Week 2

Topics: recap linear algebra; variation of parameters; power series expansion; the Wronskian

Review from Week 1:

- **Integrating factor:** To solve $ay' + by = h$, multiply by

$$\mu(x) = \exp\left(\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right),$$

  then LHS $= (\mu y)' = h =$ RHS, so solve by integrating and then dividing by μ.

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads

$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right).$$

1. Show that the function $\ell : V \to V$ given by the formula $\ell(y) = a(x)y' + b(x)y$ has the properties required for linearity, and the kernel of ℓ is a one-dimensional subspace of V. Find a vector (a function) $f \in V$ such that $\ker \ell = \mathrm{span}\{f\}$.

2. Let V be the three-dimensional vector space of polynomials of degree no greater than 2, with basis $\mathfrak{B} = \{1, x, x^2\}$. Let ℓ be the linear differential operator

$$\ell(y) = (x+1)y' - 2y.$$

(a) Write down the matrix L that represents ℓ with respect to the basis $\mathfrak{B}$, and find a basis for the kernel of L.

(b) Then find, by algebraic methods, the general solution to $\ell(y) = -2x$.

(c) Is there any element $h(x) \in V$ for which $\ell(y) = h(x)$ cannot be solved?

3. Let V be the space of smooth functions and

$$\ell(y) = ay' + by.$$

for smooth functions a and b. The space of smooth functions is not finite-dimensional s.t. we cannot simply write down a matrix for ℓ, but from **Problem Set 1** we know how to find a vector (a function!) f that spans its kernel.

(a) Prove that if y_p is any **particular solution** to $\ell(y) = h$, then the general solution is $y = Cf + y_p$ for some constant C.

(b) Use this theorem and a bit of guesswork to find the general solution to

$$3xy' - y = \log x + 1.$$

4. Suppose that we want to solve $\ell(y) = h$ (where $\ell(y) = ay' + by$) and we have already found a function f that spans the kernel of ℓ. Use the variation of the parameters idea to find a particular solution to $\ell(y) = h$ (the expression that you get still depends on f, of course).

(a) Show that we can get a formula for the derivative g' and thereby (at least in principle) find a solution to the inhomogeneous equation $\ell(y) = h$.

(b) Apply this approach to the equation

$$xy' + 2y = x.$$

Notice that you could also solve this equation by using an integrating factor.

5. (a) Using the ratio test, find the radius of convergence R about $x_0 = 0$ for $x \mapsto f(x) = e^x$ and $x \mapsto f(x) = \log(1 + x)$.

(b) Explain why the function f with values $f(x) = \exp\left(-\frac{1}{x}\right)$ if $x > 0$ and $f(x) = 0$ otherwise is not analytic at $x_0 = 0$.

Answers and Solutions.

1. We may check *both* linearity conditions at once by considering,

$$\begin{aligned}
\ell\left(y_1+Ky_2\right) &= a(x)\left(y_1+Ky_2\right)'+b(x)\left(y_1+Ky_2\right), \text{ by def. of } \ell, \\
&= a(x)\left(y_1'+Ky_2'\right)+b(x)\left(y_1+Ky_2\right), \text{ properties of the derivitave}, \\
&= \left(a(x)y_1'+b(x)y_1\right)+K\left(a(x)y_2'+b(x)y_2\right), \text{ rearranging}, \\
&= \ell\left(y_1\right)+K\ell\left(y_2\right), \text{ by def. of } \ell.
\end{aligned}$$

This proves linearity. Next, recall that

$$\ker\ell = \{f \in V \mid \ell(f) = 0\}.$$

Thus, we want to solve

$$\ell(f) = a(x)f' + b(x)f = 0.$$

From problem **3**, we recall that the general solution of $l(f) = 0$ is

$$f(x) = f(x_0)\exp\left[-\int_{x_0}^{x}\frac{b(t)}{a(t)}\,dt\right]$$

Now, observe that $f(x_0)$ is an arbitrary constant, so that the kernel of l is the span of the single function $x \mapsto \exp\left[-\int_{x_0}^{x}\frac{b(t)}{a(t)}\,dt\right]$. This shows that $\dim\ker\ell = 1$.

2. (a) Recall that a linear transformation is determined by what it does to basis, and so we compute,

$$\begin{aligned}
\ell(1) &= (x+1)\cdot(1)' - 2\cdot 1 = (-2)\cdot 1,\\
\ell(x) &= (x+1)\cdot(x)' - 2\cdot x = (x+1) - 2x = (1)\cdot 1 + (-1)\cdot x,\\
\ell(x^2) &= (x+1)\cdot(x^2)' - 2\cdot x^2 = 2x(x+1) - 2x^2 = (2)\cdot x.
\end{aligned}$$

The matrix L for ℓ with respect to the basis $\mathfrak{B}$ is then given by plugging in the coefficients (in brackets above) along each column as follows,

$$L = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

To find a basis for $\ker\ell$ we must solve,

$$L \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that this question also wants us to find the general solution to $\ell(y) = -2x$.

(b) To this end, we shall solve first the following equation

$$L \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix},$$

where we have written the function $-2x$ in terms of the basis $\mathfrak{B}$ on the RHS. Row reduce,

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2\to -R_2]{R_1\to -\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1\to R_1+\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $t = c$ be our free variable. Then R_1 tells us that $a = 1+t$ and R_2 says that $b = 2+2t$. Thus, the solution in vector form is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1+t \\ 2+2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

Converting back to polynomials, we see that a particular solution is

$$y_p = 1 + 2x,$$

and the kernel is given by

$$\ker\ell = \mathrm{span}\{1 + 2x + x^2\} = \mathrm{span}\{(1+x)^2\}.$$

How did we find the kernel? Choosing $t = 0$ and $t = 1$ as two particular solutions, we observe their linear independence and conclude that $\ker\ell$ can be at most one dimensional. But the difference of these two particular solutions is non zero and in the kernel!

(c) Since $\ker(\ell) \neq \{0\}$, there must be such an element. To give an explicit example, since the the bottom row of the matrix L is zero, the system $\ell(y) = h(x)$ is impossible to solve if $h(x) = x^2$.

3. (a) Let $g = y - y_p$ and consider,

$$\begin{aligned}\ell(g) &= \ell(y - y_p),\\ &= \ell(y) - \ell(y_p), \text{ by linearity of } \ell,\\ &= h(x) - h(x), \text{ by assumption},\\ &= 0.\end{aligned}$$

Thus, $g \in \ker\ell = \text{span}\{f\}$, and therefore

$$\begin{aligned}g &= Cf,\\ y - y_p &= Cf,\\ y &= y_p + Cf.\end{aligned}$$

This proves the theorem.

(b) For the next part of the question we first find $\ker\ell$ where $\ell(f) = 3xf' - f$, and solve

$$\begin{aligned}3xf' - f &= 0,\\ \int \frac{f'}{f}(t)\, dt &= \int \frac{1}{3t}\, dt,\\ \log|f| &= \frac{1}{3}\log|x| + K,\\ |f|(x) &= C\exp\left[\frac{1}{3}\log|x|\right],\\ |f|(x) &= C|x|^{1/3}.\end{aligned}$$

We choose $x \mapsto f(x) = x^{1/3}$. Now we need to find a particular solution y_p. This requires some inspired guesswork. Since the RHS of the equation we wish to solve involves $\log x$ and the constant function 1 we will guess that y_p is a linear combination of these functions

$$y_p = A + B\log x.$$

To find the constants A, B, plug this guess for y_p in the original equation

$$\begin{aligned}3xy_p' - y_p &= \log x + 1,\\ 3x(A + B\log x)' - (A + B\log x) &= \log x + 1,\\ 3B - A - B\log x &= \log x + 1.\end{aligned}$$

Equating coefficients of $1, \log x$ on both sides we find

$$\begin{aligned}3B - A &= 1,\\ -B &= 1\end{aligned}$$

which tells us that $A = -4$, $B = -1$. Thus,

$$y_p = -4 - \log x,$$

and the general solution is

$$y = y_p + Cf = -4 - \log x + Cx^{1/3}.$$

4. (a) Plug $y(x) = g(x)f(x)$ into the equation $\ell(y) = h$,

$$\begin{aligned} ay' + by &= h, \\ a(gf)' + b(gf) &= h, \\ a(g'f + gf') + bgf &= h, \\ ag'f + g(af' + bf) &= h, \\ ag'f &= h, \text{ since } af' + bf = 0, \\ g' &= \frac{h}{af}, \\ g(x) &= \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt. \end{aligned}$$

Thus,

$$y(x) = f(x)g(x) = f(x)\int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt$$

(b) In this case $a(x) = x, b(x) = 2, h(x) = x$. First we find $f \in \ker \ell$

$$\begin{aligned} af' + bf &= 0, \\ xf' + 2f &= 0, \\ \int_{x_0}^{x} \frac{f'}{f}(t)\, dt &= -\int_{x_0}^{x} \frac{2}{t} dt, \\ \log|f| &= -2\log|x| + C, \\ |f| &= C\exp\left[-2\log|x|\right] = Cx^{-2}. \end{aligned}$$

Choose $f = x^{-2}$. Using our formula from part (1)

$$\begin{aligned} y(x) = f(x)g(x) &= f(x)\int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt \\ &= x^{-2}\int_{x_0}^{x} \frac{t}{t \cdot t^{-2}}\, dt, \\ &= x^{-2}\int_{x_0}^{x} t^2\, dt, \\ &= \frac{1}{3}x + \frac{C}{x^2}. \end{aligned}$$

5. (a) Plug $y(x) = g(x)f(x)$ into the equation $\ell(y) = h$,

$$\begin{aligned}
ay' + by &= h, \\
a(gf)' + b(gf) &= h, \\
a(g'f + gf') + bgf &= h, \\
ag'f + g(af' + bf) &= h, \\
ag'f &= h, \text{ since } af' + bf = 0, \\
g' &= \frac{h}{af}, \\
g(x) &= \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt.
\end{aligned}$$

Thus,

$$y(x) = f(x)g(x) = f(x) \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt$$

(b) In this case $a(x) = x, b(x) = 2, h(x) = x$. First we find $f \in \ker \ell$

$$\begin{aligned}
af' + bf &= 0, \\
xf' + 2f &= 0, \\
\int_{x_0}^{x} \frac{f'}{f}(t)\, dt &= -\int_{x_0}^{x} \frac{2}{t} dt, \\
\log|f| &= -2\log|x| + C, \\
|f| &= C \exp\left[-2\log|x|\right] = Cx^{-2}.
\end{aligned}$$

Choose $f = x^{-2}$. Using our formula from part (1)

$$\begin{aligned}
y(x) = f(x)g(x) &= f(x) \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)}\, dt \\
&= x^{-2} \int_{x_0}^{x} \frac{t}{t \cdot t^{-2}}\, dt, \\
&= x^{-2} \int_{x_0}^{x} t^2\, dt, \\
&= \frac{1}{3}x + \frac{C}{x^2}.
\end{aligned}$$

6. (a) Recall that the Taylor series for e^x about $x_0 = 0$ is

$$1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

and the ratio test says that for a series

$$S = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then the series S converges (also if the limit if > 1 then the series diverges). Thus, consider

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{x}{n} \right| < 1,$$

and this holds for *any* value of x. In case we see that the radius of convergence is $R = \infty$.

For the function $\log(1+x)$ write

$$\begin{aligned} \log(1+x) &= \int_0^x \frac{1}{1+t}\, dt, \\ &= \int_0^x \left[1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots\right] dt, \text{ geometric series,} \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n+1}\frac{1}{n}x^n \cdots \end{aligned}$$

and consider

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = \lim_{n\to\infty} |x| \left| \frac{n}{n+1} \right| = |x|.$$

Thus, the radius of convergence for $\log(x+1)$ is $R = 1$.

(b) For $x > 0$, one proves by induction (recall the principle of induction and carry out the argument!) that for all $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \exp\left(-\frac{1}{x}\right)\left(\frac{c_1}{x^{n+1}} + \frac{c_2}{x^{n+2}} + \cdots + \frac{c_n}{x^{2n}}\right)$$

for some constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Given any $n \in \mathbb{N}$, we also see that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq \frac{x^{2n+2}}{(2n+2)!}$$

for all $x \geq 0$. This implies, that for all $n \in \mathbb{N}$, we have

$$0 \leq e^{-x} \leq \frac{(2n+2)!}{x^{2n+2}}$$

Proceeding as in class, we now see that for any $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \lim_{\substack{h \to 0, \\ h>0}} \frac{1}{h} \left(\frac{c_1}{h^{n+1}} \pm \frac{c_2}{h^{n+2}} \pm \cdots + \frac{c_n}{h^{2n}} \right) e^{-\frac{1}{h}} \\ &\leq \lim_{\substack{h \to 0, \\ h>0}} \frac{1}{h} \left(\frac{c_1}{h^{n+1}} \pm \frac{c_2}{h^{n+2}} \pm \cdots + \frac{c_n}{h^{2n}} \right) \frac{(2n+2)!}{(1/h)^{2n+2}} = 0 \end{aligned}$$

and that this implies that $f^{(n)}(0) = 0$ (can you explain how this follows?). Hence $f^{(n)}(0) = 0$ for *all* $n = 0, 1, 2, 3, \ldots$, which means that the Taylor series for f is the zero function and this can not equal f since $\exp\left[-1/x\right]$ is not zero for any small positive value $x > 0$!