A. 3-2

56. For which values of the constants $a, b, \ldots, m$ are the given vectors linearly independent?

$$\begin{bmatrix} a \\ b \\ c \\ d \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} e \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} f \\ g \\ h \\ i \\ j \\ 1 \end{bmatrix}, \quad \begin{bmatrix} k \\ m \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Sol Rearrange these four vectors:

$$\vec{v_1} = \begin{bmatrix} e \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} k \\ m \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} a \\ b \\ c \\ d \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v_4} = \begin{bmatrix} f \\ g \\ h \\ i \\ j \\ 1 \end{bmatrix}$$

every vector $\vec{v_i}$, $i \geq 2$ has an nonzero entry where the preceding vectors all have a zero.

So by theorem 3.2.5, no matter what $a, b, \ldots, m$ are, the vectors are always linearly independent.

(In other words: let $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} = \vec{0}$, since $\vec{v_1}, \vec{v_2}, \vec{v_3}$ all have 0 in the 6^{th} entry but $\vec{v_4}$ does not, c_4 must be 0.

Then $c_1\vec{v_1}+c_2\vec{v_2}+c_3\vec{v_3}=\vec{0}$.

And similarly, we get c_3, c_2, c_1 must be 0 one by one.

So they are linearly independent)

Therefore $a, b, \dots, m$ can be any value in $\mathbb{R}$.

3-3

33. A subspace V of $\mathbb{R}^n$ is called a *hyperplane* if V is defined by a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0,$$

where at least one of the coefficients c_i is nonzero. What is the dimension of a hyperplane in $\mathbb{R}^n$? Justify your answer carefully. What is a hyperplane in $\mathbb{R}^3$? What is it in $\mathbb{R}^2$?

Sol. Since V is defined by

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$$

the kernel of V is $\left\{ \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right\}$.

Denote the vector by A.

So V is kernel of the linear transformation

$T_A: \mathbb{R}^n \to \mathbb{R}$, $T_A = A\vec{x}$ $(V = \ker(T))$

Since $\operatorname{im}(T) = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right\}$ where at least one c_i is not 0, $\dim(\operatorname{im}(T)) = 1$

So by rank-nullity theorem, $\underline{\dim(\ker(T))=n-1}$

so $\underline{\dim(V)=n-1}$.

Therefore the dimension of hyperplane is $n-1$.

So in $\mathbb{R}^3$, a hyperplane is a plane.

in $\mathbb{R}^2$, a hyperplane is a line.

63. Consider two subspaces V and W of $\mathbb{R}^n$, where V is contained in W. In Exercise 62 we learned that $\dim(V) \leq \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

Hint: argue that every basis of V must be a basis of W

Pf. Let $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$ be a basis of V.

Assume $\dim(V) = \dim(W)$

By definition of basis, $\dim V = \underline{\dim W = m}$ ①

Since $V \subseteq W$, $\underline{\vec{V_1}, \vec{V_2}, \ldots, \vec{v_m} \subseteq W}$ ②

Since $\{\vec{v_1}, \vec{v_2}, \vec{v_m}\}$ is a basis of V,

they are linearly independent. ③

Since ①, ②, ③, by theorem 3.3.4, $\{\vec{v_1}, \vec{V_2}, \ldots, \vec{U_m}\}$ is a basis of W.

So arbitrary element $\vec{w} \in W$ is a linear combinition of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}$, and since $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}\}$ is also a basis of V, $\vec{w} \in V$, so $\boxed{W \subseteq V}$

And since $V \subseteq W$, $V = W$

4-1

Let V be the space of all infinite sequences of real numbers. See Example 5. Which of the subsets of V given in Exercises 12 through 15 are subspaces of V?

12. The arithmetic sequences [i.e., sequences of the form $(a, a+k, a+2k, a+3k, \ldots)$, for some constants a and k]

Sol. yes.

Denote the set of all arithmetic sequences as S. We know $S \subseteq V$

① Let $a, k = 0$, the sequence $(0, 0, \dots, 0, \dots)$ $\in S$, which is the 0 element in V.

② Let m, n be two arbitrary element of S

Denote m, n by $(m_0, m_0+k, m_0+2k, \ldots)$

$(n_0, n_0+k, n_0+2k, \ldots)$ respectively

So $m+n = \big((m_0+n_0), (m_0+n_0)+(2k),$

$(m_0+n_0)+2(2k), \ldots\big)$

is also in S

③ Let r be an arbitrary scalar,

x be an arbitary element of S

Denote x by $(x_0, x_0+k, x_0+2k, x_0+3k, \ldots)$

So $rx = (rx_0, rx_0+(rk), rx_0+2(rk),$

$rx_0+3(rk), \ldots)$

also $\in S$

Therefore it is a subspace of V.

Find a basis for each of the spaces V in Exercises 16 through 36, and determine its dimension.

28. The space of all 2×2 matrices A that commute with

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of the space. Denote the space by S.

So $AB = BA$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

So $\underline{c=0,\ a=d}$.

Therefore the space $S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

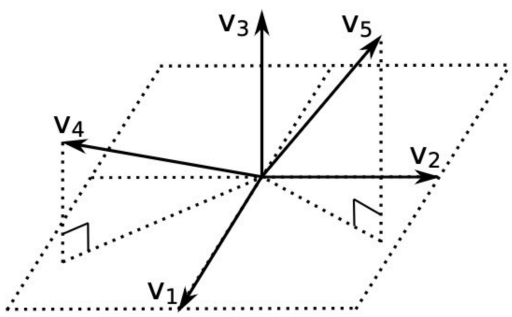
$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are linearly independent,

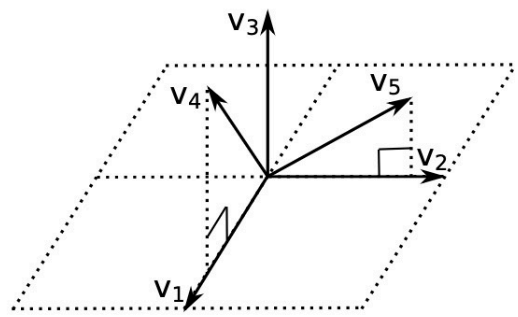
$\underline{\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} \text{ is a basis of } S}$

So $\underline{\dim S = 2}$

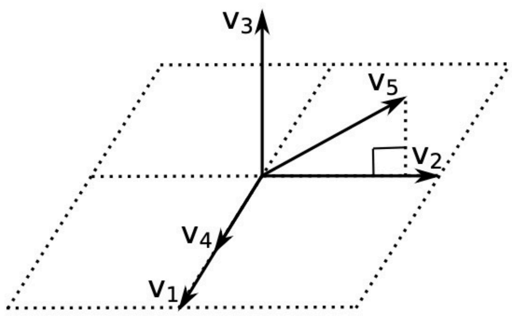
Part A Problem 6. Let $\mathbf{v}_1, \dots, \mathbf{v}_5$ be vectors in $\mathbb{R}^3$, as shown in the four figures below. In each figure, find *all* linearly dependent sets consisting of three of these five vectors, or else state that there are none if this is the case. *No justification needed.* (Note that in each of these figures, $\mathbf{v}_1$ and $\mathbf{v}_2$ span the displayed plane, $\mathbf{v}_3$ points "up" and is perpendicular to this plane, and for any other vector *not* in the plane, we draw a dotted vertical line indicating its position above the plane.)



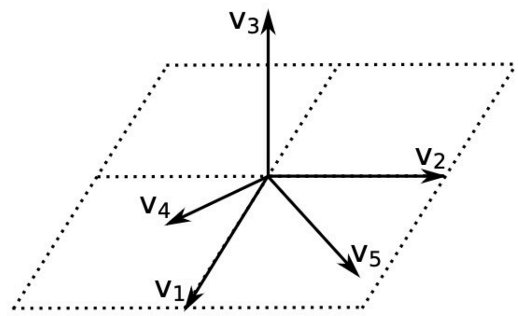
(a)



(b)



(c)



(d)

Sol. (a) $\{V_1, V_2, V_3\}$, $\{V_1, V_2, V_5\}$, $\{V_1, V_2, V_4\}$

$\{V_1, V_3, V_5\}$, $\{V_1, V_3, V_4\}$

$\{V_2, V_3, V_4\}$, $\{V_2, V_3, V_5\}$

$\{V_3, V_4, V_5\}$, $\{V_2, V_4, V_5\}$, $\{V_1, V_4, V_5\}$

(num: 10)

(b) $\{V_1, V_2, V_3\}$, $\{V_1, V_2, V_5\}$, $\{V_1, V_2, V_4\}$

$\{V_1, V_3, V_5\}$, $\{V_2, V_3, V_4\}$, $\{V_3, V_4, V_5\}$,

$\{V_2, V_4, V_5\}$, $\{V_1, V_4, V_5\}$ (num: 8)

(c) $\{V_1, V_2, V_3\}$, $\{V_1, V_2, V_5\}$, $\{V_1, V_3, V_5\}$,
$\{V_2, V_3, V_4\}$, $\{V_3, V_4, V_5\}$, $\{V_2, V_4, V_5\}$

(num: 6)

(d) $\{V_1, V_2, V_3\}$, $\{V_1, V_3, V_5\}$, $\{V_1, V_3, V_4\}$
$\{V_2, V_3, V_4\}$, $\{V_2, V_3, V_5\}$, $\{V_3, V_4, V_5\}$

Part B (25 points)

Problem 1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ be a list of vectors in V, and consider the list $Y = (T(\mathbf{x}_1), \ldots, T(\mathbf{x}_k))$ of vectors in W. Determine whether the following statements are true or false. If true, provide a proof. If false, provide a counter-example.

(a) If X is linearly independent, then Y is also linearly independent.
(b) If Y is linearly independent, then X is also linearly independent.

(a) False.

Counterexample: $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$

$\vec{x} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$

T is a linear transformation since for any arbitrary $\vec{a}, \vec{b} \in \mathbb{R}^2, k \in \mathbb{R}$

$T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{0}$

$T(k\vec{a}) = kT(\vec{a}) = \vec{0}$

$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are linearly independent

But $Y = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ are linearly dependent.

(b) Assume Y is linearly independent

this means whenever $c_1T(x_1)+c_2T(x_2)+\dots+c_kT(x_k)=0$, $c_1=c_2=\dots=c_k=0_W$. ①

we want to show X is linearly independent.

Assume $d_1x_1+d_2x_2+\dots+d_kx_k=0_V$

So $T(d_1x_1+d_2x_2+\dots+d_kx_k)=T(0_V)=0_W$

by theorem 3.1.6 ($0_V\in\ker T$)

So $d_1T(x_1)+d_2T(x_2)+\dots+d_kT(x_k)=0_W$

by the property of linear transformation.

Since ①, $d_1=d_2=\dots=d_k=0_V$

So $d_1=d_2=\dots=d_k=0_V$ whenever $d_1x_1+d_2x_2+\dots+d_kx_k=0_V$

Therefore X is linearly independent by definition.

Problem 2.

(a) Find[1] a linear transformation $T:\mathbb{R}^5\to\mathbb{R}^3$ such that

$$\ker(T)=\{\vec{x}\in\mathbb{R}^5 : x_1=5x_2 \text{ and } x_3=7x_4\} \qquad \text{and} \qquad \operatorname{im}(T)=\{\vec{x}\in\mathbb{R}^3 : x_1=x_3\}.$$

(b) Is the linear transformation you found in part (a) unique? Justify your claim.

[1]What does "find" mean? Should you go look in your closet? In this context, "find" means to explicitly describe or construct, as in "produce a concrete example and prove that it works." Here you're asked to find a function, and functions are typcially defined by specifing their source and target and a rule for converting inputs to outputs. In this case you're *given* the source and target, so you just need to specify a rule. So your answer should be something lke "Let T be the function defined by $T(\vec{x})=??$ for all $\vec{x}\in\mathbb{R}^5$." Your job is to decide what ?? should be.

(a) Since $T: \mathbb{R}^5 \to \mathbb{R}^3$, it can be represented by a standard matrix, by the key theorem.

So if $T(\vec{x}) = \vec{0}$, $A\vec{x} = \vec{0}$. This means the kernel is the solution set to the system $A\vec{x} = \vec{0}$

Since $\ker(T) = \{\vec{x} \in \mathbb{R}^5 : x_1 = 5x_2, x_3 = 7x_4\}$,

$$\text{rref}(A) = \begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\text{im}(T) = \{x^3 \in \mathbb{R}^3 : x_1 = x_3\}$

$$= \left\{ r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| \, r, s \in \mathbb{R} \right\}$$

a basis of $\text{im}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ since it spans $\text{im}(T)$ by definition and it is linearly independent.

And by theorem 3.2.4, a basis of $\text{im}(T)$ is all columns of A omitth redundant ones.

So we can perform some elementary transformations on rref(A) to get A such that the columns except redundant ones are $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

Consider $\begin{bmatrix}1 & -5 & 0 & 0 & 0\\0 & 0 & 1 & -7 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ +I

$$\to \boxed{A = \begin{bmatrix}1 & -5 & 0 & 0 & 0\\0 & 0 & 1 & -7 & 0\\1 & -5 & 0 & 0 & 0\end{bmatrix}}$$

In A, column 2, 4, 5 are redundant, so basis of A is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

Therefore the linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$ determined by $T = A\vec{x}$ where $A = \begin{bmatrix}1 & -5 & 0 & 0 & 0\\0 & 0 & 1 & -7 & 0\\1 & -5 & 0 & 0 & 0\end{bmatrix}$ is one satisfying the requirement.

(b) The linear transformation is not unique since we only need to satisfy:

① $\text{rref}(A) = \begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

② a basis of $\text{im}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

But this not the only basis.

For example: $\left\{ \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is also a basis.

Since elementary transformations do not change $\text{rref}(A)$, we can make another A that satisfy the requirements

example: $\begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \times 5 \\ \\ + \text{I} \times 5 \end{array}$

$\rightarrow A' = \begin{bmatrix} 5 & -25 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 5 & -25 & 0 & 0 & 0 \end{bmatrix}$

$T_{A'} = A'\vec{x}$ is also a linear transformation satisfying the requirements. And it represents a different linear transformation since $A' \neq A$.

Problem 3. Let X and Y be vector spaces.

(a) Consider a basis $\mathfrak{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of X. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be any vectors (not necessarily a basis, or even distinct) in Y. Prove that there exists a unique linear transformation $T : X \to Y$ such that $T(\mathbf{x}_i) = \mathbf{y}_i$ for all $1 \le i \le n$.

(b) Let U and V be subspaces of X and Y respectively such that $\dim(U) + \dim(V) = \dim(X)$. Prove that there exists a linear transformation $T_{U,V} : X \to Y$ such that $\ker(T_{U,V}) = U$ and $\operatorname{im}(T_{U,V}) = V$. (Hint: use part (a). You might also want to try to generalize the method you used to solve Problem 2.)

(c) In the map $T_{U,V}$ that you found in part (b) unique? Justify your answer.

(a) Proof First we prove the linear transformation exists.

Since $\mathfrak{B}$ is a basis of X, any vector $x \in X$ is a linear combination of vectors of $\mathfrak{B}$.

Let $\vec{v}$ be any vector in X

Write v as a linear combination of the vectors in $\mathfrak{B}$: $v = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (1)

Now we Define $T: X \to Y$

by $v \mapsto c_1T(x_i) + c_2T(x_2) + \ldots + c_nT(x_n)$

where $T(x_i) = y_i$

First we prove this is a valid linear transformation:

Let v_1, v_2 be two arbitrary elements of X

k be an arbitrary scalar.

So $v_1 = d_1 T(x_1) + d_2 T(x_2) + \dots + d_n T(x_n)$

$v_2 = e_1 T(x_1) + e_2 T(x_2) + \dots + e_n T(x_n)$

for some scalars $d_1, \dots, d_n, e_1, \dots, e_n$

Then $T(v_1 + v_2) =$

$(d_1 + e_1) T(x_1) + (d_2 + e_2) T(x_2) + \dots + (d_n + e_n) T(x_n)$

$= T(v_1) + T(v_2)$

$T(kv_1) = kd_1 T(x_1) + kd_2 T(x_2) + \dots + kd_n T(x_n)$

$= kT(v_1)$

So this is a linear transformation

Now we prove this is unique.

Let $T' : X \to Y$ be a linear transformation

such that $T'(x_i) = y_i$ for all $x \in \mathcal{B}$, and fixed $y_1, \dots, y_n$.

Let v be an arbitrary element of V

Since $\mathcal{B}$ is a basis of V,

② $v = d_1 x_1 + d_2 x_2 + \dots + d_n x_n$ for some scalars $d_1, d_2, \dots, d_n$

By the definition of linear transformation,

$T'(v) = d_1 T(x_1) + d_2 T(x_2) + \dots + d_n T(x_n)$

By definition 4.1.3, the coordinate of v under the same basis is unique, so in ① and ②,

$(c_1, c_2, \dots, c_n) = (d_1, d_2, \dots, d_n)$

Therefore $T = T'$ since v is arbitrary.

Then we have finished the proof.

(b) Denote $\dim(X)$ by n

Let $\{u_1, u_2, \dots, u_k\}$ be a basis of U.

Since $U \leq X$ is a subspace, we can extend this basis to a basis of X:

$\mathcal{B} = \{u_1, u_2, \dots, u_k, w_{k+1}, w_{k+2}, \dots, w_n\}$

So $\dim(U) = k$, since $\dim(U) + \dim(V) = \dim(X)$,
$\dim(V) = n-k$

Let $\{v_1, v_2, \dots, v_{n-k}\}$ be a basis of V.

Therefore we can map every w_{k+j} to v_j, $1 \leq j \leq n-k$.

Now we define $T: X \to Y$ as:

for any element $x \in X$, write x as a linear combination of vectors in basis $\mathcal{B}$:

$x = a_1 u_1 + a_2 u_2 + \dots + a_k u_k + a_{k+1} w_{k+1} + \dots + a_n w_n$

Let $T_{U,V}(u_i) = 0$, for $i = 1, 2, \dots, k$

and $T_{U,V}(w_{k+j}) = v_j$ for $j = 1, 2, \dots, n-k$

and define $T_{U,V}(x) = a_1 T_{U,V}(u_1) + a_2 T_{U,V}(u_2) + \ldots + a_k T_{U,V}(u_k) + a_{k+1} T_{U,V}(w_{k+1}) + \ldots + a_n T_{U,V}(w_n)$

$(= a_{k+1} v_1 + \ldots + a_n v_{n-k})$

Similar to (a), we know this linear transformation is valid since $x \in X$ which is a vector space

And $\mathrm{im}(T_{U,V}) = \{ a_{k+1} v_1 + a_{k+2} v_2 + \ldots + a_n v_{n-k} \mid a_{k+1}, a_{k+2}, \ldots, a_n$ are arbitrary scalars$\}$

whose basis is $\{v_1, v_2, \ldots, v_{n-k}\}$

So $\boxed{\mathrm{im}(T_{U,V}) = V}$

Let u be an arbitrary element of U

Since we map every u_i in basis of U to 0_V, $T(u) = 0 + 0 + \ldots + 0_V = 0_V$

and let w be an arbitrary element in $X \setminus U$, then at least one of $a_k, a_{k+1}, \ldots, a_n \neq 0$, so $T(w) \neq 0_V$. so $\boxed{\ker(T_{U,V}) = U}$

Hence $T_{U,V}$ is the linear transformation meeting the requirements.

(3) This is not unique because the construction of $T_{U,V}$ is based on the basis of V we arbitrarily choosed. If we choose a different basis of V, then $T_{U,V}(w_{k+j}) = v_j$ is different and therefore the whole linear transformation can be a different one, still meeting requirements.

Problem 4. Let U, V, and W be finite-dimensional vector spaces, and let $T : U \to V$ and $S : V \to W$ be linear transformations. Determine whether the following statements are true or false, and provide a proof of your claim.

(a) $\mathrm{rank}(S \circ T) \le \mathrm{rank}(S)$.
(b) $\mathrm{rank}(S \circ T) \le \mathrm{rank}(T)$.
(c) $\mathrm{nullity}(S \circ T) \ge \mathrm{nullity}(T)$.
(d) $\mathrm{nullity}(S \circ T) \ge \mathrm{nullity}(S)$.

(a) True.

$\mathrm{rank}(S \circ T) = \dim(\mathrm{im}(S \circ T))$

$\mathrm{rank}(S) = \dim(\mathrm{im}(S))$

by worksheet 10.

since $\mathrm{im}(S \circ T) \subseteq \mathrm{im}(S)$

since $S \circ T$ maps vectors to the same target, but by composing T, $S \circ T$ maps at most as many vectors as S do, from the source.

$\dim(\mathrm{im}\, S \circ T) \le \dim(\mathrm{im}(S))$

So $\mathrm{rank}(S \circ T) \le \mathrm{rank}(S)$

(b) True.

We can view this as two stages

stage ①: $T : U \to V$

So $\mathrm{rank}(T) = \dim(\mathrm{im}(T)) = u - \dim(\ker(T))$

by rank-nullity theorem

stage ②: $S' : \mathrm{im}(T) \to W$

So $\mathrm{rank}(S \circ T) = \dim(\mathrm{im}(S'))$

$= \mathrm{im}(T) - \dim(\ker(S'))$

$= u - \dim(\ker(T)) - \dim(\ker(S'))$

So $\mathrm{rank}(S \circ T) \le \mathrm{rank}(T)$ since $\dim(\ker(S')) \ge 0$

(c) True.

By rank nullity theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim(U)$$

$$\text{rank}(S \circ T) + \text{nullity}(S \circ T) = \dim(U)$$

So $\text{rank}(T) + \text{nullity}(T) = \text{rank}(S \circ T) + \text{nullity}(S \circ T)$

Since in (b) we have proved $\text{rank}(S \circ T) \le \text{rank}(T)$

so $\text{nullity}(S \circ T) \geq \text{nullity}(T)$

(d) False.

Consider counterexample:

T is surjective, $S(V) = 0_W$,

So $\text{nullity}(S) = \dim(V)$

$\text{nullity}(S \circ T) = \dim(U)$

Consider if $\dim(U) < \dim(V)$

Then $\text{nullity}(S \circ T) < \text{nullity}(S)$.

Problem 5. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *symmetric* if $A^\top = A$, and *skew-symmetric* if $A^\top = -A$. Let Sym_n and Skew_n denote the set of all symmetric matrices and the set of all skew-symmetric matrices in $\mathbb{R}^{n\times n}$, respectively.

(a) Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the map defined by $T(A) = A + A^\top$. Prove that T is linear.
(b) Prove that $\ker(T) = \mathrm{Skew}_n$ and $\mathrm{im}(T) = \mathrm{Sym}_n$.
(c) Prove that Sym_n and Skew_n are subspaces of $\mathbb{R}^{n\times n}$.
(d) Find $\dim(\mathrm{Sym}_n)$ and $\dim(\mathrm{Skew}_n)$.

(a) Proof Let A_1, A_2 be two arbitrary matrices in $\mathbb{R}^{n\times n}$

then $T(A_1+A_2) = A_1 + A_1^T + A_2 + A_2^T$

$= (A_1+A_2) + (A_1+A_2)^T$ by matrix addition rule.

Let k be an arbitrary scalar,

then $T(kA_1) = kA_1 + kA_1^T$

$= k(A_1+A_1^T) = kT(A_1)$

So T is linear by definition

(b) Let $A \in \ker(T)$ be arbitrary

then $A + A^T = 0$, $A = -A^T$, so $A^T = -A$, so A is skew-symmetric.

so $A \in \mathrm{Skew}_n$. Therefore $\ker(T) \subseteq \mathrm{Skew}_n$

Let $B \in \mathrm{Skew}_n$ be arbitrary

Then $B^T = -B$, $B^T + B = 0$, so $T(B) = 0$,

and therefore $B \in \ker(T)$, so $\mathrm{Skew}_n \subseteq \ker(T)$

Therefore $\ker(T) = \mathrm{Skew}_n$

let $C \in im(T)$ be arbitrary

So there exists $M \in \mathbb{R}^{n\times n}$ such that $M+M^T = C$.

so $C^T = M^T + M = M + M^T = C$

so $C \in Sym_n$. Therefore $\underline{im(T) \subseteq Sym_n}$

let $\underline{D \in Sym_n}$ be arbitrary

So $D = D^T$

Consider N whose every entry is half of the entry of D at the corresponding location

then $N = N^T \in \mathbb{R}^{n\times n}$

And $N + N^T = N^T + N = D$

so $T(N) = D$, $\underline{D \in im(T)}$

Therefore $\underline{Sym_n \subseteq im(T)}$

Therefore $im(T) = Sym_n$.

(c) First we prove $Skew_n$ is a subspace of $\mathbb{R}^{n\times n}$.

① every element of $Skew_n$ is $n\times n$ matrix, therefore is an element of $\mathbb{R}^{n\times n}$.

so $\underline{Skew_n \subseteq \mathbb{R}^{n\times n}}$

② Since for $0 \in \mathbb{R}^{n\times n}$, $0^T = 0 = -0^T$, $0 \in Skew_n$.

(here 0 means $n\times n$ matrice of all entries as 0)

③ Let A, B be two arbitrary elements of $Skew_n$. Then $A^T = -A$, $B^T = -B$

$$A+B = -A^T - B^T$$
$$= -(A^T + B^T)$$
$$= -(A+B)^T, \quad (A+B)^T = -(A+B)$$

So $A+B \in Skew_n$

④ Let A be an arbitrary element of $Skew_n$. and k be an arbitrary scalar.

Then $A^T = -A$

$$kA = -kA^T = -(kA)^T$$

so $(kA)^T = -kA$. $kA \in Skew_n$

So $kA \in Skew_n$

By ①, ②, ③, ④, $Skew_n$ is a subspace of $\mathbb{R}^{n\times n}$.

Then we prove Sym_n is a subspace of $\mathbb{R}^{n\times n}$.

① every element of Sym_n is $n\times n$ matrix, therefore is an element of $\mathbb{R}^{n\times n}$.

so $\underline{Sym_n \subseteq \mathbb{R}^{n\times n}}$

② Since for $0 \in \mathbb{R}^{n\times n}$, $0^T = 0$, $0 \in Sym_n$.

(here 0 means $n\times n$ matrice of all entries as 0)

③ Let A, B be two arbitrary elements of Sym_n. Then $A^T = A$, $B^T = B$

$$A+B = A^T + B^T$$
$$= (A^T + B^T)$$

So $A+B \in Sym_n$

④ Let A be an arbitrary element of Sym_n and k be an arbitrary scala

Then $A^T = A$

$$kA = kA^T = (kA)^T$$

so $kA \in Skew_n$

By ①, ②, ③, ④, $Skew_n$ is a subspace of $\mathbb{R}^{n\times n}$.

(d) Since $\mathbb{R}^{n\times n} = \left\{ a_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} + \cdots + a_{n^2} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \middle| a_1, a_2, \ldots, a_{n^2} \in \mathbb{R} \right\}$

The basis of $\mathbb{R}^{n\times n}$ is the n^2 many matrices.

So $\dim(\mathbb{R}^{n\times n}) = n^2$

For Sym_n, all entries below the diagonal.

must be mirror to corresponding entries above the diagonal.

So half of (all entries except the ones on diagonal)

+ all entries on diagonal

are free variables.

Other entries are determined by them.

So $\boxed{\dim(\mathrm{Sym}_n)} = \frac{(n^2-n)}{2} + n = \boxed{\frac{n^2+n}{2}}$

And similarly, for Skew_n, all entries below the diagonal.

must be the negation of corresponding entries above

And note: all entries on the diagonal must be 0 so as to make $A^T = -A$.

So number of free variables are

half of (all entries except the ones on diagonal)

So $\boxed{\dim(\mathrm{Skew}_n) = \frac{n^2-n}{2}}$