MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 2, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:

Section 1.3: 26, 34, 48;
Section 2.1: 6, 38, 44, 46.

Solution.

[1.3.26] Since $Ax = b$ has a unique solution, the rank of A must be $m = 3$ (Example 3c in Section 1.3). Thus, $Ax = c$ has either exactly one solution or no solutions (Example 4d in Section 1.3).

[1.3.34] (a) $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$. $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$. $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$.

(b) $B\vec{e}_1 = \vec{v}_1$. $B\vec{e}_2 = \vec{v}_2$. $B\vec{e}_3 = \vec{v}_3$.

[1.3.48] (a) Assume $\vec{x}_h$ is a solution of $A\vec{x} = \vec{0}$. Then, $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h$ using algebraic rules for $A\vec{x}$. Hence by assumption and algebraic rules for vectors,

$$A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}.$$

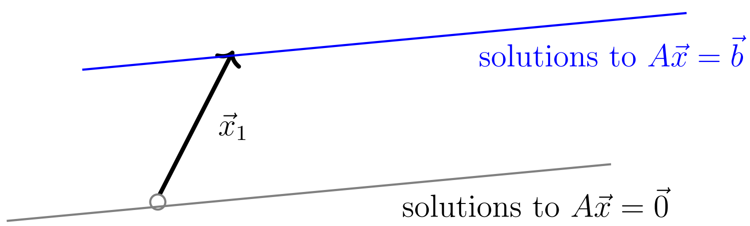
We can conclude that $\vec{x}_1 + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$.

(b) Assume $\vec{x}_2$ is a solution of $A\vec{x} = \vec{b}$. Then, $A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1$ using algebraic rules for $A\vec{x}$. Hence by assumption and algebraic rules for vectors,

$$A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0}.$$

We can conclude that $\vec{x}_2 - \vec{x}_1$ is a solution to $A\vec{x} = \vec{0}$.

(c)



Solution. [**2.1.6**] Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

For any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$$T\vec{x} = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}\ .$$

Thus, there exists a matrix A satisfying Definition 2.1.1. We conclude that T is linear.

[**2.1.38**] $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2v_1 - v_2.$

[**2.1.44**] Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and

$$A = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

Note that

$$T(\vec{x}) = \vec{v} \times \vec{x} = \begin{bmatrix} -v_3x_2 + v_2x_3 \\ v_3x_1 - v_1x_3 \\ -v_2x_1 + v_1x_2 \end{bmatrix} = A\vec{v}$$

so T is linear.

[**2.1.46**] $T(\vec{e}_1) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ap + cq \\ ar + cs \end{bmatrix}$, and
$T(\vec{e}_2) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} bp + dq \\ br + ds \end{bmatrix}$,

so the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$ is $\begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix}$

Part B (25 points)

Let X and Y be sets. Recall that a **function** f **from** X **to** Y is a rule which assigns a unique element $f(x) \in Y$ to each element $x \in X$. We call X the **domain** or **source** of f, we call Y the **codomain** or **target space** of f, and we write $f : X \to Y$ to indicate that f is a function from X to Y. The **image** of f is defined to be the set $\text{im}(f) = \{f(x) \mid x \in X\}$. (Note that the codomain and the image of f are not necessarily equal.) We say that the function $f : X \to Y$ is:

- **surjective** or **onto** if for all $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$;
- **injective** or **one-to-one** if for all $y \in \text{im}(f)$ there exists at most one $x \in X$ such that $f(x) = y$;
- **bijective** if f is both injective and surjective.

Problem 1. In parts (a) – (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

(a) the function $f : [0, 4] \to [0, 18]$ defined by $f(x) = x^2 + 2$;
(b) the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x - 5$;
(c) the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = 2x^2 + 5y^2$;
(d) the function $q : \mathbb{N} \to \mathbb{N}$ defined by $q(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$

Solution.

(**a**) It is injective but not surjective. Suppose $f(x) = f(y)$ for some $x, y \in [0, 4]$. Then $x^2 + 2 = y^2 + 2$. This implies $x^2 = y^2$, and $x = y$ since $x, y \in [0, 4]$. So, f is injective.

We have $0 \in [0, 18]$, but there exists no $x \in [0, 4]$ such that $f(x) = 0$. The equation $x^2+2 = 0$ has no real solutions. So, f is not surjective.

(b) The function is both injective and surjective. Suppose $g(x) = g(y)$ for some $x, y, \in \mathbb{R}$. Then $2x - 5 = 2y - 5$. This gives $x = y$. So, g is injective. Let $z \in \mathbb{R}$. Consider $\frac{z+5}{2} \in \mathbb{R}$. Since $g(\frac{z+5}{2}) = z$, we see that every real number is in the mage of g. So, g is onto.

(c) The function is not injective since $h(1, 1) = h(-1, -1) = 7$. It is not surjective either since there are no $x, y \in \mathbb{R}$ such that $h(x, y) = -1$. The equation $2x^2 + 5y^2 = -1$ has no solutions in real numbers since left hand side is greater than or equal to zero.

(d) The function is not injective since $q(3) = q(6)$. It is surjective since given $n \in \mathbb{N}$ we have $2n \in \mathbb{N}$ and $q(2n) = n$. So, every element in $\mathbb{N}$ is in the image of q.

Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define the **direct image** or **forward image of** A **under** f to be the set

$$f[A] = \{f(x) \mid x \in A\}.$$

Similarly, if $B \subseteq Y$ then we define the **preimage** or **inverse image of** B **under** f to be the set

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$

Note that $f[X] = \mathrm{im}(f)$.

Problem 2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

(a) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.
(b) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$.
(c) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f^{-1}[f[A]] = A$.
(d) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f[X \setminus A] = Y \setminus f[A]$.
(e) For every bijective function $f : X \to Y$ and all $A, B \subseteq X$, we have $f[A\cap B] = f[A] \cap f[B]$.

Solution.

(**a**) **False**. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = x^2$. Let $A = \{1, 2, 3, \cdots\}$ and
$B = \{-1, -2, -3, \cdots\}$. We have $A \cap B = \emptyset$, but $f[A] \cap f[B] = \{1, 4, 9, 16, 25, \cdots\}$.

(b) True. Let $f : X \to Y$ be a function such that $A, B \subseteq X$ and $f[A] \cap f[B] = \emptyset$. Suppose now that $A \cap B \neq \emptyset$. Then there exists $x \in A \cap B$, so $f(x) \in f[A]$ and $f(x) \in f[B]$. This gives us $f(x) \in f[A] \cap f[B]$, which gives a contradiction. So, we must have $A \cap B = \emptyset$.
$B = \{-1, -2, -3, \cdots\}$.

(c) False. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = x^2$. Let $A = \{1, 2, 3, \cdots\}$. Then, $f^{-1}[f[A]] = A \cup B \neq A$, where $B = \{-1, -2, -3, \cdots\}$.

(d) False. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = x^2$. Let $A = \{1, 2, 3, \cdots\}$. Then, $4 \in f[X \setminus A]$ but $4 \notin Y \setminus f[A]$.

(e) True. Let $z \in f[A] \cap f[B]$. Then $z = f(x), z = f(y)$ for some $x \in A, y \in B$. We have $z = f(x) = f(y)$. Since f is injective this gives us $x = y$. So, $x \in A \cap B$. Hence $z = f(x) \in f[A \cap B]$. This shows $f[A] \cap f[B] \subseteq f[A \cap B]$. Let $z \in f[A \cap B]$. Then there exists $x \in A \cap B$ such that $z = f(x)$. Since $x \in A$ we have $z \in f[A]$. Similarly, since $x \in B$ we have $z \in f[B]$. So, $z \in f[A] \cap f[B]$. This shows $f[A \cap B] \subseteq f[A] \cap f[B]]$. Hence, $f[A \cap B] = f[A] \cap f[B]$.

We call a function $T : \mathbb{R}^m \to \mathbb{R}^n$ a *linear transformation* if it satisfies:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$; and
(2) $T(k\vec{x}) = kT(\vec{x})$ for all vectors $\vec{x} \in \mathbb{R}^m$ and all scalars $k \in \mathbb{R}$.

(Note that this definition differs from the one given in Section 2.1 of the textbook.)

Problem 3.

(a) Prove that for every function $f : \mathbb{R} \to \mathbb{R}$, if $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, prove that every function $f : \mathbb{R} \to \mathbb{R}$ that preserves scalar multiplication is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.)
(b) Give an example to show that the argument you gave in part (a) cannot work in 2 dimensions. That is, explicitly describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is *not* a linear transformation but has the property that $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Remember to prove that your example works!

Solution.

(a) Let $x, y \in \mathbb{R}$. Since $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$, we have $f(x + y) = (x + y)f(1) = xf(1) + yf(1) = f(x) + f(y)$. So, $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(b) Let $f((x, y)) = \begin{cases} (y, 0), & \text{if } y \neq 0; \\ (x, 0), & \text{if } y = 0. \end{cases}$

If $c = 0$, then $f(c\vec{x}) = f(\vec{0}) = \vec{0} = 0f(\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

Let $c \in \mathbb{R} \setminus \{0\}$. Then $cy = 0 \iff y = 0$. So, $f(c(x, y)) = f(cx, cy) = \begin{cases} (cy, 0), & \text{if } y \neq 0; \\ (cx, 0), & \text{if } y = 0. \end{cases}$

We also have $cf((x, y)) = \begin{cases} (cy, 0), & \text{if } y \neq 0; \\ (cx, 0), & \text{if } y = 0. \end{cases}$

So, $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. On the other hand, since $f((1, 0)) + f((-1, -2)) = (1, 0) + (-2, 0) = (-1, 0) \neq (-2, 0) = f((0, -2))$, f does not preserve vector addition, and therefore is not a linear transformation.

(b) (Alternate Solution): Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(\vec{x}) = \begin{cases} (0, 0) & \text{if } \vec{x} = \vec{0}; \\ (\|\vec{x}\|, \|\vec{x}\|) & \text{if } \vec{x} \neq \vec{0} \text{ and } 0 \leq \theta < \pi; \\ (-\|\vec{x}\|, -\|\vec{x}\|) & \text{if } \vec{x} \neq \vec{0} \text{ and } \pi \leq \theta < 2\pi, \end{cases}$$

where θ is the angle $\vec{x}$ makes with the positive x-axis. One can check that this function also preserves scalar multiplication but not vector addition.

(b) (Another Alternate Solution): Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(\vec{x}) = \vec{x}$ if $x_1 = x_2$, and $f(\vec{x}) = \vec{0}$ otherwise. One can check that this function also works. [There are many possible solutions to this problem].

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, suppose that f preserves addition).

(a) Prove that $f(0) = 0$.
(b) Prove that for all $x \in \mathbb{R}$, $f(-x) = -f(x)$.
(c) Use induction to prove that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(nx) = nf(x)$.
(d) Prove that for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, $f(mx) = mf(x)$.
(e) (**RECREATIONAL**) Prove that for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$, $f(qx) = qf(x)$.

Remark: It will perhaps come as a surprise that the property in (c)-(e) cannot be extended to include arbitrary real scalars. That is, there exist functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ but also $f(cx) \neq cf(x)$ for some $c, x \in \mathbb{R}$. Put yet another way, there exist functions $f : \mathbb{R} \to \mathbb{R}$ that preserve addition but not scalar multiplication. *This* fact is actually rather difficult to prove!)

Solution.

(a) We have $f(0) = f(0+0) = f(0) + f(0)$. By subtracting $f(0)$ from both sides, we get $f(0) = 0$.

(b) Let $x \in \mathbb{R}$. We have $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$. Hence, $f(-x) = -f(x)$.

(c) For $n = 1$ the statement is obvious. Let $k \geq 1$, and suppose $f(kx) = kf(x)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$, and consider $f((k+1)x)$. We have $f((k+1)x) = f(kx + x) = f(kx) + f(x)$. By substituting $f(kx) = kf(x)$, we get $f((k+1)x) = kf(x) + f(x) = (k+1)f(x)$. This means the statement is true for $k+1$. Hence, by induction the statement is true for every integer $n \geq 1$.

(d) In part (c) we proved the claim for positive integers. For $n = 0$, the statement follows from (a) easily. Let m be a negative integer. Then $-m$ is a positive integer and by (c) we have $f((-m)x) = -mf(x)$ for all $x \in \mathbb{R}$. Then $0 = f(0) = f(mx + (-m)x) = f(mx) + f((-m)x) = f(mx) - mf(x)$ for all $x \in \mathbb{R}$. This gives $f(mx) = mf(x)$ for all $x \in \mathbb{R}$. So, the statement is true for every negative integer m as well. Hence, $f(mx) = mf(x)$ for all $x \in \mathbb{R}$ and for all $m \in \mathbb{Z}$.