(* Complement: "MathHygine"
Principle of Mathematical Induction: $[S(1) \wedge (\forall n\in\mathbb{N}, S(n) \Rightarrow S(n+1))] \Rightarrow (\forall m\in\mathbb{N}, S(m))$

$$\left|\begin{array}{l} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{array}\right| \xrightarrow{\text{take numbers}} \begin{array}{cc} & \text{columns} \\ \text{rows} & \begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix} \end{array}$$

matrix
read: (three by four)

1. $A = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & & \ddots & \\ & & & a_{34} \end{bmatrix}$ is called a 3×4 matrix, (row col)

2. matrix $A = B$ if same size and $\forall i,j,\ a_{ij} = b_{ij}$

3. if $A = n\times n$, A is called a $\boxed{\text{square matrix}}$ and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the $\boxed{\text{main diagonal}}$ of A

4. A square matrix A is called $\boxed{\text{diagonal}}$ p.t. all its entries above and below the diagonal are 0.
   ex. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
   i.e. $a_{ij} = 0$ whenever $i \neq j$.
   ex $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$

5. A is called $\boxed{\text{upper triangular}}$ p.t. all its entries below the main diagonal are 0.
   ex $\begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}$ $\boxed{\text{Lower triangular}}$: ~ above the main diagonal are 0.

Def Vectors and vector spaces

A matrix with only one column is called a column vector, or simply a $\boxed{\text{vector}}$.

The entries of a vector are called its $\boxed{\text{components}}$

The set of all column vectors with n components is denoted by $\boxed{\mathbb{R}^n}$

We will refer to $\mathbb{R}^n$ as a $\boxed{\text{vector space}}$

(A matrix with only one row: row vector
In this text, we refer to vectors as column vectors unless otherwise stated
下一章会说 preference for column vectors 的 apparent reason)

ex $\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix}$ is a vector in $\mathbb{R}^4$

$[1\ 5\ 5\ 3\ 7]$ is a row vector with 5 components.

$\begin{array}{cc} & m \\ n & [\ \vdots\ \vdots\ \vdots\ ] \end{array}$

Note that the m columns of $n\times m$ matrix are vectors in $\mathbb{R}^n$ but not $\mathbb{R}^m$.

(each vector in m vectors have n components)

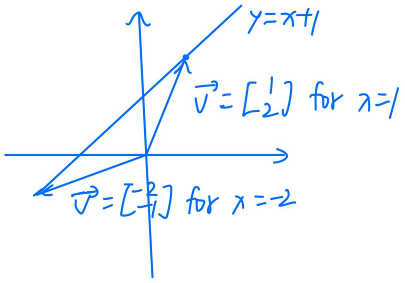
Standard representation of vectors

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ (in Cartesian plane)

(in $\mathbb{R}^3$: defined analogously
$\vdots$
$\mathbb{R}^n$)

When considering an infinite set of vectors, the arrow representation becomes impractical. In this case, sensible to represent $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ simply by the point (x,y), the head of the standard arrow representation of $\vec{v}$.

ex the set of all vectors $\vec{v} = \begin{bmatrix} x \\ x+1 \end{bmatrix}$ where x is arbitrary can be represented as the line $y = x+1$. 但 for a few special values of x we may still use arrow representation



Consider the system

$$\left|\begin{array}{l} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{array}\right|$$

The matrix which contains the coefficients of the variables in the system is called its $\boxed{\text{coefficient matrix}}$

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$$

By contrast the matrix

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

which displays all numerical info in the system is called $\boxed{\text{augmented matrix}}$

For the sake of clarity, we will often indicate the position of the equal signs in the equations by a dotted line

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$$

我们可以把之前的对equation的操作用在matrix上，并且将answer represented as a vector.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$$

ex

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2+t-4r \\ t \\ 2+r \\ 3+r \\ r \end{bmatrix}$$

这一个equation容易解是因为: coeff of the leading variable in each equa.

(P1) The Leading coefficient is always 1

(P2) The leading variable in each equa 不在其他equa出现.

(P3) The leading variables 以 natural order 出现 (x_1, x_3, x_4 而非 x_3, x_2, x_4)

当一个linear system有这样三条性质后就非常容易解

因而我们希望总是把linear system reduce至满足P_1, P_2, P_3.

ex.

$$(P_1)\ \begin{bmatrix} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{bmatrix} \begin{array}{l} \div 2 \\ \\ \\ \\ \end{array} \Rightarrow$$

$$(P_2)\ \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{bmatrix} \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{bmatrix} \begin{array}{l} +(IV) \\ \\ -(IV) \\ \\ \end{array} (P_3) \Rightarrow$$

(P2)

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & -2 & 4 & -6 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \times(-\frac{1}{2}) \\ \\ \end{array}$$

(P1)

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix} \begin{array}{l} (-III) \\ \\ \\ (-III) \end{array} \Rightarrow$$

(P2)

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

只要top → down，凭借这三个原则就可以完成消元获得满足(P_1, P_2, P_3)的reduced matrix 而解出linear system.

$$\left| \begin{array}{l} x_1 = 2 - x_2 - x_5 \\ x_3 = 4 + x_5 \\ x_4 = 3 + 2x_5 \end{array} \right| \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2-2t-3r \\ t \\ 4+r \\ 3+2r \\ r \end{bmatrix}$$

我们可以总结解linear system的algorithm.

From top to down, move on to the i^{th} equation: $cx_j + \dots = b$

⇒ ① Divide by c ⇒ $x_j + \dots = \frac{b}{c}$

② Eliminate x_j from all other equations above and below.

③ Proceed to next equation

④ Check: if $0 = \text{non-}0$ ⇒ inconsistent ⇒ no sol. ⇒ end

⑤ rearrange equations 以使leading variables in natural order.

$\boxed{\text{Reduced row-echelon form}}$ 行阶梯矩阵 otb P16

我们定义满足P_1, P_2, P_3的matrix为 $\boxed{\text{rref}}$

更严谨地说: 一个matrix is said to be in rref if.

① 若一row有non-0 entries, the first non-0 entry must be 1, called $\boxed{\text{the leading 1}}$ $\boxed{\text{pivot}}$

② 若一col中有pivot, 则col中其他entries必须为0.

③ 若一row中有pivot, 则它上面每一row必须也有pivot 且在它左侧.

(③ ⇒ rows of 0's 必须在bottom of matrix)

Elementary row operations 初等行变换 otb P16

之前我们对linear system中equations的三种operations用在matrix上，这三个operation统称 $\boxed{\text{elementary row operations}}$.

即 (1) Divide a row by a non-zero scalar

(2) Substract a multipite of a row from another.

(3) Swap two rows.

这种使用elementary row operations将matrix化为rref的解linear system的algorithm 叫做 $\boxed{\text{Gauss-Jordan elimination}}$