Problem Set 5

Due: Tue, March 10, 6.00 PM

Topics: continuation of metric spaces

1. In view of our midterm exam, review first and second order linear ODE as well as all the notions and methods we introduced to solve them.

Definition: Let (M, d) be a metric space. We say that a sequence $(x_j)_{j\in\mathbb{N}}$ in M (that is, $x_j \in M$ for all $j \in \mathbb{N}$) converges in M if there exists a point $y \in M$ such that for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ s.t. $d(x_j, y) \leq \varepsilon$ whenever $j \geq N$.

Definition: Let (M, d) be a metric space. We call $(x_j)_{j\in\mathbb{N}}$ a Cauchy sequence if for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ s.t. $d(x_j, x_k) \leq \varepsilon$ whenever $j, k \geq N$.

2. Consider $\mathbb{R}^n$ equipped with the following metrics:

a) $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0;\infty)$, defined through $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$.

b) $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0;\infty)$, defined through $d_2(x, y) = \big(\sum_{i=1}^n |x_i - y_i|^2\big)^{1/2}$.

c) $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0;\infty)$, defined through $d_\infty(x, y) = \max_{i\in\{1,\dots,n\}} |x_i - y_i|$.

Prove that in each case a Cauchy sequence $(x_j)_{j\in\mathbb{N}}$ in $(\mathbb{R}^n, d_j)$ is also a convergent sequence in $(\mathbb{R}^n, d_j)$, for $j = 1, 2, \infty$.

Definition: Let (M, d) be a metric space. A subset $C \subset M$ is called *closed* in M if for every sequence $(x_n)_{n\in\mathbb{N}}$ s.t. $x_n \in C$ for all $n \in \mathbb{N}$ and s.t. $\lim_{n\to\infty} x_n = x \in M$, we have that $x \in C$. Put in words: C contains all of its limit points.

3. Let (M, d) be a metric space. Prove the following:

a) Show that the whole space M is a closed subset of M.

b) Show that $B_\delta(\mathbf{y}_0) = \{\mathbf{y} \in M : d(\mathbf{y}_0, \mathbf{y}) \leq \delta\}$ is a closed subset of (M, d). We call $B_\delta(\mathbf{y}_0)$ the *closed ball of radius δ around $\boldsymbol{y}_0 \in M$*.

c) Choose $(M, d) = (C([a;b];\mathbb{R}^n), d_\infty)$ and fix $\delta \geq 0$, $\mathbf{y}_0 \in \mathbb{R}^n$. Define the set $A = \{g \in C([a;b];\mathbb{R}^n) : \sup_{x\in[a;b]} d_2(g(\mathbf{x}), \mathbf{y}_0) \leq \delta\}$. Show that $A \subset C([a;b];\mathbb{R}^n)$ is a closed subset of $C([a;b];\mathbb{R}^n)$.

4. Let $p \geq 1$ be a real number. The space (real) ℓ^p of sequences is defined by

$$\ell^p = \Big\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\, \forall\, n \in \mathbb{N}, \text{ and } \sum_{n=1}^{\infty} |x_n|^p < \infty \Big\}$$

Given $x, y \in \ell^p$, define $d_p : \ell^p \times \ell^p \to \mathbb{R}$ through

$$d_p(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

We also allow for the case $p = \infty$ and define in that case

$$\ell^\infty = \Big\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\, \forall\, n \in \mathbb{N}, \text{ and } \sup_{n\in\mathbb{N}} |x_n| < \infty \Big\}, \quad d_\infty(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n|.$$

In this exercise, we want to prove that $d_p : \ell^p \times \ell^p \to \mathbb{R}$ defines a metric:

(a) Recall from analysis that a function f is called *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all x, y in the domain of f and $t \in [0;1]$. f is called *convave* if $-f$ is convex. A smooth function is convex if its second derivative is non-negative. Prove that $\log : (0;\infty) \to \mathbb{R}$ is concave.

(b) For $1 \leq p < \infty$, prove that for all $x, y \geq 0$, we have $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ where $q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. This inequality is called *Young's inequality.*
Hint: Recall that $a = \exp(\log(a))$ *and use that the logarithm is concave!*

(c) Let $q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Here we allow for the case $p = 1$ and $q = \infty$. Show that

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}$$

for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p, y = (y_n)_{n\in\mathbb{N}} \in \ell^q$. This is *Hölder's inequality.*

Hint: As in class for $p = 2$ *divide first by the right hand side. Use Young's inequality and proceed as in the case* $p = 2$.

(d) Show that d_p is a metric for every $p \geq 1$ and also for $p = \infty$. The difficult part here is to prove the triangle inequality in the cases $p > 1$. To do this, use that $|a+b|^p \leq (|a| + |b|)(|a+b|)^{p-1}$ and apply Hölder's inequality from the previous part in a suitable way.