Friday: in basement (on syllabus)

OH: Tu 9:15am
Th 4pm
Fri 1:30pm
(Evan) Mon 6pm
(Yuyang) Th 1pm

First hw due Friday (7pm) (every week 2 assignments)

Today: compactness in $\mathbb{R}^n$ & general metric spaces

Thm closed subsets of cpt sets are cpt.

Pf let $C \subseteq K$, C closed, K cpt.
Let $\{U_\alpha\}$ be an open cover of C.
$\Rightarrow \{U_\alpha\} \cup \{C^c\}$ is an open cover of K
so it has a finite subcover
then is a finite cover of C by discarding $\{C^c\}$

Thm Finite intersection property

If $\{K_\alpha\}$ is a collection of cpt sets s.t. the intersection of any finite # of them is not empty

$\Rightarrow \bigcap_\alpha K_\alpha \neq \emptyset$ (Rmk: false without cpt
ex: $X = \mathbb{R}$, $K_n = \{[n,\infty) \mid n \in \mathbb{N}\}$)

Pf Suppose for contra: $\bigcap_\alpha K_\alpha = \emptyset$
$\Rightarrow \underline{\bigcup_\alpha K_\alpha^c = X}$
So $\{K_\alpha^c\}$ is an open cover of X.

Pick arbitrary $\alpha_0 \in I$
$\{K_\alpha^c\}$ is an open cover of K_{α_0}
By cptness, $\exists$ finite subcover $K_{\alpha_0} \subseteq K_{\alpha_1}^c \cup \dots \cup K_{\alpha_n}^c$

$\Rightarrow K_{\alpha_0}^c \cup \dots \cup K_{\alpha_n}^c = X$

$\Rightarrow K_{\alpha_0} \cap \dots \cap K_{\alpha_n} = \emptyset$

Thm cptness $\Rightarrow$ seq. compactness

Let K be a cpt set (def of seq. cpt.)

$\Rightarrow \forall (x_n)_{n=1}^\infty$ be a seq. in K, $\exists$ subseq that conv. to a pt. in K

Pf First suppose $\{x_n \mid n = 1, 2, \dots\}$ has no limit pt. in K (前面已经 done)
Thus $\forall p \in K$, $\exists \delta_p > 0$ s.t. $B_{\delta_p}(p)$ has at most 1 pt. of $\{x_n\}$
Note $\{B_{\delta_p}(p)\}_{p \in K}$ is an open cover of K
So by cptness, $\exists p_1, \dots, p_n$ s.t. $K \subseteq B_{p_1}(p_1) \cup \dots \cup B_{p_n}(p_n)$
So the seq. take at most n values
因而存在 const subseq.

Compactness in $\mathbb{R}^n$

Thm nested seq. property

Let $(I_n = [a_n, b_n])_n$ be a nested seq. of intervals.
s.t. $I_1 \supset I_2 \supset I_3 \supset \dots$

$\Rightarrow \bigcap_n I_n \neq \emptyset$

Pf So $(a_n)_n$ is a increasing & bdd seq.
Let $x = \sup a_n$
$\Rightarrow a_n \leq x \ \forall n$ by def
exercise: pf $x \leq b_n \ \forall n$
So $x \in \bigcap [a_n, b_n]$ $\square$

Rmk the statement is false if intervals not closed
ex: $\bigcap (0, \frac{1}{n}] = \emptyset$

Def A box in $\mathbb{R}^d$ is a set of the form
$B = [a_1, b_1] \times \dots \times [a_d, b_d]$

Corollary nested box property

Let $(B_n)_n$ be a seq. of nested boxes $\Rightarrow \bigcap_n B_n \neq \emptyset$

Pf Say $B_n = [a_1^{(n)}, b_1^{(n)}] \times \dots \times [a_d^{(n)}, b_d^{(n)}]$
$\forall i = 1, \dots, d$, I_i nested $\Rightarrow \exists x_j \in \bigcap I_j^{(n)}$
$\Rightarrow (x_1, \dots, x_d) \in \bigcap B_n$ $\square$

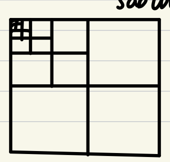
Thm Every closed box in $\mathbb{R}^n$ is cpt.

Pf Let $B_0 = [a_0, b_0] \times \dots \times [a_d, b_d]$
Suppose for contra that $\exists$ an open cover $\{U_\alpha\}$ of B_0 without a finite subcover.
Divide the box into 2^d subboxes
(ex: $d=2$, $[a_1, \frac{a_1+b_1}{2}] \times [a_2, \frac{a_2+b_2}{2}]$)

$\Rightarrow$ At least one of the 2^d subboxes 无 finite subbox
Call it $\boxed{B_1}$ (否则整个也有, contra)
Recursively cut the interval. For $i \in \mathbb{N}$, let B_{i+1} be the subbox without finite subcover

By nested box property $\bigcap_i B_i \neq \emptyset$
Since the B_i are getting smaller, $\bigcap_i B_i = \{\vec{x}\}$ for some $\vec{x} \in \mathbb{R}^d$
Any open cover of that point surely have a finite subcover.

conv. to
$\bigcap_i B_i = (a_1, \dots, a_d)$



Thm (Heine-Borel)

A subset of $\mathbb{R}^d$ is cpt iff it is closed & bounded

pf ① cpt $\Rightarrow$ closed & bdd

② bdded $\Rightarrow$ contained in a box B. (cpt)

and closed subset of a cpt set is cpt. □

ex let $l^\infty(\mathbb{N})$ denote all bounded seq. of natural numbers.

def: l^∞ metric (也稱 sup norm distance)

$$d\Big((a_1, a_2, \dots), (b_1, b_2, \dots)\Big) = \sup_n |a_n - b_n|$$

Consider $B = \{(a_1, a_2, \dots) \in l^\infty \mid d((a_1, a_2, \dots), (0, 0, \dots)) \leq 1\}$

exercise B is closed and bounded

(但是 B 不 cpt)