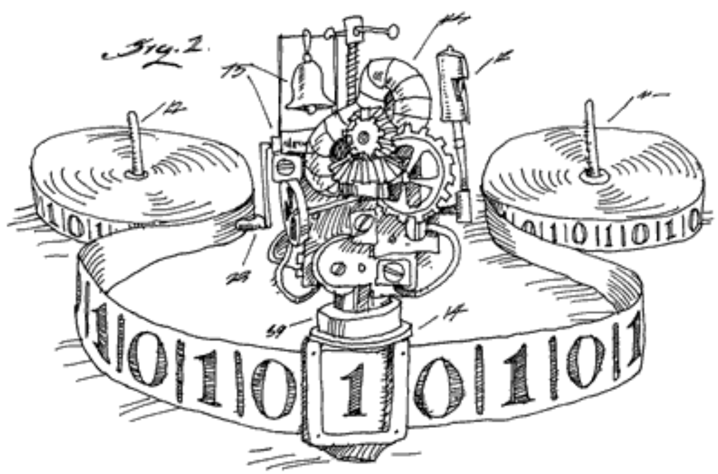
EECS 376: Foundations of Computer Science

Lecture 14 - Introduction to Complexity



More practical classification of problems

- Previously, we classified problems like this:
  - **Decidable:** solvable in finite time
  - **Undecidable:** not solvable in finite time

- "For practical purposes,
  the difference between polynomial and exponential
  is often more crucial than
  the difference between finite and non-finite."

- Jack Edmonds

(defined complexity class P, 1965)

- **Today and next class:** another classification
  - **P:** solvable in polynomial time
  - **NP-hard:** not likely solvable in polynomial time

Plan for this part of the course

Lecture 1:

- Define **P** and **NP**

Lecture 2

- Define **NP-hard** and **NP-complete.**
- Show the first NP-complete problem: SAT

Lectures 3 - 4

- Show many NP-complete problems via reductions

Lectures 5 - 6

- Show many methods to solve efficiently NP-hard Problems

Class **P:**

problems we can solve "fast"

Exponential vs. Polynomial

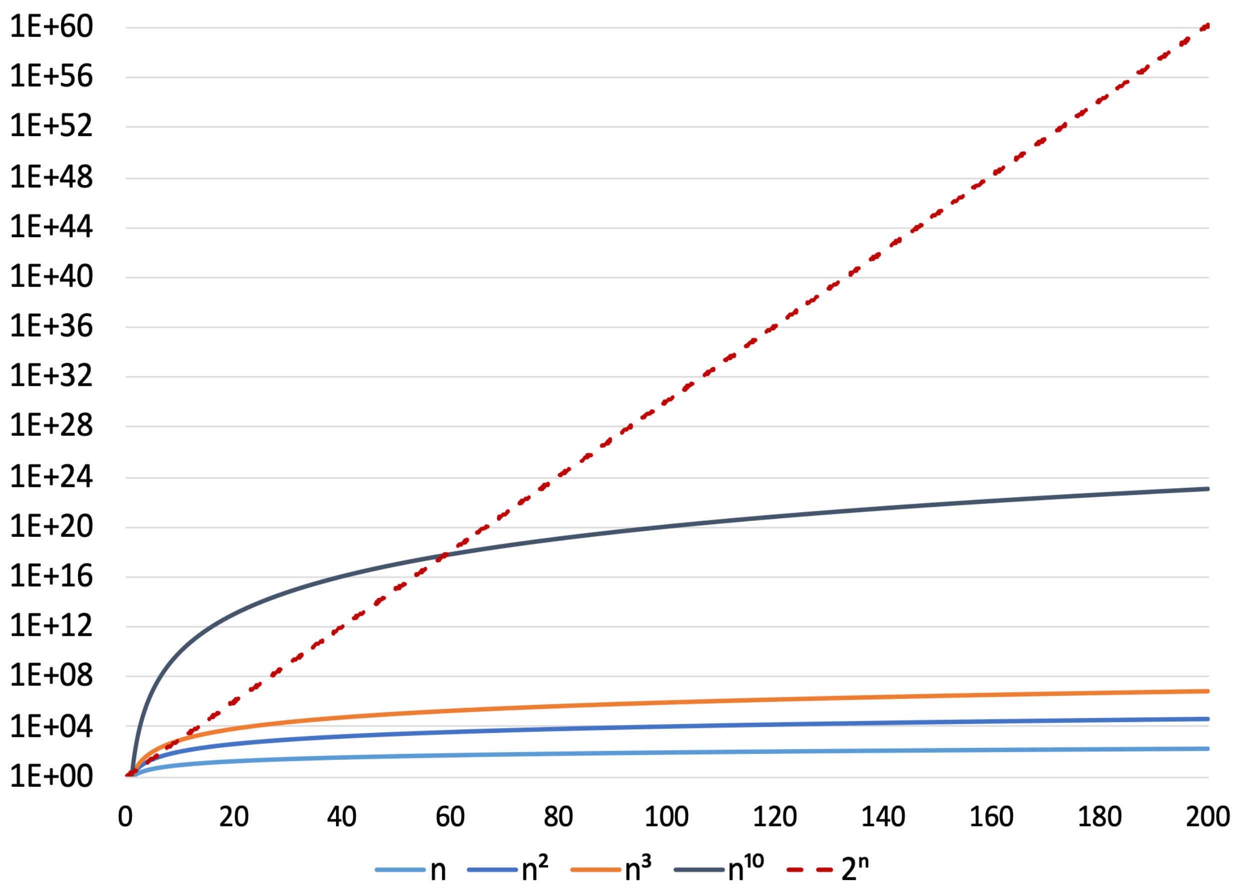
A regular Mac Pro computer performs about 10^{12} operations/sec

	n=10	**n=35**	**n=60**	**n=85**
n^2	100 < 1 sec	1225 < 1 sec	3600 < 1 sec	7225 < 1 sec
n^3	1000 < 1 sec	43k < 1 sec	216k < 1 sec	614k < 1 sec
2^n	1024 < 1 sec	34×10^9 < 1 sec	> 4 years	> 120 million years

"Efficient": running time polynomial in input size

Exponential vs. Polynomial

- Consider this plot with an **adjusted scaled**



The Complexity Class **P**

Definition:

- **P** is the set of all decision problems that can be decided in polynomial time. $\emptyset$

Formally:

- For any problem **L,** an efficient decider **Decide-L** for **L** is such that
  - **x** is a "yes" instance ⇔ **Decide-L**(**x**) accepts
  - **x** is a "no" instance ⇔ **Decide-L**(**x**) rejects (follows from above)
  - **Decide-L**(**x**) runs in poly(|**x**|) time

What we already know

- **P** is the set of all decision problems that have efficient deciders

Questions:

- Why polynomial? Why not $O(n^3)$? Why not $O(n)$?
- Why decision problems only?

Why do we use **Polynomial time** to capture the notion of efficiency?

- It is a robust definition.

- **Composable:**
  - If my program calls a polynomial number of polynomial-time algorithms, then my program runs in polynomial time
    - *Proof idea*: $(n^k)^{k'} = n^{k \cdot k'}$ is also polynomial.

- **Model-independent:** by Church-Turing thesis
  - A problem is solvable in polynomial time on a TM if and only if
  - it is solvable in polynomial time any computer.

Why do we restrict ourselves to only **Decision problems**?

- Decision problems are simpler
    - They also fit with the **language** formulation discussed in previous lectures

- To show that some problems are solvable in polytime,
    - Usually, via **binary search,**
      it is enough to check if the decision version is solvable in polytime
    - Let's see examples…

Decision version of problems

Shortest path

- Search version:
  - Given a graph and vertices s,t,
    what is the length of the shortest path from s to t?
- Decision version:
  - Given a graph, vertices s,t, and a **budget b**,
    is there a path from s to t of length **at most b**?

GCD

- Search version:
  - Given two numbers x and y, what is gcd(x,y)?
- Decision version:
  - Given two numbers x and y, and a **threshold b**, is gcd(x,y) **at most b**?

For these problems, if you can solve the decision version,
you solve the search version too. How?

Example: Solving search problems using decision problems

Suppose we have M where

- $M(x, y, b)$ accepts if $\gcd(x, y) \leq b$.
- M runs in polynomial time in the input size
  - Input size: log(x)+log(y)+log(b)

Goal: compute $\gcd(x, y)$ in polynomial time, i.e., poly(log(x)+log(y))

Bad approach:

- For $i = 0, \ldots, \min\{x, y\}$: if $M(x, y, i)$ accepts, return i.
- What's wrong?
  - It is correct, but…
  - It takes $\Omega(\min\{x, y\})$ iterations. Not polynomial in log(x)+log(y)

Good approach: **binary search** in the range of $[0, \ldots, \min\{x, y\}]$.

- Total time: poly(log(x)+log(y))

The Complexity Class **P**

Definition:

- **P** is the set of all decision problems that can be decided in polynomial time.

Formally:

- For any problem **L,** an efficient decider **Decide-L** for **L** is such that
  - **x** is a "yes" instance ⇔ **Decide-L**(**x**) accepts
  - **x** is a "no" instance ⇔ **Decide-L**(**x**) rejects (follows from above)
  - **Decide-L**(**x**) runs in poly(|**x**|) time

- **P** is the set of all decision problems that have an efficient decider non deterministic polynomial

Class NP:

problems we can verify "fast"

Common mistakes:

NP does **not** stand for "Not Polynomial"

What does **verify** mean?

- **Example 1:** Given a sudoku puzzle, is there a solution?
- **Answer:** Yes.
- **Reply:** We are not convinced (i.e. you could be lying to us).
- **Reply:** Now we are convinced.

			2	6		7		1
6	8			7			9	
1	9				4	5		
8	2		1				4	
		4	6		2	9		
	5				3		2	8
		9	3				7	4
	4			5			3	6
7		3		1	8			

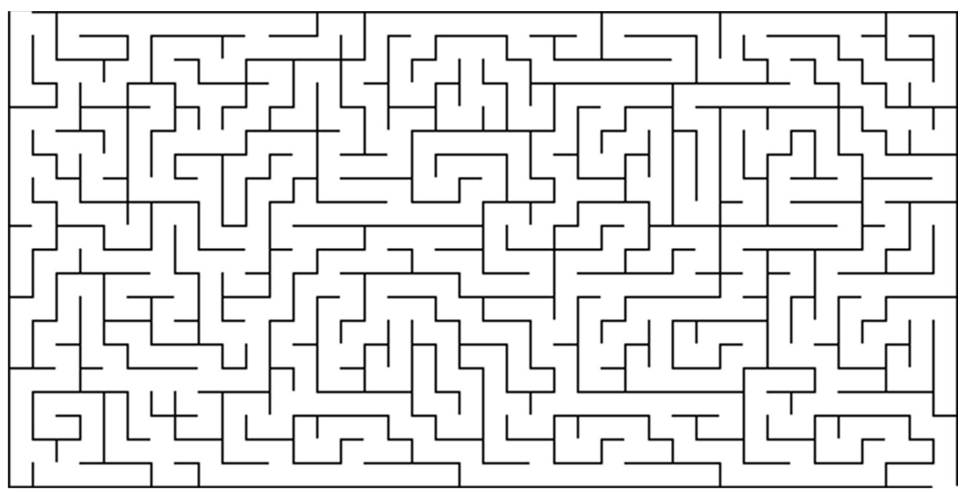
4	3	5	2	6	9	7	8	1
6	8	2	5	7	1	4	9	3
1	9	7	8	3	4	5	6	2
8	2	6	1	9	5	3	4	7
3	7	4	6	8	2	9	1	5
9	5	1	7	4	3	6	2	8
5	1	9	3	2	6	8	7	4
2	4	8	9	5	7	1	3	6
7	6	3	4	1	8	2	5	9

If there is no solution, we will not ever be convinced either.

What does **verify** mean?

Consider a maze.

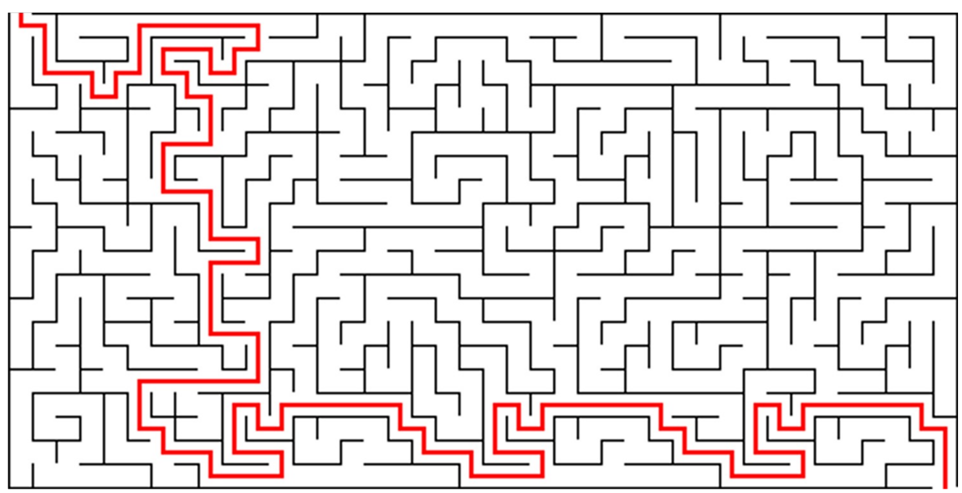
It might be hard to solve…



What does **verify** mean?

But if you give me the solution,

I can verify that it's a valid solution.



If there is no solution, we will not ever be convinced either.

The Complexity Class **NP**

Definition:

- **NP** is the set of all decision problems whose yes-instances can be verified in polynomial time.

Common mistakes: NP does **not** stand for "Not Polynomial"

- NP stands for "Nondeterministic Polynomial"
- We will not talk about non-determinism in this class though.

"A better name would have been **VP: verifiable** in polynomial time."

-Clyde Kruskal

The Complexity Class **NP**

Definition:

- **NP** is the set of all decision problems whose yes-instances can be verified in polynomial time.

Formally:

- For any problem **L**, an efficient verifier **Verify-L** for **L** is such that
  - **x** is a "yes" instance $\Leftrightarrow$ $\exists$**C** **Verify-L**(**x**, **C**) accepts
  - **x** is a "no" instance $\Leftrightarrow$ $\forall$**C** **Verify-L**(**x**, **C**) rejects (follows from above)
  - **Verify-L**(**x**, **C**) runs in poly(|**x**|) time

- **NP** is the set of all decision problems that have efficient verifiers
- If **Verify-L**(**x**, **C**) accepts, then **C** is called a certificate.

Intuition:

- If the input has a solution,
  then we can efficiently verify that given some *additional information*
- If there is *no solution*,
  then no additional information (even maliciously produced) could convince us

Nondeterministic Turing Machines

Definition:

A nondeterministic Turing machine is defined in the expected way. At any point in a computation, the machine may proceed according to several possibilities. The transition function for a nondeterministic Turing machine has the form

$$\delta: Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}).$$

The computation of a nondeterministic Turing machine is a tree whose branches correspond to different possibilities for the machine. If some branch of the computation leads to the accept state, the machine accepts its input.

otherwise: $\forall c$, $V(x, c)$ 都 rej

$\Rightarrow$ no.

The Running Time of a Deterministic Turing Machine

Definition:

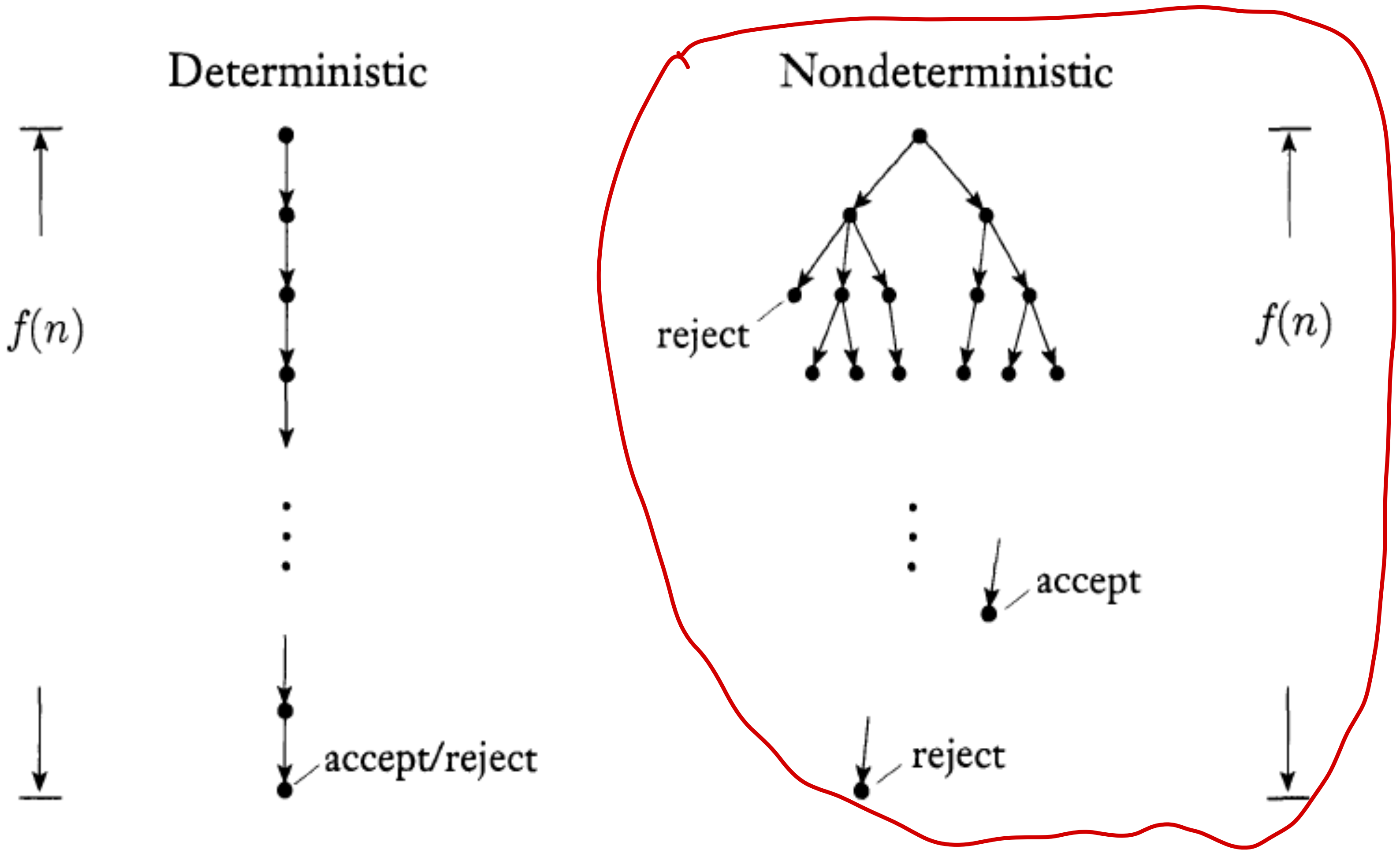
Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

The Running Time of a Nondeterministic Turing Machine

Definition:

Let N be a nondeterministic Turing machine that is a decider. The ***running time*** of N is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that N uses on any branch of its computation on any input of length n, as shown in the following figure.

Measuring Deterministic and Nondeterministic Running Time



Equivalent Definition of NP

Theorem:
A language is in NP if and only if some nondeterministic polynomial time Turing machine decides it.

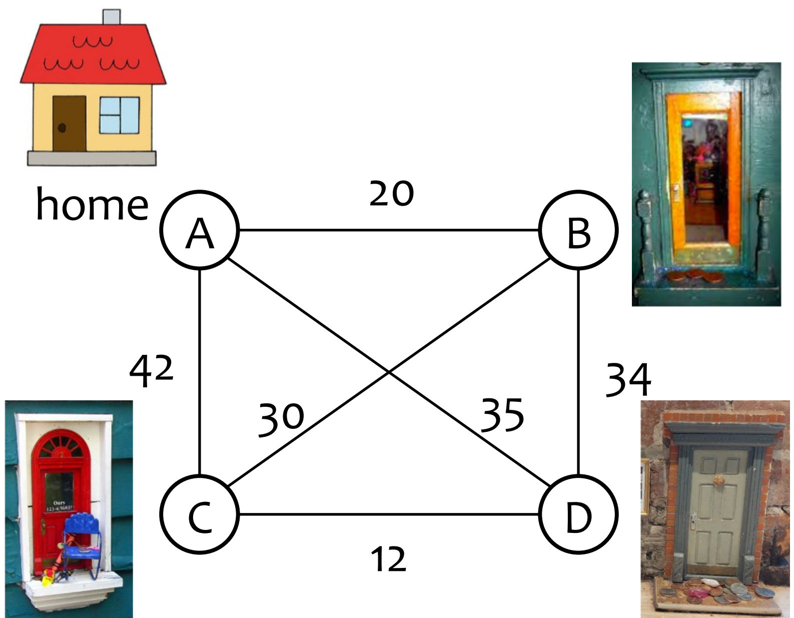
Prove that a problem is in **NP:**

Showing efficient verifiers

Traveling Salesperson Problem (TSP)

Input: n vertices, distances between each pair of vertices, budget b

Output: Is there a length ≤**b** cycle containing every vertex exactly once?



b=100

TSP is in NP

Recall:

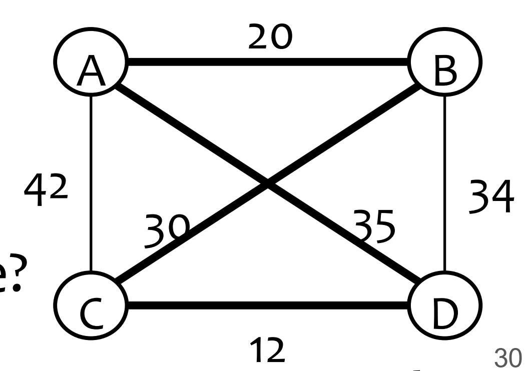
NP is the set of all decision problems **L** that have efficient verifiers **Verify-L**

- **x** is a "yes" instance ⇔ **∃C** **Verify-L**(**x, C**) accepts
- **Verify-L**(**x, C**) runs in poly(|**x**|) time

Example: TSP:

- **Certificate C:** Length ≤**b** cycle with all vertices.
- Efficient verifier **Verify-TSP**(**⟨G, b⟩**, **C**):
  - Is **C** a cycle in **G**?
  - Does **C** contain every vertex in **G** exactly once?
  - Do the edge weights of **C** add up to ≤**b**?

a "yes" instance



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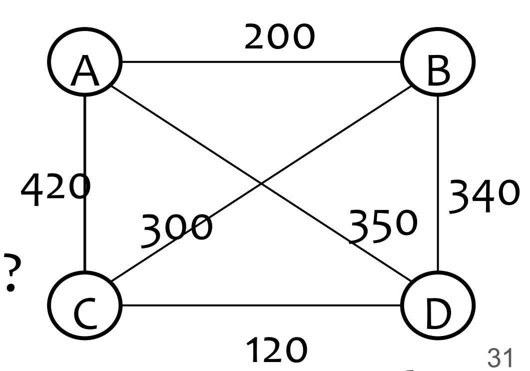
a "no" instance



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TSP is in NP

Recall:

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Example: TSP:

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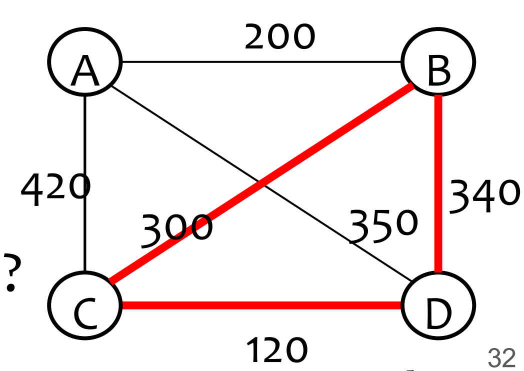
a "no" instance



b=100

To show that a problem is in **NP**, you need to specify (e.g. for the HW):

1. Certificate
2. Efficient verifier
3. Proof of correctness of verification algorithm

Example: TSP:

1. **Certificate C:** Length ≤**b** cycle containing every vertex.
2. Efficient verifier: **Verify-TSP**(**⟨G, b⟩**, **C**):
   ~~Is C a cycle in G?~~
   Does C contain every vertex in G exactly once?
   Do the edge weights of C add up to ≤**b**?
   Accept if all 3 answers are "yes"
   (you'd need to analyze the running time)
3. TSP "yes" instance ⇒ exists a length ≤**b** cycle containing every vertex ⇒ ∃C **Verify-TSP**(**⟨G, b⟩**, **C**) accepts

   TSP "no" instance ⇒ no length ≤**b** cycle containing every vertex ⇒ ∀C **Verify-TSP**(**⟨G, b⟩**, **C**) rejects

Useful fact: Certificate has poly length

Reminder:

- For any problem **L,** an efficient verifier **Verify-L** for **L** is such that
  - **x** is a "yes" instance ⇔ ∃**C** **Verify-L**(**x, C**) accepts
  - **Verify-L**(**x, C**) runs in poly(|**x**|) time

- If **Verify-L**(**x, C**) accepts, then **C** is called a certificate.

Claim: without loss of generality, we can assume |**C**| ≤ poly(|**x**|).

Proof:

- **Verify-L**(**x, C**) runs in poly(|**x**|) time.
- It reads only poly(|**x**|) symbols of **C**. So, we can remove the rest.

Subset Sum is in NP

Input: a set S of integers and a target t.

Output: Is there a subset of the integers in S whose sum is exactly t?

Prove that Subset Sum is in **NP**.

- Certificate: a subset **C** ⊆ S whose sum is t.
- Verifier: **Verify**(**S, t, C**): Check that **C** ⊆ S and the sum of **C** is t.
- Analysis:
  - (**S, t**) is a "yes" instance ⇔ ∃**C** **Verify**(**S,t, C**) accepts
  - **Verify**(**x, C**) runs in poly(**|S| log t**) time

Terminology on Satisfiability (SAT)

A **Boolean formula** Φ is made up of:

- "literals": variables and their negations (e.g. $x, y, z, \neg x, \neg y, \neg z$)
- OR: $\vee$
- AND: $\wedge$

Example:

$$\Phi 1 = (x \vee y) \wedge (\neg y \vee x \vee \neg z) \wedge (\neg x \vee (y \wedge \neg z))$$

Φ is ***satisfiable*** if

- $\exists$ a true/false assignment **A** to the variables so that Φ(**A**) = true
- For example, $\Phi 1$ is satisfiable.
    - Assign $x = F$, $y = T$, $z = F$

SAT is in NP

Input: A Boolean formula Φ

Output: Is Φ *satisfiable*?

Prove that SAT is in **NP**.

- Certificate: a true/false assignment **C** to variables where Φ(**C**) = true
- Verifier: **Verify**(Φ, **C**): Check that Φ(**C**) = true
- Analysis:
  - Φ is a "yes" instance $\Leftrightarrow$ **∃C** **Verify**(Φ**,** **C**) accepts
  - **Verify**(Φ**,** **C**) runs in poly($|\Phi|$) time

THE Major Open Problem in Computer Science

$$P \stackrel{?}{=} NP$$

"Is every efficiently verifiable problem also efficiently solvable?"

"If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in 'creative leaps', no fundamental gap between solving a problem and recognizing the solution once it's found."

- Scott Aaronson

Two Possible Worlds