HW 3

DUE FRIDAY SEPTEMBER 13 AT 7PM

Problem A: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that a function $f : X \to Y$ is Lipschitz with constant C if for any $x, y \in X$, we have

$$d_2(f(x), f(y)) \leq C d_1(x, y).$$

(1) Show that Lipschitz maps are uniformly continuous, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d_1(x_1, x_2) < \delta$ then $d_2(f(x_1), f(x_2)) < \epsilon$.

(2) Let $f_n : X_1 \to X_2$ be Lipschitz maps with common Lipschitz constant C. Suppose that the f_n converge uniformly to f, i.e. for all $\epsilon > 0$ there exists $N > 0$ such that for all $n > N$, and all $x \in X_1$,

$$d_2(f_n(x), f(x)) < \epsilon.$$

Is f Lipschitz? What if we only assume that the f_n are Lipschitz (without giving a common Lipshitz constant)?

Problem B: We say that a metric space X is connected if it cannot be written as $X = A \cup B$ where A and B are nonempty disjoint open subsets of X.

(1) Show that if $f : X \to Y$ is a continuous function between metric spaces X and Y, then $f(X)$ is connected if X is connected.

(2) Conclude that if $f : X \to \mathbb{R}$ and X is a connected metric space, then f admits all *intermediate* values $m \in (\inf f, \sup f)$. That is, for any such m, there exists $x_0 \in X$ such that $f(x_0) = m$.

Problem C: Let $f : X \to Y$ be a continuous bijective (one-to-one and onto) mapping between metric spaces X and Y.

(1) Suppose that X is compact. Show that the inverse function $f^{-1} : Y \to X$ is also continuous.

(2) Give an example to show that the requirement that X is compact is necessary.

Problem D: Let f be a real valued function defined on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$). Recall that the directional derivative $D_v f(p)$ of f at p

in the direction v is vector

$$D_v f(p) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

if this limit exists.

(1) If $c \in \mathbb{R}$ and $D_v f(p)$ exists, prove that $D_{cv} f(p)$ exists and $D_{cv} f(p) = c \cdot D_v f(p)$.

(2) For $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{|xy|}$$

and $v = (1, 0)$, $v' = (0, 1)$, show that $D_v f(0, 0)$ and $D_{v'} f(0, 0)$ exist but $D_{v+v'} f(0, 0)$ does not exist.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove that $D_v f(0, 0)$ exists for every $v = (a, b) \in \mathbb{R}^2$, vanishing if $v = 0$ and equal to

$$\frac{ab^2}{a^2 + b^2}$$

otherwise.

Remark 1. This formula for $D_v f(0, 0)$ is not linear in v.

Remark 2. Using polar coordinates, it is easy to see that f is continuous at $(0, 0)$.

Problem E: Give the statement of the Baire Category Theorem (from Worksheet 1). (Test yourself by seeing if you can write it down from memory!)

Problem F: Submit a writeup of Problem B from Worksheet 2.

Bonus problem: A metric space (X, d) is said to be uniformly disconnected if there is $\epsilon_0 > 0$ so that no pair of distinct points $x, y \in X$ can be connected by an ϵ_0-chain, where an ϵ_0-chain connecting x and y is a sequence of points

$$x = x_0, x_1, \ldots, x_m = y$$

satisfying

$$d(x_i, x_{i+1}) \leq \epsilon_0 d(x, y).$$

(1) Show that the Cantor set is uniformly disconnected.
(2) Show that a metric space (X, d) is uniformly disconnected if and only if there is an ultrametric d' on X for which there is some $C > 1$ such that

$$d'(x, y)/C \leq d(x, y) \leq Cd'(x, y).$$

An ultrametric is a metric which satisfies the following improvement of the triangule inequality:

$$d(x, z) \leq \max(d(x, y), d(y, z))$$

for all x, y, z. The discrete metric, where the distance between any pair of distinct points is 1, is an example of an ultrametric. Many other more interesting and important examples exist.

For a hint on the bonus, see office door. But try without first.