Math 217 Worksheet 9: Subspaces and Linear Independence (§3.2)

DEFINITION: A **subspace** of a vector space V is a subset W satisfying:

- The zero element $\mathbf{0}_V$ of V is in W;
- If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$ ("W is closed under addition");
- If $w \in W$, then $\lambda w \in W$ for all scalars $\lambda \in \mathbb{R}$ ("W is closed under scalar multiplication").

Problem 1. Using the *definition*, decide whether each of the given *subsets* is a *subspace* of $\mathbb{R}^3$.

(a) The solution set of the system of linear equations $\{x + y + z = 0, 3x - z = 0\}$.

(b) The kernel of the matrix* $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$. (Why is this even a *subset* of $\mathbb{R}^3$?)

(c) The set of all columns of the matrix $\begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 2 & 0 & -1 & 0 & -1 \\ 3 & 1 & 0 & 0 & -2 \end{bmatrix}$.

(d) The span of the columns of the matrix $\begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 2 & 0 & -1 & 0 & -1 \\ 3 & 1 & 0 & 0 & -2 \end{bmatrix}$.

(e) The set $\{c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 \mid c_1, c_2, c_3 \geq 0\}$.

(f) The plane in $\mathbb{R}^3$ with equation $x + y + z = 1$.

(g) A line through the origin in $\mathbb{R}^3$.

Solution:

(a) Yes, subspace. We check

- $\vec{0} \in W$, because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies both equations.
- For arbitrary $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in W, we need to check that $\vec{v}_1 + \vec{v}_2 \in W$. Note that $\vec{v}_i$ in W means that $x_i + y_i + z_i = 0$ and $3x_i - z_i = 0$ for $i = 1, 2$. Adding these equations we get

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0 \quad \textit{and} \quad 3(x_1 + x_2) - (z_1 + z_2) = 0,$$

so also $\vec{v}_1 + \vec{v}_2 \in W$.

*This is an abuse of terminology meaning "The kernel of the linear transformation T_A with matrix A."

- For arbitrary $\vec{v} \in W$, we need to check that $\lambda\vec{v} \in W$ for all scalars λ. If $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, then $x_1 + y_1 + z_1 = 0$ and $3x_1 - z_1 = 0$, so multiplying by λ we have $\lambda x_1 + \lambda y_1 + \lambda z_1 = 0$ *and* $3\lambda x_1 - \lambda z_1 = 0$, so also $\lambda\vec{v} \in W$.

(b) This is just (a) rephrased, if you write out what it means to be in the kernel of A. That is, $\ker A = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\} = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 0 \text{ and } 3x - z = 0\}$, which is the set we had in (a). So, yes, subspace. More generally, the kernel of any matrix is subspace.

(c) NO! Not a subspace. Not closed under addition. The sum of the first two columns is not one of the five columns.

(d) YES, subspace! More generally, the SPAN of any vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is always a subspace. Proof:

- $\vec{0}$ is in the span (since we can take all the coefficients in the linear combination to be 0).
- Closed under addition: Take two arbitrary elements in the span. Each is a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Adding them together and grouping "like terms," we see that the sum is also a linear combination of $\{\vec{v}_1, \ldots, \vec{v}_n\}$.
- Closed under scalar multiplication: Take an arbitrary element in Span $\{\vec{v}_1, \ldots, \vec{v}_n\}$. This is a linear combination of the $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Multiplying by any scalar, we still have a linear combination of $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

(e) Not a subspace. Not closed under multiplication by negative scalars.

(f) Not a subspace. Does not contain $\vec{0}$.

(g) Subspace. We can write it $\{t\vec{m} \mid t \in \mathbb{R}\}$ for some "slope vector" $\vec{m} \in \mathbb{R}^3$. This is closed under addition (since $t_1\vec{m} + t_2\vec{m} = (t_1 + t_2)\vec{m}$) and scalar multiplication (since $\lambda(t\vec{m}) = (\lambda t)\vec{m}$), and contains zero (since $0 = 0\vec{m}$).

Problem 2. Proof practice with Subspaces. Let A be an $d \times p$ matrix.

(a) Prove that the solution set of the system of linear equations $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^p$.

(b) Prove that the kernel of $T_A : \mathbb{R}^p \to \mathbb{R}^d$ is a subspace of $\mathbb{R}^p$. [HINT: Use (a) to do it quickly!]

(c) Let $T : V \to W$ be any linear transformation. Prove that the kernel of T is a subspace of V. [HINT: This is more abstract than (b). You'll need to use the *definitions* of linear transformation and kernel.]

(d) Let $T : V \to W$ be any linear transformation. Prove that the image of T is a subspace of W. [HINT: An arbitrary element of $\operatorname{im} T$ can be written $T(v)$ for some $v \in V$]

(e) Prove that the image of the linear transformation $T_A : \mathbb{R}^p \to \mathbb{R}^d$ is a subspace of $\mathbb{R}^d$. [HINT: Use something you've already proved to do it quickly!]

Solution:

(a) We show the solution set satisfies the three properties of a subspace:

- $\vec{0}$ is in the solution set, since $A\vec{0} = \vec{0}$.
- The solution set is closed under addition: If $\vec{x}$ and $\vec{y}$ are solutions, then $A\vec{x} = A\vec{y} = 0$. So also $A\vec{x} + A\vec{y} = 0$, and using the distributive property of matrix multiplication, also $A(\vec{x} + \vec{y}) = \vec{0}$, which says that $\vec{x} + \vec{y}$ is a solution.
- The solution set is closed under scalar multiplication: If $\vec{x}$ is solution, then $A\vec{x} = 0$. Take any scalar c. So also $A(c\vec{x}) = cA(\vec{x}) = c\vec{0} = \vec{0}$, which says that $c\vec{x}$ is a solution.

(b) The kernel of T_A is, by definition, the set of all vectors $\vec{x}$ such that $T_A(\vec{x}) = 0$. Then by definition of T_A, this is the set of all vectors $\vec{x}$ such that $A\vec{x} = 0$—that is, the solution set we proved in (a) is a subspace!

(c) We show ker T satisfies the three defining properties of a subspace.

- Check is $0_V \in \ker T$? Yes, since $T(0_V) = 0_W$ for any linear transformation.
- Check if $x, y \in \ker T$, is $x + y \in \ker T$? Yes, since $T(x + y) = T(x) + T(y) = 0 + 0 = 0$ (using the definition of linearity).
- Check if $x \in \ker T$, is $\lambda x \in \ker T$ for any scalar λ? Yes: $T(\lambda x) = \lambda T(x) = \lambda 0 = 0$ (using definition of linearity).

(d) We show im T satisfies the three defining properties of a subspace.

- Check is $0_W \in \operatorname{im} T$? Yes, since $T(0_V) = 0_W$ for any linear transformation.
- Check if $x, y \in \operatorname{im} T$, is $x+y \in \operatorname{im} T$? Take arbitrary $x, y \in \operatorname{im} T$. This means we can write $x = T(v)$ and $y = T(w)$ for some v, w in the source. So $x + y = T(v) + T(w) = T(v + w)$. So $x + y$ is in the image since the vector $v + w$ goes to $x + y$ under T.
- Take arbitrary $T(v)$ in the image. Then for any scalar c, $cT(v) = T(cv)$ is also in the image.

(e) Here, we can just apply (d) as this is just one special kind of linear transformation and (d) proves it in general. The power of abstract theory!

DEFINITION. A set of vectors $\{v_1, v_2, \dots v_d\}$ in a vector space V is **linearly independent** if whenever

$$c_1v_1 + c_2v_2 + \cdots + c_dv_d = 0$$

for some scalars c_i, it follows that $c_1 = c_2 = \cdots = c_d = 0$.

DEFINITION. A **relation** on a set of vectors $\{v_1, v_2, \dots v_d\}$ is a linear combination

$$c_1v_1 + c_2v_2 + \cdots + c_dv_d$$

that equals zero. We say the relation is **trivial** if all coefficients $c_i = 0$.

OBSERVE: the set of vectors $\{v_1, v_2, \dots, v_d\}$ is **linearly dependent** if there is a non-trivial relation on them, and **linearly independent** if the *only* relation on them is the trivial relation.

Problem 3: Relations and linear independence.

(a) Using the *definition above*, explain why the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^3$ are linearly independent.

(b) Using the book's method ("redundant vectors"), check whether $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ linearly independent or not.

(c) Find a relation on $\{\vec{v}_1, \vec{v}_2\}$ from (b). Is there a *non-trivial relation* on $\{\vec{v}_1, \vec{v}_2\}$?

(d) Now, consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Which is the first "redundant" vector? Write it as a linear combination of the preceding vectors.

(e) Write a nontrivial linear relation on $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, with $\vec{v}_i$ as in (d). [HINT: Use (d)!]

(f) Discuss with your group why our definition of linear independence above (which is the one you must memorize and use in proofs!) is equivalent to the book's.

(g) *Using the definition,* show that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is **linearly dependent**.

(h) *Using the definition,* prove that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

[PROOF TECHNIQUE: To show vectors are linear independent, use first sentence "Let $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = 0$ be a relation." Now do some math to show the relation is trivial.]

Solution:

(a) Suppose that $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In other words, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This forces all the c_i to be zero, so our arbitrary relation is trivial—that is, the only relation is the trivial relation. By defintion, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are linearly independent.

(b) Most definitely linearly independent! Note, $\vec{v}_2 \neq k\vec{v}_1$ for any k. Thus it is not a linear combination of the vector(s) preceding it, and is not redundant. Therefore the vectors are linearly independent.

(c) We have only the trivial relation $0\vec{v}_1 + 0\vec{v}_2 = 0$, because these vectors are linearly independent (as we checked using the book's definition in (b)). We can also check directly that there is no non-trivial relation: If there were a non-trivial relation, say $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ with not both c_i zero, note then that neither c_i is zero in this case (since $\vec{v}_i \neq 0$ for $i = 1, 2$). So we could write $\vec{v}_2 = \frac{c_1}{c_2}\vec{v}_1$. But inspecting the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, we see that this is not so (as we checked in (b)).

To prove $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent using our definition above, suppose there were a non-trivial relation, say $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ with not both c_i zero. This can be written as the matrix equation $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, or

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of equation in three variables of rank 2, which we can check using the method of row reduction from Chapter 1 has exactly one solution, namely $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) From (b) we know that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, so the first vector that could be redundant is $\vec{v}_3$. We note that $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2$; thus it is redundant and we have found our linear combination all at once!

(e) Given (d), we can take $1\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 + 0\vec{v}_4 = \vec{0}$.

(g) The vectors are not linearly independent because $1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$ is a non-trivial relation on them.

(h) Suppose $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ is a linear relation on the vectors. This gives us a system of linear equations

$$\begin{bmatrix} c_1 + c_2 + c_3 & = & 0 \\ c_1 \quad\;\; + c_3 & = & 0 \\ \qquad\quad c_3 & = & 0 \end{bmatrix},$$

and to show that the relation is trivial we must prove that the only solution to this system is $c_1 = c_2 = c_3 = 0$. The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$ and we can row reduce it to see that the rank is 3, so that there is a *unique* solution, namely $\vec{0}$.

Problem 4. Subspaces of Coordinate spaces. Describe all subspaces of the vector space $\mathbb{R}^3$. What do they look like geometrically? How many are there? Can you classify them into different "types?" What are their dimensions? Repeat for $\mathbb{R}$ and $\mathbb{R}^2$. What about $\mathbb{R}^4$?

Solution: There are infinitely many but they come in three types: subspaces of $\mathbb{R}^3$ are the whole $\mathbb{R}^3$, planes through the origin, lines through the origin, and the origin. These have dimensions 3, 2, 1, 0, respectively. The subspaces of $\mathbb{R}^2$ are the whole $\mathbb{R}^2$, lines through the origin, and the origin. These have dimensions 2, 1, 0, respectively. There are only two subspaces of $\mathbb{R}$, namely $\{0\}$ and $\mathbb{R}$. For $\mathbb{R}^4$, we have the whole $\mathbb{R}^4$, "3-planes" through the origin, planes through the origin, lines through the origin, and the origin. These have dimensions 4, 3, 2, 1, 0, respectively.

Problem 5. Prove that the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent if and only if $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$. What if we assume *only* that $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$?

Solution: Assume $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. Now if $\vec{v}_1 = a\vec{v}_2$ for some scalar a, then there is a non-trivial relation $\vec{v}_1 - a\vec{v}_2 = 0$, contrary to the linear independence of $\{\vec{v}_1, \vec{v}_2\}$. Similarly, if $\vec{v}_2 = a\vec{v}_1$, we also conclude that $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Conversely, assume that the vector $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$. To show $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent, assume we have a relation $a\vec{v}_1 + b\vec{v}_2 = 0$. If $a \neq 0$, then $\vec{v}_1 = \frac{-b}{a}\vec{v}_2$ and if $b \neq 0$, then $\vec{v}_2 = \frac{-a}{b}\vec{v}_1$, contradicting the hypothesis. Thus $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. QED.

The proof does not work if we assume only that $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$. For example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a scalar multiple of $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but $\vec{0}$ *is* a scalar multiple of $\vec{v}_1$. In this case, even though $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$, the vectors $\{\vec{v}_1, \vec{v}_2\}$ are linearly *dependent.*

Problem 6. Prove that any set $\{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}\}$ of three vectors in $\mathbb{R}^2$ is linearly dependent.

[Hint: You need to show that there is a non-trival relation on the vectors. Write out what this means and then do some math to show that a non-trivial relation must exist. Note: you only need to show it exists....this is a case where that's easier than actually writing one out.]

Solution: Consider a potential relation $c_1 \begin{bmatrix} a_1 \\ v_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + c_3 \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = 0$. We can rewrite this as a matrix equation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A non-trivial relation exists if and only if this system has a non-trivial solution. Since this is a homogeneous system with at least one free variable (at most two pivots but three total variables), we know that there must be solutions. So there must be a non-trivial relation on the vectors.