Week 6

Topics: Local existence & uniqueness for 1st order ODE II: completeness, contraction argument

Review from Week 5:

- Our goal is to solve the general first order ODE initial value problem

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

- Our strategy is to find a continuous function $\mathbf{y} \in C((x_0 - \epsilon; x_0 + \epsilon); \mathbb{R}^n)$ such that

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \mathbf{y}(s))\, ds$$

- The appropriate setting to generalize our fixed point strategy is that of *metric spaces*. A metric space (M, d) is a set M together with a function $d : M \times M \to [0; \infty)$ such that for all $x, y, z \in M$ we have that

$$\begin{aligned} &1)\ d(x,y) \geq 0 \text{ and } d(x,y) = 0 \leftrightarrow x = y, \\ &2)\ d(x,y) = d(y,x), \\ &3)\ d(x,y) \leq d(x,z) + d(z,y). \end{aligned}$$

1. Metric spaces for 1st order ODE.

Most important for the standard local existence and uniqueness theorem for 1st order ODE are spaces of continuous functions. To this end, fix $a < b, a, b \in \mathbb{R}$ and set

$$C([a;b]) = C([a;b];\mathbb{R}) = \big\{ f : [a;b] \to \mathbb{R} : f \text{ continuous in } [a;b] \big\}$$

Recall that f is continuous at the point $x_0 \in [a;b]$ iff for all $\varepsilon > 0$ there exists a $\delta = \delta_\varepsilon > 0$ s.t. $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$. f is continuous in the interval $[a;b]$ if it is continuous at each $x_0 \in [a;b]$.

(a) Define $d_\infty : C([a;b]) \times C([a;b]) \to \mathbb{R}$ through

$$d_\infty(f,g) = \sup_{x \in [a;b]} |f(x) - g(x)|.$$

Prove that d_∞ defines a metric on $C([a;b])$.

(b) Set $C([a;b];\mathbb{R}^n) = \big\{ f : [a;b] \to \mathbb{R}^n : f \text{ continuous in } [a;b] \big\}$ and define (with slight abuse of notation) $d_\infty : C([a;b];\mathbb{R}^n) \times C([a;b];\mathbb{R}^n) \to \mathbb{R}$ in this case through

$$d_\infty(f,g) = \sup_{x \in [a;b]} \Big(\sum_{i=1}^{n} |f_i(x) - g_i(x)|^2 \Big)^{1/2}.$$

Prove that d_∞ defines a metric on $C([a;b];\mathbb{R}^n)$.

2. Convergence and Cauchy sequences in metric spaces.

In $(\mathbb{R}, |\cdot|)$, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty \in \mathbb{R}$ iff for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_\infty| \le \varepsilon \quad \text{for all} \quad n \ge N.$$

A Cauchy sequence is a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m| \le \varepsilon \quad \text{for all} \quad n, m \ge N.$$

(a) Suppose that (M, d) is a general metric space. Can you formulate an analogous definition for a convergent and for a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ ($x_n \in M$ for all $n \in \mathbb{N}$) in (M, d)?

(b) Show that every Cauchy sequence in a metric space (M, d) is bounded.

(c) Show that every convergent sequence in $\mathbb{R}$ is a Cauchy sequence. What about the same statement in a general metric space?

(d) Recall that every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$. Is this true in a general metric space as well?

3. Completeness of $C(a;b])$. Prove that $C([a;b])$ with the metric from problem **1** is complete: if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(C([a;b]), d_\infty\big)$, then there exists a limit $f_\infty \in C([a;b])$ such that $\lim_{n\to\infty} f_n = f_\infty$ in $C([a;b])$. To prove this theorem, proceed as follows:

(a) First, we have to find a good candidate f_∞ for our limit. To this end, fix $x \in [a;b]$ and prove that the sequence $(f_n(x))_{n\in\mathbb{N}}$ **of real numbers** is a Cauchy sequence in $\mathbb{R}$. Consequently, what function $f_\infty : [a;b] \to \mathbb{R}$ is a natural candidate for our limit?

(b) We have to prove that our candidate f_∞ is a continuous function in $[a;b]$. Prove that for any $x, y \in [a;b]$ and $n, m \in \mathbb{N}$, it holds true that

$$|f_\infty(x) - f_\infty(y)| \leq |f_\infty(x) - f_m(x)| + |f_\infty(y) - f_m(y)| + |f_n(x) - f_n(y)| + 2d_\infty(f_n, f_m).$$

How does this help?

(c) Knowing that $f_\infty \in C([a;b])$, it remains to prove that $\lim_{n\to\infty} f_n = f_\infty$ in $\big(C([a;b]), d_\infty\big)$. It may be useful to observe that for all $n, m \in \mathbb{N}$, it holds true that

$$|f_\infty(x) - f_n(x)| \leq |f_\infty(x) - f_m(x)| + d_\infty(f_n, f_m).$$

4. The convergence criterion: Banach's fixed point theorem. Let (M, d) be a complete metric space with $M \neq \emptyset$. Suppose $T : M \to M$ is a map with the property that

$$d(T(x), T(y)) \leq c d(x, y).$$

for all $x, y \in M$ and a constant $0 \leq c < 1$. We call such a map a *contraction*. Prove that T has a unique fixed point $x \in M$. This means that there exists a unique element $x \in M$ such that $T(x) = x$. To prove this fact, proceed as follows:

(a) Pick some fixed $x_1 \in M$ (it is here where we use that $M \neq \emptyset$). Define $(x_n)_{n \in \mathbb{N}}$ by $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$. Prove that the resulting sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in M, due to the contraction property of T.

(b) Conclude that the sequence from the previous part has a limit $x \in M$.

(c) Prove that x is a fixed point of T. Prove that it is the unique fixed point of T.

Answers and Solutions.

1.

(a) Since every continuous function on a closed and bounded interval attains its maximum, it is clear that $d_\infty(f,g) < \infty$ for all $f, g \in C([a;b])$. To check the remaining three properties of a metric, let's observe that

$$0 \le d_\infty(f,g) \ \text{ and } \ d_\infty(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = 0 \leftrightarrow f(x) = g(x)\ \forall x \in [a;b] \leftrightarrow f = g$$

and that

$$d_\infty(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = \sup_{x\in[a;b]} |g(x) - f(x)| = d_\infty(g,f).$$

It thus only remains to verify the triangle inequality. Here we use the triangle inequality on $\mathbb{R}$ and estimate

$$\begin{aligned} d_\infty(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| &\le \sup_{x\in[a;b]} \Big(|f(x) - h(x)| + |h(x) - g(x)|\Big) \\ &\le \sup_{x\in[a;b]} |f(x) - h(x)| + \sup_{x\in[a;b]} |h(x) - g(x)| = d_\infty(f,h) + d_\infty(h,g). \end{aligned}$$

(b) The proof is very similar and I leave this as an exercise. To carry out the details, recall the definition of the metric $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0;\infty)$, defined in class and on problem set **4**, and then proceed as in the previous part (a).

2.

(a) Let (M, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in M converges if and only if there exists some $y \in M$ such that for all $\varepsilon > 0$ there exists $N = N_\varepsilon \in \mathbb{N}$ so that for all $n \geq N$ it holds true that

$$d(x_n, y) \leq \varepsilon.$$

In this case we call $y \in M$ *the limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, and we introduce the notation $y = \lim_{n\to\infty} x_n$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in M converges is called a Cauchy sequence if for all $\varepsilon > 0$ there exists $N = N_\varepsilon \in \mathbb{N}$ so that for all $n, m \geq N$ it holds true that

$$d(x_n, x_m) \leq \varepsilon.$$

(b) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. To get the intuition why it is bounded, try to visualize the following proof: notice that, by the Cauchy property, we can find $N \in \mathbb{N}$ such that (for instance) $d(x_n, x_m) \leq 1$ for all $n, m \geq N$. Set $z = x_N$, then

$$d(z, x_m) \leq K = \max\{1, d(z,x_1), \ldots, d(z, x_{N-1})\} < \infty$$

for all $m \in \mathbb{N}$, which means that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence: it is contained in the ball of radius K around the point $z \in M$.

(c) Suppose $(x_n)_{n\in\mathbb{N}}$ is convergent in the metric space (M, d) (the case $(\mathbb{R}, |\cdot|)$ is special case). Let $x \in M$ denote its limit. Given $\varepsilon > 0$, we choose $N \in \mathbb{N}$ s.t. $d(x_n, x) \leq \varepsilon/2$ for all $n \geq N$. Then, the triangle inequality implies that

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq \varepsilon.$$

Hence, the sequence is a Cauchy sequence.

(d) This is not true in general. By worksheet **5**, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers, $x_n \in \mathbb{Q} \subset \mathbb{R}$ for all $n \in \mathbb{N}$, such that, as a sequence in $\mathbb{R}$, we have that $\lim_{n\to\infty} x_n = \sqrt{2}$. The Greeks noticed a while ago that $\sqrt{2}$ is not a rational number. But then, we see that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(\mathbb{Q}, |\cdot|)$ of rational numbers, but not convergent in $(\mathbb{Q}, |\cdot|)$ - its limit lies outside of $(\mathbb{Q}, |\cdot|)$. We call a space in which every Cauchy sequence converges a *complete metric space*. The metric space $(\mathbb{Q}, |\cdot|)$ is thus not complete.

3.

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C([a;b]), d_\infty)$. This implies the following: for every $x \in I = [a;b]$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers is a Cauchy sequence, because for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$, it holds true that

$$|f_n(x) - f_m(x)| \leq d_\infty(f_n, f_m) \leq \varepsilon.$$

Hence, by completeness of $\mathbb{R}$, for every $x \in I$, there exists a limit $y = y_x = \lim_{n\to\infty} f_n(x)$. We define a function $f : I \to \mathbb{R}$ through these limits, that is $f(x) = y_x$. f is our candidate for the limit of our sequence $(f_n)_{n\in\mathbb{N}}$ in $(C([a;b]), d_\infty)$.

(b) The inequality follows from the triangle inequality in $(C([a;b]), d_\infty)$. Verify this! Once we are done with that, let's fix an arbitrary $x \in I$ and some $\varepsilon > 0$. The Cauchy property of $(f_n)_{n\in\mathbb{N}}$ implies that for all $m \geq N$, for a suitable large $N \in \mathbb{N}$, it holds true

$$|f(x) - f(y)| \leq |f(x) - f_m(x)| + |f(y) - f_m(y)| + |f_N(x) - f_N(y)| + \varepsilon/2.$$

Since f_N is a continuous map, there exists $\delta > 0$ s.t. for all $y \in I$ with $|x - y| \leq \delta$, it holds true that $|f_N(x) - f_N(y)| \leq \varepsilon/2$. Thus, we have proved that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $y \in I$ with $|x - y| \leq \delta$, it holds true that

$$|f(x) - f(y)| \leq |f(x) - f_m(x)| + |f(y) - f_m(y)| + \varepsilon, \ \textit{for all } m \geq N.$$

Since $m \geq N$ is arbitrary and $\lim_{m\to\infty} f_m(x) = f(x), \lim_{m\to\infty} f_m(y) = f(y)$, it follows that for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $y \in I$ with $|x - y| \leq \delta$, it holds true that

$$|f(x) - f(y)| \leq \varepsilon.$$

But this means that $f \in C(I)$.

(c) The inequality follows, as in the previous part, by the triangle inequality and the definition of the metric in $C(I)$. By the Cauchy property of $(f_n)_{n\in\mathbb{N}}$, we find that for all $m \geq N$, for a suitable $N \in \mathbb{N}$, it holds true that

$$|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + \varepsilon, \ \textit{for all } m \geq N.$$

Once more by the definition of the limit function f, this implies that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have *for all* $x \in I$ that

$$|f(x) - f_n(x)| \leq \varepsilon$$

Since the last inequality is true for all $x \in I$, it implies that $d_\infty(f, f_n) \leq \varepsilon$, hence the claim.

4.

(a) We apply iteratively the triangle inequality

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq cd(x_{n+m-1}, x_{n+m-2}) + cd(x_{n+m-2}, x_{n+m-3}) + \cdots + cd(x_n, x_{n-1}) \\
&\leq c^{n+m-1} d(x_2, x_1) + c^{n+m-2} d(x_2, x_1) + \cdots + c^{n-1} d(x_2, x_1) \\
&\leq c^{n-1} d(x_2, x_1) \big(1 + c + \cdots + c^m\big) \\
&\leq c^{n-1} d(x_2, x_1) \sum_{j=0}^{\infty} c^j = c^{n-1} d(x_2, x_1) \frac{1}{1-c} \to 0
\end{aligned}$$

as $n \to \infty$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Notice that we used crucially that $c < 1$.

(b) We assume that (M, d) is complete, so that $(x_n)_{n\in\mathbb{N}}$, as a Cauchy sequence, has a limit $x \in M$.

(c) We apply the triangle inequality to see that

$$d(T(x), x) \leq d(T(x_n), T(x)) + d(x_{n-1}, x) \leq cd(x_n, x) + d(x, x_{n-1}) \to 0$$

as $n \to \infty$. Hence $T(x) = x$. The fixed point is unique, because if $T(x) = x, T(y) = y$, then

$$d(x, y) = d(T(x), T(y)) \leq cd(x, y) < d(x, y),$$

and therefore $d(x, y) = 0$, that is $x = y$!