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Math 217 Worksheet 23: Eigenbases and Diagonalization (§7.1, §7.2, §7.3)

Definition: Let $V \xrightarrow{T} V$ be a linear transformation.
A basis $\mathcal{B}$ of V is an **eigenbasis** for T if every element of the basis is an eigenvector of T.

Problem 1. Warm-up. For each transformation T below, investigate whether or not T has an *eigenbasis.* If so, find one. Are there others? Try to think geometrically when possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by *dilation by 3.*

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by *rotation counterclockwise by* $\frac{\pi}{4}$ around the z-axis.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by *reflection over the line* $y = 2x$.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by *projection onto a* 2*-dimensional subspace* $V = span(\vec{v}_1, \vec{v}_2)$.

(e) $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(g) = 2g'$. [Hint: Think about degrees.]

Solution:

(a) Yes, the standard basis is an eigenbasis. In fact, any basis is an eigenbasis for T.

(b) No. All eigenvectors are parallel to the z-axis. We can find at most one linearly independent eigenvector in this case.

(c) Yes. One example is $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$. Other examples are $\left\{a\begin{bmatrix}1\\2\end{bmatrix}, b\begin{bmatrix}-2\\1\end{bmatrix}\right\}$, for $a \neq 0$ and $b \neq 0$.

(d) Yes. One example is $\{v_1, \vec{v}_2, \vec{v}_3\}$, where $\vec{v}_3$ is any non-zero vector in $V^{\perp}$. One specific such $\vec{v}_3$ could be $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$, the cross product of $\vec{v}_1$ and $\vec{v}_2$ if you know about cross product. Other examples are $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \pi\vec{v}_3\}$, or any other basis for V together with a normal vector to V.

(e) No. If p has degree at least 1 then $T(p)$ has degree smaller than p. Thus p is not an eigenvector if its degree is at least 1. All eigenvectors are constant functions (for eigenvalue 0). Hence there is no eigenbasis for T.

Theorem A: Let $T : V \to V$ be a linear transformation of a finite dimensional vector space. A basis $\mathcal{B}$ for V is an eigenbasis if and only if $[T]_{\mathcal{B}}$ is a diagonal matrix. In this case, the elements on the diagonal will be eigenvalues for T.

Definition: The linear transformation $T : V \to V$ of a finite dimensional vector space V is **diagonalizable** if there exits a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal.

Problem 2. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space.

(a) Verify Theorem A for each of the transformations in each of Problem 1 (a), (c), and (d) by finding the matrix of T with respect to the eigenbasis you found.

(b) Prove Theorem A. [HINT: Scaffold the proof first. What are the two things to show?]

Solution:

(a) The matrices for linear transformations in Problem 1 (a), (c), and (d) are

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. They are all diagonal. And in all cases, the elements on the diagonal are eigenvalues of T (of the corresponding eigenvectors in $\mathcal{B}$.

(b) First suppose that $\mathcal{B} = (b_1, \ldots, b_n)$ is an eigenbasis for T. Thus, there are $\lambda_1, \ldots, \lambda_n$ such that $T(b_i) = \lambda_i b_i$ for $i = 1, \ldots, n$. Then

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(b_1)]_{\mathcal{B}} & \ldots & [T(b_n)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} [\lambda_1 b_1]_{\mathcal{B}} & \ldots & [\lambda_n b_n]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{e}_1 & \ldots & \lambda_n \vec{e}_n \end{bmatrix},$$

which is diagonal. Note that the elements on the diagonal are the eigenvalues of T.

Next suppose $\mathcal{B} = (b_1, \ldots, b_n)$ is a basis such that $[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \vec{e}_1 & \ldots & \lambda_n \vec{e}_n \end{bmatrix}$. It follows that $T(b_i) = \lambda_i b_i$ for $i = 1, \ldots, n$, thus b_i is an eigenvector ($b_i \neq \vec{0}$). Therefore $\mathcal{B}$ is an eigenbasis.

Theorem B: A linear transformation $T : V \to V$ is **diagonalizable** if and only if its matrix $[T]_{\mathcal{A}}$ with respect to *some* basis $\mathcal{A}$ (equivalently, *every* basis) $\mathcal{A}$ of V is *similar to* a diagonal matrix.

Problem 3. Let $\sigma : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by $A \mapsto A - A^\top$.

(a) Is σ diagonalizable? If so, find an eigenbasis $\mathcal{B}$ and diagonal matrix $[\sigma]_{\mathcal{B}}$ witnessing this fact. [HINT: One eigenvalue is easy to find; another eigenvalue is 2.]

(b) Let $\mathcal{E}$ be the basis $(E_{11}, E_{12}, E_{21}, E_{22})$ for $\mathbb{R}^{2\times 2}$. Find $[\sigma]_{\mathcal{E}}$.

(c) Find both change of basis matrices between the basis $\mathcal{E}$ and your eigenbasis $\mathcal{B}$ from (a). [HINT: One is easier to find than the other.]

(d) Find a matrix S witnessing the similarity of $[\sigma]_{\mathcal{B}}$ and $[\sigma]_{\mathcal{E}}$ guaranteed by Theorem B.

Solution:

(a) Yes. An eigenbasis is $\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ and

$$[\sigma]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b)

$$[\sigma]_{\mathcal{E}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c)

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$S_{\mathcal{B}\to\mathcal{E}} = S_{\mathcal{E}\to\mathcal{B}}^{-1}.$$

(d) Let $S = S_{\mathcal{B}\to\mathcal{E}}$. Then $[\sigma]_{\mathcal{B}} = S^{-1}[\sigma]_{\mathcal{E}}S$.

Definition: An $n \times n$ matrix A is **diagonalizable** if the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_A(\vec{x}) = A\vec{x}$ is a diagonalizable linear transformation.

Theorem C: A matrix A is **diagonalizable** if and only if A is similar to a diagonal matrix.

Problem 4. Decide whether or not each matrix below is *diagonalizable*. For those that are, find an invertible matrix S witnessing its similarity to a diagonal matrix (guaranteed by Theorem C).

(a) $A = \begin{bmatrix} \cos(\frac{\pi}{8}) & -\sin(\frac{\pi}{8}) \\ \sin(\frac{\pi}{8}) & \cos(\frac{\pi}{8}) \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$. [Hint: One eigenvector is easy. For the other, try $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.]

(c) Explain why Theorem C follows from Theorem B.

Solution:

(a) No. There is no eigenbasis.

(b) Yes. $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ is an eigenbasis for T_B. And $S^{-1}BS = S^{-1}[T_B]_{\mathcal{E}}S = [T_B]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} S$, where $S = S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) A is diagonalizable if and only if T_A is diagonalizable (by definition). On the other hand, A is the standard matrix of T_A, i.e. $A = [T_A]_{\mathcal{E}}$. By Theorem A, T_A is diagonalizable if and only if A is similar to a diagonal matrix.

***Problem 5.** Prove Theorem B. [Hint: Compare $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{B}}$ where $\mathcal{B}$ is an eigenbasis.]

Solution:

If T is diagonalizable then there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal. For any basis $\mathcal{A}$, we have $[T]_{\mathcal{A}} = S^{-1}[T]_{\mathcal{B}}S$, where $S = S_{\mathcal{A}\to\mathcal{B}}$. Thus $[T]_{\mathcal{A}}$ is similar to a diagonal matrix.

Conversely, assume $[T]_{\mathcal{A}}$ is similar to a diagonal matrix B. Let S be an invertible matrix such that $[T]_{\mathcal{A}} = SBS^{-1}$. Assume that $S = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$, where $n = \dim(V)$. Since S is invertible, its columns are linearly independent and form a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$. As $L_{\mathcal{B}}^{-1}$ is an isomorphism, the set $\{L_{\mathcal{A}}^{-1}(\vec{v}_1), \dots, L_{\mathcal{A}}^{-1}(\vec{v}_n)\}$ is a basis of V—the column vectors $\vec{v}_1, \dots \vec{v}_n$ are the $\mathcal{A}$-coordinate column vectors of the vectors in this basis. We set $\mathcal{B} = (L_{\mathcal{A}}^{-1}(\vec{v}_1), \dots, L_{\mathcal{A}}^{-1}(\vec{v}_n))$, then $S = S_{\mathcal{B}\to\mathcal{A}}$. It follows that $[T]_{\mathcal{B}} = B$, which is diagonal.

Definition: Let λ be an eigenvalue of a linear transformation $T : V \longrightarrow V$.
The **geometric multiplicity** of λ of T is the dimension of the λ-eigenspace E_λ.

Problem 6. For each transformation in Problems 1 (a), (c), and (d), find the eigenspace and geometric multiplicity for each eigenvalue you found.

Solution: The λ-eigenspace $E_\lambda = \{v \in V \mid T(v) = \lambda v\}$. For problem 1

(a) the 3-eigenspace is $\mathbb{R}^3$, so the geometric multiplicity of 3 is 3;

(c) The 1-eigenspace is the line $y = 2x$, which we could also write $\mathrm{Span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$, so its geometric multiplicity is 1. The -1-eigenspace is the perrpendicular line $y = -\frac{1}{2}x$, which we could also write $\mathrm{Span}(\begin{bmatrix} -2 \\ 1 \end{bmatrix})$, so the geometric multiplicity of -1 is 1;

(d) The 1-eigenspace is V, so the geometric multiplicity of 1 is 2. The 0-eigenspace is $V^\perp$, so the geometric multiplicity of 0 is 1.

Problem 7. Let $V \xrightarrow{T} V$ be a linear transformation of a vector space V. Fix $c \in \mathbb{R}$, and consider the map $\phi : V \to V$ defined by $\phi(v) = T(v) - cv$.

(a) Show that ϕ is a linear transformation. Another notation for ϕ is $T - c\mathrm{Id}_V$. Do you see why?

(b) The scalar c is an eigenvalue if and only if $\ker\phi$ is *not* trivial. Explain.

(c) Now assume that c is an eigenvalue of T. Prove that $\ker\phi = E_c$, the c-eigenspace of T.

(d) Prove Theorem D below.

(e) Rephrase Theorem D below to describe the eigenspaces and geometric multiplicities for eigenvalues of a matrix A. Your answer should be a theorem from the book.
[Hint: Consider the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A.]

Theorem D: Let λ be an eigenvalue of a linear transformation $T : V \longrightarrow V$. Then

(i) The λ-eigenspace of λ of T is equal to the kernel of the transformation $T - \lambda\mathrm{Id}_V$; and

(ii) The geometric multiplicity of λ is the nullity of $T - \lambda\mathrm{Id}_V$.

Solution:

(a) We verify ϕ satisfy the definition of a linear transformation. Indeed, for every $u, v \in V$, $\phi(u+v) = T(u+v) - c(u+v) = T(u) - cu + T(v) - cv = \phi(u) + \phi(v)$. Moreover, for every $k \in \mathbb{R}$, $\phi(ku) = T(ku) - cku = kT(u) - kcu = k\phi(u)$. Therefore, ϕ is a linear transformation.

(b& c) We have

$$\ker \phi = \{v \in V : \phi(v) = \vec{0}\} = \{v \in V : T(v) = cv\} = E_c.$$

This is non-zero if and only if c is an eigenvalue. When c is an eigenvalue, this is precisely the c-eigenspace.

(d) Applying (a)-(c) for the eigenvalue λ, note that $T - \lambda \mathrm{Id}_V$ is just another way to denote ϕ. The geometric multiplicity of the eigenvalue λ of T is, by definition, $\dim(E_\lambda)$. By (c), E_λ is the kernel of ϕ, so its dimension is the nullity of $T - \lambda \mathrm{Id}_V$.

(e) Theorem D in the case of coordinate space becomes: if λ is an eigenvalue of an $n \times n$ matrix A, then $E_\lambda = \ker(A - \lambda I_n)$.

Problem 8. Compute the geometric multiplicities of all eigenvalues you found for the transformations in Problems 3 and 4. [HINT: Use Theorem D and rank-nullity to do it quickly!]

Solution:

(3) For the eigenvalue 0, geometric multiplicity is the dimension of the 0 eigenspace, or kernel, of σ. Looking at the matrix $[\sigma]_{\mathcal{B}}$, we see the rank is 1, so the kernel has dimension 3. For the eigenvalue 2, the geometric multiplicity is the dimension of the 2-eigenspace, or kernel of $\sigma - 2I$. In $\mathcal{B}$-coordinates, the matrix of $\sigma - 2I$ is

$$[\sigma]_{\mathcal{B}} = \begin{bmatrix} 2-2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

which has rank 3. So gemu(2)=1 by rank nullity.

(4) Only (b) has eigenvalues; these are 1 and 4. Compute: the geometric multiplicity of 1 is the dimension of the kernel of $A - 1I_2 = \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}$, which is 1 by rank nullity. Similarly, the geometric multiplicity of 4 is the dimension of the kernel of $A - 4I_2 = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$, which is 1 by rank nullity.

Problem 9. For each part below, Use Theorem d to construct linear transformations $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ given by $T_A(\vec{x}) = A\vec{x}$ with the stated properties.

(a) T_A has eigenvalues $1, 2, 3$, and 4.

(b) T_A has eigenvalues 2 and 4 of geometric multiplicities 1 and 3, respectively, and the standard basis is *not* an eigenbasis of A. [HINT: What does Theorem 1 say about $[T]_{\mathcal{E}}$ if $\mathcal{E}$ is not an eigenbasis?]

(c) T_A has eigenvalue 1 of geometric multiplicity 1.

Solution:

(a) There are many correct answers. The easiest are diagonal matrices with $1, 2, 3, 4$ on the diagonal, such as $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

(b) There are many correct answers. We can't use a diagonal matrix because the standard basis is not an eigenbasis. One correct answer is $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. To check this works, we compute $A - 2I_4$ and see that its rank is 3, so its nullity is 1. And we compute $A - 4I_4$ and see that its rank is 1, so its nullity is 3.

(c) There are many correct answers. One correct answer is $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. We see that $A - I_4$ has rank 3, hence nullity 1.