

WORKSHEET 1

Problem A: Let (X, d) be a metric space. Recall that a set U is said to be open if for all $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

- (1) Show that the open sets according to this definition define a topology on X .
- (2) Show that a sequence (x_n) in X converges to x_∞ in this topology if and only if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n \geq N$ then

$$d(x_n, x_\infty) < \epsilon.$$

- (3) Show that for all $x \in X$ and all $r > 0$, the "open ball"

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

is in fact an open set by the definition above.

- (4) Show that for all $x \in X$ and all $r > 0$, the "closed ball"

$$\{y \in X : d(x, y) \leq r\}$$

is in fact a closed set by the definition above.

- (5) Give an example of a metric space (X, d) , a point x , and an $r > 0$ such that

$$\{y \in X : d(x, y) \leq r\}$$

is not the closure of $B_r(x)$.

$\mathcal{T} = \text{Topo of open sets with the defn above}$

1) $\emptyset \in \mathcal{T}$ sure is!

$X \in \mathcal{T}$ sure is!

$I = \text{indexing for } \{U_i\}_{i \in I}$

b) $\bigcup_{i \in I} U_i$
 fix $x \in \bigcup_{i \in I} U_i$, $\exists n \in I$ s.t. $x \in U_n$
 so, $\exists r_x > 0$ s.t. $B_{r_x}(x) \subseteq U_n \subseteq \bigcup_{i \in I} U_i$

c) $\bigcap_{i=1}^n U_i$
 fix $x \in \bigcap_{i=1}^n U_i$, $\forall 1 \leq i \leq n$, $\exists r_i$ s.t. $B_{r_i}(x) \subseteq U_i$
 let $R = \{r_i : 1 \leq i \leq n\}$, R is finite so has max.
 $\min(R) = \beta$, so by construction
 $\forall 1 \leq i \leq n$ $B_\beta(x) \subseteq U_i \Rightarrow B_\beta(x) \subseteq \bigcap_{i=1}^n U_i$

4) let C denote the closed ball
 let $p \in C$, so $d(x, p) \leq r$
 so $B_{d(x, p)}(p) \subseteq C$
 So C is open. C is closed.
 explanation: for any $k \in B_{d(x, p)}(p)$
 $d(k, p) < d(x, p) - r$
 $d(x, k) \geq d(x, p) - d(k, p) > r$
 so $k \notin C$
 So $B_{d(x, p)}(p) \subseteq C$

2) (\Rightarrow) $x_n \rightarrow x_\infty$ in this topology.
 Let $\epsilon > 0$. Since $x_n \rightarrow x_\infty$, there exists
 $N > 0$ s.t. $\forall n \geq N$, $x_n \in B_\epsilon(x_\infty)$. This is
 $d(x_n, x_\infty) < \epsilon$.
 (\Leftarrow) Sps for all $\epsilon > 0$, $\exists N > 0$ s.t. for $n \geq N$
 $d(x_n, x_\infty) < \epsilon$. Let U be an open neighborhood
 of x_∞ . Then, $\exists \epsilon > 0$ s.t. $B_\epsilon(x_\infty) \subseteq U$. Thus,
 $\exists N > 0$ s.t. for $n \geq N$, $d(x_n, x_\infty) < \epsilon$, i.e. $x_n \in U$.
 conclude that $x_n \rightarrow x_\infty$.

5) Consider the set $X = \{1, 2\}$ with
 metric $d(x, y) = |x - y|$ given by
 $d(1, 1) = 0$, $d(1, 2) = 1$, $d(2, 1) = 1$, $d(2, 2) = 0$.
 Let $x = 1$, $r = 1$.
 $B_1(1) = \{y \in X : d(1, y) < 1\} = \{1\}$ and $\overline{B_1(1)} = \{1, 2\}$
 but $\{y \in X : d(1, y) \leq 1\} = \{1, 2\}$ for $n \geq 1$.

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Problem B: Recall that a metric space is called complete if every Cauchy sequence converges.

The Baire Category Theorem states the following:

Theorem 1. *Let (X, d) be a complete metric space, and let $(U_n)_{n=1}^{\infty}$ be a sequence of open dense sets in X . Then*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X .

- (1) Show that this is false without the assumption of completeness by considering the rationals with the usual metric.
- (2) Prove the Baire Category Theorem as follows.
 - (a) Show that it suffices to show that for all $x_0 \in X$ and $r_0 > 0$ the intersection contains a point of the ball $B_{r_0}(x_0)$.
 - (b) With r_i, x_i already having been defined, show that you can pick $x_{i+1} \in X, 0 < r_{i+1} < r_i/2$ so that

$$\overline{B_{r_{i+1}}(x_{i+1})} \subset B_{r_i}(x_i) \cap U_{i+1}.$$

- (c) Show that the sequence (x_i) is a Cauchy sequence.
 - (d) Show that the limit of (x_i) is a point of $\bigcap_{n=1}^{\infty} U_n$ contained in $B_{r_0}(x_0)$.

Problem C: Suppose that (X, d) is a (non-empty) complete metric space in which every point is a limit point. Use the Baire Category Theorem to show X is uncountable.

$$\forall a \in X, \forall \varepsilon > 0 \exists c \in U_i \text{ s.t. } d(c, a) < \varepsilon$$

Let $i: \mathbb{N} \rightarrow \mathbb{Q}$ bijection

$$\{U_i\} = \{\mathbb{Q} \setminus \{q_i\} \mid q_i \in \mathbb{Q}\}$$

is countable and
can be sequenced
has intersection \emptyset

$$\mathbb{N} \rightarrow \{U_i\}$$

now

$$U_q = \mathbb{Q} \setminus \{q\}$$

Define U_q
index over \mathbb{Q}

$$\text{Then, } \bigcap_{q \in \mathbb{Q}} U_q = \mathbb{Q} \setminus \left(\bigcup_{q \in \mathbb{Q}} \{q\} \right) = \mathbb{Q} \setminus \mathbb{Q} = \emptyset$$

Let X be top. space.
 $A \subseteq X$ is dense, if $\bar{A} = X$.

Let $y \in X$. Then, for every
open nbhd U of y , $U \cap A \neq \emptyset$

Let $y \in X$. Then $\forall \varepsilon > 0, \exists x \in A$
s.t. $d(x, y) < \varepsilon$

Since U_{i+1} is dense in X ,

$$U_i \cap B_r(x) \neq \emptyset \quad \forall x \in X, \forall r > 0$$

We can find x_i, r_i s.t.

$$B_{r_i}(x_i) \cap U_{i+1} \neq \emptyset$$

$$\exists x_{i+1} \in B_{r_i}(x_i) \cap U_{i+1}$$

$$\exists r_{i+1} = \min\left\{\varepsilon, \frac{r_i}{2}\right\}$$

$$B_{r_{i+1}}(x_{i+1}) \subset B_{r_i}(x_i) \cap U_{i+1}$$

$$\bigcap_{i \in \mathbb{N}} X \setminus \{x_i\}$$

Let $\varepsilon > 0$. Since $r_n \rightarrow 0$,

$\exists N > 0$ s.t. for $n \geq N$, $r_n < \varepsilon$.

Then, for $j, k \geq N$,

$$d(x_j, x_k) < r_n < \varepsilon$$

$$\text{since } d(x_j, x_k) < d(x_{j-1}, x_k)$$

Since x_i is limit
point, the set
is dense
in X

$$\bigcap_{n=1}^{\infty} X \setminus \{x_i\}$$