Week 4

Topics: Abel's formula; 2nd order ODE with constant coefficients; variation of parameters

Review from Weeks 2 & 3:

- **Homogeneous case:** If $h = 0$, then the solution with initial value $y(x_0) = y_0$ reads
$$y(x) = y_0 \exp\left(- \int_{x_0}^{x} \frac{b(t)}{a(t)} \, dt \right).$$

- **General solution to 1st order ODE:** Let $\ell(y) = ay' + by$, then the kernel $\ker(\ell)$ is one-dimensional. Suppose it is spanned by some function f. Then the general solution to $\ell(y) = h$ has the form $y = y_p + Cf$ for some particular solution y_p and some constant $C \in \mathbb{R}$. We can find a particular solution using the **variation of parameters**. This yields a particular solution of the form $x \mapsto y_p(x) = f(x) \int_{x_0}^{x} \frac{h(t)}{a(t)f(t)} \, dt$

- **Analytic functions:** A function $x \mapsto f(x)$ is real-analytic at x_0 if its Taylor series T_{f,x_0} converges in some interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, with $\varepsilon > 0$, and is equal to f in that interval.

- **Wronskian:** A quick way to check linear independence of two functions y_1 and y_2 is to use the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. If $W(y_1, y_2)(x_0) \neq 0$ for some x_0, then y_1 and y_2 are linearly independent.

- **Existence of solutions:** For $\ell(y) = ay'' + by' + cy$, we consider the initial value problem is $\ell(y) = 0$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1$. Under the usual assumptions on a, b, c, we have seen that solutions exist and every solution may be written in the form $y = y_0 f_1 + y_1 f_2$, where f_1 satisfies $f_1(0) = 1$, $f_1'(0) = 0$ and f_2 satisfies $f_2(0) = 0$, $f_2'(0) = 1$.

1. Start with the second-order equation $\ell(y) = ay'' + by' + cy = 0$.

(a) Show that the Wronskian W of any two independent solutions f_1 and f_2 of this equation satisfies the first order equation $aW' + bW = 0$. As a consequence, what is the formula for W in terms of a and b? The result is called **Abel's Formula**.

(b) Knowing the Wronskian and one vector f_1 in the kernel of ℓ, can you find a second vector f_2, independent of f_1, in the kernel of ℓ?

2. Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$ and consider the equation $ay'' + by' + cy = 0$.

(a) If $x \mapsto y(x) = e^{rx}$ is a solution to the first order homogeneous ODE above, can you find an algebraic equation that is solved by r?

(b) Using this approach, find two independent solutions to the equation

$$y'' - y' - 2y = 0.$$

3. Now, we would also like to solve the second order *inhomogeneous* equation

$$\ell(y) = ay'' + by' + cy = h.$$

where still $a, b, c \in \mathbb{R}$ are constants, but h is a function which in general is not just constant.

(a) Find a particular solution to the equation

$$l(y) = y'' + y' - 2y = x - 2x^3,$$

by guessing that y_p is in the space $V = \text{span}\{1, x, x^2, x^3\}$.

(b) Use the previous result to find the general solution to $\ell(y) = h$ where $h(x) = x - 2x^3$.

4. Suppose our Ansatz $y(x) = e^{rx}$ leads only to one solution to the ODE. In this case, the only (real) solution to $ar^2 + br + c = 0$ is $r = -b/2a$ (why is this so?).

(a) We know that one solution is $x \mapsto f_1(x) = e^{rx}$. Use the Wronskian to find the second solution in the kernel of $\ell(y) = ay'' + by' + c$.

(b) As an application, solve the IVP $y'' + 4y' + 4y = 0$ with $y(0) = 1, y'(0) = -1$.

5. Suppose $r = \alpha + i\beta$ is a complex solution to $ar^2 + br + c = 0$.

(a) Using the fact that $y(x) = e^{rx}$ is a solution to the equation $ay'' + by' + cy = 0$, find two *real*, linearly independent solutions y_1, y_2 by using Euler's formula.

(b) As an example, find the general solution to $y'' + 2y' + 5y = 0$.

6. **Second Order Variation of Parameters.** Use the second order variation of parameters method discussed in class to find a particular solution of $ay'' + by' + cy = h$ as follows.

(a) Start to derive the relation $c_1' f_1' + c_2' f_2' = \frac{h}{a}$.

(b) We now have a pair of equations for c_1' and c_2', namely

$$\begin{cases} c_1' f_1 + c_2' f_2 & = 0, \\ c_1' f_1' + c_2' f_2' & = \frac{h}{a} \end{cases}$$

Solve this system for c_1', c_2' using the Wronskian $W(x) = \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right)$.

(c) Solve for c_1, c_2 and write down the final solution $x \mapsto y_p(x) = c_1(x)f_1(x) + c_2(x)f_2(x)$.

(d) As an example, use this method to find once more the general solution to $y'' + y' - 2y = x - 2x^3$. Compare your result with the solution to problem 3.

Answers and Solutions.

1. (a) Start with the formula for the Wronskian,

$$W = f_1 f_2' - f_1' f_2$$

and its derivative

$$\begin{aligned} W' &= (f_1 f_2')' - (f_1' f_2)', \\ &= f_1 f_2'' + f_1' f_2' - f_1' f_2' - f_1'' f_2, \\ &= f_1 f_2'' - f_1'' f_2. \end{aligned}$$

Now consider the combination $aW' + bW$

$$\begin{aligned} aW' + bW &= a(f_1 f_2'' - f_1'' f_2) + b(f_1 f_2' - f_1' f_2), \\ &= -f_1 \cdot (cf_2) + f_2 \cdot (cf_1)\,, \text{ since } \ell(f_1) = \ell(f_2) = 0, \\ &= 0. \end{aligned}$$

We know how to solve first order ODEs and Abel's formula reads

$$W(x) = W(x_0) \exp\left(- \int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt \right)$$

(b) By the definition of the Wronskian of f_1 and f_2, we have $W = f_1 f_2' - f_1' f_2$. Thus, we may solve this for f_2 using the method of *variation of parameters* applied to the equation

$$ay' + by = h,$$

where $a = f_1$, $b = -f_1'$, and $h = W$. As we have seen before, the solution is

$$x \mapsto y(x) = f(x)g(x) = f(x) \int_{x_0}^{x} \frac{h(t)}{f(t)a(t)}\, dt$$

where f is a solution to the associated homogeneous equation $af' + bf = 0$, i.e.

$$\begin{aligned} f(x) &= \exp\left(- \int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt \right), \\ &= f_1(x). \end{aligned}$$

Thus

$$f_2(x) = f_1(x) \int_{x_0}^{x} \frac{W(t)}{f_1(t)^2}\, dt dt.$$

2. (a) Plug $y(x) = e^{rx}$ into the equation $ay'' + by' + cy = 0$:

$$\begin{aligned} ay'' + by' + cy &= a\left(e^{rx}\right)'' + b\left(e^{rx}\right)' + c\left(e^{rx}\right), \\ &= ar^2\left(e^{rx}\right) + br\left(e^{rx}\right)' + c\left(e^{rx}\right), \\ &= e^{rx}\left(ar^2 + br + c\right), \\ &= 0 \end{aligned}$$

Since $e^{rx} \neq 0$, we conclude that r is a root of the quadratic equation

$$ar^2 + br + c = 0.$$

(b) In this case $a = 1, b = -1, c = -2$ and the associated quadratic equation is

$$r^2 - r - 2 = 0.$$

This factors as $(r-2)(r+1) = 0$ and therefore $r = 2, -1$. The two corresponding solutions are given by

$$\begin{aligned} y_1(x) &= e^{2x}, \\ y_2(x) &= e^{-x} \end{aligned}$$

These are easily confirmed to be independent solutions using the Wronskian

$$\begin{aligned} W(y_1, y_2)(x) &= (y_1 y_2' - y_1' y_2)(x) \\ &= -e^{2x}e^{-x} - 2e^{2x}e^{-x} \\ &= -3e^{x}, \\ &\neq 0. \end{aligned}$$

Note that the general solution is therefore

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{-x}.$$

3. (a) First we compute the matrix L for ℓ with respect to the basis $\{1, x, x^2, x^3\}$ given above.

$$\begin{aligned}
\ell(1) &= (1)'' + (1)' - 2(1) = -2, \\
\ell(x) &= (x)'' + (x)' - 2(x) = 1 - 2x, \\
\ell(x^2) &= (x^2)'' + (x^2)' - 2(x^2) = 2 + 2x - 2x^2, \\
\ell(x^3) &= (x^3)'' + (x^3)' - 2(x^3) = 6x + 3x^2 - 2x^3.
\end{aligned}$$

Therefore,

$$L = \begin{bmatrix} -2 & 1 & 2 & 0 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

To find a particular solution to the equation $y'' + y' - 2y = x - 2x^3$, write $x - 2x^3$ with respect to the given basis and reduce

$$\left[\begin{array}{cccc|c} -2 & 1 & 2 & 0 & 0 \\ 0 & -2 & 2 & 6 & 1 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow[\text{trust me!}]{\text{row operations}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(b) To find the general solution we need to find $\ker \ell$ and solve

$$y'' + y' - 2y = 0.$$

The associated quadratic equation is

$$r^2 + r - 2 = 0,$$

which factors as $(r+2)(r-1) = 0$ and so $r = 1, -2$. The two corresponding solutions are

$$\begin{aligned}
x \mapsto y_1(x) &= e^x, \\
x \mapsto y_2(x) &= e^{-2x}.
\end{aligned}$$

Check that these two solutions are indeed linearly independent! Next, recall that the general solution is of the form $y = y_p + f$ where $f \in \ker \ell$ and therefore

$$y = \frac{7}{2} + 4x + \frac{3}{2}x^2 + x^3 + c_1 e^x + c_2 e^{-2x}$$

4. (a) Choose $x_0 = 0$ and evaluate,

$$\begin{aligned} f_2(x) &= f_1(x) \int_{x_0}^{x} \frac{W(t)}{f_1(t)^2} dt, \\ &= e^{rx} \int_0^x \frac{e^{2rx}}{(e^{rx})^2} dt, \\ &= e^{rx} \int_0^x 1 \cdot dt, \\ &= xe^{rx}. \end{aligned}$$

Thus, a second solution is $f_2(x) = xe^{rx}$. Therefore, the general solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Note: We automatically know that f_1, f_2 are independent functions since the Wronskian is already prescribed $W(x) = e^{2rx} \neq 0$.

(b) To solve the initial-value problem $y'' + 4y' + 4y = 0$ with $y(0) = 1, y'(0) = -1$, consider the associated quadratic equation

$$r^2 + 4r + 4 = 0.$$

This factors as $(r+2)^2 = 0$ and there is only one root $r = -2$. By part (1) we automatically know the general solution

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

To find c_1, c_2 use the initial values $y(0) = 1, y'(0) = -1$.

$$y(0) = c_1 e^{-2 \cdot 0} + c_2 \cdot 0 \cdot e^{-2 \cdot 0} = c_1 = 1.$$

$$y' = -2c_1 e^{-2x} + c_2(1 - 2x)e^{-2x},$$

and,

$$\begin{aligned} y'(0) &= -2c_1 e^{-2 \cdot 0} + c_2(1 - 2 \cdot 0)e^{-2 \cdot 0}, \\ &= -2c_1 + c_2, \\ &= -2 \cdot 1 + c_2, \text{ since } c_1 = 1, \\ &= -1. \end{aligned}$$

Therefore $c_1 = c_2 = 1$ and the solution is

$$y = e^{-2x} + xe^{-2x}.$$

5. (a) Consider

$$\begin{aligned} y &= e^{rx} = e^{(\alpha+i\beta)x}, \\ &= e^{\alpha x} e^{i\beta x}, \\ &= e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right), \text{ by Euler's formula,} \\ &= e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x), \\ &= y_1 + iy_2, \end{aligned}$$

where we have set $y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$. Since $\ell(y) = ay''+by'+cy$ is a (complex) *linear* operator, this implies

$$\begin{aligned} \ell(y) &= \ell(y_1 + iy_2), \\ &= \ell(y_1) + i\ell(y_2), \\ &= 0. \end{aligned}$$

Since a complex number equals 0 $\iff$ its real and imaginary parts vanish, we find,

$$\begin{aligned} \ell(y_1) &= 0, \\ \ell(y_2) &= 0. \end{aligned}$$

Thus, $y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$ are both *real* solutions to $\ell(y) = 0$! Furthermore, these are easily confirmed to be independent solutions using the Wronskian! Thus, our general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Note that we also write this as

$$y = e^{\alpha x}\left(c_1\cos(\beta x) + c_2\sin(\beta x)\right).$$

(b) The associated quadratic equation is

$$r^2 + 2r + 5 = 0.$$

By the quadratic formula we find,

$$\begin{aligned} r &= \frac{-2 \pm \sqrt{4 - 4\cdot 5}}{2}, \\ &= \frac{-2 \pm 4i}{2}, \\ &= -1 \pm 2i, \end{aligned}$$

and so $\alpha = -1$ and $\beta = \pm 2$ from our general formula above. Note that we may choose either sign for β and this will not change our general solution (why?)! Thus, the general solution is

$$y = e^{-x}\left(c_1\cos(2x) + c_2\sin(2x)\right).$$

6. (a) First compute (assuming $c_1' f_1 + c_2' f_2 = 0$)

$$\begin{aligned} y' &= (c_1' f_1 + c_2' f_2) + (c_1 f_1' + c_2 f_2'), \\ &= c_1 f_1' + c_2 f_2', \end{aligned}$$

Next

$$y'' = (c_1' f_1' + c_2' f_2') + (c_1 f_1'' + c_2 f_2'')$$

Plug these results into $\ell(y) = ay'' + by' + cy = h$ s.t.

$$\begin{aligned} \ell(y) &= ay'' + by' + cy, \\ &= a\left[(c_1' f_1' + c_2' f_2') + (c_1 f_1'' + c_2 f_2'')\right] + b\left[c_1 f_1' + c_2 f_2'\right] + c\left[c_1 f_1 + c_2 f_2\right], \\ &= c_1\left[a f_1'' + b f_1' + c f_1\right] + c_2\left[a f_2'' + b f_2' + c f_2\right] + a\left[c_1' f_1' + c_2' f_2'\right], \\ &= a\left[c_1' f_1' + c_2' f_2'\right], \text{ since } f_1, f_2 \in \ker \ell, \\ &= h \end{aligned}$$

Thus

$$c_1' f_1' + c_2' f_2' = \frac{h}{a}.$$

(b) Write the equations

$$\begin{aligned} c_1' f_1 + c_2' f_2 &= 0, \\ c_1' f_1' + c_2' f_2' &= \frac{h}{a} \end{aligned}$$

in matrix form, that is

$$\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{h}{a} \end{bmatrix}$$

Since

$$W(x) = \det \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} = \exp\left(-\int_{x_0}^{x} \frac{b(t)}{a(t)}\, dt\right) \neq 0,$$

the coefficient matrix for this system is invertible and the solution is

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} f_2' & -f_2 \\ -f_1' & f_1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{h}{a} \end{bmatrix}.$$

Thus,

$$c_1' = -\frac{f_2 h}{W a}, \quad c_2' = \frac{f_1 h}{W a}$$

(c) We just integrate the previous equations for c_1' and c_2' which gives

$$c_1 = -\int_{x_0}^{x} \frac{f_2(t)h(t)}{a(t)W(t)}dt, \quad c_2 = \int_{x_0}^{x} \frac{f_1(t)h(t)}{a(t)W(t)}dt$$

Thus, we find a particular solution of the form

$$y_p(x) = -f_1(x)\int_{x_0}^{x} \frac{f_2(t)h(t)}{a(t)W(t)}dt + f_2(x)\int_{x_0}^{x} \frac{f_1(t)h(t)}{a(t)W(t)}dt.$$

(d) This is part of problem set **4**. To double check your result, compare it to the solution to problem **3**.