Part A

Solve the following problems from Bretscher:

5.4: 27, 31.

5.5: 15, 23, 32(a, b, c, d).

5.4

27. Consider an inconsistent linear system $A\vec{x} = \vec{b}$, where A is a 3×2 matrix. We are told that the least-squares solution of this system is $\vec{x}^* = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Consider an orthogonal 3×3 matrix S. Find the least-squares solution(s) of the system $SA\vec{x} = S\vec{b}$.

Sol The normal equation of $SA\vec{x} = S\vec{b}$ is

$$(SA)^T SA\vec{x} = (SA)^T S\vec{b}$$

$$\Rightarrow A^T S^T S A\vec{x} = A^T S^T S\vec{b} \text{ by Thm 5.3.9(c)}$$

So $A^T(S^T S)A\vec{x} = A^T(S^T S)\vec{b}$

Since S is orthogonal, $S^T = S^{-1}$

Therefore the normal equation is

$$A^T A\vec{x}^* = A^T\vec{b}$$

which is the same with the normal equation of $A\vec{x} = \vec{b}$

So the least-square solution of $SA\vec{x} = S\vec{b}$

is the same as that of $A\vec{x} = \vec{b}$ which is $\left\{ \begin{bmatrix} 7 \\ 11 \end{bmatrix} \right\}$

31. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0,3)$, $(1,3)$, $(1,6)$, using least squares. Sketch the solution.

Sol

$$\begin{cases} c_0 = 3 \\ c_0 + c_1 = 3 \\ c_0 + c_1 = 6 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}}_{\vec c} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}}_{\vec b}$$

The normal equation of $A\vec c = \vec b$

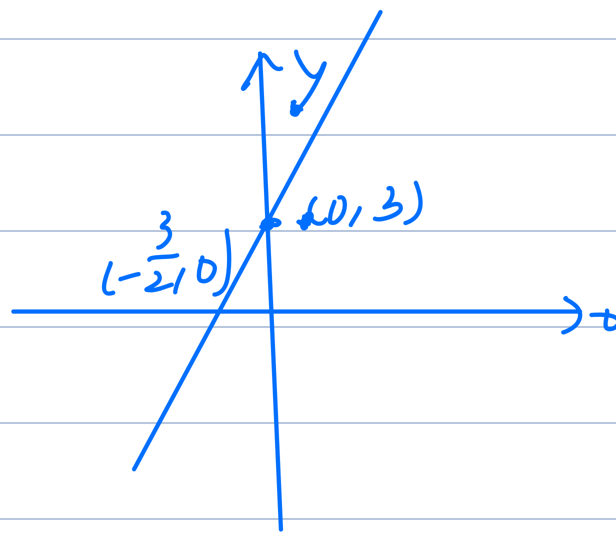
is $A^TA\vec c = A^T\vec b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \vec c = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \vec c = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

$$\vec c = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 9 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$$



5-5

15. For which values of the constants b, c, and d is the following an inner product in $\mathbb{R}^2$?

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$$

Hint: Be prepared to complete a square.

Sol By symmetry:

$$x_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2 = x_1y_1 + by_1x_2 + cy_2x_1 + dx_2y_2$$

$$\Rightarrow b = c$$

By positive define:

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1^2 + (b+c)x_1x_2 + dx_2^2 > 0$$

whenever $x_1, x_2 \neq 0$.

So $x_1^2 + 2bx_1x_2 + dx_2^2 > 0$

$$(x_1 + bx_2)^2 - (b^2x_2^2 - dx_2^2) > 0$$

$$(x_1 + bx_2)^2 + (d - b^2)x_2^2 > 0$$

$$\Rightarrow d > b^2$$

Therefore the condition is $b = c$ and $d > b^2$

23. In the space P_1 of the polynomials of degree ≤ 1, we define the inner product

$$\langle f, g \rangle = \frac{1}{2}\big(f(0)g(0) + f(1)g(1)\big).$$

Find an orthonormal basis for this inner product space.

Consider $(1, x)$ as a basis

$$\|1\| = \sqrt{\langle 1, 1 \rangle}^2 = \sqrt{1} = 1$$

$$\Rightarrow \vec{u_1} = 1$$

$$\vec{u_2} = \frac{x - \langle x, 1 \rangle 1}{\| x - \langle x, 1 \rangle 1 \|}$$

$$\langle x, 1 \rangle = \frac{1}{2}$$

$$\left\|x - \frac{1}{2}\right\| = \sqrt{\left(\left(\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right) + \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2}\right)\right)^2} = \frac{1}{2}$$

$$\Rightarrow \vec{u_2} = 2x - 1$$

So an orthogonal basis is $(1, 2x-1)$.

32. In the space $C[-1, 1]$, we introduce the inner product

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)dt.$$

a. Find $\langle t^n, t^m\rangle$, where n and m are positive integers.

b. Find the norm of $f(t) = t^n$, where n is a positive integer.

c. Applying the Gram–Schmidt process to the standard basis $1, t, t^2, t^3$ of P_3, construct an orthonormal basis $g_0(t), \ldots, g_3(t)$ of P_3 for the given inner product.

d. Find the polynomials $\dfrac{g_0(t)}{g_0(1)}, \ldots, \dfrac{g_3(t)}{g_3(1)}$. (Those are the first few *Legendre polynomials,* named after the great French mathematician Adrien-Marie Legendre, 1752–1833. These polynomials have a wide range of applications in math, physics, and engineering. Note that the Legendre polynomials are normalized so that their value at 1 is 1.)

Sol

a. $\langle t^n, t^m\rangle = \frac{1}{2}\int_{-1}^{1} t^{n+m}\,dt$

$$= \frac{1}{2}\left[\frac{1}{n+m+1}t^{n+m+1}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{n+m+1} - \frac{(-1)^{n+m+1}}{n+m+1}\right)$$

$$= \begin{cases} 0, & \text{if } n+m \text{ is even} \\ \frac{1}{n+m+1}, & \text{if } n+m \text{ is odd} \end{cases}$$

b. $\|f(t)\| = \sqrt{\langle f(t), f(t)\rangle^2} = \sqrt{\langle t^n, t^n\rangle^2}$

$= \sqrt{\frac{1}{2n+1}}$

c. $g_0(t) = \frac{1}{\|1\|} = \frac{1}{\sqrt{1}} = 1$

Since $\langle t, 1\rangle = \frac{1}{2}\int_{-1}^{1} t\,dt = 0$

So $g_1(t) = \frac{t - \langle t,1\rangle 1}{\|t - \langle t,1\rangle 1\|} = \frac{t}{\sqrt{\frac{1}{3}}} = \sqrt{3}\,t$

$g_2(t) = \frac{t^2 - \langle t^2, 1\rangle 1 - \langle t^2, t\rangle t}{\|t^2 - \langle t^2,1\rangle 1 - \underbrace{\langle t^2, t\rangle t}_{=0}\|}$

$= \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \sqrt{\frac{90}{8}}(t^2 - \frac{1}{3}) = \frac{3\sqrt{5}}{2}(t^2 - \frac{1}{3})$

$= \frac{\sqrt{5}}{2}(3t^2 - 1)$

$g_3(t) = \frac{t^3 - \sum_{i=1}^{3}\langle t^3, g_i(t)\rangle}{\|t^3 - \sum_{i=1}^{3}\langle t^3, g_i(t)\rangle\|} = \frac{t^3 - \langle t^3, \sqrt{3}t\rangle \sqrt{3}t}{\|t^3 - \langle t^3, \sqrt{3}t\rangle\sqrt{3}t\|}$

$= \frac{t^3 - \frac{3}{5}t}{\|t^3 - \frac{3}{5}t\|} = \frac{1}{\sqrt{\frac{4}{175}}}(t^3 - \frac{3}{5}t)$

$= \frac{\sqrt{7}}{2}(5t^3 - 3t)$

d. $\frac{g_0(t)}{g_0(1)} = 1$, $\frac{g_1(t)}{g_1(1)} = t$

$\frac{g_2(t)}{g_2(1)} = \frac{3t^2 - 1}{2}$, $\frac{g_3(t)}{g_3(1)} = \frac{5t^3 - 3t}{2}$

Part B

Problem 1. Consider the four points $(2,4,6)$, $(1,3,2)$, $(1,1,0)$ and $(1,2,3)$ in $\mathbb{R}^3$.

(a) Write a matrix equation that, *if it were consistent*, could be used to find the coefficients A, B, C in the equation of a plane of the form $z = Ax + By + C$ that contains all four points.

(b) Show that the matrix equation from (a) is, in fact, inconsistent.

(c) Now write a matrix equation that can be used to find the least-squares solution to the equation you wrote in (a). Fully simplify any matrix products that occur in your equation, but *do not (yet) attempt to solve the equation.*

(d) Now, solve your equation using methods taught in this course. (You can use a matrix calculator to check your answer, but you must be able to solve this problem by hand.)

(e) **(Recreational:)**[1] Use 3-D graphing software such as GeoGebra or Desmos 3D to plot the four points and graph the "plane of best fit" through them.

(a)
$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

(b) By using Gauss-Jordan elimination

$$\begin{bmatrix} 2 & 4 & 1 & 6 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

So the equation is inconsistent.

(c) Let $\begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ be A, $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ be x, $\begin{bmatrix} 6 \\ 2 \\ 0 \\ 3 \end{bmatrix}$ be b.

So the normal equation is $A^TA\vec{x} = A^Tb$

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 5 \\ 14 & 30 & 10 \\ 5 & 10 & 4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 36 \\ 11 \end{bmatrix}$$

Therefore the normal equation is $\underline{\begin{bmatrix} 7 & 14 & 5 \\ 14 & 30 & 10 \\ 5 & 10 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 17 \\ 36 \\ 11 \end{bmatrix}}$

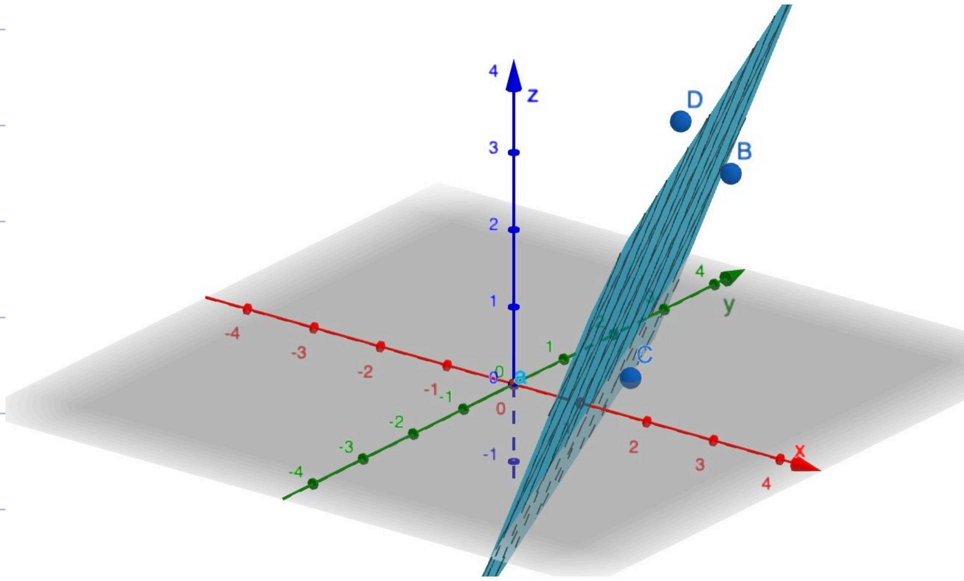
(d) We perform Gauss-Jordan elimination on

$$\left[\begin{array}{ccc|c} 7 & 14 & 5 & 17 \\ 14 & 30 & 10 & 36 \\ 5 & 10 & 4 & 11 \end{array}\right] \xrightarrow{\div 7} \begin{bmatrix} 1 & 2 & \frac{5}{7} & \frac{17}{7} \\ 14 & 30 & 10 & 36 \\ 5 & 10 & 4 & 11 \end{bmatrix} \begin{array}{l} \\ -14\times I \rightarrow \\ -5\times I \end{array}$$

$$\begin{bmatrix} 1 & 2 & \frac{5}{7} & \frac{17}{7} \\ 0 & 2 & 0 & 2 \\ 0 & 0 & \frac{3}{7} & -\frac{8}{7} \end{bmatrix} \begin{array}{l} \\ \div 2 \\ \times\frac{7}{3} \end{array} \longrightarrow \begin{bmatrix} 1 & 2 & \frac{5}{7} & \frac{17}{7} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix} \begin{array}{l} -2\times II \\ -\frac{5}{7}\times II \\ \longrightarrow \end{array}$$

$$\text{rref}(A^TA \,\vdots\, A^Tb) = \begin{bmatrix} 1 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix}, \text{ So } \begin{bmatrix} A \\ B \\ C \end{bmatrix}^* = \begin{bmatrix} \frac{7}{3} \\ 1 \\ -\frac{8}{3} \end{bmatrix}$$

(e)



Problem 2.

(a) Which of the following is an inner product in $\mathcal{P}_2$? Explain.

(i) $\langle f, g\rangle = f(1)g(2) + f(2)g(1) + f(3)g(3)$

(ii) $\langle f, g\rangle = f(1)g(1) + f(2)g(2) + f(3)g(3)$

(b) Let $V = C^\infty[-1,1]$, the vector space of smooth functions on the interval $[-1,1]$. Which of the following is an inner product in V? Explain.

(i) $\langle f, g\rangle = \int_{-1}^{1} x f(x)g(x)\,dx$

(ii) $\langle f, g\rangle = \int_{-1}^{1} x^2 f(x)g(x)\,dx$

(a) (i) $f = x^2-4x+3$. $f(1) = f(3) = 0$

So $\langle f, f\rangle = 2f(1)f(2) + f^2(3) = 0$

f is no zero vector but $\langle f, f\rangle = 0$,

so the function is not positive-define, therefore not an inner product.

(ii) It is inner product.

1. It is symmetric: $\forall f, g \in \mathcal{P}_2$, $\langle f, g\rangle = f(1)g(1) + f(2)g(2) + f(3)g(3) = g(1)f(1) + g(2)f(2) + g(3)f(3)$

$= \langle g, f \rangle$

2. It is linear in both positions

$\forall a, b \in \mathbb{R}$, $f, g, h \in P_2$,

$\langle af+bg, h \rangle = (af(1)+bg(1))h(1)$

$+(af(2)+bg(2))h(2) + (af(3)+bf(3))h(3)$

$= af(1)h(1) + bg(1)h(1) + af(2)h(2) + bg(2)h(2)$

$+ af(3)h(3) + bf(3)h(3)$

$= a\langle f, h \rangle + b\langle g, h \rangle$

By symmetric, it is also linear in the second position.

3. It is positive-defined.

$\forall f \in P_2$, with $f \neq 0$, $\langle f, f \rangle = f^2(1) + f^2(2) + f^2(3) > 0$

(b) (i) is not an inner product because it is not positive-define.

Consider $f = \sin x \in C^\infty[-1,1]$, $\int_{-1}^{1} x f^2(x)\, dx = 0$

while f is not 0.

(ii) is an inner product.

1. it is symmetric because

$\forall f, g \in C^\infty[-1,1]$, $\langle f, g\rangle = \int_{-1}^{1} x^2 f(x) g(x)\, dx$

$= \int_{-1}^{1} x^2 g(x) f(x)\, dx = \langle g, f\rangle$

2. it is linear in both positions

because $\forall f, g, h \in C^\infty[-1,1]$, $a, b \in \mathbb{R}$

$\langle af + bh, g\rangle = \int_{-1}^{1} x^2 (af + bh) g\, dx$

$= a\int_{-1}^{1} x^2 fg\, dx + b\int_{-1}^{1} x^2 hg\, dx$

$= a\langle f, g\rangle + b\langle h, g\rangle$

By symmetric, it is also linear in the second position.

3. it is positive-defined

$\forall f \in C^\infty[-1,1]$, with f is not 0 everywhere, $\langle f, f\rangle = \int_{-1}^{1} x^2 f^2(x)\, dx$

Problem 3. Let $V = C^\infty\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the vector space of smooth functions on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and consider the inner product defined by $\langle f, g\rangle = \displaystyle\int_{-\pi/2}^{\pi/2} f(x)g(x)\sin^2(x)\,dx$. (You do not need to show that this is an inner product, but make sure that you would be able to do so if it were an exam question!) Let $W = \operatorname{span}(1, x, x^2)$.

In what follows, you may feel free to use an online integral calculator (e.g. Wolfram Alpha) to evaluate any difficult integrals[2], but make sure that your work shows clearly *what integrals* you are computing, and how you are making use of the results. Results may be expressed using either exact expressions (e.g., $\pi/\sqrt{2}$) or decimal approximations (e.g., 2.2214), but if you use decimal approximations, please retain at least four digits' worth of precision.

(a) Compute each of the following.
  (i) $\langle 1, x\rangle$
  (ii) $\|1\|$
  (iii) $\|x\|$

(b) Find a basis $\mathcal{U}$ for the subspace W that is orthonormal relative to the given inner product.

(c) Let $h \in C^\infty\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the function defined by $h(x) = e^x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Compute $\operatorname{proj}_W h$.

(d) **(Recreational:)** Repeat parts (a)–(c), this time using the simpler inner product
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(e) **(Recreational:)** Use graphing software (e.g., Desmos) to plot the function h and the two different projections you found in (c) and (d) over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, all on the same axes. How do these three functions compare? Which of the two projections does a "better job" of approximating h (and in what sense is it "better"?) What are some situations in which you might choose to use one inner product rather than the other?

(a) (i) $\langle 1, x\rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin^2(x)\,dx = 0$ since

$x\sin^2(x)$ is an odd function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\|1\| = \sqrt{\langle 1, 1\rangle} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(x)\,dx = \frac{\pi}{2}$

(iii) $\|x\| = \sqrt{\langle x, x\rangle} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\sin^2(x)\,dx = \frac{1}{24}\pi(6+\pi)^2$

$\approx \frac{\pi}{24}(4.4335)$

(b) We apply the Gram-Schdmit Process:

$$u_1 = \frac{1}{\|1\|} = \frac{2}{\pi}$$

$$u_2 = \frac{x - \langle x, \vec{u_1}\rangle \vec{u_1}}{\|x - \langle x, \vec{u_1}\rangle \vec{u_1}\|} = \frac{x - 0}{\|x - 0\|} = \frac{24x}{\pi(6+\pi)^2} \approx 0.0914x$$

$$u_3 = \frac{x^2 - \langle x^2, u_1\rangle u_1 - \langle x^2, u_2\rangle u_2}{\|x^2 - \langle x^2, u_1\rangle u_1 - \langle x^2, u_2\rangle u_2\|}$$

Since $\langle x, x^2\rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \sin^2(x)\,dx$ is 0 because $x^3\sin^2(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is odd function,

$\langle x^2, u_2\rangle = 0$.

So $u_3 = \dfrac{x^2 - \langle x^2, u_1\rangle u_1}{\|x^2 - \langle x^2, u_1\rangle u_1\|}$

$$\langle x^2, u_1\rangle = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\sin^2(x)\,dx = \frac{2}{\pi}\cdot\frac{1}{24}\pi(6+\pi)^2$$

$$= \frac{1}{12}(6+\pi)^2$$

So $$u_3 = \frac{x^2 - \frac{(6+\pi)^2}{6\pi}}{\|x^2 - \frac{(6+\pi)^2}{6\pi}\|} = \frac{x^2 - \frac{(6+\pi)^2}{6\pi}}{9 + \frac{18}{\pi} + \frac{5}{4}\pi - \frac{\pi^2}{4} - \frac{\pi^3}{36} - \frac{\pi^4}{72} + \frac{\pi^5}{160}}$$

$$\approx \frac{x^2 - 4.4335}{15.8876}$$

So an orthonormal basis is (u_1, u_2, u_3)

(c) $\mathrm{proj}_w h = \langle h, u_1\rangle u_1 + \langle h_1, u_2\rangle u_2 + \langle h, u_3\rangle u_3$

$$= \left(\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin^2(x)\, dx\right) u_1 + \left(\frac{24}{\pi(6+\pi)^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x e^x \sin^2 x\, dx\right) u_2$$

$$+ \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \left(x^2 - \frac{(6+\pi)^2}{6\pi}\right) \sin^2 x\, dx\right) u_3$$

$$\approx \underline{1.7581\, u_1 + 0.2492\, u_2 - 8.1687\, u_3}$$

$$= 1.1192 + 0.0226x - 8.1687\left(\frac{x^2 - 4.4335}{15.8876}\right)$$

$$= 3.3472 + 0.0226x - 0.5142x^2$$