28. Find the inverse of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 22 \\ -16 \\ 8 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 13 \\ -3 \\ 9 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 8 \\ -2 \\ 7 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$ by theorem (1.3.1)

So by the key theorem, the standard matrix of T is $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$

So by theorem 2.4.5

we find rref$(A \vdots I_n)$

$$\left[\begin{array}{cccc:cccc} 22 & 13 & 8 & 3 & 1 & 0 & 0 & 0 \\ -16 & -3 & -2 & -2 & 0 & 1 & 0 & 0 \\ 8 & 9 & 7 & 2 & 0 & 0 & 1 & 0 \\ 5 & 4 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \times \frac{1}{22}$$

$$\rightarrow \left[\begin{array}{cccc:cccc} 1 & \frac{13}{22} & \frac{4}{11} & \frac{3}{22} & \frac{1}{22} & 0 & 0 & 0 \\ -16 & -3 & -2 & -2 & 0 & 1 & 0 & 0 \\ 8 & 9 & 7 & 2 & 0 & 0 & 1 & 0 \\ 5 & 4 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ +16 \times I \\ -8 \times I \\ -5 \times I \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & \frac{13}{22} & \frac{4}{11} & \frac{3}{22} & \frac{1}{22} & 0 & 0 & 0 \\ 0 & \frac{71}{11} & \frac{42}{11} & \frac{2}{11} & \frac{8}{11} & 1 & 0 & 0 \\ 0 & \frac{47}{11} & \frac{45}{11} & \frac{10}{11} & -\frac{4}{11} & 0 & 1 & 0 \\ 0 & \frac{23}{22} & \frac{13}{11} & \frac{7}{22} & -\frac{5}{22} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \div \frac{71}{11} \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & \frac{13}{22} & \frac{4}{11} & \frac{3}{22} & \frac{1}{22} & 0 & 0 & 0 \\ 0 & 1 & \frac{42}{71} & \frac{2}{71} & \frac{8}{71} & \frac{11}{71} & 0 & 0 \\ 0 & \frac{47}{11} & \frac{45}{11} & \frac{10}{11} & -\frac{4}{11} & 0 & 1 & 0 \\ 0 & \frac{23}{22} & \frac{13}{11} & \frac{7}{22} & -\frac{5}{22} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -II \times \frac{13}{22} \\ \\ -II \times \frac{47}{11} \\ -II \times \frac{23}{22} \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & \frac{1}{71} & \frac{17}{142} & -\frac{3}{142} & -\frac{13}{142} & 0 & 0 \\ 0 & 1 & \frac{42}{71} & \frac{2}{71} & \frac{8}{71} & \frac{11}{71} & 0 & 0 \\ 0 & 0 & \frac{111}{71} & \frac{56}{71} & -\frac{60}{71} & -\frac{47}{71} & 1 & 0 \\ 0 & 0 & \frac{40}{71} & \frac{41}{142} & -\frac{49}{142} & -\frac{23}{142} & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -II \times \frac{47}{11} \\ -II \times \frac{23}{22} \end{array}$$

$$\xrightarrow{\text{tons of elimination}} \quad \mathrm{rref}([A \,\vdots\, I]) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 9 & -25 \\ 0 & 1 & 0 & 0 & -2 & 5 & -22 & 60 \\ 0 & 0 & 1 & 0 & 4 & -9 & 41 & -112 \\ 0 & 0 & 0 & 1 & -9 & 17 & -80 & 222 \end{array}\right]$$

So The inverse matrix of A is

$$A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & -80 & 222 \end{bmatrix}$$

And $T^{-1}(\vec{x}) = A^{-1}\vec{x}$, for all $\vec{x} \in \mathbb{R}^4$.

30. For which values of the constants b and c is the following matrix invertible?

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Denote the matrix by A.

By theorem 2.4.3, A is invertible iff

$\operatorname{rref}(A) = I_n$

So we find $\operatorname{rref}(A)$

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\times(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} {\scriptstyle +b\times I} \rightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{bmatrix} {\scriptstyle +c\times II}$$

$$\rightarrow \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore A is not invertible regardless of the value of b, c.

42. A square matrix is called a *permutation matrix* if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are I_n and

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

By by elementary transformations (changing row order), any permutation matrice can be transformed into I_n, its rref is always I_n, so it is invertible by theorem 2.4.3

Denote the arbitrary permutation matrix by A.

And by theorem 2.4.5, rref $[A \vdots I_n] = [I_n \vdots B]$, then B is the inverse matrix of A

Since finding the rref(A) is just changing its row order, every row of B is chosen from I_n unrepeatedly, so B is always a permutation matrix by the definition.

GOAL *Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.*

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

6. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

The problem is to find all $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{0}$

Therefore it is to find $\operatorname{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{-1 \times I} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-II}$$

$$\operatorname{rref}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

Therefore the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

so $\ker(A) = \left\{ t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle|\, t \in \mathbb{R} \right\}$

$= \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$ by definition of span

For each matrix A in Exercises 14 through 16, find vectors that span the image of A. Give as few vectors as possible. Use paper and pencil.

14. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Sol. T determined by A is

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} x_3$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (x_1 + 2x_2 + 3x_3),$$

$x_1, x_2, x_3 \in \mathbb{R}$

So $\underbrace{\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).}$ by definition of span.

Part B (25 points)

Problem 1. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. As on HW 3, we define T^k to be the k-fold composition of T with itself,

$$= \underbrace{T \circ T \circ T \circ \cdots \cdots \circ T}_{k \text{ times}}.$$

Let A be the standard matrix of T, by which we mean the unique $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(a) Prove that for all k, the standard matrix for T^k is the matrix A^k. [Hint: induction works nicely.]

(b) We define T to be **nilpotent** if there exists some $k \in \mathbb{N}$ such that T^k is the zero transformation. Prove that if T is nilpotent, then A is not invertible.

(c) Prove that if T is nilpotent, then $A - I_n$ **is** invertible.
[Hint: try multiplying out $(A - I_n)(I_n + A + A^2 + \cdots + A^{k-1})$ and see what you get.]

(a) We prove it by induction on k

Base case: $k=1$. $T^1(\vec{x}) = A\vec{x} = A^1\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

So base case is true

Inductive case: Assume $k=n$, $T^n(\vec{x}) = A^n\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

We want to show for $k=n+1$,

$T^{n+1}(\vec{x}) = A^{n+1}\vec{x}$

Since by the composition rule,

$T^{n+1}(\vec{x}) = T \circ T^n(\vec{x}) = T(A^n\vec{x})$

$= A \cdot (A^n\vec{x}) = (A \cdot A^n)\vec{x}$ by Theorem 2.3.6 (associativity)

$= A^{n+1}\vec{x}$

So inductive step is true.

Therefore we have proved that for all k, the standard matrix for T^k is the matrix A^k.

(b) Assume T is a nilpotent.

Then for some positive integer k, $T^k(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$

Since $T^k(\vec{x}) = A^k\vec{x}$ by key Theorem and (a)

$A^k\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.

This means for all $\vec{x} \in \mathbb{R}^n$, $A^k\vec{x} = \vec{0}$ has a solution. ①

We can write the system as the matrix form

$[\, A^k \,\vdots\, \vec{0} \,]$ by Theorem 1.3.11

Assume A is invertible for contradiction

then A^2 is also invertible since

$(A^{-1} \cdot A^{-1}) \cdot A^2 = A^{-1}(A^{-1}A) \cdot A = A^{-1} I_n A = A^{-1}A = I_n$

by Theorem 2.3.6.

Similarly, A^k is invertible since $(\underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}}) \cdot A^k = I_n$

Then $\operatorname{rank}(A^k) = n$ by theorem 2.4.3

So $\operatorname{rref}(A^k) = I_n$ by definition of rank

Then $\operatorname{rref}([\, A^k \,\vdots\, \vec{0} \,]) = [\, I_n \,\vdots\, \vec{0} \,]$. It contains

n rows of $[\cdots 1 \_ \,\vdots\, 0]$ which means the

system is inconsistent, contradicting with ①

Therefore A is not invertible if T is a nilpotent.

(c) Assume T is a nilpotent.

Then for some positive integer k, $T^k(\vec{x}) = \vec{0}$ for all $x \in \mathbb{R}^n$

Case 1. $k=1$, $T(\vec{x}) = \vec{0}$ for all $x \in \mathbb{R}^n$

Then $T(\vec{e_1}), T(\vec{e_2}) \dots, T(\vec{e_n}) = \vec{0}$

So $A = [T(\vec{e_1})\ T(\vec{e_2}) \dots T(\vec{e_n})] = [\vec{0}\ \vec{0} \dots \vec{0}]$

by theorem 2.1.2

So $A - I_n = -I_n$, $\text{rref}(A - I_n) = I_n$

so $(A - I_n)$ is invertible by theorem 2.4.3

Case 2 $k \geq 2$

Take $B = I_n + A + A^2 + \dots + A^{k-1}$

Then $(I_n - A)B = (I_n - A)(I_n + A + \dots + A^{k-1})$

$= (I_n + A + A^2 + \dots + A^{k-1}) - (A + A^2 + \dots + A^k)$

$= I_n - A^k$ by theorem 2.3.6, 2.3.7

Since $A^k \vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$,

$A^k \cdot \vec{e_1} = \vec{0}$, $A^k \cdot \vec{e_2} = \vec{0}$, ..., $A^k \cdot \vec{e_n} = \vec{0}$

So A^k is a zero matrix by theorem 2.1.2

So $(I_n - A)B = I_n$

Similarly, $B(I_n - A) = I_n + A + A^2 + \dots + A^{k-1}$

$- I_n - A - A^2 - A^3 - \dots - A^k$

$= I_n - A^k = I_n$

So by definition of invertible matrix,

$I_n - A$ is invertible

So by theorem 2.4.3, $\mathrm{rref}(I_n - A) = I_n$

So $\mathrm{rref}(A - I_n) = \mathrm{rref}(-(I_n - A))$

$= I_n$

So by theorem 2.4.3,

$(A - I_n)$ is also invertible.

Since all cases hold true that $(A - I_n)$ is inverible, the statement is true

Problem 2. Let V be any vector space, and let S be any set. Let $\mathcal{F}(S, V)$ denote the set of all functions from S to V. (Note: we are not assuming $S \subseteq V$ here, just that S is some set. S is not assumed to be a vector space, but it could be. Similarly, the functions in $\mathcal{F}(S, V)$ are not assumed to be linear transformations, although it is possible that some of them might be.)

For any functions $f, g \in \mathcal{F}(S, V)$ we can define their *sum* to be the function $f + g$ given by the formula $(f + g)(s) = f(s) + g(s)$, where s is any element in S. Similarly, for any scalar $c \in \mathbb{R}$ and any function $f \in \mathcal{F}(S, V)$ we define the function cf to be given by the formula $(cf)(s) = c(f(s))$ for all $s \in S$.

(a) Prove that $\mathcal{F}(S, V)$ is a vector space. **Note:** For this problem you must *explicitly prove* that each of the vector space properties VS1-8 from Worksheet 6 is true. (These proofs should be very short but are not skippable.)
(b) Is $0_{\mathcal{F}(S,V)}$ the same element as 0_V? If not, explain how they are different.
(c) We could similarly define $\mathcal{F}(V, S)$ to be the set of all functions from V to S. Would $\mathcal{F}(V, S)$ also a vector space? Why or why not?
(d) The familiar vector spaces $\mathcal{P}, \mathcal{P}_n$ and $\mathcal{C}^\infty$ (all from Worksheet 6) are all subsets of $\mathcal{F}(S, V)$ for some S and V. What are S and V for each of these functions?

(a) Let f, g, h be three arbitrary elements (functions) in $F(S, V)$

Then

① $f+g$ is also in $F(S,V)$

since $f(x)+g(x)$ is also well defined for all $x \in S$. (closure)

② for all $x \in S$

$$(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$$
$$= f(x)+g(x)+h(x)$$

(VS-1)

③ for all $x \in S$

$f(x)+g(x) = g(x)+f(x)$ (VS-2)

④ Consider the function $\varphi(x) = 0_V$ for all $x \in S$.

then all any $f(x)$, $f(x)+\varphi(x) = \varphi(x)+f(x) = f(x)+0 = f(x)$

(VS-3)

⑤ For any $f(x) \in F(S,V)$, consider $n(x) = -f(x)$ for all $x \in S$

then $n(x)+f(x) = 0_V$. such $n(x)$ always exists (VS-4)

Let α, β be arbitrary scalars in $\mathbb{R}$

then

(6) $\alpha(f+g) = \alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x) = \alpha f + \alpha g$

($\boxed{VS-5}$)

(7) $(\alpha+\beta)f = \alpha f(x) + \beta f(x) = \alpha f + \beta f$

($\boxed{VS-6}$)

(8) $\alpha(\beta f(x)) = \alpha \cdot \beta \cdot f(x) = (\alpha\beta) f(x)$ ($\boxed{VS-7}$)

for all $x \in S$

(9) Consider function $\lambda(x) = 1$ for all $x \in S$, in $F(S, V)$

Then for any $f \in F(S,V)$, $\lambda(x) \cdot f(x) = f(x)$.

($\boxed{VS-8}$)

Therefore $F(S, V)$ is a vector space..

(b) $0_{F(S,V)}$ is a function $f(x) = 0_V$ for all $x \in S$.

0_V is a vector in V.

So they are different element, though the value of $0_{F(S,V)}$ is always 0_V.

(c) It is not ensured to be a vector space.

Consider take S which is not a vector space.

(explicit example: consider $S=\{1,2\}$, $\underbrace{1+2}_{=3} \notin S$)

Then take a function $f \in F(V,S)$

that $f(\vec{v})=1$ for all $\vec{v} \in V$

take another function $g \in F(V,S)$

that $g(\vec{v})=2$ for all $\vec{v} \in V$

By the definition of vector space,

$f+g = f(\vec{v})+g(\vec{v})$ for all $\vec{v}$

$=3$ should also be an

element of $F(V,S)$

but the function $f+g=3$ does not exist.

$\therefore$ Therefore it violates the definition of a vector space.

Therefore $F(V,S)$ is not ensured to be a vector space.

(d) P: all polynomial functions of all degrees from $\mathbb{R}$ to $\mathbb{R}$.

So $P \subseteq F(\mathbb{R},\mathbb{R})$

$S=\mathbb{R}$, $V=\mathbb{R}$

P_n: all polynomial functions of at most n degrees from $\mathbb{R}$ to $\mathbb{R}$.

So $P \subseteq F(\mathbb{R}, \mathbb{R})$

$S = \mathbb{R}$, $V = \mathbb{R}$

C^∞: all infinitely-differentiable functions from $\mathbb{R}$ to $\mathbb{R}$

so $C^\infty \subseteq F(\mathbb{R}, \mathbb{R})$

$S = \mathbb{R}$, $V = \mathbb{R}$.

Problem 3. Let $\mathcal{P}$ be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ in the variable t, and for each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be (as usual) the subset of $\mathcal{P}$ consisting of all polynomial functions of degree at most n. (We already know that $\mathcal{P}_n$ is also a vector space.) Also let $T : \mathcal{P} \to \mathcal{P}$ be the map defined by $T(p)(t) = p'(t) + p(0)$ for each $p \in \mathcal{P}$ and for all $t \in \mathbb{R}$.

(a) Show that T is a linear transformation.

(b) Let $n \in \mathbb{N}$, and let $T_n : \mathcal{P}_n \to \mathcal{P}_n$ be defined by $T_n(p)(t) = p'(t) + p(0)$, so that T_n is just T with both domain and codomain restricted to $\mathcal{P}_n$. Is T_n injective? Is T_n surjective?

(c) Is T injective? Is T surjective?

(a) Let p_1, p_2 be two arbitrary functions in P.

So $T(p_1+p_2) = (p_1+p_2)'(t) + (p_1+p_2)(0)$

$= p_1'(t) + p_2'(t) + p_1(0) + p_2(0)$ by differentiation rule.

$= (p_1'(t) + p_1(0)) + (p_2'(t) + p_2(0))$

$= T(p_1) + T(p_2)$ ①

Let k be an arbitrary scalar in $\mathbb{R}$

So $T(kp_1) = kp'(t) + kp(0) = k(p'(t) + kp(0))$

$= kT(p_1)$ ②

By ①②, T is a linear transformation)

(b) ① T_n is not surjective.

Consider $p(t) = x^n$. This is a function in P_n which is the target

But there is no function in P_n that can be mapped to $p(t)$ by $T(n)$, since by taking derivative, the degree of any element of P_n can at most be $n-1$ by its definition.

② T_n is not injective.

Consider $p_1(t) = 1+2t+t^2$
$p_2(t) = 2+t+t^2$

then, $p_1'(t) + p_1(0) = 3+2t$
$p_2'(t) + p_2(0) = 3+2t$
for all $t \in \mathbb{R}$

This mean $T_n(p_1) = T_n(p_2)$
but $p_1 \neq p_2$, so the function is not injective.

(c) ① T is not injective

By the same counterexample in (b) ②, T is not injective.

② T is surjective.

Let p be an arbitrary polynomial function in $\mathcal{P}$.

Then $p(t) = k + a_1 t + a_2 t^2 + \dots + a_m t^m$

for some integer m, and

real numbers $k, a_1, a_2 \dots a_m$

for all $t \in \mathbb{R}$

Consider $q(t) = kt + \frac{1}{2}a_1 t^2 + \frac{1}{3}a_2 t^3 + \dots + \frac{1}{m+1} a_m t^{m+1}$ $\in \mathcal{P}$

Then $q'(t) + q(0) = \underbrace{k + a_1 t' + a_2 t^2 + \dots + a_m t^m}_{= p(t)}$

Since p ($\in \mathcal{P}$) is arbitrary, we can always find the q ($\in \mathcal{P}$) so that $p = T(q)$

Therefore T is surjective

Problem 4. We denote by $\mathbb{R}^{n \times n}$ the vector space of all $n \times n$ matrices. Let A be an $n \times n$ matrix, and define the function $L_A : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ by $L_A(B) = AB$ for all $B \in \mathbb{R}^{n \times n}$. (Note carefully: this is *not* the same function as T_A. While both can be described informally as "multiplication by A", the two functions L_A and T_A have different domains and codomains. Make sure you understand this distinction before beginning to work on this problem!)

(a) Show that L_A is a linear transformation.

(b) Show that the matrix A is invertible if and only if the linear transformation L_A is invertible.

Now let $\mathcal{F}$ be the set of all functions from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$, and define the function $L : \mathbb{R}^{n \times n} \to \mathcal{F}$ by $L(A) = L_A$.

(c) Show that L is injective.

(d) Is L surjective? Be sure to justify your claim.

(a) Let B, C be two arbitrary vector in $\mathbb{R}^{n\times n}$

So $L_A(B+C) = A(B+C) = AB + AC$ by theorem 2.3.7

$= L_A(B) + L_A(C)$

Let $k \in \mathbb{R}$ be an arbitrary scalar.

So $L_A(kB) = A\cdot(kB) = k(AB)$ by theorem 2.3.5

$= kL_A(B)$

Therefore, L_A is a linear transformation.

(b) ① We prove: if A is invertible, then L_A is invertible

Assume A is invertible, and denote its inverse matrix by A^{-1}.

Let $B_1, B_2 \in \mathbb{R}^{n\times n}$

Assume $L_A(B_1) = L_A(B_2)$

Then $AB_1 = AB_2$

Multiplying both sides by A^{-1},

$A^{-1}(AB_1) = A^{-1}(AB_2)$

So $(A^{-1}A)B_1 = (A^{-1}A)B_2$ by theorem 2.3.6

Then $I_nB_1 = I_nB_2$

$B_1 = B_2$

So L_A is invertible

(2) We prove: if L_A is invertible, then A is invertible

Assume L_A is invertible

Since L_A is invertible (Denote the inverse function of L_A as L_A^{-1}) it is surjective and injective by the definition of invertible function.

So consider $I_n \in \mathbb{R}^{n\times n}$

Since L_A is surjective, there exists some matrix $C \in \mathbb{R}^{n\times n}$ such that $L_A(C) = I_n$,

So $AC = I_n$

By Collary 2.15, C is the inverse matrix of A

Therefore A is invertible.

Since (1), (2), we have proved the if-and-only-if statement.

(c) Proof

Assume $A, B \in \mathbb{R}^{n\times n}$ such that $L(A) = L(B)$

So for all $C \in \mathbb{R}^{n\times n}$, $L_A(C) = L_B(C)$ by definition

Consider taking $C = I_n$

We have $L_A(I_n) = L_B(I_n)$

So $AI_n = BI_n$, $A = B$.

So $L(A) = L(B)$ implies $A = B$

Therefore the function is injective.

(d) L is not surjective.

Consider $f = I_n$, the function that maps every $A \in \mathbb{R}^{n \times n}$ to I_n in F.

Assume for contradiction that for some $A \in \mathbb{R}^{n \times n}$

$$L(A) = L_A = f$$

Then $A \cdot C = I_n$ for all $C \in \mathbb{R}^{n \times n}$

Consider $C = 0_{n \times n}$

then $A \cdot \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = I_n$, which is impossible since any matrix multiplying the matrix $0_{n \times n}$ gets $0_{n \times n}$.

Problem 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as follows:

$$T = \mathrm{Rot}_{-80^\circ} \circ \mathrm{Proj}_y \circ \mathrm{Rot}_{35^\circ},$$

where Rot_θ is counter-clockwise rotation by θ, and Proj_y is projection onto the y-axis.

(a) Sketch $\mathrm{im}(T)$ in $\mathbb{R}^2$. Indicate the angle between $\mathrm{im}(T)$ and the x-axis.
(b) Sketch $\ker(T)$ in $\mathbb{R}^2$. Indicate the angle between $\ker(T)$ and the x-axis.
(c) Let $T_{\phi,\theta} := \mathrm{Rot}_\phi \circ \mathrm{Proj}_y \circ \mathrm{Rot}_\theta$. For which ϕ and θ is $\mathrm{im}(T_{\phi,\theta}) = \ker(T_{\phi,\theta})$?

(a) For any vector $\vec{v} \in \mathbb{R}^2$

What T does: ① rotate $\vec{v}$ by 35° (counter-clockwise(ly))

This rotation does not change the whole image

So $\mathrm{im}(\mathrm{Rot}35^\circ)$ is still $\mathbb{R}^2$.

② Project $\left(\mathrm{Rot}_{35^\circ}(\vec{v})\right)$ on y-axis

This squeezes all the vectors onto the line $x=0$

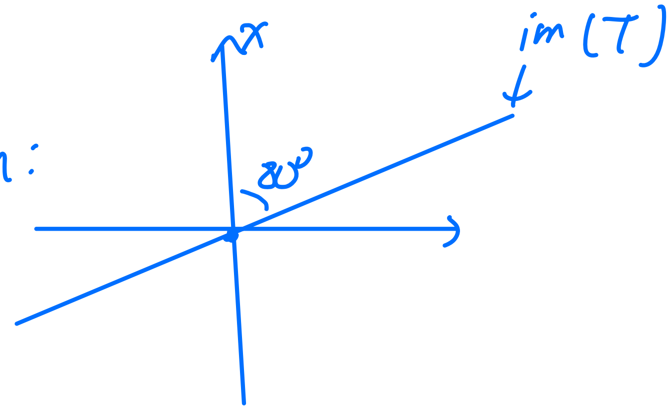
So $\mathrm{im}(\mathrm{Proj}_y \circ \mathrm{Rot}35^\circ) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| \, y \in \mathbb{R} \right\}$

③ Rotate $(\mathrm{Proj}_y \circ \mathrm{Rot}_{35^\circ}(\vec{v}))$ by 80° clockwisely.

This changes the direction of the graph of image to the line 80° from y-axis.

So $\mathrm{im}(T) = \{ \vec{v} \in \mathbb{R}^2 \mid \vec{v}$ is 80° from y-axis clockwisely$\}$

So the sketch:



The angle between im(T) and x-axis is $90° - 80° = 10°$

(b) ① Since $Rot_{-80°}$ does not change the position of $\vec{0} \in \mathbb{R}^2$,

$ker(T) = ker(Proj_y \circ Rot_{35°})$

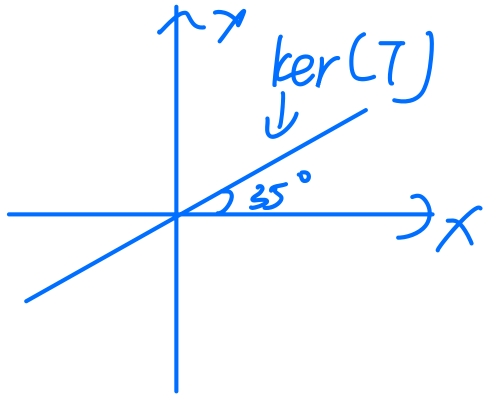
② Since $Proj_y$ maps every previous vectors on x-axis to $\vec{0} \in \mathbb{R}^2$,

$ker(T) = \left\{ \vec{w} \in \mathbb{R}^2 \middle| Rot_{35°}(\vec{w}) = \begin{bmatrix} x \\ 0 \end{bmatrix}, x \in \mathbb{R} \right\}$

③ Since $Rot_{35°}$ rotates every vector by 35° counterclockwisely,

$ker(T) = \{ \vec{w} \in \mathbb{R}^2 \mid \vec{w}$ is 35° counterclockwisely from x-axis$\}$

So sketch:



the angle between ker(T) and x-axis is 35°

(c) $\mathrm{im}(T) = \{\vec{v} \in \mathbb{R}^2 \mid \vec{v}$ is $-\varphi$ from y-axis clockwisely$\}$

$\ker(T) = \{\vec{w} \in \mathbb{R}^2 \mid \vec{w}$ is θ from x-axis counter-clockwisely$\}$

So in order that $\mathrm{im}(T) = \ker(T)$,

it suffices: $-\varphi = \frac{\pi}{2} - \theta + k\pi,\ k \in \mathbb{Z}$

$\Longrightarrow \theta - \varphi = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$