WORKSHEET 2

Problem A: Let us start with the interval $C = [0, 1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. This leaves us with the set

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

formed of 2 closed subintervals. Having constructed

$$C_1 \supset C_2 \supset \cdots \supset C_n$$

where C_n is the union of 2^n subintervals each of length $\frac{1}{3^n}$, we construct C_{n+1} as follows: To obtain C_{n+1} we remove the middle third of each of the 2^n intervals that form C_n. This leaves us with a union of 2^{n+1} intervals each of length $\frac{1}{3^{n+1}}$. Let

$$C = \bigcap_{n=1}^{\infty} C_n.$$

This is called the (middle thirds) Cantor set.

(1) Show that C is non-empty and compact.
(2) Show that every point in C is a limit point.
(3) Conclude using the last worksheet that C is uncountable.
(4) Show that C cannot contain any interval (a, b).
(5) What is the total length of C_n? What would be a reasonable definition of the length of C?

Problem B: Motivated by the above, it would be grand to have a measure function that tells us how big or small a subset of $\mathbb{R}^d$ is. This would be a function from the set $\mathcal{P}(\mathbb{R}^d)$ of subsets of $\mathbb{R}^d$ into $[0, \infty]$, say

$$m : \mathcal{P}(\mathbb{R}^d) \to [0, \infty].$$

We would like this function to satisfy the following properties:

a) If $E_1, E_2, \ldots$ is a countable collection of disjoint subsets of $\mathbb{R}$, then
$$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n).$$
b) If E is congruent to F (i.e. F can be obtained from E by applying rigid motions: translations, rotations, or reflections) then we should have that $m(E) = m(F)$.

c) $m([0,1)^d) = 1$.

The bad news is that no such function can exist, and here's why (at least when $d = 1$). Let us define an equivalence relation between elements of $[0,1)$ as follows: We say $x \sim y$ if $x - y$ is a rational number. Let N be a subset of $[0,1)$ that contains exactly one element of each equivalence class (the existence of such a N requires invoking the axiom of choice). Now let

$$R = [0,1) \cap \mathbb{Q},$$

and for each $r \in R$ define the set

$$N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}.$$

(Basically N_r is just the translate of N by r units to the right, except that we move the part that sticks out of the interval $[0,1)$ one unit to the left).

(1) Show that $[0,1)$ is the disjoint union of N_r for $r \in R$.
(2) Show that if a measure function satisfying a), b) and c) above exists, then $m(N) = m(N_r)$ for every $r \in R$.
(3) Arrive at a contradiction.

Remark: One might think that possibly relaxing condition a) to cover only *finitely* many disjoint sets E_n, i.e.

$$m(\cup_{n=1}^{N} E_n) = \sum_{n=1}^{N} m(E_n).$$

would resolve the contradiction. Unfortunately, the Banach-Tarski paradox tells us that this is not enough to resolve this issue.

Banach-Tarski tells us that we can split the unit ball in $\mathbb{R}^3$ into finitely many (actually 5 is sufficient) many disjoint pieces, apply rigid motions to those pieces and then reassemble them to obtain two copies of the unit ball.

Conclusion: The problem with the above wishlist is that we insisted on being able to measure *every* subset of $\mathbb{R}^d$. We have shown that this is impossible. The solution is to be content with a measure function that is defined on some but not all subsets. Such subsets will be called measurable subsets.