HW 4

DUE FRIDAY SEPTEMBER 20 AT 7PM (BONUS 24 HOURS LATER)

Problem A: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be satisfy

$$F(tx) = tF(x)$$

for all all positive real numbers t and all $x \in \mathbb{R}^n$. Assume F is differentiable at the origon. Show F is linear.

Problem B: Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $(1 \leq i \leq m, 1 \leq j \leq n)$ exist are are bounded on A. Show that f is continuous on A.

Problem C: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation:

$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

(1) Calculate Df and $\det Df$.
(2) Let $S = [1, 2] \times [0, \pi/2]$. Find $f(S)$ and sketch it.
(3) Show that f is a homeomorphism from S on $f(S)$ and compute the inverse function f^{-1}.
(4) Compute Df^{-1} and $\det Df^{-1}$.
(5) What relation can you find between Df and Df^{-1}?

Problem D: Give an example of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that, at the origin, all directions derivatives exist and are zero, but F is not differentiable at the origin.

Problem E: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}.$$

(1) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at 0.
(2) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x, y) \neq 0$.
(3) Show that $f \in C^1(\mathbb{R}^2)$.
(4) Show that

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

exist everywhere on $\mathbb{R}^2$, but they are not equal at $(x, y) = 0$.

Bonus: Recall that an ultametric space is a metric space where one has the following stronger than usual form of the triangle inquality:

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

(1) Show that, in an ultrametric space, open balls are closed.
(2) Show that, in an ultrametric space, if two balls intersect, one of the two must be contained in the other.
(3) Show that, in an ultrametric space, every point of a ball is the center of the ball. That is, if $y \in B_r(x)$, then $B_r(x) = B_r(y)$.
(4) Let G be a connected weighted undirected graph. (The weighting is the assignment of a positive number to each edge). Let $V(G)$ be the set of vertices.

    Given a path in the graph (a sequence of adjacent edges), define the length of the path to be the largest weight of an edge crossed by the path.

    Given $v, w \in V(G)$, define $d(v, w)$ to be the smallest length of a path from v to w.

    Show that d is an ultrametric on $V(G)$.
(5) Show that any finite ultrametric arises as in the previous part.

Just for fun (don't hand in): Imagine you have an electric car, and you live in a country that provides free charging stations, and you're not in a hurry. Why might you end up thinking about an ultrametric?