Last time we proved:

if f in $C^2 \Rightarrow \dfrac{\partial^2 f}{\partial x_k \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_k}$

(任意二阶 partial 可换序)

Rmk f in $C^{k+1} \Longleftrightarrow$ all partials of f in C^k

因而:

ex if f is C^3

$$\Rightarrow \frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z} f = \frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{\partial}{\partial y} f = \frac{\partial}{\partial z}\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \dots$$

Corollary 如果 $f: A \subseteq \mathbb{R}^r \to \mathbb{R}$ is C^r, then $\forall\, 2 \le m \le r$

$$\frac{\partial^m f}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_m}} = \frac{\partial^m f}{\partial x_{j_{\pi(1)}} \cdots \partial x_{j_{\pi(m)}}} \quad \text{for any permutation } \pi \in S_m$$

(即 f $C^r \Rightarrow f$ 的 r-order 内 partial derivative 可以随意换序.)

Pf 作为 exercise

这里引入一个 notation:

Def Multi-index notation

一个 n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ s.t. each $\alpha_j \in \mathbb{Z}_{\ge 0}$

If α is a multi-index, define its degree (or order)

$$|\alpha| = \sum_j \alpha_j$$

and write $\alpha! = \prod_j \alpha_j!$ (note: $0! = 1$)
$= \alpha_1! \alpha_2! \dots \alpha_n!$

For $\mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

For $f: \mathbb{R}^n \to \mathbb{R}$, $\partial^\alpha f = \left(\dfrac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{\alpha_n} f$

每个运算即 α_i 对 x_i 运算
然后各个结果相乘

ex $f: \mathbb{R}^2 \to \mathbb{R}$

$$\partial^{(2,1)} f = \left(\frac{\partial}{\partial x_1}\right)^2 \left(\frac{\partial}{\partial x_2}\right)^1 f = \frac{\partial^3}{\partial x_1 \partial x_1 \partial x_2} f$$

$(3,3,0)! = 3!3!0! = 36$

Chain Rule

recall: 1-dim

if $f: A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$
$g: B \to \mathbb{R}$ (A,B open)

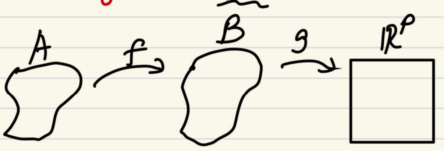
$$\Rightarrow \frac{d}{dx}(g \circ f)(x) = g'(f(x)) \cdot f'(x)$$

can view: 1×1 matrix, 1×1 matrix

(if $g'(f(x))$ & $f'(x)$ exist)

now: multi-dim

Suppose $f: A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$ (A,B open)
$g: B \to \mathbb{R}^p$



$\Rightarrow$ If f is diffble at x_0 & g is diffble at $f(x_0)$
then $g \circ f$ is diffble at x_0 且

$$\underbrace{D(g\circ f)(x_0)}_{p\times n} = \underbrace{Dg(f(x_0))}_{p \times m} \cdot \underbrace{Df(x_0)}_{m \times n}$$

Pf For h small, define:

$$R_f(h) = \frac{f(x_0+h) - f(x_0) - Df(x_0)\cdot h}{|h|} \quad \text{(the remainder)}$$

Since f diffble, $|R_f(h)| \to 0$ as $|h| \to 0$

Letting $y_0 = f(x_0)$, for k small enough we have

$$R_g(k) = \frac{g(y_0+k) - g(y_0) - Dg(y_0)k}{|k|}$$

Again, $|R_g(k)| \to 0$ as $|k| \to 0$ (by g diffble at y_0)

Now set $A = Dg(y_0) \cdot Df(x_0)$

and define $R_{g\circ f}(h) = \dfrac{g\circ f(x_0+h) - g \circ f(x_0) - Ah}{|h|}$

WTS: $|R_{g\circ f}(h)| \to 0$ as $|h| \to 0$

Note:

$$R_{g\circ f}(h) = \frac{g(f(x_0) + Df(x_0)h + |h|R_f(h)) - g(f(x_0)) - Ah}{|h|}$$

Set $k = Df(x_0)h + |h|R_f(h)$ since $|R_f| \to 0$ as $|h| \to 0$, $|R_f| < 1$ for h small enough

note: for h small enough, $|k| \le \|Df(x_0)\| \cdot |h| + |h|$ $(= \text{const} \cdot |h|)$ $(*)$

$$\Rightarrow R_{g\circ f}(h) = \frac{g(y_0+k) - g(f(x_0)) - Ah}{|h|}$$

$$= \frac{g(y_0) + Dg(y_0)k + |k|R_g(k) - g(f(x_0)) - Ah}{|h|}$$

$$= \frac{Dg(y_0)\left(\cancel{Df(x_0)h} + |h|R_f(h)\right) + |k|R_g(k) - \cancel{Ah}}{|h|}$$

$$= \frac{|h|Dg(y_0)R_f(h) + |k|R_g(h)}{|h|}$$

So $|R_{g\circ f}(h)| \le \|Dg(y_0)\| \cdot |R_f(h)| + \dfrac{|k|}{|h|}|R_g(k)|$

Using $(*)$ and $|R_g|, |R_f| \to 0$, we get $|R_{g\circ f}(h)| \to 0$ as $|h| \to 0$

Taylor's formula

Recall (Binomial Thm)

$$(x_1+x_2)^k = \sum_{a=0}^{k} \binom{k}{a} x_1^a x_2^{k-a}$$

$$= \sum_{a+b=k} \frac{k!}{a!\,b!} x_1^a x_2^b$$

Using multi-index notation:

$$= \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$

Lemma Multinomial Thm

For any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $k\in\mathbb{N}$

$$(x_1+\dots+x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$

Proof sketch

$\frac{k!}{\alpha!}$ = the # of ways to divide a set of size k into disjoint subsets of size $\alpha_1,\dots,\alpha_n$

There are $k!$ ways to order the set.

Say the 1^{st} α_1 go to the 1^{st} subset, the 2^{nd} α_2 to the 2^{nd} subset, etc.

$$\underbrace{\quad}_{\alpha_1}\underbrace{\quad}_{\alpha_2}\cdots$$

And there are $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$ ways to get the same result.

Def A subset of $\mathbb{R}^n$ is convex if it contains the whole line segment of any two points in it.

i.e. $\forall x,y\in G \Rightarrow tx+(1-t)y\in G$ for all $t\in[0,1]$

Taylor's Thm

Let $G\subseteq\mathbb{R}^n$ be open and convex

Let $f: G\to\mathbb{R}$ be C^{k+1} and $a\in G$

$\Rightarrow \forall x\in G,$

$$f(x) = \sum_{|\alpha|\le k} \left(\frac{1}{\alpha!}\partial^\alpha f(a)\right)(x-a)^\alpha + R_{a,k}(x)$$

where $R_{a,k} = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(c)}{\alpha!}(x-a)^\alpha$ for some c on the line from a to x.