HW 2

DUE FRIDAY SEPTEMBER 6

For hints see office door. But try without the hints first.

Problem A: If $\|\cdot\|$ is a norm on a vector space V, show that $d(x, y) = \|x - y\|$ defines a metric on V.

Problem B: Let $T : V_1 \to V_2$ be a linear map between normed vector spaces. The norm on V_i will be denoted $\|\cdot\|_i$. Define

$$\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = \sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2.$$

This is either a non-negative real number or infinity. The linear map is called bounded if it is not infinity. Show that T is continuous if and only if it is bounded.

Problem C: Give an example of an unbounded linear map.

Problem D: Given an example of a sequence (T_i) of diagonalizable 2 by 2 real matrices whose eigenvalues stay bounded but for which $\|T_i\| \to \infty$. (Here the matrices define linear maps from $\mathbb{R}^2$ to itself, and we use the Euclidean norm on $\mathbb{R}^2$.)

Problem E: Show that if a subset of a metric space is totally bounded, then it is also separable (i.e. there exists a countable dense subset).

Problem F: Let X be define as infinitely many copies of $[0, 1]$ with all their left endpoints glued together, with the natural metic d.

Formally, we can first define $\hat{X} = \mathbb{N} \times [0, 1]$, and define an equivalence relation on $\hat{X}$ by $(i, x) \sim (j, y)$ if and only if $(i, x) = (j, y)$ or $x = y = 0$. Let X be the set of equivalence classes, and define a metric d by setting $d([(i, x)], [(j, y)])$ to be $|x| + |y|$ if $i \neq j$ and $|x - y|$ if $i = j$. You should convince yourself that this makes sense but don't have to write this up.

Prove that (X, d) is bounded but not totally bounded.

Problem G: Let c_0 be the subspace of $\ell^\infty(\mathbb{N})$ of sequences that converge to zero, with the sup metric. Show that a subset Q of c_0 is totally bounded if and only if it is bounded and for all $\epsilon > 0$ there exists $N > 0$ such that for all $(x_n) \in Q$ and all $n \geq N$ we have $|x_n| < \epsilon$.

Bonus problem: A map $f : X \to Y$ between metric spaces is called an isometric embedding if

$$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

for all $x_1, x_2 \in X$. If such a map exists we say X embedds isometrically in Y.

Show that every seperable metric space embedds isometrically into $\ell^\infty(\mathbb{N})$.