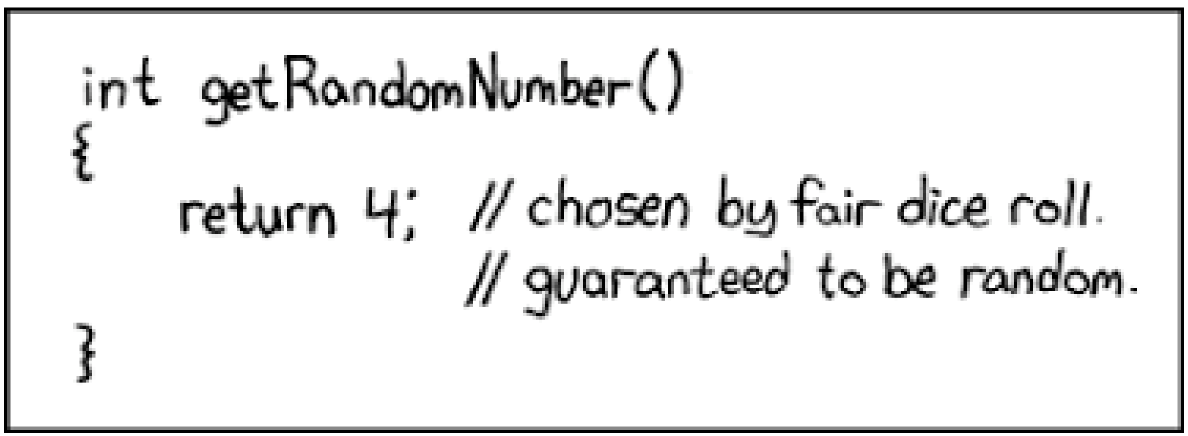
More Randomized Algorithms



When is Randomization Necessary?

my "bonus" lecture next week

- In the areas of online algorithms / cryptography / games / etc. when the input is hidden, we can often **prove** randomization is necessary.

- In the "standard" setting (input given upfront), whether or not randomization is necessary is a **big open problem**. Most researchers seem to believe *every* randomized algorithm can be derandomized.

- When randomization isn't necessary, it's still useful for getting simpler and/or faster-in-practice algorithms, and it provides inspiration for designing deterministic algorithms.

Tools from last time

An ***indicator*** random variable X has 2 possible outcomes: 0 and 1.

Expected value of an indicator r.v.: **E[X] = Pr[X=1].**

Linearity of Expectation: For any (not necessarily independent!) random variables N_i:

$$\mathbb{E}\Big[\sum_i N_i\Big] = \sum_i \mathbb{E}[N_i].$$

Markov's Inequality: If X is a <u>*non-negative*</u> random variable and $a > 0$, then $\Pr[X \geq \underset{2 \cdot E[X]}{\cancel{a}}] \leq \underset{1/2}{\cancel{\mathbb{E}[X]/a}}$.

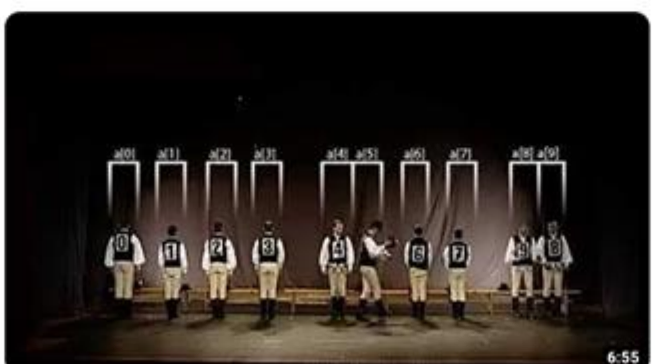
QuickSort

QuickSort is a commonly used randomized sorting algorithm.

1. Pick an array element as the **“pivot”**.
2. Compare pivot to each element to partition list into **two parts**: elements **less that pivot** and elements **greater than pivot**.
3. **Recurse** on both parts of list.

How do you choose the **pivot**?

We will analyze a common strategy: **choose uniformly at random!**



Quick sort with Hungarian, folk dance

153K views • 10 years ago

kamal yassin

https://www.youtube.com/watch?v=3San3uKKHgg

From EECS 281 on quicksort:

Time Analysis

- Cost of partitioning N elements: $O(N)$
- Worst case: pivot always leaves one side empty
  - $T(N) = N + T(N - 1) + T(0)$
  - $T(N) = N + T(N - 1) + c$ [since $T(0)$ is $O(1)$]
  - $T(N) \sim N^2/2 \Rightarrow O(N^2)$ [via substitution]
- Best case: pivot divides elements equally
  - $T(N) = N + T(N / 2) + T(N / 2)$
  - $T(N) = N + 2T(N / 2) = N + 2(N / 2) + 4(N / 4) + \ldots + O(1)$
  - $T(N) \sim N \log N \Rightarrow O(N \log N)$ [master theorem or substitution]
- ~~Average case~~: $\boxed{\text{tricky}}$ ← We have the background for this now
  - Between $2N \log N$ and $\sim 1.39\, N \log N \Rightarrow O(N \log N)$

For any input, expected running time is

Quicksort

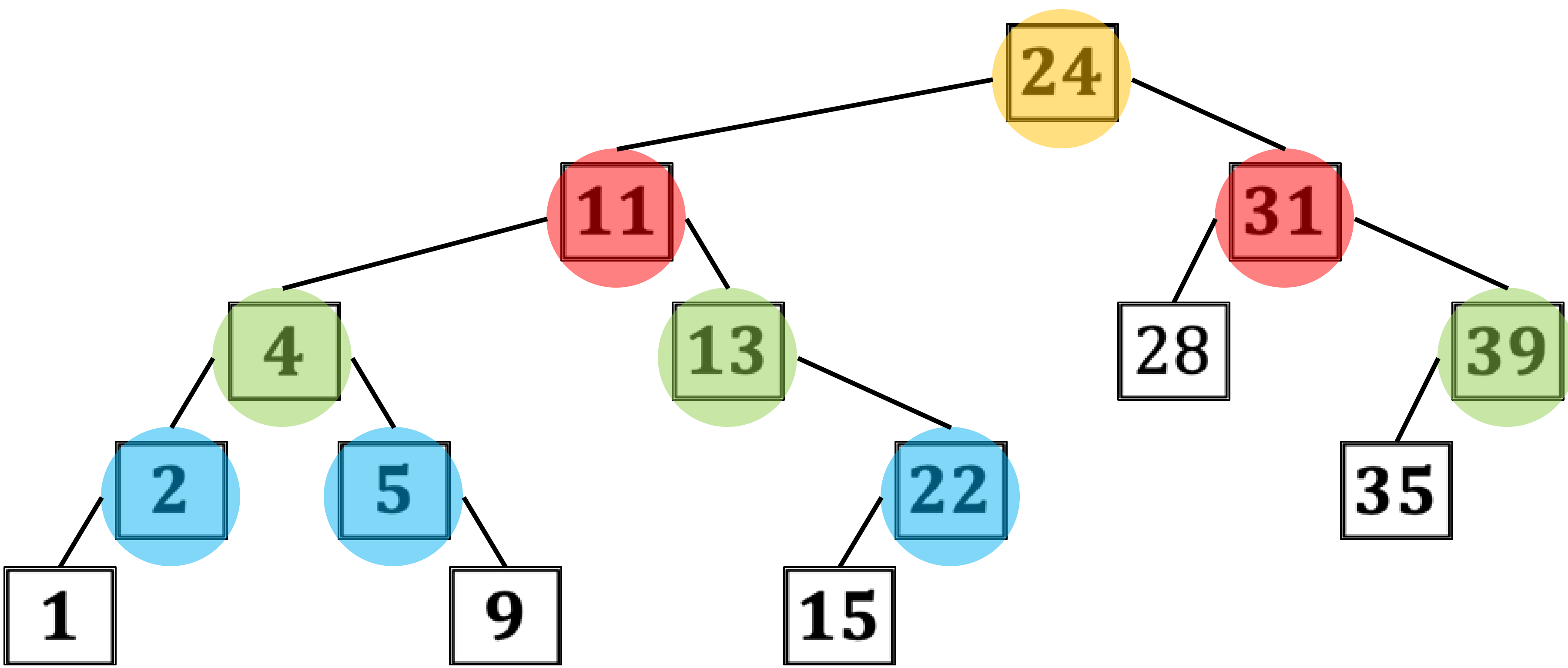
Our goal: Prove that *for any input*, the expected running time of QuickSort is O(n log n).

5	28	22	11	1	35	39	24	2	4	13	9	31	15
5	22	1	11	2	4	13	9	15	**24**	28	35	39	31
5	1	2	4	9	**11**	22	13	15	**24**	28	**31**	35	39
1	2	4	5	9	**11**	**13**	15	22	**24**	28	**31**	**35**	**39**
1	**2**	**4**	**5**	**9**	**11**	**13**	**15**	**22**	**24**	**28**	**31**	**35**	**39**

Analysis of Quicksort

Question: Which pairs of elements were compared during the execution of the algorithm?

ancestors and descendents



Analysis of Quicksort

* **Idea:** Fix all the randomness at the beginning. Randomly assign **priorities** to each element, pick the one with smallest priority when choosing a pivot.

5	28	22	11	1	35	39	24	2	4	13	9	31	15
10	11	13	2	9	6	5	1	7	3	8	12	4	14

5	22	11	1	2	4	13	9	15	24	28	35	39	31
10	13	2	9	7	3	8	12	14		11	6	5	4

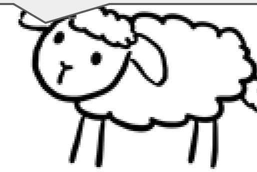
5	1	2	4	9	11	22	13	15	24	28	31	35	39
10	9	7	3	12		13	8	14				6	5

1	2	4	5	9	11	13	22	15	24	28	31	35	39
9	7		10	12			13	14					

1	2	4	5	9	11	13	15	22	24	28	31	35	39

Analysis of Quicksort

Remember, A is the *sorted* array, not the original array



* **Example:** Sorted sequence A with priorities:

						i					j		
1	2	4	5	9	11	13	15	22	24	28	31	35	39
9	7	3	10	12	2	8	14	13	1	11	4	6	5

* **Question:** Can you tell if A[i] and A[j] were compared during the algorithm, just by looking at the priorities?

Compared iff A[i] or A[j] has smallest priority in subarray A[i]....A[j]

Pf. $\Rightarrow$ (contrapositive) A[k] has smallest priority in A[i]... A[j].
When A[k] becomes pivot, A[i] and A[j] go to separate subarrays and never compared.

$\Leftarrow$ Only elts in A[i]...A[j] can split A[i] and A[j]. A[i] or A[j] is pivot first among A[i].. A[j]. At this time they are compared.

Analysis of Quicksort

Our goal: Prove that *for any input*, the expected running time of QuickSort is O(n log n).

We'll apply the method of <u>indicator random variables</u> + <u>linearity of expectation</u> from last time

Analysis of Quicksort

Let **X** be the number of comparisons made by QuickSort.

Goal: calculate E[X] (since running time of Quicksort = O(X))

Let **Xij** be an indicator r.v. for whether A[i] and A[j] are compared.

Observe that $\mathbf{X} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{X_{ij}}$.

$E[X_{ij}] = Pr[X_{ij}=1]$

Let's use linearity of expectation to calculate E[X].

(question from 2 slides ago will be useful)

$$E[X] = E\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum \sum \underbrace{Pr[X_{ij}=1]}_{2/(j-i+1)}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

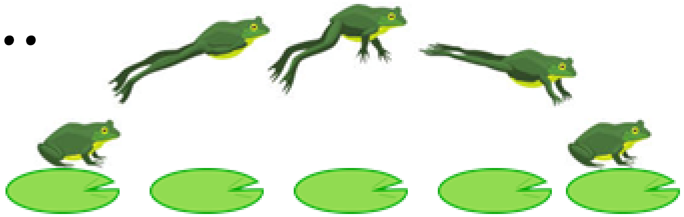
$i=1$

$$\leq 2n \underbrace{\sum_{j=2}^{n} \frac{1}{j}}_{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \;\; \Theta(\log n)}$$

$$= O(n \log n)$$

Now for a randomized *data structure*…

Skip Lists



A ***dictionary*** is an abstract data type (ADT) that supports insert, find, and delete operations.

Simple implementations: linked list, array

no in-order traversal complicated bookkeeping

Faster implementations: hash tables (various types), balanced binary search trees (AVL, red-black, etc.)

Discovered by William Pugh, 1989

Skip lists are **simpler to implement** (no need for rotations, invariants, bookkeeping)

A **skip list** is like an embellished version of a **linked list**, with expected O(log n) search time (instead of O(n))

Usages of Skip Lists

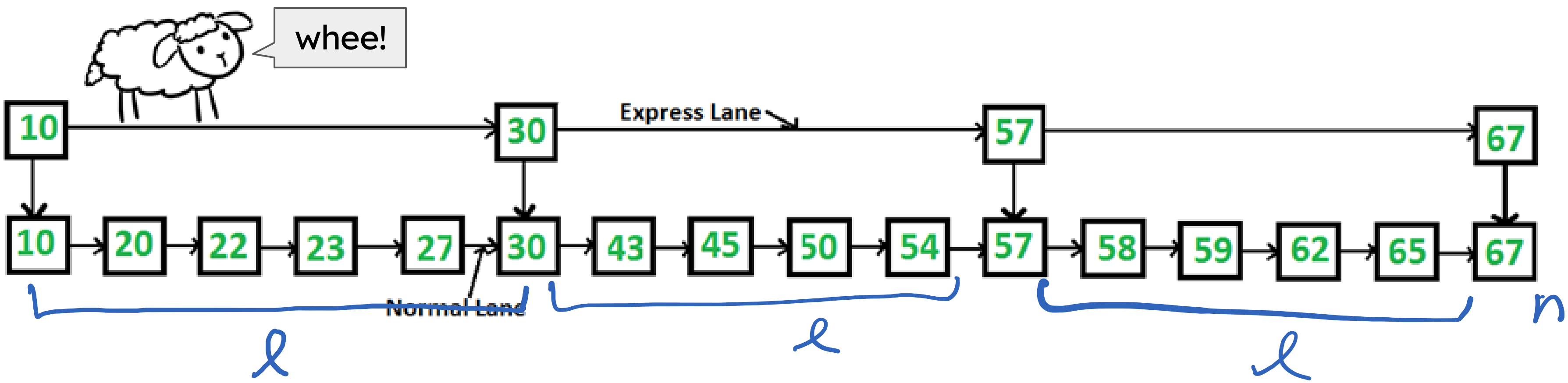
Usages [edit]

List of applications and frameworks that use skip lists:

- Apache Portable Runtime implements skip lists.[9]
- MemSQL uses lock-free skip lists as its prime indexing structure for its database technology.
- MuQSS, for the Linux kernel, is a cpu scheduler built on skip lists.[10][11]
- Cyrus IMAP server offers a "skiplist" backend DB implementation[12]
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.[*citation needed*]
- The "QMap" key/value dictionary (up to Qt 4) template class of Qt is implemented with skip lists.[13]
- Redis, an ANSI-C open-source persistent key/value store for Posix systems, uses skip lists in its implementation of ordered sets.[14]
- Discord uses skip lists to handle storing and updating the list of members in a server.[15]
- RocksDB uses skip lists for its default Memtable implementation.[16]

Warm-up: 2-level Linked List

A linked list with an "express lane"



Worst-case running time for search?

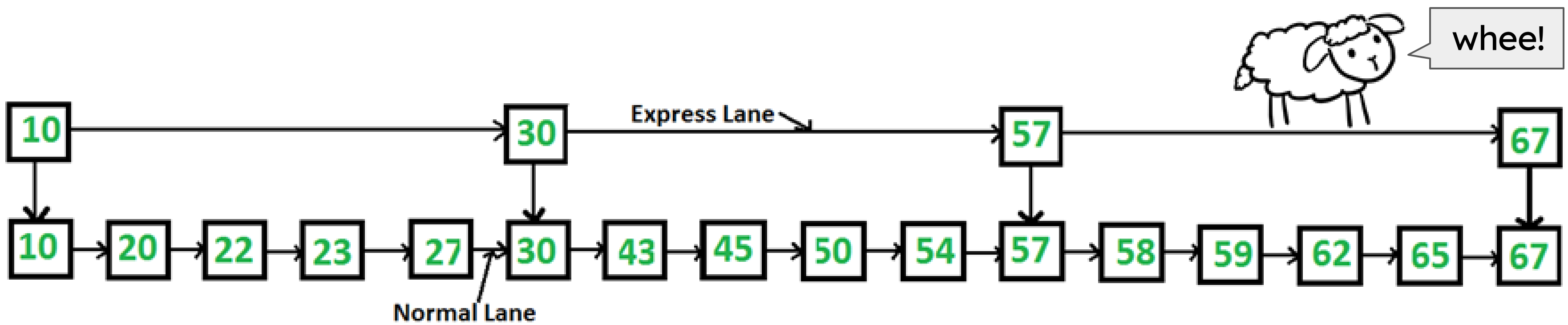
$O\left(\ell + \frac{n}{\ell}\right)$ $\quad \ell = \frac{n}{\ell} \Rightarrow \ell = \sqrt{n}$

How many elements should be in the express lane?

$\sqrt{n}$

Warm-up: 2-level Linked List

A linked list with an "express lane"



Deterministically keeping express lane roughly evenly spaced amidst insertions/deletions would require some bookkeeping…

Instead let's promote elements to the express lane randomly!

With what probability should we promote each element?

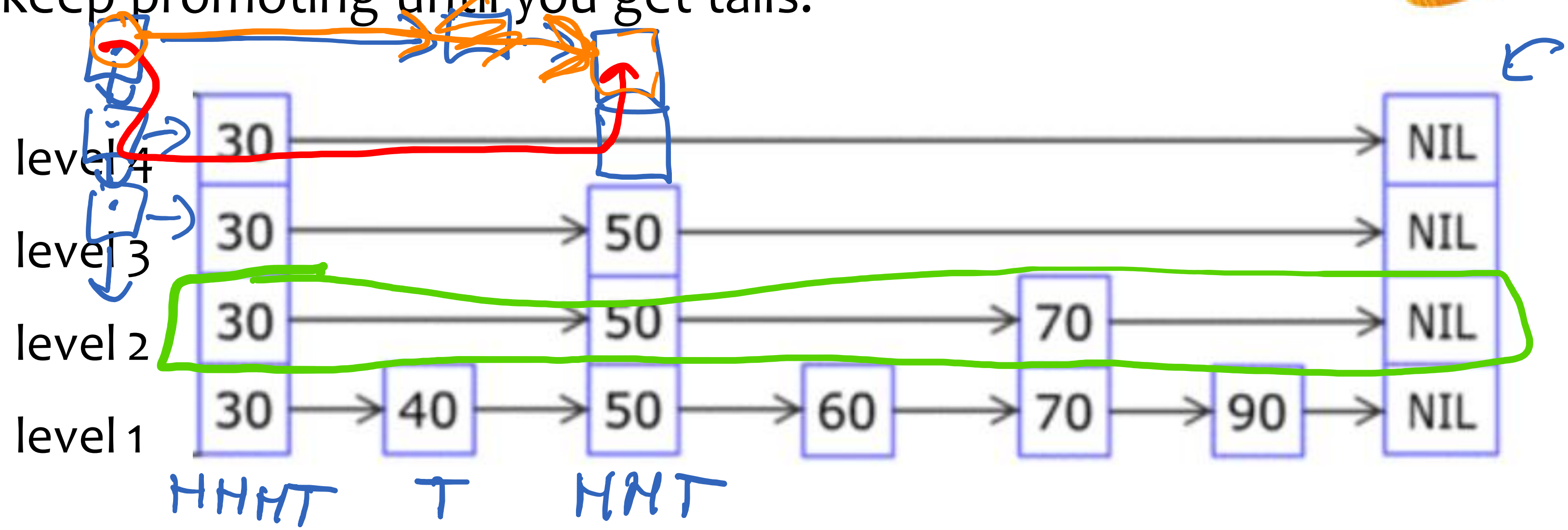
$\frac{1}{\sqrt{n}}$

Idea of a Skip List: Add more express lanes!

(Roughly log n of them)

Put every element in **level 1**
Promote about ½ of the elements to **level 2**
Promote about ½ of the elements in **level 2** to **level 3**
Promote about ½ of the elements in **level 3** to **level 4**
⋮

How to accomplish this: For each element, flip a coin and keep promoting until you get tails.



Skip Lists

How do we <u>insert</u> an element?

(<u>delete</u> and <u>find</u> are very similar to <u>insert</u>)

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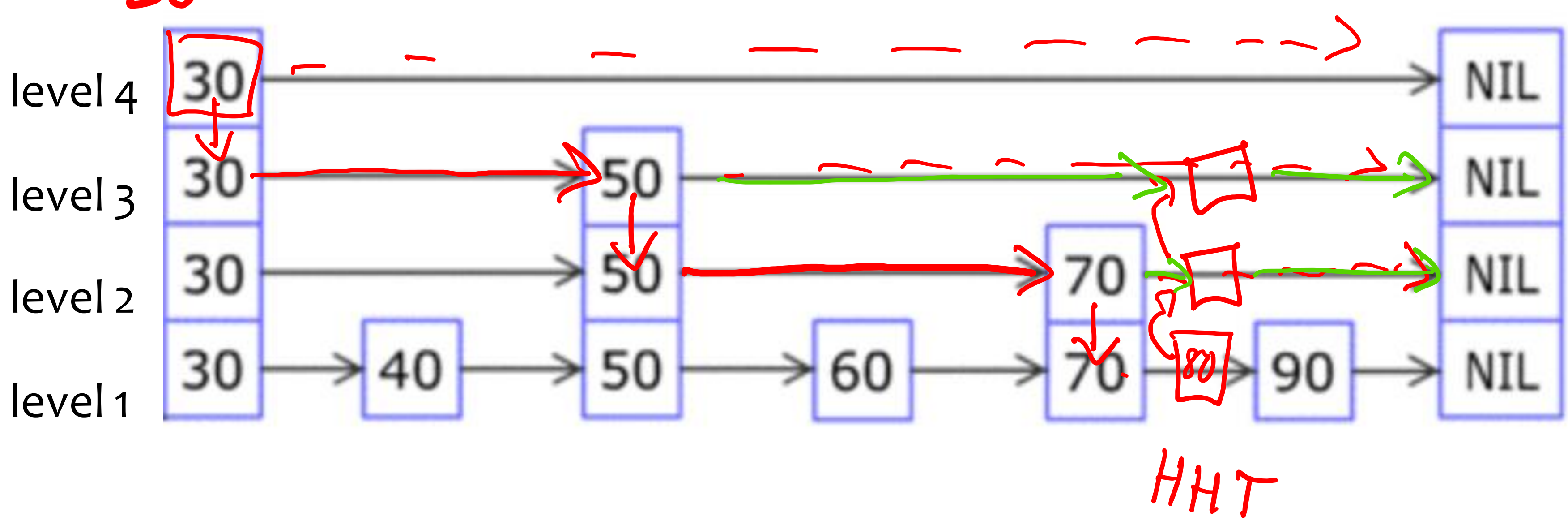


Skip Lists

We will prove: *For any fixed sequence of n operations* (insert, delete, find), the expected running time per operation is O(log n).

Note: Our analysis will rely on the fact that the choice of operations cannot respond to the random choices of the algorithm.

(Imagine the sequence of operations is fixed ahead of time.)

Properties of Skip Lists

Goal #1: Show that the expected number of levels is O(log n).

Suppose we have a skip list containing elements $\mathbf{x_1, x_2}$**…**

Question 1: In expectation, how many elements are on level **i**?

What tool should we use to answer this question?

Fix a level **i**. Let **X** be the number of elements on level **i**.

Indicator $X_j = \begin{cases} 1 & \text{if } x_j \text{ on level } i \\ 0 & \text{o.w.} \end{cases}$

Pr i-1 heads in a row.

$$E[X] = E\left[\sum_j X_j\right] = \sum_j E[X_j] = \sum_j \Pr[X_j = 1] = \sum_j \frac{1}{2^{i-1}} \leq \frac{n}{2^{i-1}}$$

Properties of Skip Lists

Goal #1: Show that the expected number of levels is O(log n).

Fix a level **i**. Let **X** be the number of elements on level **i**.

Question 2: What is the probability level **i** has at least one element?

What tool should we use to answer this question? Markov: X non-neg

$\Pr[X \geq a] \leq E[X]/a$

$\Pr[X \geq 1] \leq E[X] \leq \frac{n}{2^{i-1}}$

Properties of Skip Lists

Goal #1: Show that the expected number of levels is $O(\log n)$.

Question 3: In expectation, how many levels have ≥ 1 element?

Let **Yk** be an indicator that **level $\log_2 n$ + k** has ≥ 1 element.

$$E[\#\text{ levels above }\log_2 n] = E\Big[\sum_{k=1}^{\infty} Y_k\Big]$$

$$= \sum_{k=1}^{\infty} E[Y_k] \qquad \text{(linearity of expectation)}$$

$$= \sum_{k=1}^{\infty} \Pr[Y_k = 1] \qquad \text{(expectation of indicator)}$$

$$\leq \sum \frac{n}{2^{\log_2 n + k - 1}}$$

$$\leq \sum_{k=1}^{\infty} \frac{\cancel{n}}{\cancel{n} \cdot 2^{k-1}} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots \leq 2$$

Properties of Skip Lists

Goal #2: Show that in expectation each operation (insert, delete find) touches a constant number of elements per level.

Combining both goals, by linearity of expectation we've proved the original goal:

> **Original Goal:** *For any fixed sequence of n operations* (insert, delete, find), the expected running time per operation is O(log n).

Properties of Skip Lists

Goal #2: Show that in expectation each operation (insert, delete find) touches a constant number of elements per level.

Say we are inserting (or deleting or finding) **y.**

Fix a level **i.** Let **Z** be the number of elements touched on level **i.**

Let **Zj** be an indicator variable for whether **ej** is touched (on level **i**)

$E[\mathbf{Z}] = 2 + E[\Sigma_j \mathbf{Zj}]$

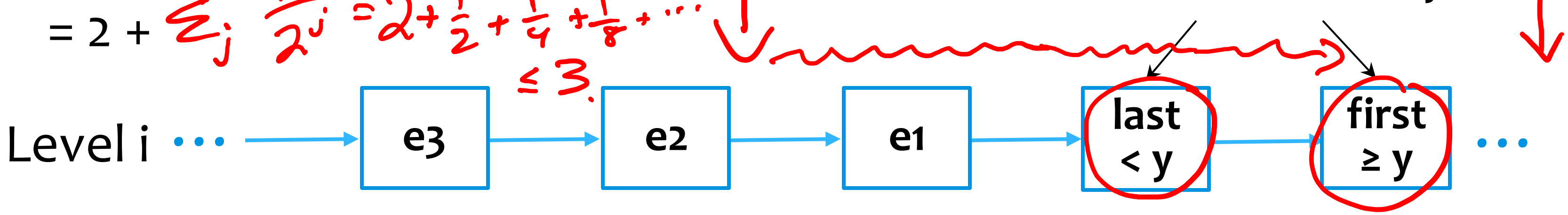
$= 2 + \Sigma_j E[\mathbf{Zj}]$ (linearity of expectation)

$= 2 + \Sigma_j \Pr[\mathbf{Zj}=1]$ (expectation of indicator)

$= 2 + \Sigma_j \frac{1}{2^j} = 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

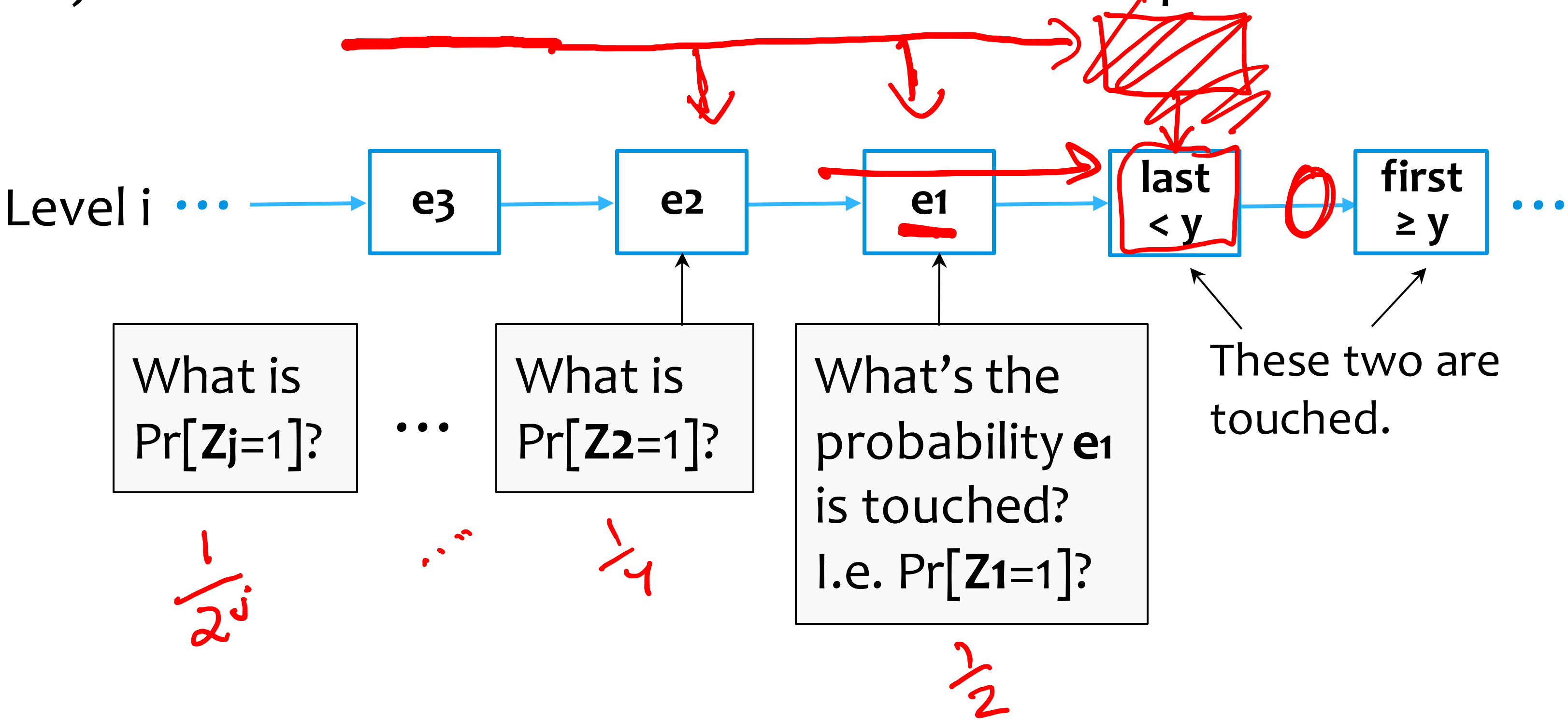
$\leq 3.$

Claim: These two are touched. Why?



Properties of Skip Lists

Goal #2: Show that in expectation each operation (insert, delete find) touches a <u>constant</u> number of elements per level.



What we showed

Goal #1: Show that the expected number of levels is O(log n).

Goal #2: Show that in expectation each operation (insert, delete find) touches a constant number of elements per level.

Original Goal: *For any fixed sequence of n operations* (insert, delete, find), the expected running time per operation is O(log n).

High-level takeaway:

- Often, deterministic data structures use a lot of bookkeeping to maintain the desired properties.

- With randomization, we stop micromanaging our data structure and use random choices that satisfy the desired properties on average.