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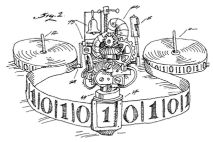
EECS 376: Foundations of Computer Science

Lecture 05 - Dynamic Programming 2



Dynamic Programming Review

- **Step 1:**
  Write a ***recursive formulation*** of the solution.
  Bound the number of ***distinct subproblems*** that ever appear in your formulation
- **Step 2:**
  Create a table representing distinct subproblems.
  Fill in the table from the **bottom-up.**
- **Runtime:** (#subproblem × time per subproblem)

Longest Common Subsequence

Motivation: DNA Comparison

- Your DNA is a (*long*) string over {A, T, C, G}.
- "Humans and chimps are 98.9% similar."
  - X: ACCGGTCGAGTGCGCGGAAGCCGGCCGAA
  - Y: GTCGTTCGGAATGCCGTTGCTCTGTAA
- The length of the *longest common subsequence* between two genomes is a measure of *similarity*.

Longest Common Subsequence

- Given strings $X[1..m]$ and $Y[1..n]$
- **Goal:** find the *length* of a ***longest common subsequence*** of X and Y
  - A **subsequence** of X is a string obtainable from X by deleting chars (may not be consecutive)
  - A **common subsequence** of X and Y is a subsequence of both X and Y
- **Example:**
  - "CT" is a common subsequence of "CGATG" and "CATGT".
  - **Q:** What's the longest?
- **Q:** What's a brute force solution?
  - Each character of X and Y is either deleted or not: Runtime: $O(2^{m+n})$

Recurrence for LCS: First, define the function

- **Def**: $LCS(i, j)$ = *length* of a LCS of $X[1..i]$ and $Y[1..j]$.
  - $i = 0$ means X is the empty string
  - $j = 0$ means Y is the empty string
- **Goal**: $LCS(m, n)$ (compute the length of LCS, will compute LCS itself soon)
- **Example:** Suppose X = "ATGCC" and Y = "TAGC".
  - **Q:** What's $LCS(1,0)$? 0
  - **Q:** What's $LCS(5,3)$? 2 TG or AG
  - **Q:** What's $LCS(4,4)$? 3 TGC or AGC
  - **Q:** What's $LCS(5,4)$? 3 TGC or AGC

Recurrence for LCS

- **Def**: $LCS(i, j)$ = *length* of a LCS of $X[1..i]$ and $Y[1..j]$.
  - $i = 0$ means X is the empty string
  - $j = 0$ means Y is the empty string
- **Goal**: return $LCS(m, n)$
- What is a recursion for $LCS(i, j)$?

$$LCS(i,j) = \begin{cases} ? & \text{if } i = 0 \text{ or } j = 0 \\ \text{Pause and Think...} & \text{if } X[i] = Y[j] \\ & \text{if } X[i] \neq Y[j] \end{cases}$$

Recurrence for LCS

Def: $LCS(i, j)$ = *length* of a LCS of $X[1..i]$ and $Y[1..j]$.

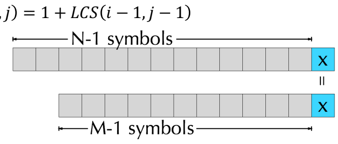
(所有 op sol 中的 char 数都一样, 因而可以放心删

那么有某个optimal sol 中包含

因而把它delete掉 (即 i,j--)

length ++

- **Case 1:** $X[i] = Y[j]$ (ends with the same character)
  - **Example:** $X[1..i]$ = "CTGCA" and $Y[1..j]$ = "TCGA"
  - **Claim.** There exists an optimal solution OPT that matches X[i] and Y[j]. 然后看下一个
  - **Proof by contradiction.**
    - Suppose for contradiction: all optimal sol OPT do not match X[i]=Y[j] (Say, X[i] = "A").
    - OPT + "A" is an LCS too. But OPT + "A" is longer than OPT.
    - So, OPT is not optimal, contradiction.
  - $LCS(i, j) = 1 + LCS(i - 1, j - 1)$



因为这是最后一个字符，如果两个都在，那么它们一定一样的

那么X[i]和Y[j]中到有个不在optimal sol中

Recurrence for LCS

Def: $LCS(i,j)$ = *length* of a LCS of $X[1..i]$ and $Y[1..j]$.

$$LCS(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ 1 + LCS(i-1, j-1) & X[i] = Y[j] \\ ? & X[i] \neq Y[j] \end{cases}$$

- **Case 2:** $X[i] \neq Y[j]$ (end with different characters)
  - **Example:** $X[1..i]$ = "GTCA" and $Y[1..j]$ = "GTC"
  - So, either X[i] or Y[j] is not part of LCS
  - **Q:** How do we know which one?
    - Try both! And take the better one

选取better one：即到底哪个在optimal sol中（可以都不在）。即：去掉X[i]大

- $LCS(i,j) = \max\{LCS(i-1,j), LCS(i,j-1)\}$ 还是去掉Y[j]大

Recurrence for LCS

- **Def**: $LCS(i,j)$ = *length* of a LCS of $X[1..i]$ and $Y[1..j]$.
  - $i = 0$ means X is the empty string
  - $j = 0$ means Y is the empty string
- **Goal**: return $LCS(m,n)$
- We have: $LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ 1 + LCS(i-1,j-1) & \text{if } X[i] = Y[j] \\ \max\begin{Bmatrix} LCS(i-1,j), \\ LCS(i,j-1) \end{Bmatrix} & \text{if } X[i] \neq Y[j] \end{cases}$
- **Q:** How many subproblems does this recurrence generate? mn $(|X||Y|)$
- **Q**: how much time does it take per subproblem? $O(1)$

Example

Table of subproblems: **2-dimensional!**

Task: compute LCS of
- "APOCRYPHAL"
- "POLYPEPTIDE"

Col: prefix of "APOCRYPHAL"

Row: prefix of "POLYPEPTIDE"

LCS between empty strings is 0

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0										
O	0										
L	0										
Y	0										
P	0										
E	0										
P	0										
T	0										
I	0										
D	0										
E	0										

Table Filling

$$LCS(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ 1 + LCS(i-1,j-1) & X[i] = Y[j] \\ \max\begin{Bmatrix} LCS(i-1,j), \\ LCS(i,j-1) \end{Bmatrix} & X[i] \neq Y[j] \end{cases}$$

- $LCS(X,Y)$

table = **2D-array**, indexed from 0 to |X| and 0 to |Y|

table$[i,j]$ stores LCS(X[1 … i], Y[1 … i])

For $i = 0, \ldots, |X|$: table$[i, 0] = 0$ Base Case

For $j = 0, \ldots, |Y|$: table$[0, j] = 0$

For i, j **in which order?**

…

Return ??

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0										
O	0										
L	0										
Y	0										
P	0										
E	0										
P	0										
T	0										
I	0										
D	0										
E	0										

Table Filling

$$LCS(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ 1 + LCS(i-1,j-1) & X[i] = Y[j] \\ \max\begin{Bmatrix} LCS(i-1,j), \\ LCS(i,j-1) \end{Bmatrix} & X[i] \neq Y[j] \end{cases}$$

$O(|X||Y|)$ operations total

- $LCS(X,Y)$

table = **2D-array**, indexed from 0 to |X| and 0 to |Y|

table$[i,j]$ stores LCS(X[1 … i], Y[1 … i])

For $i = 0, \ldots, |X|$: table$[i, 0] = 0$ Base Case

For $j = 0, \ldots, |Y|$: table$[0, j] = 0$

For $i = 1, \ldots, |X|$: $O(|X||Y|)$ rounds of loop

  For $j = 1, \ldots, |Y|$:

    if $X[i] = Y[j]$ **then**

      table$[i][j] \leftarrow 1 +$ table$[i-1][j-1]$

    else

      table$[i][j] \leftarrow \max\{$table$[i-1][j]$, table$[i][j-1]\}$

Return table$[|X|, |Y|]$ $O(1)$ operations/loop

Longest Common Subsequence, via Backtracking

Input: strings X and Y

Output: a longest common subsequence of the strings

function LCS(X[1 … |X|], Y[1 … |Y|])

table = **2D-array**, indexed from 0 to |X| and 0 to |Y|

table$[i,j]$ stores LCS(X[1 … i], Y[1 … j])

$s = \varepsilon$ ◁ the empty string

i = |X|, j = |Y|

while i>0 and j>0 **do**

  if X[i] = Y[j] **then**

    s = X[i] + s

    $i = i - 1$, $j = j - 1$

  else if table[i][j − 1] > table[i − 1][j] **then**

    j=j−1

  else

    i=i−1

return s

Example

Task: compute LCS of
- "APOCRYPHAL"
- "POLYPEPTIDE"

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	1	1	1	1	1	?			
O	0										
L	0										
Y	0										
P	0										
E	0										
P	0										
T	0										
I	0										
D	0										
E	0										

=1+0 max(1,0)

Example

Task: compute LCS of
- "APOCRYPHAL"
- "POLYPEPTIDE"

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	1	1	1	1	1	1	1	1	1
O	0	0									
L	0										
Y	0										
P	0										
E	0										
P	0										
T	0										
I	0										
D	0										
E	0										

?=1+0 =1+0 =max(1,0)

考虑了X中的P和每个Y中的P刚match时的情况

Example

Task: compute LCS of
- "APOCRYPHAL"
- "POLYPEPTIDE"

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	1	1	1	1	1	1	1	1	1
O	0	0	1	2							
L	0			1+1							
Y	0										
P	0										
E	0										
P	0										
T	0										
I	0										
D	0										
E	0										

Example

Task: compute LCS of
- "APOCRYPHAL"
- "POLYPEPTIDE"

Q: What is the length of LCS?
A: 4

Q: What is the LCS itself?
A: POYP

Exercise: Think how to edit the program so that it can return the LCS itself.

		A	P	O	C	R	Y	P	H	A	L
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	1	1	1	1	1	1	1	1	1
O	0	0	1	2	2	2	2	2	2	2	2
L	0	0	1	2	2	2	2	2	2	2	3
Y	0	0	1	2	2	2	3	3	3	3	3
P	0	0	1	2	2	2	3	4	4	4	4
E	0	0	1	2	2	2	3	4	4	4	4
P	0	0	1	2	2	2	3	4	4	4	4
T	0	0	1	2	2	2	3	4	4	4	4
I	0	0	1	2	2	2	3	4	4	4	4
D	0	0	1	2	2	2	3	4	4	4	4
E	0	0	1	2	2	2	3	4	4	4	4

Time and Space Complexity

- How efficient is this algorithm? Computing a single table entry requires a constant number of operations. Since there are $(N+1)\cdot(M+1)$ entries, constructing the table takes $O(NM)$ time and requires $O(NM)$ space.
- Backtracking also does a constant number of operations per entry on the path, and the path length is at most $N + M + 1$, so backtracking takes $O(N + M)$ time.
- Thus, the total complexity of this algorithm is $O(NM)$ in both time and space.

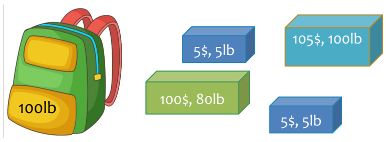
0-1 KNAPSACK

0-1 Knapsack Problem

Input:
- n items $t_1, \ldots t_n$: each with value v_i and weight w_i.
- W: the total capacity of the 0-1 knapsack.
- All v_i, w_i, W are integers.

$t_i = \{v_i, w_i\}$ value weight

Output: A set of items $S \subseteq \{1, \ldots, n\}$ such that
- (not too heavy) $\sum_{i \in S} w_i \le W$
- (max value) $\sum_{i \in S} v_i$ as large as possible



$Knapsack(\{t_1, \ldots, t_n\}, W)$

Recursive Strategy: Take t_n?

Reject: $val_{rej} = Knapsack(\{t_1, \ldots, t_{n-1}\}, W)$

Accept: $val_{acc} = v_n + Knapsack(\{t_1, \ldots, t_{n-1}\}, W - w_n)$

reject: 不拿ti，不损失weight，取i-1个物品时的最优解.

- Do not know whether optimal solution actually has t_n
  - Again, try both and take the better one.
- **The recurrence:**

$$Knapsack(\{t_1, \ldots, t_n\}, W) = \max \begin{cases} Knapsack(\{t_1, \ldots, t_{n-1}\}, W) \\ v_n + Knapsack(\{t_1, \ldots, t_{n-1}\}, W - w_n) \end{cases}$$

Valid only when $W \ge w_n$

- **The base cases:**
  - $Knapsack(\emptyset, W') = 0$ for all W'
  - $Knapsack(\{t_1, \ldots, t_i\}, 0) = 0$ for all i

Accept: 拿ti，但是要牺牲一点weight，因而是fill满 W-wi 的最优解 + 重为wi 的 ti 的 vi

- What is the size of the table?
- What is time spent per cell?

t_1: $V_1 = 3, W_1 = 1$; t_2: $V_2 = 5, W_2 = 4$; t_3: $V_3 = 6, W_3 = 4$

$W = 7$

	i=0	i=1	i=2	i=3
w=0	0	0	0	0
W=1	0	m(0,3) =3	m(3,-∞) =3	m(5,-∞)=5
w=2	0	3	3	5
w=3	0	3	3	5
w=4	0	3	m(3,4) =4	5
w=5	0	3	max(3,5) =5	m(5,3+6)=9 d[2,1]=3
W=6	0	3	5	9
W=7	0	3	5	9

Knapsack Solution

- $Knapsack(\{t_1, \ldots, t_n\}, W)$ — O(nW) ops

table = **2D-array**, indexed from 0 to n and 0 to W

table$[i, W']$ stores subproblem Knapsack $(\{t_1, \ldots, t_i\}, W')$

For j = $0, \ldots, n$:
- table$[j, 0] = 0$ — Base Case 1: no weight to carry → no value

For k = $0, \ldots, W$:
- table$[0, k] = 0$ — Base Case 2: no items to choose from → no value

For j = $1, \ldots, n$: — O(nW) loops

For k = $1, \ldots, W$:

reject = table[j − 1, k]

accept = $(v_j + \text{table}[j-1, k - w_j])$ if $k \ge w_j$ else $-\infty$

table[j, k] = max{reject, accept}

Return table[n, W] — O(1) operations/loop

Is this polynomial in input size?

- What is the "input size" in term of n and W?
- Is there poly$(n, \log W)$-time algorithm?
  - No, unless $P = NP$.
  - You will learn about "$P\ vs.\ NP$" after the midterm.

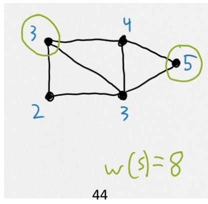
Wrap Up

- We have seen more examples of dynamic programming
- **Next**:
  - Using graphs (instead of 1D/2D tables) in dynamic programming algo.
  - For: shortest path problems.

BONUS: MAXIMUM WEIGHT INDEPENDENT SET OF TREES

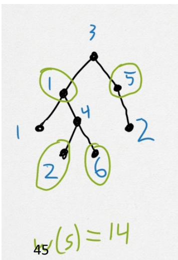
Max Weight Independent Set

- **Input:** Graph G with a weight w_v on each **node** v
- A set of nodes **S** is **independent** if no edges between nodes in **S.**
- **Output:** Independent set **S** maximizing $\sum_{s \in S} w_s$
- NP-Hard ☹



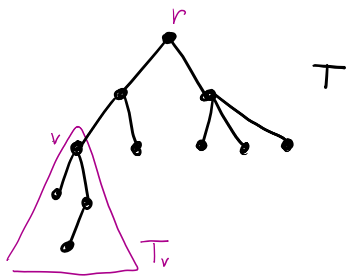
Max Weight Independent Set **On Trees**

- **Input:** Tree T with a weight w_v on each **node** v
- A set of nodes **S** is **independent** if no edges between nodes in **S.**
- **Output:** Independent set **S** maximizing $\sum_{s \in S} w_s$
- Admits linear-time algo!



Coming up with the recurrence

- MWIS(v): max weight independent set of subtree T_v rooted at v
- **Goal**: MWIS(r) where r is the root of T

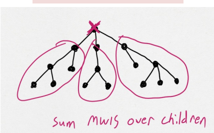


Coming up with the recurrence

- MWIS(v): max independent set of subtree T_v rooted at v
- **Goal**: MWIS(r) where r is the root of T

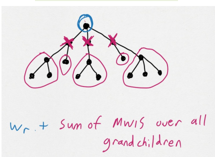
Recursive Strategy:
Consider the root node r.
Include r in S?

Reject root r



MWIS$(r) = \Sigma_{v \text{ child of } r} MWIS(v)$

Accept root r



MWIS$(r) = w_r + \Sigma_{v \text{ grandchild of } r} MWIS(v)$

Max Weight Independent Set **On Trees**

- MWIS(T) assume T nonempty
  Computes max possible total weight of MWIS
  (Could easily tweak algorithm to also output that independent set)

r = root node of T
table = array, indexed by tree nodes memo[v] stores $MWIS$ of subtree rooted at v

For each node v in T, **in which order???**

....

Return table[**r**]

Max Weight Independent Set **On Trees**

- MWIS(T) assumes T is nonempty
  Computes max possible total weight of MWIS
  (Could easily tweak algorithm to also output that independent set)

r = root node of T
table = array, indexed by tree nodes memo[v] stores $MWIS$ of subtree rooted at v

For each node v in T, in ascending tree order — process all descendants before v itself

reject ← $\sum_{c \text{ child of } v}$ table[c] — don't take v_i
accept ← $w_v + \sum_{g \text{ grandchild of } v}$ table[g] — take v_i, reject its children
table[v] ← max{reject, accept}

Return table[**r**]

n loops
Ops/loop **depends on # of children/grandchildren**
Tree of branching factor **b** has $O(nb^2)$ ops total

Max Weight Independent Set **On Trees**

- **Today**:
  – $O(nb^2)$ time
- **Challenge**:
  – $O(n)$ time
  – **Hint: Same algorithm. Smarter analysis.**
  Show that $\sum_{v \in T} |\text{grandchildren}(v)| = O(n)$

Upshot:
Dynamic Programming on Trees

Many hard problems
usually become easy on trees
via dynamic programming

Longest Palindromic Substring
(Another classic coding problem)

- Given a string $X[1..n]$
- **Goal:** Find the length of the longest palindrome in X.
  - A ***palindrome*** is a string that's equal to its reverse
- **Example:** "aba", "aca", and "ada" are the longest palindromes in X = "abacada" (**Note:** "aacaa" doesn't count!)
- **Q:** What's a brute force algorithm?
  - Try every substring $X[i..j]$

Recurrence for PAL

- Given a string $X[1..n]$
- Let $PAL(i,j)$ be a <u>*Boolean*</u> (T/F) value for whether $X[i..j]$ is a palindrome

Sometimes the problem we solve might seem quite "different"

$$PAL(i,j) = \begin{cases} X[i] = X[j] & j \le i+1 \\ X[i] = X[j] \textbf{ and } PAL(i+1, j-1) & j > i+1 \end{cases}$$

Q: Given this recurrence, how do we find the length of a longest palindrome?