MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 7, SOLUTIONS

Part A (15 points)

Solve the following problems from the book:
Section 4.3: 14, 28, 60

Solution.

14. Write

$$u_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Then

$$T(u_1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0u_1 + 0u_2 + 0u_3 + 0u_4$$

$$T(u_2) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0u_1 + 0u_2 + 0u_3 + 0u_4$$

$$T(u_3) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} = 0u_1 + 0u_2 + 3u_3 + 0u_4$$

$$T(u_4) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} = 0u_1 + 0u_2 + 0u_3 + 3u_4$$

and we conclude that $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

28. Write $p_1(t) = 1, p_2(t) = t-1, p_3(t) = (t-1)^2$. Then

$$T(p_1(t)) = p_1(2t-1) = 1 = p_1(t)$$
$$T(p_2(t)) = p_2(2t-1) = (2t-1) - 1 = 2t-2 = 2p_2(t)$$
$$T(p_3(t)) = p_3(2t-1) = ((2t-1)-1)^2 = (2t-2)^2 = 4(t-1)^2 = 4p_3(t)$$

and therefore $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

60.

(a) We write

$$\vec{b}_1 = 1\vec{a}_1 + 0\vec{a}_2$$
$$\vec{b}_2 = 1\vec{a}_1 + 1\vec{a}_2$$

and therefore $S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) We could express each of the $\vec{a}_k$ in terms of the basis $\mathfrak{B}$, but instead we will just compute the inverse of our answer in (a).

$$S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(c) The relationship between the matrices is $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} S_{\mathfrak{B}\to\mathfrak{A}}$, or, explicitly,

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Section 5.1: 6, 17, 26

Solution.

6. We calculate $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} = \frac{-3}{\sqrt{10}\sqrt{54}} = -\frac{1}{2\sqrt{15}}$, so $\theta = \cos^{-1}\left(-\frac{1}{2\sqrt{15}}\right)$. [Numerically, this is approximately 1.7 radians, or about 97.42°, though you don't need to write this for full credit here.]

17. We must find a basis for the set of all vectors $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that

$$\begin{aligned} w + 2x \quad +3y + 4z \quad &= 0 \\ 5w + 6x \quad +7y + 8z \quad &= 0 \end{aligned}$$

Solving this system, we find that the general solution is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ and therefore a basis of $W^{\perp}$ is

$$\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

26. Let W be the subspace spanned by $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$. To begin, we need an orthonormal basis for W. The vectors are already orthogonal, as can be readily checked, so we need only divide each one by its magnitude. Both vectors have a magnitude of 7, so an orthonormal basis for W is

$$(\vec{u}_1, \vec{u}_2) = \left(\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} \right)$$

Now we project $\vec{v} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto both $\vec{u}_1$ and $\vec{u}_2$:

$$\operatorname{proj}_{\vec{u}_1} \vec{v} = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 = 77\vec{u}_1 = \begin{bmatrix} 22 \\ 33 \\ 66 \end{bmatrix}$$
$$\operatorname{proj}_{\vec{u}_2} \vec{v} = (\vec{u}_2 \cdot \vec{v})\vec{u}_2 = -7\vec{u}_2 = \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}$$

and therefore $\operatorname{proj}_W = \begin{bmatrix} 22 \\ 33 \\ 66 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}$.

Part B (25 points)

Problem 1. Let W be an n-dimensional vector space with ordered bases $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.

(a) Prove that $S_{\mathcal{C}\to\mathcal{A}} = S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{C}\to\mathcal{B}}$.
(b) Show that $S_{\mathcal{C}\to\mathcal{A}}\ S_{\mathcal{B}\to\mathcal{C}}\ S_{\mathcal{A}\to\mathcal{B}} = I_n$.

Solution. Thinking about both $S_{\mathcal{C}\to\mathcal{A}}$ and $S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{C}\to\mathcal{B}}$ as linear transformations from $\mathcal{C}$-coordinate space to $\mathcal{A}$-coordinate space, it suffices to show that they take the same values for all inputs. Let $[v]_{\mathcal{C}}$ be an arbitrary element in the source $\mathbb{R}^n$. We know that

$$S_{\mathcal{C}\to\mathcal{A}}[v]_{\mathcal{C}} = [v]_{\mathcal{A}}.$$

On the other hand, we also know that

$$S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{C}\to\mathcal{B}}[v]_{\mathcal{C}} = S_{\mathcal{B}\to\mathcal{A}}[v]_{\mathcal{B}} = [v]_{\mathcal{A}}.$$

Since these two transformations (or matrices representing transformations) of coordinate space take the same value at every inputed vector, they are the same transformation.

Alternate Solution: For (a), note that the three change of basis matrices are defined by

$$\begin{aligned} S_{\mathcal{C}\to\mathcal{B}} &= L_{\mathcal{B}} \circ L_{\mathcal{C}}^{-1} \\ S_{\mathcal{B}\to\mathcal{A}} &= L_{\mathcal{A}} \circ L_{\mathcal{B}}^{-1} \\ S_{\mathcal{C}\to\mathcal{A}} &= L_{\mathcal{A}} \circ L_{\mathcal{C}}^{-1} \end{aligned}$$

Then the right-hand side of the identity we are to prove is

$$\begin{aligned} S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{C}\to\mathcal{B}} &= (L_{\mathcal{A}} \circ L_{\mathcal{B}}^{-1}) \circ (L_{\mathcal{B}} \circ L_{\mathcal{C}}^{-1}) \\ &= L_{\mathcal{A}} \circ (L_{\mathcal{B}}^{-1} \circ L_{\mathcal{B}}) \circ L_{\mathcal{C}}^{-1} \\ &= L_{\mathcal{A}} \circ L_{\mathcal{C}}^{-1} \\ &= S_{\mathcal{C}\to\mathcal{A}} \end{aligned}$$

For (b), we have

$$\begin{aligned} S_{\mathcal{C}\to\mathcal{A}}\ S_{\mathcal{B}\to\mathcal{C}}\ S_{\mathcal{A}\to\mathcal{B}} &= (S_{\mathcal{C}\to\mathcal{A}}\ S_{\mathcal{B}\to\mathcal{C}})\ S_{\mathcal{A}\to\mathcal{B}} \\ &= S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{A}\to\mathcal{B}} \\ &= I_n \end{aligned}$$

where we first used the result from (a), and then the fact (proved in a worksheet) that $S_{\mathcal{B}\to\mathcal{A}}^{-1} = S_{\mathcal{A}\to\mathcal{B}}$.

Problem 2. Let f_1, f_2, f_3 be the smooth functions defined by

$$f_1(x) = \sin 2x, f_2(x) = \cos 2x, f_3(x) = e^{3x}$$

and consider the subspace $V \subseteq C^\infty(\mathbb{R})$ spanned by the basis $\mathcal{B} = (f_1, f_2, f_3)$. (You may assume without proof that these three functions are linearly independent.) Now consider the linear transformation $D : V \to V$ defined by differentiation, i.e. for any function $g \in V$, $D(g)(x) = \frac{dg}{dx}$.

(a) Find $[D]_\mathcal{B}$.
(b) Give a geometric interpretation of the matrix $[D]_\mathcal{B}$. That is, how does it act on $\mathbb{R}^3$?

Solution.

(a) We observe that $D(f_1)(x) = 2\cos 2x = 2f_2(x)$, $D(f_2)(x) = -2\sin 2x = -2f_1(x)$, and $D(f_3)(x) = 3e^{3x} = 3f_3(x)$, and conclude that

$$[D]_\mathcal{B} = \begin{bmatrix} 0 & -2 & \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) $[D]_\mathcal{B}$ acts on $\mathbb{R}^3$ by rotating around the z-axis 90° in the positive sense (i.e., counterclockwise, as you look down at the xy-plane from the positive z-axis), dilating vectors in the xy-plane by a factor of 2, and stretching vectors in the z-direction by a factor of 3. Another way of describing this is that it deforms a unit sphere into an ellipsoid with a circular cross-section of radius 2 in the xy-plane, and a diameter of 6 in the z-direction, and then rotates the ellipsoid 90° around the z-axis.

Problem 3. Let V be a vector space with ordered bases $\mathcal{B} = (b_1, \dots, b_n)$ and $\mathcal{C} = (c_1, \dots, c_n)$. Let $T : V \to V$ be a linear transformation, with $B = [T]_\mathcal{B}$ and $C = [T]_\mathcal{C}$. Give a proof or counterexample for each of the following statements:

(a) For all integers $k \geq 1$, B^k and C^k are similar.
(b) $\ker(B) = \ker(C)$.
(c) $\dim(\ker(B)) = \dim(\ker(C))$.

Solution.

(a) This is true. We prove this by induction on k. Let $S = S_{\mathcal{B}\to\mathcal{C}}$ be the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. We will show by induction that $B^k = S^{-1}C^kS$ for all $k \geq 1$.

For the base case, recall that by the change of basis theorem for matrices, if $S = S_{\mathcal{B}\to\mathcal{C}}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, then $B = S^{-1}CS$.

For the inductive step, assume that $B^{k-1} = S^{-1}C^{k-1}S$. Multiply both sides on the left by B:

$$\begin{aligned} B^k &= B\ (S^{-1}C^{k-1}S) \\ &= (S^{-1}CS)\ (S^{-1}C^{k-1}S) \\ &= S^{-1}C(S\,S^{-1})C^{k-1}S \\ &= S^{-1}C\,C^{k-1}S \\ &= S^{-1}C^kS. \end{aligned}$$

So B^k and C^k are similar. This completes the proof by induction.

ALTERNATIVE PROOF: We know that C^k represents the transformation T^k in the basis $\mathcal{C}$ since $[T \circ S]_{\mathcal{C}} = [T]_{\mathcal{C}}[S]_{\mathcal{C}}$ for any transformations. Likewise B^k represents the transformation T^k in the basis $\mathcal{B}$. So the two matrices, representing the same matrix in different bases, must be similar: $B^k = S^{-1}C^kS$ where $S = S_{\mathcal{B}\to\mathcal{C}}$.

(b) This is false! For a counter example, consider the vector space $V = \mathcal{P}_1$ and the transformation $T = \frac{d}{dx}$. With respect to the basis $\mathcal{B} = (1, x)$, the $\mathcal{B}$ matrix is $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. With respect the the basis $\mathcal{C} = (x + 1, x)$, the $\mathcal{C}$ matrix is $C = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. The kernel of B is span $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, but the kernel of C is span $\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Another basis is $\mathcal{D} = (x, 1)$: the $\mathcal{D}$ matrix is $D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which has kernel Span $\vec{e}_2$.

(c) This is true. Both $\ker B$ and $\ker C$ are isomorphic to $\ker T$, and hence to each other. To see this, observe that $S_{\mathcal{B}\to\mathcal{C}}$ takes the subspace $\ker B$ isomorphically to $\ker C$, and $S_{\mathcal{C}\to\mathcal{B}}$ is the inverse map, so $\ker C \cong \ker T$.

Isomorphic vector spaces always have the same dimension. This follows from the following argument: any injective linear transformation maps a linearly independent set in the source onto a linearly independent set in the target with the same number of elements; any surjective linear transformation maps a spanning set in the source onto a spanning set in the target; therefore any isomorphism maps a basis in the source onto a basis in the target with the same number of elements.

Problem 4. Let $T : U \to W$ be a linear transformation between vector spaces U and W. Suppose that $\mathcal{B} = (u_1, u_2, \ldots, u_k)$ is a basis for the source U and $\mathcal{C} = (w_1, w_2, \ldots, w_d)$ is a basis for the target W. As usual, let $L_{\mathcal{B}}$ denote the coordinate isomorphism $U \to \mathbb{R}^k$ and let $L_{\mathcal{C}}$ denote the coordinate isomorphism $W \to \mathbb{R}^d$.

(a) Show that there exists a linear transformation $T' : \mathbb{R}^k \to \mathbb{R}^d$ such that $T' \circ L_{\mathcal{B}} = L_{\mathcal{C}} \circ T$. [HINT: A diagram showing four vector spaces and four maps between them, similar to those immediately before and after Definition 4.3.1 in the textbook, might be useful.]

(b) Let $[T]_{(\mathcal{B},\mathcal{C})}$ denote the standard matrix of the transformation T' you described in (a). Prove that for all $u \in U$,

$$[T(u)]_{\mathcal{C}} = [T]_{(\mathcal{B},\mathcal{C})}[u]_{\mathcal{B}}.$$

(c) Describe, with explanation, the columns of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ in terms of the bases $\mathcal{B}$ and $\mathcal{C}$.

Solution.

(a) Define $T' = L_{\mathcal{C}} \circ T \circ L_{\mathcal{B}}^{-1}$. Then by construction, $T' \circ L_{\mathcal{B}} = L_{\mathcal{C}} \circ T$, as required.

(b) We have

$$\begin{aligned} [T]_{(\mathcal{B},\mathcal{C})}[u]_{\mathcal{B}} &= [T]_{(\mathcal{B},\mathcal{C})}L_{\mathcal{B}}(u) \\ &= T'(L_{\mathcal{B}}(u)) \\ &= L_{\mathcal{C}}(T(u)) \\ &= [T(u)]_{\mathcal{C}} \end{aligned}$$

(c) By the Key Theorem, the i^{th} column of $[T]_{(\mathcal{B},\mathcal{C})}$ is given by $T'(\vec{e_i})$, where $\vec{e_i}$ is the i^{th} standard basis vector in $\mathbb{R}^k$. But $\vec{e_i} = L_{\mathcal{B}}(u_i)$, so

$$T'(\vec{e_i}) = T'(L_{\mathcal{B}}(u_i)) = L_{\mathcal{C}}(T(u_i)) = [T(u_i)]_{\mathcal{C}}$$

Thus, the i^{th} column of $[T]_{(\mathcal{B},\mathcal{C})}$ consists of the $\mathcal{C}$-coordinates of the image under T of the i^{th} basis vector of $\mathcal{B}$.

Note: it is also possible to solve (b) by first using the Key Theorem to compute $[T]_{(\mathcal{B},\mathcal{C})}$ and then applying it to (b).

Problem 5. Let f_1, f_2, f_3 be the functions defined by

$$f_1(x) = \sin x,\ \ f_2(x) = \cos x,\ \ f_3(x) = e^x,$$

which you may assume without proof are linearly independent. Consider the subspace V of $\mathcal{C}^\infty$ spanned by the set $\{f_1, f_2, f_3\}$. Recall from Calculus that every function in V may be expressed as a Taylor series that converges for all real numbers. For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots,$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

Let $T : V \to \mathcal{P}_3$ be the linear transformation that assigns to each function $f \in V$ the third-degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ for f, a polynomial approximation to f.

(a) Find a basis $\mathcal{C}$ for $\mathcal{P}_3$ such that

$$[T(f_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, [T(f_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [T(f_3)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Let $\mathcal{C}$ be as in (a), and let $\mathcal{B} = (f_1 + f_2, f_1 - f_2, f_3 + f_1)$. Find $[T]_{(\mathcal{B},\mathcal{C})}$ (see Problem 4).

Solution.

(a) By inspection, if we choose $\mathcal{C} = (p_0, p_1, p_2, p_3)$ where

$$p_0(x) = 1$$
$$p_1(x) = x$$
$$p_2(x) = \frac{x^2}{2}$$
$$p_3(x) = \frac{x^3}{3!}$$

then

$$T(f_1) = x - \frac{x^3}{3!} = p_1 - p_3$$
$$T(f_2) = 1 + \frac{x^2}{2} = p_0 - p_2$$

$$T(f_3) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} = p_0 + p_1 + p_2 + p_3$$

and therefore

$$[T(f_1)]_\mathcal{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, [T(f_2)]_\mathcal{C} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [T(f_3)]_\mathcal{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

as required.

(b) We follow the procedure outlined in the previous problem. The first column of $[T]_{(\mathcal{B},\mathcal{C})}$ will be

$$[T(f_1 + f_2)]_\mathcal{C} = [T(f_1)]_\mathcal{C} + [T(f_2)]_\mathcal{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Similarly the second and third columns will be found from

$$[T(f_1 - f_2)]_\mathcal{C} = [T(f_1)]_\mathcal{C} - [T(f_2)]_\mathcal{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$[T(f_3 + f_1)]_\mathcal{C} = [T(f_3)]_\mathcal{C} + [T(f_1)]_\mathcal{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

so $[T]_{(\mathcal{B},\mathcal{C})} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

Problem 6. Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

(a) Show that for all $\vec{v} \in V$, $A\vec{v} \in V$.
(b) Find a basis for $V^\perp$, and show that for all $\vec{w} \in V^\perp$, $A\vec{w} \in V^\perp$.
(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_\mathcal{B}$ is diagonal, and write the matrix $[T]_\mathcal{B}$ explicitly.
(d) Calculate $[T^{10}]_\mathcal{B}$. [Hint: Leave numbers like 7^{13} in that form; do not attempt to multiply them out.]
(e) Calculate $[T^{10}]_\mathcal{E}$. [Hint: Leave the entries as numerical expressions; do not attempt to simplify.]

Solution.

(a) Call $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We calculate

$$A\vec{u} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -78 \\ -52 \end{bmatrix} = -26\vec{u}$$

It follows that if $\vec{v} \in V$, so that $\vec{v} = k\vec{u}$ for some $k \in \mathbb{R}$, then $A\vec{v} = -26k\vec{u} \in V$.

(b) If we write $w = \begin{bmatrix} x \\ y \end{bmatrix}$ then $\vec{w} \in V^\perp$ if and only if $3x + 2y = 0$, or equivalently $x = -\frac{2}{3}y$. Thus $V^\perp$ is spanned by $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. We now calculate

$$A \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -78 \\ 117 \end{bmatrix} = 39 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

which shows that $A\vec{w} \in V^\perp$.

(c) If we choose $\mathcal{B} = \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right)$ then

$$[T]_{\mathcal{B}} = \begin{bmatrix} -26 & 0 \\ 0 & 39 \end{bmatrix}$$

This in not the only answer: the two vectors can be scaled by any non-zero scalar.

(d)

$$[T^{10}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{10} = \begin{bmatrix} 26^{10} & 0 \\ 0 & 39^{10} \end{bmatrix} = 13^{10} \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix}$$

(e)

$$\begin{aligned}
[T^{10}]_{\mathcal{E}} &= S_{\mathcal{B}\to\mathcal{E}} [T^{10}]_{\mathcal{B}} S_{\mathcal{E}\to\mathcal{B}} \\
&= \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} 13^{10} \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}^{-1} \\
&= 13^{10} \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \\
&= 13^9 \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \cdot 2^{10} & 2 \cdot 2^{10} \\ -2 \cdot 3^{10} & 3 \cdot 3^{10} \end{bmatrix} \\
&= 13^9 \begin{bmatrix} 9 \cdot 2^{10} + 4 \cdot 3^{10} & 6 \cdot 2^{10} - 6 \cdot 3^{10} \\ 6 \cdot 2^{10} - 6 \cdot 3^{10} & 4 \cdot 2^{10} + 9 \cdot 3^{10} \end{bmatrix}
\end{aligned}$$

You don't have to multiply out the three matrices for full credit.