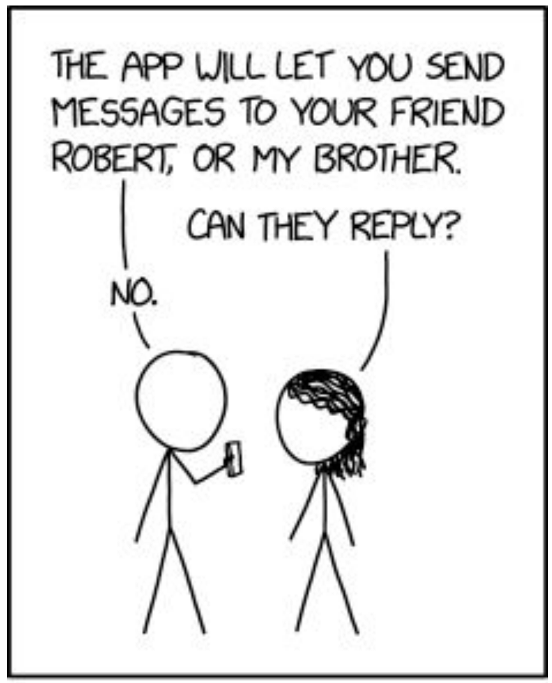
RSA:
Public-Key Encryption and Signatures



MY NEW SECURE TEXTING APP ONLY ALLOWS PEOPLE NAMED ALICE TO SEND MESSAGES TO PEOPLE NAMED BOB.

What is Public-Key Encryption?

Last time we showed that Alice and Bob, who have never met, can establish a shared secret over a public channel.
(That was Diffie-Hellman *key exchange*, not public-key *encryption*)

Public-key encryption: Alice can publicly publish a key that allows *anyone* to send her secret messages.

Alice doesn't need to communicate individually to Bob for him to be able to send her secret messages.

A very powerful idea!

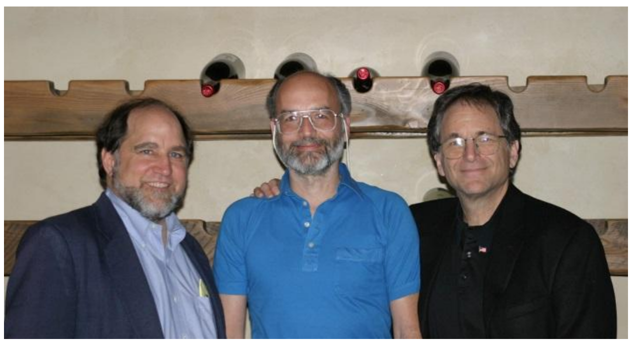
Public-Key Encryption Analogy

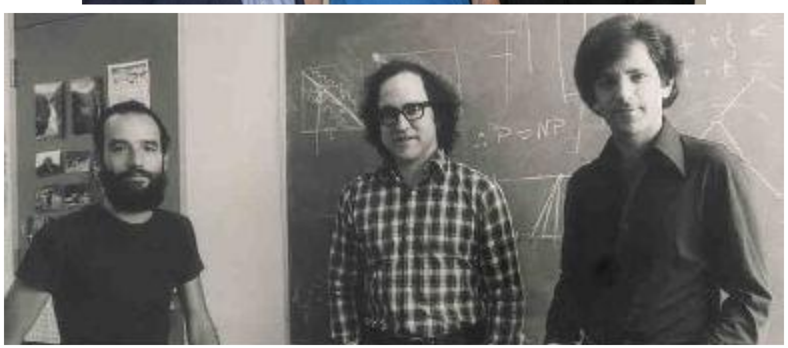
- Central Tower Emperor leaves a box of open locks out in the street.
- North Tower Emperor uses one of them to lock the gift and send it.

Today: RSA

First* public-key encryption

First digital signature scheme





Adi Shamir Ron Rivest Len Adleman (1977)

Also discovered by Clifford Cocks at British Intelligence in 1973; classified until 1997.

*Diffie-Hellman can also be made into PKE but people didn’t realize it at the time (see HW)

RSA

Here’s an idea

That doesn’t work, try again

Here’s another idea

That still doesn’t work, try again

⋮

Apparently this happened 42 times

Another "Hard Problem": Factoring

Factoring Assumption: Given a number n = pq where p and q are randomly chosen secret primes, there is no *efficient* algorithm for finding p,q.

The best known attack for RSA is to solve **factoring.**

(Recall that the best known attack for Diffie-Hellman is to solve **discrete log.**)

Factoring (like discrete log):

- is conjectured to be NP-intermediate.
- has a polynomial-time *quantum* algorithm [Shor, 1994]

RSA Factoring Challenge

In 2005, J. Franke et al. **won $20,000 for showing:**

n=310741824049004372135075003588856793003734602284272754572016148823206
440580815045563468296717232867824379162728380334154710731085019195485
29007337724822783525742386454014691736602477652346609

is the product of

p=1634733645809253848443133883865090859841783670033092312181110852389
333100104508151212118167511579

and

q=1900871281664822113126851573935413975471896789968515493666638539088
027103802104498957191261465571

RSA Factoring Challenge

RSA $100,000 challenge (defunct): factor n into two large primes:

n=135066410865995223349603216278805969938881475605667027524
4851438515265106048595338339402871505719094417982072821644715
5137368041970396419174304649658927425623934102086438320211O3
729587257623585096431105640735015081875106765946292055636855
294752135008528794163773285339061097505443349998111500569772
36890927563

MIT time capsule

1999

2019

$$2^{2^{79685186856218}}$$

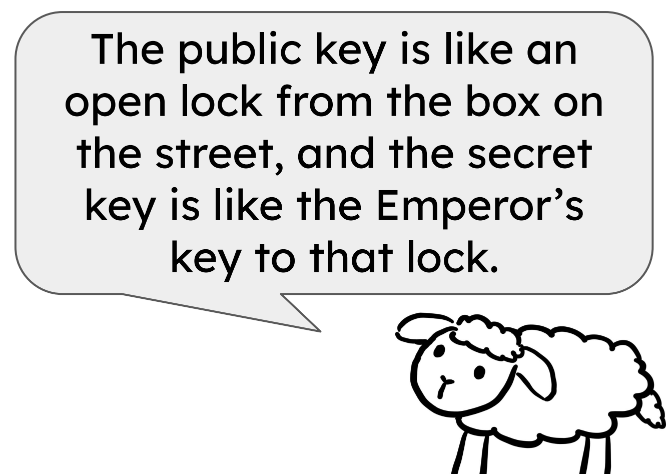
(mod product of two 1024-bit primes)

Goal for Public-Key Cryptography

Create a poly-time encryption algorithm **E** (using **public key**).
Create a poly-time decryption algorithm **D** (using **secret key**).

Requirement: For any message **m**, **D**(**E**(**m**)) = **m**.
i.e. **D** is the (left) inverse of **E**.

Goal: Find a function **E** that is easy to evaluate, hard to invert, but easy to invert with a “trapdoor” (secret key).



Preview of the Structure of RSA

One-time pad

key = **k**

E is "add **k** (mod 26)"

D is "subtract **k** (mod 26)"

E is easy to invert
(i.e. **D** is easy to find given **E**)

RSA

public key = **(n, e)**, secret key = **d**

E is "take e^{th} power (mod **n**)"

D is "take d^{th} power (mod **n**)"

We'll carefully choose **e**, **n**, **d** so that **E**, **D** are inverses, i.e. $m^{ed} \equiv m \pmod{n}$

Moreover, **E** will be hard to invert without knowing **d** (i.e. **D** will be hard to find)

To find such **e**, **n**, **d**, **Fermat's Little Theorem** will be useful…

Fermat's Little Theorem (1640)

Fermat's Little Theorem (FLT):

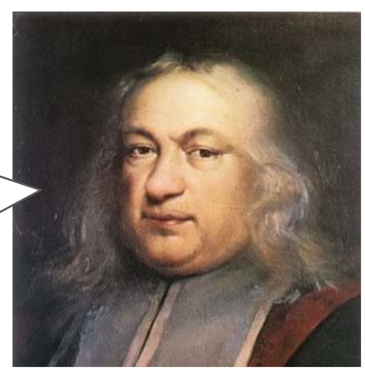
If p is prime, then for any $a, k \in \mathbb{Z}$,

$$a^{\underbrace{1+k(p-1)}} \equiv a \pmod{p}.$$

any number ≡ 1 (mod p-1)

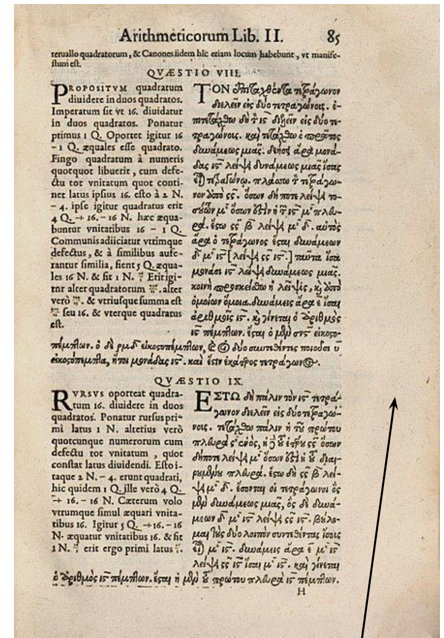
E.g. $3^{13} \equiv 3 \pmod{7}$

I never would have predicted this would be so useful one day!



*proof on last slide and in course notes

(Not to be confused with Fermat's Last Theorem)


Arithmeticorum Lib. II. 85

QVÆSTIO VIII.

QVÆSTIO IX.

"It is impossible… for any number which is a power greater than the second to be written as the sum of two like powers. I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain."

Public-Key Encryption: Attempt #1

How?

(Supposedly) secret information is **bold**

I pick:

1. large random prime p.
2. e, $\boldsymbol{d}$ so that $e \cdot \boldsymbol{d} \equiv 1 \pmod{p-1}$.

(p, e)

$c = \boldsymbol{m}^e \pmod{p}$

I compute $c = \boldsymbol{m}^e \pmod{p}$

A Bob wants to send Alice a message $\boldsymbol{m} < p$

I decrypt by taking $c^{\boldsymbol{d}} \pmod{p}$, which I claim is $\boldsymbol{m}$.

Why?

d is supposed to be secret but I can compute it! hehehe

The Issue with Attempt #1

Alice calculated d from (p, e) by solving $e \cdot d \equiv 1 \pmod{p-1}$ where the modulus $p-1$ was **public!**

We would like this modulus to be private to Alice.

Then Alice could still compute d from e and the **private modulus**, but Eve couldn't!

An extension of Fermat's Little Theorem will help…

Euler's Theorem (1763)

(A special case of) Euler's Theorem:

If $n = p \cdot q$ is the product of two <u>*distinct*</u> primes, then for any $a, k \in \mathbb{Z}$:

$$a^{\underbrace{1+k(p-1)(q-1)}_{\text{any number} \equiv 1 \ (\text{mod } (p-1)(q-1))}} \equiv a \pmod{n}.$$

E.g. setting n = 2 · 5 = 10, a = 4, k = 3, we have $4^{13} \equiv 4 \pmod{10}$

I, too, never would have predicted this would be so useful one day!

*proof in course notes

RSA: Public-Key Encryption

Secret information is **bold**

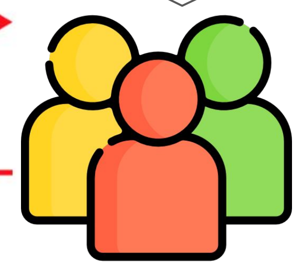
I pick:

1. large random primes $\boldsymbol{p}$, $\boldsymbol{q}$.
2. Set $n = \boldsymbol{p} \cdot \boldsymbol{q}$.
3. e coprime to $(\boldsymbol{p}-1)(\boldsymbol{q}-1)$, and pick $\boldsymbol{d}$ so that $e \cdot \boldsymbol{d} \equiv 1 \pmod{(\boldsymbol{p}-1)(\boldsymbol{q}-1)}$.

I compute $c = \boldsymbol{m}^e \pmod{n}$

(n, e)

$c = \boldsymbol{m}^e \pmod{n}$



A Bob wants to send Alice a message $\boldsymbol{m} < n$

I decrypt by taking $c^{\boldsymbol{d}} \pmod{n}$, which I claim is $\boldsymbol{m}$.

Why?

RSA Example

* Set $n = p \cdot q = 3 \cdot 17 = 51$ (the primes are *secret*)
* Generate matching public/private key pair $(e, d) = (3,11)$
  * $e \cdot d \equiv 1 \pmod{32}$
  * E.g., pick e coprime to 32 and compute inverse d using EEA
* Alice sends $(n, e) = (51,3)$ to Bob
* To send $m = 4$, Bob sends the ciphertext:

$$m^e \equiv 4^3 \equiv 13 \pmod{51}$$

* After receiving $c = 13$, Alice computes:

$$c^d \equiv 13^{11} \equiv 4 \pmod{51}$$

RSA Security

Why can't Eve figure out m?

Well… because we assume that.

RSA assumption: For randomly chosen m, there is no *efficient* algorithm that given $\underbrace{n, e, m^e \pmod{n}}_{\text{what Eve knows}}$, finds m.

Best known attack: Factor n to find $p, q,$ then compute d by solving $e \cdot d \equiv 1 \pmod{(p-1)(q-1)}$ using Extended Euclid.

One-Time Pad Cons from last time

- **Con 1:** It's insecure to use the same key twice.

We didn't fix this one

- **Con 2:** Alice and Bob must privately agree on the secret key beforehand.

Diffie-Hellman and RSA fix this one

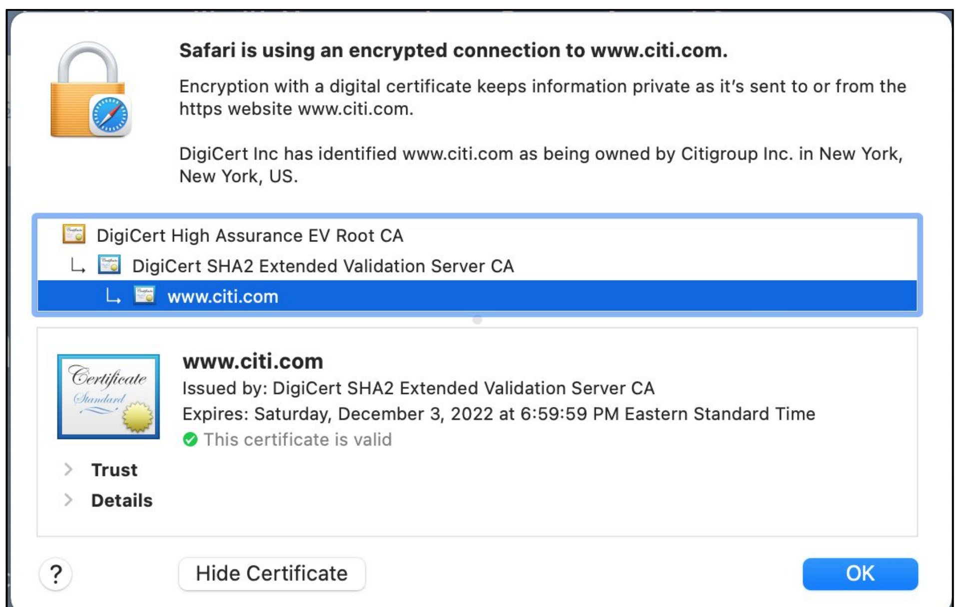
The RSA protocol you just saw suffers from **Con 1** because the encryption algorithm is **deterministic**.
E.g. Eve can tell if the same message is sent twice.

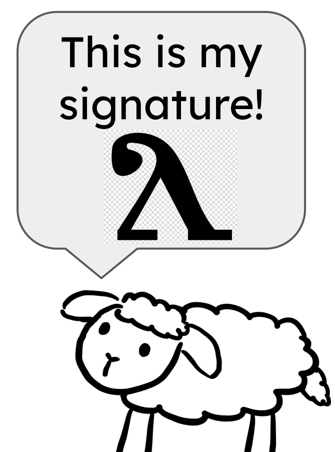
This can be fixed by inserting some **randomization** into the encryption algorithm. (We won't show.)

RSA can also be used for "Signatures"

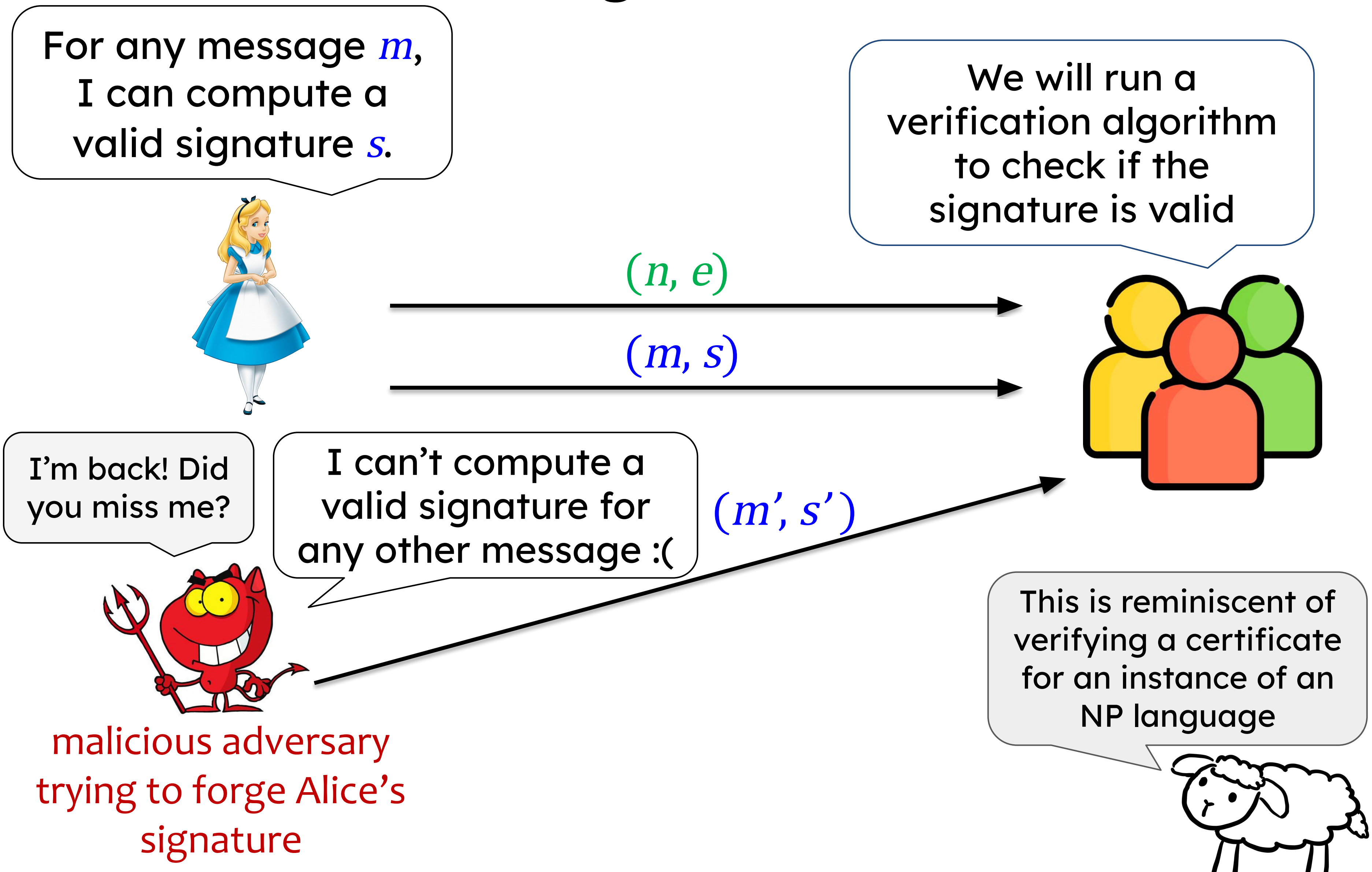
Goal of a signature:

1. Alice publishes a public key (n, e).
2. Alice OR a malicious "forger" sends a public message m with a "signature" s.
3. Anyone can check whether the same entity did both 1. and 2.





RSA Signature Goal



RSA Signatures: Run RSA "Backwards"

I pick:

1. large random primes $\boldsymbol{p}$, $\boldsymbol{q}$.
2. Set $n = \boldsymbol{p} \cdot \boldsymbol{q}$.
3. e coprime to $(\boldsymbol{p}-1)(\boldsymbol{q}-1)$, and pick $\boldsymbol{d}$ so that $e \cdot \boldsymbol{d} \equiv 1 \pmod{(\boldsymbol{p}-1)(\boldsymbol{q}-1)}$.

I want to sent message m.

I compute signature $s = m^{\boldsymbol{d}} \pmod{n}$

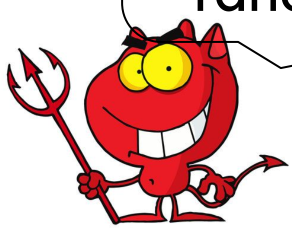
Secret information is **bold**

Verification algorithm: check that $s^e \equiv m \pmod{n}$

(n, e)

$(m, s = m^{\boldsymbol{d}} \pmod{n})$

I can't seem to compute a valid signature for a randomly chosen m'...



(m', s')

RSA assumption: For randomly chosen m, there is no *efficient* algorithm that given n, e, $\underbrace{m^e}_{m'} \pmod{n}$, finds $\underbrace{m}_{m'^{\boldsymbol{d}} \pmod{n}}$.

What if the forger gets to choose m’ based on s’?

Then the forger can successfully forge Alice’s signature by first picking s' and then letting $m' = s'^{e}$ (mod n)!

This can be fixed by Alice applying some wild function H to m and sending $H(m)$, $s = H(m)^{d}$ $(mod\ n)$ (We won’t show.)

Proof of Fermat's Little Theorem

* **Lemma:** If p is prime and $a \not\equiv 0 \pmod p$, then the set of numbers $\{a, 2a, 3a, \dots, (p-1)a\} \pmod p$ is the same set as $\{1, \dots, p-1\}$.
  1) For every $i \in \{1, \dots, p-1\}$, ia is not a multiple of p since p does not divide either i or a (***Euclid's lemma***). Thus, each $ia \pmod p \in \{1, \dots, p-1\}$.
  2) For every $i, j \in \{1, \dots, p-1\}, i \neq j$, $(j-i)a$ is not a multiple of p. Thus, there are no collisions: $ia \not\equiv ja \pmod p$.
* **Then:** Since the sets are the same, their products are too:
  * $a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdots (p-1) \pmod p$
  * Hence $a^{p-1} \equiv 1 \pmod p$. ($\{1, \dots, p-1\}$ all have inverses mod p, so multiply both sides by $1^{-1} \cdot 2^{-1} \cdots (p-1)^{-1} \pmod p$)