上次: $\mathbb{R}^n$中, cpt $\Longleftrightarrow$ seq. cpt. $\Longleftrightarrow$ closed & bounded

But it is not true in general metric space (eg: ℓ^∞)

Today: fix it for general metric space.

closed $\xrightarrow{\text{stronger}}$ complete (i.e. every Cauchy seq. conv.)

bdd $\xrightarrow{\text{stronger}}$ totally bdd.

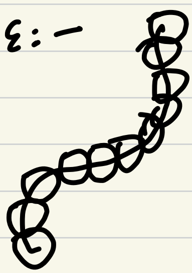
Def Totally bounded.

X a metric space.

$E \subseteq X$ 称为 ttl. bdd. if:

$\forall \varepsilon > 0$, 都$\exists$一个 finite cover of E by balls of radius ε

ε: —



即: $\exists x_1, \dots, x_n \in E$ s.t.

$$E \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

In any MS, ttl. bdd. $\Longrightarrow$ bdd

In $\mathbb{R}^d$, ttl. bdd. $\Longleftrightarrow$ bdd 且 closed $\Longleftrightarrow$ complete

eg: the unit ball in ℓ^∞ is not ttl. bdd.

Thm X be a MS.
for $E \subseteq X$, TFAE:
(1) E cpt.
(2) E seq. cpt.
(3) E complete & ttl. bdd.

(Rmk1: 即 generally, cpt $\Leftrightarrow$ seq. cpt. $\Leftrightarrow$ complete & ttl. bdd.)

(Rmk2 (exercise): if X complete, then $E \subseteq X$ is closed iff it is complete.)

(Rmk3: 对于 general topological space, 并不 true)

Pf. $1 \Rightarrow 2$: Already done
$2 \Rightarrow 3$ seq. cpt. implies complete & ttl. bdd.
Pf of ttl. bdd.
Fix $\varepsilon > 0$
Pick $p_1 \in E$
Given $p_1, \dots, p_{n-1}$:
if $E \subseteq \bigcup_{i=1}^{n-1} B_\varepsilon(p_i)$, stop
otherwise pick $p_n \in E$ s.t. $p_n \notin \bigcup_{i=1}^{n-1} B_\varepsilon(p_i)$
Claim This process must stop.
if not, we will get seq $(p_n)_{n=1}^\infty$ with $d(p_i, p_j) > \varepsilon \ \forall i \neq j$
Then it is not Cauchy thus not conv. conflicting seq. cpt.
So eventually will have $E \subseteq \bigcup_{i=1}^{n} B_\varepsilon(p_i)$

Pf of complete

Let (x_n) be a Cauchy seq. in E

Since E seq. cpt., $\exists$ subseq. $(x_{n_r}) \to p$ for some $p \in E$

Claim $(x_n) \to p$ also.

Given $\varepsilon > 0$

Pick N s.t. $\forall n, m \geq N$, $d(x_n, x_m) < \frac{\varepsilon}{2}$

Pick $n_k \geq N$ s.t. $d(x_{n_k}, p) < \frac{\varepsilon}{2}$

$\Rightarrow \forall m \geq N$, $d(x_m, p) \leq d(x_m, x_{n_k}) + d(x_{n_k}, p) < \varepsilon$

$3 \Rightarrow 2$ ttl. bdd. & complete implies seq. cpt.

Let (x_n) be a seq. in E

WTS: (x_n)有一 conv. subseq.

Fix a finite cover of E by balls of radius 1

One of them must have infly many terms of (x_n)

Call it B_1

Given $B_1, \ldots, B_{n-1}$ st. there are infly many m with $x_m \in \bigcap_{i=1}^{n-1} B_i$

Let radius$(B_i) = \frac{1}{2^{i-1}}$

Find a finite cover of E by balls of radius $\frac{1}{2^{n-1}}$

By pigeonhole principle, one of them, call it B_n, $\exists$ infly many m s.t. $x_m \in \bigcap_{i=1}^{n} B_i$

Now pick n_1 with $x_{n_1} \in B_1$,

n_2 with $x_{n_2} \in B_1 \cap B_2$ with $n_2 > n_1$

$\vdots$

n_k with $x_{n_k} \in \bigcap_{i=1}^{k} B_k$ with $n_k > n_{k-1}$

For $\forall k_1, k_2 \geq k$,

$x_{k_1}, x_{k_2} \in B_k$ so $d(x_{k_1}, x_{k_2}) \leq 2 \cdot \frac{1}{2^{k-1}} = \frac{1}{2^{k-2}}$

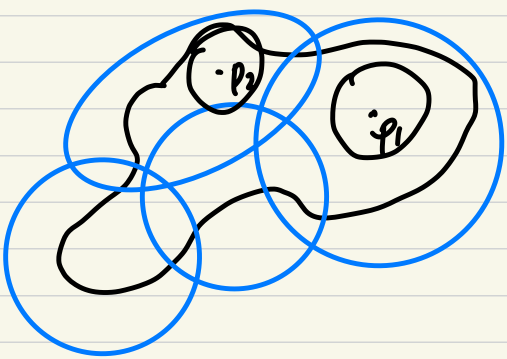
So (x_{n_k}) is Cauchy, by completeness it conv.

$2 \Rightarrow 1$ Seq cpt. implies cpt.

Intermediate Lemma (Leasbegue covering thm)

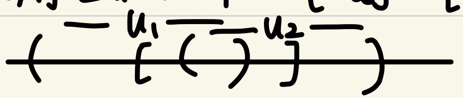
Let $\{U_\alpha\}$ be an open cover of any subset of a cpt set.

$\Rightarrow \exists \varepsilon > 0$ s.t. $\forall p \in E$, $\exists \alpha$ s.t. $B_\varepsilon(p) \subseteq U_\alpha$



Rmk：表达了一件事，即如果E cpt，那么E的open cover中任意个 U_α 和E边缘及其他 U_β 边缘都有一定的uniformly large distance

ex $[0,1] \subseteq \mathbb{R}$ is cpt. $\{U_\alpha\} = \{(-1, \frac{2}{3}), (\frac{1}{3}, 2)\}$



Take $\varepsilon = \frac{1}{6}$, works.

Pf of Lebesgue covering thm:

suppose not then $\forall n \in \mathbb{N}$, 都 $\exists p_n$ s.t. $\forall \alpha,\ B_{\frac{1}{n}}(p_n) \not\subseteq U_\alpha$

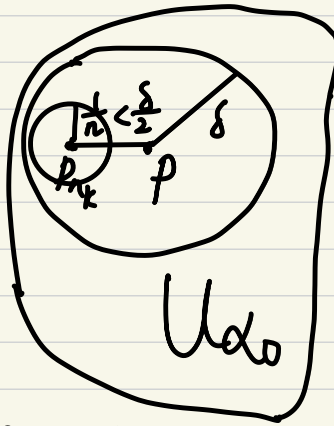
$\Longrightarrow$ (p_n) 有一个 conv. subseq. $(p_{n_k}) \longrightarrow p \in E$ for some p

但 $p \in U_{\alpha_0}$ for some α_0, 由于 U_{α_0} open $\Longrightarrow \exists \delta$ s.t. $\boxed{B_\delta(p) \subseteq U_{\alpha_0}}$

而由于 (p_{n_k}) conv. Take k_0 s.t. $d(p, p_{n_k}) < \frac{\delta}{2}$ whenever $k \geq k_0$

$\Longrightarrow \delta - \frac{1}{n_k} < \frac{\delta}{2}$

$\Longrightarrow B_{\frac{1}{n_k}}(p_{n_k}) \subseteq B_\delta(p) \subseteq U_{\alpha_0}$, contradicts



$\square$

然后回到 $2 \Longrightarrow 1$ 的 pf:

Let $\{U_\alpha\}$ be an open cover of E

By lemma, $\exists \varepsilon > 0$ s.t. $\forall p$, $\exists \alpha$ with $B_\varepsilon(p) \subseteq U_\alpha$

Since seq. cpt $\Longrightarrow$ ttl. bdd.

$\exists p_1, \dots, p_N$ s.t. $E \subseteq \bigcup_{i=1}^{N} B_\varepsilon(p_i)$

And for each p_i, $\exists \alpha_i$ s.t. $B_\varepsilon(p_i) \subset U_{\alpha_i}$

$$\Longrightarrow E \subseteq \bigcup_{i=1}^{N} U_{\alpha_i}$$

$\square$