WORKSHEET 3

Definitions: An interval I is a subset of $\mathbb{R}$ of the form $[a,b]$, $[a,b)$, $(a,b]$, or (a,b) where $a,b \in \mathbb{R}$. The length of I is defined to be $|I| := b-a$.

A *box* in $\mathbb{R}^d$ is a Cartesian product of intervals

$$B = I_1 \times \cdots \times I_d$$

and its *volume* is defined to be

$$|B| = |I_1| \times \cdots \times |I_d|.$$

An *elementary set* is any subset of $\mathbb{R}^d$ which is the union of a finite number of boxes.

Problem A: Show that if $E, F \subset \mathbb{R}^d$ are elementary sets, then the union $E \cup F$, the intersection $E \cap F$, the set theoretic difference $E \setminus F$, and the symmetric difference $E \Delta F = (E \setminus F) \cup (F \setminus E)$ are also elementary. Also, if $x \in \mathbb{R}^d$, then the translate $E+x := \{y+x : y \in E\}$ is also elementary.

Problem B: Show that any elementary set E can be expressed as the finite union of disjoint boxes. (Hint: Start with $d=1$.)

Definition: Let E be an elementary set. The above question allows to write

$$E = B_1 \cup B_2 \cup \cdots \cup B_n,$$

where $B_1, \ldots, B_n$ are disjoint. We define the elementary measure of E as

$$m(E) := |B_1| + |B_2| + \cdots + |B_n|.$$

Problem C: Show that $m(E)$ is well-defined in the sense that if E can be expressed in two ways as a union of disjoint boxes $B_1, \ldots, B_n$ and $B_1', \ldots, B_m'$, then

$$|B_1| + |B_2| + \cdots + |B_n| = |B_1'| + |B_2'| + \cdots + |B_m'|.$$

There's more than one approach you can take, but here you should use the following approach: First prove that for an interval I in $\mathbb{R}$,

$$|I| = \lim_{N \to \infty} \frac{1}{N} \# \left(I \cap \frac{1}{N}\mathbb{Z} \right),$$

where $\#S$ denotes the number of elements of a set S. And more generally for a box B, prove that

$$|B| = \lim_{N\to\infty} \frac{1}{N^d} \# \left(B \cap \frac{1}{N}\mathbb{Z}^d \right).$$

Here $\frac{1}{N}\mathbb{Z}^d = \{\frac{k}{N} : k \in \mathbb{Z}^d\}$. Use this to give an alternative definition of $m(E)$ for an elementary set that does not rely on its decomposition into disjoint boxes.

Problem D: Show that the following holds

(1) If $E_1, \dots, E_n$ are disjoint elementary sets, then

$$m(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^{n} m(E_i)$$

Recall that this is called finite additivity.

(2) If $E \subset F$ are two elementary sets, then

$$m(E) \leq m(F).$$

This property is called monotonicity.

(3) Show that if $E_1, E_2, \dots, E_n$ is an arbitrary finite collection of elementary sets, then

$$m(E_1 \cup \cdots \cup E_n) \leq m(E_1) + \cdots + m(E_n).$$

This is called finite subadditivity.

Why is this unsatisfactory? Of course, the main problem with this measure is that we can only measure relatively simple sets (namely the elementary sets). For example, we cannot measure the area of a disc.

Definition: One might be tempted to generalize this measure naively as follows: For an arbitrary set $E \subset \mathbb{R}^d$, define

$$m_{\text{pixel}}(E) = \lim_{N\to\infty} \frac{1}{N^d} \# \left(E \cap \frac{1}{N}\mathbb{Z}^d \right).$$

This is not a particurlary satisfactory definition either. One reason (out of many) is the following:

Problem E: Find a subset E of $\mathbb{R}$ such that both $m_{\text{pixel}}(E)$ and $m_{\text{pixel}}(E + x)$ exist, but $m_{\text{pixel}}(E) \neq m_{\text{pixel}}(E + x)$ for some $x \in \mathbb{R}$.