MATH110 Spring 2020 HW5

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Problem 2

We take as given that a Cauchy sequence $(x_j^{(i)})_{j\in\mathbb{N}}$ with $x_j^{(i)} \in \mathbb{R}$ converges to $y_j^{(i)} \in \mathbb{R}$. This means that every Cauchy sequence $(x_j)_{j\in\mathbb{N}}$ with $x_j \in \mathbb{R}^n$ and with the jth sequence's ith element defined by $x_j^{(i)}$ converges to a point y, defined by setting the ith element equal to $y^{(i)}$; in plain English, the n-dimensional real sequence converges to the point defined by where the n 1-dimensional real sequences converge to. In the previous problem set (Problem Set 4, Problem 3), we showed for the three metric spaces $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$, $(\mathbb{R}^n, d_\infty)$, if each 1-dimensional real sequence converges, the entire n-dimensional sequence converges. Since the condition holds true here for all three metric spaces, any Cauchy sequence $(x_j)_{j\in\mathbb{N}}$ defined in the set of one of those three metric spaces is also a convergent sequence in that metric space.

Problem 3

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in M$ such that $\forall n \in \mathbb{N}, x_n \in M$ and $\lim_{n\to\infty} x_n = x \in M$. Then $x \in M$ is trivially true, and we conclude that M is a closed set.

2. Define $B_\delta(y_0) = \{y \in M : d(y_0, y) \leq \delta\}$. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}, x_n \in B_\delta(y_0)$ and let $\lim_{n\to\infty} x_n = x \in M$.

$$\begin{aligned} d(x, y_0) &\leq d(x, x_n) + d(x_n, y) \\ &\leq d(x, x_n) + \delta \\ &\leq \delta \end{aligned}$$

   where the last line follows from the definition of $x = \lim_{n\to\infty} x_n$ i.e. $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}$ such that $\forall n > N$, $d(x_n, x) < \epsilon$. Intuitively, this means that if we go far enough in the sequence, we can bound any possible distance between x and x_n, meaning that $x \in B_\delta(y_0)$.

3. Define a constant function $y \in C([a, b]; \mathbb{R}^n)$ that returns y_0 over the interval $[a, b]$. We see that the set A is equivalent to $B_\delta(y_0)$ if we view $C([a, b]; \mathbb{R}^n)$ as M. Since we've already shown that $B_\delta(y_0)$ is closed, we conclude that A is also closed.

Problem 4

1. Consider $f(x) = \log(x)$. $f(x)$ is concave because $-f(x)$ is convex, as seen by $-\frac{d^2}{dx^2}\log(x) = \frac{1}{x^2} > 0$ for $x \in (0, \infty)$.

2.

$$
\begin{aligned}
xy &= \exp(\log(xy)) \\
&= \exp(\frac{1}{p}\log(x^p) + \frac{1}{q}\log(y^q)) \\
&\leq \exp(\log(\frac{1}{p}x^p + \frac{1}{q}y^q)) \\
&\leq \frac{1}{p}x^p + \frac{1}{q}y^q
\end{aligned}
$$

3. For brevity, define $X = (\sum_n |x_n|^p)^{1/p}$ and $Y = (\sum_n |y_n|^q)^{1/q}$.

$$
\begin{aligned}
\sum |x_n y_n| &\leq XY \\
\sum \frac{x_n y_n}{XY} &\leq 1
\end{aligned}
$$

We can bound the LHS in the following manner:

$$
\begin{aligned}
\sum \frac{x_n y_n}{XY} &\leq \sum \frac{|x_n|}{X}\frac{|y_n|}{Y} \\
&\leq \sum \frac{1}{p}(\frac{|x_n|}{X})^p + \frac{1}{q}(\frac{|y_n|}{Y})^q \\
&= \frac{1}{p}\sum \frac{|x_n|^p}{\sum |x_n|^p} + \frac{1}{q}\sum \frac{|y_n|^q}{\sum |y_n|^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}
$$

Thus we conclude that $\sum \frac{x_n y_n}{XY} \leq 1 \Leftrightarrow \sum |x_n y_n| \leq XY$

4. Define $d_p \stackrel{\text{def}}{=} (\sum_n |x_n - y_n|^p)^{1/p}$. We show that d_p meets the three criteria of a metric.

   (a) $d_p(x,y) \geq 0$ because each element $|x_n - y_n|$ is non-negative and the sum of non-negative elements is non-negative. We also note that $d_p(x,y) = 0 \Leftrightarrow x = y$ because a sum of non-negative elements is 0 if and only if each element is zero, and if each element is zero, $x_n = y_n$ for all n.

   (b) $d_p(x,y) = (\sum_n |x_n - y_n|^p)^{1/p} = (\sum_n |y_n - x_n|^p)^{1/p} = d_p(y,z)$. Thus d is symmetric.

   (c) Our goal is to show that $d_p(x,y) \leq d_p(x,z) + d_p(z,y)$. Starting with $d(x,y)^2$, we see that

$$
\begin{aligned}
d_p(x,y)^2 &\overset{\text{def}}{=} \sum_n |x_n - y_n|^p \\
&= \sum_n |x_n - z_n + z_n - y_n||x_n - y_n|^{p-1} \\
&\le \sum_n (|x_n - z_n| + |z_n - y_n|)|x_n - y_n|^{p-1} \\
&\le \Big[(\sum_n |x_n - z_n|^p)^{1/p} + (\sum_n |z_n - y_n|^p)^{1/p}\Big](\sum_n |x_n - y_n|^p)^{\frac{p-1}{p}} \\
\frac{\sum_n |x_n - y_n|^p}{(\sum_n |x_n - y_n|^p)^{\frac{p-1}{p}}} &\le (\sum_n |x_n - z_n|^p)^{1/p} + (\sum_n |z_n - y_n|^p)^{1/p} \\
(\sum_n |x_n - y_n|^p)^{1/p} &\le d_p(x,z) + d_p(z,y) \\
d_p(x,y) &\le d_p(x,z) + d_p(z,y)
\end{aligned}
$$