Worksheet 2: Linear Equations and Matrices (§1.1–1.2)

Problem 1. Consider the linear system

$$\begin{aligned} 4z - 9 &= 3x + 6y \\ 3z - 1 &= w + 2x + y \\ 3z - 3x &= 2w - 1 \\ w + 4x + 5y + 7 &= 9z \end{aligned}$$

(In answering the questions below, write the variables w, x, y, z in their natural order).

(a) Write the *coefficient matrix* of this linear system.

(b) Write the *augmented matrix* of this linear system.

(c) Find the *reduced row echelon form (rref)* of the augmented matrix of this linear system.

(d) Express the solution set of this linear system as a set of vectors. Your solution should have the form

$$\{\vec{c} + t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \mid t_1, \ldots, t_k \in \mathbb{R}\},$$

where $\vec{c} \in \mathbb{R}^4$ and $t_1, \ldots, t_k$ are parameters. (You'll have to figure out what k should be).

Solution:

(a) $\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$

Problem 2. Consider a matrix A, which we transform into the matrix

$$B = \begin{bmatrix} 0 & a & 0 & 0 & b \\ c & 0 & d & 0 & e \\ 0 & 0 & 0 & 1 & f \end{bmatrix}$$

by a sequence of elementary row operations. Assume $B = \mathrm{rref}(A)$.

(a) What can we say about the constants a through f? What is the first column of A?

Solution: $a = 1$, $c = 0$, $d = 1$, and b, e, f can be anything. The first column of A is $\vec{0}$.

(b) We (temporarily!*) define the *rank* of a matrix to be the number of leading 1s in its reduced row echelon form. What is the rank of the matrix A?

Solution: 3

(c) If $\left[A \mid \vec{0}\right]$ is the augmented matrix of a linear system, is the system consistent? If so, how many solutions does it have?

Solution: Yes, the system would be consistent, with infinitely many solutions.

(d) Now, going further, suppose that $\left[A \mid \vec{v}\right]$ (where $\vec{v} \neq \vec{0}$) is the augmented matrix for a linear system. Is the system consistent? If so, how many solutions are there?

Solution: Again, the system would be consistent with infinitely many solutions.

(e) Continue to suppose that $\left[A \mid \vec{v}\right]$ (where $\vec{v} \neq \vec{0}$) is the augmented matrix of a linear system. Can you change the last row of B so that the resulting linear system has no solutions? How does your answer depend on $\vec{v}$? Can you change the last row of B to ensure that there is a *unique* solution?

Solution: By changing the last row of B to have all zeros, we could make the system have no solution, as long as the row operations that change A into B do not also change $\vec{v}$ into a vector whose third component is zero. But there is no way to make the system have a unique solution, since B has more columns than rows.

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*Eventually we will define $\mathrm{rank}(A) = \dim \mathrm{im}(A)$, but we don't know what "dim" or "im" mean yet (although you're welcome to guess, if you can!).

Mathematical Proofs

Problem 3. DeMorgan's Law states that if A and B are sets[†], then $(A \cup B)^C = A^C \cap B^C$.

(a) Draw a Venn diagram to understand DeMorgan's Law and why it should be true.

(b) Start to think about how you might *prove* DeMorgan's Law. What definitions would you need to understand? What kind of logical structure does the theorem have? How does this help you scaffold your proof? How would the proof begin? — and how should it end?

(c) A useful strategy is to think of all the different ways that we can restate the conclusion we are trying to prove. For example, one technique for showing that two sets X and Y are equal is to show, separately, that both $X \subseteq Y$ and $Y \subseteq X$.[‡] Using only this technique, outline a proof of DeMorgan's Law.

(d) Refine your outline of the proof of DeMorgan's Law by incorporating the following proof techniques:

  (i) The standard way to prove that $X \subseteq Y$ is to say "Let x be an element in X" and then somehow show that x also belongs to Y.

  (ii) The standard way to prove that an element x belongs to the intersection $Z \cap W$ is to show separately that $x \in Z$ and that $x \in W$.

(e) Prove DeMorgan's Law by filling in this outline to obtain a complete argument.

Solution: We must show that

$$(A \cup B)^C \subseteq A^C \cap B^C \quad \text{AND} \quad A^C \cap B^C \subseteq (A \cup B)^C.$$

To show $(A \cup B)^C \subseteq A^C \cap B^C$, take arbitrary $x \in (A \cup B)^C$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. This can be written $x \in A^C$ and $x \in B^C$, and therefore we conclude that $x \in A^C \cap B^C$. This shows $(A \cup B)^C \subseteq A^C \cap B^C$.

To show $A^C \cap B^C \subseteq (A \cup B)^C$, suppose $x \in A^C \cap B^C$. This means that $x \in A^C$ and $x \in B^C$, which implies $x \notin A$ and $x \notin B$. In other words, $x \notin A \cup B$, and therefore $x \in (A \cup B)^C$. That is, $A^C \cap B^C \subseteq (A \cup B)^C$.

We have shown $(A \cup B)^C \subseteq A^C \cap B^C$ and $A^C \cap B^C \subseteq (A \cup B)^C$, which means that $(A \cup B)^C = A^C \cap B^C$ as desired.

Problem 4. Define (for now[§]) a *linear function* from $\mathbb{R}$ to $\mathbb{R}$ to be a function

$$f : \mathbb{R} \to \mathbb{R}$$

for which $f(t) = mt$ for some constant m. *Using only this definition*[¶], prove that for any linear function $f : \mathbb{R} \to \mathbb{R}$ and any real constants a and b, $f(ax + by) = af(x) + bf(y)$ for all $x, y \in \mathbb{R}$.

[†] both contained in the same larger set called "the universe"

[‡] As you read about in the *Joy of Sets*.

[§] we will soon introduce the general definition of a linear function; the definition here should be used only for this exercise.

[¶] Careful! In the past you may have used *linear* to mean "polynomial of degree one" (as in *linear*, *quadratic*, *cubic*, etc.) In this class we will use a different definition that vastly generalizes the one given in this problem.

Solution: Let f be a linear function, and let $m \in \mathbb{R}$ be such that $f(t) = mt$ for all $t \in \mathbb{R}$. Suppose a, b, x, y are any real numbers. Then

$$f(ax + by) \;=\; m \cdot (ax + by) \;=\; a \cdot (mx) + b \cdot (my) \;=\; af(x) + bf(y).$$

Problem 5. It is a famous theorem of Pythagoras that $\sqrt{2}$ is an irrational number.

(a) One way to prove this theorem is by *contradiction*: in this method of proof, we *assume* that the theorem is false, and from that assumption derive a logical absurdity (such as $0 = 1$ or "P and not P" where P is some statement). A usual first line in such a proof is "Suppose for contradiction that [the theorem] is false." What is the first line here?

Solution: Suppose for contradiction that $\sqrt{2}$ is rational.

(b) Complete the proof that $\sqrt{2}$ is irrational. You may use the following facts without proof:

> *Fact 1:* Every positive rational number can be written in the form m/n where m and n are positive integers with no common divisors.
>
> *Fact 2:* The square of any odd integer is odd.

Solution: Suppose for contradiction that $\sqrt{2}$ is rational. Then by Fact 1 we can write $\sqrt{2} = m/n$ where m and n are positive integers having no common divisors. Then $2 = m^2/n^2$, so $2n^2 = m^2$. This implies that m is even (since if m is odd, then m^2 is odd too by Fact 2, and so could not be a multiple of an even number). So we can write $m = 2k$ where k is an integer. Then $2n^2 = (2k)^2 = 4k^2$, so $n^2 = 2k^2$, which implies that n is even by Fact 2. But now m and n are both multiples of 2, contradicting our assumption that m and n have no common divisors. Therefore $\sqrt{2}$ is irrational.