MATH110 Spring 2020 HW8

Rylan Schaeffer

April 16th, 2020

Problem 2

Disclaimer: The below math has infinities. I know that I should write limits e.g. $\lim_{a\to\infty} f(a)$, but this takes additional typing and introducing new variables, so I omit the limits for brevity e.g. $f(\infty)$. We start by reminding ourselves of the definitions of H^1 and $\tilde{H}^1$:

$$H^1 \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}, dx) : \exists g \in L^2(\mathbb{R}, dx) \text{ s.t. } f(b) - f(a) = \int_a^b g(x)dx\}$$

and

$$\tilde{H}^1 \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}, dx) : \exists g \in L^2(\mathbb{R}, dx) \text{ s.t. } \forall h \in C_c^\infty(\mathbb{R}), \int_{\mathbb{R}} dx f(x)h'(x) = -\int_{\mathbb{R}} g(x)h(x)\}$$

We will show that $H^1 = \tilde{H}^1$ by showing any element in one set is also an element of the other set. Choose $f \in H^1$ and let g be its directional derivative. By the definition of H^1:

$$f(x) - f(-\infty) = \int_{-\infty}^x g(u)du$$

Recall that because h has compact support, $h(\infty) = h(-\infty) = 0$. Combined with the Fundamental Theorem of Calculus, we see that that:

$$\int_{\mathbb{R}} \frac{d}{dx}[h(x)(f(x) - f(-\infty))] = h(\infty)f(\infty) - h(\infty)f(-\infty) = 0$$

But we also see that

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d}{dx}[h(x)(f(x) - f(-\infty))] &= \int_{\mathbb{R}} \frac{d}{dx}[h(x)\int_{-\infty}^x g(u)du] \\
&= \int_{\mathbb{R}} h'(x)\int_{-\infty}^x g(u)du + \int_{\mathbb{R}} h(x)g(x)dx \\
&= \int_{\mathbb{R}} h'(x)(f(x) - f(-\infty)) + \int_{\mathbb{R}} h(x)g(x)dx \\
&= \int_{\mathbb{R}} h'(x)f(x) - f(-\infty)\int_{\mathbb{R}} h'(x) + \int_{\mathbb{R}} h(x)g(x)dx \\
&= \int_{\mathbb{R}} h'(x)f(x) + \int_{\mathbb{R}} h(x)g(x)dx
\end{aligned}$$

where the last line follows because by the Fundamental Theorem of Calculus, $\int_{\mathbb{R}} h'(x) = h(\infty) - h(-\infty) = 0$. Combining the two, see that $H^1 \subset \tilde{H}^1$:

$$0 = \int_{\mathbb{R}} h'(x)f(x) + \int_{\mathbb{R}} h(x)g(x)dx \iff \int_{\mathbb{R}} h'(x)f(x) = -\int_{\mathbb{R}} h(x)g(x)$$

The proof runs in the opposite direction as well, assuming the last equality and working up, allowing us to see that $\tilde{H}^1 \subset H^1$. Thus we conclude that H^1 is the same space as $\tilde{H}^1$.

Problem 3

Let $\phi, \psi \in H^1$. We want to show that the linear differential operator $l = i\partial_x$ with domain $D_1 = \{f \in H^1([0,1]; dx) : f(0) = f(1) = 0\}$ is a symmetric operator.

$$\begin{aligned}
\langle l\phi, \psi \rangle &\stackrel{\text{def}}{=} \int_0^1 dx l(\phi)\overline{\psi} \\
&= i[\psi(1)\phi(1) - \psi(0)\phi(0) - \int_0^1 \phi\partial_x\overline{\psi}] \\
&= i[\psi(1) * 0 - \psi(0) * 0 - \int_0^1 \phi\partial_x\overline{\psi}] \\
&= -i\int_0^1 \phi\partial_x\overline{\psi} \\
&= \langle \phi, l\psi \rangle
\end{aligned}$$

Thus we conclude that l is symmetric. If we change the domain to $D_2 = \{f \in H^1([0,1]; dx) : f(0) = 0\}$, then l is no longer symmetric because nothing eliminates $i\psi(1)\phi(1)$.

Problem 4

Let $\phi, \psi \in H^2$. We want to show that the linear differential operator $l = \partial_x^2$ with domain $D_1 = \{f \in H^2([0,1]; dx) : f(0) = f(1) \text{ and } f'(0) = f'(1)\}$ is a symmetric operator.

$$\begin{aligned}
\langle l\phi, \psi \rangle &\stackrel{\text{def}}{=} \int_0^1 dx - \partial_x^2\phi\overline{\psi} \\
&= -\overline{\psi}\phi|_0^1 + \overline{\psi'}\phi'|_0^1 - \int_0^1 dx\phi\partial_x^2\overline{\psi} \\
&= 0 + 0 - \int_0^1 dx\phi\partial_x^2\overline{\psi} \\
&= 0 + 0 + \int_0^1 dx\phi\overline{-\partial_x^2\psi} \\
&= \langle \phi, l\psi \rangle
\end{aligned}$$

To find the eigenvalues and eigenfunctions of l, we consider:

$$l(\phi) = \lambda\phi \Leftrightarrow -\partial_x^2\phi = \lambda\phi$$

From this second order differential equation, we see that the eigenfunctions are $\phi(x) = ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x}$ for $a, b \in \mathbb{C}$. To determine valid eigenvalues, we need to ensure the boundary conditions are met:

$$\begin{aligned}
\phi(0) &= \phi(1) \\
a + b &= ae^{i\sqrt{\lambda}} + be^{-i\sqrt{\lambda}} \\
&\Rightarrow \\
\cos(\sqrt{\lambda}) &= 1 \\
&\Rightarrow \\
\sqrt{\lambda} &= 2\pi k \quad \forall k \in \mathbb{Z} \\
\lambda &= (2\pi k)^2
\end{aligned}$$

I find that the derivative boundary conditions $\phi'(0) = \phi'(1)$ don't affect the valid eigenvalues. The spectrum of l is thus $\sigma(l) = \{(2\pi k)^2 : k \in \mathbb{Z}\}$. Moving on, if we consider the domain $D_2 = \{f \in H^2([0,1]; dx) : f(0) = f(1) = 0\}$, we see that l is still symmetric:

$$\begin{aligned}
\langle l\phi, \psi\rangle &\overset{\text{def}}{=} \int_0^1 dx \partial_x^2 \phi \overline{\psi} \\
&= \phi\overline{\psi}|_0^1 - \int_0^1 dx \partial_x \phi \partial_x \overline{\psi} \\
&= \phi\overline{\psi}|_0^1 + \int_0^1 dx \partial_x \phi \overline{-\partial_x \psi} \\
&= \int_0^1 dx \phi \overline{-\partial_x^2 \psi} \\
&= \langle \phi, l\psi\rangle
\end{aligned}$$

The operator hasn't changed, so the form of the eigenfunctions doesn't either i.e. $\phi(x) = ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x}$, but the boundary condition $\phi(0) = \phi(1) = 0$ now requires that $a + b = 0$ since:

$$\begin{aligned}
0 &= \phi(0) \\
0 &= a + b \\
b &= -a \\
0 &= \phi(1) \\
&= ae^{i\sqrt{\lambda}} - ae^{-i\sqrt{\lambda}} \\
0 &= ai\sin(\sqrt{\lambda}) + ai\sin(\sqrt{\lambda}) \\
&\Rightarrow \\
\sin(\sqrt{\lambda}) &= 0 \\
\sqrt{\lambda} &= \pi k \quad \forall k \in \mathbb{Z} \\
\lambda &= (\pi k)^2
\end{aligned}$$

The spectrum is thus $\sigma(l) = \{(\pi k)^2 : k \in \mathbb{Z}\}$.

Problem 5

Let $\psi \in D_T$. We start by considering $T\psi$:

$$
\begin{aligned}
T\psi &= \sum_{j=1}^{\infty} \langle T\psi, \phi_j \rangle \phi_j && \text{Expansion into orthonormal basis} \\
&= \sum_{j=1}^{\infty} \langle \psi, T\phi_j \rangle \phi_j && \text{Defn. of a symmetric operator} \\
&= \sum_{j=1}^{\infty} \langle \psi, \lambda_j \phi_j \rangle \phi_j && \text{Defn. eigenvalue-eigenfunction} \\
&= \sum_{j=1}^{\infty} \overline{\lambda_j} \langle \psi, \phi_j \rangle \phi_j && \\
&= \sum_{j=1}^{\infty} \lambda_j \langle \psi, \phi_j \rangle \phi_j && \text{Eigenvalues of symmetric operators are real}
\end{aligned}
$$