Math 217 – Midterm 1

Winter 2024

Solutions

Student ID Number: ______________________________ Section: __________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	16	16	13	10	7	12	8	8	10	100
Score:										

1. Complete each partial sentence into a precise definition for, or precise mathematical characterization of, the *italicized* term in each part:

   For full credit, please remember to include all appropriate quantifiers, and write out fully what you mean instead of using shorthand phrases such as "preserves" or "closed under."

   (a) (4 points) Let V and W be vector spaces. A function $f : V \to W$ is called a *linear transformation* if ...

   Solution: if f satisfies

   - $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$
   - $f(c\vec{v}) = cf(\vec{v})$ for all $\vec{v} \in V$ and $c \in \mathbb{R}$

   (b) (4 points) Let X and Y be sets. A function $g : X \to Y$ is called *injective* if ...

   Solution: if for any $x, y \in X$ such that $x \neq y$, then $g(x) \neq g(y)$.

   (c) (4 points) Let $V \subseteq \mathbb{R}^n$ be a vector space. The *dimension* of V is ...

   Solution: the number of elements in a basis $\mathcal{B}$ for V.

   (d) (4 points) Let $T : V \to W$ be a linear transformation. The *rank* of T is ...

   Solution: the dimension of the image of T.

2. State whether each statement is True or False, and justify your answer with either a **short proof** or an **explicit counterexample**.

   (a) (4 points) There exists a matrix $A \in \mathbb{R}^{5\times 9}$ such that $\dim \ker A = 3$.

   Solution: This is false. Suppose that there is a matrix $A \in \mathbb{R}^{5\times 9}$ such that $\dim \ker A = 3$. Then, by the rank-nullity theorem, $\dim \ker A + \dim \operatorname{im} A = 9$, which implies $\dim \operatorname{im} A = 6$. However, since the $\operatorname{im} A$ is spanned by the columns of A (this is a theorem from a worksheet), $\dim \operatorname{im} A$ cannot exceed 5. This is a contradiction.

   (b) (4 points) If $T : V \to W$ is a linear transformation, and $U \subseteq W$ is a subspace, then $T^{-1}[U]$ is a subspace of V. (Recall that if $f : A \to B$ is any function and $S \subseteq B$, then we define $f^{-1}[S] = \{x \in A : f(x) \in S\}$.)

Solution: This is true. In order to show $T^{-1}[U]$ is a subspace of V, we have to check $0_V \in T^{-1}[U]$, $T^{-1}[U]$ is closed under addition, and $T^{-1}[U]$ is closed under scalar multiplication. To begin with, we have $T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V) = 2T(0_V)$. Hence $T(0_V) = 0_W \in U$ since U is a subspace of W. Thus, $0_V \in T^{-1}[U]$. Furthermore, to show $T^{-1}[U]$ is closed under addition, take arbitrary elements $x, y \in T^{-1}[U]$. Then, by the definition of inverse image, $T(x), T(y) \in U$. Since U is a subspace and T is linear, $T(x+y) = T(x)+T(y) \in U$, thus $x + y \in T^{-1}[U]$. Lastly, for any $x \in T^{-1}[U]$ and $c \in \mathbb{R}$, we know $cT(x) \in U$ again because U is a subspace. Since T is linear, $cT(x) = T(cx)$, and so $cx \in T^{-1}[U]$. Therefore, $T^{-1}[U]$ is a subspace of V.

NOTE: The set $T^{-1}[U]$ is defined *even when the function T is not invertible, so that there is no function T^{-1}*. Any solution that uses T^{-1} as a function is only valid in the case in which T is an invertible function, which need not be true for this problem.

(c) (4 points) If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$ satisfy $\ker(AB) = \ker(B) = \{\vec{0}\}$, then $\ker(A) = \{\vec{0}\}$.

Solution: This is true. Let $v \in \ker A$. Then, $Av = \vec{0}$ by definition of the kernel. We claim that v has to be $\vec{0}$. In order to do this, let us look at conditions for B. Since $\ker B = \{\vec{0}\}$, the map $T_B(\vec{x}) = B\vec{x}$ is injective (this is a problem from a worksheet). Furthermore, by the rank-nullity theorem, this implies $\dim \operatorname{im} B = n$. Hence, $T_B(\vec{x})$ is surjective, and therefore the matrix B is invertible. Now, observe that

$$Av = AB(B^{-1}v) = \vec{0}.$$

Since $\ker AB = \{\vec{0}\}$, $B^{-1}v = \vec{0}$. Thus, $v = \vec{0}$.

(d) (4 points) Suppose that

$$c_1v_1 + \cdots + c_nv_n = 0$$

is a relation on a set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V. If the relation above is trivial, then $\{v_1, \ldots v_n\}$ is a linearly independent set.

Solution: This is false. The statement doesn't say that the trivial relation is the "only" relation that the set of vectors can have. For example, $\{\vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subset \mathbb{R}^2$ is linearly dependent, but $0 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2 + 0 \cdot (\vec{e}_1 + \vec{e}_2) = 0$ is a valid relation.

3. Suppose A is a 3×5 matrix that has been transformed by a sequence of elementary row operations into the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & k^2-1 & k-1 \end{bmatrix},$$

where k is a constant. Assume that A is the augmented matrix of the linear system $\mathcal{S}$.

(a) (2 points) Find all values of k such that R is in reduced row echelon form.

Solution: $k=\sqrt{2}, k=-\sqrt{2}, k=1$ are the only three solutions.

(b) (3 points) Find all values of k such that $\mathcal{S}$:

(i) has no solutions

Solution: $k=-1$ is the only case in which $\mathcal{S}$ has no solutions.

(ii) has exactly one solution

Solution: This is impossible – there is always at least one free variable, regardless of the value of k.

(iii) has a solution set that is a line

Solution: We need $k\neq 1$ and $k\neq -1$. Anything else will do.

(c) (4 points) Assuming $k=2$, find the solution set of $\mathcal{S}$, expressed in parametric vector form.

Solution:

(d) (4 points) Now suppose A is the standard matrix of a linear transformation T.

(i) If $T:\mathbb{R}^n\to\mathbb{R}^m$, find the values of n and m.

Solution: Since A is a 3×5 matrix, $n=5$ and $m=3$.

(ii) Find $T(\vec{e}_3)$, assuming $T(\vec{e}_1)=\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and $T(\vec{e}_2)=\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

Solution: Let R_1, R_2, and R_3 be the first three columns of R and A_1, A_2, and A_3 be the first three columns of A. Now, from the matrix given above, we have $R_3 = -3R_1 + 2R_2$. Note that we proved in a worksheet that the same relation holds for the columns of A (i.e., $A_3 = -3A_1 + 2A_2$). By the Key Theorem, $A_i = T(\vec{e}_i)$ for each i. Thus,

$$T(e_3) = -3T(e_1) + 2T(e_2) = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 5 \end{bmatrix}$$

4. Consider the following functions:

- Let $f\colon \mathbb{R}^2 \to \mathcal{P}_3$ be defined by $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^3 + bx$.
- Let $g\colon \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $g(p) = p'$.
- Let $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ be defined by $h(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

(a) (2 points) Find $h \circ g \circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

(b) (4 points) The composition $h \circ g \circ f$ is a linear transformation. (You do not need to verify this.) What is its standard matrix?

(c) (4 points) Is $h \circ g \circ f$ invertible? Justify your answer.

Solution:

(a) Since $h \circ g \circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = h\left(g\left(f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right)\right)$, we evaluate the function step-by-step as follows:

- $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x^3$
- $g \circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3x^2$
- $h \circ g \circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(b) In order to use the Key Theorem, we need $h \circ g \circ f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

- $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x$

- $g \circ f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 1$
- $h \circ g \circ f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$

Hence, the standard matrix for $h \circ g \circ f$ is $\begin{bmatrix}0 & 1\\3 & 1\end{bmatrix}$.

(c) The standard matrix of $h \circ g \circ f$ is invertible its determinant is nonzero (alternatively: because its rank is 2; because its rref is I_2). Consequently, by a worksheet problem, $h \circ g \circ f$ is itself invertible as well.

5. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices with real entries. Let

$$A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Let L be a function from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ defined by

$$L(X) = AX - XA \quad \text{for all} \quad X \in \mathbb{R}^{2\times 2}.$$

Note that L is a linear transformation (you do not need to verify this fact).

(a) (4 points) Find a basis of the kernel of L.

(b) (3 points) Find the dimension of the image of L.

Solution:

(a) We will show $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$ is a basis for $\ker L$. Let $X = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ be an element of $\ker L$. Then, since $L(X) = 0$, we have

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} - \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = 0$$

$$\begin{bmatrix}c & d\\a & b\end{bmatrix} - \begin{bmatrix}b & a\\d & c\end{bmatrix} = 0.$$

Hence, this gives $a = d$ and $b = c$. We can now rewrite

$$X = \begin{bmatrix}a & c\\c & a\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

Therefore, $\mathcal{B}$ spans $\ker L$. $\mathcal{B}$ is linearly independent since $a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = 0$ implies $a = b = 0$.

(b) By the rank-nullity theorem, $\dim\ker L + \dim\operatorname{im} L = \dim(\text{Source}) = 4$. By the previous part, we know $\dim\ker L = 2$. Therefore, $\dim\operatorname{im} L = 2$.

6. Let $\theta \in \mathbb{R}$ be a fixed angle (measured in radians). Suppose $R : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation about the origin by θ, and $P : \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal projection onto the y-axis. Let $T = P \circ R$.

   (a) (4 points) Find a basis of $\operatorname{im}(T)$ and a basis of $\ker(T)$. Your answer may include the variable θ. (No justification needed.)

   (b) (4 points) Find the standard matrix of T. Your answer may include the variable θ.

   (c) (4 points) Find all angles θ in the interval $[0, 2\pi]$ such that T^2 is the zero map on $\mathbb{R}^2$.

Solution:

(a) Since R is bijective, $\operatorname{im} R = \mathbb{R}^2$. Furthermore, P projects $\mathbb{R}^2$ onto the y-axis. Thus, $\dim\operatorname{im} T = 1$ and a basis for $\operatorname{im} T$ is $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. By the rank-nullity theorem, $\dim\ker T = 1$, so we need to identify one non-zero vector in $\ker T$ to form a basis. Geometrically, if a vector $\vec{v}$ has $-\theta$ with the positive x-axis, then $R(\vec{v})$ is on the x-axis, and so $P \circ R(\vec{v}) = 0$. Thus, $\left\{\begin{bmatrix}\cos(-\theta)\\\sin(-\theta)\end{bmatrix}\right\}$ is a basis for $\ker T$.

(b) By the key theorem, we have to identify $T(\vec{e}_1)$ and $T(\vec{e}_2)$ to find the standard matrix of T.

- $T(\vec{e}_1) = P \circ R\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = P\left(\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\right) = \begin{bmatrix}0\\\sin\theta\end{bmatrix}$
- $T(\vec{e}_2) = P \circ R\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = P\left(\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}\right) = \begin{bmatrix}0\\\cos\theta\end{bmatrix}$

Thus, $\begin{bmatrix}0 & 0\\\sin\theta & \cos\theta\end{bmatrix}$ is the standard matrix of T.

(c) We know that $\operatorname{im} T$ is y-axis. The goal to have $T \circ T$ a zero map. Considering the fact that $T = P \circ R$ and P maps every vectors on the x-axis to the zero vector, the rotation has to map the y-axis to the x-axis. Only possible angles are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

This can also be checked by doing matrix algebra. Let $A = \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix}$ be the standard matrix of T. Then, the standard matrix of T^2 is A^2.

$$A^2 = \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

Now, if T^2 is the zero map, A^2 must be 0. This gives $\sin\theta\cos\theta = 0$ and $\cos^2\theta = 0$. Note that the first equation gives $\sin\theta = 0$ or $\cos\theta = 0$, but if $\sin\theta = 0$, then $\cos\theta \neq 0$ which doesn't satisfy the second equation. Thus $\cos\theta = 0$. The solutions for $\cos\theta = 0$ in $[0, 2\pi]$ is $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

7. Let U and V both be subspaces of the vector space W.

   (a) (4 points) Prove or disprove: $U \cap V$ is a subspace of W.

Solution: This is true. Since U and V are subspaces of W, both U and V contain the zero vector in W, so $U \cap V$ contains the zero vector in W. To see that $U \cap V$ is closed under vector addition and scalar multiplication, let $x, y \in U \cap V$ and $c \in \mathbb{R}$. Then $x + y \in U$ and $x + y \in V$ since both U and V are closed under vector addition, and $cx \in U$ and $cx \in V$ since both U and V are closed under scalar multiplication. So $x + y \in U \cap V$ and $cx \in U \cap V$. This shows that $U \cap V$ is a subspace of W.

   (b) (4 points) Prove or disprove: $U \cup V$ is a subspace of W.

Solution: This depends on U and V, but need not be true in general. For instance, let $W = \mathbb{R}^2$, and let $U = \text{Span}(\vec{e}_1)$ be the x-axis and $V = \text{Span}(\vec{e}_2)$ the y-axis. Then U and V are subspaces of W, but $U \cup V$ is not a subspace of W since, e.g., $\vec{e}_1 + \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U \cup V$, so $U \cup V$ is not closed under vector addition.

Remark: In fact, $U \cup V$ is a subspace of W if and only if $U \subseteq V$ or $V \subseteq U$.

8. (a) (4 points) Prove that for all linear transformations $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$, the composite map $T \circ S$ is not surjective.

Solution: Let $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$ be linear transformations. By Rank-Nullity,

$$\dim \operatorname{im}(T) \;=\; \dim(\mathbb{R}^4) - \dim\ker(T) \;\leq\; \dim(\mathbb{R}^4) \;=\; 4.$$

Since $\operatorname{im}(T \circ S) \subseteq \operatorname{im}(T)$, this implies $\dim \operatorname{im}(T \circ S) \leq 4$. But the codomain of $T \circ S$ is $\mathbb{R}^5$, which has dimension 5, so the image of $T \circ S$ is a proper subspace of its codomain and therefore $T \circ S$ is not surjective.

(b) (4 points) Prove that for every linear transformation $U : \mathbb{R}^4 \to \mathbb{R}^4$, there exist linear transformations $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$ such that $U = S \circ T$.

Solution: Let $U : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, and let $A \in \mathbb{R}^{4\times 4}$ be the standard matrix of U. Let $B = \begin{bmatrix} A & \vec{0} \end{bmatrix}$ be the 4×5 matrix whose columns are the four columns of A followed by a column of zeros, and let $C = \begin{bmatrix} I_4 \\ 0 \cdots 0 \end{bmatrix}$ be the 5×4 matrix whose jth column is $\vec{e}_j \in \mathbb{R}^5$. Then $A = BC$, so if we let $S_B : \mathbb{R}^5 \to \mathbb{R}^4$ and $T_C : \mathbb{R}^4 \to \mathbb{R}^5$ be defined by $S_B(\vec{x}) = B\vec{x}$ and $T_C(\vec{x}) = C\vec{x}$, then $U = S_B \circ T_C$.

Solution: Just like the previous solution, except let

$$B = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

is the standard matrix of U.

Solution: Let $U : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, and let $u_1, \ldots, u_4$ be its component functions, so for all $\vec{x} \in \mathbb{R}^4$ we have

$$U(\vec{x}) = \begin{bmatrix} u_1(\vec{x}) \\ u_2(\vec{x}) \\ u_3(\vec{x}) \\ u_4(\vec{x}) \end{bmatrix}.$$

Now define the linear maps $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$ by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} u_1(\vec{x}) \\ u_2(\vec{x}) \\ u_3(\vec{x}) \\ u_4(\vec{x}) \\ 0 \end{bmatrix}.$$

Then $S \circ T = U$.

Solution: Let $U : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation. For each $n \in \mathbb{N}$ and $i \le n$, write $\vec{e}_i^{(n)}$ for the ith standard basis vector in $\mathbb{R}^n$. Using Problem 3 of HW 5, let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be the unique linear transformation such that $T(\vec{e}_i^{(4)}) = \vec{e}_i^{(5)}$ for each $1 \le i \le 4$, and let $S : \mathbb{R}^5 \to \mathbb{R}^4$ be the unique linear transformation such that $S(\vec{e}_5^{(5)}) = \vec{0}$ and $S(\vec{e}_i^{(5)}) = U(\vec{e}_i^{(4)})$ for each $1 \le i \le 4$. Then for each $1 \le i \le 4$, we have

$$(S \circ T)\big(\vec{e}_i^{(4)}\big) \;=\; S\big(T(\vec{e}_i^{(4)})\big) \;=\; S\big(\vec{e}_i^{(5)}\big) \;=\; U\big(\vec{e}_i^{(4)}\big).$$

It follows, again by Problem 3 of HW 5, that $S \circ T = U$.

9. Consider three vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ in a vector space V.

   (a) (6 points) Prove that if $\vec{z} \neq \vec{0}$, $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$, and $\vec{x}+\vec{y} \notin \operatorname{Span}(\vec{x}, \vec{z})$, then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

Solution: Let V be a vector space, with $\vec{x}, \vec{y}, \vec{z} \in V$. Assume $\vec{z} \neq \vec{0}$, $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$, and $\vec{x} + \vec{y} \notin \operatorname{Span}(\vec{x}, \vec{z})$. Let $a, b, c \in \mathbb{R}$ be arbitrary, and assume $a\vec{x} + b\vec{y} + c\vec{z} = \vec{0}$. If $a \neq 0$, then

$$x = \left(\frac{-b}{a}\right)\vec{y} + \left(\frac{-c}{a}\right)\vec{z} \;\in\; \operatorname{Span}(\vec{y}, \vec{z}),$$

so we must have $a = 0$. Now if $b \neq 0$, then $\vec{y} = (\frac{-c}{b})\vec{z}$ and therefore

$$\vec{x} + \vec{y} = \vec{x} + \left(\frac{-c}{b}\right)\vec{z} \;\in\; \operatorname{Span}(\vec{x}, \vec{z}),$$

so we must have $b = 0$. By now our equation $a\vec{x} + b\vec{y} + c\vec{z} = \vec{0}$ has been reduced to $c\vec{z} = \vec{0}$. Since $\vec{z} \neq \vec{0}$, it follows that $c = 0$ as well. We have shown $a = b = c = 0$, and conclude that $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

   (b) (4 points) Suppose instead that $\vec{y} \in \operatorname{Span}(\vec{x}, \vec{z})$ but $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$. Fully justify your answers to the questions below.

      (i) Must $\vec{y}$ be a scalar multiple of $\vec{z}$?

Solution: Yes! Suppose $\vec{y} \in \operatorname{Span}(\vec{x}, \vec{z})$, and fix $a, b \in \mathbb{R}$ such that $\vec{y} = a\vec{x} + b\vec{z}$. If $a \neq 0$ then $\vec{x} = \frac{1}{a}(\vec{y} - b\vec{z}) \in \operatorname{Span}(\vec{y}, \vec{z})$, so we must have $a = 0$. But then $\vec{y} = b\vec{z}$.

      (ii) Must $\vec{z}$ be a scalar multiple of $\vec{y}$?

Solution: No! For instance, we could have $V = \mathbb{R}^2$, $\vec{x} = \vec{e}_1$, $\vec{y} = \vec{0}$, and $\vec{z} = \vec{e}_2$.