HW 5

DUE FRIDAY SEPTEMBER 27 AT 7PM (BONUS 24 HOURS LATER)

Problem A: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = (\exp(x^2 + 2y^2), \sin(z^2 - y^2) \cdot (x^2 + 2z^2), (x^2 + y^2 + z^2)^9).$$

Explain why F is differentiable, and then prove why $DF(x, y, z)$ always has zero determinant. You may not actually compute any derivatives your solution.

Problem B: Suppose

$$F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$$

and

$$G : B \subset \mathbb{R}^m \to A \subset \mathbb{R}^n$$

are both differentiable and are inverses of each other (A, B open). Show that $n = m$ and that, for all pairs $a \in A, b \in B$ with $F(a) = b$

$$DG(b) = DF(a)^{-1}.$$

Problem C: Give an example of a differentiable homeomorphism from $\mathbb{R}$ to itself whose inverse is not differentiable at every point. (For your example, you need to find only a single point where the inverse isn't differentiable.)

Problem D: Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin. Show

$$\lim_{h\to 0} \lim_{k\to 0} F(h, k) = \lim_{k\to 0} \lim_{h\to 0} F(h, k),$$

assuming all these limits exit. Give an example where F is not continuous, both double limits exist, but the two double limits are not equal.

Just for fun (don't hand in): Give an example where $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin but $\lim_{h\to 0} \lim_{k\to 0} F(h, k)$ does not exist.

Just for fun (don't hand in): Also note that for $a_{n,m} = 2^{n-m}$ it is not true that

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m}.$$

Problem E: If F is a function of 4 variables, how many terms (in general) are in the degree 10 Taylor series of F? (Degree 10 means you use multi-indices α of degree at most 10.) You do not need to show your work; just give the final answer.

Problem F: Suppose that $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with A open and connected and $Df(a) = 0$ for all $a \in A$. Show that F is constant.

Problem G: Let $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be C^k. Show that

$$\partial^k(f_1 \cdots f_m) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_m} f_m.$$

Problem H: Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}. Show that the Taylor polynomial of degree k centered at $x_0 \in \mathbb{R}^n$ is the best polynomial approximation of $f(x)$ near x_0 in the following sense: Suppose that $P(x)$ is a polynomial of degree k. Then

$$P(x) - f(x) = o(|x - x_0|^k)$$

if and only if P is the Taylor polynomial of degree k centered at x_0. (Recall that a quantity Q is $o(|x - x_0|^k)$ if $\lim_{x \to x_0} \frac{Q}{|x-x_0|^k} = 0$.)

Problem I: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x^2 + y^2)$, so F is differentiable.

(1) Prove
$$x \frac{\partial F}{\partial y} = y \frac{\partial F}{\partial x}.$$

(2) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are differentiable. Define $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ via
$$\phi(x, y, z) = (f(h(x), g(x, y), z), g(y, z)).$$
Find a formula for the matrix of the derivative $D\phi(p) : \mathbb{R}^3 \to \mathbb{R}^2$ at an arbitrary point $p \in \mathbb{R}^3$ in terms of the partial derivatives of f, g, and h at p.

(3) In *(ii)*, compute $D\phi(1, 1, 1)$ when $f(x, y, z) = x^2 + yz$, $g(x, y) = y^3 + xy$, and $h(x) = e^x$. Do this in two ways: using your general formula in *(ii)* and also by explicitly computing ϕ in this case and directly computing the Jacobian matrix from this.

Problem J: Problem 2(a) on page 63 of the text.

Problem K: Find the 3rd order Taylor series of $F(x, y) = e^{x+y^2}$ abou the origin.

Bonus: A real symmetric n by n matrix A is called positive definite if $x^T Ax > 0$ for all $x \in \mathbb{R}^n$.

(1) Show that a real symmetric matrix is positive definite if and only if it is invertible and, for all non-zero x, the angle between Ax and x is less than 90 degrees.
(2) Show that a real symmetric matrix is positive definite if and only if all its eigenvalues are positive.
(3) Let A_d be the top left d by d minor of A. Show that if A is positive definite, so is each $A_d, 1 \leq d \leq n$.
(4) Prove that A is positive definite if and only if $\det(A_d) > 0$ for all $1 \leq d \leq n$.

You may use without proof that a real symmetric matrix has an orthonormal basis of eigenvectors. A hint for the last part is available on the office door, but as always try it without first!