Week 5

Topics: Local existence & uniqueness for 1st order ODE I: basics on metric spaces

Review from Week 4:

- **Second order constant coefficient equations:** Let $a, b, c \in \mathbb{R}$ be constants such that $a \neq 0$. We would like to solve the second order linear equation:

$$ay'' + by' + cy = 0$$

  Solutions are of form $x \mapsto y(x) = e^{rx}$, where r is a root of the equation $ar^2 + br + c = 0$

- **Case I: two real roots.** If $r = r_1, r_2$ are two distinct real roots then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- **Case II: double root.** If there is only one root r then

$$y(x) = c_1 e^{r\,x} + c_2 x e^{r\,x}$$

- **Case III: complex root.** If there is a complex root $r = \alpha + i\beta$ then

$$y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

1. **General 1st Order ODE.** In the next few lectures, we want to start to discuss an important function space method to prove the existence of solutions to general first order ODE. More precisely, we consider the following: Let $f \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ be continuous, $n \in \mathbb{N}$, and let $(x_0, \mathbf{y}_0) \in \mathbb{R} \times \mathbb{R}^n$. The initial value problem related to an (not necessarily linear) *ordinary differential equation of first order* reads

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

   (a) One can recast our familiar 2nd linear order ODE $\ell(y) = ay'' + by' + cy = h$ as a (vector-valued) 1st order ODE. To this end, set $y_1 = y$ and $y_2 = y'$. Can you reformulate $\ell(y) = h$ as a 1st order ODE for the vector-valued function $\mathbf{y} = (y_1, y_2)$ with values in $\mathbb{R}^2$?

   (b) Show that a continuous function $\mathbf{y} \in C((x_0 - \epsilon; x_0 + \epsilon); \mathbb{R}^n)$ is a solution to the above initial value problem if and only if

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} f(s, \mathbf{y}(s))\, ds$$

   for all $x \in (x_0 - \epsilon; x_0 + \epsilon)$.

2. An example of a fixed point argument. Suppose you want to prove the statement that there exists a real number $x \in \mathbb{R}$ such that $x^2 = 2$, i.e. that $\sqrt{2} \in \mathbb{R}$ exists. This is not a trivial question and below is an argument that uses the fact that every bounded, monotone sequence in $\mathbb{R}$ has a limit in $\mathbb{R}$.

(a) Verify that $x^2 = 2$ if and only if $x = f(x)$ for the function $f : (0; \infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{1}{2}x + \frac{1}{x}.$$

(b) Define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = f(x_n)$. Use induction to prove that $2 \leq x_n^2$ and $x_n \leq 2$ for all $n \in \mathbb{N}$.

(c) Prove that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. Conclude the theorem.

3. First examples of metric spaces.

(a) Consider the set $\ell^1 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\right\}$ and define

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Prove that $d_1 : \ell^1 \times \ell^1 \to \mathbb{R}$ defines a metric.

(b) Consider the set $\ell^2 = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$ and define

$$d_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}.$$

Prove that $d_2 : \ell^2 \times \ell^2 \to \mathbb{R}$ defines a metric.

4. Metric spaces for 1st order ODE.

Most important for the standard local existence and uniqueness theorem for 1st order ODE are spaces of continuous functions. To this end, fix $a < b$, $a, b \in \mathbb{R}$ and set

$$C([a;b]) = C([a;b];\mathbb{R}) = \big\{ f : [a;b] \to \mathbb{R} : f \text{ continuous in } [a;b] \big\}$$

Recall that f is continuous at the point $x_0 \in [a;b]$ iff for all $\varepsilon > 0$ there exists a $\delta = \delta_\varepsilon > 0$ s.t. $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$. f is continuous in the interval $[a;b]$ if it is continuous at each $x_0 \in [a;b]$.

(a) Define $d_\infty : C([a;b]) \times C([a;b]) \to \mathbb{R}$ through

$$d_\infty(f,g) = \sup_{x \in [a;b]} |f(x) - g(x)|.$$

Prove that d_∞ defines a metric on $C([a;b])$.

(b) Set $C([a;b];\mathbb{R}^n) = \big\{ f : [a;b] \to \mathbb{R}^n : f \text{ continuous in } [a;b] \big\}$ and define (with slight abuse of notation) $d_\infty : C([a;b];\mathbb{R}^n) \times C([a;b];\mathbb{R}^n) \to \mathbb{R}$ in this case through

$$d_\infty(f,g) = \sup_{x \in [a;b]} \Big(\sum_{i=1}^{n} |f(x) - g(x)|^2 \Big)^{1/2}.$$

Prove that d_∞ defines a metric on $C([a;b];\mathbb{R}^n)$.

Answers and Solutions.

1. (a) With $y_1' = y_2$ and $y_2' = y'' = -by_2/a - cy_1/a$, the linear system can be written as

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

for

$$f(x, \mathbf{y}(x)) = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a}(x) & -\frac{b}{a}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

(b) On the one hand, if $\mathbf{y} \in C((x_0 - \varepsilon; x_0 + \varepsilon))$ solves the initial value problem, then we know (by definition of the notion of solution) that $\mathbf{y}$ is continuously differentiable in the interval $(x_0 - \varepsilon; x_0 + \varepsilon)$ with $\mathbf{y}'(x) = f(x, \mathbf{y}(x))$. The fundamental theorem of calculus

$$\mathbf{y}(x) - \mathbf{y}(x_0) = \mathbf{y}(x) - \mathbf{y}_0 = \int_{x_0}^{x} ds\ f(s, \mathbf{y}(s)).$$

On the other hand, if $\mathbf{y} \in C((x_0 - \varepsilon; x_0 + \varepsilon))$ is such that

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} ds\ f(s, \mathbf{y}(s))$$

holds true, then by the fundamental theorem of calculus, $\mathbf{y} \in C^1((x_0 - \varepsilon; x_0 + \varepsilon))$ is actually continuously differentiable and its derivative is apparently given by

$$\mathbf{y}'(x) = f(x, \mathbf{y}(x))$$

for all $x \in (x_0 - \varepsilon; x_0 + \varepsilon)$. Since $\mathbf{y}(x_0) = \mathbf{y}_0$, $\mathbf{y}$ is a solution of the initial value problem.

2. (a) It follows from $\frac{1}{2}x + \frac{1}{x} = f(x) = x$ that $\frac{1}{2}x^2 = 1$ which is equvialent to $x^2 = 2$.

(b) Since $x_1 = 2$, it is clear that $x_1 \leq 2$ and $x_1^2 \geq 4$, so let's use induction and go from n to $n+1$. We have on the one hand that

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n} \leq 1 + \frac{1}{x_n} \leq 2,$$

because $x_n \leq 2$ and because $x_n^2 \geq 2$ implies that $x_n \geq 1$ (more precisely, we should also prove inductively first that $x_n \geq 0$ for all $n \in \mathbb{N}$; I leave this as a simple exercise – let me know if you have any questions about it!).
Similarly, we have that

$$x_{n+1}^2 \geq 2 \;\leftrightarrow\; \frac{(x_n^2+2)^2}{4x_n^2} \geq 2 \;\leftrightarrow\; (x_n^2+2)^2 \geq 8x_n^2 \;\leftrightarrow\; (x_n-2)^2 \geq 0.$$

(c) We observe that

$$x_{n+1} - x_n = \frac{2 - x_n^2}{2x_n} \leq 0,$$

by the previous part.

(d) The previous parts imply that $(x_n)_{n\in\mathbb{N}}$ is monotonically decreasing and bounded. From calculus/real analysis we recall that such a sequence has a limit in $\mathbb{R}$, let's denote it by $x_\infty \in \mathbb{R}$. By continuity of the function f, we have that

$$x_\infty = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x_\infty).$$

Thus, $x_\infty^2 = 2$, i.e. $x_\infty = \sqrt{2}$.

3. Both parts follow as on problem set **4**, I leave it as an exercise to go through the arguments carefully. Two facts that we used in class are

$$1)\ |a+b| \leq |a| + |b| \qquad 2)\ |ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$$

for all real numbers $a, b \in \mathbb{R}$. To prove the triangle inequality in case b), we showed in particular that

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}$$

for all sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^2$. Let's recall the main steps of the proof here: without loss of generality assume that all sequence elements are non-negative and assume that the right hand side in the previous inequality is non-zero. Then define $(c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}} \in \ell^2$ through

$$c_n = a_n \left(\sum_{n=1}^{\infty} a_n^2\right)^{-1/2}, d_n = b_n \left(\sum_{n=1}^{\infty} b_n^2\right)^{-1/2}.$$

We then find that

$$\sum_{n=1}^{\infty} c_n d_n \leq \frac{1}{2}\sum_{n=1}^{\infty} c_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} d_n^2 = \frac{1}{2} + \frac{1}{2} = 1,$$

which proves the claim after multiplying both sides with $\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}$.

4. (a) We abbreviate $I = [a; b]$. We find that

- $d_\infty(f, g) \geq 0$ and $d_\infty(f, g) = \sup_{x \in I} |f(x) - g(x)| = 0 \leftrightarrow f(x) = g(x) \; \forall \, x \in I \leftrightarrow f = g$.
- $|f(x) - g(x)| = |g(x) - f(x)| \; \forall \, x \in I$ so that $d_\infty(f, g) = d_\infty(g, f)$.
- $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \; \forall \, x \in I$ so that
  $d_\infty(f, g) \leq \sup_{x \in I} \big(|f(x) - h(x)| + |h(x) - g(x)|\big) \leq d_\infty(f, h) + d_\infty(h, g)$.

Notice that we used the triangle inequality for real numbers in proving the triangle inequality for d_∞.

(b) With the notation from problem set **4**, we find that

- $d_\infty(f, g) \geq 0$ and $d_\infty(f, g) = \sup_{x \in I} d_2(f(x), g(x)) = 0 \leftrightarrow f_i(x) = g_i(x) \; \forall \, x \in I$,
  and $\forall \, i = 1, \ldots, n \leftrightarrow f = g$.
- $|f_i(x) - g_i(x)| = |g_i(x) - f_i(x)| \; \forall \, x \in I, i = 1, \ldots, N$ so that $d_\infty(f, g) = d_\infty(g, f)$.
- $d_2(f(x), g(x)) \leq d_2(f(x), h(x)) + d_2(h(x), g(x)) \;\; \forall \, x \in I$
  so that $d_\infty(f, g) \leq d_\infty(f, h) + d_\infty(h, g)$.

Notice that we used the triangle inequality for the metric space $(\mathbb{R}^n, d_2)$ from problem set **4** in proving the triangle inequality for d_∞.