Math 217 Worksheet 10: Bases and Dimension (§3.3)

IMPORTANT DEFINITIONS: Let $\{v_1, v_2, \ldots v_d\}$ be a set of vectors in a vector space V.

The set **spans** V if every $v \in V$ is a linear combination of the vectors $v_1, \ldots, v_d$.

The set of vectors is **linearly dependent** if there is a non-trivial relation—that is, a relation

$$c_1v_1 + c_2v_2 + \cdots + c_dv_d = 0$$

where at least one $c_i \neq 0$. The set is **linearly independent** if it is not linearly dependent.

The set of vectors is a **basis** for V if it is *both* linearly independent *and* spans V.

The **dimension** of V is the number of elements in a basis (by a theorem, all bases have the same number of elements).

All the above concepts can be defined for infinite sets (see Problem 5). It is not much harder, but in Math 217 we focus mostly on finite dimensional vector spaces.

Problem 1. Practice with definitions. Determine whether the given set $\mathcal{S}$ *spans* the given vector space W. If not, describe the *subspace spanned by* $\mathcal{S}$. Is the set $\mathcal{S}$ *linearly independent*? If not, find a *relation* on $\mathcal{S}$. Finally, find a basis for W, and determine the dimension of W.

(a) $W = \mathbb{R}^4$, $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 + \vec{e}_1\}$

(b) $W = \mathbb{R}^4$, $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_2 + \vec{e}_1\}$

(c) W is the image of the linear map $\mathbb{R}^4 \xrightarrow{T} \mathbb{R}^3$ sending $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x-y \\ 2y+z \\ x+y+z \end{bmatrix}$, $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

[HINT: Remember a theorem describing $\operatorname{im} T$ in terms of the standard matrix of T.]

(d) W is the plane $\{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{u} = 0\}$ where $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution:

(a) The vectors are linearly independent and span W. So $\mathcal{S}$ is a basis. The dimension is 4.

(b) These are not linearly independent because the last vector is redundant. A relation is $\vec{e}_1 + \vec{e}_2 - (\vec{v}_1\vec{e}_2) = 0$. So they are not a basis. Also, they do not span all of $\mathbb{R}^4$, since $\vec{e}_4$ is not a linear combination of them. For a basis, we can take $\{\vec{e}_1, \ldots, \vec{e}_4\}$. The dimension is 4.

(c) The image is spanned by the columns of the matrix, so the elements of $\mathcal{S}$ do span the image. However, they are not linearly independent: we have the non-trivial relation $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$. So S is not a basis. We need only remove the redundant ones to get a basis. Any two of the three columns will be a basis. So the dimension is 2.

(d) W is the plane (though the origin) with equation $x-y=0$. Since both vectors in S are on it, and are linearly independent, they must form a basis for W provided W is dimension two. Intuitively, W is dimension 2 because is is a plane. To prove this, we can use the rank-nullity theorem. We can also think of W as the kernel of the linear map $\mathbb{R}^3 \to \mathbb{R}$ sending $\vec{v} = [x\ y\ z]^T$ to $\vec{v} \cdot \vec{u} = x - y$. The image is all of $\mathbb{R}$, so has dimension one. So by rank-nullity theorem, the dimension of the kernel is 2.

Problem 2. Bases for $\mathbb{R}^{m\times n}$. Consider the vector space $\mathbb{R}^{2\times 2}$. Consider the elements

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Explain, using the definition, why $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ spans $\mathbb{R}^{2\times 2}$.

(b) Explain, using the definition, why $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is linearly independent.

(c) Explain, using the definition, why $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $\mathbb{R}^{2\times 2}$. What is $\dim \mathbb{R}^{2\times 2}$?

(d) Is the set $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ a basis for $\mathbb{R}^{2\times 2}$.

(e) What is the dimension of $\mathbb{R}^{m\times n}$? Describe a basis.

Solution:

(a) Every matrix is a linear combination of these: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$.

(b) If $aE_{11}+bE_{12}+cE_{21}+dE_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-trivial relation, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $a = b = c = d = 0$.

(c) This set is both linearly independent and spans $\mathbb{R}^{2\times 2}$.

(d) Yes. It spans $\mathbb{R}^{2\times 2}$ because we can write each matrix as a linear combination of these. To see this, we just need to observe we can write each E_{ij} as a linear combination of these. Also, they are linearly independent: if

$$a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

is a relation, then $\begin{bmatrix} a+c & a+b+c \\ b+c & d \end{bmatrix} = 0$, which point you can show that $a = b = c = d$ so the relation is trivial.

(e) $\mathbb{R}^{m\times n}$ has dimension mn and the analogous basis, consisting of matrices with exactly one non-zero entry in the ij-th spot, is a basis.

Problem 3. Consider the vector space $\mathcal{P}$ of polynomials (of all degrees). Remember the elements of $\mathcal{P}$ are familiar *functions* from $\mathbb{R}$ to $\mathbb{R}$; you have been looking at their graphs in previous courses.

(a) Explain* why the set $\{1, x, x^2, x^3, \ldots, x^n\}$ is linearly independent for any $n \in \mathbb{N}$. We have a symbol for the subspace it spans. What is it?

(b) Find a basis for the vector space $\mathcal{P}_n$, and determine its dimension. Find a different basis.

*you may use the fact from previous math courses that a polynomial function of degree d has at most d zeros

Solution:

(a) Suppose that $\sum_{i=0}^{n} c_i x^i$ is the ZERO function from $\mathbb{R}$ to $\mathbb{R}$. Note that *every* real number is a therefore a root of the polynomial $\sum_{i=0}^{n} c_i x^i$, so there are infinitely many roots. But if the c_i are not all zero, then we would have a polynomial of some degree $d \leq n$ with infinitely many zeros. This is is impossible unless we have the zero polynomial, with all $c_i = 0$. So the set is linearly independent.

Here is different (nicer?) argument of linear independence, due to W23 Math 217 Section 8 student Andrew Kiesling. Suppose $a_0 + a_1 + \cdots + a_n x^n = 0$ is a non-trivial relation (so the a_i's here are some scalars in $\mathbb{R}$). Without loss of generality, we can assume $a_n \neq 0$. Taking the n-th derivative of both sides, we have $n!a_n = 0$, but this forces $a_n = 0$, a contradiction!

Their span is the vector space $\mathcal{P}_n$ of polynomials of degree at most n.

(b) A basis for $\mathcal{P}_n$ is $\{1, x, x^2, x^3, \ldots, x^n\}$, so $\mathcal{P}_n$ is $n+1$ dimensional. Another basis is $\{1, x+1, x^2+x, x^3+x^2, \ldots, x^n + x^{n-1}\}$.

Problem 4. Rank and Nullity. Let A be a $d \times n$ matrix with columns $\vec{C}_1, \ldots, \vec{C}_n$.

(a) Prove that the following are equivalent:

  (i) $[a_1\ a_2\ \ldots a_n]^\top$ is a solution to $A\vec{x} = 0$.

  (ii) $[a_1\ a_2\ \ldots a_n]^\top$ is in the kernel of the transformation T_A given by multiplication by A.

  (iii) $a_1\vec{C}_1 + a_2\vec{C}_2 + \cdots + a_n\vec{C}_n = 0$ is a relation on the columns.

(b) Suppose the system $A\vec{x} = 0$ has solution $[1\ 2\ 3\ 4\ 5]^T$. Are the columns of A linearly independent? What is the maximal possible rank of A given this information? [HINT: Use (a).]

(c) Now instead, suppose $\{\vec{C}_1, \ldots, \vec{C}_n\}$ is linearly independent. What does this say about the system of equations $A\vec{x} = 0$? [HINT: Use (a).]

(d) Now, suppose $\vec{D}_1, \ldots, \vec{D}_n$ be the columns of rref(A). Prove that $a_1\vec{C}_1 + a_2\vec{C}_2 + \cdots + a_n\vec{C}_n = 0$ is a relation on the columns of A if and only if $a_1\vec{D}_1 + a_2\vec{D}_2 + \cdots + a_n\vec{D}_n = 0$ is a relation on the columns of rref(A). [HINT: Use (a), and remember that solutions of any system of linear equations are the same as the solutions of the corresponding row-reduced form of the system.]

(e) Suppose now that A is 4×5 and that rref(A) has pivots in columns 1, 3 and 5 only. Write down all you can about rref(A), using letters in the places you can't determine the entries.

(f) With A the 4×5 matrix as in (e), find a basis for the image of T_A.
[HINT: Recall the important theorem from WS 8: *The image of T_A is spanned by the columns of A.*†]

(g) With the 4×5 matrix as in (e), how many "free variables" are there when writing out the solution space of $A\vec{x} = 0$. What does this say about the dimension of the kernel of T_A?

(h) Discuss the following important theorem with your group, to make sure everyone understand the **statements.** Then prove it. [HINT: You've basically already proved it all!]

Theorem. Let $T_A : \mathbb{R}^n \to \mathbb{R}^d$ be the linear transformation given by multiplication by A.

- The columns of A corresponding to pivot columns of rref(A) form a *basis* for $\operatorname{im} T_A$.
- The dimension of the image of T_A equals the rank of A.
- The dimension of the source of T_A equals $\dim(\ker T_A) + \dim(\operatorname{im} T_A)$.

†The image of T_A is often called the "image of A" for short. Because $\operatorname{im} T_A$ is the same as the subspace of $\mathbb{R}^d$ spanned by the columns of A, it is also sometimes called the "column space" of A.

Solution:

(a) All three of these conditions can be written as $A\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$. So all three are equivalent.

(b) If $A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = 0$, this tells us that $C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 = 0$. This is a non-trivial relation on the columns, so they are not linearly independent. So $A\vec{x}$ has a non-trivial solution, and the rank must be less than 5.

(c) $A\vec{x} = 0$ has only the trivial solution.

(d) The solutions to $A\vec{x} = 0$ are the same as the solutions to $rref A\vec{x} = 0$. So a relation $a_1C_1 + a_2C_2 + \cdots + a_nC_n = 0$ means that $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is a solution to $A\vec{x} = 0$. This is the same as being a solution to $\text{rref}A\vec{x} = 0$. In turn, this is equivalent to $a_1D_1 + a_2D_2 + \cdots + a_nD_n = 0$.

(e) $\begin{bmatrix} 1 & a & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) Columns C_1, C_3, C_5 are a basis. There are no relations on these because there are no relations on D_1, D_3, D_5 (which is the same as $\vec{e}_1, \vec{e}_2, \vec{e}_3$) and we know the relations are the same from (d). Since D_2 is a multiple of D_1, it is also true that C_2 is a multiple of C_1. Similarly, since D_4 is a linear combination of D_1 and D_3, it is also true that C_4 is a linear combination of C_1 and C_3.

(g) There are two. The dimension of the kernel is the total number of variables minus the number of pivots. In this case, that's $5 - 3 = 2$.

Problem 5: An infinite dimensional space. Consider the vector space $\mathcal{P}$ of polynomials.

(a) Explain why no finite set of polynomials spans $\mathcal{P}$. [HINT: Think about *degree*.]

(b) Explain what it means that the infinite set $\mathcal{S} = \{1, x, x^2, \cdots\}$ spans $\mathcal{P}$.
[CAUTION! There is no such thing as an "infinite sum" of vectors.]

(c) Is the set $\mathcal{S}$ in (b) linearly independent? By definition, this means that every relation on any finite subset of the elements in $\mathcal{S}$ is trivial.
[CAUTION! There is no such thing as an "infinite linear combination" of vectors.]

(d) What is the dimension of the vector space $\mathcal{P}$? Describe two *different* bases.

Solution:

(a) Suppose some finite set $\mathcal{S}$ of polynomials spans $\mathcal{P}$. There is a maximal degree of the elements in this set, call it D. Note that a linear combination of polynomials of degree $\leq D$ can not have degree $D + 1$. So the polynomial x^{D+1} is not in the span of $\mathcal{S}$, and so $\mathcal{S}$ does not span $\mathcal{P}$.

(b) The infinite set $\{1, x, x^2, x^3, \ldots, \}$ spans $\mathcal{P}$ because every polynomial is a (finite) linear combination of these functions.

(c) Yes! If we had a relation on the elements of $\mathcal{S}$, it a relation on a finite subset $\{1, x, x^2, x^3, \ldots, x^n\}$. In Problem 3, we showed there is only the trivial relation.

(d) $\mathcal{P}$ is an infinite dimensional vector space! There are many bases. One is $\{1, x, x^2, x^3, \ldots, \}$. Another is $\{1, x, \frac{1}{2}x^2, \frac{1}{3}x^3, \ldots, \}$. Another is $\{1, x + 1, x^2 + 2, x^3 + 3, \ldots, \}$.