Math 217 Worksheet: Using Linear Algebra to Prove Trig Formulas

Theorem: *Let* $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^d \xrightarrow{T_B} \mathbb{R}^m$ *be a composition of two linear transformations given by left multiplication by matrices A and B, respectively. Then the composition $T_B \circ T_A$ is left multiplication by the matrix BA.*

A.

1. Discuss the source and target of the maps T_A, T_B and $T_B \circ T_A$, as well as the dimensions of the matrices A, B and BA.
2. Prove the theorem.

B. For any angle α, let $\rho_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by the map given by counterclockwise rotation through α.

1. Explain, using a geometric argument, why ρ_α is a linear transformation.
2. Explain, using a geometric argument, why $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta}$.
3. For any θ, find the matrix for ρ_θ.
4. Find the matrix for the linear transformation $\rho_{\alpha+\beta}$ in two different ways: one way should use the Theorem above, and one should not.
5. Use your answer to (4) to prove formulas[1] for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

C.

1. Let T be rotation counterclockwise through $\pi/2$. Explain why T^4 is the identity map.
2. $T = \rho_{2\pi/200}$. Explain why T^{200} is the identity map.
3. Fix a positive integer N. Find a 2×2 matrix A such that $A^N = I_2$ but no smaller positive power of A is the identity matrix.

D. Prove that if B is an $n \times n$ matrix such that $B^d = I_n$, then B is invertible.

[1] Now you know where these crazy formulas come from! Now you don't to memorize these formulas! You can derive them using matrix multiplication if you forget!