Final Exam

Graded

Student

Qiulin Fan

Total Points

80 / 100 pts

Question 1

Definitions **16** / 16 pts

1.1 Span **4** / 4 pts

✔ **+ 4 pts** Correct

4 the set of

1.2 Diagonalizable **4** / 4 pts

✔ **+ 4 pts** Equivalent correct definition: "There exists an eigenbasis for T" (or something that means the same as this)

1.3 Eigenvalue **4** / 4 pts

✔ **+ 4 pts** Correct

1.4 Subspace **4** / 4 pts

✔ **+ 4 pts** Correct

Question 2

True / False **14** / 16 pts

2.1 2(a) **2** / 2 pts

✔ **+ 2 pts** Correct

2.2 2(b) **2** / 2 pts

✔ **+ 2 pts** Correct (False)

2.3 2(c) **2** / 2 pts

✔ **+ 2 pts** Correct (False)

2.4 2(d) **2** / 2 pts

✔ **+ 2 pts** Correct (False)

2.5 2(e) **0** / 2 pts

✔ **+ 0 pts** Incorrect (False)

2.6 2(f) **2** / 2 pts

✔ **+ 2 pts** Correct (True)

2.7 2(g) **2** / 2 pts

✔ **+ 2 pts** Correct (True)

2.8 2(h) **2** / 2 pts

✔ **+ 2 pts** Correct (False)

Question 3

Determinants (calculation) — **8** / 10 pts

3.1 **det(A + B)** — **2** / 2 pts

- ✔ **+ 2 pts** Correct (Not Enough Information)

3.2 **det(B^T AB)** — **2** / 2 pts

- ✔ **+ 2 pts** Correct (answer: 45)

3.3 **det(...)** — **2** / 2 pts

- ✔ **+ 2 pts** Correct answer (1/5)

3.4 **|det(QA^2)|, Q orthogonal** — **2** / 2 pts

- ✔ **+ 2 pts** Correct (answer: 25)

3.5 **det(rref(B))** — **0** / 2 pts

- ✔ **+ 0 pts** No credit or no submission

Question 4

Transformation on 2x2 matrices — **12** / 12 pts

4.1 **Find B-matrix** — **2** / 2 pts

- ✔ **+ 2 pts** Correct

4.2 **Find [T(A)]_B** — **4** / 4 pts

- ✔ **+ 4 pts** Correct (no DJ with (a))

4.3 **Compute T([1 1 \\ 1 1])** — **3** / 3 pts

- ✔ **+ 1 pt** Correct
- ✔ **+ 1 pt** multiplied basis vectors by e-vals or B vector by B matrix
- ✔ **+ 1 pt** wrote input correctly as LC of basis

4.4 **Basis for ker(T) and im(T)** — **3** / 3 pts

- ✔ **+ 1 pt** basis for ker
- ✔ **+ 2 pts** basis for im

Question 5

"Eigencoordinates" 7 / 8 pts

Non-eigen approach

✔ **+ 8 pts** Sets up correct system of equations, solves correctly, expresses answer in correct form

---

General errors

✔ **− 1 pt** Expresses answer as a column vector in $\mathbb{R}^2$ (or some other kind of column vector) instead of as an element of V

5 this is an element of $\mathbb{R}^2$, not a vector in V

Question 6

Orthogonal and symmetric **2** / 8 pts

6.1 **Diagonalizable, eigenvalues are ± 1, orthogonal eigenspaces** **2** / 4 pts

✔ **+ 1 pt** T diagonalizable by S.T.

✔ **+ 1 pt** showed evals can only be pm1

3 why?

6.2 **Reflection** **0** / 4 pts

✔ **+ 0 pts** No credit or no submission

Question 7

Inner product space with finite et **8** / 10 pts

7.1 **projection** **5** / 5 pts

✔ **+ 5 pts** Correct

7.2 **S^\perp^\perp = Span(S)** **3** / 5 pts

✔ **+ 3 pts** Got the right setup, but incorrect one of the containment.

1 There is a logical gap here

Question 8

Orthonormal bases for two subspace **7** / 10 pts

8.1 **Express A^T A** **4** / 4 pts

✔ **+ 4 pts** Correct

8.2 **Show [T]_U = A^T A** **1** / 4 pts

✔ **+ 1 pt** Definition of $[T]_{\mathcal{U}}$

2 How?

8.3 **Find det(T)** **2** / 2 pts

✔ **+ 1 pt** $\det\left(A^{\top} A\right)=\det\left(A^{\top}\right) \det(A)$

✔ **+ 1 pt** $\frac{1}{4}$

Question 9

A similar to A^2 **6** / 10 pts

9.1 **Induction** **5** / 5 pts

✔ **+ 5 pts** Correct

9.2 **If Diagonalizable then...** **1** / 5 pts

✔ **+ 1 pt** Some initial correct step or idea

Math 217 – Final Exam

Winter 2024

Time: 120 mins.

1. Answer each question in the space provided; we have left some pages blank if you need more space, but please indicate when you do so.
2. Ask us if you need more paper.
3. Your solutions will be graded for clarity, precision, and the correct use of mathematical notation.
4. You must solve all problems using methods that have been taught in this course.
5. You are free to quote results from the worksheets, the textbook, or homework as a step in proving something else, but indicate clearly when you are doing so. **Exception:** if an entire problem is asking you to reprove a result from class or homework, we expect you to reproduce a proof.
6. **No calculators,** notes, or other outside assistance allowed.
7. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
8. Even if a problem states that no justification is necessary, you may provide explanations if you wish; this could result in partial credit for an incorrect final answer.

Student ID Number: 58848733 Section: 005

Qiulin Fan

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	16	16	10	12	8	8	10	10	10	100
Score:										

Blank page

1. Complete each of the below phrases into a precise definition or equivalent mathematical characterization of the **boldfaced** term. For full credit, use precise mathematical terms and write clear, complete mathematics and English, and be sure to define/explain any symbols you introduce.

   (a) (4 points) For a subset $S = \{v_1, v_2, \ldots, v_k\}$ of a vector space V, the **span of** S is ...

   4 all possible linear combinations of vectors in S.

   i.e. $\{a_1v_1 + a_2v_2 + \ldots + a_kv_k \mid a_1, a_2, \ldots, a_k \in \mathbb{R}\}$

   (b) (4 points) For a linear transformation $T : V \to V$, where V is a finite-dimensional vector space, we say that T **is diagonalizable** if ...

   there exists some basis of V such that

   ~~there exists~~ invertible matrix S, for some $S^{-1}[T]_{\mathcal{B}}S$ is a diagonal matrix. (i.e., $[T]_{\mathcal{B}}$ is similar to a diagonal matrix.

   (c) (4 points) If $T : V \to V$ is a linear transformation, a scalar $k \in \mathbb{R}$ is called an **eigenvalue of** T if ...

   for some nonzero vector $\vec{v} \in V$, $T(\vec{v}) = k\vec{v}$.

   (d) (4 points) If V is a vector space, a subset $W \subseteq V$ is called a **subspace of** V if ...

   ① $\vec{0} \in W$

   ② W is closed under addition $(\forall \vec{v}, \vec{w} \in W, \vec{v}+\vec{w} \in W)$

   ③ W is closed under scalar multiplication $(\forall k \in \mathbb{R}, \vec{v} \in W, k\vec{v} \in W)$

Blank page

2. Decide whether each statement is True or False.

   Then neatly write **T** if true or **F** if false in the box provided.

   No justification required. No partial credit. Each is worth 2 points.

   (a) (2 points) If $A \in \mathbb{R}^{m\times n}$ is an $m \times n$ matrix with both orthonormal columns and orthonormal rows, then A is square.

   T

   $[\cdots]$

   (b) (2 points) If V is a finite-dimensional vector space, $T : V \to V$ is a linear transformation, $\mathcal{B}$ is an eigenbasis for T, and $\mathcal{C}$ is the result of applying the Gram–Schmidt algorithm to $\mathcal{B}$, then $[T]_\mathcal{C}$ is diagonal.

   $[T]_C =$

   ~~T~~ F

   (c) (2 points) If A is a symmetric real matrix, then every eigenbasis for A is orthogonal.

   F

   $[\quad]$

   (d) (2 points) The sum of any two diagonalizable $n \times n$ matrices is diagonalizable.

   F

   $A = S^{-1}\cdot$

(Problem 2 continues on the next page...)

Blank page

(Continuation of Problem 2. Decide whether each statement is True or False. Then neatly write **T** if true or **F** if false in the box provided. **No justification required. No partial credit.** Each is worth 2 points.)

(e) (2 points) An $n \times n$ matrix A is diagonalizable if and only if $A + I_n$ is diagonalizable.

T

$(x-\lambda_1)^m$

$\det(A - xI_n)$

$\det(A - (x+1)I_n)$

(f) (2 points) For every vector space V and linear map $T : V \to V$, every eigenvector of T belongs to $\text{im}(T) \cup \ker(T)$.

T

(g) (2 points) If n is odd and A is an $n \times n$ matrix that is skew-symmetric ($A^\top = -A$), then $\det(A) = 0$.

T

$\det(A^T) = \det(A)$

$= \det(-A)$

$= (-1)^n \det(A)$

(h) (2 points) If a and b are distinct eigenvalues of a transformation $T : V \to V$ then $E_a \cup E_b$ is a subspace of V.

F

Blank page

3. Let $A, B \in \mathbb{R}^{4\times 4}$ and suppose that $\det A = 5$, $\det B = -3$. Find each of the following determinants if possible. Otherwise, write "not enough information."

(a) (2 points) $\det(A + B)$

not enough information..

(b) (2 points) $\det(BAB^\top)$

$= \det(B)\det(A)\det(B^T)$

$= (-3) \times 5 \times (-3) = 45$

(c) (2 points) $\det\left(-\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} A^{-1}\right)$

$= \det(-C)\det(A^{-1})$

$= \frac{\det(C)}{\det A} = \frac{1}{5}$

(d) (2 points) $|\det(QA^2)|$ where $Q \in \mathbb{R}^{4\times 4}$ is an orthogonal matrix

$= |\det Q| \cdot |\det(A^2)|$

$\det Q = \pm 1$ since Q is orthogonal

$\Rightarrow |\det(Q)| = 1$

$\Rightarrow$ it is $|\det(A^2)| = |(\det A)^2| = 25$

(e) (2 points) $\det(\text{rref}(B))$

not enough information..

Blank page

4. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation, and suppose

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

is an eigenbasis of $\mathbb{R}^{2\times 2}$ for T with corresponding eigenvalues 0, 1, 2, and 3, in the given order.

(a) (2 points) Find the $\mathcal{B}$-matrix of T.

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(T_1)]_{\mathcal{B}} & \sim & [T(T_4)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) (4 points) If the matrix $A \in \mathbb{R}^{2\times 2}$ has $\mathcal{B}$-coordinates $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, what is $[T(A)]_{\mathcal{B}}$?

$$[T(A)]_{\mathcal{B}} = [T]_{\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 12 \end{bmatrix}$$

(c) (3 points) Compute $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$.

Note that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} \cancel{8}\,4 & 0 \\ 4 & 0 \end{bmatrix}$

So $\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 4 \end{bmatrix}$

So $\left[T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \\ 12 \end{bmatrix}$

(Problem 4 continues on the next page...)

So $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + 12\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 1 \\ 6 & 4 \end{bmatrix}$$

Blank page

This is a continuation of Problem 4. For reference, $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is a linear transformation for which $\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right)$ is an eigenbasis, with corresponding eigenvalues 0, 1, 2, and 3, in the given order.

(d) (3 points) Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

Since $\mathcal{B}$ is an eigenbasis, the four elements are linearly independent

Since $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \in E_1$, $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \in E_2$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in E_3$ (all not $\vec{0}$)

they are not in $\ker(T)$

and conversely since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in E_0$ and it is not 0 vector, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \ker T$

So $\dim(\ker T) = 1$, $\dim(\operatorname{im} T) = 3$

So a basis for $\ker T$ is $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$, a basis for $\operatorname{im}(T)$ is $\left(\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right)$

5. (8 points) Let V be a vector space with basis $\mathcal{B} = (v_1, v_2)$. Let $\mathcal{C} = (2v_1 - v_2, v_1 + 4v_2)$ be a second basis for V. (You do not have to verify that $\mathcal{C}$ is a basis.) Find all nonzero vectors $v \in V$ and all $\lambda \in \mathbb{R}$ such that $[v]_\mathcal{B} = \lambda [v]_\mathcal{C}$. (Express your answers in terms of v_1 and v_2.) let $\vec{v} \in V$ be arbitrary, ~~So $\vec{v} = a\vec{v_1} + b\vec{v_2}$~~ for some $a, b \in \mathbb{R}$

Assume $[v]_\mathcal{B} = \lambda [v]_\mathcal{C}$

so $\vec{v} = a(2v_1 - v_2) + b(v_1 + 4v_2)$

$= (2a + b)v_1 + (4b - a)v_2$

then $\begin{bmatrix} 2a+b \\ 4b-a \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow a = 4b - \lambda b \Rightarrow 8b - 2\lambda b + b = \lambda(4b - \lambda b)$

Case 1° $b = 0$

$\Rightarrow 4b - a = b$ and $2a + b = 0$

$\Rightarrow a = 4b$ and $a = \frac{-1}{2}b$

$\Rightarrow a = b = 0$

~~Since $\vec{0}$ is not eigenvector, this is impossible.~~

$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \vec{0}$, drop

Case 2° $b \neq 0$

$\Rightarrow 8 - 2\lambda + 1 = 4\lambda - \lambda^2$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$

So $\begin{bmatrix} 2a+b \\ 4b-a \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$

$\Rightarrow b = a$

~~So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis of E_3.~~

~~Therefore $\lambda = 3$ is the only eigenvalue~~

So the only possible λ is 3 and $\vec{v} \in \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

5

Blank page

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation whose standard matrix $A \in \mathbb{R}^{n\times n}$ is symmetric.

   (a) (4 points) Show that T is diagonalizable, that its only possible eigenvalues (real or complex) are ± 1, and that $(E_1)^\perp = E_{-1}$.

① by the spectral theorem, since $[T]_\varepsilon = A$ is symmetric, T is orthogonally diagonalizable.

② Since T is orthogonal linear transformation, its standard matrix A is orthogonal matrix
since the only possible eigenvalues of are A are ± 1 (A has no complex eigenvalue, by the spectral Thm)

~~③ Since A is orthogonal, $A^T = A^{-1}$ and since A is symmetric, $A^T = A$ $\Rightarrow A = A^{-1}$~~

~~③ $E_1 = \ker(A - I_n)$~~
~~$E_{-1} = \ker(A + I_n)$~~
~~if $\vec{v} \in E_1 \Rightarrow A - I_n = \vec{0} \Rightarrow A = I_n$~~

③ Since T is diagble, A has an eigenbasis $\mathcal{B}$
And since the only possible eigenvalues are ± 1, any $\vec{v} \in \mathcal{B}$ is either in E_1 or E_{-1}

So $(E_1)^\perp = E_{-1}$

   (b) (4 points) Find a subspace $V \subseteq \mathbb{R}^n$ such that $T = \text{refl}_V$, reflection through V. Be sure to include justification.

Since A is symmetric, there $\exists$ an orthogonal eigenbasis

Blank page

7. Let V be a finite dimensional inner product space, with inner product $\langle -, - \rangle$, and let $S \subset V$ be a finite set. .

(a) (5 points) Let $x \in (S^\perp)^\perp$. Prove that $\text{proj}_{\text{Span(S)}} x = x$.

Pf. Since $x \in (S^\perp)^\perp$, ~~$x \in S$, so $\text{proj}_{\text{span}(S)} x = x$, $x \in \text{span}(S)$. Therefore $\text{proj}_{\text{span}(S)} x = x$.~~

$\forall \vec{v} \in S^\perp, \langle \vec{v}, x \rangle = 0$

We factor $\vec{x}$ as its projection on span(S) and ~~the distance between~~ $x^\perp = x - \text{proj}_{\text{span}} x$ ~~$\Rightarrow x \in S^\perp$~~ $\in S^\perp$

Since $x = \underbrace{\text{proj}_{\text{span}(S)} x + \vec{x}^\perp}_{x^{\parallel} + x^\perp} = \text{proj}_{\text{span}(S)} x + 0 = \text{proj}_{\text{span}(S)} x$.

(b) (5 points) Prove $(S^\perp)^\perp = \text{Span}(S)$.

~~Let $\vec{v} \in (S^\perp)^\perp$~~

~~So for any vectors $\vec{w} \in S^\perp$, $\langle \vec{v}, \vec{w} \rangle = 0$~~

~~Since $\vec{w} \in S^\perp$, for any vector $\vec{a} \in S$, $\langle \vec{w}, \vec{a} \rangle = 0$~~

Denote S as $S = \{\vec{w_1}, \vec{w_2}, \dots, \vec{w_n}\}$

① Assume $\vec{v} \in S^\perp$, the for any $\vec{w} \in S$,

$\langle \vec{v}, \vec{w} \rangle = 0$.

Take arbitrary $\vec{a} \in \text{span}(S) \Rightarrow \vec{a} = a_1 \vec{w_1} + \dots + a_n \vec{w_n}$ for some $a_1, \dots, a_n \in \mathbb{R}$

~~$S^\perp = (\text{Span}(S))^\perp$~~ $\Rightarrow \langle \vec{a}, \vec{v} \rangle = a_1 \langle \vec{w_1}, \vec{v} \rangle + \dots + a_n \langle \vec{w_n}, \vec{v} \rangle$

$= 0 + \dots + 0 = 0$

So $\vec{a} \in (S^\perp)^\perp$

Therefore $\text{span}(S) \subseteq (S^\perp)^\perp$

② Assume $\vec{v} \in (S^\perp)^\perp$, then take arbitrary $\vec{w} \in (\text{Span}(S))^\perp \Rightarrow \langle \vec{v}, \vec{w} \rangle = 0$

$\Rightarrow \vec{v} \in \text{Span} S$

$\Rightarrow (S^\perp)^\perp \subseteq \text{Span}(S)$

Therefore $(S^\perp)^\perp = \text{Span}(S)$

Blank page

8. Let V and W be subspaces of $\mathbb{R}^4$. Suppose that $(\vec{v}_1, \vec{v}_2)$ is an orthonormal basis of V and $(\vec{w}_1, \vec{w}_2)$ is an orthonormal basis of W. Let

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{w}_1 & \vec{w}_2 \\ | & | & | & | \end{bmatrix}.$$

(a) (4 points) Express the matrix $A^\top A$ in terms of

$$a = \vec{v}_1 \cdot \vec{w}_1, \quad b = \vec{v}_1 \cdot \vec{w}_2, \quad c = \vec{v}_2 \cdot \vec{w}_1, \quad \text{and} \quad d = \vec{v}_2 \cdot \vec{w}_2.$$

$$A^TA = \begin{bmatrix} - \vec{v}_1^T - \\ - \vec{v}_2^T - \\ - \vec{w}_1^T - \\ - \vec{w}_2^T - \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{w}_1 & \vec{w}_2 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 & \vec{v}_1\cdot\vec{w}_1 & \vec{v}_1\cdot\vec{w}_2 \\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2 & \vec{v}_2\cdot\vec{w}_1 & \vec{v}_2\cdot\vec{w}_2 \\ \vec{w}_1\cdot\vec{v}_1 & \vec{w}_1\cdot\vec{v}_2 & \vec{w}_1\cdot\vec{w}_1 & \vec{w}_1\cdot\vec{w}_2 \\ \vec{w}_2\cdot\vec{v}_1 & \vec{w}_2\cdot\vec{v}_2 & \vec{w}_2\cdot\vec{w}_1 & \vec{w}_2\cdot\vec{w}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ a & c & 1 & 0 \\ b & d & 0 & 1 \end{bmatrix}$$

(b) (4 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T(\vec{x}) = \text{proj}_V(\vec{x}) + \text{proj}_W(\vec{x}) \quad \text{for all} \quad \vec{x} \in \mathbb{R}^4.$$

Suppose that $\mathcal{U} = (\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2)$ is a basis of $\mathbb{R}^4$. Show that $[T]_\mathcal{U} = A^\top A$.

$$[T]_\mathcal{U} = \begin{bmatrix} | & | & | & | \\ [T(v_1)]_\mathcal{U} & [T(v_2)]_\mathcal{U} & [T(\vec{w}_1)]_\mathcal{U} & [T(\vec{w}_2)]_\mathcal{U} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ [\text{proj}_V(v_1) + \text{proj}_W(\vec{v}_1)]_\mathcal{U} & \cdots & [\text{proj}_V(\vec{w}_2) + \text{proj}_W(\vec{w}_2)]_\mathcal{U} \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ a & c & 1 & 0 \\ b & d & 0 & 1 \end{bmatrix} = A^TA.$$

(c) (2 points) Suppose that $\det(A) = \frac{1}{2}$. Find the determinant of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined in part (b).

$$\det T = \det([T]_\mathcal{U}) = \det(A^TA) = \det(A^T)\cdot\det(A)$$
$$= \det(A)^2 = \frac{1}{4}$$

Blank page

9. (a) (5 points) Prove that for every square matrix A, if A is similar to A^2 then A is similar to A^{2^k} for every $k \in \mathbb{N}$.

Let A be arbitrary square matrix

Assume A is similar to A^2

So $A = S^{-1}A^2S$ for some invertible ~~matrix~~ matrix S

~~Then $A^2 = S^{-2}A^4S^2 \Rightarrow A^4$~~

Now we prove the statement by induction on $k \in \mathbb{N}$.

~~Let $k \in \mathbb{N}$ be arbitrary~~

~~$A^{2^k} = (A^2)\cdot(A^2)\cdots(A^2) = S^{-2}A$~~

Base case: $k=1$. A is similar to $A^{2^1} = A^2$ ~~trivially~~

Inductive step: Assume A is similar to A^{2^k} ~~for any $k \in \mathbb{N}$ and $k \leq k$~~ ~~$k \in \mathbb{N}$ and $n \leq k$~~ for $k \in \mathbb{N}$.

we want to show: A is similar to $A^{2^{k+1}}$, ~~so that A is similar to~~

here continued

Since A is similar to A^{2^k}

Then $A^{2^k} = B^{-1}AB$ for some invertible matrix B

So $A^{2^{(k+1)}} = A^{2\cdot 2^k} = (A^{2^k})^2 = A^{2^k}\cdot A^{2^k}$

$= (B^{-1}AB)(B^{-1}AB) = B^{-1}A^2B$

is similar to A^2

Since A^2 is similar to A,

$A^{2^{(k+1)}}$ is similar to A by transitivity of similarity

Therefore A is similar to A^{2^k} for every $k \in \mathbb{N}$ by induction.

(b) (5 points) Prove that for every *diagonalizable* square matrix A, if A is similar to A^2 then in fact $A = A^2$.

Assume A is diagonalizable and similar to A^2

So A is similar to some diagonal matrix D

Blank page

Blank page

$A^4 = S^2 A^2 S^{-2}$

$A^8 = S$

A $(u_1, \ldots, u_n)$

$A^{2(k+1)}$ E_1

$A^2 \cdot A^{2k}$

$\begin{bmatrix} -3 & 0 \\ -3 & 3 \end{bmatrix}$

$A^{2(k+1)}$

$+ \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$

$A^{2 \cdot 2^k}$

$a \vec{v} \in S$

$\langle a, u \rangle = 0$

$(A^{2k})(A^{2k})$

$= \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$

$\langle \vec{v}, \vec{v} \rangle$

$A^{2k} = S^{-1} A S$

$\langle a - v, v \rangle =$

$-6 \quad 1$

$-6 \quad 4$ S^{-1}

S

$\text{span}(S)$

$\langle w + a, u + w \rangle = \langle x, u \rangle$

$\langle w, w \rangle + \langle a, \vec{v} \rangle = 0$

$= \langle x, v \rangle + \langle \; \rangle$

$A = S^{-1} D S$

$A^2 = M^{-1} A M$

$= M^{-1} S^{-1} D S M$

$A = A^2$

$A = S^{-1} A^2 S$

$A = M^{-1} D M$