Recall sufficient condition for f diff'ble at x_0

Thm $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, A open

If all $\dfrac{\partial f_i}{\partial x_j}$ exist near x_0 and is ctn at x_0

(i.e. $f \in C^1$)

$\Rightarrow f$ is diff'ble at x_0

Higher-order derivatives

let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, A open

(with component $f_i: A \to \mathbb{R}$ $(i = 1, \dots, m)$)

Inductively define: (component-wise)

$$\frac{\partial^r}{\partial x_{j_r}, \dots, \partial x_{j_1}} f_i = \frac{\partial}{\partial x_{j_r}}\left[\frac{\partial^{r-1}}{\partial x_{j_{r-1}} \cdots \partial x_{j_1}} f_i\right]$$

as a r^{th} partial

ex $f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right), \quad \frac{\partial^2 f}{(\partial x_1)^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)$$

Def C^r

A function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class C^r if all partitials of order at most r $\exists$ and ctn.

Def C^∞

f is C^∞ if $\forall r \in \mathbb{N}$, it is C^r

ex $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = a + b_1 x_1 + b_2 x_2 + c_{1,1} x_1^2 + c_{1,2} x_1 x_2 + c_{2,2} x_2^2$$

$(a_1, b's, c's \in \mathbb{R})$

$$\frac{\partial^2}{\partial x_1 \partial x_2} f = \frac{\partial}{\partial x_1}\left(b_2 + c_{1,2} x_1 + 2c_{2,2} x_2\right) = c_{1,2}$$

$$\frac{\partial^2}{\partial x_2 \partial x_1} f = \frac{\partial}{\partial x_2}\left(b_1 + c_{1,2} x_2 + 2c_{1,1} x_1\right) = c_{1,2}$$

Thm (Mixed partials) Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ (A open) be in C^2, then $\forall x_0 \in A$,

$$\Rightarrow \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} f(x_0) = \frac{\partial^2}{\partial x_{j_2} \partial x_{j_1}} f(x_0)$$

(Wrong pf) for $n=2, m=1$:

$$\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x_2}(x_1+h, x_2) - \frac{\partial f}{\partial x_2}(x_1, x_2)}{h}$$

$$= \lim_{h \to 0} \frac{\lim_{k \to 0} \frac{f(x_1+h, x_2+k) - f(x_1+h, x_2)}{k} - \lim_{k \to 0} \frac{f(x_1, x_2+h) - f(x_1, x_2)}{k}}{h}$$

$$= \lim_{h\to 0}\lim_{k\to 0} \frac{f(x_1+h, x_2+k) - f(x_1+h, x_2) - f(x_1, x_2+k) + f(x_1,x_2)}{hk}$$

$$= \lim_{k\to 0}\lim_{h\to 0} \frac{f(x_1+h, x_2+k) - f(x_1, x_2+k) - f(x_1+h, x_2) + f(x_1,x_2)}{hk}$$

$$= \lim_{k\to 0} \frac{\lim_{h\to 0}\frac{f(x_1+h, x_2+k) - f(x_1, x_2+h)}{h} - \lim_{h\to 0}\frac{f(x_1+h, x_2) - f(x_1, x_2)}{h}}{k}$$

$$= \lim_{k\to 0} \frac{\frac{\partial f}{\partial x_1}(x_1, x_2+k) - \frac{\partial f}{\partial x_1}(x_1,x_2)}{k}$$

$$= \frac{\partial f}{\partial x_2 \partial x_1} \qquad D?$$

Question: (1) Can we actually do this? No.

(2) Where we use ctnity of C^2? not yet.

Actual Proof

WLOG Let $m=1$, $n=2$ $(f: \mathbb{R}^2 \to \mathbb{R})$

Define $G(h,k) = f(x_1+h, x_2+k) - f(x_1+h, x_2)$
$\qquad - f(x_1, x_2+k) + f(x_1, x_2)$

Let $\varphi(t) = f(x_1+h, t) - f(x_1, t)$

MVT $\Rightarrow \exists t_0$ s.t. $\varphi(x_2+k) - \varphi(x_2) = k\varphi'(t_0)$

i.e. $G(h,k) = k\left(\frac{\partial}{\partial x_2} f(x_1+h, t_0) - \frac{\partial}{\partial x_2} f(x_1, t_0)\right)$

Now apply MVT to $s \mapsto \frac{\partial}{\partial x_2} f(s, t_0)$

to get a s_0 s.t.

$$G(h,k) = kh \frac{\partial^2}{\partial x_1 \partial x_2} f(s_0, t_0)$$

A similar argument gives : $\exists s_0', t_0'$ s.t.

$$G(h,k) = kh \frac{\partial^2}{\partial x_2 \partial x_1} f(s_0', t_0')$$

Thus we get $\frac{\partial^2 f}{\partial x_1 \partial x_2}(s_0, t_0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(s_0', t_0')$

Let $h, k \to 0 \Rightarrow (s_0, t_0), (s_0', t_0') \to (x_1, x_2)$

So ctnity gives the result.