WORKSHEET 5

Problem A: Show that the open and closed balls of radius r in $\mathbb{R}^d$ are both Jordan measurable, and that their Jordan measure is $c_d r^d$ for some constant $c_d > 0$ that only depends on the dimension.

Problem B (optional): Establish the bound $\left(\frac{2}{\sqrt{d}}\right)^d \leq c_d \leq 2^d$.

Problem C: Let $E \subset \mathbb{R}^d$ be bounded. Show that both E and its closure $\overline{E}$ have the same Jordan outer measure.

Problem D: Let $E \subset \mathbb{R}^d$ be bounded. Show that E and its interior have the same Jordan inner measure.

Problem E: Show that E is Jordan measurable if and only if the topological boundary $\partial E = \overline{E} \setminus E^\circ$ has Jordan outer measure 0.

Recall: To define the Riemann integral of a bounded function f on an interval $[a, b] \subset \mathbb{R}$, we first recall the notion of a partition $\mathcal{P}$, which is a set of points

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b.$$

The norm of the partition is $\Delta\mathcal{P} = \max_{1 \leq k \leq n} x_k - x_{k-1}$, and we denote by $\Delta x_k = x_k - x_{k-1}$. For each such partition, we define to quantities:

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} \Delta x_k \inf_{[x_{k-1}, x_k]} f, \quad \text{and} \quad U(f, \mathcal{P}) = \sum_{k=1}^{n} \Delta x_k \sup_{[x_{k-1}, x_k]} f.$$

We define the lower and upper Darboux integrals respectively as

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(f, \mathcal{P}), \quad \textit{and} \quad \overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

where the extrema above are taken over all partitions of the interval $[a, b]$. We say that f is Riemann integrable if the above two numbers are equal. We define the common value as the Riemann (or Darboux) integral of f.

Problem F: Let $[a, b]$ be an interval and let $f : [a, b] \to \mathbb{R}$ be a bounded nonnegative function. Show that f is Riemann integrable if and only if the set $E := \{(x, t) : x \in [a, b] : 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbb{R}^2$.