = f(x_n)$. Use induction to prove that $2 \leq x_n^2$ and $x_n \leq 2$ for all $n \in \mathbb{N}$.

   (c) Prove that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. Conclude the theorem.

9. First examples of some metric spaces.

(a) Consider the set $\ell^1 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\right\}$ and define

$$d_1(\mathbf{x},\mathbf{y}) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Prove that $d_1 : \ell^1 \times \ell^1 \to \mathbb{R}$ defines a metric.

(b) Consider the set $\ell^2 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$ and define

$$d_2(\mathbf{x},\mathbf{y}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}.$$

Prove that $d_2 : \ell^2 \times \ell^2 \to \mathbb{R}$ defines a metric.