Metric spaces

Def A metric $\underline{d}$ on a set X is a func.

$$d: X \times X \to \mathbb{R}$$

satisfying:

(a) symmetry. $d(x,y) = d(y,x) \quad \forall x,y$

(b) positivity. $d(x,y) \geq 0$ and $=0$ iff $x=y$

(c) triangle ineq. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z$

Then (X,d) is called a metric space.

ex

(a) $(\mathbb{R}, d(x,y) = |x-y|)$

(b) $(\mathbb{R}, d(x,y) = |\int_x^y e^{-t} dt|)$

(c) $(\mathbb{R}^m, d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|)$

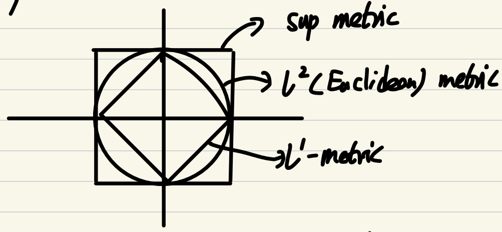
$= \sqrt{\sum (x_i - y_i)^2}$ l^2-metric

(d) $(\mathbb{R}^m, d(\vec{x},\vec{y}) = \max |x_i - y_i|)$

$\|\vec{x}-\vec{y}\|_\infty$, 称为 l^∞-metric

(e) $(\mathbb{R}^m, d(\vec{x},\vec{y}) = \max |x_i - y_i|)$ l^1-metric

$(\mathbb{R}^2)$



(f) $C[0,1] = \{ f: [0,1] \to \mathbb{R} \mid f \text{ cts} \}$

$d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|$

(g) $(C[0,1], d(f,g) = \int_0^1 |f(t) - g(t)| dt)$

Def (X,d) be a metric space

(a) For $\varepsilon > 0$, the ε-neighborhood of $x_0 \in X$ is

$$B_\varepsilon(x_0) = \{ x \mid d(x,x_0) < \varepsilon \}$$

(b) $\Omega \subseteq X$ is open in X if:

$\forall x_0 \in X, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x_0) \subseteq \Omega$

$C \subseteq X$ is closed in X if

$C^c \subseteq X$ is open in X

Lemma $\Omega \subseteq \mathbb{R}^m$ is open using the Euclidean metric

$\Leftrightarrow$ it is open using the sup metric

Pf let $B_\varepsilon^{Euc}(x_0)$ be the Euclidean ball

$B_\varepsilon^{sup}(x_0)$ be the sup ball

note: $|\vec{x}| \leq \|\vec{x}\| \leq \sqrt{m} |\vec{x}|$

$\Rightarrow B_\varepsilon^{sup}(x_0) \subseteq B_\varepsilon^{Euc}(x_0) \subseteq B_{\sqrt{m}\varepsilon}^{sup}(x_0)$

Def If $E \subseteq X$ where X is a metric space

$p \in X$ is a limit. pt. of E

if $\forall \varepsilon, (B_\varepsilon(p) \cap E) \setminus \{p\} \neq \emptyset$

Lemma E is closed iff $E' \subseteq E$

Pf E closed $\Rightarrow X \setminus E$ open

$\Rightarrow \forall p \in X \setminus E, \exists \varepsilon$ s.t. $B_\varepsilon(p) \subseteq X \setminus E \Rightarrow p$ is not a lim pt.

$E' \subseteq E \Rightarrow \forall p \in X \setminus E, \exists \varepsilon$ s.t. $B_\varepsilon(p) \subseteq X - E$

$\Rightarrow X \setminus E$ open $\Rightarrow E$ closed.

Def If X is a metric space, $E \subseteq X$

closure of E:

$$\bar{E} = E \cup E'$$

ex $E = (0,1] \Rightarrow \bar{E} = E' = [0,1]$

$E = (0,1] \cup \{2\} \Rightarrow E' = [0,1], \bar{E} = [0,1] \cup \{2\}$

Lemma Let X be a metric space, $E \subseteq X$

$\Rightarrow$ (a) $\bar{E}$ is closed

(b) $E = \bar{E}$ iff E is closed

(c) if $E \subseteq F$ 且 F closed $\Rightarrow \bar{E} \subseteq F$

(Rmk: 即 $\bar{E}$ 为含 E 的 smallest closed set)

Pf let $q \in (\bar{E})^c$

$\Rightarrow \exists \delta > 0$ s.t. $B_\delta(q) \cap E = \emptyset$

$\Rightarrow (\bar{E})^c$ is open $\Rightarrow \bar{E}$ is closed

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Lemma let $E \neq \emptyset \subseteq \mathbb{R}$ be bounded above

$\Rightarrow \sup E \in \bar{E}$

(in par., if E closed then $\sup E \in E$, 即 $\sup E = \max E$)

Pf If $y \in E \Rightarrow y \in \bar{E}$

If $y \notin E \Rightarrow \forall \varepsilon\ \exists x \in E$ st $y-\varepsilon < x < y$

$\Rightarrow B_{\varepsilon}(y) \cap E = \phi$

$\Rightarrow y \in E' \subseteq \bar{E}$ $\square$

Compactness

Def An open cover of E in metric space X

is $\{G_\alpha\}_{\alpha \in I}$ of open subsets s.t. $E \subset \bigcup_\alpha E_\alpha$

(index set)

線 E compact if $\forall$ open cover of E 都有 finite subcover.

Thm Cpt subsets of metric spaces are closed & bounded.

(def of bdd:

$\exists x_0, r$ s.t. $E \subset B_r(x_0)$)

Pf Let $K \subset X$ be cpt

let $p \in K$

consider $\{q : d(p,q) > \frac{1}{n}\}_{n=1}^{\infty}$

By cptness, $\exists n$ s.t $K \subseteq \{q : d(p,q) > \frac{1}{n}\}$

$\Rightarrow B_{\frac{1}{n}}(q) \cap K \setminus \{q\} = \phi$

$\Rightarrow X \setminus K$ open $\Rightarrow K$ closed

Bdd consider $\{p \mid d(p,q) < n\}_{n=1}^{\infty}$