WORKSHEET 4

The main caveat of elementary measure is that it only allows us to measure elementary sets, which is a fairly restrictive family of sets. Building on the old intuition (going back at least to Archimedes) we can lower bound (respectively upper bound) the measure of a set by approximating it from within (respectively without) by an elementary set, i.e. if A and B are elementary and $A \subset E \subset B$, then the measure of E (if it exists) should be sandwiched between that of A and B.

Definitions: Let $E \subset \mathbb{R}^d$ be a bounded set.

- The *Jordan inner measure* $\underline{m}_J(E)$ of E is defined as
$$\underline{m}_J(E) = \sup_{A \subset E, A \,\text{elementary}} m(A).$$
Here $m(A)$ is the elementary measure of A.
- The *Jordan outer measure* $\overline{m}_J(E)$ of E is defined as
$$\overline{m}_J(E) = \inf_{A \supset E, A \,\text{elementary}} m(A).$$
- If $\underline{m}_J(E) = \overline{m}_J(E)$, we say that E is Jordan measurable, and call the common value $m(E)$ (the Jordan measure of E).
  By convention, we do not consider unbounded sets to be Jordan measurable.

Problem A: Assume that $E \subset \mathbb{R}^d$ is bounded. Show that the following are equivalent:

a) E is Jordan measurable.
b) For every $\epsilon > 0$, there exists elementary sets A and B such that $A \subset E \subset B$ and $m(B \setminus A) \leq \epsilon$.
c) For every $\epsilon > 0$, there exists an elementary set A such that $\overline{m}_J(E \Delta A) \leq \epsilon$.

Problem B: Deduce that every elementary set E is Jordan measurable and that its Jordan measure is the same as its elementary measure. In particular, $m(\emptyset) = 0$.

Problem C: Let E, F be Jordan measurable sets. Clearly $m(E) \geq 0$. Show that

(1) $E \cup F, E \cap F, E \setminus F$, and $E \Delta F$ are all Jordan measurable.
(2) (Finite additivity) If E and F are disjoint, then $m(E \cup F) = m(E) + m(F)$.

(3) (Monotonicity) If $E \subset F$, then $m(E) \leq m(F)$.
(4) (Finite subadditivity) $m(E \cup F) \leq m(E) + m(F)$.
(5) (Translation invariance) for any $x \in \mathbb{R}^d$, $m(E + x) = m(E)$.

Problem D: Let B be a closed box of $\mathbb{R}^d$ and $f : B \to \mathbb{R}$ a continuous function.

(1) Show that the graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in $\mathbb{R}^{d+1}$ and that it has Jordan measure 0. *Hint: Use that f is uniformly continuous.*
(2) Show that the set $\{(x, t) : x \in B, 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.

From this we conclude that some familiar sets like triangles in $\mathbb{R}^2$ and balls in $\mathbb{R}^d$ are Jordan measurable.

Problem E: Is $\mathbb{Q} \cap [0, 1]$ Jordan measurable? What are its inner and outer Jordan measure?