Week 12

Topics: self-adjointness, Laplace operator, Hermite's operator

Review from Week 11:

- **Boundary value problems:** motivated by the vibrating wire, our goal is to set up a mathematical framework that deals with eigenvalue problems in infinite dimensions of the form:

$$\begin{cases} y'' &= \lambda y \\ y(0) &= y(L) = 0. \end{cases}$$

- **Compatibility:** If we want to define a differential operator $\ell(y) = ay'' + by' + cy$ on some $L^2(I; dx)$ space, we need at least to make sure that $\ell(y) \in L^2(I; dx)$. A function $y \in L^2(I; dx)$ such that $\ell(y) \in L^2(I; dx)$ is called *compatible* with ℓ.

- **Regularity:** Functions $y \in L^2(I; dx)$ need not be regular (differentiable) in the usual sense. To make nevertheless sense of expressions like y', y'', etc., one can introduce the notion of *weak derivatives*. We say that $\zeta \in L^2(I; dx)$ is the weak derivative of y if and only if

$$y(x) = y(x_0) + \int_{x_0}^{x} \zeta(s)\, ds.$$

If y has such a weak derivative, we write symbolically $\zeta \equiv y'$. The space of functions with weak derivatives in the $L^2(I; dx)$ sense is denoted by $H^1(I; dx)$, the Sobolev space of first order. If $y \in H^1(I; dx)$ such that also $y' \in H^1(I; dx)$, we write $y \in H^2(I; dx)$, etc. Having introduced Sobolev functions, it makes sense to define $\ell(y) = -y''$ on $H^2(I; dx)$, which is a strictly bigger set of functions than, for instance, $C^2(I)$.

- **Integration by parts in H^1:** If $\phi, \psi \in H^1(I; dx)$, then also $\phi\psi \in H^1(I; dx)$ and the weak derivative is given by $(\phi\psi)' = \phi'\psi + \psi'\phi$. In particular, we can integrate by parts as usual.

- **Eigenvalues and Spectrum:** If $T : D_T \to \mathcal{H}$ is a linear operator with domain $D_T \subset \mathcal{H}$ we say that $0 \neq \phi \in D_T$ is an eigenvector of T with eigenvalues $\lambda \in \mathbb{C}$ if and only if $T\phi = \lambda\phi$. Notice that there are two subtle details in this definition: $\phi \neq 0$ is not the zero vector and $\phi \in D_T$ must be in the domain of T. The spectrum $\sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of T } \} \subset \mathbb{C}.$$

- **Symmetric operators:** An operator $T : D_T \to \mathcal{H}$ is called symmetric if and only if

$$\langle \psi, T\phi \rangle_{\mathcal{H}} = \langle T\psi, \phi \rangle_{\mathcal{H}} \qquad \forall\, \psi, \phi \in D_T.$$

Symmetric operators have real eigenvalues and eigenvectors to different eigenvalues are orthogonal. In finite dimensions, symmetric operators admit an orthonormal basis of eigenvectors. This is called the *spectral theorem for symmetric matrices*. One of our next goals is to generalize this to infinite dimensions.

1. Boundary Conditions. Consider the operator $\ell = i\partial_x$ with domains $D_\alpha \subset H^1([0;1];dx)$ and $D_1 \subset H^1([0;1];dx)$ defined below. Prove the following.

(a) $D_\alpha = \{\psi \in H^1([0;1];dx) : \psi(0) = \alpha\psi(1)\}$ where $\alpha \in \mathbb{C}$ denotes a constant with the property that $|\alpha| = 1$. Show that ℓ is symmetric on D_1.

Math Fact: ℓ defined on D_1 is a self-adjoint operator.

see week 11 answers

(b) $D_1 = \{\psi \in H^1([0;1];dx) : \psi(0) = \psi(1) = 0\}$. With the remarks about symmetric and self-adjoint operators from the lecture, argue why ℓ is symmetric, but why it is not self-adjoint on D_2. As a reality check, explain again why the spectral theorem does not hold true for (ℓ, D_1).

see week 11 answers

2. Laplace Operator with Periodic Boundary Conditions. By the Laplace operator we mean the operator that acts in one dimension as $\ell(y) = -y''$ on subspaces of $H^2(I; dx)$. Let's choose $I = [0; 1]$. The Laplace operator with *periodic boundary conditions* has the domain

$$D_{\text{pbc}} = \{\psi \in H^2(I; dx) : \psi(0) = \psi(1) \text{ and } \psi'(0) = \psi'(1).\}$$

(a) Prove that D_{pbc} is a vector space and that ℓ is symmetric on D_{pbc}.
Math Fact: ℓ defined on D_{pbc} is a self-adjoint operator.

1) we have to check that the boundary cond. are preserved: e.g. $(\alpha\psi+\beta\varphi)(0) = \alpha\psi(0)+\beta\varphi(0) = (\alpha\psi+\beta\varphi)(1)$, etc.

2) $\int -\psi''\overline{\varphi} = -\psi'\overline{\varphi}\Big|_0^1 + \int\psi'\overline{\varphi'}$
$= -\psi'\overline{\varphi}\Big|_0^1 + \psi\overline{\varphi'}\Big|_0^1 + \int\psi\overline{(-\varphi'')} \overset{\text{pbc.}}{=} \int\psi\overline{(-\varphi'')}$

(b) Prove that $\ell \geq 0$ is a non-negative operator. What is its lowest eigenvalue?

- by (a): $\int -\psi''\psi = \int\psi'\cdot\overline{\psi'} = \int|\psi'|^2 \geq 0$
- since $\psi(x) = 1 \ \forall x \in [0,1]$ is in D_{pbc} and $\psi \neq 0 \rightsquigarrow -\psi'' = \ell(\psi) = 0$, hence $\lambda_{\min} = 0$!

(c) With the remarks about complete eigenbases from the lecture, argue why ℓ has a complete orthonormal eigenbasis of $L^2(I; dx)$. Compute all eigenvalues and eigenvectors.

- we find $\sigma(\ell) = \{(2\pi p)^2 \mid p \in \mathbb{Z}\}$ and eigenfunctions $\varphi_p(x) = e^{-2\pi p x}$
- since $\lambda_0 = 0 \leq \lambda_1 = 4\pi^2 \leq \lambda_2 = 4\pi^2 \leq \ldots \leq \lambda_p = 4\pi^2p^2 \xrightarrow{|p|\to\infty} \infty$,

the fact from the lecture says φ_p is complete ONB

3. Hermite's operator 1. We have defined Hermite's operator ℓ on $L^2(\mathbb{R}; e^{-x^2/2}dx)$ through

$$\ell(y) = -e^{x^2/2}\partial_x\big(e^{-x^2/2}\partial_x y\big).$$

(a) Prove that $\mathbb{R} \ni x \mapsto x^n \in D_\ell$, for every $n \in \mathbb{N}$.

- we compute $\ell(x^n) = -e^{x^2/2}\left(-x e^{-x^2/2}(n x^{n-1}) + e^{-x^2/2} n(n-1)x^{n-2}\right)$

$= n(n-1)x^{n-2} - nx^n$; consider now $m \in \mathbb{N}$

$$\int_{\mathbb{R}} x^{2m} e^{-x^2/2}\,dx \leq C \int_{\mathbb{R}} \frac{x^{2m}}{1+x^{2m+2}} < \infty$$

because $\int_1^\infty \frac{1}{x^2} < \infty$ (C is some constant in $\mathbb{R}$)

(b) Prove that any polynomial lies in the domain D_ℓ. Give an explicit example of a function $\phi : \mathbb{R} \to \mathbb{R}$ which is not compatible with ℓ.

- the domain of ℓ is a <u>subspace</u> of L^2, hence all polynomials are in D_ℓ by a) ;
- define $\varphi(x) = e^{x^2/2}$, then

$\ell(\varphi) = -e^{x^2/2}\,\partial_x(1+x) = -e^{x^2/2}$ and

$$\int_{\mathbb{R}} |e^{x^2/2}|^2 e^{-x^2/2}\,dx = \int_{\mathbb{R}} e^{x^2/2}\,dx = \infty$$

4. Hermite's operator 2. Consider again Hermite's operator $\ell(y) = -e^{x^2/2}\partial_x\big(e^{-x^2/2}\partial_x y\big)$.

(a) Let $a < b \in \mathbb{R}$. Show that for all $\psi, \phi \in D_\ell$, it holds true that

$$(*) = \int_a^b dx\ \ell(\psi)(x)\overline{\phi}(x)e^{-x^2/2} = + e^{-a^2/2}(\partial_x\psi)(s)\overline{\phi}(a) - e^{-b^2/2}(\partial_x\psi)(b)\overline{\phi}(b) + \int_a^b dx\ (\partial_x\psi)(x)(\overline{\partial_x\phi})(x)e^{-x^2/2}.$$

integration by parts:

$$(*) = \int_a^b dx\ -\partial_x\left(e^{-x^2/2}\partial_x\psi\right)\overline{\phi} = -e^{-\frac{x^2}{2}}\partial_x\psi\ \partial_x\overline{\phi}\Big|_a^b + \int_a^b dx\ \partial_x\psi\ \partial_x\overline{\phi}\ e^{-x^2/2}$$

(b) Using the facts from the lecture, prove that ℓ is a symmetric operator.

- letting $a \to -\infty$, $b \to \infty$ proves $(*) = \int_{\mathbb{R}} dx\ \partial_x\psi\ \partial_x\overline{\phi}\ e^{-x^2/2}$ and by symmetry of the argument in ψ and ϕ, we find $\langle \ell(\psi), \phi\rangle_{L^2(\mathbb{R}, e^{-x^2/2}dx)} = \langle \psi, \ell(\phi)\rangle_{L^2(\mathbb{R}, e^{-\frac{x^2}{2}}dx)}$

(c) Prove that ℓ is a positive semi–definite operator.

- by b):

$$\underset{\psi=\phi}{(*) =} \int_{\mathbb{R}} dx\ \underbrace{|\partial_x\psi|^2}_{\geq 0}\ \underbrace{e^{-x^2/2}}_{\geq 0} \geq 0$$