# Worksheet 25: Diagonalization over the Real Numbers (§7.3)

**Theorem.** Let $V \xrightarrow{T} V$ be a linear transformation, where $V$ is an $n$-dimensional vector space. The transformation $T$ has an eigenbasis (or, equivalently, is diagonalizable) if and only if the sum of the geometric multiplicities of its eigenvalues is $n$.

**Corollary.** The transformation $T$ has an eigenbasis if and only if every eigenvalue satisfies $\text{almu}(\lambda) = \text{gemu}\ (\lambda)$, and $T$ does not have any non-real eigenvalues (see Problem 1c below).

**Problem 1.** To *diagonalize* a matrix $A$ means to find a diagonal matrix $D$ and an invertible matrix $S$ such that $A = SDS^{-1}$.

(a) Explain why only square matrices with eigenbases can be diagonalized.

(b) Find the almu and gemu* of each eigenvalue of the matrix $P = \begin{bmatrix} 2 & 1 & 3 & \pi \\ 0 & 2 & 2 & 9/7 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Do the same for the matrix $Q = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -5 \end{bmatrix}$.

(c) For the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we could say that $A$ has *non-real* eigenvalues. Why? What are the eigenvalues? [We will return to *diagonalization over* $\mathbb{C}$ on Worksheet 26.]

(d) Use the theorem above to determine whether or not $P$, $Q$, or $A$ is diagonalizable (over $\mathbb{R}$).

(e) Diagonalize any diagonalizable matrices (over $\mathbb{R}$) discovered in (d).

**Solution:**

(a) To diagonalize a matrix is essentially the same as finding an eigenbasis. Eigenvectors are defined only for transformations $V \to V$, with the same source and target, so the matrix in any basis is square. Given a matrix $A$, we think of $A \in \mathbb{R}^{n\times n}$ as the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. If $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis, then $[T]_\mathcal{B}$ is diagonal, and if $S = S_{\mathcal{B}\to\mathcal{E}}$ is the change of basis matrix, then the columns of $S$ are precisely $\vec{v}_1, \ldots, \vec{v}_n$. In this case, $A = [T]_\mathcal{E} = S[T]_\mathcal{B}S^{-1}$.

(b) For $P$, the char poly is $(x-2)^2(x+1)^2$, so the eigenvalues are 2 and $-1$ and both have algebraic multiplicity two. We compute the geometric multiplicity of each eigenvalue by rank-nullity:

$$P - 2I_4 = \begin{bmatrix} 0 & 1 & 3 & \pi \\ 0 & 0 & 2 & 9/7 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

*Use Rank-Nullity to do this quickly; you need not find the full eigenspace!

has rank 3, hence nullity 1. This means gemu$(2) = 1$. And

$$P + 1I_4 = \begin{bmatrix} 3 & 1 & 3 & \pi \\ 0 & 3 & 2 & 9/7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank 3, hence nullity 1. This means gemu$(-1) = 1$.

For $Q$, the char poly is $-(x-1)^2(x+5)$ so the eigenvalues are 1 and $-5$, which have algebraic multiplicities 2 and 1, respectively. The geometric multiplicities are the same, which we see by computing the nullity of the matrices $Q - 1I_3$ and $Q + 5I_3$:

$$Q - I_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & -6 \end{bmatrix}, \qquad Q + 5I_3 = \begin{bmatrix} 6 & 0 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix},$$

which have nullity 2 and 1, respectively.

(c) The char poly of $A$ is $x^2 + 1$, which has no real roots, so no real eigenvalues. But it does have complex roots: $\pm i$.

(d) By the Theorem, only $Q$ is diagonalizable. Indeed, $A$ has no real eigenvalues, so is not diagonalizable. For $P$, the sum of the gemus is 2, and not 4, so the Theorem tells us that $P$ is not diagonalizable. For $Q$, the sum of the gemus for $Q$ is $2 + 1 = 3$, and $Q$ is diagonalizable by the Theorem. Alternatively, the corollary also tells us $Q$ is diagonalizable, since the algebraic and geometric multiplicities of each eigenvalue are equal.

(e) To find an eigenbasis for $Q$, we compute the eigenspaces of the eigenvalues. The 1-eigenspace is the kernel of $Q - I_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & -6 \end{bmatrix}$, which has basis $(\vec{e}_1, \vec{e}_2)$. The $-5$-eigenspace is the kernel of $Q + 5I_3 = \begin{bmatrix} 6 & 0 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ which has basis $\vec{v} = \begin{bmatrix} 1 \\ \frac{7}{3} \\ -2 \end{bmatrix}$ So $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{v})$ is an eigenbasis for $Q$. We have

$$Q = [Q]_{\mathcal{E}} = S_{\mathcal{B}\to\mathcal{E}}[Q]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}},$$

which becomes $Q = SDS^{-1}$ where $S = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{v} \end{bmatrix}$ and $D$ is the diagonal matrix with diagonal elements $1, 1, -5$.

**Problem 2.** Let $\mathcal{P}_2$ be the vector space of polynomial functions of degree at most 2, and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by $T(f)(x) = f(x) + (1-x)f'(x) + 2x^2 f''(x)$.

(a) Find the characteristic polynomial of $T$. Find the almu and gemu of each eigenvalue.

(b) Determine whether or not $T$ is diagonalizable; if so, find a basis $\mathcal{B}$ of $\mathcal{P}_2$ in which $T$ is diagonal.

**Solution:**

(a) In the basis $\mathcal{E} = (1, x, x^2)$, we have $[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, so the characteristic polynomial of $T$ is $\det\big(tI - [T]_{\mathcal{E}}\big) = t(t-1)(t-3)$. So the eigenvalues of $T$ are $0, 1$, and $3$, each with algebraic multiplicity one. Since the geometric multiplicity of an eigevalue is a positive number at most the algebraic multiplicity, it follows that each eigenvalue has geometric multiplicity one as well.

(b) Since the geometric and algebraic multiplicities of $T$ are equal for each eigenvalue, the Corollary tells us that $T$ is diagonalizable with $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ for an appropriate eigenbasis $\mathcal{B}$. By finding vectors in the eigenspaces $\ker(\lambda I - T)$ for each eigenvalue $\lambda$ of $T$, we obtain the eigenbasis $\mathcal{B} = (1, 1-x, 1+2x+3x^2)$.

**Problem 3: Proof of the Theorem.** Let $V \xrightarrow{T} V$ be a linear transformation, where $V$ is an $n$-dimensional vector space. Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of $T$. Recall Theorem D on Worksheet 25: *Eigenvectors with distinct eigenvalues are linearly independent.*

(a) Prove that $E_{\lambda_1} \cap E_{\lambda_2} = \{0_V\}$. [Hint: This follows from Thm D, though it is not hard to prove directly.]

(b) Let $\{v_1, \dots, v_{\beta_1}\}$ be a basis for $E_\lambda$ and $\{w_1, \dots, w_{\beta_2}\}$ be a basis for $E_{\lambda_2}$. Show that $\{v_1, \dots, v_{\beta_1}, w_1, \dots, w_{\beta_2}\}$ is linearly independent.

(c) Let $\mathcal{B}_k$ be a basis of the eigenspace $E_{\lambda_k}$. Prove that $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_r$ is linearly independent.

(d) Prove the Theorem.

(e) Show the Corollary follows from the Theorem. [Hint: Use Thm C from WS 24: $\operatorname{almu}(\lambda_i) \geq \operatorname{gemu}(\lambda_i)$.]

**Solution:**

(a) It follows immediately from the definition of eigenspace that $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$ whenever $\lambda_1 \neq \lambda_2$, since if $v \in E_{\lambda_1} \cap E_{\lambda_2}$ where $\lambda_1 \neq \lambda_2$, then $\lambda_1 v = T(v) = \lambda_2 v$, which implies $(\lambda_2 - \lambda_1)v = 0$, so $v = 0$.

(b) Suppose we have a relation

$$a_1 v_1 + \dots + a_{\beta_1} v_{\beta_1} + b_1 w_1 + \dots + b_{\beta_2} w_{\beta_2} = 0. \tag{1}$$

Then

$$a_1 v_+ \dots + a_{\beta_1} v_{\beta_1} = -b_1 w_1 - \dots - b_{\beta_2} w_{\beta_2} \in E_{\lambda_1} \cap E_{\lambda_2} = \{0_V\}.$$

So $a_1 v_+ \dots + a_{\beta_1} v_{\beta_1} = 0$ is a relation on $\{v_1, \dots, v_{\beta_1}\}$, the basis for $E_{\lambda_1}$. This forces $a_i = 0$ for all $i$. Similarly, $b_i = 0$ for all $i$. So the relation (**??**) is trivial, completing the proof.

(c) Write $\mathcal{B}_k = \{v_{k1}, \dots, v_{km_k}\}$ for each $1 \leq k \leq r$. Suppose we have a relation

$$\sum_{k=1}^{r} \sum_{j=1}^{n_k} c_{kj} v_{kj} = 0. \tag{2}$$

For each $1 \le k \le r$, write $\vec{w}_k = \sum_{j=1}^{m_k} c_{kj} v_{kj}$, so each $w_k$ is a $\lambda_k$ eigenvalue, and (**??**) becomes $w_1 + \cdots + w_r = 0$. From Theorem D (WS 24), eigenvectors with distinct eigenvalues are linearly independent, so this is possible only if each $w_k$ is 0. So $w_k = \sum_{j=1}^{m_k} c_{kj} v_{kj} = 0$, which implies that $c_{kj} = 0$ for all $k, j$ since each $\mathcal{B}_k$ is a basis for $E_{\lambda_k}$. So our relation (**??**) is trivial and hence $\mathcal{B}$ is linearly independent.

(d) If $\beta_i$ is the geometric multiplicity of $\lambda_i$, then there are $\beta_i$ elements in the basis $\mathcal{B}_i$ and a total of $\sum \beta_i$ linearly independent element in $V$ that are eigenvectors. If these total $n$ in number, they must be an eigenbasis. On the other hand, if we have an eigenbasis, then the vectors in it must partition out into groups, depending which eigenvalue they belong to. As vectors in $E_\lambda$ they are also linear independent, so there are at most $\beta_i$ in the $\lambda_i$-eigenspace. The only way to get a full set of $n$ linearly independent eigenvectors, then, is that $\sum \beta_i = n$.

(e) The corollary follows since, using the notation of (d), $\sum_{i=1}^{r} \operatorname{gemu}(\lambda_i) \le \sum_{i=1}^{r} \operatorname{almu}(\lambda_i) \le n$. So we can get the needed equality for the theorem only when the inequality $\operatorname{gemu}(\lambda_i) \le \operatorname{almu}(\lambda_i)$ is an equality for each $i$.

**Problem 4.** Let $A$ be an $n \times n$ matrix with only one eigenvalue. Prove that if $A$ is a diagonalizable, then $A$ is already a diagonal matrix.

**Solution:** Suppose $A$ is a diagonalizable $n \times n$ matrix with only one eigenvalue, $\lambda$. Let $P$ be an invertible matrix such that $A = P\lambda I_n P^{-1}$. Then using the fact that scalar matrices commute with all other matrices (of the same square size), we have

$$A \;=\; P\lambda I_n P^{-1} \;=\; \lambda I_n P P^{-1} \;=\; \lambda I_n,$$

showing that $A$ is diagonal.

Another way to think about this is that for $A$ to be diagonalizable, $\lambda$ must have geometric multiplicity $n$. Thus $\dim\ker(\lambda I_n - A) = n$, which implies $\lambda I_n - A = 0$.

**Problem 5.** Determine whether the following matrices or linear transformations are diagonalizable. Diagonalize the given matrix or transformation, if possible, or explain why this is impossible.

(a) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

(d) $T : \mathcal{P}_3 \to \mathcal{P}_3$ is the differentiation operator on the space $\mathcal{P}_3$ of polynomials of degree $\le 3$.

**Solution:**

(a) The char poly is $x^2 + 3x - 10 = (x-5)(x+2)$. So the eigenvalues are 5 and $-2$, both of algebraic and geometric multiplicity 1. So $A$ is diagonalizable. To find an eigenbasis, we find each eigenspace:

$$\ker(A - 5I_2) = \ker \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} = \mathrm{Span}(\begin{bmatrix} 3 \\ 4 \end{bmatrix}), \quad \text{and}$$

$$\ker(A + 2I_2) = \ker \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix}).$$

So an eigenbasis $\mathcal{B} = (\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$, and $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$. Thus

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}^{-1}.$$

(b) $A$ is not diagonalizable, since $A$ has only one eigenvalue ($\lambda = 1$), and the corresponding eigenspace is only 1-dimensional.

(c) $A$ is not diagonalizable. The eigenvalues of $A$ are $\lambda = 3$ and $\lambda = 2$, each with an algebraic multiplicity of 2. The geometric multiplicity of 2 is 2, but the geometric multiplicity of 3 is only 1 (since rank $(A - 3I_4) = 3$), meaning that there are not enough eigenvectors corresponding to $\lambda = 3$ to complete a basis for $\mathbb{R}^4$.

(d) $T$ is not diagonalizable, since $T$ has only one eigenvalue ($\lambda = 0$), and the corresponding eigenspace is only 1-dimensional.

**Problem 6.** Let $A$ be a $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$.

(a) Show that the characteristic polynomial of $A$ is $x^2 - \mathrm{tr}(A)x + \det(A)$.

(b) Show that $\mathrm{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.

**Solution:**

(a)

$$\det\left(xI_2 - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det \begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix} = (x-a)(x-d) - bc = x^2 - (a+d)x + ad - bc.$$

(b) The characteristic polynomial factors as $(x-\lambda_1)(x-\lambda_2)$. Multiplying out, we get $x^2 - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2$. Comparing to (a), the result follows.

**Problem 7.** For (a) - (f), prove the stated conclusion or show it is false by giving a counterexample: *For all $n \times n$ matrices $A$ and $B$, if $A$ and $B$ are similar then they have the same ...*

(a) characteristic polynomial

(b) eigenvalues

(c) eigenvectors

(d) determinant

(e) trace

(f) reduced row echelon form

**Solution:** (a), (b), (d), and (e) are true. For (d), we have

$$\det(PAP^{-1}) \;=\; (\det P)(\det A)(\det P^{-1}) \;=\; (\det P)(\det A)(\det P)^{-1} \;=\; \det A.$$

For (e), using the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\operatorname{tr}(PAP^{-1}) \;=\; \operatorname{tr}(AP^{-1}P) \;=\; \operatorname{tr}(A).$$

For (a), the characteristic polynomial of $PAP^{-1}$ is

$$\det(xI - PAP^{-1}) \;=\; \det(PxIP^{-1} - PAP^{-1}) \;=\; \det(P(xI-A)P^{-1}) \;=\; \det(xI - A),$$

which is the characteristic polynomial of $A$. Finally, (b) follows from (a) since the eigenvalues of a matrix are just the roots of its characteristic polynomial.

For counterexamples, it is easy to see geometrically that two reflections over different lines in $\mathbb{R}^2$ will be similar but will have different eigenvectors. And the projection matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are similar but have different rref. (Intuitively, rref has nothing to do with similarity).

**Problem 8.** For each of (a) – (f), find two $n \times n$ matrices that are *not* similar to each other, but nevertheless have the same ...

(a) characteristic polynomial

(b) eigenvalues

(c) eigenvectors

(d) determinant

(e) trace

(f) reduced row echelon form

Then show that two *diagonalizable* $n \times n$ matrices with the same characteristic polynomial *are* similar to each other.

**Solution:** The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are not similar to each other, since the first is diagonalizable and the second is not. But they have the same characteristic polynomial, eigenvalues, trace, determinant, and rref. For (c), $I_2$ and $2I_2$ have the same eigenvectors but different eigenvalues, so they are not similar.

If $A$ and $B$ are two diagonalizable $n \times n$ matrices with the same characteristic polynomial, and hence the same eigenvalues, then $A$ and $B$ are both similar to the same $n \times n$ diagonal matrix $D$ whose diagonal entries are the common eigenvalues of $A$ and $B$, and therefore $A$ and $B$ are similar to each other.

**Problem 9.** In each part below, either give an example of a square matrix $A$ with the stated properties, or else explain why this is impossible:

(a) invertible but not diagonalizable

(b) diagonalizable but not invertible

(c) neither diagonalizable nor invertible

(d) both diagonalizable and invertible

**Solution:**

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Problem 10.** For $S = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix}$, consider the linear transformation $\gamma_S : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ sending $A$ to $S^{-1}AS$. Find the characteristic polynomial, the eigenvalues, and for each eigenvalue, its algebraic and geometric multiplicity. Then find a basis for each eigenspace. Does $\gamma_S$ have an eigenbasis?

**Solution:**

For this, we need to pick a basis $\mathcal{B}$ and find the $\mathcal{B}$-basis. Say we pick $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. We then compute $T(E_{11}) = \begin{bmatrix} -1 & 7 \\ 0 & 0 \end{bmatrix}, T(E_{12}) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -E_{12}, T(E_{21}) = \begin{bmatrix} -7 & 49 \\ -1 & 7 \end{bmatrix} T(E_{22}) = \begin{bmatrix} 0 & -7 \\ 0 & 1 \end{bmatrix}$. We then arrange this into the $\mathcal{B}$-matrix:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -7 & 0 \\ 7 & -1 & 49 & -7 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 7 & 1 \end{bmatrix}.$$

We compute its char poly by computing $\det[T]_{\mathcal{B}} - xI_4$ by Laplace expansion along the second column. We get

$$(-1-x)(1-x)(-1-x)(x-1) = (x-1)^2(x+1)^2.$$

This shows that the eigenvalues are $1, 1, -1, -1$ (that is, $\pm 1$ each with algebraic multiplicity 2.

Note that we could have taken any basis $\mathcal{A}$. If we are clever, we might notice that $I_2$ and $S$ are both eigenvectors (with eigenvalue 1), so they are both excellent choices for (part of) a basis. Also, $E_{12}$ is an eigenvector of eigenvalue -1. These three at least will be very easy to find coordinates for in $\mathcal{A}$, and the $\mathcal{A}$-matrix will have lots of zero. To complete the argument this way, we still need to find a fourth matrix not in the span of $I_2, S$ and $E_{12}$ and find the $\mathcal{A}$ coordinates of its image under $T$. Try it and see if you get the same characteristic polynomial! (of course you have to, since the char poly doesn't depend on the choice of the basis.

**Problem 11.*** Prove Theorem C from WS 24: *If $\lambda$ is an eigenvalue of a linear transformation $T : V \to V$, with $V$ finite dimensional, then gemu($\lambda$) $\leq$ almu($\lambda$).* [HINT: If $(v_1, \ldots, v_d)$ is a basis for $E_\lambda$, extend to a basis $(v_1, \ldots, v_d, v_{d+1}, \ldots, v_n)$ for $V$. Compute the char poly of $T$ using the block diagonal matrix $[T]_{\mathcal{B}}$.]

**Solution:** See the proof in the textbook of Theorem 7.3.6.