MATH110 Spring 2020 HW4

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Problem 1

We first solve the homogeneous equation, considering solutions of form e^{rx}:

$$y'' + y' - 2y = 0 \Rightarrow r^2 + r - 2 = 0 \Rightarrow r = -2, 1 \Rightarrow y_n(x) = c_1 e^{-2x} + c_2 e^x$$

Let $f_1(x) = e^{-2x}$ and $f_2(x) = e^x$. We use the Wronskian to confirm that the two functions are linearly independent:

$$W = e^{-2x}e^x - (-2e^{-2x}e^x) = 3e^{-x} \neq 0$$

From lecture, we know that c_1 has the following form:

$$\begin{aligned}
c_1(x) &= -\int_{t=x_0}^{t=x} \frac{f_2(t)h(t)}{a(t)W(t)} dt \\
&= -\int_{t=x_0}^{t=x} \frac{e^x(x-2x^3)}{3e^{-x}} dt \\
&= -\frac{1}{3}\Big(\int_{t=x_0}^{t=x} xe^{2x} - \int_{t=x_0}^{t=x} 2x^3e^{2x}\Big) \\
&= -\frac{1}{3}\Big(\frac{1}{4}e^{2x}(2x-1) - \frac{1}{4}e^{2x}(4x^3 - 6x^2 + 6x - 3)\Big) \\
&= \frac{1}{12}e^{2x}(4x^3 - 6x^2 + 4x - 2)
\end{aligned}$$

And we know that c_2 has the following form

$$\begin{aligned}
c_2(x) &= \int_{t=x_0}^{t=x} \frac{f_1(t)h(t)}{a(t)W(t)} dt \\
&= \int_{t=x_0}^{t=x} \frac{e^{-2x}(x-3x^2}{a(t)3e^{-x}} dt \\
&= -\frac{1}{3}\Big(\int_{t=x_0}^{t=x} xe^{-x} - \int_{t=x_0}^{t=x} 3x^2e^{-x}dt\Big) \\
&= -\frac{1}{3}\Big(-e^{-x}(x+1) - -2e^{-x}(x^3 + 3x^2 + 6x + 6)\Big) \\
&= \frac{1}{3}e^{-x}(-2x^3 - 6x^2 - 11x - 11)
\end{aligned}$$

This yields the particular solution:

$$
\begin{aligned}
y_p &= c_1 f_1 + c_2 f_2 \\
&= x^3 + \frac{3x^2}{2} + 4x + \frac{7}{2}
\end{aligned}
$$

With the general solution, with $k_1, k_2 \in \mathbb{R}$:

$$y_g = y_p + k_1 f_1 + k_2 f_2 = x^3 + \frac{3x^2}{2} + 4x + \frac{7}{2} + k_1 f_1 + k_2 f_2$$

Problem 2

Some of my work relies on the properties that $|a - b| = |b - a|$ and $|a + b| < |a| + |b|$.

a

1. Because $|x_i - y_i| \geq 0$, with equality only when $x_i = y_i$, each term in the sum $d_1(x, y)$ must be non-negative. Since the sum of non-negative terms is non-negative, we conclude $d_1(x, y) \geq 0$. Since the sum of non-negative numbers is zero only when every term is zero, we also conclude that $d_1(x, y) = 0$ only when $\forall i \in \{1, ..., n\}$, $x_i - y_i = 0$

2. $$d_1(x, y) \overset{\text{def}}{=} (\sum_{i=1}^{n} |x_i - y_i|) = (\sum_{i=1}^{n} |y_i - x_i|) \overset{\text{def}}{=} d_1(y, x)$$

3. Note that $|x_i - y_i| = |x_i - z_i + z_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$, where the inequality follows from the triangle inequality, which states that $|a + b| \leq |a| + |b|$. Since each element on the LHS is less than or equal to each element on the RHS, the sum on the LHS must be less than or equal to the sum on the RHS.

b

1. Because $|x_i - y_i|^2 \geq 0$, with equality only when $x_i = y_i$, each term in the sum $d_2(x, y)$ must be non-negative. Since the sum of non-negative terms is non-negative, we conclude $d_2(x, y) \geq 0$. Since the sum of non-negative zeros is zero only when every term is zero, we also conclude that $d(x, y) = 0$ only when $\forall i \in \{1, ..., n\}$, $x_i - y_i = 0$

2. $$d_2(x, y) \overset{\text{def}}{=} (\sum_{i=1}^{n} |x_i - y_i|^2)^{1/2} = (\sum_{i=1}^{n} |y_i - x_i|^2)^{1/2} \overset{\text{def}}{=} d_2(y, x)$$

3. Our goal is to show that $d_2(x,y) \leq d_2(x,z) + d_2(z,y)$. Define $a_i = x_i - z_i$ and $z_i - y_i$, which means that $a_i + b_i = x_i - y_i$. Starting with the LHS, showing the Triangle Inequality holds is equivalent to showing that:

$$
\begin{aligned}
d_2(x,y) &\leq d_2(x,z) + d_2(z,y) \\
d_2(x,y)^2 &\leq d_2(x,z)^2 + d_2(z,y)^2 + 2d_2(x,z)d_2(z,y) \\
\sum_i |a_i + b_i|^2 &\leq \sum_i |a_i|^2 + \sum_i |b_i|^2 + 2(\sum_i a_i^2)^{1/2}(\sum_i b_i^2)^{1/2} \\
\sum_i a_i b_i &\leq (\sum_i a_i^2)^{1/2}(\sum_i b_i^2)^{1/2} \\
\frac{1}{(\sum_i a_i^2)^{1/2}(\sum_i b_i^2)^{1/2}} \sum_i a_i b_i &\leq 1 \\
\sum_i \frac{a_i}{(\sum_i a_i^2)^{1/2}} \frac{b_i}{(\sum_i b_i^2)^{1/2}} &\leq 1
\end{aligned}
$$

Using the properties that $\sum_i a_i b_i \leq \sum_i |a_i b_i|$ and that $|a_i b_i| \leq \frac{1}{2}a_i^2 + \frac{1}{2}b_i^2$, we see that the above equality holds:

$$\sum_i \frac{a_i}{(\sum_i a_i^2)^{1/2}} \frac{b_i}{(\sum_i b_i^2)^{1/2}} \leq \frac{1}{2}\sum_i \frac{a_i^2}{\sum_j a_j^2} + \frac{1}{2}\sum_i \frac{b_i^2}{\sum_j b_j^2} = \frac{1}{2} + \frac{1}{2} = 1$$

c

(a) Same reason as 2a1 and 2b1. The max of a set of non-negative numbers is non-negative, with equality if and only if the max of a set of absolute values is zero, implying all differences are zero and thus $x = y$.

(b)

$$d_\infty(x,y) \stackrel{\text{def}}{=} \max_{i \in \{1,\ldots,n\}} |x_i - y_i| = \max_{i \in \{1,\ldots,n\}} |y_i - x_i| \stackrel{\text{def}}{=} d_\infty(y,x)$$

(c)

$$d_\infty(x,y) = \max_i |x_i - y_i| = \max_i |x_i - z_i + z_i - y_i| \leq \max_i |x_i - z_i| + |z_i - y_i| \leq \max_i |x_i - z_i| + \max_i |z_i - y_i|$$

$$d_\infty(x,y) \leq d_\infty(x,z) + d_\infty(z,y)$$

Problem 3

a

I first show that if each sequence of real numbers converges, then the sequence of vectors converges. Assume that $\forall i \in \{1,...,n\}, \forall \epsilon^i > 0, \exists N^i \in \mathbb{N}$ such that $\forall j \geq N^i, d(x_j^i, y^i) \leq \epsilon^i$. We want to show that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall j \geq N, d(x_j, y) \leq \epsilon$. Fix ϵ. For each sequence, choose $\epsilon^i = \frac{\epsilon}{n}$ and take $N^* = \max\{N^i\}_{i=1}^n$, which we know exists because each sequence individually converges. Then $\forall j \geq N^*$, $d(x_j, y) = \sum_{j=1}^n d(x_j^i, y^i) \leq \sum_{j=1}^n \frac{\epsilon}{n} = \epsilon$.

I then show the opposite direction, that if the sequence of vectors converges, then each sequence of real numbers converges. Because the sequence of vectors converges, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$d_1(x_j, y) \leq \epsilon \Rightarrow \sum_i |x_j^i - y^i| \leq \epsilon$. By the non-negativity of absolute value, this means that the largest term in the summation is bounded by ϵ. Since we can find an N for any ϵ, and since N ensures $|x_j^i - y^i| \leq \epsilon$, each sequence $(x_j^i)_{j\in\mathbb{N}}$ converges.

b

I'm going to be hand-wavy since intuitively, it's the same proof. If each element of the vector is a sequence that converges, we know that for any bound ϵ on the distance, at some index in the sequence every element of the vector is below the bound. Then the entire vector has converged to be below some other bound, which depends on the sum of the element-wise bounds. But since neither the element-wise distance bounds were specified nor the vector distance bound was specified, I can play with one until I get the other. More specifically, I can fix the element-wise distance bounds and find the corresponding vector distance bound, or I can fix the vector distance bound and find the element-wise distance bounds.

c

Same answer as b.

Problem 4

To show that $C([a;b])$ is a vector space, we define scalar multiplication and element-wise addition. Let $f_1, f_2 \in C([a;b])$ and let $c_1, c_2 \in \mathbb{R}$. Define scalar multiplication as $c_1 f_1$ and element-wise addition as $f_1 + f_2$. $c_1 f_1 + c_2 f_2 \in C([a;b])$ because the sum of two continuous functions is continuous and the scalar multiple of a continuous function is also continuous. Thus $C([a;b])$ is a vector space. I now prove that $(C([a;b]), d_\infty)$ is a metric space.

(a) $d_\infty(f,g) \geq 0$ is satisfied since $\forall x$, $|f(x) - g(x)| \geq 0$ due to the absolute value function. We then need to show $d_\infty(f,g) = 0 \leftrightarrow f = g$. Going left to right, assume $f = g$. Then $|f - g| = 0 \Rightarrow d_\infty(f,g) = 0$. Going now right to left, assume $d_\infty(f,g) = 0$. For purposes of contradiction, assume $f \neq g \Rightarrow \exists\, x$ s.t. $f(x) \neq g(x) \Rightarrow |f(x) - g(x)| > 0 \Rightarrow d_\infty(f,g) > 0$, which contradictions the assumption that $f \neq g$.

(b)
$$d_\infty(f,g) \overset{\text{def}}{=} \sup_{x\in[a,b]} \{|f(x) - g(x)|\} = \sup_{x\in[a,b]} \{|g(x) - f(x)|\} \overset{\text{def}}{=} d_\infty(fg, f)$$

(c) We want to show that $d_\infty(f,g) \leq d_\infty(f,p) + d_\infty(p,g)$. Dropping the supremum's domain for brevity, we see that

$$d_\infty(f,g) = \sup\{|f-g|\} = \sup\{|f-p+p-g|\} \leq \sup\{|f-p|+|p+g|\} \leq \sup\{|f-p|\}+\sup\{|p+g|\} = d_\infty(f,p)+d_\infty(p,g)$$