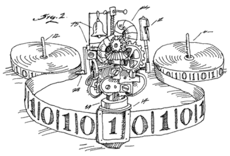
EECS 376: Foundations of Computer Science

Lecture 15 - Cook-Levin Theorem and Satisfiability



Plan in this part of the course

Lecture 1:
- Define **P** and **NP**

Lecture 2: (today)
- Define **polynomial-time mapping reduction**
- Define **NP-hard** and **NP-complete.**
- Show the first NP-complete problem: SAT

Lectures 3 – 4:
- Show many NP-complete problems via reductions

The Complexity Class **P**

Definition:
P is the set of all decision problems that can be decided in polynomial time.

Formally:
- For any problem **L,** an efficient decider **Decide-L** for **L** is s.t.
  - **x** is a "yes" instance ⇔ **Decide-L**(**x**) accepts
  - **x** is a "no" instance ⇔ **Decide-L**(**x**) rejects (follows from above)
  - **Decide-L**(**x**) runs in poly(|**x**|) time
- **P** is the set of all decision problems that have efficient deciders

Example: is gcd(x,y) ≤ b?

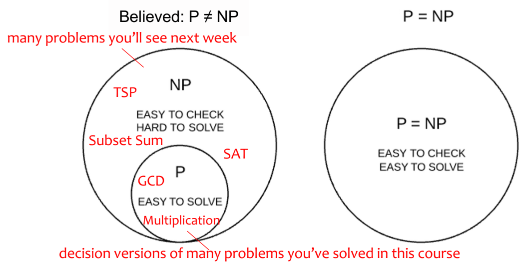
The Complexity Class **NP**

Definition:
NP is the set of all decision problems whose yes-instances can be verified in polynomial time.

Formally:
- For any problem **L,** an efficient verifier **Verify-L** for **L** is s.t.
  - **x** is a "yes" instance ⇔ ∃**C** **Verify-L**(**x, C**) accepts
  - **x** is a "no" instance ⇔ ∀**C** **Verify-L**(**x, C**) rejects (follows from above)
  - **Verify-L**(**x, C**) runs in poly(|**x**|) time
- **NP** is the set of all decision problems that have efficient verifiers
- If **Verify-L**(**x, C**) accepts, then **C** is called a certificate.

Example: Subset Sum, TSP, SAT
Quiz: Explain why **P** ⊆ **NP.**

Two Possible Worlds



Polynomial-time mapping reduction
(also called **Karp reduction**)

Note: a different type of reduction from **Turing reduction**

Polynomial-time mapping reduction from **A** to **B** (denoted **A** $\leq_p$ **B**)

Defn: A $\leq_p$ **B** if there is a poly-time-computable function **f** where

x is a yes-instance of **A** ⇔ **f(x)** is a yes-instance of **B.**

In words, given any instance of **A, in polynomial time** we can construct an instance of **B** whose **yes/no answer is the same.**

Instance **x** of **A** → fast conversion machine → Instance **f(x)** of **B** with same answer

"Problem **B** is at least as hard as Problem **A**"

Polynomial-time mapping reduction from **A** to **B** (denoted **A** $\leq_p$ **B**)

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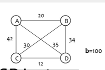
Instance **x** of **A** → fast conversion machine → Instance **f(x)** of **B** with same answer

$(x_1 \vee \overline{x}_2 \vee x_4) \wedge$
$(x_2 \vee x_3 \vee \overline{x}_{42})$

E.g. SAT Instance



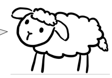
TSP Instance

Polynomial-time mapping reduction from A to B
(denoted A $\leq_p$ B)

Defn: **A** $\leq_p$ **B** if there is a poly-time-computable function **f** where

x is a yes-instance of **A** ⇔ **f(x)** is a yes-instance of **B**.

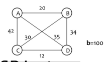
Difference from Turing's reduction
1. Polynomial time
2. No "flipping" the answer
3. To solve A, the reduction makes only **one** call to B



Instance **x** of **A** →  → Instance **f(x)** of **B** with same answer

$(x_1 \vee \overline{x}_2 \vee x_4)\wedge$
$(x_2 \vee x_3 \vee \overline{x}_{42})$

fast conversion machine



E.g. SAT Instance TSP Instance

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NP-hardness
and
NP-completeness

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NP-Hardness and NP-Completeness

Informal definition: A problem **L** is **NP**-hard if it is at least as hard as EVERY problem in **NP**.

Formal definition: A problem **L** is NP-hard if: for EVERY problem **X** in NP, **X** $\leq_p$ **L.**

Showing that some problem is NP-hard seems very strong.

But we will do it soon!

A problem **L** is NP-complete if
- **L ∈ NP**
- **L** is **NP**-hard

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Exercise

Recall: **A** $\leq_p$ **B** if
- there is a poly-time-computable function f where
- x in **A** ⇔ f(x) in **B.**

A problem **L** is NP-hard if for EVERY problem **X** in NP, **X** $\leq_p$ **L.**

Exercise: Suppose **A** $\leq_p$**B.**
1. If **B** ∈ **P,** then **A** ∈ **P.**
   - Given an instance **x** for **A,** compute an instance **f(x)** for **B** in **poly time.**
   - As **B** ∈ **P**, we can decide **f(x) in poly time.** Then, return the same answer for **A.**
2. If **A** is NP-hard, then **B** is NP-hard.
   - **Fact:** if **X** $\leq_p$ **A** and **A** $\leq_p$**B**, then **X** $\leq_p$ **B.** (see HW.)
   - For any X ∈ **NP**, **X** $\leq_p$ **A.** (**A** is NP-hard). So, for any X ∈ **NP**, **X** $\leq_p$ **B.** (**B** is NP-hard)
3. Suppose L is NP-complete. Then, L ∈ **P** iff **P** = **NP**
   - Suppose L ∉ **P**. As L ∈ **NP**, then **P** ≠ **NP.**
   - Suppose L ∈ **P**. As L is **NP-hard,** every NP-problem **X** is in **P.** So, **NP** ⊆ **P.**

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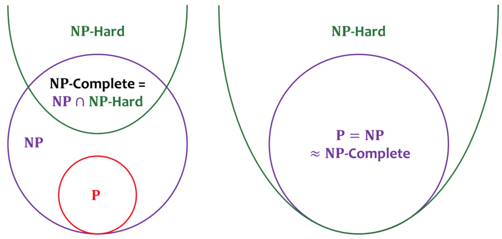
Two Possible Worlds

P ≠ NP P = NP



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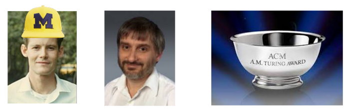

The Cook-Levin Theorem (1971):
SAT is NP-complete

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Terminology on Formulas

A **Boolean formula** Φ is made up of:
- "literals": variables and their negations (e.g. x, y, z, ¬x, ¬y, ¬z)
- OR: ∨
- AND: ∧

Example:

Φ1 = (x ∨ y) ∧ (¬y ∨ x ∨ ¬z) ∧ (¬x ∨ (y∧¬z))

Φ is ***satisfiable*** if
- ∃ a true/false assignment **A** to the variables so that Φ(**A**) = true
- For example, Φ1 is satisfiable.
  - Assign x = F, y = T, z = F

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Satisfiability (SAT)

Input: A Boolean formula Φ
Output: Is Φ *satisfiable*?

Last time, we showed that SAT is in **NP**.
- Certificate: a true/false assignment **C** to variables where Φ(**C**) = true
- Verifier: **Verify**(Φ, **C**): Check that Φ(**C**) = true

To prove that SAT is **NP**-complete, remain to prove that SAT is **NP**-hard

A problem **L** is NP-complete if
- **L ∈ NP**
- **L** is **NP**-hard

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Proving SAT is NP-hard

Goal: for every problem **L** in NP, **L** $\leq_p$ **SAT**.

Fix a problem **L** in NP.
Let x be an instance of **L** of size $|x| = n$.

There is an efficient verifier V s.t.

- x is a "yes" instance $\Leftrightarrow \exists C\ V(x, C)$ accepts
- $V(x, C)$ runs in n^k time (k is a constant)

Important:
$\varphi_{V,x}$ depends on V and x.
But not on C

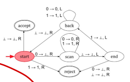
Will show: in poly(n) time, can construct a formula $\varphi_{V,x}$ s.t.

- $\exists C\ V(x, C)$ accepts $\Leftrightarrow \varphi_{V,x}$ is satisfiable
- So, **L** $\leq_p$ **SAT**. Done.

Recall defn: **A** $\leq_p$ **B** if
there is a poly-time-computable function **f** where
x is a yes-instance of **A** $\Leftrightarrow$ **f(x)** is a yes-instance of **B**.

Overview of Formula Construction

x: instance of size n.
$V(x, C)$: a TM that runs in n^k steps



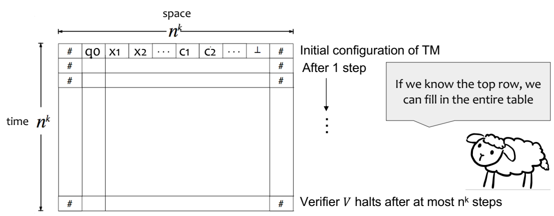
Goal: construct formula $\varphi_{V,x}$ s.t.

$$\exists C\ V(x, C) \text{ accepts} \Leftrightarrow$$
$$\exists \text{ assignment } A\ \varphi_{V,x}(A) = \text{true}$$

(i.e. $\varphi_{V,x}$ is satisfiable)

Variables of $\varphi_{V,x}$ is based on the **execution tableau** of V

Execution Tableau: Visualizing the execution of TM



If a row of the tableau says "#011$q_5$0001#" it means:

- The tape says: "0110001"
- We are in state q_5
- The head is pointing to the symbol right after the state, i.e. 011**0**001

Symbols in tableau consists of $S = \{0,1\} \cup \{\#, \$, \perp\} \cup Q$

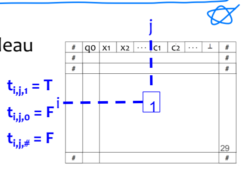
Q: set of the states of V

Overview of Formula Construction

Goal: construct formula $\varphi_{V,x}$ s.t.

$$\exists C\ V(x, C) \text{ accepts} \Leftrightarrow$$
$$\exists \text{ assignment } A\ \varphi_{V,x}(A) = \text{T}$$

- **Variables** of $\varphi_{V,x}$ are $t_{i,j,s}$ for all $i, j \leq n^k$ and each symbol s
- **Intention:** $t_{i,j,s} = \text{T}$ *iff symbol in cell* (i, j) *of the tableau is "s"*
- Assignment of $\varphi_{V,x} \Leftrightarrow$ Values in tableau



Overview of Formula Construction

Assignment of $\varphi_{V,x} \Leftrightarrow$ Values in tableau

$$\varphi_{V,x} = \varphi_{start} \wedge \varphi_{cell} \wedge \varphi_{accept} \wedge \varphi_{step}$$

1. φ_{start} fixes **the value of x** in the **first row** of the tableau
2. φ_{accept} checks if the tableau contains the **accepting state q_{accept}**
3. φ_{cell} checks that there is exactly **one symbol per cell**
4. φ_{step} checks that the tableau is valid according to **transition rules of V**

If we can ensure (1) – (4), we have
$\exists C\ V(x, C)$ accepts $\Leftrightarrow \exists$ assignment $A\ \varphi_{V,x}(A) = \text{T}$

(1) φ_{start} fixes **the value of x** in the **first row** of the tableau

φ_{start} enforces the starting configuration

#	q_0	x	\$	c	⊥	⊥	#

- Initial state q_0,
- Input x, |x| = n; certificate C, |C| = m $\leq n^k$
- \$ - a special symbol that separates x and c
- **WE DO NOT KNOW c (!!)**, so we leave a "placeholder"

$\varphi_{start} = t_{1,1,\#} \wedge t_{1,2,q_0} \wedge t_{1,3,x_1} \wedge t_{1,4,x_2} \wedge \dots \wedge t_{1,n+2,x_n} \wedge t_{1,n+3,\$} \wedge$ — This fixes the first n+3 symbols

$(t_{1,n+4,1} \vee t_{1,n+4,0} \vee t_{1,n+4,\perp}) \wedge (t_{1,n+5,1} \vee t_{1,n+5,0} \vee t_{1,n+5,\perp}) \wedge \dots$ — Placeholders for $\approx n^k$ symbols

(c_1 can be either 1 or 0 or ⊥)

Note: The size of φ_{start} is $O(n^k) = \text{poly(n)}$

(2) φ_{accept} checks if the tableau contains the **accepting state q_{accept}**

$$\varphi_{accept} = \bigvee_{1 \leq i,j \leq n^k} t_{i,j,q_{accept}}$$

Note: The size of φ_{accept} is $O(n^{2k}) = \text{poly(n)}$

(3) φ_{cell} checks that there is exactly **one symbol per cell**

cannot contain two different symbols

contains at least one symbol

every cell

$$\varphi_{cell} = \bigwedge_{1 \leq i,j \leq n^k} \left[\bigvee_{\sigma \in S} t_{i,j,\sigma} \wedge \bigwedge_{\sigma \neq \tau \in S} \overline{(t_{i,j,\sigma} \wedge t_{i,j,\tau})} \right]$$

What do these parts mean, in English?

Note: The size of φ_{cell} is $O(n^{2k}) = \text{poly(n)}$

(4) φ_{step} checks that the tableau is valid according to **transition rules of V**

Definition: A 2x3 "window" is *valid* if it could appear in a valid tableau

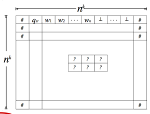
Example transition rule: $q_4 \xrightarrow{1 \to 0,\ R} q_5$

0	q_4	1
0	0	q_5

~~0~~	~~q_4~~	~~1~~
~~0~~	~~1~~	~~q_5~~

Theorem:
The whole tableau is valid if and only if **every 2x3 window is valid.**

Proof Idea:
TM can only move 1 left/right step at each time.



Exercise after class: see why **2x2 windows** do not work

(4) φ_{step} checks that the tableau is valid according to **transition rules of V**

$$\varphi_{step} = \bigwedge_{1 \le i,j \le n^k} \varphi_{window,i,j}$$

$\varphi_{window,i,j} = (t_{i,j,0} \wedge t_{i,j+1,q4} \wedge t_{i,j+2,1} \wedge t_{i+1,j,0} \wedge t_{i+1,j+1,0} \wedge t_{i+1,j+2,q5}) \vee (\ldots) \vee \ldots)$

Example of 2x3 valid window

0	q_4	1
0	0	q_5

More valid windows:

0	1	1
0	1	1

nothing changes if head isn't around

0	1	1
q_3	1	1

head could enter from the side

(4) φ_{step} checks that the tableau is valid according to **transition rules of V**

$$\varphi_{step} = \bigwedge_{1 \le i,j \le n^k} \varphi_{window,i,j}$$

$$\varphi_{\text{window},i,j} = \bigvee_{\substack{\binom{s_1,s_2,s_3}{s_4,s_5,s_6} \\ \text{valid 2x3 window}}} \left(\begin{array}{c} t_{i,j,s_1} \wedge t_{i,j+1,s_2} \wedge t_{i,j+2,s_3} \wedge \\ t_{i+1,j,s_4} \wedge t_{i+1,j+1,s_5} \wedge t_{i+1,j+2,s_6} \end{array} \right)$$

Note: the size of $\varphi_{window,i,j}$ is $O(|S|^6) = O(1)$.
So, the size of φ_{step} is $O(n^{2k})$

Conclusion: Formula Construction

Assignment of $\varphi_{V,x}$ ⇔ Values in tableau

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Proving SAT is NP-hard

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- So, **L** $\le_p$ **SAT**. Done.

Wrap Up

- Define **polynomial-time mapping reduction**
- Define **NP-hard** and **NP-complete.**
- Show the first NP-complete problem: SAT

- SAT ∈ **P** iff **P** = **NP**
  - Assuming **P** ≠ **NP**, no efficient algorithm for **SAT.**

- **Next week:**
  - Assuming **P** ≠ **NP**, no efficient algorithm for **many other problems**