Week 11

Topics: compatibility and regularity, recap linear algebra, boundary conditions, Laplacians

Review from Week 10:

- **Boundary value problems:** motivated by the vibrating wire, our goal is to set up a mathematical framework that deals with eigenvalue problems in infinite dimensions of the form:

$$\begin{cases} y'' &= \lambda y \\ y(0) &= y(L) = 0. \end{cases}$$

- **Compatibility:** If we want to define a differential operator $\ell(y) = ay'' + by' + cy$ on some $L^2(I; dx)$ space, we need at least to make sure that $\ell(y) \in L^2(I; dx)$. A function $y \in L^2(I; dx)$ such that $\ell(y) \in L^2(I; dx)$ is called *compatible* with ℓ.

- **Regularity:** Functions $y \in L^2(I; dx)$ need not be regular (differentiable) in the usual sense. To make nevertheless sense of expressions like y', y'', etc., one can introduce the notion of *weak derivatives*. We say that $\zeta \in L^2(I; dx)$ is the weak derivative of y if and only if

$$y(x) = y(x_0) + \int_{x_0}^{x} \zeta(s)\, ds.$$

If y has such a weak derivative, we write symbolically $\zeta \equiv y'$. The space of functions with weak derivatives in the $L^2(I; dx)$ sense is denoted by $H^1(I; dx)$, the Sobolev space of first order. If $y \in H^1(I; dx)$ such that also $y' \in H^1(I; dx)$, we write $y \in H^2(I; dx)$, etc. Having introduced Sobolev functions, it makes sense to define for instance $\ell(y) = -y''$ on $H^2(I; dx)$, which is a strictly bigger set of functions than, for instance, $C^2(I)$.

- **Integration by parts in H^1:** If $\phi, \psi \in H^1(I; dx)$, then also $\phi\psi \in H^1(I; dx)$ and the weak derivative is given by $(\phi\psi)' = \phi'\psi + \psi'\phi$. In particular, we can integrate by parts as usual.

1. Regularity 2. Let's continue our discussion with a similar example on $L^2((-\frac{1}{2};\frac{1}{2});dx)$ as last time and with $\ell(y) = -y''$ for $y \in C^2((-\frac{1}{2};\frac{1}{2}))$. Last week's problem suggests that we can make sense of $\ell(y)$ as long as y and y' have L^2 derivatives so that the fundamental theorem of calculus applies to them. To provide another example that supports this intuition, let's analyse the function $x \mapsto \phi(x) = x\mathbf{1}_{[0;\frac{1}{2}]}(x)$.

(a) Show that $\phi \notin C^1((-\frac{1}{2};\frac{1}{2}))$, but that ϕ' exists almost everywhere in $(-\frac{1}{2};\frac{1}{2})$ and that $\phi, \phi' \in L^2((-\frac{1}{2};\frac{1}{2});dx)$. Moreover, if you had to, what "value" would you assign to the second derivative $\phi''(0)$ at $x_0 = 0$? What "value" would you assign to $\phi''(x)$ for $x \neq 0$?

- note $\phi'(x) = 0$ if $x<0$ and 1 if $x>0$
- at $x_0 = 0$ ϕ is <u>not</u> differentiable; clearly $\phi, \phi' \in L^2$
- if we have to assign a value to ϕ'' at 0, then "$\phi'' = \infty$"

(b) Suppose that, as in the previous problem, we can still make sense of $\ell(\phi) = -\phi''$. More precisely, suppose there exists a function[1] $\delta \in L^2((-\frac{1}{2};\frac{1}{2});dx)$ such that

$$\phi'(x) = \phi'(x_0) + \int_{x_0}^{x} ds\, \delta(s).$$

With the discussion in class about absolutely continuous functions and the FTC, use integration by parts to show that under this assumption, it holds true that

$$\int_{-1/2}^{1/2} dx\, \psi(x)\delta(x) = \psi(0)$$

for all $\psi \in C^2((-\frac{1}{2};\frac{1}{2}))$. Use the Fourier series expansion from problem set 6 and Parseval's theorem to prove that then $\delta \notin L^2((-\frac{1}{2};\frac{1}{2});dx)$. Thus, it *does not* make sense to give meaning to $\ell(\phi)$ in the L^2 sense – functions that have discontinuity jumps in their first derivatives, like ϕ' has, are examples of functions that are *not compatible* with ℓ!

- see class notes (April 7)

[1] In physics jargon, this would correspond to the *famous Dirac δ function!*

2. Linear Algebra Recap. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric or $A \in \mathbb{C}^{n\times n}$ Hermitian. Prove the following statements.

(a) A is symmetric (or Hermitian) iff for all $x, y \in \mathbb{K}^n$ it holds true that $\langle Ax, y\rangle = \langle x, Ay\rangle$.

"$\Rightarrow$": $\langle Ax, y\rangle = \sum_{i,j} a_{ij} x_i \overline{y}_j = \sum_{i,j} x_i \overline{a}_{ji} \overline{y}_j = \langle x, Ay\rangle$

"$\Leftarrow$", pick $x = e_i = (0, 0, \ldots, \underbrace{1}_{i\text{-th pos.}}, 0, \ldots)$, $y = e_j$

$\Rightarrow \langle Ax, y\rangle = a_{ij}$ and $\langle x, Ay\rangle = \overline{a}_{ji}$

(b) The eigenvalues of A are real.

$\lambda = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle$

$= \langle v, Av\rangle = \overline{\lambda} \Rightarrow \lambda = \overline{\lambda}$

(c) Eigenvectors to different eigenvalues of A are orthogonal.

$(\lambda_2 - \lambda_1)\langle v_2, v_1\rangle \underset{\lambda_1 \in \mathbb{R}}{=} \langle Av_2, v_1\rangle - \langle v_2, Av_1\rangle$

$= 0$

$\Rightarrow \langle v_2, v_1\rangle = 0$ if $\lambda_2 \neq \lambda_1$

(d) Orthogonal vectors are linearly independent.

If $\lambda_1 \phi_1 + \cdots + \lambda_n \phi_n = 0 \quad | \langle \cdot, \phi_k\rangle$

$\Rightarrow \lambda_k \langle \phi_k, \phi_k\rangle = 0 \Rightarrow \lambda_k = 0$.

(we assume here $\|\phi_k\| = 1$ for each k)

3. Eigenvalues are Sensitive to Domains. Consider the operator $\ell = i\frac{d}{dx}$ with the following domains $D_1, D_2, D_3, D_4 \subset H^1([0;1];dx)$ and determine for each case the eigenvalues of ℓ.

(a) $D_1 = \{\psi \in H^1([0;1];dx) : \psi(0) = \psi(1) = 0\}$.

$$i\frac{d}{dx}\psi = \lambda\psi \Rightarrow \psi(x) = \psi(0)\, e^{-i\lambda x}$$

$$\Rightarrow \psi \equiv 0 \Rightarrow \sigma(\ell) = \emptyset$$

(b) $D_2 = \{\psi \in H^1([0;1];dx) : \psi(0) = 0\}$.

- same as part (a)

(c) $D_3 = \{\psi \in H^1([0;1];dx) : \psi(0) = \psi(1)\}$.

$$\psi(0) = \psi(1) \Rightarrow e^{-i\lambda} = 1$$

$$\Rightarrow \cos(\lambda) = 1, \quad \sin(\lambda) = 0$$

$$\Rightarrow \lambda = 2\pi k, \; k \in \mathbb{N}$$

(d) $D_4 = H^1([0;1];dx)$.

- $\psi(0)$ can be arbitrary here, so $\lambda \in \mathbb{C}$ is eigenvalue

$$\rightsquigarrow \sigma(\ell) = \mathbb{C} \; !$$

4. Boundary Conditions 2. Consider again the operator $\ell = i\frac{d}{dx}$, this time with the domains $D_\alpha \subset H^1([0;1];dx)$ and $D_1 \subset H^1([0;1];dx)$ defined below. Prove the following.

(a) $D_\alpha = \{\psi \in H^1([0;1];dx) : \psi(0) = \alpha\psi(1)\}$ where $\alpha \in \mathbb{C}$ denotes a constant with the property that $|\alpha| = 1$. Show that ℓ is symmetric on D_α.
Math Fact: ℓ defined on D_α is a self-adjoint operator.

$$\langle i\partial_x \psi, \varphi\rangle = \int_0^1 i\psi'(s)\bar{\varphi}(s)\,ds$$

$$\underset{\text{int. by parts}}{=} i\psi(s)\bar{\varphi}(s)\Big|_0^1 - i\int_0^1 \psi(s)\bar{\varphi}'(s)\,ds$$

$$= i\left(\psi(1)\bar{\varphi}(1) - \psi(0)\bar{\varphi}(0)\right) + \langle \psi, i\partial_x\varphi\rangle$$

$$= i\left(\underbrace{\alpha\bar{\alpha}}_{=|\alpha|^2=1}\psi(0)\bar{\varphi}(0) - \psi(0)\bar{\varphi}(0)\right) + \langle \psi, i\partial_x\varphi\rangle$$

$$= \langle \psi, i\partial_x\varphi\rangle$$

(b) $D_1 = \{\psi \in H^1([0;1];dx) : \psi(0) = \psi(1) = 0\}$. With the remarks about symmetric and self-adjoint operators from the lecture, argue why ℓ is symmetric, but why it can not be self-adjoint on D_1.

- we try to use our rule of thumb and try to find the largest domain in H^1 such that

$$(*) \quad \langle i\partial_x\psi, \varphi\rangle = \langle \psi, i\partial_x\varphi\rangle \quad \text{for all } \psi \in D_1.$$

- now, let $\varphi \in H^1$ be arbitrary, then

$$\langle i\partial_x\psi, \varphi\rangle = \underbrace{i\left(\psi(1)\bar{\varphi}(1) - \psi(0)\bar{\varphi}(0)\right)}_{=0 \text{ if } \psi \in D_1!} + \langle \psi, i\partial_x\varphi\rangle$$

$\Rightarrow$ $(*)$ is true for all $\psi \in D_1$ and $\varphi \in \underline{\underline{H^1}}$

$\Rightarrow$ $D_{\ell^*} = H^1 \neq D_1 \Rightarrow \ell$ is not self-adjoint

5. Laplace Operator with Periodic Boundary Conditions. By the Laplace operator we mean the operator that acts in one dimension as $\ell(y) = -y''$ on subspaces of $H^2(I; dx)$. Let's choose $I = [0; 1]$. The Laplace operator with *periodic boundary conditions* has the domain

$$D_{\text{pbc}} = \{\psi \in H^2(I; dx) : \psi(0) = \psi(1) \ \text{ and } \ \psi'(0) = \psi'(1).\}$$

(a) Prove that D_{pbc} is a vector space and that ℓ is symmetric on D_{pbc}.
Math Fact: ℓ defined on D_{pbc} is a self-adjoint operator.

(b) Prove that $\ell \geq 0$ is a non-negative operator. What is its lowest eigenvalue?

(c) With the remarks about complete eigenbases from the lecture, argue why ℓ has a complete orthonormal eigenbasis of $L^2(I; dx)$. Compute all eigenvalues and eigenvectors.