Math 110, spring 2020

Problem Set 4

Due: Tue, March 3, 6.00 PM

Topics: linear second order ODE, basics on metric spaces

1. Practice the variation of parameters by finding the general solution to the second order linear ordinary differential equation $y'' + y' - 2y = x - 2x^3$.

Definition: A metric space (M, d) is a set M together with a function $d : M \times M \to [0; \infty)$ s.t. that for all $x, y, z \in M$ we have that

$$\begin{aligned} &1)\ d(x,y) \geq 0 \ \text{ and } \ d(x,y) = 0 \leftrightarrow x = y, \\ &2)\ d(x,y) = d(y,x), \\ &3)\ d(x,y) \leq d(x,z) + d(z,y). \end{aligned}$$

2. Prove that $\mathbb{R}^n$ equipped with each of the following functions defines a metric space:

a) $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0; \infty)$, defined through $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$.

b) $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0; \infty)$, defined through $d_2(x, y) = \big(\sum_{i=1}^n |x_i - y_i|^2 \big)^{1/2}$.

c) $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0; \infty)$, defined through $d_\infty(x, y) = \max_{i \in \{1,\dots,n\}} |x_i - y_i|$.

Definition: Let (M, d) be a metric space. We say that a sequence $(x_j)_{j\in\mathbb{N}}$ in M (that is, $x_j \in M$ for all $j \in \mathbb{N}$) converges to a point $y \in M$ if and only if for all $\varepsilon > 0$ there exists some $N = N_\varepsilon \in \mathbb{N}$ such that $d(x_j, y) \leq \varepsilon$ whenever $j \geq N$.

3. Consider the metric spaces $(\mathbb{R}^n, d_1), (\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ from problem **2**. Prove that in each case a sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ converges to a point $\mathbf{y} \in \mathbb{R}^n$ if and only, for each $i \in \{1, \dots, n\}$, the sequence $(\mathbf{x}_j^{(i)})_{j\in\mathbb{N}}$ of real numbers converges to the real number $\mathbf{y}^{(i)}$. Here, $\mathbf{x}_j^{(i)}$ and $\mathbf{y}^{(i)}$ denote the i-th components of the vectors $\mathbf{x}_j = (\mathbf{x}_j^{(1)}, \mathbf{x}_j^{(2)}, \dots, \mathbf{x}_j^{(n)}) \in \mathbb{R}^n$ and $\mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(n)}) \in \mathbb{R}^n$, respectively.

4. Let $C([a; b])$ denote the set of all continuous functions $f : [a; b] \to \mathbb{R}$. Show that $C([a; b])$ is a real vector space. Define $d_\infty : C([a; b]) \times C([a; b]) \to [0; \infty)$ through

$$d_\infty(f, g) = \sup_{x\in[a;b]} |f(x) - g(x)|.$$

Prove that $\big(C([a; b]), d_\infty\big)$ is a metric space.