# MATH 217 - LINEAR ALGEBRA
# HOMEWORK 1, SOLUTIONS

## Part A (10 points)

Solve the following problems from the book:
**Section 1.1:** 20, 32, 34, 44
**Section 1.2:** 12, 36, 44

**Solution.**

**1.1.20**: We label the equations in the natural order, i.e. the top equation is labeled as I, the middle equation is II, and the bottom equation is III. Subtracting I from II and I from III gives:

$$\left|\begin{array}{llll} x & +y & -z & =2 \\ x & +2y & +z & =3 \\ x & +y & +(k^2-5)z & =k \end{array}\right| \to \left|\begin{array}{llll} x & +y & -z & = 2 \\ & +y & +2z & = 1 \\ & & (k^2-4)z & = k-2 \end{array}\right|.$$

We will be able to find a unique/none/infinitely many solutions to the system depending on the value of $k$:

(a) If $k=2$ then equation III reads $0\,z=0$, and the equation provides no information, i.e. $z$ can have arbitrary values. Therefore, there are *infinitely* many solutions (parametrized by $t\in\mathbb{R}$) given by:
$$x=1+3t,\ y=1-2t,\ z=t.$$

(b) If $k=-2$ then equation III reads $0\,z=-4$. Thus the system is inconsistent and it has *no* solutions.

(c) If $k\neq\pm2$ then equation 3 simplifies to $(k+2)z=1$. Hence, the *unique* solution to the system is:
$$z=\frac{1}{k+2},\ y=\frac{k}{k+2},\ x=\frac{k+5}{k+2}.$$

**1.1.32**: Write $f(t)=a+bt+ct^2$. If $(1,p)$, $(2,q)$, and $(3,r)$ lie on the graph of $f$, then $p,q,r$ satisfy the linear equations

$$\left|\begin{array}{ccccccc} a & + & b & + & c & = & p \\ a & + & 2b & + & 4c & = & q \\ a & + & 3b & + & 9c & = & r \end{array}\right|$$

Solving this linear system by row reducing its augmented matrix, we find that

$$a=3p-3q+r,\quad b=-\frac{5}{2}p+4q-\frac{3}{2}r,\quad \text{and}\quad c=\frac{1}{2}p-q+\frac{1}{2}r,$$

so $f(t)=(3p-3q+r)+(-\frac{5}{2}p+4q-\frac{3}{2}r)t+(\frac{1}{2}p-q+\frac{1}{2}r)t^2$. This solution is valid for all values of $p,q,r$.

**1.1.34**: The graph of the polynomial $f(t) = a+bt+ct^2$ goes through the point $(1,1)$ if $f(1) = 1$, i.e. $a+b+c = 1$. Similarly, the graph goes through $(2,0)$ if $f(2) = 0$, i.e. $a+2b+4c = 0$. The integral $\int_1^2 f(t)\,dt = -1$ translates to $\int_1^2 (a+bt+ct^2)\,dt = -1$. By integration this means that $(at + \frac{1}{2}bt^2 + \frac{1}{3}ct^3)|_1^2 = -1$, i.e. that $a + \frac{3}{2}b + \frac{7}{3}c = -1$. So we must solve the system

$$\begin{bmatrix} a & +b & +c = & 1 \\ a & +2b & +4c = & 0 \\ a & +\frac{3}{2}b & +\frac{7}{3}c = & -1 \end{bmatrix}.$$

Using the first equation to eliminate the $a$ variable in the second and third equations yields

$$\begin{bmatrix} a & +b & +c = & 1 \\ a & +2b & +4c = & 0 \\ a & +\frac{3}{2}b & +\frac{7}{3}c = & -1 \end{bmatrix} \to \begin{bmatrix} a & +b & +c = & 1 \\ & b & +3c = & -1 \\ & \frac{1}{2}b & +\frac{4}{3}c = & -2 \end{bmatrix}.$$

Using the second equation to eliminate the $b$ variable in the first and third equations yields

$$\begin{bmatrix} a & +b & +c = & 1 \\ & b & +3c = & -1 \\ & \frac{1}{2}b & +\frac{4}{3}c = & -2 \end{bmatrix} \to \begin{bmatrix} a & & -2c = & 2 \\ & b & +3c = & -1 \\ & & -\frac{1}{6}c = & -\frac{3}{2} \end{bmatrix}.$$

From the third equation, $c = 9$, so from the second equation, $b = -28$, and from the first equation, $a = 20$. Therefore, the only polynomial $f(t)$ of degree $\le 2$ whose graph goes through the points $(1,1)$ and $(2,0)$, and that satisfies $\int_1^2 f(t)\,dt = -1$ is $f(t) = 20 - 28t + 9t^2$.

**1.1.44**: The line through the points $(1,1,1)$ and $(3,5,0)$ is defined by the parametric equations

$$x = 1 + 2t$$
$$y = 1 + 4t$$
$$z = 1 - t$$

Solving $t = 1 - z$ and substituting into the first two equations, we get

$$x = 1 + 2(1 - z)$$
$$y = 1 + 4(1 - z)$$

which is a linear system with the desired properties. Rewriting these equations produces the equivalent linear system

$$x + 2z = 3$$
$$y + 4z = 5.$$

**1.2.12**: The sequence of steps to transform the augmented matrix of the system to its reduced row-echelon form is:

$$\left[\begin{array}{cccccc|c} 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ -2 & 1 & 6 & 0 & -6 & -12 & 0 \\ 0 & 1 & -3 & 0 & 1 & 5 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & -3 & 0 & 8 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 1 & -3 & 0 & 1 & 5 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{cccccc|c} 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 0 & -6 & 0 & 0 & 10 & 0 \\ 0 & 0 & 6 & 1 & 3 & -9 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 0 & -6 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{3}{2} & 0 & \frac{7}{2} & \frac{7}{2} & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solving for the leading variables in each row of the reduced row-echelon form gives:

$$\begin{aligned} x_4 &= -3x_5 - x_6, \\ x_3 &= \frac{5}{3}x_6, \\ x_2 &= -3x_3 - x_5 + 5x_6 = -x_5, \\ x_1 &= -\frac{7}{2}x_5 - x_6. \end{aligned}$$

The infinitely many solutions of the system are given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ \frac{5}{3} \\ -1 \\ 0 \\ 1 \end{bmatrix} t \qquad \text{for } s, t \in \mathbb{R}.$$

**1.2.36**: The vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ is perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ if and only if $x+3y-z=0$. Therefore the set of all vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is the solution set of this linear equation, which is the set $\left\{ y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \ : \ y, z \in \mathbb{R} \right\}$.

**1.2.44**: Let us represent the four question marks in the diagram by the variables $w$, $x$, $y$, and $z$, ordered counterclockwise beginning with $w$ representing the volume of traffic between Winthrop Street and Mt. Auburn Street. Then the information in the diagram leads to the linear system

$$\begin{aligned} w + z &= 120 + 150 \\ 300 + z &= 320 + y \\ 300 + y &= 400 + x \\ 100 + w + x &= 250, \end{aligned}$$

which has augmented matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 270 \\ 0 & 0 & -1 & 1 & 20 \\ 0 & -1 & 1 & 0 & 100 \\ 1 & 1 & 0 & 0 & 150 \end{bmatrix}.$$

Row-reducing this matrix using Gaussian elimination, we obtain its reduced row echelon form

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 270 \\ 0 & 1 & 0 & -1 & -120 \\ 0 & 0 & 1 & -1 & -20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This shows that the linear system does not have a unique solution, so we cannot say exactly how much traffic there was on each block. One possible scenario is

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 70 \\ 80 \\ 180 \\ 200 \end{bmatrix}.$$

In general, the highest and lowest possible traffic volumes at each point are given by:

$$\begin{aligned} 0 \leq w &\leq 150 \\ 0 \leq x &\leq 150 \\ 100 \leq y &\leq 250 \\ 120 \leq z &\leq 250 \end{aligned}$$

## Part B (50 points)

The following part covers material that is in the "Joy of Sets" and "Mathematical Hygiene" handouts, available on the Canvas site. Some of the material in these handouts, but not all of it, is recapped here.

### 1. Logical Connectives.

Every mathematical statement is either true or false. Starting from given mathematical statements, we can use logical operations to form new mathematical statements which are again either true or false. Let $P$ and $Q$ be two statements. Here are four basic logical constructions:

- The statement "$P$ *and* $Q$" is true exactly when both $P$ and $Q$ are true statements.
- The statement "$P$ *or* $Q$" is true exactly when at least one (possibly both!) of $P$ or $Q$ is true.
- The statement "*if* $P$ *then* $Q$" is true exactly when $Q$ is true or $P$ is false. The shorthand notation for *"if $P$ then $Q$"* is $P \implies Q$, read "$P$ implies $Q$."
- The statement "$P$ *if and only if* $Q$" is true exactly when both $P \implies Q$ and $Q \implies P$ are true statements. The shorthand notation for "$P$ *if and only if* $Q$" is $P \iff Q$.

**Problem 1.** Decide whether the following statements are true or false. *Briefly* justify your answers.[1]

(a) 2 is even or 3 is odd.
(b) If the Riemann Hypothesis is true, then 217 is not a prime number.
(c) $\frac{d}{dx}(x^2) = 2x$ if and only if $\tan(\pi/6) = \sqrt{3}$.

[1] Don't work too hard here. For instance, sufficient justification for claiming that "If 13 is prime then $\sqrt{2}$ is rational" is false could be: "FALSE by the meaning of 'if. . . then', since '13 is prime' is true, but '$\sqrt{2}$ is rational' is false." (In particular, you would *not* need to prove that 13 is prime or that $\sqrt{2}$ is irrational!)

(d) If the set of even prime numbers is infinite, then 10 is even and $10^{10}$ is odd.[2]
(e) If every right triangle in $\mathbb{R}^2$ has two acute angles, then every real number has a positive cube root.

**Solution.**

(a) TRUE, since the number 2 is even (Alternatively: TRUE, since the number 3 is odd).
(b) TRUE, because the conclusion "217 is not a prime number" of the implication is true, which makes the entire implication true no mater what the truth value of its premise is. (In fact, nobody knows for sure whether the Riemann Hypothesis is true or false – it has been an open problem since it was first conjectured in 1859).
(c) FALSE by the meaning of "if and only if," because "$\frac{d}{dx}(x^2) = x^2$" is true but "$\sin(\pi/6) = \sqrt{3}/2$" is false.
(d) TRUE, since the premise "If the set of even prime numbers is infinite" of the implication is false (there is only one even prime number), which makes the entire implication *true*.
(e) FALSE, by the meaning of "if...then," since the premise "every right triangle in $\mathbb{R}^2$ has two acute angles" of the implication is true, but the conclusion "every real number has a positive cube root" of the implication is false.

## 2. Quantifiers.

Starting from a mathematical statement or predicate which involves a variable, we can form a new one by quantifying the given variable.

- The quantifier "for all" indicates that something is true about every element in a given set and is abbreviated $\forall$. It is often appropriate to read "for all" as "for every" or "for each". For example, the truth value of $x^2 > 0$ depends on the value of $x$. So the quantified statement "$\forall x \in \mathbb{R},\ x^2 > 0$" is false, since it fails for $x = 0$.
- The quantifier "there exists" indicates that something is true for at least one element in a given set and is abbreviated $\exists$. It is often read as "for some", where "some" is not necessarily plural. For example, the truth value of $x^2 = 0$ depends on the value of $x$. So the quantified statement "$\exists x \in \mathbb{R}$ such that $x^2 = 0$" is true, since it holds for $x = 0$.
- The abbreviation "s.t." stands for "such that". (Yes, mathematicians can be lazy!)
- Since "for all" and "for some" are different quantifiers, it is very important that you never just write "for", since this would be ambiguous! (For instance, how should we interpret "$x^2 > 0$ for $x \in \mathbb{R}$". It's true if we mean "for some" but false if we mean "for all.") *Usually, but not always,* "for" by itself means "for all"; but it's always best to help out your reader (ahem, *grader*) by making your quantifiers explicit!

**Problem 2.**

(a) Let $P(x)$ be a statement whose truth value depends on $x$. An *example* is a value of $x$ that makes $P(x)$ true, and a *counterexample* is a value of $x$ that makes $P(x)$ false. Fill in the blank spaces with "is true", "is false", or "nothing" as appropriate:

| | "$\forall x, P(x)$" | "$\exists x$ s.t. $P(x)$" |
|---|---|---|
| An example proves | | |
| A counterexample proves | | |

[2]By convention, the connectives "and" and "or" bind more strongly than do the "if ... then" and "if and only if" connectives. This means that you should read the statement in Problem 1(d) as "If (the set of prime numbers is finite), then (10 is even and $10^{10}$ is odd)" rather than "(If the set of prime numbers is finite, then 10 is even) and ($10^{10}$ is odd)". *Negation*, which is signified by the word "not," binds even more strongly than "and" and "or" do.

Determine whether each of the given statements is true or false, and briefly justify your answer (as you did for Problem 1).

(b) Every prime number is even or odd.
(c) Every prime number is even or every prime number is odd.
(d) There exists $n \in \mathbb{Z}$ such that for every $x \in \mathbb{R}$, $n < x$.
(e) For every $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n < x$.
(f) Some squares are rectangles.
(g) For every nonnegative real number $a$, there exists a unique[3] real number $x$ such that $x^2 = a$.

**Solution.**

(a)

| | "$\forall x, P(x)$" | "$\exists\ x$ s.t. $P(x)$" |
|---|---|---|
| An example proves | nothing | is true |
| A counterexample proves | is false | nothing |

**Solution.**

(a) TRUE, since every prime number is an integer, and every integer is either even of odd.
(b) FALSE by the meaning of *or*, since "every prime number is even" is false, and also "every prime number is odd" is false.
(c) FALSE, since given any integer $n$, $x = n - 1$ is a real number that does not satisfy $n < x$.
(d) TRUE, since there is no real number that is less than every integer.
(e) TRUE, since *all* squares are rectangles, so in particular some of them are.
(f) FALSE. For instance, 1 is a nonnegative real number, but there does not exist a *unique* real number $x$ such that $x^2 = 1$, since both $(-1)^2 = 1$ and $1^2 = 1$.

## 3. Negation.

The *negation* of a statement $P$, denoted "*not* $P$," is a statement that is true whenever $P$ is false and false whenever $P$ is true. There may be many different ways to formulate the negation of $P$, but all of them will be logically equivalent. Note that the negation of the if-then statement "$P \implies Q$" is "$P$ and not $Q$," as $P \implies Q$ is false if and only if $P$ is true and $Q$ is false.

**Problem 3.** Formulate the negation of each of the statements below in a meaningful way (these statements have been recycled from Problems 1 and 2). Note: just writing "It is not the case that ..." before each statement will not receive credit, as that does not help the reader understand the meaning of the negation. (No justification is needed – you may just write the negation).

(a) 2 is even or 3 is odd.
(b) If the Riemann Hypothesis is true, then 217 is not a prime number.
(c) $\frac{d}{dx}(x^2) = 2x$ if and only if $\tan(\pi/6) = \sqrt{3}$.
(d) If the set of even prime numbers is infinite, then 10 is even and $10^{10}$ is odd.
(e) If every right triangle in $\mathbb{R}^2$ has two acute angles, then every real number has a positive cube root.
(f) There exists $n \in \mathbb{N}$ such that for every $x \in \mathbb{R}$, $x < n$.
(g) Some squares are rectangles.

[3] The statement "there exists unique $x \in X$ such that $P(x)$" means that there is one and only one element in the set $X$ having property $P$. For those who like fancy symbolisms, this is sometimes abbreviated "$\exists! x \in X$ s.t. $P(x)$."

**Solution.**

(a) 2 is not even and 3 is not odd.
(b) The Riemann Hypothesis is true and 217 is a prime number.
(c) Either $\frac{d}{dx}(x^2) = 2x$ and $\tan(\pi/6) \neq \sqrt{3}$, or else $\frac{d}{dx}(x^2) \neq 2x$ and $\tan(\pi/6) = \sqrt{3}$.
(d) The set of even prime numbers is infinite and either 10 is not even or $10^{10}$ is not odd.
(e) Every right triangle in $\mathbb{R}^2$ has two acute angles and some real number does not have a positive cube root.
(f) For all $n \in \mathbb{N}$, there is $x \in \mathbb{R}$ such that $x \geq n$.
(g) No squares are rectangles.

## 4. Converse and Contrapositive.

There are two additional logical statements that can be formed from a given "if-then" statement:

- The *converse* of the statement $P \implies Q$ is the statement $Q \implies P$. The converse may be true or false, independent of the truth value of the original "if-then" statement. To see this, compare the truth tables for both statements:

| $P$ | $Q$ | $P \implies Q$ | $Q \implies P$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ |

  The last two columns do not coincide.

- The *contrapositive* of the statement $P \implies Q$ is the statement not $Q \implies$ not $P$. The original "if-then" statement and its contrapositive have the *same* truth value. To see this, compare the truth tables for both statements:

| $P$ | $Q$ | $P \implies Q$ | not $Q$ | not $P$ | not $Q \implies$ not $P$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

  The columns corresponding to $P \implies Q$ and not $Q \implies$ not $P$ coincide.

**Problem 4.** Write both the converse and the contrapositive of the following "if-then" statements.

(a) If something can think, then it exists[4].
(b) If $p$ is an irrational number, then $p^2$ is an irrational number.
(c) If $n > 2$ is a natural number such that the Collatz sequence beginning with $n$ does not eventually reach 1, then $n^2 + 1$ is prime.

**Solution.**

(a) Converse: If something exists, then it can think. Contrapositive: If something does not exist, then it cannot think.

[4] You may recognize this as a paraphrase of "I think, therefore I am", which is itself a translation of "*Cogito, ergo sum*", an axiom used by the 17th century mathematician in his work of philosophy *Discourse on the Method*.

(b) Converse: If $p^2$ is an irrational number, then $p$ is an irrational number. Contrapositive: If $p^2$ is not an irrational number [alternatively: if $p^2$ is a rational number] then $p$ is not an irrational number [alternatively: then $p$ is a rational number].
(c) Converse: If $n^2 + 1$ is prime, then $n > 2$ is a natural number such that the Collatz sequence beginning with $n$ does not eventually reach 1. Contrapositive: If $n^2 + 1$ is not prime, then $n > 2$ is either not a natural number, or the Collatz sequence beginning with $n$ does eventually reach 1.

## 5. Sets.

A *set* is a container with no distinguishing feature other than its contents. The objects contained in a set are called the *elements* of the set. We write $a \in S$ to signify that the object $a$ is an element of the set $S$. The number of elements in a set $S$ is called the *cardinality* of the set, and is denoted by $|S|$.

Since a set has no distinguishing feature other than its contents, there is a unique set containing no elements which is called the *empty set* and is denoted $\emptyset$. Some other very common sets are the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}$ of all integers, the set $\mathbb{Q}$ of all rational numbers, the set $\mathbb{R}$ of all real numbers, and the set $\mathbb{C}$ of all complex numbers.

There are two important ways to specify a set.

- *Enumeration.* One can list the contents of the set, in which case the set is denoted by enclosing the list in curly braces. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- *Comprehension.* One can describe the contents of the set by a property of its elements. If $P(a)$ is a property of the object $a$, then the set of all objects $a$ such that $P(a)$ is true is denoted by $\{a \mid P(a)\}$, or equivalently $\{a : P(a)\}$. For example,

$$\mathbb{Q} \; = \; \{x \mid x = \tfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

  Comprehension can also be used together with functions. For instance, $\{n^2 : n \in \mathbb{N}\}$ is the set of all perfect squares, and $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is the set of all reciprocals of natural numbers.

Let $X$ and $S$ be sets. We say that $S$ is a *subset* of $X$ if $a \in S \Longrightarrow a \in X$ holds for all objects $a$. We write $S \subseteq X$ to signify that $S$ is a subset of $X$. This means that $S$ is a set each of whose elements also belongs to $X$. The subset of $X$ consisting of all elements $a$ of $X$ such that property $P(a)$ holds true is denoted $\{a \in X \mid P(a)\}$ or $\{a \in X : P(a)\}$.

**Problem 5.**

(a) Give common English descriptions of the following sets:
  (1) $\{n \in \mathbb{N} \mid \text{there exist } a \in \mathbb{N} \text{ such that } n = 2a - 1\}$.
  (2) $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1 \text{ and } a \geq 0\}$.
(b) Use set comprehension notation to give a description of each of the following sets:
  (i) The unit sphere in $\mathbb{R}^3$.
  (ii) The set of all integer multiples of $\sqrt{2}$.
(c) Determine whether each of the following statements is true or false (no justification necessary):

| | | | |
|---|---|---|---|
| (i) $\sqrt{2} \in \mathbb{R}$ | (iii) $\{\sqrt{2}\} \in \mathbb{R}$ | (v) $\emptyset \in \mathbb{R}$ | (vii) $\emptyset \in \emptyset$ |
| (ii) $\sqrt{2} \subseteq \mathbb{R}$ | (iv) $\{\sqrt{2}\} \subseteq \mathbb{R}$ | (vi) $\emptyset \subseteq \mathbb{R}$ | (viii) $\emptyset \subseteq \emptyset$ |

**Solution.**

(a) (i) The set of odd natural numbers. (ii) The upper half of the unit disk in the real plane.

(b) (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. (ii) $\{n\sqrt{2} : n \in \mathbb{Z}\}$.

(c)

| | | | |
|---|---|---|---|
| (i) T | (iii) F | (v) F | (vii) F |
| (ii) F | (iv) T | (vi) T | (viii) T |

## 6. Set Operations.

Starting from given sets, we can use set operations to form new sets.

- Given sets $X$ and $Y$, the *intersection* of $X$ and $Y$ is defined as
$$X \cap Y = \big\{a \mid a \in X \text{ and } a \in Y\big\}.$$
- Given sets $X$ and $Y$, the *union* of $X$ and $Y$ is defined as
$$X \cup Y = \big\{a \mid a \in X \text{ or } a \in Y\big\}.$$
- Given sets $X$ and $Y$, the *difference* of $X$ and $Y$, denoted $X \setminus Y$ or $X - Y$, is the set
$$\{x \in X \mid x \notin Y\}.$$
- Given a set $Y$ inside some larger set $X$, the *complement* of $Y$ with respect to $X$, denoted $Y^C$, is $X \setminus Y$. (The larger set $X$, sometimes referred to as the *universe*, is often suppressed in the notation).

**Problem 6.** For each rational number $q$, let $q\mathbb{N} = \{qm \mid m \in \mathbb{N}\}$, so that we have $q\mathbb{N} \subseteq \mathbb{Q}$.

(a) Use enumeration to describe each of the following sets (listing at least the first six elements of each set, in order from smallest to largest): $\frac{1}{2}\mathbb{N}$, $\frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \cap \frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \setminus \frac{1}{3}\mathbb{N}$, and $(3\mathbb{N})^C$ (where the complement is taken inside $\mathbb{N}$).

(b) What is the smallest natural number $n$ such that every set from part (a) is contained in $\frac{1}{n}\mathbb{N}$? (Alternatively, if you think no such $n$ exists, explain why.)

**Solution.**

(a) $\frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$
$\frac{1}{3}\mathbb{Z} = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots\}$
$\frac{1}{2}\mathbb{Z} \cap \frac{1}{3}\mathbb{Z} = \mathbb{Z} = \{0, 1, 2, 3, 4, 5, \ldots\}$
$\frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, \ldots\}$
$\frac{1}{2}\mathbb{Z} \setminus \frac{1}{3}\mathbb{Z} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \ldots\}$
$(3\mathbb{Z})^C = \{1, 2, 4, 5, 7, 8, 10, \ldots\}$

(b) $n = 6$

**Problem 7 (Recreational Problem).** [5] According to legend, Abraham Lincoln once said:

> "You can fool all the people some of the time, and some of the people all the time,
> but you cannot fool all the people all the time."

[5] Recreational problems may come up from time to time and exist for your amusement and edification, but they are optional and will not be graded. Handle with care. These problems are appropriate if (and only if) you need an additional challenge after finishing all of the other problems.

Form an intelligible negation of this statement.

*Hint*: Sometimes when you are dealing with a complex or potentially ambiguous statement in natural language, you can use logic to diagram the statement and remove any ambiguity. In this case, it will help to use a two-variable predicate, say $F(x,t)$, which intuitively says "you can fool person $x$ at time $t$." Then Honest Abe's statement can be translated into logic as follows:

$$\exists t \forall x F(x,t) \wedge \exists x \forall t F(x,t) \wedge \neg \forall t \forall x F(x,t).$$

Much better, no?[6] *Now* form the negation.

**Solution.** In logical notation, the negation is:

$$\forall t \exists x \neg F(x,t) \vee \forall x \exists t \neg F(x,t) \vee \forall t \forall x F(x,t).$$

Translated (roughly) back into English, this becomes: "There's always someone you can't fool, or no one can be fooled all the time, or everyone can be fooled all the time." Doesn't quite have the same ring...

[6]You can probably guess from this that $\wedge$ is the logical symbol for "and" and $\neg$ is the logical symbol for "not". Symbols like this, and also $\forall$ and $\exists$, can safely appear in your scratch work but should never appear in your written proofs, which need to be intelligible to human beings!