Spectral Thm 内容 (结论：A是orthogonally diagable的)

$\boxed{\exists\, S\in\mathbb{R}^{n\times n}\ \text{s.t.}\ S^T = S^{-1}\ (\text{orthonormal cols}),\ \text{and}\ S^{-1}AS\ \text{is diagonal.}}$

(Equivalently, S的cols为 $\mathbb{R}^n$ 的一个orthonormal basis，且为A的eigenvectors.)

iff

A is symmetric

意义：普通的 diagonalization：$D = S^{-1}AS$.

S的cols为V的一个eigenbasis $(\vec{v}_1, \dots, \vec{v}_n)$

而 orthogonal diagonalization，其 $D = S^{-1}AS$ 中的S的cols不仅是eigenbasis of V，而且还是一个orthonormal eigenbasis

这意味着一共n种不同的eigenvectors，它们都是 $\boxed{\text{orthogonal}}$ 的.(可化为orthonormal).

也就是说：$\forall\, E_{\lambda_i}$ 和 E_{λ_j},

$E_{\lambda_i} \perp E_{\lambda_j}$ ☆

Pf of the Spectral Thm.

Claim ①: if A orthogonally diagble

⇒ then A is symmetric.

A orthogonally diagble

$\Rightarrow \exists S,\ S^{-1}AS = D$ 且 $S^{-1} = S^T$

$\Rightarrow A = SDS^{-1}$

$A^T = (SDS^{-1})^T = (S^{-1})^T D^T S^T$

$= S D S^T = SDS^{-1} = A$

⇒ A symmetric.

Claim ②, if $A\in\mathbb{R}^{n\times n}$ is symmetric

then A is orthogonally diagble over $\mathbb{R}$

This Claim is divided into 3 parts.

Claim ②. pt.(1): if $A\in\mathbb{R}^{n\times n}$ is symmetric,

then A 只有 real eigenvalues. ☆

Claim ② pt(2): if $A\in\mathbb{R}$ is symmetric,

then A is orthogonally diagble

whenever it is diagble

Claim ②. pt.(3): if $A\in\mathbb{R}$ is symmetric,

then A is diagble

Let A be symmetric,

pt(1): by Fund Thm of Alegbra,

$\chi_T(x) = 0$ 有n个 complex roots, 包含重复

let λ be any complex eigenvalue of T

那么 $\exists\, \vec{z}\in\mathbb{C}^n,\ A\vec{z} = \lambda\vec{z}$

$\Rightarrow \bar{\vec{z}}$ 为 T的 $\bar{\lambda}$-eigenvector

$\Rightarrow A\bar{\vec{z}} = \bar{\lambda}\bar{\vec{z}}$

两边取T $\Rightarrow (A\bar{\vec{z}})^T = \bar{\lambda}(\bar{\vec{z}})^T$

$\Rightarrow \bar{\vec{z}}^T A^T = \bar{\lambda}\bar{\vec{z}}^T$

$\xrightarrow{\text{A symmetric}} \bar{\vec{z}}^T A = \bar{\lambda}\bar{\vec{z}}^T$

$\xrightarrow{\text{multiply } \vec{z} \text{ on the right}} \bar{\vec{z}}^T(A\vec{z}) = \bar{\lambda}(\bar{\vec{z}}\cdot\vec{z})$

$\Rightarrow \bar{\vec{z}}^T(\lambda\vec{z}) = \bar{\lambda}(\bar{\vec{z}}\cdot\vec{z})$

$\lambda(\bar{\vec{z}}\cdot\vec{z}) = \bar{\lambda}(\bar{\vec{z}}\cdot\vec{z})$

Since $\bar{\vec{z}}\cdot\vec{z} = \sum_i(\bar{z}_i z_i) = \sum_i |z_i|^2 > 0$

λ

$\Rightarrow \lambda = \bar{\lambda}$

⇒ 每个 complex eigenvalue 都 $\in\mathbb{R}$

pt(2): consider λ_1, λ_2 $(\lambda_1\neq\lambda_2)$ 及任其 $\vec{v}_1\in E_{\lambda_1}\ (\neq\vec{0})$, $\vec{v}_2\in E_{\lambda_2}\ (\neq\vec{0})$

$\Rightarrow \lambda_1\vec{v}_1\cdot\vec{v}_2 = A(\vec{v}_1)\cdot\vec{v}_2$

$= (A\vec{v}_1)\cdot(\vec{v}_2)$

$= (A\vec{v}_1)^T(\vec{v}_2)$

$= \vec{v}_1^T A\vec{v}_2$

$= \vec{v}_1^T(\lambda_2\vec{v}_2) = \lambda_2\vec{v}_1\cdot\vec{v}_2$

Since $\lambda_1\neq\lambda_2 \Rightarrow \vec{v}_1\cdot\vec{v}_2 = 0 \Rightarrow \vec{v}_1\perp\vec{v}_2$.

Pf(3):

Prove by induction

Base case: 1×1 matrix 都是 diagonal 且 symmetric 的

Inductive step: if 每个 symmetric $\mathbb{R}^{n\times n}$ matrix 都是 orthogonally diagable 的,
那么每个 $\mathbb{R}^{(n+1)\times(n+1)}$ matrix 也是

Pf. 令 $A \in \mathbb{R}^{(n+1)\times(n+1)}$ 为 symmetric 的

令 λ 为 A 的任意 eigenvalue.
$\vec{u}$ 为其一个 unit eigenvector

$\Rightarrow$ complete $U = (\vec{u}, \vec{u_1}, \dots, \vec{u_n})$
为 $\mathbb{R}^{n+1}$ 的一个 orthonormal basis.

则考虑 $Q = \begin{bmatrix} | & | & & | \\ u & u_1 & \cdots & u_n \\ | & | & & | \end{bmatrix}$, Q 为 orthogonal matrix

$(Q^T = Q^{-1})$

注意, Q 为 $S_{U\to\varepsilon}$, 因而 $Q^{-1} = Q^T = S_{\varepsilon\to U}$.

$\Rightarrow Q^T A Q = [T]_U$

注意: ① $(Q^T A Q)^T = Q^T A^T Q = Q^T A Q$

因而 $Q^T A Q$ 是 symmetric 的

并且: $Q^T A Q \vec{e_1} = Q^T (A\vec{u})$

$= Q^T \lambda \vec{u}$

$= \lambda S_{\varepsilon\to U} \vec{u}$

$= \lambda \vec{e_1} = \begin{bmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

因而 $Q^T A Q = \begin{bmatrix} \lambda & \vec{0}^T \\ \vec{0} & B \end{bmatrix}$ for some $B \in \mathbb{R}^{n\times n}$
且 B 为 symmetric 的.

by inductive hypothesis, B orthogonally diagable

因而 $B = RDR^T$ for some orthogonal R
以及 diagonal D

$\Rightarrow Q^T A Q = \begin{bmatrix} \lambda & \vec{0} \\ \vec{0} & B \end{bmatrix} = \begin{bmatrix} \lambda & \vec{0} \\ \vec{0} & RDR^T \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R^T \end{bmatrix}$

note: $\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}$ diagonal,
$\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$ orthogonal
(因而 $\begin{bmatrix} 1 & 0 \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}^T$
$= \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}^{-1}$)

$\Rightarrow$ 我们完成了对 A 的 orthogonal diagonalization.