Online Algorithms

Admin

Exam Review Session led by Daphne:
Wednesday April 24th, 7-9pm, BBB 1670.

HW11 is due Tuesday 4/23 at 8pm.

Reminder: Filling out the course evaluations is 1% of your grade, which is otherwise covered by the final exam. Remember to submit receipt on Gradescope.

It's the end of 376 as we know it

If you enjoyed this course:

Fall '24:

475: Introduction to Cryptography
477: Introduction to Algorithms
498: Advanced Data Structures
498/598: Algorithms for Machine Learning and Data Science
572: Randomness and Computation
575: Advanced Cryptography
598: Machine Learning Theory

Winter '25:

475: Introduction to Cryptography
477: Introduction to Algorithms
574: Computational Complexity Theory
598: Machine Learning Theory
598: Graph Algorithms (taught by me)

Also, consider applying to become an IA for 376

*recently some courses were renamed "CSE" instead of "EECS" so try both names

**When registering for grad-level courses, uncheck the box for "open courses only"

It's the end of 376 as we know it

If you really enjoyed this course:

Consider trying research in theoretical computer science (TCS)!

What is research?

Pushing the boundaries of human knowledge.

What is TCS research?

Well, it's pretty similar to the experience of working on 376 HW

… except you're working on the same problem for months/years and nobody knows the solution.

How to get involved in research:

- Research programs at UM:
  cse.engin.umich.edu/academics/undergraduate/undergraduate-resources-and-opportunities/undergraduate-research/
- Apply to **REU** programs
- Look up the research of professors at UM and reach out to them

A puzzle game made by UM alums!

Online Algorithms

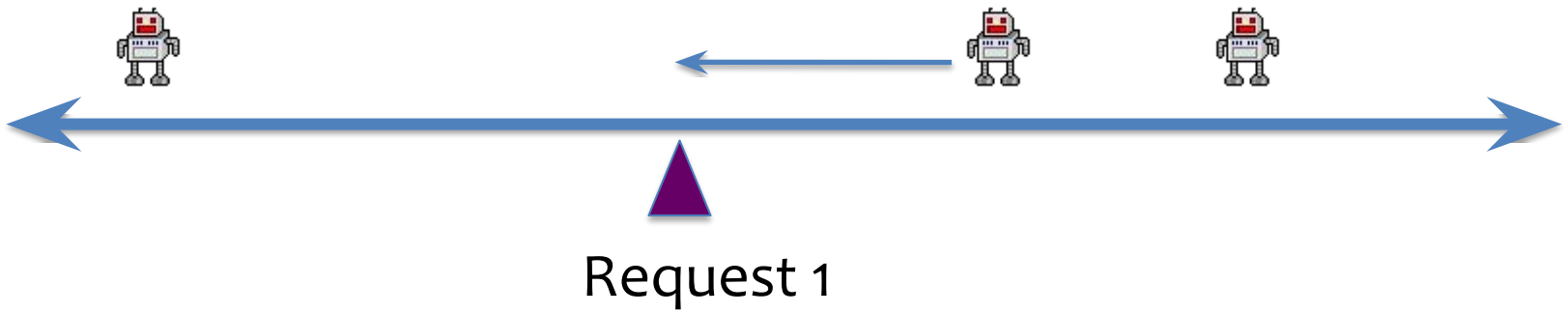
The input is revealed over time.

The algorithm needs to make decisions with partial information!

E.g. ridesharing, stock market, caching, ...

A classic problem in online algorithms:

The k-Server Problem



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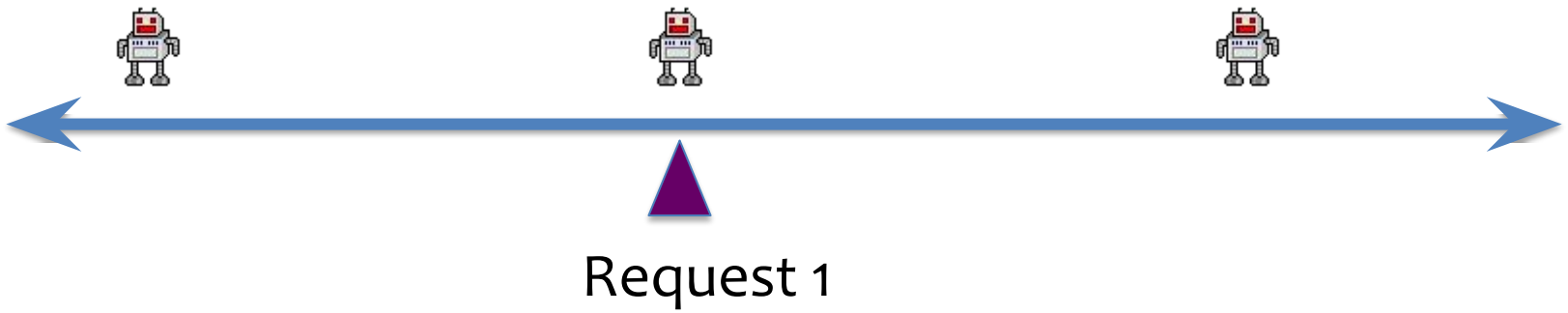

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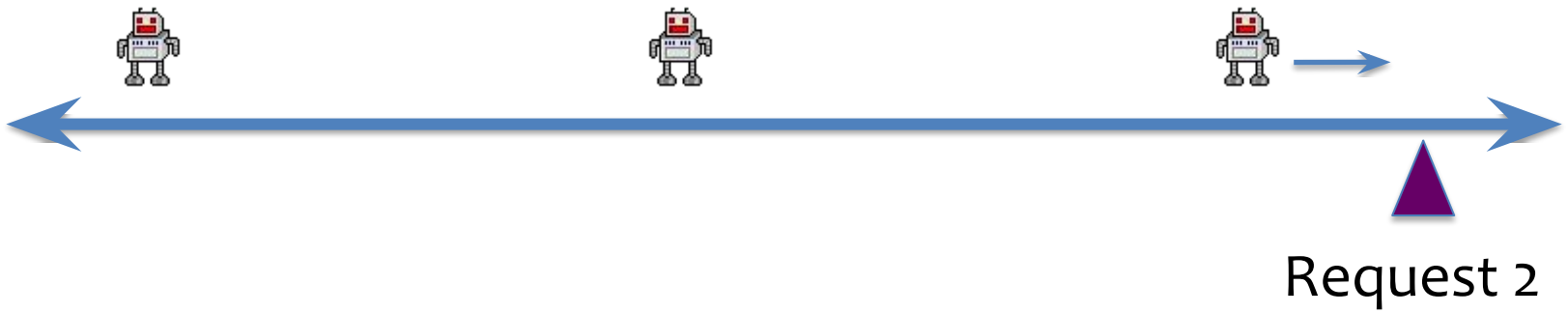

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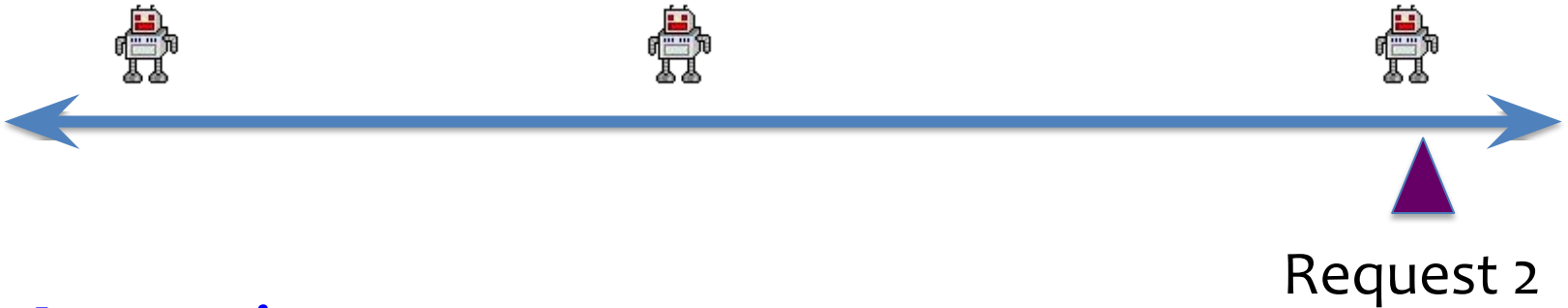


Assumptions:

- *k* identical robots. Any robot can service any request.
- Requests arrive one-at-a-time. A request must be serviced before the next request arrives.

Goal: Minimize total travel distance of all robots

Competitive Ratio

We measure the quality of an online algorithm by its ***competitive ratio***.

An online algorithm for the k-Server problem is **c-*competitive*** if its cost is *at most* **c** times the cost of an optimal ***offline*** algorithm i.e. an algorithm that *knows the entire request sequence in advance.*

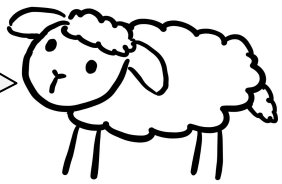
$$\mathbf{ALG} \leq \mathbf{c} \cdot \mathbf{OPT}, \quad \mathbf{c} > 1$$

Cost of our algorithm

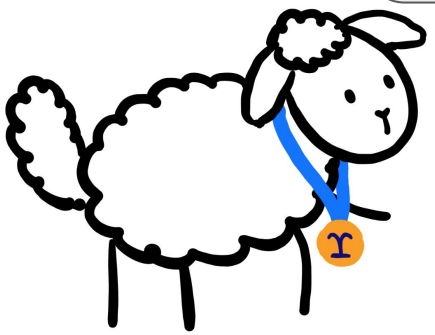
Cost of optimal ***offline*** algorithm (can see future)

c is called the ***competitive ratio*** (smaller is better here).

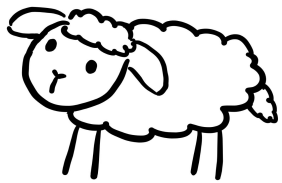
This is similar to approximation algorithms, but here OPT can see the future and ALG can't



Check out my greedy algorithm! Always dispatch the robot closest to the request point.



For your algorithm, there's a request sequence of length n, whose competitive ratio goes to infinity as n goes to infinity!



The Double Coverage (DC) Algorithm

The DC Algorithm (Chrobak, Karloff, Payne, Vishwanathan, 1990)**:**

- The closest robots on both sides of the request (if they exist) move towards the request at equal speed.
- Both stop moving once one of them reaches the request.
- If two or more robots are co-located, only one will move.

We will show: If there are k robots, DC is k-competitive!

It turns out k is the *best possible* competitive ratio for a deterministic algorithm. But you can do better with randomization (we won't show).

The Double Coverage (DC) Algorithm

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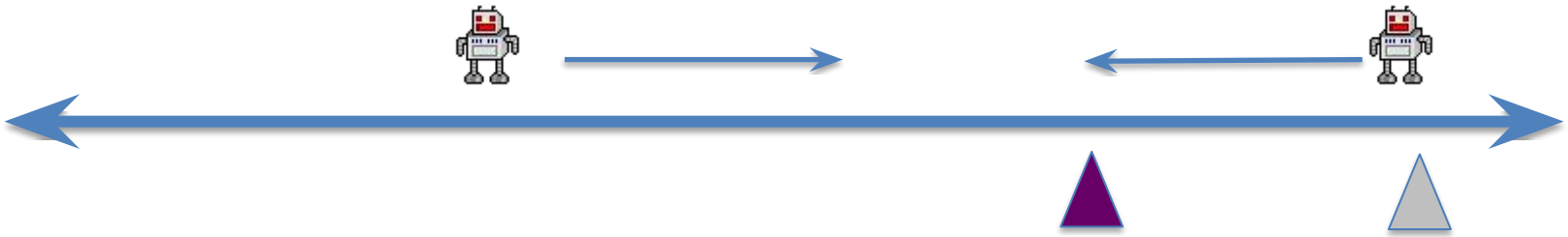

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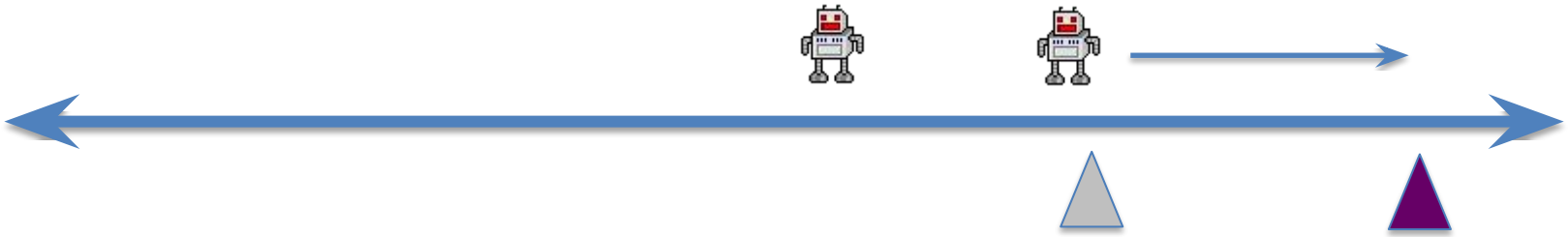

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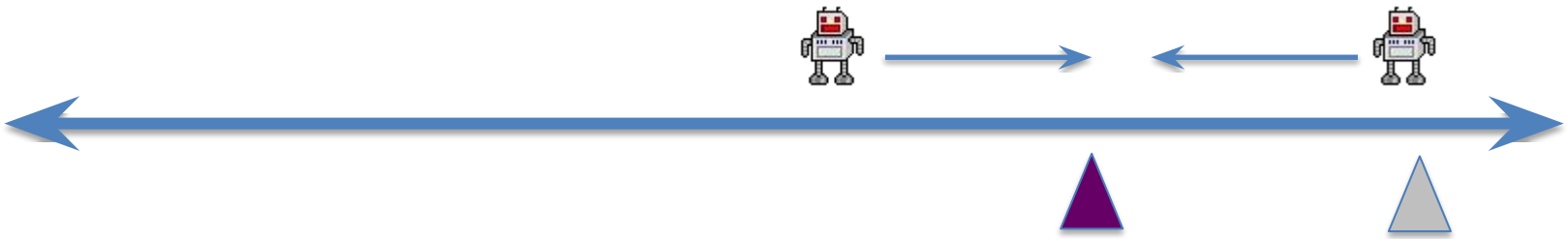
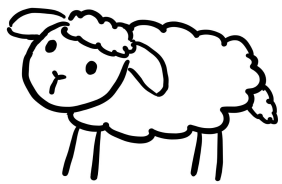


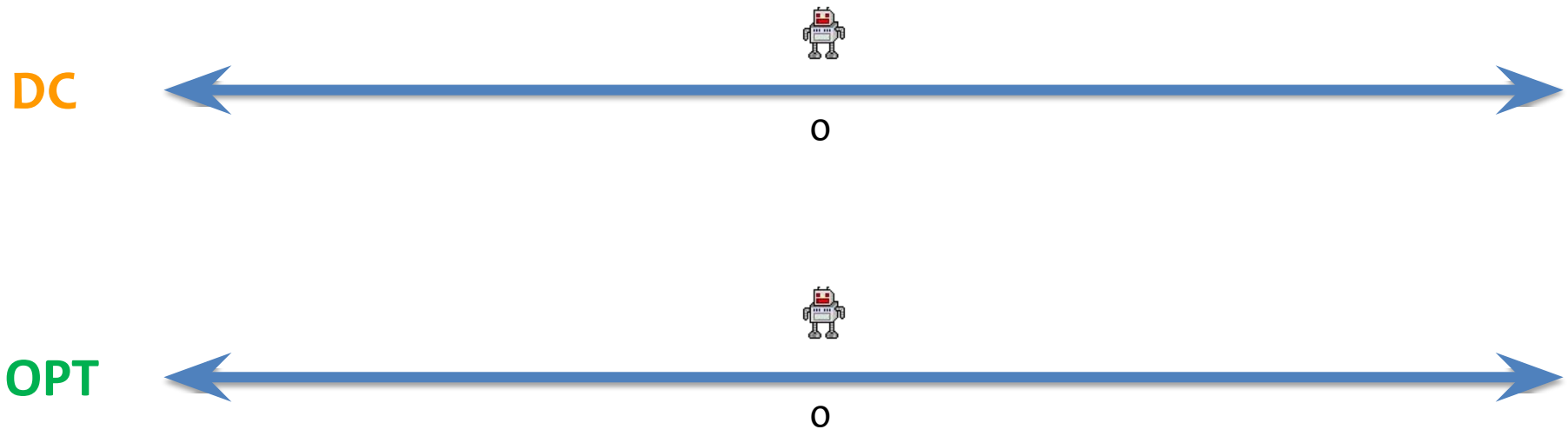
The Double Coverage (DC) Algorithm

The Double Coverage (DC) Algorithm

DC certainly looks better than the greedy algorithm

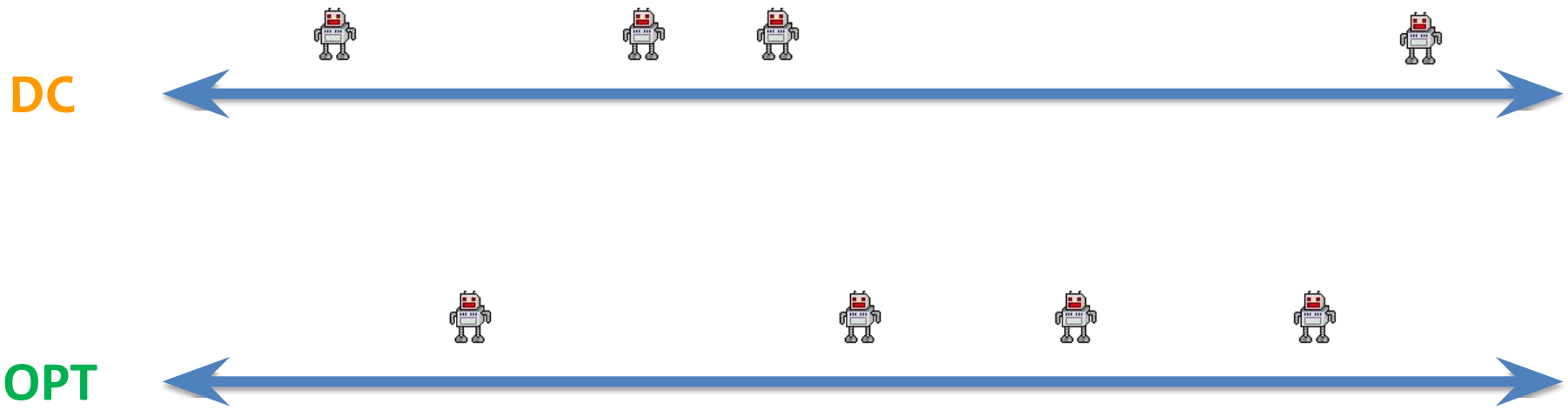


Goal: Show DC is k-competitive



Assumption: Initially, **DC** and **OPT** have all of their robots at the origin…

Goal: Show DC is k-competitive



… but over time, as they service requests, the robots spread out and the configurations of **DC** and **OPT** may start looking different

Claim: Without loss of generality, we can assume **OPT** only moves 1 robot per request. **Why?**

A Potential Function Argument

To prove that **DC** is k-competitive, we will use a **potential function!**

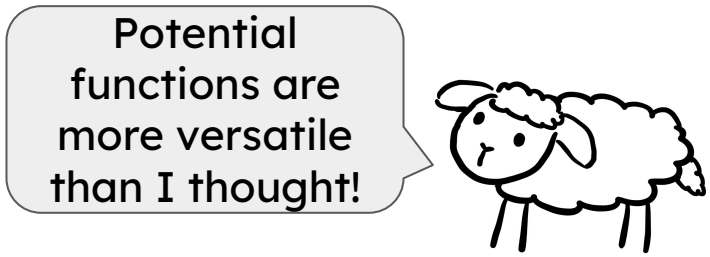
Previously: We used a potential function S_i to measure the ***running time*** of an algorithm.

⇒ Smaller S_i value means the algorithm is closer to termination.

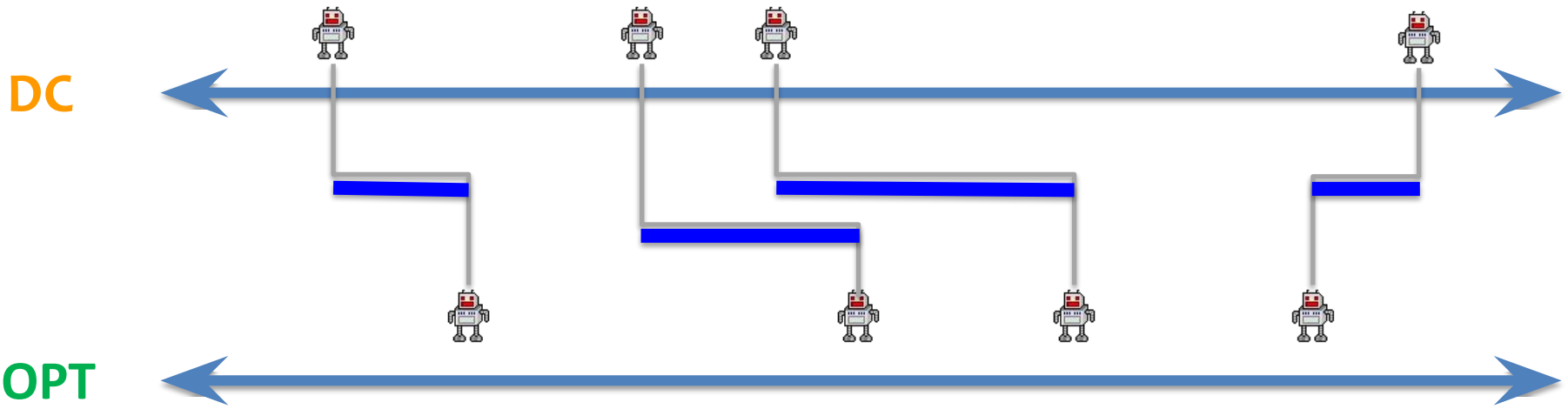
Now: We will use a potential function S_i to measure the ***competitive ratio*** of the **DC** algorithm.

⇒ Smaller S_i value means **DC** is in a "better configuration" ≈ closer to **OPT**.

(Here, S_i can go up and down over time.)



Potential Function: Attempt #1

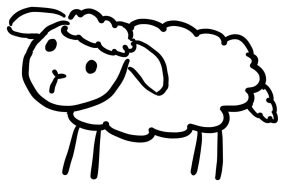


M = "match" the robots and take the total length of the **"horizontal legs"**

Potential function (attempt #1): S_i = **M**

(i.e. value of **M** right after i^{th} request has been serviced.)

This is a way of measuring how close DC is to OPT



How will we use our potential function?

Observations:

- $S_0 = 0$.
- $S_i \geq 0$ for all i.

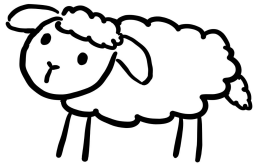
therefore $\Rightarrow$

total amount S_i *ever decreases*

$\leq$

total amount S_i *ever increases*

For each request, we measure the change in potential right after OPT moves and then again after DC moves!



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Key properties we want to show:

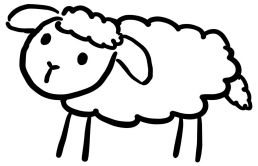
- For any given request, if **OPT** moves its server a distance of **d** then S_i **increases by ≤ k · d** (or decreases).
- For any given request, if **DC** moves its server(s) a total distance **d** then S_i **decreases by ≥ d**.

Consequence of key properties:

- total amount S_i *ever increases* ≤ **k · OPT**
- total amount S_i *ever decreases* ≥ **DC**

⇒ **DC ≤ k·OPT** (as desired)

For each request, we measure the change in potential right after OPT moves and then again after DC moves!



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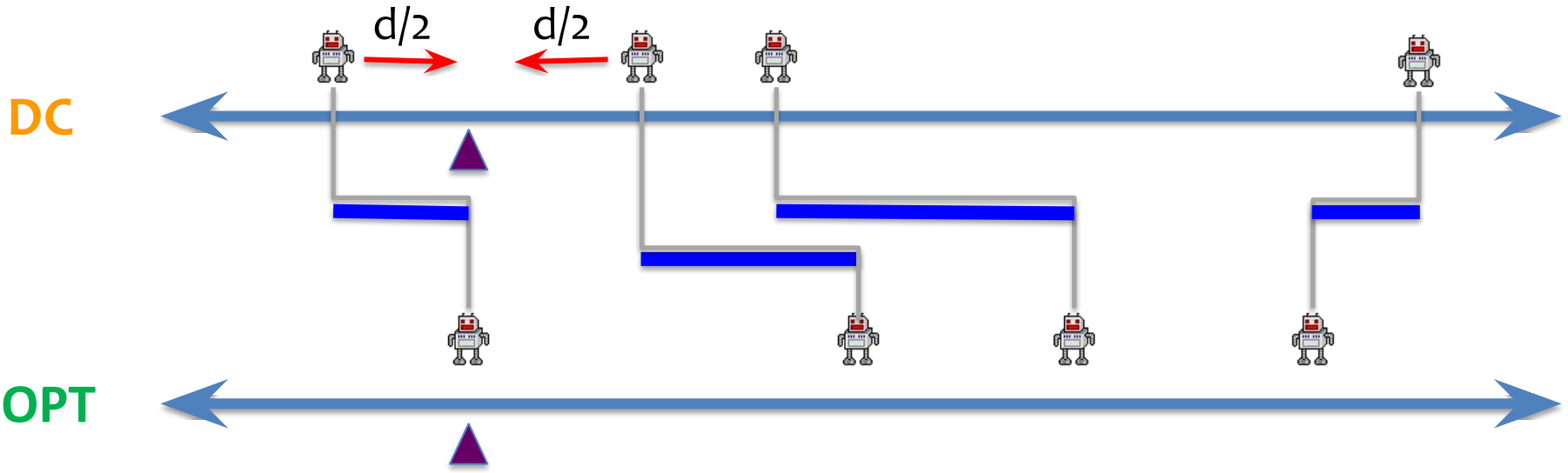
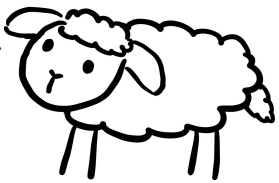


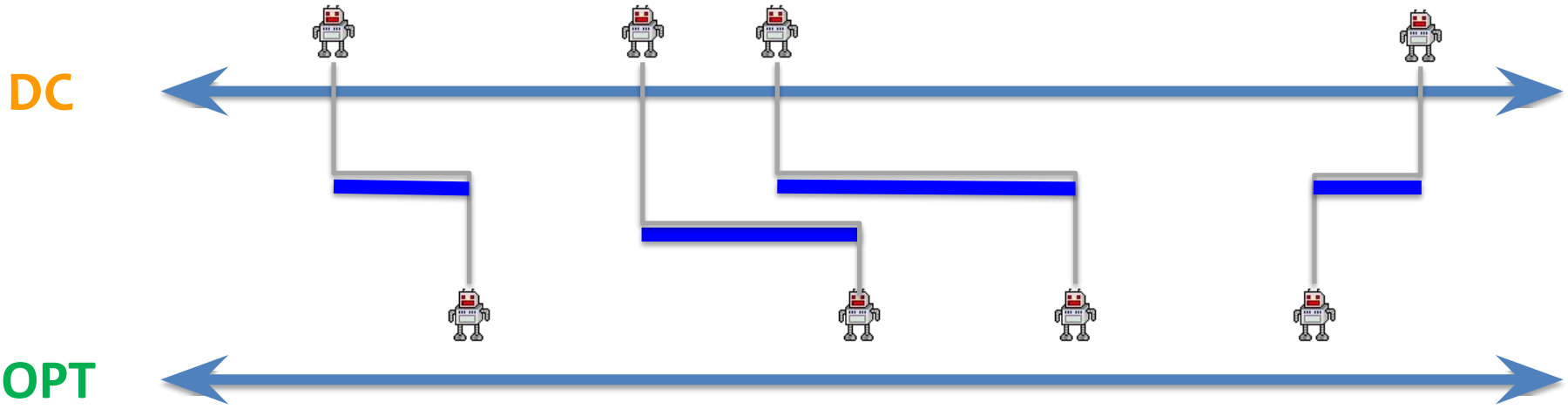
S_i (attempt #1) = **M** (= total length of "horizontal legs")

Want: if **DC** moves its server(s) a total distance **d** then S_i **decreases by ≥ d**.

oops, we need a new potential function



A New and Improved Potential Function

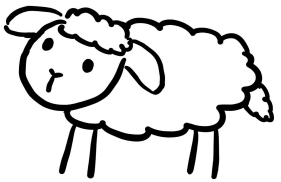


M = total length of the **"horizontal legs"**

Σ_{DC} = Sum of all pairwise distances between **DC**'s servers

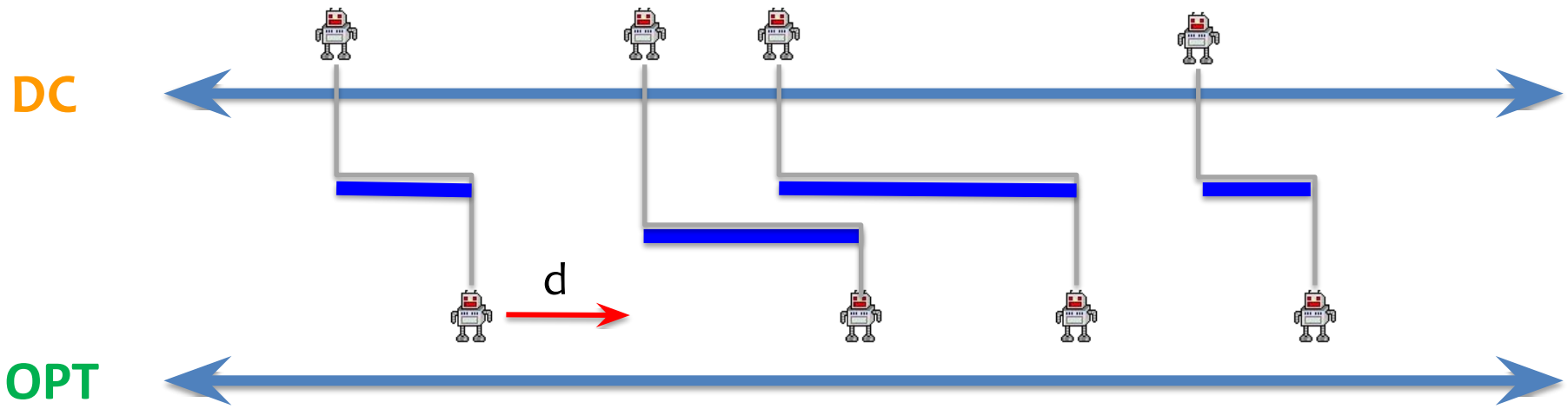
New potential function: $S_i = k \cdot M + \Sigma_{DC}$

What planet did this potential function come from?



Proving the Key Properties

Want to show: If **OPT** moves its server a distance of **d** then S_i **increases by ≤** $k \cdot d$ (or decreases)**.**

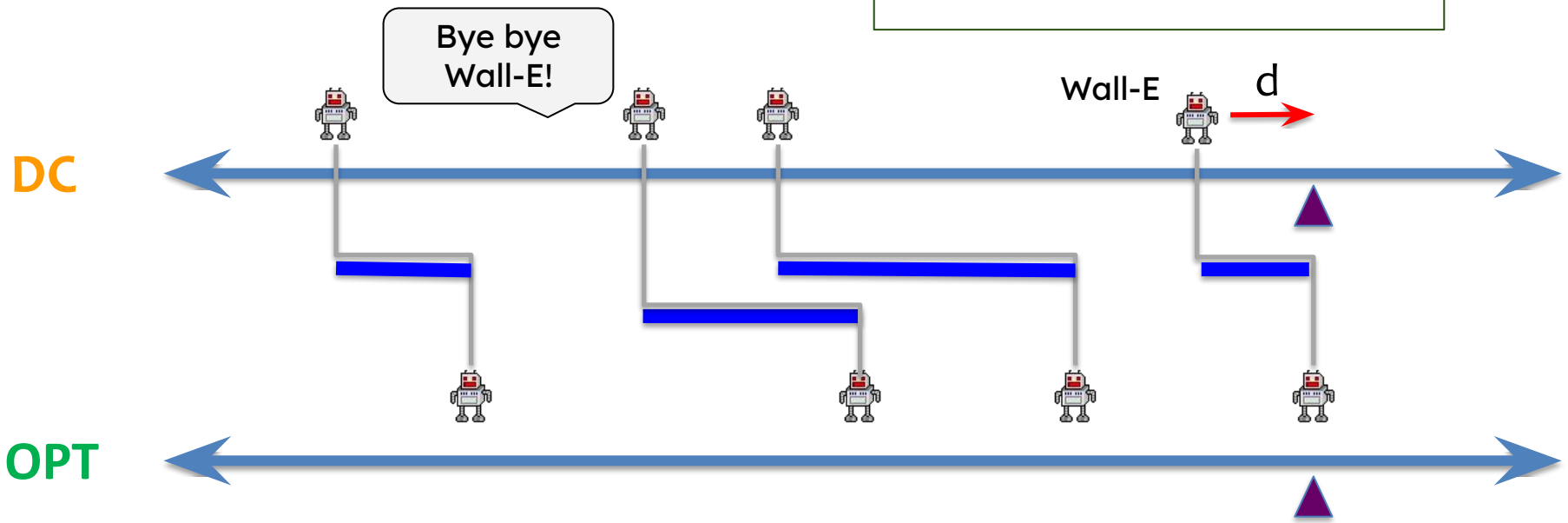


Potential function: $S_i = k \cdot M + \Sigma_{DC}$

Proving the Key Properties

Want to show: If DC moves its server(s) a total distance **d** then S_i **decreases by ≥ d.**

Case 1: DC moves 1 robot



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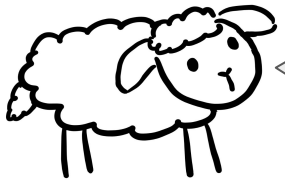
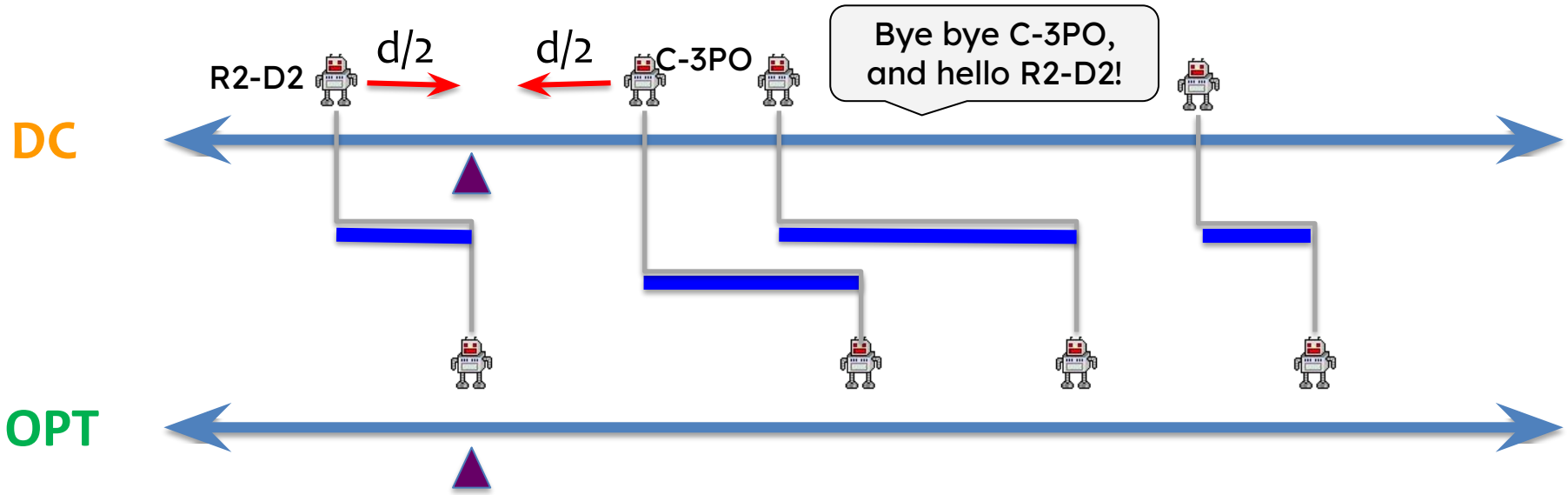


Note that OPT already has a robot at the request

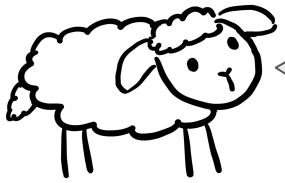
Proving the Key Properties

Want to show: If DC moves its server(s) a total distance **d** then S_i **decreases by ≥ d.**

Case 2: DC moves 2 robots



First, how does Σ_{DC} change?



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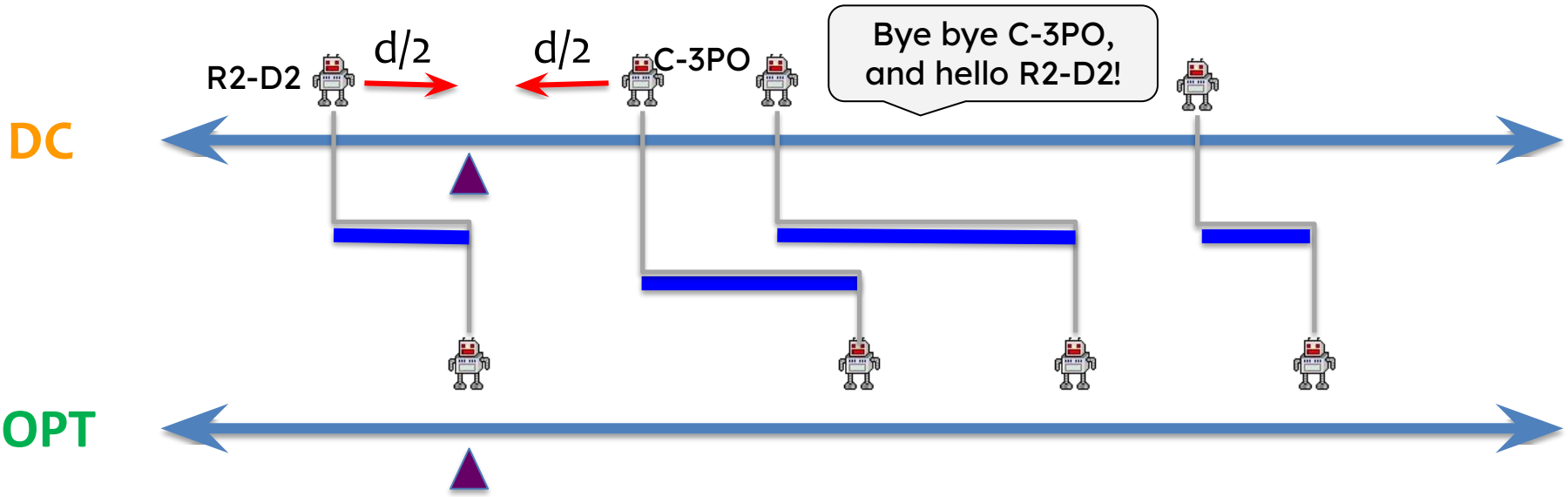

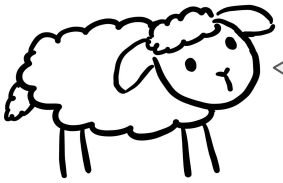

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Second, how does **M** change?

How we used our potential function

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therefore ⟹

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