MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 5, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 3.2: 56
Section 3.3: 33, 63 (Hint: argue that every basis of V must also be a basis of W).
Section 4.1: 12, 28

Solution.

3.2.56 : Recall that for $A \in \mathbb{R}^{m\times n}$, the set of its column vectors is linearly independent if and only if $\operatorname{rank}(A) = n$. Consider

$$A := \begin{bmatrix} e & k & a & f \\ 1 & m & b & g \\ 0 & 1 & c & h \\ 0 & 0 & d & i \\ 0 & 0 & 1 & j \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the columns of A are linearly independent for any values of $a, b, \dots, m$.

3.3.33 : Consider the hyperplane $V \subset \mathbb{R}^n$ defined by the following linear equation

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0.$$

By assumption, at least one of the coefficients is non-zero. Let $k \leq n$ be the smallest number such that $c_k \neq 0$. Then we may rewrite the above linear equation as

$$x_k + d_{k+1}x_{k+1} + \ldots + d_nx_n = 0,$$

where $d_i := c_i/c_k$. It is easy to see that any solution $\mathbf{x} = (x_1, \dots, x_n)$ to this equation can be expressed as

$$\mathbf{x} = x_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_{k-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_{k+1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -d_{k+1} \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -d_n \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= x_1\mathbf{v}_1 + \ldots + x_{k-1}\mathbf{v}_{k-1} + x_{k+1}\mathbf{v}_k + \ldots + x_n\mathbf{v}_{n-1}.$$

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is clearly linearly independent, and hence, forms a basis for V. Therefore, $\dim(V) = n - 1$. A hyperplane in $\mathbb{R}^3$ defined by the equation

$$ax + by + cz = d$$

forms a two-dimensional plane. Similarly, a hyperplane in $\mathbb{R}^2$ defined by the equation

$$ax + by = c$$

forms a one-dimensional line.

3.3.63: Let $d := \dim(V) = \dim(W)$. Choose a basis $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ for V. Since $V \subseteq W$, we see that $\mathfrak{B} \subset W$. Since $\mathfrak{B}$ is a linearly independent set of $d = \dim(W)$ vectors, $\mathfrak{B}$ must be a basis for W by Theorem 3.3.4.c.

4.1.12: Let W be the set of all arithmetic sequences of V. There are three things that need to be checked:

- $0 \in W$, since clearly the zero sequence
$$(0, 0, 0, 0, \dots) = (0, 0 + 0, 0 + 2(0), 0 + 3(0), \dots)$$
is an arithmetic sequence.
- Let $(a, a + k, a + 2k, a + 3k, \dots), (b, b + l, b + 2l, b + 3l, \dots) \in W$ for some constants a, b, k and l. Then the sum
$$\begin{aligned}(a, a + k,&a + 2k, a + 3k, \dots) + (b, b + l, b + 2l, b + 3l, \dots) \\ &= (a + b, (a + b) + (k + l), (a + b) + 2(k + l), (a + b) + 3(k + l), \dots)\end{aligned}$$
is also an arithmetic sequence, W is closed under addition.
- Let $(a, a + k, a + 2k, a + 3k, \dots) \in W$ and $r \in \mathbb{R}$. Then the scalar product
$$r(a, a + k, a + 2k, a + 3k, \dots) = (ra, ra + rk, ra + 2(rk), ra + 3(rk), \dots)$$
is also an arithmetic sequence. That is, W is also closed under scalar multiplication.

Thus W is a subspace.

4.1.28: Let $V \subset \mathbb{R}^{2\times 2}$ denote the space of matrices which commute with

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V.$$

Observe

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a + b \\ c & c + d \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ c & d \end{bmatrix}.$$

Hence, we have

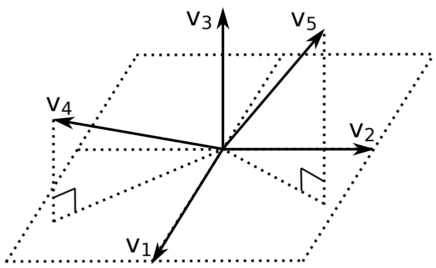
$$\begin{cases} a = a + c \\ a + b = b + d \\ c + d = d \end{cases} \iff \begin{cases} c = 0 \\ a = d \end{cases}.$$

Thus

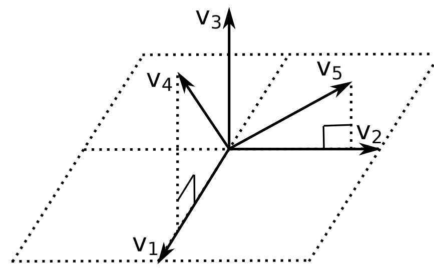
$$V = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

We conclude that $\dim(V) = 2$.

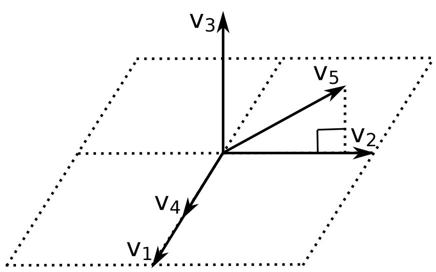
Part A Problem 6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_5$ be vectors in $\mathbb{R}^3$, as shown in the four figures below. In each figure, find *all* linearly dependent sets consisting of three of these five vectors, or else state that there are none if this is the case. *No justification needed.* (Note that in each of these figures, $\mathbf{v}_1$ and $\mathbf{v}_2$ span the displayed plane, $\mathbf{v}_3$ points "up" and is perpendicular to this plane, and for any other vector *not* in the plane, we draw a dotted vertical line indicating its position above the plane.)



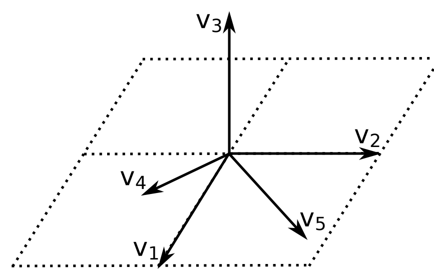
(a)



(b)



(c)



(d)

Solution.

(a) There are no linearly dependent subsets.
(b) $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$.
(c) $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$.
(d) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$, $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$, $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5\}$.

Part B (25 points)

Problem 1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ be a list of vectors in V, and consider the list $Y = (T(\mathbf{x}_1), \ldots, T(\mathbf{x}_k))$ of vectors in W. Determine whether the following statements are true or false. If true, provide a proof. If false, provide a counter-example.

(a) If X is linearly independent, then Y is also linearly independent.
(b) If Y is linearly independent, then X is also linearly independent.

Solution.

(a) **FALSE.** Counter-example: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T((x, y)) = (x, 0)$. Then $X := ((1, 0), (0, 1))$ is linearly independent, but $Y = (T(1, 0), T(0, 1)) = ((1, 0), (0, 0))$ is linearly dependent.

(a) **TRUE.** Observe:

$$c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = 0 \implies T(c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k) = 0 \implies c_1\mathbf{y}_1 + \ldots + c_k\mathbf{y}_k = 0.$$

Since $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is linearly independent, the last equality implies that $c_1 = \ldots = c_k = 0$.

Problem 2.

(a) Find[1] a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ such that

$$\ker(T) = \{\vec{x} \in \mathbb{R}^5 \; : \; x_1 = 5x_2 \text{ and } x_3 = 7x_4\} \qquad \text{and} \qquad \operatorname{im}(T) = \{\vec{x} \in \mathbb{R}^3 \; : \; x_1 = x_3\}.$$

(b) Is the linear transformation you found in part (a) unique? Justify your claim.

Solution.

(a) We observe that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 \\ x_3 - 7x_4 \\ x_1 - 5x_2 \end{bmatrix}$ has both of the desired properties:

- First, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \ker T$ if and only if $x_1 - 5x_2 = 0$ and $x_3 = 7x_4 = 0$.

[1] What does "find" mean? Should you go look in your closet? In this context, "find" means to explicitly describe or construct, as in "produce a concrete example and prove that it works." Here you're asked to find a function, and functions are typcially defined by specifying their source and target and a rule for converting inputs to outputs. In this case you're *given* the source and target, so you just need to specify a rule. So your answer should be something lke "Let T be the function defined by $T(\vec{x}) = ??$ for all $\vec{x} \in \mathbb{R}^5$." Your job is to decide what ?? should be.

- Second, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right)$ has the form $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$, where $a = x_1 - 5x_2$ and $b = x_3 - 7x_4$. So $\operatorname{im} T \subseteq \{\vec{x} \in \mathbb{R}^3 : x_1 = x_3\}$. Moreover, every vector of the form $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$ is equal to $T\left(\begin{bmatrix} a \\ 0 \\ b \\ 0 \\ 0 \end{bmatrix}\right)$, which shows that $\{\vec{x} \in \mathbb{R}^3 : x_1 = x_3\} \subseteq \operatorname{im} T$.

(b) Definitely not! Notice that if T is any linear transformation with the desired properties, then kT (for any fixed $k \in \mathbb{R}$) is another (different) linear transformation with the same image and kernel. This is not the only way to produce multiple linear transformations with the same image and kernel; we could also multiply the 1st and 3rd components of $T(\vec{v})$ (as defined above) by a scalar a and multiply the 2nd component of $T(\vec{v})$ by a different scalar b.

Problem 3. Let X and Y be vector spaces.

(a) Consider a basis $\mathfrak{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of X. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be any vectors (not necessarily a basis, or even distinct) in Y. Prove that there exists a unique linear transformation $T : X \to Y$ such that $T(\mathbf{x}_i) = \mathbf{y}_i$ for all $1 \leq i \leq n$.

(b) Let U and V be subspaces of X and Y respectively such that $\dim(U) + \dim(V) = \dim(X)$. Prove that there exists a linear transformation $T_{U,V} : X \to Y$ such that $\ker(T_{U,V}) = U$ and $\operatorname{im}(T_{U,V}) = V$. (Hint: use part (a). You might also want to try to generalize the method you used to solve Problem 2.)

(c) In the map $T_{U,V}$ that you found in part (b) unique? Justify your answer.

Solution.

(a) Any vector $\mathbf{x} \in X$ can be expressed as

$$\mathbf{x} = c_1\mathbf{x}_1 + \ldots c_n\mathbf{x}_n$$

for some unique set of constants $c_1, \ldots, c_n \in \mathbb{R}$. Define a transformation $T : X \to Y$ by

$$T(\mathbf{x}) := c_1\mathbf{y}_1 + \ldots + c_n\mathbf{y}_n.$$

First, we prove that T is linear:

- Let $\mathbf{v} = v_1\mathbf{x}_1 + \ldots + v_n\mathbf{x}_n$ and $\mathbf{w} = w_1\mathbf{x}_1 + \ldots + w_n\mathbf{x}_n$ be vectors in X. Then

$$\begin{aligned} T(\mathbf{v} + \mathbf{w}) &= T((v_1 + w_1)\mathbf{x}_1 + \ldots + (v_n + w_n)\mathbf{x}_n) \\ &= (v_1 + w_1)\mathbf{y}_1 + \ldots + (v_n + w_n)\mathbf{y}_n \\ &= v_1\mathbf{y}_1 + \ldots + v_n\mathbf{y}_n + w_1\mathbf{y}_1 + \ldots + w_n\mathbf{y}_n \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

(b) Let $c \in \mathbb{R}$. Then

$$\begin{aligned} T(c\mathbf{x}) &= T(cx_1\mathbf{x}_1 + \ldots + cx_n\mathbf{x}_n) \\ &= cx_1\mathbf{y}_1 + \ldots + cx_n\mathbf{y}_n \\ &= cT(\mathbf{x}). \end{aligned}$$

Next, we prove that T is unique. Suppose $T' : X \to Y$ is another linear transformation such that $T'(\mathbf{x}_i) = \mathbf{y}_i$ for each i. Then by linearity, we must have

$$\begin{aligned} T'(\mathbf{x}) &= T'(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n) \\ &= c_1T'(\mathbf{x}_1) + \ldots + c_nT'(\mathbf{x}_n) \\ &= c_1\mathbf{y}_1 + \ldots + c_n\mathbf{y}_n \\ &= T(\mathbf{x}). \end{aligned}$$

(c) Choose a basis $\mathfrak{B}_0 = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for U. Complete $\mathfrak{B}_U$ into a basis $\mathfrak{B}_X = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{x}_1, \ldots, \mathbf{x}_r\}$ for X. Lastly, choose a basis $\mathfrak{B}_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for V. By part (a), there exists a unique linear transformation $T : X \to Y$ such that $T(\mathbf{u}_i) = 0$ and $T(\mathbf{x}_i) = \mathbf{v}_i$.

We first prove that $\ker(T) = U$.

- Let $\mathbf{u} = a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \in U$. Then $T(\mathbf{u}) = a_1 \cdot 0 + \ldots + a_n \cdot 0 = 0$. Thus, $\mathbf{u} \in \ker T$, and therefore $U \subseteq \ker(T)$.
- Let $\mathbf{x} = b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k + c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r \in \ker(T)$. Then $0 = T(\mathbf{x}) = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent, we have $c_1 = \ldots = c_r = 0$. Hence,

$$\begin{aligned} \mathbf{x} &= b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k + c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r \\ &= b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k + 0 \in U. \end{aligned}$$

Thus, $\ker(T) \subseteq U$.

We next prove that $\operatorname{im}(T) = V$.

- Let $\mathbf{x} = b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k + c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r \in X$. Then $T(\mathbf{x}) = 0 + c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r \in V$. Thus, $\operatorname{im}(T) \subseteq V$.
- Let $\mathbf{v} = c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r \in V$. Define $\mathbf{x} := c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r$. Then $T(\mathbf{x}) = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{v}$. Thus, $V \subseteq \operatorname{im}(T)$.

(1) It is not hard to see that there are as many possible definitions of $T_{U,V}$ as there are choices for basis $\mathfrak{B}_U$, $\mathfrak{B}_X$ and $\mathfrak{B}_V$ for U, X and V respectively, with the only restriction being that $\mathfrak{B}_U \subset \mathfrak{B}_X$. Thus, there are infinitely many different ways to define $T_{U,V}$ such that $\ker(T_{U,V}) = U$ and $\operatorname{im}(T_{U,V}) = V$.

Problem 4. Let U, V, and W be finite-dimensional vector spaces, and let $T : U \to V$ and $S : V \to W$ be linear transformations. Determine whether the following statements are true or false, and provide a proof of your claim.

(a) $\operatorname{rank}(S \circ T) \leq \operatorname{rank}(S)$.
(b) $\operatorname{rank}(S \circ T) \leq \operatorname{rank}(T)$.
(c) $\operatorname{nullity}(S \circ T) \geq \operatorname{nullity}(T)$.
(d) $\operatorname{nullity}(S \circ T) \geq \operatorname{nullity}(S)$.

Solution.

(a) **TRUE.** Let $\mathbf{w} = S \circ T(\mathbf{u})$ for some $\mathbf{u} \in U$. Then $\mathbf{w} = S(T(\mathbf{u}))$, where $T(\mathbf{u}) \in V$. Thus, $\operatorname{im}(S \circ T) \subseteq \operatorname{im}(S)$. The inequality follows.

(b) **TRUE.** By the Rank-Nullity Theorem, we have

$$\operatorname{rank}(T) = \dim(U) - \operatorname{nullity}(T)$$

and

$$\operatorname{rank}(S \circ T) = \dim(U) - \operatorname{nullity}(S \circ T).$$

The desired inequality follows from Part (c).

(c) **TRUE.** Let $\mathbf{u} \in \ker(T)$, so that $T(\mathbf{u}) = 0$. Then $S \circ T(\mathbf{u}) = S(0) = 0$. Thus, $\mathbf{u} \in \ker(S \circ T)$. We conclude $\ker(T) \subseteq \ker(S \circ T)$, and the inequality follows.

(d) **FALSE.** Counter-example: Consider $T : \mathbb{R} \to \mathbb{R}^{1000}$ and $S : \mathbb{R}^{1000} \to \mathbb{R}$ defined by $T(u) = \vec{0}$ and $S(\mathbf{v}) = 0$. Then $\operatorname{nullity}(S \circ T) = 1$ while $\operatorname{nullity}(S) = 1000$.

Problem 5. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *symmetric* if $A^\top = A$, and *skew-symmetric* if $A^\top = -A$. Let Sym_n and Skew_n denote the set of all symmetric matrices and the set of all skew-symmetric matrices in $\mathbb{R}^{n\times n}$, respectively.

(a) Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the map defined by $T(A) = A + A^\top$. Prove that T is linear.
(b) Prove that $\ker(T) = \mathrm{Skew}_n$ and $\operatorname{im}(T) = \mathrm{Sym}_n$.
(c) Prove that Sym_n and Skew_n are subspaces of $\mathbb{R}^{n\times n}$.
(d) Find $\dim(\mathrm{Sym}_n)$ and $\dim(\mathrm{Skew}_n)$.

Solution.

(a) We check the two axioms of linearity.

- Let $A, B \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned} T(A+B) &= (A+B) + (A+B)^\top \\ &= A + B + A^\top + B^\top \\ &= A + A^\top + B + B^\top \\ & T(A) + T(B). \end{aligned}$$

- Let $A \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} T(cA) &= (cA) + (cA)^\top \\ &= cA + c(A^\top) \\ &= c(A + A^\top) \\ & cT(A). \end{aligned}$$

(b) Observe

$$T(A) = 0 \iff A + A^\top = 0 \iff A^\top = -A.$$

Hence, $\ker(T) = \mathrm{Skew}_n$. To show that $\operatorname{im}(T) \subseteq \mathrm{Sym}_n$, we check that

$$(A + A^\top)^\top = A^\top + (A^\top)^\top = A^\top + A.$$

Lastly, let $A \in \mathrm{Sym}_n$. Then

$$\begin{aligned} T\left(\frac{1}{2}A\right) &= \frac{1}{2}A + \left(\frac{1}{2}A\right)^\top \\ &= \frac{1}{2}A + \frac{1}{2}(A^\top) \\ &= \frac{1}{2}A + \frac{1}{2}A \\ &= A. \end{aligned}$$

Hence, $A \in \mathrm{im}(T)$.

(c) Claim follows from Part (b), since the kernel and image of a linear transformation are subspaces.

(d) We first find $\dim(\mathrm{Skew}_n)$. Let $A = [a_{i,j}] \in \mathrm{Skew}_n$. Then we have

$$a_{j,i} = -a_{i,j}.$$

Thus, the diagonal entries of A are 0, and the lower triangular entries of A are negatives of the corresponding upper triangular entires of A, which are arbitrary. For $1 < i \leq n$ and $1 \leq j < i$, define $A_{i,j} \in \mathbb{R}^{n \times n}$ as the matrix with its (i,j)th entry equal to 1 and (j,i)th entry equal to -1, and all other entries equal to 0. It is easy to see that the set of matrices $\mathfrak{B} := \{A_{i,j}\}$ is a basis for Skew_n. Since $\mathfrak{B}$ contains $1 + 2 + \ldots + (n-1) = n(n-1)/2$ elements, we have

$$\dim(\mathrm{Skew}_n) = \frac{n(n-1)}{2}.$$

By Part (b) and the Rank-Nullity Theorem, we also have

$$\begin{aligned} \dim(\mathrm{Sym}_n) &= \dim(\mathbb{R}^{n \times n}) - \frac{n(n-1)}{2} \\ &= n^2 - \frac{n(n-1)}{2} \\ &= \frac{n(n+1)}{2}. \end{aligned}$$