HW 6

DUE FRIDAY OCTOBER 4 AT 7PM (BONUS 24 HOURS LATER)

For hints see office door. But try without the hints first.

Problem A: Let $f : [a, b] \to \mathbb{R}^n$ be continuous on the closed interval $[a, b] \subset \mathbb{R}$ and differentiable on (a, b).

(1) Show that there is a $c \in (a, b)$ such that
$$|f(a) - f(b)| \leq |f'(c)| \cdot |a - b|.$$
(2) Give an example when $n = 2$ to show that it is possible the inequality is strict for all $c \in (a, b)$. (In particular, the Mean Value Theorem does not hold for f.)

Problem B: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation
$$f(x, y) = (e^x \cos y, e^x \sin y).$$
(1) Show that f is one-to-one on the set
$$A = \{(x, y) : x \in \mathbb{R}, 0 < y < 2\pi\}.$$
(2) What is $B = f(A)$?
(3) If g is the inverse function of f restricted to A, find $Dg(0, 1)$.
(4) What is $f(\mathbb{R}^2)$?
(5) Show that the Jacobian matrix of f is nonsingular for any $(x, y) \in \mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood on which f is one-to-one. Nonetheless, show that f is not one-to-one on $\mathbb{R}^2$.
(6) Find an explicit formula for the inverse function g of f in the neighborhood of $(0, 1)$. Use this formula to check your answer in part (3).

Problem C: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and locally invertible. Show that the image of f is open, and that that a global inverse for f exists defined on the image.

Remark: The analogue of Problem C is false in higher dimensions. You should pause to note what your solution uses that wouldn't be available in higher dimensions.

Just for fun (don't hand in): Give an example of a continuous $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is locally invertible but not injective.

Problem D: Say that $f : \mathbb{R}^m \to \mathbb{R}$ is C^∞, and there is a number $W > 0$ such that for all $x \in B_r(0)$ and all α we have

$$|\partial^\alpha f(x)| \leq W^{|\alpha|}.$$

Show that

$$\lim_{k\to\infty} \sum_{|\alpha|\leq k} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha = f(x)$$

for $x \in B_r(0)$. Show also that the infinite series

$$\sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha$$

converges absolutely, so that without any ambiguity we can write

$$f(x) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha.$$

Just for fun (don't hand in): $|\partial^\alpha f(x)| \leq W^{|\alpha|}$ is not optimal. Can you phrase a natural assumption that is closer to optimal?

Problem E: If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is C^1 and has a local minimum at x, prove $Df(x) = 0$. (Points x with $Df(x) = 0$ are called critical points.)

Problem F: Let $f : A \to \mathbb{R}$ be a C^2 function, $A \subset \mathbb{R}^n$ open, and let $x \in A$. The Hessian $Hf(x)$ of f at x is the n-by-n symmetric matrix with entry (j, k) equal to $\partial^{e_i+e_j} f(x)$.

A symmetric matrix S is called positive definite if $x^T Sx > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$, in which case we write $S > 0$. If the same condition holds with a non-strict inequality $x^T Sx \geq 0$ we say S is positive semi-definite and write $S \geq 0$. The negative of a positive (semi-)definite matrix is called negative (semi-)definite, and we similarly write $S < 0$ or $S \leq 0$.

Assume that A is convex, and $Df(x_0) = 0$. Prove that:

(1) if $Hf(x) \geq 0$ for all $x \in A$ then $f(x) \geq f(x_0)$ for all $x \in A$.
(2) if $Hf(x) > 0$ for all $x \in A$ then $f(x) > f(x_0)$ for all $x \in A$.
(3) if $Hf(x) \leq 0$ for all $x \in A$ then $f(x) \leq f(x_0)$ for all $x \in A$.
(4) if $Hf(x) < 0$ for all $x \in A$ then $f(x) < f(x_0)$ for all $x \in A$.
(5) if $Hf(x_0) \not\geq 0$ then f does not have a local min at x_0.
(6) if $Hf(x_0) \not\leq 0$ then f does not have a local max at x_0.

You only need to submit your proofs for parts (1) and (5); you don't have to write up the rest.

Problem G: Submit a writeup for Problem E on worksheet 5.

Bonus: Let $M_n(\mathbb{R})$ denote the set of n by n real matrices. It has a topology and metric by identifying with $\mathbb{R}^{n^2}$ using the entries of the metric.

(1) For any $A \in M_n(\mathbb{R})$, show that
$$\lim_{K\to\infty} \sum_{k=0}^{K} \frac{A^k}{k!}$$
exists. This limit is denoted
$$\exp(X) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(2) Compute exp of the following matrices
$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}.$$

(3) Show that $\exp(A + B) = \exp(A)\exp(B)$ when A and B commute.

(4) Prove that exp is differentiable at the origin and compute its derivative there.

(5) Is exp surjective?

(6) Is exp injective?