Qiulin Fan **ma217-w24 (section 005)**

This is the Matrix Operations Gateway test. Passing on this test is SEVEN or more correct answers from the eight problems on the test.

Problem 1. **(1 point)**

Consider the matrix A, with

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\operatorname{rank}(A) =$ ___

Solution: The rank of a matrix is just the number of leading ones in its reduced row echelon form. Here,

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\operatorname{rank}(A) = 4$.

Answer(s) submitted:

- 4

submitted: (correct)
recorded: (correct)

Problem 2. **(1 point)**

Consider the system of equations given by $A\vec{x} = \vec{b}$, with

$$\operatorname{rref}(A|\vec{b}) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

How many solutions are there to the system?
Number = ___
(Enter the number of solutions, or **inf** or **infinity** for infinitely many solutions.)

Solution: We know that a system $A\vec{x} = \vec{b}$ is inconsistent, that is, has no solutions, if and only if $\operatorname{rref}(A|\vec{b})$ has a row $[\,0\cdots 0 \mid 1\,]$.

Here,

$$\operatorname{rref}(A|\vec{b}) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so we know there is at least one solution. Then, if there are no free variables we know that solution is unique. From the reduced row echelon form we see that there is no column in the row-reduced form of the coefficient matrix that does not have a leading one, so there are no free variables. Therefore, there must be a unique solution to the system: the number of solutions is 1.

Answer(s) submitted:

- 1

submitted: (correct)
recorded: (correct)

Problem 3. **(1 point)**

If $A = \begin{bmatrix} 0 & 8 & 16 & -3 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$, find $\mathrm{rref}(A)$.

$\mathrm{rref}(A) = \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}$

Solution: We want to use a sequence of row operations to transform A so that the first non-zero entry in any row is a 1, all such leading 1s are to the right and below previous leading 1s, and all values above and below leading 1s are zero. In this case, we have

$$A = \begin{bmatrix} 0 & 8 & 16 & -3 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}.$$

One way to reduce this to reduced row-echelon form is by first swapping row 1 with row 4, to get

$$\begin{bmatrix} 0 & 8 & 16 & -3 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 0 & 8 & 16 & -3 \end{bmatrix}.$$

Then, by adding -2 times row 2 to row 4,

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 0 & 8 & 16 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Then adding -4 times row 1 to row 3 gives

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 4 & 0 & -8 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

adding 3 times row 3 to row 4 gives

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and finally multiplying row 2 by $1/4$ gives

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is the reduced row-echelon form of A, $\mathrm{rref}(A)$.

Answer(s) submitted:

- $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Problem 4. **(1 point)**

If $A = \begin{bmatrix} 0 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ and $k = 2$, then

$AB + kC = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We have $AB = \begin{bmatrix} -6 & -2 \\ -2 & 1 \end{bmatrix}$ and $kC = \begin{bmatrix} 2 & -4 \\ -2 & 2 \end{bmatrix}$, so that

$$AB + kC = \begin{bmatrix} -6 & -2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ -4 & 3 \end{bmatrix}.$$

Answer(s) submitted:

- $\begin{bmatrix} -4 & -6 \\ -4 & 3 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Problem 5. **(1 point)**

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ -8 & -4 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find rref$(A)$.

rref$(A) = \begin{bmatrix} __ & __ & __ \\ __ & __ & __ \\ __ & __ & __ \\ __ & __ & __ \\ __ & __ & __ \end{bmatrix}$

Solution: We want to use a sequence of row operations to transform A so that the first non-zero entry in any row is a 1, all such leading 1s are to the right and below previous leading 1s, and all values above and below leading 1s are zero. In this case, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ -8 & -4 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One way to reduce this to reduced row-echelon form is by first swapping row 3 with row 5, to get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ -8 & -4 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ -8 & -4 & 0 \end{bmatrix}.$$

Then, by adding 2 times row 4 to row 5,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ -8 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & -2 & 0 \end{bmatrix}.$$

Then swapping row 2 with row 4 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ -4 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \\ -4 & -2 & 0 \end{bmatrix},$$

adding -3 times row 3 to row 4 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \\ -4 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -4 & -2 & 0 \end{bmatrix},$$

adding 2 times row 2 to row 5 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -4 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and finally adding -2 times row 1 to row 2 gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is the reduced row-echelon form of A, rref(A).

Answer(s) submitted:

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Problem 6. **(1 point)**

Suppose that for the matrix equation $A\vec{x} = \vec{b}$ we know that

$$\text{rref}(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 & 1 & | & -5 \\ 0 & 1 & 0 & 5 & | & 2 \\ 0 & 0 & 1 & -1 & | & -4 \end{bmatrix}.$$

Write the general solution in the form $\vec{x} = \vec{u} + t\vec{v}$, where t is any real number.

$$\vec{x} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + t \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

Solution: From $\text{rref}(A|\vec{b})$ we can read the coefficients of the simplified system of equations we are solving. With $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, we have

$$x_1 + 1x_4 = -5$$
$$x_2 + 5x_4 = 2$$
$$x_3 - 1x_4 = -4,$$

So we see that, identifying the free variable x_4 by the parameter t,

$$\vec{x} = \begin{bmatrix} -5 - 1x_4 \\ 2 - 5x_4 \\ -4 + 1x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -5 \\ 1 \\ 1 \end{bmatrix}.$$

Answer(s) submitted:

- $\begin{bmatrix} -5 \\ 2 \\ -4 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} -1 \\ -5 \\ 1 \\ 1 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Problem 7. **(1 point)**

Consider the system of equations given by $A\vec{x} = \vec{b}$, with

$$\text{rref}(A|b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & | & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & -4 \end{bmatrix}.$$

How many solutions are there to the system?
Number = ___
(Enter the number of solutions, or **inf** or **infinity** for infinitely many solutions.)

Solution: We know that a system $A\vec{x} = \vec{b}$ is inconsistent, that is, has no solutions, if and only if $\text{rref}(A|\vec{b})$ has a row $[\, 0 \cdots 0 \mid 1]$.

Here,

$$\text{rref}(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & | & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & -4 \end{bmatrix},$$

so we know there is at least one solution. Then, if there are no free variables we know that solution is unique. From the reduced row echelon form we see that there is no column in the row-reduced form of the coefficient matrix that does not have a leading one, so there are no free variables. Therefore, there must be a unique solution to the system: the number of solutions is 1.

Answer(s) submitted:

- 1

submitted: (correct)
recorded: (correct)

Problem 8. **(1 point)**

Solve the linear system

$$\begin{aligned} 3x+6y &= -3 \\ 2x+5y &= 0 \\ 2x+4y &= -2 \end{aligned}$$

$x =$ ___
$y =$ ___

Solution: We can easily solve this by row-reduction. Writing the augmented matrix $A|b$, we have

$$A|b = \begin{bmatrix} 3 & 6 & | & -3 \\ 2 & 5 & | & 0 \\ 2 & 4 & | & -2 \end{bmatrix},$$

so that by multiplying row 1 by $1/3$ we have

$$\begin{bmatrix} 3 & 6 & -3 \\ 2 & 5 & 0 \\ 2 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 2 & 4 & -2 \end{bmatrix}.$$

Then adding -2 times row 1 to row 3,

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 2 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and finally adding -2 times row 1 to row 2,

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and we can see that $x = -5$ and $y = 2$.

Answer(s) submitted:

- -5
- 2

submitted: (correct)
recorded: (correct)

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