WORKSHEET 1

Problem A: Let (X, d) be a metric space. Recall that a set U is said to be open if for all $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

(1) Show that the open sets according to this definition define a topology on X.

(2) Show that a sequence (x_n) in X converges to x_∞ in this topology if and only if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n \geq N$ then

$$d(x_n, x_\infty) < \epsilon.$$

(3) Show that for all $x \in X$ and all $r > 0$, the "open ball"

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

is in fact an open set by the definition above.

(4) Show that for all $x \in X$ and all $r > 0$, the "closed ball"

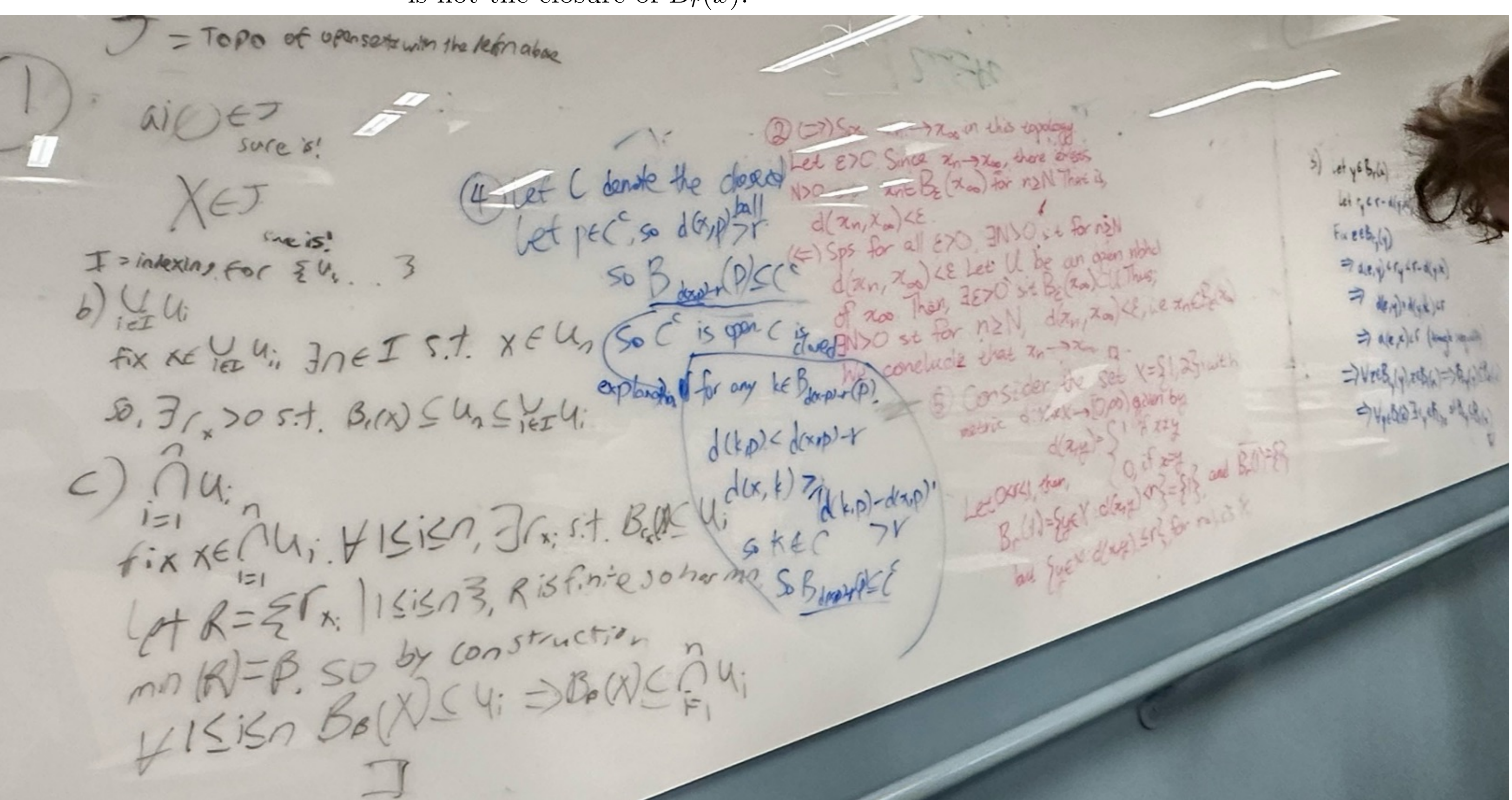
$$\{y \in X : d(x, y) \leq r\}$$

is in fact a closed set by the definition above.

(5) Give an example of a metric space (X, d), a point x, and an $r > 0$ such that

$$\{y \in X : d(x, y) \leq r\}$$

is not the closure of $B_r(x)$.



Problem B: Recall that a metric space is called complete if every Cauchy sequence converges.

The Baire Category Theorem states the following:

Theorem 1. *Let (X, d) be a complete metric space, and let $(U_n)_{n=1}^{\infty}$ be a sequence of open dense sets in X. Then*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X.

(1) Show that this is false without the assumption of completeness by considering the rationals with the usual metric.

(2) Prove the Baire Category Theorem as follows.

  (a) Show that it suffices to show that for all $x_0 \in X$ and $r_0 > 0$ the intersection contains a point of the ball $B_{r_0}(x_0)$.

  (b) With r_i, x_i already having been defined, show that you can pick $x_{i+1} \in X, 0 < r_{i+1} < r_i/2$ so that

$$\overline{B_{r_{i+1}}(x_{i+1})} \subset B_{r_i}(x_i) \cap U_{i+1}.$$

  (c) Show that the sequence (x_i) is a Cauchy sequence.

  (d) Show that the limit of (x_i) is a point of $\bigcap_{n=1}^{\infty} U_n$ contained in $B_{r_0}(x_0)$.

Problem C: Suppose that (X, d) is a (non-empty) complete metric space in which every point is a limit point. Use the Baire Category Theorem to show X is uncountable.

$\forall a \in X, \forall \varepsilon > 0\ \exists c \in U_i$ s.t.

$d(c, a) < \varepsilon$

Let $i : \mathbb{N} \to \mathbb{Q}$ bijection

$\{U_i\} = \{\mathbb{Q} / \{q_i\} \mid q \in \mathbb{Q}\}$

is countable and can be sequenced. has intersection $\emptyset$

$\mathbb{N} \to \{U_i\}$

$n \in \mathbb{N}$ S_n

Define $U_q = \mathbb{Q} - \{q\}$, index over $\mathbb{Q}$.

Then, $\bigcap_{q \in \mathbb{Q}} \mathbb{Q} \setminus \{q\} = \mathbb{Q} \setminus \left(\bigcup_{q \in \mathbb{Q}} \{q\} \right)$

$= \mathbb{Q} \setminus \mathbb{Q} = \emptyset$.

Let X be top. space. $A \subseteq X$ is dense, if $\bar{A} = X$.

Let $y \in X$. Then, for every open nbhd U of y, $U \cap A \neq \emptyset$.

Let $y \in X$. Then $\forall \varepsilon > 0, \exists x \in A$ s.t $d(x, y) < \varepsilon$.

Since U_{i+1} is dense in X,

$U_i \cap B_r(x) \neq \emptyset\ \forall x \in X, \forall r > 0$

We can find x_i, r_i st

$B_{r_i}(x_i) \cap U_{i+1} \neq \emptyset$.

$\exists x_{i+1} \in B_{r_i}(x_i) \cap U_{i+1}$

$\exists r_{i+1} = \min\{\varepsilon, \frac{r_i}{2}\}$

$\hookrightarrow B_{r_{i+1}}(x_{i+1}) \subset B_{r_i}(x_i) \cap U_{i+1}$

x_i U_i

ε $X \to x_i$

$(x_i \in B_{r_i} \cap U_i)$

Let $\varepsilon > 0$. Since $r_n \to 0$, $\exists N > 0$ s.t for $n \geq N$, $r_n < \varepsilon$.

Then, for $j, k \geq N$,

$d(x_j, x_k) < r_n < \varepsilon$

since $d(x_j, x_k) < d(x_{j-1}, x_k)$

Since x_1 is limit point, the set is dense in X

$\bigcap_{n=1}^{\infty} X \setminus \{x_1\}$

$X \setminus \bigcup_{n=1}^{\infty} \{x_1\}$