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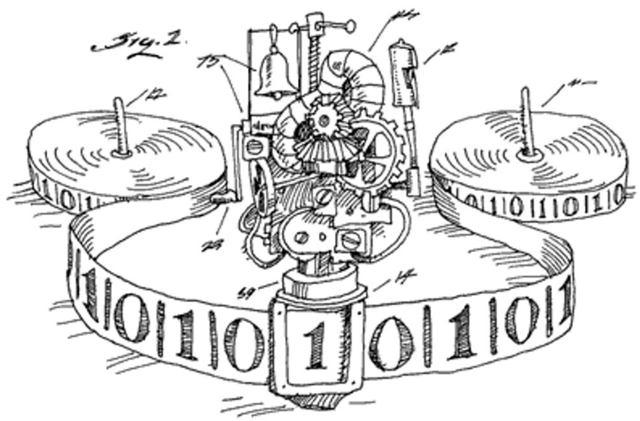
EECS 376: Foundations of Computer Science

Lecture 03 - Divide and Conquer 2



Another example of divide and conquer:

Closest Pair of Points

Closest Pair Data Structures: Applications

The following algorithms and applications can be implemented efficiently using our new closest pair data structures, or involve closest pair computation as important subroutines.

- Dynamic minimum spanning trees
- Two-optimization heuristics in combinatorial optimization
- Straight skeletons and roof design
- Ray-intersection diagram
- Other collision detection applications
- Hierarchical clustering
- Traveling salesman heuristics
- Greedy matching
- Constructive induction
- Gröbner bases

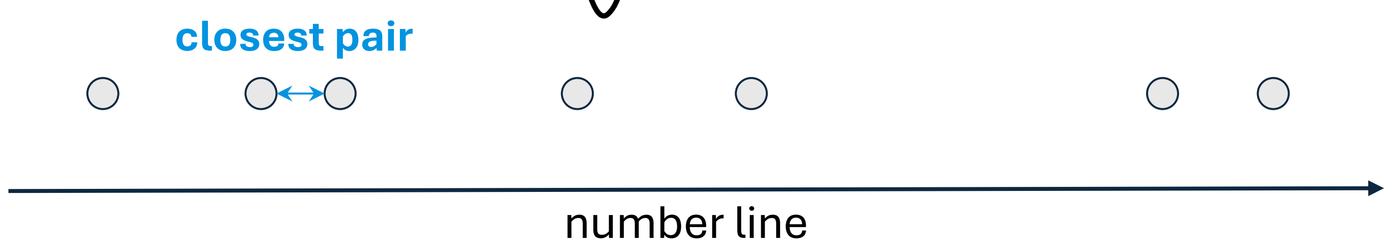
David Eppstein, Information & Computer Science, UC Irvine, .

Warm-up: Closest Pair of Points in 1-D

* **Problem:** Given n real numbers $x_1, x_2, \ldots, x_n$, find $i \neq j$ with the smallest $|x_i - x_j|$.

* **Solution:**
  * Sort the points. $(n \log n)$
  * Go over the list and compute the distances between the adjacent points. (n)
  * Return the pair of adjacent points with the min distance.
* **Runtime:**

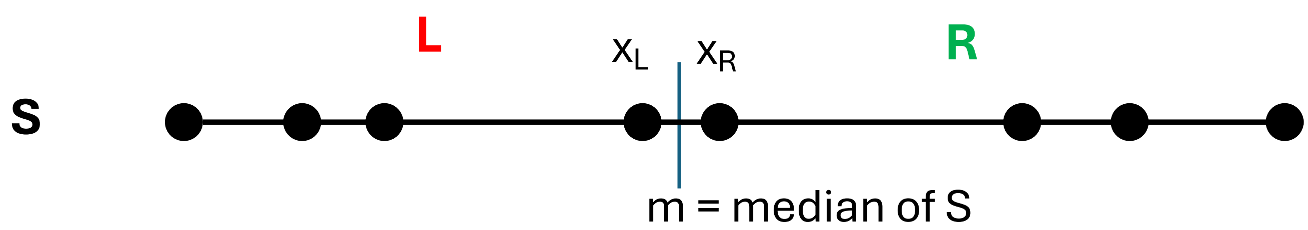
$= n \log n$



Warm-up: Closest Pair in 1D

- You’re given a set of $n \geq 2$ *distinct* points on a line.
- **Goal:** Find minimum distance between any pair of points
- **Q:** Can you think of a fast algorithm?
  - (1) Sort the points in increasing order as $(p_1, p_2, \dots, p_n)$ — $O(n \log n)$ time
  - (2) Scan the list of sorted points; return $\min\limits_{1 \leq i < n} \{p_{i+1} - p_i\}$. — $O(n)$ time

Building intuition: Divide and Conquer 1D



Not any better than before, but again we're here to build intuition.

Algorithm: Closest-Pair(set of n points S)

- Find the median m (There is an O(n) algorithm…)
- Split the points according to m to L and R
- δ_L = dist(Closest-Pair(L)) ①
- δ_R = dist(Closest-Pair(R)) ②
- Find the **maximal** element $x_L \in L$ $|x_L - x_R|$ ③
- Find the **minimal** element $x_R \in R$ ↓ compare ①②③
- **Return**: the pair that lies within distance $\min(\delta_L, \delta_R, |x_L - x_R|)$

Runtime Analysis:

- $T(n) = 2T(n/2) + O(n)$

Formally: Consider the recurrence relation $T(n) = kT(n/b) + O(n^d)$, when $k, b > 1$. Then:

$$T(n) = \begin{cases} O(n^d) & \text{if } (k/b^d) < 1 \\ O(n^d \log n) & \text{if } (k/b^d) = 1 \\ O(n^{\log_b k}) & \text{if } (k/b^d) > 1 \end{cases}$$

$$\frac{k}{b^d} = \frac{2}{2^1} = 1 \Rightarrow O(n \log n)$$

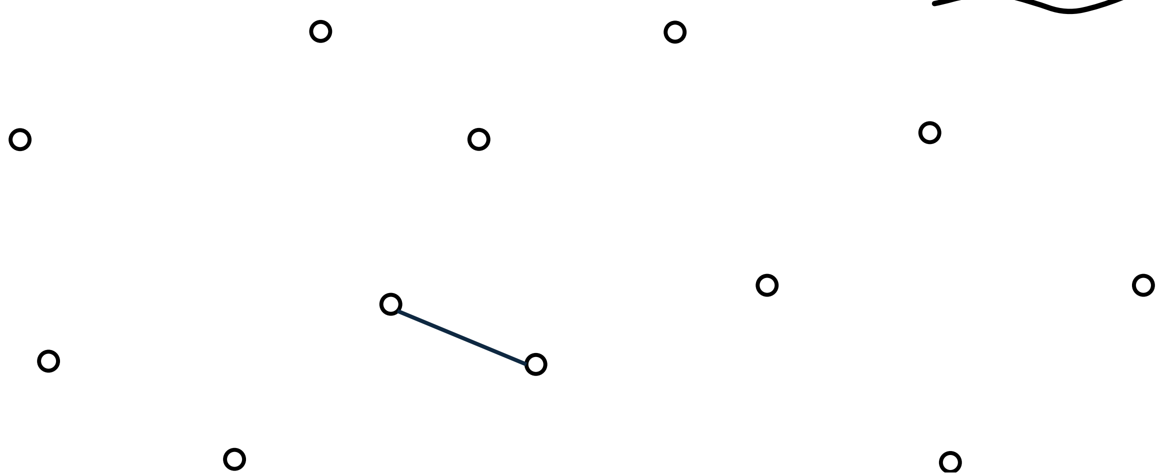
$O(n)$ Median Selection Algorithms

- An algorithm called "quickselect", was developed by Tony Hoare who also invented the similarly-named quicksort in 1959-1960. A recursive algorithm can find any element (not just the median). This algorithm has an *average* performance of $O(n)$.
- Another algorithm called PICK was originally developed in 1973 by the mouthful of Blum, Floyd, Pratt, Rivest, and Tarjan. It has $O(n)$ performance in the worst case.

Manuel Blum, Robert W. Floyd, Vaughan Pratt, Ronald L. Rivest, Robert E. Tarjan. (1973). "Time bounds for selection" (PDF). Journal of Computer and System Sciences. **7** (4): 448–461

Closest Pair in 2D

- Given a set of $n \geq 2$ points in the *plane*.
- **Goal:** Find ***minimum distance*** between any pair of points.
- A point $p = (x_p, y_p)$ is represented by a pair of numbers.
- (Pythagorean Theorem) $dist(p, q) = \sqrt{\left(x_p - x_q\right)^2 + \left(y_p - y_q\right)^2}$.
- **How fast is the trivial algorithm for this problem?** $O(n^2)$



Finding the 2-D Closest Pair of Points With Divide and Conquer

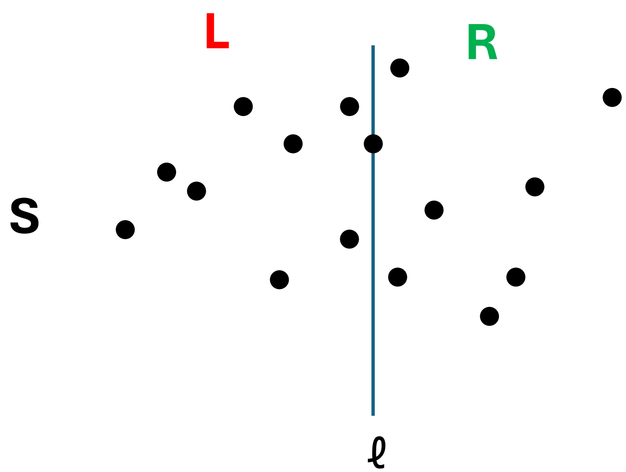
Input: A list of n points in $\mathbb{R}^2$: $[(x_1,y_1), (x_2,y_2), \ldots , (x_n,y_n)]$

Output: The closest pair of points

Goal: O(n log n) algorithm

$\boxed{T(n) = 2T(n/2)+O(n)}$

Working backwards: We want a "mergesort" recurrence relation!

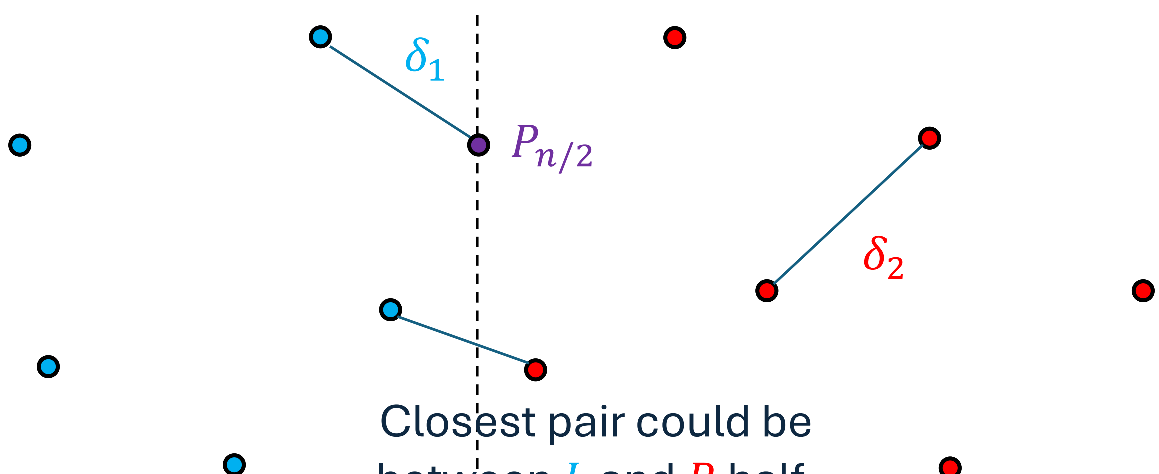


Formally: Consider the recurrence relation $T(n) = kT(n/b) + O(n^d)$, when $k, b > 1$. Then:

$$T(n) = \begin{cases} O(n^d) & \text{if } \left(k/b^d\right) < 1 \\ O(n^d \log n) & \text{if } \left(k/b^d\right) = 1 \\ O(n^{\log_b k}) & \text{if } \left(k/b^d\right) > 1 \end{cases}$$

Divide and Conquer?

ClosestPair$(P_1, \ldots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // $P_{n/2}$: median in x-coord
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left n/2 pts
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right n/2 pts
...What comes next? *// ...how?*
...Just return $min\{\delta_1, \delta_2\}$*?*

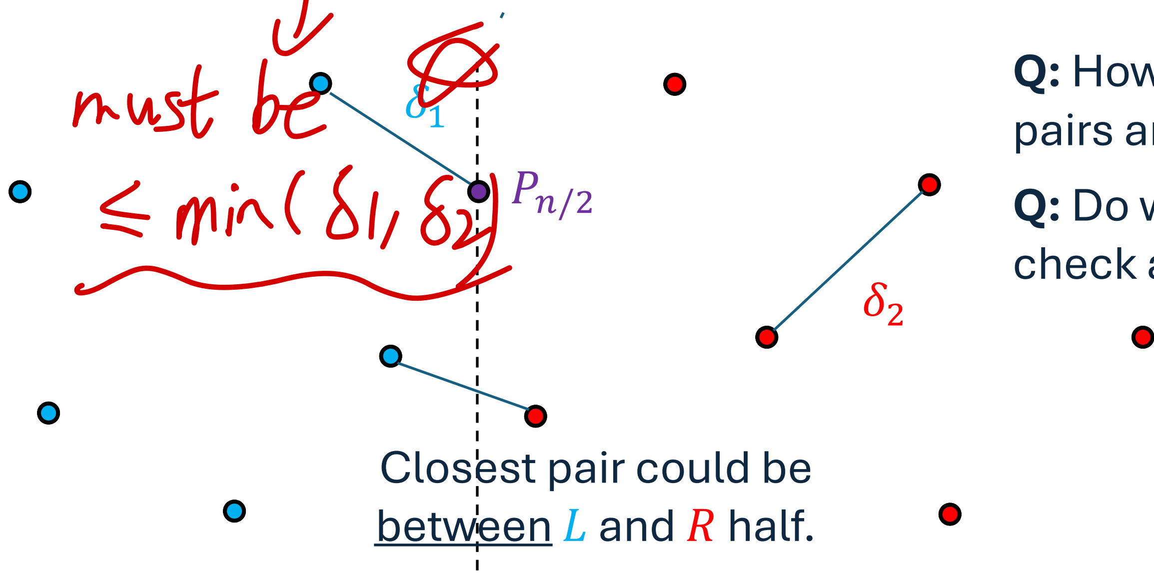


Closest pair could be
<u>between</u> L and R half.

Divide and Conquer?

ClosestPair$(P_1, \ldots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // $P_{n/2}$: median in x-coord
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left n/2 pts
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right n/2 pts
$\boldsymbol{\delta_3} \leftarrow$ **min distance between** L and R half // *...how??*
return $\min\{\delta_1, \delta_2, \delta_3\}$

must be $\leq \min(\delta_1, \delta_2)$



Closest pair could be <u>between</u> L and R half.

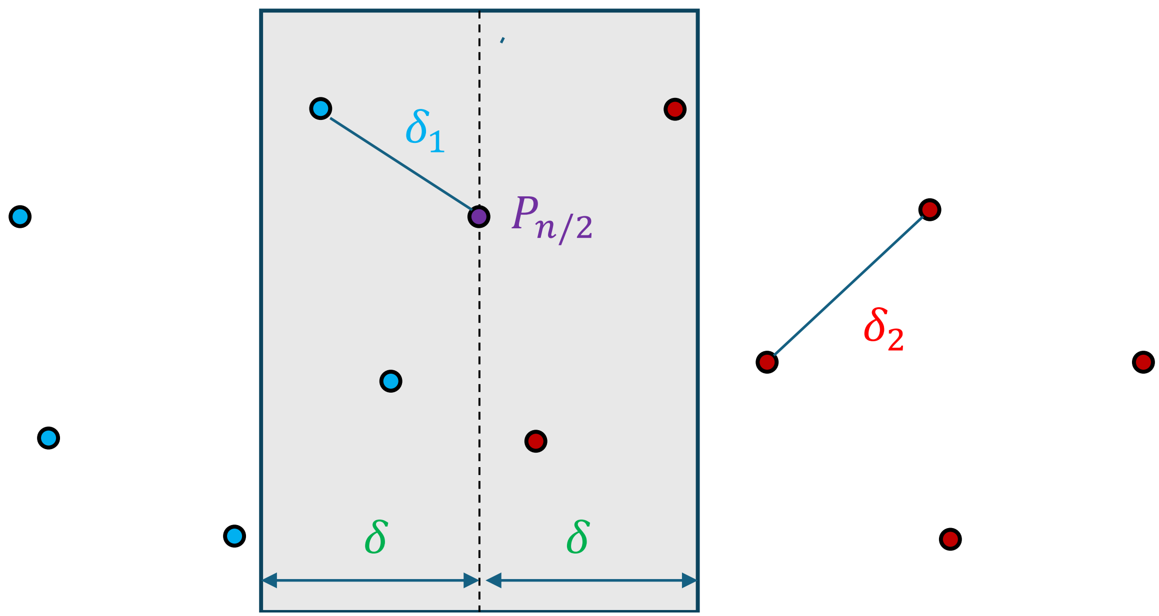
Q: How many blue/red pairs are there?

Q: Do we need to check all of them?

Key Insight: The δ-strip

ClosestPair$(P_1, \ldots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right
need to know min dist between L and R // ...look at δ-strip

- Let $\delta = \min\{\delta_1, \delta_2\}$.
- **Observation:**
  - If a closest pair is between blue and red,
  - then their x-coord are within δ of $P_{n/2}$'s x-coord (the "δ-strip").

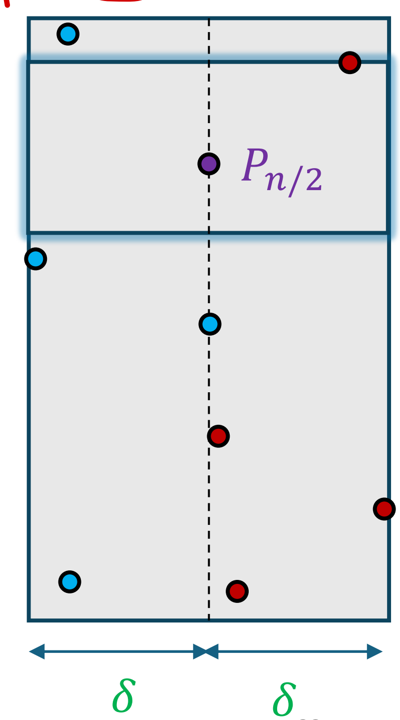


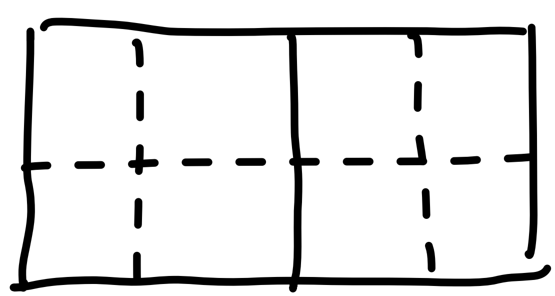
Closest blue/red pair is $\delta \times 2\delta$ rectangle!

从下往上！只需计算每个点以它 y-coord 为底的 $\delta \times 2\delta$

ClosestPair$(P_1, \ldots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right
need to know min dist between L and R // ...look at δ-strip

- Let $\delta = \min\{\delta_1, \delta_2\}$. square之间的点的距离即可道
- Consider any pair of points (p, q) in δ-strip.
- **Observe:** If diff in y-coord of p and $q > \delta$, then (p, q) is not closest
- **So:** if (p, q) is closest, then they are in $\delta \times 2\delta$ rectangle!
- **Next:** there are $O(1)$ points in $\delta \times 2\delta$ rectangle. Why?





Properties of the δ-strip

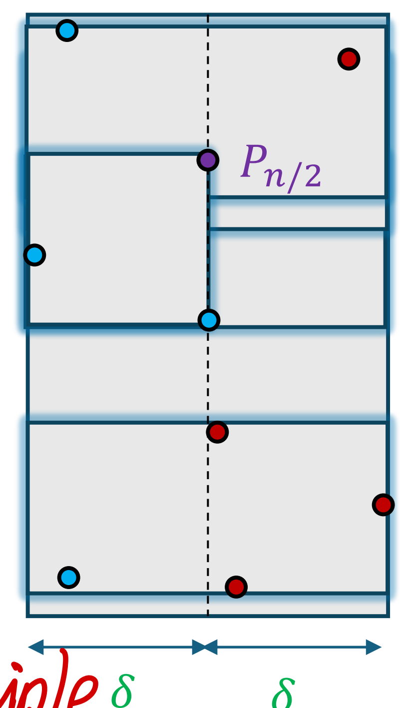
ClosestPair$(P_1, \dots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right
need to know min dist between L and R // ...look at δ-strip

- Let $\delta = \min\{\delta_1, \delta_2\}$.

- **Q:** How many blue pts can there be in a $\delta \times \delta$ square?
- **A:** 4 = O(1)

- **Q:** How many pts can there be in a $\delta \times 2\delta$ rectangle?
- **A**: 8 = O(1)

≤ 8, by pigeon hole principle



How to find a closest red/blue pair:
Slide a $\delta \times 2\delta$ rectangle!

Pause and Think

ClosestPair$(P_1, \dots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if n = 2 then return dist(P_1, P_2) // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right
Let $(P_1', P_2', \dots, P_m')$ be points in the δ-strip, // m≤n sorted *by y-coordinate* $O(n \log n)$
$\boldsymbol{\delta_3} \leftarrow ???$ **// O(n) distances computed**
return min$\{\delta_1, \delta_2, \delta_3\}$

$(\leq 8n) \Rightarrow O(n)$

Analysis of ClosestPair

ClosestPair$(P_1, \ldots, P_n)$: // $n \geq 2$ pts in the plane, x-sorted asc.
if n = 2 then return dist(P_1, P_2) // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median
$\delta_1 \leftarrow$ **ClosestPair**(L) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R) // min dist on right $O(n \log n)$
Let $(P'_1, P'_2, \ldots, P'_m)$ be points in the δ-strip, // m≤n sorted *by y-coordinate*
$\boldsymbol{\delta_3 \leftarrow \min_{1 \leq i < m, 1 \leq c \leq 7} \{dist(P'_i, P'_{i+c})\}}$ // ≤7m distances computed $+ O(n)$
$=$
$O(n \log n)$
return min$\{\delta_1, \delta_2, \delta_3\}$

- **Runtime:** let $T(n)$ be the runtime of **ClosestPair** on n points.
- $T(n) = 2T(n/2) + O(n \log n) = \boldsymbol{O(n \log^2 n)}$
- How can we improve this to $T(n) = 2T(n/2) + O(n)$?

Sort all points from the beginning

Sorted by x	Sorted by y

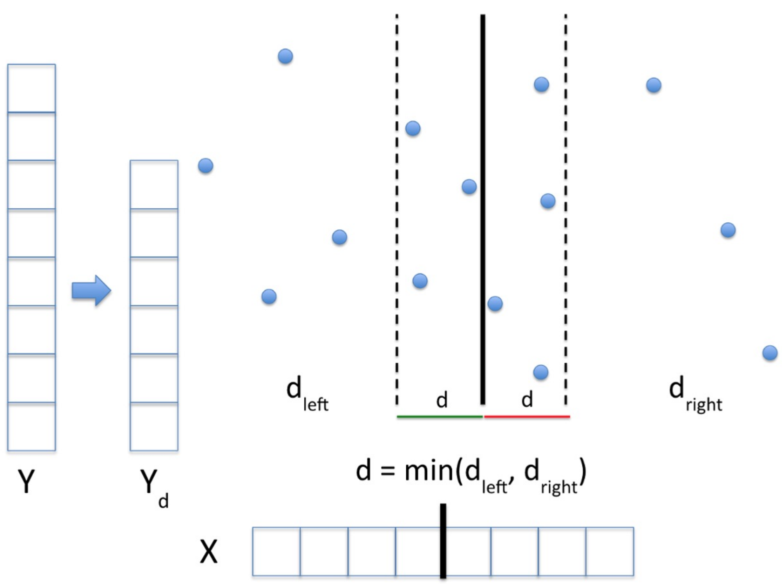
ClosestPair$(P_1, \dots, P_n, P'_1, \dots, P'_n)$: // $n \geq 3$ pts in the plane
if $n \leq 3$ then return min dist among P_1, P_2, P_3 // base case
$(L, R) \leftarrow$ partition points by $P_{n/2}$ // split by median x-coordinate
$\delta_1 \leftarrow$ **ClosestPair**(L (sorted by x), L (sorted by y)) // min dist on left
$\delta_2 \leftarrow$ **ClosestPair**(R (sorted by x), R (sorted by y)) // min dist on right
$\delta_3 \leftarrow$ min distance in δ-strip // details in notes
return $\min\{\delta_1, \delta_2, \delta_3\}$

- **Runtime:**
  - Sort points in x and y: $O(n \log n)$.
  - Recursive algo: $T(n) = 2T(n/2) + O(n) = O(n \log n)$.
- **Total time**: $O(n \log n)$

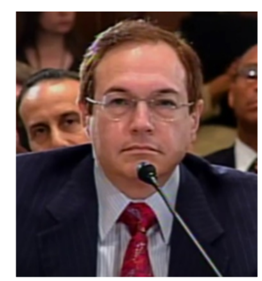
Total running time

- Sort once by x-coordinate: O(n log n)
- Sort once by y-coordinate: O(n log n)
- Recursive algorithm: T(n) = 2T(n/2) + O(n)

T(2) = O(1)



Discovered in 1976 by
Jon Louis Bentley and Michael Ian Shamos



Proof of Master Theorem

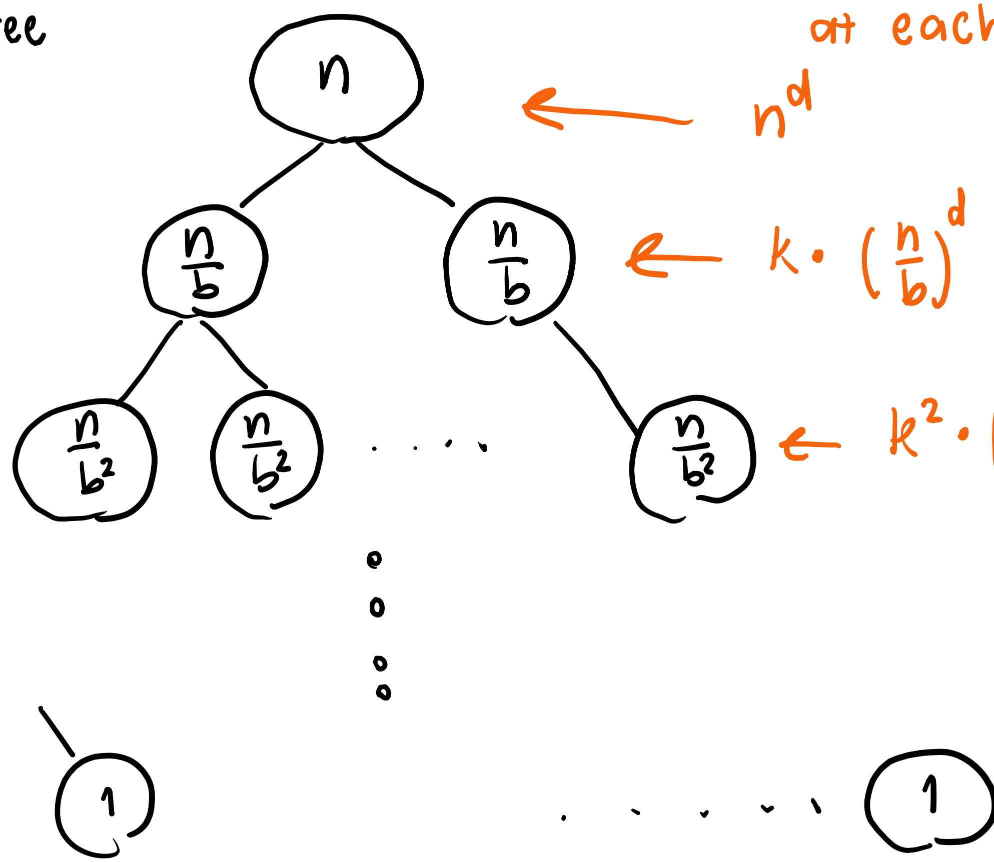
$$T(n) = kT(n/b) + O(n^d)$$

Recursion Tree

Total time outside recursion at each level

height

$h = \lg_b n$



n^d

$k \cdot \left(\frac{n}{b}\right)^d$

$k^2 \cdot \left(\frac{n}{b^2}\right)^d$

$k^h \left(\underbrace{\frac{n}{b^h}}_{1}\right)^d$

So total time over all levels $= n^d \left(1 + \frac{k}{b^d} + \left(\frac{k}{b^d}\right)^2 + \ldots \left(\frac{k}{b^d}\right)^h \right)$

Proof of Master Theorem

$$T(n) = kT(n/b) + O\left(n^d\right)$$

- $T(n) = n^d\left(1 + \frac{k}{b^d} + \left(\frac{k}{b^d}\right)^2 + \cdots + \left(\frac{k}{b^d}\right)^h\right)$
  - where height $h = \log_b n$
- By properties of geometric series, $T(n) = \begin{cases} O\left(n^d\right) & \text{if } k < b^d \\ O\left(n^d \log n\right) & \text{if } k = b^d \\ O\left(n^{\log_b k}\right) & \text{if } k > b^d \end{cases}$

Conclude: Divide-and-Conquer Algorithms

Main Idea:

1. **Divide** the problem into smaller sub-problems (creative step)
2. **Conquer** (solve) each sub-problem *recursively* (easy step)
3. **Combine** the solutions (creative step)

Examples: Merge-sort, Closest pair in 2d, Karatsuba's algorithm

Note:

- Not every recurrence is captured by the Master Theorem;
- good enough for us in this class (can derive more general versions)

Next: Dynamic Programming

(very efficient recursive algorithms when the number subproblems is small)

Bonus: Ultimate way to solve recursions

- Guess and check.
- How formally?
  1. Guess ☺ (maybe by drawing the recursion tree)
  2. Check via induction

Examples:

- $T(n) = n - 1 + \frac{2}{n-1} \sum_{r=n/2}^{n} T(r-1)$
- $T(n) = T\left(n^{1/2}\right) + 1$
- $T(n) = T(0.2n) + T(0.7n) + n$

Example How to Use Induction

- **Analysis:** For some c, $T(n) \leq cn$ for all $n \geq 1$.
- **Proof:** By induction on n.

Inductive Step:

Assume $T(n') \leq cn'$ for all $n' < n$.

$$T(n) = n - 1 + \frac{2}{n-1} \sum_{r=n/2}^{n} T(r-1)$$

$$\leq n - 1 + \frac{2}{n-1} \sum_{r=n/2}^{n} c(r-1)$$

$$\leq n - 1 + \frac{3c}{4} n + 2$$

$\leq cn$ (by choice of $c \geq 8$)