P2

(a) Inner product space ✓

① $\langle f,g\rangle = \langle g,f\rangle$

② linearity : $\langle af_1+bf_2, g\rangle$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(af_1(t)+bf_2(t))g(t)\,dt$$

$$= \frac{a}{\pi}\int_{-\pi}^{\pi} f_1(t)g(t)dt + \frac{b}{\pi}\int_{-\pi}^{\pi} f_2(t)g(t)dt$$

$$= a\langle f_1,g\rangle + b\langle f_2,g\rangle$$

③ positive define; $\langle f,f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(t)\,dt$

$f^2(t) > 0 \Rightarrow \langle f,f\rangle > 0$

(b) 同 (a), inner product space ✓

(c) 不是 inner prod space : can diverge, $\infty \notin \mathbb{R}$

P4 find two different inner products in $\mathbb{R}^2$

① (dot product) $\langle x,y\rangle = x_1y_1 + y_1y_2$

here e_1, e_2 are orthonormal

② different weight : $\langle x,y\rangle = 3x_1y_1 + 2x_2y_2$

P5 Every finite dimensional inner product space has an orthonormal basis.

Take : $u_1 = \frac{v_1}{\sqrt{\langle v_1,v_1\rangle}}$

$$u_2 = \frac{v_2 - \langle v_2,u_1\rangle u_1}{\| u_2 - \langle v_2,u_1\rangle u_1\|}$$

$$\cdots,\ u_n = \frac{v_n - \sum_{j=1}^{n-1}\langle v_n,u_j\rangle u_j}{\|v_n - \sum_{j=1}^{n-1}\langle v_n,u_j\rangle u_j\|}$$

point: generalized Gram-Schmidt

P8 Pf: 任取 orthonormal basis $\mathcal{U}$

任取 inner product $\langle -,-\rangle$

那么 $(V, \langle -,-\rangle)$ 上,

$$\langle \vec{x},\vec{y}\rangle = [\vec{x}]_{\mathcal{U}}\cdot[\vec{y}]_{\mathcal{U}}$$

Pf $\vec{x} = \sum_{i=1}^{n} a_i\vec{u_i}$ for some $a_1,\dots,a_n$

$\vec{y} = \sum_{j=1}^{n} b_j\vec{u_j}$ for some $b_1,\dots,b_n$

$$\langle\vec{x},\vec{y}\rangle = \left\langle \sum_{i=1}^{n} a_i\vec{u_i}, \sum_{j=1}^{n} b_j\vec{u_j}\right\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j \underbrace{\langle\vec{u_i},\vec{u_j}\rangle}_{=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}}$$

$$\Rightarrow = \sum_{i=1}^{n} a_ib_i = [\vec{x}]_{\mathcal{U}}\cdot[\vec{y}]_{\mathcal{U}}$$

P9 $\forall$ inner product $\langle -,-\rangle$ on $\mathbb{R}^n$,

$\exists$ $n\times n$ (symmetric) matrix A, 使 $\forall \vec{x},\vec{y}\in\mathbb{R}^n$, $\langle\vec{x},\vec{y}\rangle = \vec{x}^TA\vec{y}$

Pf Claim ①: note that:

对任意 $n\times n$ matrix A, A 的 (i,j) entry

是 $\vec{e_i}^TA\vec{e_j}$

$$A = \begin{bmatrix} | & | & & | & & | \\ A e_1 & Ae_2 & \cdots & Ae_j & \cdots & Ae_n \\ | & | & & | & & | \end{bmatrix}$$

$$\vec{e_i}^TA\vec{e_j} = \begin{array}{c} \scriptstyle i^{th} \\ [0 \cdots 1 \cdots 0] \end{array} \begin{bmatrix} | \\ A\vec{e_j} \\ | \end{bmatrix}$$

$= $ the i^{th} entry of $A\vec{e_j}$

$(=$ the $(i,j)^{th}$ entry of $A)$

② 注意到 $\langle\vec{x},\vec{y}\rangle = \left\langle \sum_{i=1}^{n} x_i\vec{e_i}, \sum_{j=1}^{n} y_j\vec{e_j}\right\rangle$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_iy_j\langle\vec{e_i},\vec{e_j}\rangle.$$

③ Consider : A, 其 (i,j) entry 为 $\langle\vec{e_i},\vec{e_j}\rangle$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j\langle\vec{e_i},\vec{e_j}\rangle$$

$$\Rightarrow = \sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} y_j a_{ij}\right)$$

$$= \vec{x}\cdot\begin{bmatrix} y_1a_{11} + y_2a_{12} + \dots + y_na_{1n} \\ \vdots \\ y_na_{n1} + y_na_{n2} + \dots + y_na_{nn}\end{bmatrix}$$

$= \vec{x} \cdot (A\vec{y})$

$= \vec{x}^T A \vec{y}$ A是 symmetric 的是因为 $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

因而可总结：

对于任意 $\mathbb{R}^n$ 上的 inner product，

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \cdots & \langle e_1, e_n \rangle \\ \langle e_1, e_2 \rangle & \ddots & & \\ \vdots & & \ddots & \\ \langle e_1, e_n \rangle & & & \langle e_n, e_n \rangle \end{bmatrix} \vec{y}$$

$$(= \vec{x} \cdot (A\vec{y}))$$

注意到：A 的 diagonal 上的 entry 都 > 0 (by positive-define)

又：A 是 symmetry 的

显然：A 是可逆的 (full rank)

实际上可理解为：把 $\vec{y}$ 伸缩变换后再和 $\vec{x}$ dot product.

$\mathbb{R}^n$ 的任何 inner product 都只是：把一个 vector 做一个 linear trans 后和另一个 dot prod.

P10：All inner products on $\mathbb{R}^2$ 都是

$\beta_A(\vec{x}, \vec{y}) = \vec{x}^T A \vec{y}$ for some ① $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

② 且 $a > 0$, (by P9) $\det A = ac - b^2 > 0$ ③

① symmetry matrix

$\Downarrow$

$\beta_A(x, y) = \beta_A(y, x)$

② $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0$

(这一个初中数学：充分条件为 $a > 0$ 且 $\underbrace{ac - b^2}_{=\det A} > 0$)

③ linearity is guaranteed by matrix multiplication.

因而这三条为充要条件.