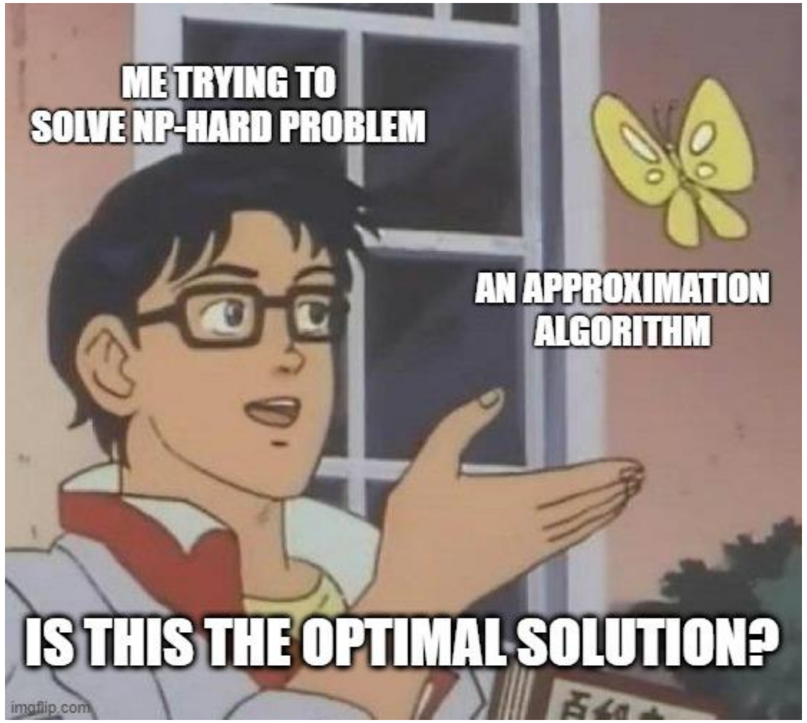
D9: Search to Decision and Approximation Algorithms



Sec 101: MW 3:00-4:00pm DOW 1018
IA: Eric Khiu

Announcement

- I will be away next week for a research conference in Mexico!
- June 12 (this Wed): I will cover everything on Randomized Algorithms
- June 17 (next Mon): Donayam will cover my discussion session
  - He will cover modular arithmetic review, very important for the cryptography unit!
- June 19 (next Wed): Juneteenth- No class, no OH

Review: Decidability and Complexity

Complete the following statements with always/ sometimes/ never accepts:

- To prove that language L is decidable, we can give a decider such that
  - When given an instance x such that $x \in L$, this machine
- To prove that language L is **in P,** we can give an efficient decider such that
  - When given an instance x such that $x \in L$, this machine
- To prove that language L is **in NP,** we can give an efficient verifier such that
  - When given an instance x such that $x \in L$, this machine
  - When given an instance x and a certificate c, this machine
  - When given an instance x such that $x \in L$ and a certificate c, this machine

Review: Decidability and Complexity

Complete the following statements with always/ sometimes/ never accepts:

- To prove that language L is decidable, we can give a decider such that
  - When given an instance x such that $x \in L$, this machine always accept.
- To prove that language L is **in P**, we can give an efficient decider such that
  - When given an instance x such that $x \in L$, this machine always accepts.
- To prove that language L is **in NP**, we can give an efficient verifier such that
  - When given an instance x such that $x \in L$, this machine sometimes accepts.
  - When given an instance x and a certificate c, this machine sometimes accepts.
  - When given an instance x such that $x \in L$ and a certificate c, this machine sometimes accepts.
- Takeaway: The certificate is not "trustworthy" or assumed to be "valid". If it were, there would be no need for a verifier; it could just ignore the certificate and accept.

Certificates ≈ Wizards

“We can think of the verifier as skeptically checking the claim of an all-powerful but devious wizard. The wizard wants to convince the verifier that x is in L, even if it is not. If the verifier is appropriately skeptical, then when x is in L, the verifier should be convinced by a suitable proof/certificate from the wizard. But when x is not in L, the verifier should not be convinced no matter what the wizard says. The challenge is to design the logic of an appropriate “skeptical verifier” so that both of these conditions hold.

One more addition: In class when we specified "Certificate: xxx", it really should have said "Valid Certificate: xxx". And then the purpose of the verification algorithm is to check if the input certificate is a valid certificate.”

- Prof. Chris Peikert, Winter 2024

Agenda

- Search to Decision
- Approximation Algorithms

Search to Decision

Notes

Starter: Decision vs Search

- Consider the following language:

$$L = \{A: A \text{ is an array of } n \text{ integers that contains } m\}$$

where m is a magic integer.

- Suppose I have a decider D that decides L, what does the output of $D(A[1, \dots, n])$ tells me? (Note: m is hard-coded in D)
- What about $D(A[1, \dots, n-1])$?

Discuss: Suppose I know that m is in A (but I still don't know what m is), how can I use D to determine the *index* of m?

```
findIndex(A):
    for idx = 1, …, n do
        if D(A[i]) accepts then return idx
```

Search to Decision

- So far, we have been reducing a decision problem to another decision problem
- **Informal proposition:** A search version of any NP-complete problem has an efficient algorithm iff the decision version does
- **Corollary:** If we have access to an efficient decider for an NP-complete language, we can construct an efficient algorithm to solve corresponding search version of the language
- This efficient algorithm is known as a **search to decision reduction**

Exercise: Subset Sum

- Recall the decision version of the subset-sum problem

$$\textsc{SubsetSum} = \{(A, t) : \exists S \subseteq A \text{ whose elements sum to } t\}$$

- Suppose D is an efficient decider that decides $\textsc{SubsetSum}$, complete the following algorithm to solve the *search* version of $\textsc{SubsetSum}$, i.e., given input (A, t), it should output a subset $S \subseteq A$ whose elements sum to t, otherwise return an $\emptyset$ (Hint: Recall Knapsack DP: Take or don't take?)

function $\textsc{SubsetSumSearch}(A = \{a_1, \dots, a_n\}, t)$:

  if $D(A, t)$ rejects **then return** $\emptyset$

  $S \leftarrow \emptyset, t' \leftarrow t$

  for $i = n, \dots, 1$ **do**

    if $\boxed{}$ rejects **then**

      $S \leftarrow \boxed{}$

      $t \leftarrow \boxed{}$

  return S

Exercise: Subset Sum

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$$\textsc{SubsetSum} = \{(A, t) : \exists S \subseteq A \text{ whose elements sum to } t\}$$

- Suppose D is an efficient decider that decides $\textsc{SubsetSum}$, complete the following algorithm to solve the *search* version of $\textsc{SubsetSum}$, i.e., given input (A, t), it should output a subset $S \subseteq A$ whose elements sum to t, otherwise return an $\emptyset$ (Hint: Recall Knapsack DP: Take or don't take?)

function $\textsc{SubsetSumSearch}(A = \{a_1, \dots, a_n\}, t)$:

    if $D(A, t)$ rejects **then return** $\emptyset$

    $S \leftarrow \emptyset, t' \leftarrow t$

    for $i = n, \dots, 1$ **do**

        if $D(\{a_1, \dots, a_{i-1}\}, t')$ rejects **then** // No subset sum to t' if exclude a_i

            $S \leftarrow S \cup \{a_i\}$ // Must include a_i in S

            $t \leftarrow t' - a_i$ // Update target sum

    return S

Search + Optimization

- Sometimes, on top of searching for *a* solution, we are interested in the *best* solution (optimization problem) from a set of possible solutions
- **Best solution:** The one that has the highest/ lowest *value*
- **Maximization:** 0-1 Knapsack
  - Solution space: set of subsets of items whose weight does not exceed the capacity
  - Value: Total value of the subset
  - Goal: Find the subset with the highest value
- **Minimization:** Minimum spanning trees
  - Solution space: set of spanning trees
  - Value: Tree weight
  - Goal: Find the spanning tree with the lowest weight

But wait a minute…

- What if I don’t know the optimal value?
  - Still search to decision!
  - Use the same decider for decision problem to find it!

Example: Knapsack Max Value

- Recall the knapsack language (decision problem)

$$\textsc{Knapsack} = \left\{(W[1,\dots,n], V[1,\dots,n), C, K) : \exists S \subseteq \{1,\dots,n\} \text{ s.t.} \sum_{i\in S} W[i] < C \text{ and } \sum_{i\in S} V[i] \geq K\right\}$$

  Note: Assume all number $W[i], V[i], C,$ and K are **non-negative integers** for simplicity.

- Suppose there exists an efficient algorithm D that decides$\textsc{Knapsack}$
- Given a knapsack instance (W, V, C), describe an efficient algorithm that uses D to determine the maximum value K^* of a set of items whose total capacity is at most C.
  - Hint: What is the upper bound for K^*?
  - Sum of values of all items! $K^* \leq \sum_{i=1}^{n} V[i]$

Example: Knapsack Max Value

- Given a knapsack instance (W, V, C), describe an efficient algorithm that uses D to determine the maximum value K^* of a set of items whose total capacity is at most C. Know: $K^* \leq \sum_{i=1}^{n} V[i]$

function FINDMAXVAL(W, V, C):

$K^* \leftarrow -\infty$

$T \leftarrow \sum_{i=1}^{n} V[i]$

for $k = 0, \dots, T$ **do**

  if $D(W, V, C, k)$ accepts **then** $K^* \leftarrow k$

  else break

return K^*

Discuss: What is wrong with this?

- **Correctness analysis:** The optimal K^* must be in the range of 0 to $\sum_{i=1}^{n} V[i]$, and the algorithm will find the largest value in the range for which D accepts

It is not efficient!

- Recall that the input size of an integer is the number of bits used to represent it
- If we have an array of size n, we often say the input size is $O(n)$
- In fact, if b_{max} is the max number of bits used to represent the element with largest value in A, then the input size is $O(b_{max} \cdot n)$- but we often take b_{max} as a constant
- But it matters here!
  - Let b_w and b_v be the max number of bits of the elements with largest value in W and V
  - Input size of (W, V, C) = $O(nb_w) + O(nb_v) + O(\log C)$
  - Computing $T = \sum_{i=1}^{n} V[i]$ takes $O(n)$
  - Upper bound of $V[i]$: $2^{b_v} - 1$ ⇒ Value of T: $O\left(n \cdot (2^{b_v} - 1)\right) = O(n \cdot 2^{b_v})$
  - Linear search over $0, \ldots, T$: $O\left(n \cdot 2^{b_v}\right)$ ⇒ Total runtime = $O(n) + O\left(n \cdot 2^{b_v}\right)$ ⇒ Not efficient!

Is there a search that runs in $O(\log(\cdot))$?

- Binary search!
- **Attempt 2:** Perform a binary search over $k = 0, \ldots, T$, calling D with different values of k until we find the highest for which D accepts
- **Take home exercise:** Try to write the algorithm

- **Correctness analysis:** Same as before
- **Runtime analysis:**
  - Input size of $(W, V, C) = O(nb_w) + O(nb_v) + O(\log C)$
  - Value of T: $O\left(n \cdot (2^{b_v} - 1)\right) = O(n \cdot 2^{b_v})$
  - Total runtime = $O(n) + O(\log_2 T) = O(n) + O\left(\log_2(n \cdot 2^{b_v})\right)$

$$= O(n) + O(\log n) + O(b_v) \Rightarrow \text{Efficient!}$$

Exercise: Knapsack Best Subset

- Recall the knapsack language (decision problem)

$$\text{KNAPSACK} = \left\{(W[1,\dots,n], V[1,\dots,n), C, K) : \exists S \subseteq \{1,\dots,n\} \text{ s.t.} \sum_{i\in S} W[i] < C \text{ and } \sum_{i\in S} V[i] \geq K\right\}$$

  Note: Assume all number $W[i], V[i], C$, and K are **non-negative integers** for simplicity.

- Suppose there exists an efficient algorithm D that decides $KNAPSACK$
- Suppose K^* is the maximum value obtainable with capacity C
- Given a knapsack instance (W, V, C, K^*), describe an <u>efficient algorithm</u> that uses D to <u>determine the set of items</u> whose total weight is at most C, and whose total value is K^*

  Hint: Recall the intuition from DP: To take or not to take?

Exercise: Knapsack Best Subset

- Given a knapsack instance (W, V, C, K^*), describe an efficient algorithm that uses D to determine the set of items whose total weight is at most C, and whose total value is K^*

function $\textsc{KnapSearch}((W, V, C, K), K^*)$:

$S \leftarrow \emptyset$

for $i \in \{1, \dots, n\}$ **do**

if $D(W[(i+1), \dots, n], V[(i+1), \dots, n], C - W[i], K^* - V[i])$ **accepts then:**

$S \leftarrow S \cup \{i\}$ // Add an item iff it is possible to obtain K^* with the remaining items (D accepts)

$C \leftarrow C - W[i]$ // Update capacity available for the remaining items

$K \leftarrow K^* - V[i]$ // Update values needed from the remaining items

return S

- **Runtime analysis:** $O(n)$

Exercise: Knapsack Best Subset

- **Correctness Analysis:** Consider an optimal knapsack S^* with optimal value K^*,
  - Suppose S^* has the same decision (take/ discard) as the first i items as S
  - Assume S^* has the different decision as S on the $(i+1)^{th}$ item (the other case is trivial)

Item	1	2	…	i	i+1	…
S*	Take	Discard	…	Take	Take	…
S	Take	Discard	…	Take	Discard	…
S'	Take	Discard	…	Take	Discard	…

  - Consider a knapsack S' that follows the first $i+1$ decisions as S
  - By construction of S, it must be that $D(W[(i+1),\ldots,n], V[(i+1),\ldots,n], C-W[i], K^*-V[i])$ accepts, which means we can still obtain K^* with S' if it follows the first $i+1$ decisions as S
  - Hence, S' is also an optimal solution $\Rightarrow$ first $i+1$ decisions of S is part of an optimal solution

Exercise: Knapsack Best Subset

- **Take home exercise:** Why wouldn't this work?

  function $\textsc{KnapSearch}((W, V, C, K), K^*)$:

  $S \leftarrow \emptyset$

  for $i \in \{1, \dots, n\}$ **do**

  if $D(W \setminus W[i], V \setminus V[i], C - W[i], K^* - V[i])$ accepts **then:**

  $S \leftarrow S \cup \{i\}$

  return S

- Note: Here $W \setminus W[i]$ means removing $W[i]$ from W
- Hint: Consider $W = [1,1,1]$, $V = [1,1,1]$, $C = 2$, $K = 2$

TL; DPA

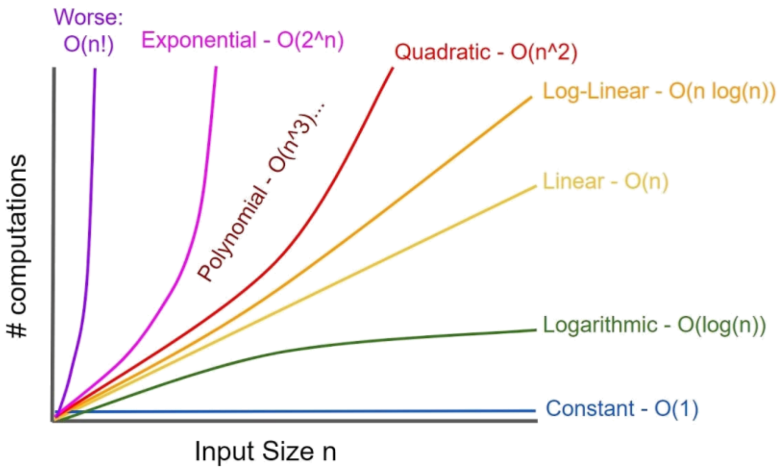
- We explored the search-to-decision reduction, which utilizer the decider of the decision-version of a problem to pinpoint the actual solution for the search-version of the same problem
- Two types of search-to-decision:
  - **Exact search:** Directly uses the decider to identify a specific solution (may need to call the decider multiple times)
  - **Optimization:** First determine the optimal value, then search for the solution that achieve this optimal value

Approximation Algorithms

Notes

Speed vs Accuracy

- Suppose you want to solve a *really hard* classification problem and there are four algorithms available to you:
  A. Runs in $O(n!)$, but the accuracy is guaranteed to be 100%
  B. Runs in $O(2^n)$, but the accuracy is *at least* 90%
  C. Runs in $O(n)$, but the accuracy is *at least* 60%
  D. Runs in $O(1)$, but there is no guarantee on the accuracy



Poll: Which one would you choose if
- this is a real-time spam message detector?
- this is an AI for identifying foes in military applications?

Approximation Algorithms

- **Motivation:** some search problems are very important (TSP, job scheduling, etc.), but if they are NP-hard, then we currently can't solve them efficiently
  - Approximation algorithms get a *close* answer, sacrificing correctness for speed
- We can define how *good* an approximation is in terms of an approximation ratio α
  - Let $val(y)$ be a function that maps the output of a function to some value
  - Let OPT be the value of an optimal solution for some search problem
- An approximate solution y is said to be an α-approximation if

$$\alpha \cdot OPT \leq val(y) \text{ for maximization problem}$$

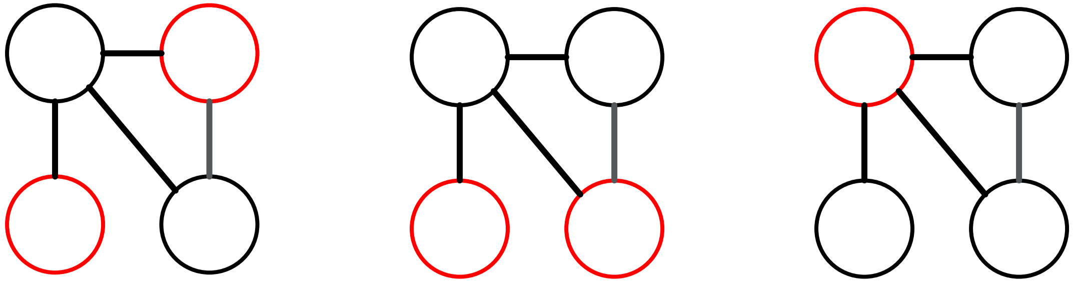
$$val(y) \leq \alpha \cdot OPT \text{ for minimization problem}$$

Concept Check

- Suppose algorithm $\mathcal{A}$ is a 2-approximation for a minimization problem. Then, for (all/ some/ no) inputs x we have $val(\mathcal{A}(x)) = 2 \cdot OPT$.
  - Note: You can assume that 2 is the **tightest value** of α
- Answer: some
  - $val(\mathcal{A}(x)) \leq 2 \cdot OPT$ for all x
  - $\mathcal{A}$ will output a solution at most $2 \cdot OPT$

Example: Independent Set

- An *independent set* of an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices for which there is no edge between any pair of vertices in S.



- The *maximum independent set* (MIS) problem is: given a graph, find an independent set of maximum size.

Example: Independent Set

- Consider the following algorithm:

function $\text{MISA}\textsc{pprox}(G = V, E)$:

1. Initialize $S \leftarrow \emptyset$
2. **while** G is not empty
3. Choose an arbitrary vertex v of G
4. $S \leftarrow S \cup \{v\}$
5. Remove v and all its neighbors (including all their incident edges) from G'
6. **return** S

- Let $U = V \setminus S$ denote the set of all vertices removed in line 5, **not including** the vertices selected for S, and let Δ be the maximum degree of *all* vertices in G. Prove that $|U| \leq |S| \cdot \Delta$.

Hint: If Δ is the max degree of all vertices, what can you say about the number of vertices added to U for each vertex added to S?

Example: Independent Set

- Consider the following algorithm:

function $\text{MISA}\small{\text{PPROX}}(G = V, E)$:

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- Let $U = V \setminus S$ denote the set of all vertices removed in line 5, **not including** the vertices selected for S, and let Δ be the maximum degree of *all* vertices in G. Prove that $|U| \leq |S| \cdot \Delta$.
  - If Δ is the max degree of all vertices, then at most Δ vertices are added to U for each vertex added to S
  - Since the algorithms adds $|S|$ vertices to S, we have $|U| \leq |S| \cdot \Delta$

Example: Independent Set

- Consider the following algorithm:

function $\text{MISA}\textsc{pprox}(G = V, E)$:

1. Initialize $S \leftarrow \emptyset$
2. **while** G is not empty
3. Choose an arbitrary vertex v of G
4. $S \leftarrow S \cup \{v\}$
5. Remove v and all its neighbors (including all their incident edges) from G'
6. **return** S

- Using the fact that $|U| \leq |S| \cdot \Delta$, prove that the algorithm is a $1/(\Delta + 1)$ approximation for $\text{MISA}\textsc{pprox}$ (WTS: $\alpha \cdot OPT \leq val(y)$)

  Hint: $V = U \cup S$ and $U \cap S = \emptyset$

Example: Independent Set

- Consider the following algorithm:

function $\text{MISA}\textsc{pprox}(G = V, E)$:

1. Initialize $S \leftarrow \emptyset$
2. **while** G is not empty
3. Choose an arbitrary vertex v of G
4. $S \leftarrow S \cup \{v\}$
5. Remove v and all its neighbors (including all their incident edges) from G'
6. **return** S

- Using the fact that $|U| \leq |S| \cdot \Delta$, prove that the algorithm is a $1/(\Delta + 1)$ approximation for $\text{MISA}\textsc{pprox}$ (WTS: $\alpha \cdot OPT \leq val(y)$)
  - Let T^* be a maximum independent set. Since $T^* \subseteq V$,

$$|T^*| \leq |V| = |U| + |S| \leq |S| \cdot \Delta + |S|$$

$$\frac{1}{\Delta + 1} |T^*| \leq |S|$$

Approximation Algorithms Proofs

- For the proof of correctness of approximation algorithms, you need to bound the value of the algorithm in terms of the optimal value.
- Usually, we need to connect multiple inequalities together to get the result. Common types of inequalities you will use are:
  - Trivial bounds that connect the optimal solution to the problem size. For example, the maximum independent set size cannot exceed the number of vertices.
  - Algorithm-related bounds that depend on what the algorithm does (e.g., choices it makes). For example, the greedy algorithm for knapsack will choose an item with the largest value.
  - Nontrivial connection to some intermediate algorithms or objects. We will usually provide some guidance in this case. For example, the metric TSP cost is compared against the cost of an MST.

Back Matter

SAT Search to Decision Reduction

$\text{SAT} = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$ ider D.

- Search objective: find an assignment for each variable $x_1, \dots, x_n$ in ϕ

1. If $D(\phi)$ returns false, output $\perp$ (the formula is unsatisfiable).
2. For each variable x_i $(1 \leq i \leq n)$ in ϕ, do the following:
   (a) Set x_i to false $(x_i = F)$. Let us denote the resulting formula (with x_i set to false) as $\phi_{x_i=F}$. Run $D(\phi_{x_i=F})$.
      i. If $D(\phi_{x_i=F})$ accepts, continue to the next iteration of the algorithm (for x_{i+1}).
      ii. If $D(\phi_{x_i=F})$ rejects, set x_i to true and continue to the next iteration of the algorithm for x_{i+1}.

SAT Search to Decision Reduction

- Runtime Analysis:
  - D runs in $O(|\phi|^k)$ for some constant k, so step 1 is efficient
  - Step 2 loops n times, which is $\leq |\phi|$, within each iteration we assign truth assignments to one variable which is linear worst case, then run D.
    $O(n \cdot (|\phi| + |\phi|^k)) = O(|\phi|^2 + |\phi|^{k+1}) = O(|\phi|^{k+1})$

1. If $D(\phi)$ returns false, output $\perp$ (the formula is unsatisfiable).
2. For each variable x_i $(1 \leq i \leq n)$ in ϕ, do the following:

   (a) Set x_i to false $(x_i = F)$. Let us denote the resulting formula (with x_i set to false) as $\phi_{x_i=F}$. Run $D(\phi_{x_i=F})$.

   i. If $D(\phi_{x_i=F})$ accepts, continue to the next iteration of the algorithm (for x_{i+1}).

   ii. If $D(\phi_{x_i=F})$ rejects, set x_i to true and continue to the next iteration of the algorithm for x_{i+1}.

Metric TSP Approx (Lecture Review)

$\text{TSP} = \{\langle G, k\rangle \mid G \text{ is an undirected, weighted, complete graph with a tour of weight} \leq k\}$

- Traveling Salesperson Problem
  - Input is a complete, weighted, undirected graph G
  - The weight of a subgraph is the sum of its edge weights
  - Goal is to find an *optimal tour*, or a Hamiltonian cycle with minimum weight

- This is very difficult to solve, so we impose the triangle inequality constraint:
  - Any three vertices in V satisfy the triangle inequality

$$w((v_1, v_2)) \leq w((v_1, v_3)) + w((v_3, v_2)).$$

  - This version of TSP is known as *Metric TSP*
  - Even Metric TSP is NP-Complete! So we present a 2-approximation

Metric TSP Approx (Lecture Review)

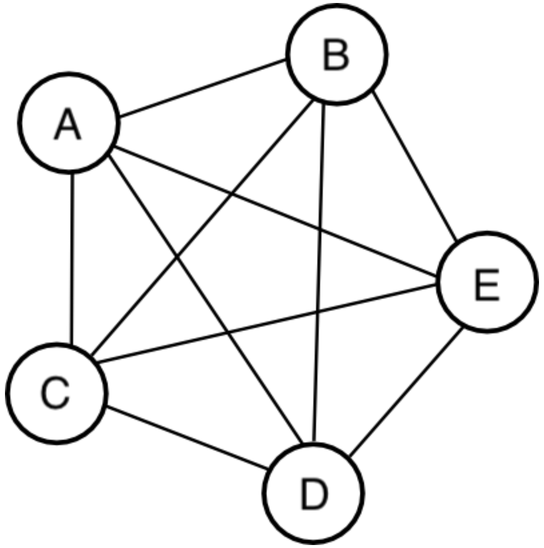
- Recall a minimum spanning tree is an undirected, connected, acyclic graph that contains all vertices in G with as little weight as possible

- The weight of the MST T is $\leq$ the weight of the optimal tour H
  - Proof: assume we have a graph where the weight of the MST is greater than the weight of the optimal tour. Removing an edge in the tour would result in a spanning tree of weight less than the MST, which is a contradiction

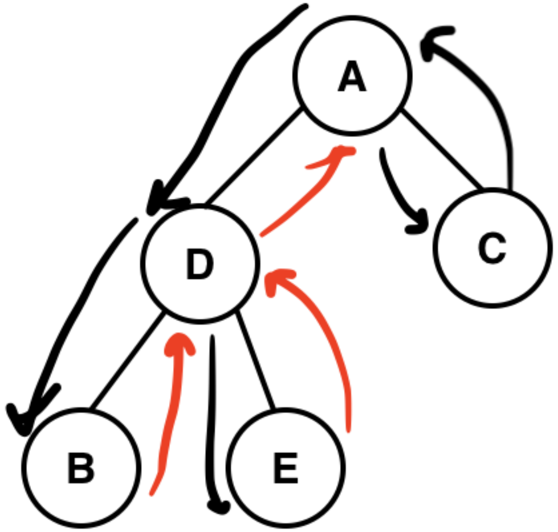
- Algorithm
  - Use Kruskal's algorithm to get T, an MST of G
  - Perform a depth-first search on the MST, but skip vertices we've already visited
    - Triangle inequality guarantees that this is better than visiting every edge twice

Example Run of TSP Approximation

- Start with a complete, undirected graph



- Find the MST and do a DFS, skipping repeated edges



Original DFS: A → D → B → D → E → D → A → C → A
Modified: A → D → B → E → C → A

Metric TSP Approx (Lecture Review)

- The weight of the MST *T* is $\leq$ the weight of the optimal tour *H*
  - Proof: assume we have a graph where the weight of the MST is greater than the weight of the optimal tour. Removing an edge in the tour would result in a spanning tree of weight less than the MST, which is a contradiction

- Algorithm
  - Use Kruskal's algorithm to get *T*, an MST of G
  - Perform a depth-first search on the MST, but skip vertices we've already visited
    - Triangle inequality guarantees that this is better than visiting every edge twice

- This gives us a Hamiltonian cycle with weight *c*
- $c \leq 2w(T)$ because we traverse each edge in *T* at most twice
- $c \leq 2w(T) \leq 2w(H)$ because $w(T) \leq w(H)$ (proved above)
- This is a 2-approximation of constrained TSP

- Worksheet Problem 8 result: Even *approximating* general TSP with a fixed α bound is NP-complete!