Problem Set 6

Due: Th, April 2, 6.00 PM

Topics: Bessel's inequality & Parseval identity, Hilbert spaces, Fourier series

1. Reading. Read sections 3.6 to 3.10 in Holland's book. Optional reading: sections 3.1, 3.2 and 3.5 (motivational background for Hilbert spaces & Fourier series).

2. Bessel's inequality and Parseval's identity. Let $(\phi_j)_{j\in\mathbb{N}}$ an orthonormal sequence in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $\psi \in H$. Prove the following:

a) Let $\mu_j \in \mathbb{C}, j = 1, \dots, N$, and set $\lambda_j = \langle \psi, \phi_j \rangle$ for all $j \in \mathbb{N}$. Prove that

$$f(\mu_1, \dots, \mu_N) \equiv \Big\|\psi - \sum_{j=1}^N \mu_j\phi_j\Big\|^2 = \Big\|\psi - \sum_{j=1}^N \lambda_j\phi_j\Big\|^2 + \sum_{j=1}^N |\lambda_j - \mu_j|^2.$$

Conclude that $\min_{(\mu_1,\dots,\mu_N)\in\mathbb{C}^N} f(\mu_1, \dots, \mu_N)$ is attained at $(\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$. This computation tells us that the 'best' approximation of ψ in the finite dimensional subspace $\operatorname{span}\{\phi_1, \dots, \phi_N\} \subset H$ is its *truncated Fourier series* $\sum_{j=1}^N \lambda_j\phi_j$.

b) Prove *Bessel's inequality* which says that for every $N \in \mathbb{N}$, we have that

$$\sum_{j=1}^N |\lambda_j|^2 \le \|\psi\|^2.$$

c) For a sequence $(c_j)_{j\in\mathbb{N}}$ in $\mathbb{C}$, prove that the limit $\sum_{j=1}^\infty c_j\phi_j$ exists in H if and only if $\sum_{j=1}^\infty |c_j|^2 < \infty$.

d) Assume now in addition that $(\phi_j)_{j\in\mathbb{N}}$ is a complete orthonormal basis of H. Given any $\psi \in H$, prove *Parseval's identity* which says that

$$\|\psi\|^2 = \sum_{j=1}^\infty |\langle\psi, \phi_j\rangle|^2.$$

We can compute the square of the norm of ψ as the sum of squares of the lengths of ψ projected orthogonally onto the direction ϕ_j, as in the Pythagorean Theorem!

3. Consider the real space $\ell^2(\mathbb{R}) \equiv \ell^2$ with inner product $\langle x, y\rangle = \sum_{j=1}^\infty x_jy_j$. Prove that ℓ^2 is a Hilbert space by proceeding as follows. Assume that $(x^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2 (recall what a Cauchy sequence is, if needed). We use the notation that $x^{(n)} \in \ell^2$ has components $x^{(n)} = \big(x_1^{(n)}, x_2^{(n)}, \dots\big)$.

(a) Fix a component-index $j \in \mathbb{N}$ and prove that the sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ of *real numbers* is a Cauchy sequence in $\mathbb{R}$. Hence, for every fixed component index $j \in \mathbb{N}$, the real sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ has a limit

$$x_j^{(\infty)} \equiv \lim_{n\to\infty} x_j^{(n)} \in \mathbb{R}.$$

Define the real sequence $x^{(\infty)}$ through its components $x^{(\infty)} = \big(x_1^{(\infty)}, x_2^{(\infty)}, \dots\big)$. $x^{(\infty)}$ is our candidate for the limit of the Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$. To show that it is indeed its limit, we need to prove that $x^{(\infty)} \in \ell^2$ and that $x^{(\infty)} = \lim_{n\to\infty} x^{(n)}$ in the sense of ℓ^2 (that is, with respect to the ℓ^2 norm).

(b) Prove that $x^{(\infty)} \in \ell^2$. What can you say about $\sum_{j=1}^N |x^{(\infty)}|^2$, for fixed $N \in \mathbb{N}$? Using the Cauchy-property of the sequence $(x^{(n)})_{n\in\mathbb{N}}$ and the previous part, can you find an upper bound on $\sum_{j=1}^N |x^{(\infty)}|^2$ that is *uniform in* N?

(c) Prove that $x^{(\infty)} = \lim_{n\to\infty} x^{(n)}$ in ℓ^2. Proceed similarly as in the previous part.

4. Functions in L^2. In the following, consider all L^2 spaces as real vector spaces.

(a) Integrate by parts to find the antiderivatives of $x \mapsto \log(x)$ and $x \mapsto (\log(x))^2$.

(b) Let $f(x) = \log(x)$. Show that $f \in L^2((0;1);dx)$.

(c) Let $f_1(x) = \log(1+x)$ and $f_2(x) = \log(1-x)$. Show that $f_1, f_2 \in L^2((-1;1);dx)$.

(d) Let $g(x) = \frac{1}{2}\log\big((1+x)/(1-x)\big)$. Show that $g \in L^2((-1;1);dx)$. What about its derivative $\partial_x g$?

5. Fourier Series. Consider the complex Hilbert space $L^2([0;1];dx)$. Define the sequence $(\varphi_p)_{p\in\mathbb{Z}}$ by $\varphi_p(x) = e^{2\pi i p x}$ for $x \in [0;1]$. Prove that $(\varphi_p)_{p\in\mathbb{Z}}$ is an orthonormal sequence in $L^2([0;1];dx)$. A fact from advanced analysis[1] is that $(\varphi_p)_{p\in\mathbb{Z}}$ *forms a complete orthonormal basis* of $L^2([0;1];dx)$. We call the basis expansion of $\psi \in L^2([0;1];dx)$ in terms of the basis vectors $(\varphi_p)_{p\in\mathbb{Z}}$ its *Fourier series*[2] and write

$$\psi(x) = \sum_{p\in\mathbb{Z}} \widehat{\psi}_p e^{2\pi i p x},$$

where the equality is interpreted in the L^2 sense and where $\widehat{\psi}_p = \langle \psi, \varphi_p \rangle$. Compute the Fourier transform of $\psi \in L^2([0;1];dx)$ defined by $\psi(x) = x$.

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[1] See for instance J.R. Higgins *Completeness and Basis Properties of Sets of Special Functions*. Notice that, compared to the lecture, we include now the case $p = 0$ in the sequence $(\varphi_p)_{p\in\mathbb{Z}}$.

[2] Compare the series expansion in terms of the φ_p with the usual Fourier series expansion from calculus by expanding $e^{2\pi i p x}$ into a sum of $\sin(2\pi p x)$ and $\cos(2\pi p x)$ terms!