Math 217: Elementary Matrices

DEFINITION: An **elementary matrix** is an $n \times n$ matrix obtained by performing *one* elementary row operation on an $n \times n$ identity matrix. There are three types:

(i). $E_{i \leftrightarrow j}$ is the $n \times n$ matrix obtained from the $n \times n$ identity matrix by switching rows i and j.

(ii). $E_{ii}(a)$ is the $n \times n$ matrix obtained from the $n \times n$ identity matrix by multiplying row i by some non-zero scalar a.

(iii). $E_{ij}(a)$ is the $n \times n$ matrix obtained from the $n \times n$ identity matrix by adding a times row i to row j for some non-zero scalar a.

Problem 1.

(a) In the case $n = 3$, write out the matrices $E_{2 \leftrightarrow 3}$, $E_{22}(\pi)$ and $E_{13}(\frac{1}{3})$.

(b) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$. Compute the three products A, where E is each of the elementary matrices in (a).

(c) Using your experience in (b), discuss with your group a way to interpret elementary row operations on a matrix A as a matrix multiplication.

(d) Discuss the theorem below. Do you see why it's true? Write out the proof in the special case A is 3×4 by checking each of the three types of row operations (i), (ii) and (iii).

Theorem: *Let A be an $n \times m$ matrix. Let B be a matrix obtained from A by an elementary row operation. Then $B = EA$ where E is the $n \times n$ elementary matrix obtained by performing the same elementary row operation on I_n.*

Problem 2.

Let A be an $n \times m$ matrix.

(a) Think about row-reducing A. There exists a $n \times n$ matrix P, which is a product of elementary matrices, such that $PA = \text{rref}(A)$. Use the Theorem above to explain why.

(b) Prove that an elementary matrix is invertible (there are three cases to check). Now prove that the matrix P in (a) is invertible.

(c) Prove that every invertible $n \times n$ matrix is a product of elementary matrices.

(d) Think about the procedure explained in the textbook for computing the inverse of an $n \times n$ matrix using row reduction. Can you explain why it works?

Problem 3.

(a) Invent a notion of an elementary column operation analogous to elementary row operations (for example, swapping *columns* i and j), and show that each of the three types of elementary matrices could just as well be obtained from performing *one* elementary column operation.

(b) Experiment with a 4×3 matrix A to show that, similar to problem 1 above, performing an elementary column operation on A is equivalent to multiplying A *on the right* (rather than the left) by an elementary matrix.

(c) State a theorem about *column* operations analogous to the theorem above for row operations.

(d) Can one find the inverse of an invertible square matrix M by performing column operations? How?