Math 217 – Midterm 2

Fall 2021

Solutions

Student ID Number: ______________________ Section: ________

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	20	14	10	12	12	10	10	100
Score:									

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $T : V \to W$ from the vector space V to the vector space W is an *isomorphism*

   Solution: The function $T : V \to W$ is an *isomorphism* if T is a bijective linear transformation.

   Solution: The function $T : V \to W$ is an *isomorphism* if T is an invertible linear transformation.

   (b) The *dimension* of the subspace V of $\mathbb{R}^n$

   Solution: The *dimension* of the subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V.

   (c) The *orthogonal projection* of the vector $\vec{x}$ in $\mathbb{R}^n$ onto the subspace W of $\mathbb{R}^n$

   Solution: The *orthogonal projection* of $\vec{x}$ onto W is the unique vector $\vec{w} \in W$ such that $\vec{x} - \vec{w} \in W^{\perp}$.

   (d) The list of vectors $(\vec{u}_1, \ldots, \vec{u}_k)$ in $\mathbb{R}^n$ is *orthonormal*

   Solution: The list of vectors $(\vec{u}_1, \ldots, \vec{u}_k)$ in $\mathbb{R}^n$ is *orthonormal* if for all integers $1 \leq i, j \leq n$ we have $\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (4 points) For all vector spaces V and W and linear transformations $T : V \to W$, if $\dim V = \dim W$ then T is an isomorphism.

   Solution: FALSE. For instance, let $V = W = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the zero map. Then T is not bijective, so T is not an isomorphism, but $\dim V = \dim W = 2$.

   (b) (4 points) For every square matrix A, we have $\dim \ker A = \dim \ker A^\top$.

   Solution: TRUE. Let $n \in \mathbb{N}$ and let A be an $n \times n$ matrix. By a theorem from class (or the text) we know $\ker(A^\top) = (\operatorname{im} A)^\perp$, so

   $$\dim \ker A^\top \;=\; \dim(\operatorname{im} A)^\perp \;=\; n - \dim \operatorname{im} A \;=\; \dim \ker A,$$

   where the last equality above follows from Rank-Nullity and the second-last follows from the fact that for any subspace V of $\mathbb{R}^n$ we have $\dim V + \dim V^\perp = n$.

   (c) (4 points) For every $m \times n$ matrix A, if either $A^\top A$ or $AA^\top$ is an identity matrix then A is orthogonal.

   Solution: FALSE. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then $AA^\top = I_2$ but A is not orthogonal because A is not even square, and orthogonal matrices must be square.

(Problem 2, Continued).

(d) (4 points) For every vector $\vec{x} \in \mathbb{R}^n$ and subspace V of $\mathbb{R}^n$, $\|\text{proj}_V(\vec{x})\| \le \|\vec{x}\|$.

> **Solution:** TRUE. Let $\vec{x} \in \mathbb{R}^n$, and write $\vec{x}^{\parallel} = \text{proj}_V(\vec{x})$ and $\vec{x}^{\perp} = \text{proj}_{V^\perp}(\vec{x})$. Then $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ and $\vec{x}^{\parallel} \cdot \vec{x}^{\perp} = 0$, so by the Pythagorean Theorem we have
>
> $$\|\vec{x}\|^2 \;=\; \|\vec{x}^{\parallel}\|^2 + \|\vec{x}^{\perp}\|^2,$$
>
> which implies $\|\vec{x}\|^2 \ge \|\vec{x}^{\parallel}\|^2$ since $\|\vec{x}^{\perp}\|^2 \ge 0$, and therefore $\|\vec{x}\| \ge \|\vec{x}^{\parallel}\|$ after taking a square root.

(e) (4 points) Let A be an $m \times n$ matrix. If the linear system $A\vec{x} = \vec{b}_0$ has a unique solution for some vector $\vec{b}_0 \in \mathbb{R}^m$, then the linear system $A\vec{x} = \vec{b}$ has a unique least-squares solution for every vector $\vec{b} \in \mathbb{R}^m$.

> **Solution:** TRUE. Suppose the linear system $A\vec{x} = \vec{b}_0$ has a unique solution. Then $\ker A = \{\vec{0}\}$, which implies $\ker(A^\top A) = \{\vec{0}\}$, so $A^\top A$ is invertible. It follows that for every $\vec{b} \in \mathbb{R}^m$, the vector $(A^\top A)^{-1}A^\top\vec{b}$ is the unique least-squares solution of the linear system $A\vec{x} = \vec{b}$.

3. Let $\mathcal{P}_2$ be the vector space of polynomial functions in the variable t of degree at most 2, and let $\mathcal{E} = (1, t, t^2)$ and

$$\mathcal{B} \;=\; (p_1, p_2, p_3) \;=\; (1 + 2t^2,\ -1 + t - 2t^2,\ 2 - 2t + 5t^2),$$

so that $\mathcal{E}$ and $\mathcal{B}$ are ordered bases of $\mathcal{P}_2$. (You may assume without proof that $\mathcal{E}$ and $\mathcal{B}$ are bases of $\mathcal{P}_2$.) *No justification is required for any part of this problem.*

(a) (4 points) Find the coordinate vectors $[p_2]_{\mathcal{E}}$ and $[p_2]_{\mathcal{B}}$.

Solution: $[p_2]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ and $[p_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) (4 points) Find the coordinate vectors $[t^2]_{\mathcal{E}}$ and $[t^2]_{\mathcal{B}}$.

Solution: $[t^2]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $[t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

(c) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{E}}$.

Solution: $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} | & | & | \\ [p_1]_{\mathcal{E}} & [p_2]_{\mathcal{E}} & [p_3]_{\mathcal{E}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 2 & -2 & 5 \end{bmatrix}$.

(d) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{E}\to\mathcal{B}}$. (You may leave your answer unsimplified.)

Solution: $S_{\mathcal{E}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{E}}^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 2 & -2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -4 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$.

4. Let $P \in \mathbb{R}^{3\times 3}$, and let $F_P : \mathbb{R}^{3\times 4} \to \mathbb{R}^{3\times 4}$ be the linear transformation defined by the rule $F_P(A) = PA$ for all $A \in \mathbb{R}^{3\times 4}$. Let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases of $\mathbb{R}^{3\times 4}$.

   (a) (6 points) *No justification is required in (i) – (iii) below.*

   (i) Let $[F_P]_{\mathcal{B}}$ be the $\mathcal{B}$-matrix of F_P. If $[F_P]_{\mathcal{B}} \in \mathbb{R}^{\ell\times m}$, what are the values of ℓ and m?

   (ii) Let $A \in \mathbb{R}^{3\times 4}$, and let $[F_P(A)]_{\mathcal{B}}$ be the $\mathcal{B}$-coordinate vector of $F_P(A)$. If $[F_P(A)]_{\mathcal{B}} \in \mathbb{R}^{j\times k}$, what are the values of j and k?

   (iii) Let $S_{\mathcal{B}\to\mathcal{C}}$ be the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. If $S_{\mathcal{B}\to\mathcal{C}} \in \mathbb{R}^{q\times r}$, what are the values of q and r?

Solution:

(i) $\ell = m = 12$.

(ii) $j = 12$ and $k = 1$.

(iii) $q = r = 12$.

   (b) (4 points) Prove that if the matrix P is invertible, then F_P is an isomorphism.

Solution: Suppose P is invertible, and define the linear map $T : \mathbb{R}^{3\times 4} \to \mathbb{R}^{3\times 4}$ by $T(A) = P^{-1}A$ for all $A \in \mathbb{R}^{3\times 4}$. Then for all $A \in \mathbb{R}^{3\times 4}$ we have

$$T(F_P(A)) \;=\; P^{-1}(PA) \;=\; (P^{-1}P)A \;=\; I_3A \;=\; A$$

and

$$F_P(T(A)) \;=\; P(P^{-1}A) \;=\; (PP^{-1})A \;=\; I_3A \;=\; A,$$

so T is an inverse of F_P. Thus F_P is an invertible linear transformation, that is, F_P is an isomorphism.

5. Let $(\vec{a}_1, \vec{a}_2)$ be a linearly independent pair of vectors in $\mathbb{R}^3$, and suppose the 3×2 matrix $A = [\, \vec{a}_1 \; \vec{a}_2 \,]$ has QR-factorization

$$A = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}}_{R} \in \mathbb{R}^{3\times 2}.$$

(Throughout this problem, your answers should not include any variables.)

(a) (4 points) Find $\|\vec{a}_1\|$ and $\vec{a}_1 \cdot \vec{a}_2$.

Solution: Recall that $R = \begin{bmatrix} \vec{a}_1 \cdot \vec{u}_1 & \vec{a}_2 \cdot \vec{u}_1 \\ \vec{a}_1 \cdot \vec{u}_2 & \vec{a}_2 \cdot \vec{u}_2 \end{bmatrix}$ and $\vec{a}_1 = \|\vec{a}_1\| \vec{u}_1$. Thus

$$\|\vec{a}_1\| = \frac{\vec{a}_1 \cdot \vec{a}_1}{\|\vec{a}_1\|} = \vec{a}_1 \cdot \vec{u}_1 = 2$$

and $\vec{a}_1 \cdot \vec{a}_2 = (\|\vec{a}_1\| \vec{u}_1) \cdot \vec{a}_2 = 2(\vec{u}_1 \cdot \vec{a}_2) = 6$.

(b) (4 points) Find a basis of the kernel of the 3×3 matrix $M = \begin{bmatrix} | & | & | \\ \vec{a}_2 & \vec{u}_1 & \vec{u}_2 \\ | & | & | \end{bmatrix}$.

Solution: First, note that since $(\vec{u}_1, \vec{u}_2)$ is orthonormal, and therefore linearly independent, $\operatorname{rank}(M) \geq 2$. Then since $\vec{a}_2 = (\vec{a}_2 \cdot \vec{u}_1)\vec{u}_1 + (\vec{a}_2 \cdot \vec{u}_2)\vec{u}_2 = 3\vec{u}_1 + 4\vec{u}_2$, we see that $\operatorname{rank}(M) = 2$, so $\dim \ker M = 1$, and $\left(\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right)$ is a basis of $\ker M$.

(c) (4 points) Find $\begin{bmatrix} — & \vec{u}_1^\top & — \\ — & \vec{a}_2^\top & — \end{bmatrix} \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$.

Solution: Using the fact that $(\vec{u}_1, \vec{u}_2)$ is orthonormal and $R = \begin{bmatrix} \vec{a}_1 \cdot \vec{u}_1 & \vec{a}_2 \cdot \vec{u}_1 \\ \vec{a}_1 \cdot \vec{u}_2 & \vec{a}_2 \cdot \vec{u}_2 \end{bmatrix}$, we have $\begin{bmatrix} — & \vec{u}_1^\top & — \\ — & \vec{a}_2^\top & — \end{bmatrix} \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 \\ \vec{a}_2 \cdot \vec{u}_1 & \vec{a}_2 \cdot \vec{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$.

6. Let V be the subspace of $\mathbb{R}^3$ defined by the equation $x_1 - x_2 - 4x_3 = 0$.

   (a) (5 points) Find an orthonormal basis $(\vec{u}_1, \vec{u}_2)$ of V and an orthonormal basis $(\vec{u}_3)$ of $V^\perp$.

   Solution: First, note that $\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}4\\0\\1\end{bmatrix}\right)$ is a basis of V and $\left(\begin{bmatrix}1\\-1\\-4\end{bmatrix}\right)$ is a basis of $V^\perp$. Applying Gram-Schmidt to these yields the orthonormal basis $(\vec{u}_1, \vec{u}_2) = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix}\right)$ of V, and the orthonormal basis $(\vec{u}_3) = \left(\frac{1}{\sqrt{18}}\begin{bmatrix}1\\-1\\-4\end{bmatrix}\right)$ of $V^\perp$.

   (b) (3 points) Find the orthogonal projection of $\vec{e}_3$ onto V; that is, find $\operatorname{proj}_V(\vec{e}_3)$.

   Solution: $\operatorname{proj}_V(\vec{e}_3) = (\vec{e}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_3 \cdot \vec{u}_2) \cdot \vec{u}_2 = \begin{bmatrix}2/9\\-2/9\\1/9\end{bmatrix}$.

   Solution: $\operatorname{proj}_V(\vec{e}_3) = \vec{e}_3 - (\vec{e}_3 \cdot \vec{u}_3)\vec{u}_3 = \begin{bmatrix}2/9\\-2/9\\1/9\end{bmatrix}$.

   (c) (4 points) Let $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ be the vectors you found in part (a), and let $A = [\vec{u}_1\ \vec{u}_2]$ be the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$. Find a least-squares solution of the linear system $A\vec{x} = \vec{b}$, where $\vec{b} = 6\vec{u}_1 - 2\vec{u}_2 + 5\vec{u}_3$.

   Solution: The least-squares solutions of $A\vec{x} = \vec{b}$ are the solutions of the normal equation $A^\top A\vec{x} = A^\top\vec{b}$. Since the columns of A are orthonormal, $A^\top A = I_2$, so the unique least-squares solution of $A\vec{x} = \vec{b}$ is

$$\vec{x}^* = A^\top\vec{b} = \begin{bmatrix} — & \vec{u}_1^\top & — \\ — & \vec{u}_2^\top & — \end{bmatrix}\vec{b} = \begin{bmatrix}\vec{u}_1 \cdot \vec{b}\\ \vec{u}_2 \cdot \vec{b}\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}.$$

   Solution: The least-squares solutions of $A\vec{x} = \vec{b}$ are the solutions of the consistent linear system $A\vec{x} = \operatorname{proj}_V(\vec{b})$. Since $\operatorname{proj}_V(\vec{b}) = 6\vec{u}_1 - 2\vec{u}_2$, the least-squares solutions of $A\vec{x} = \vec{b}$ are the solutions of the consistent linear system $A\vec{x} = 6\vec{u}_1 - 2\vec{u}_2$, which has unique solution $\begin{bmatrix}6\\-2\end{bmatrix}$.

7. Let V and W be a pair of 2-dimensional subspaces of $\mathbb{R}^3$, and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection maps onto V and W, respectively.

(a) (6 points) Prove that there is an ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix of P is

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: Let $(\vec{b}_1, \vec{b}_2)$ be an ordered basis of V, let $(\vec{b}_3)$ be a basis of $V^\perp$, and let $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$. Given arbitrary $\vec{x} \in \mathbb{R}^3$, we have $\vec{x} = \text{proj}_V(\vec{x}) + \text{proj}_{V^\perp}(\vec{x})$. Since $\text{proj}_V(\vec{x}) \in \text{Span}(\vec{b}_1, \vec{b}_2)$ and $\text{proj}_{V^\perp}(\vec{x}) \in \text{Span}(\vec{b}_3)$, we see that $\vec{x} \in \text{Span}(\mathcal{B})$. Since $\dim \mathbb{R}^3 = 3 = |\mathcal{B}|$, it follows that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

Now, since $\vec{b}_1 \in V$ we have

$$P(\vec{b}_1) = \vec{b}_1 = 1\vec{b}_1 + 0\vec{b}_2 + 0\vec{b}_3,$$

and therefore $[P(\vec{b}_1)]_{\mathcal{B}} = \vec{e}_1$. Similarly, $P(\vec{b}_2) = \vec{b}_2$, so $[P(\vec{b}_2)]_{\mathcal{B}} = \vec{e}_2$. Finally, since $\vec{b}_3 \in V^\perp$ we have $P(\vec{b}_3) = \vec{0}$, so $[P(\vec{b}_3)]_{\mathcal{B}} = \vec{0}$. We conclude that

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as desired.

(b) (4 points) Prove that the standard matrices of P and Q are similar to each other.

Solution: Let $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis of $\mathbb{R}^3$. Using part (a), fix ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ such that

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [Q]_{\mathcal{C}}.$$

Then

$$\begin{aligned} [P]_{\mathcal{E}} &= S_{\mathcal{B}\to\mathcal{E}}[P]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}} \\ &= S_{\mathcal{B}\to\mathcal{E}}[Q]_{\mathcal{C}}S_{\mathcal{E}\to\mathcal{B}} \\ &= S_{\mathcal{B}\to\mathcal{E}}(S_{\mathcal{E}\to\mathcal{C}}[Q]_{\mathcal{E}}S_{\mathcal{C}\to\mathcal{E}})S_{\mathcal{E}\to\mathcal{B}} \\ &= (S_{\mathcal{B}\to\mathcal{E}}S_{\mathcal{E}\to\mathcal{C}})[Q]_{\mathcal{E}}(S_{\mathcal{B}\to\mathcal{E}}S_{\mathcal{E}\to\mathcal{C}})^{-1}, \end{aligned}$$

which shows that $[P]_{\mathcal{E}}$ and $[Q]_{\mathcal{E}}$ are similar.

8. Let $n \in \mathbb{N}$, let V be an n-dimensional vector space, and let $A \in \mathbb{R}^{n\times n}$ be an $n \times n$ matrix.

(a) (6 points) Prove that for every ordered basis $\mathcal{B}$ of V there exists a linear transformation $T : V \to V$ such that $[T]_\mathcal{B} = A$, where $[T]_\mathcal{B}$ is the $\mathcal{B}$-matrix of T.

Solution: Assume the hypotheses, and let $\mathcal{B}$ be an ordered basis of V. Let $L_\mathcal{B} : V \to \mathbb{R}^n$ be the $\mathcal{B}$-coordinate isomorphism defined by $L_\mathcal{B}(\vec{v}) = [\vec{v}]_\mathcal{B}$ for each $\vec{v} \in V$. Let $T : V \to V$ be the linear transformation defined by

$$T(\vec{v}) \;=\; L_\mathcal{B}^{-1}\big(A[\vec{v}]_\mathcal{B}\big) \quad \text{for all } \vec{v} \in V.$$

In other words, $T = L_\mathcal{B}^{-1} \circ T_A \circ L_\mathcal{B}$, where $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map induced by A, defined by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. We claim that $[T]_\mathcal{B} = A$. There are several ways to show this, and any of the following would be sufficient:

- Using our definition of $[T]_\mathcal{B}$ as the standard matrix of $L_\mathcal{B} \circ T \circ L_\mathcal{B}^{-1}$, we have

$$[T]_\mathcal{B} \;=\; [L_\mathcal{B} \circ T \circ L_\mathcal{B}^{-1}]_\mathcal{E} \;=\; [L_\mathcal{B} \circ L_\mathcal{B}^{-1} \circ T_A \circ L_\mathcal{B} \circ L_\mathcal{B}^{-1}]_\mathcal{E} \;=\; [T_A]_\mathcal{E} \;=\; A.$$

- Writing $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$, for each $1 \le i \le n$ we have

$$[T(\vec{b}_i)]_\mathcal{B} \;=\; L_\mathcal{B}\big(T(\vec{b}_i)\big) \;=\; T_A\big(L_\mathcal{B}(\vec{b}_i)\big) \;=\; A[\vec{b}_i]_\mathcal{B} \;=\; A\vec{e}_i,$$

so $[T]_\mathcal{B} = A$ by the formula $[T]_\mathcal{B} = \begin{bmatrix} | & & | \\ [T(\vec{b}_1)]_\mathcal{B} & \cdots & [T(\vec{b}_n)]_\mathcal{B} \\ | & & | \end{bmatrix}$.

- For all $\vec{v} \in V$ we have

$$[T(\vec{v})]_\mathcal{B} \;=\; L_\mathcal{B}\big(T(\vec{v})\big) \;=\; T_A\big(L_\mathcal{B}(\vec{v})\big) \;=\; A[\vec{v}]_\mathcal{B}.$$

Since $[T]_\mathcal{B}$ is the unique matrix satisfying $[T]_\mathcal{B}[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B}$ for all $\vec{v} \in V$, it follows that $A = [T]_\mathcal{B}$.

(b) (4 points) Must it be true that for every linear transformation $T : V \to V$ there exists an ordered basis $\mathcal{B}$ of V such that $[T]_\mathcal{B} = A$? Clearly state *yes* or *no*, and prove your claim.

Solution: No, this is false! For instance, if A is the $n \times n$ zero matrix and T is the identity map on V, then for every ordered basis $\mathcal{B}$ of V we have $[T]_\mathcal{B} = I_n \neq A$. In fact, the statement is false for *every* matrix A, since if A is nonzero we could take T to be the zero map on V, in which case $[T]_\mathcal{B} = 0_{n\times n} \neq A$ for any ordered basis $\mathcal{B}$ of V.