Math 217 – Midterm 2

Fall 2019

Solutions

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	17	
5	11	
6	13	
7	10	
8	10	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The $n \times n$ matrix A is *orthogonal*

   Solution: The $n \times n$ matrix A is *orthogonal* if A is invertible and $A^{-1} = A^{\top}$.

   (b) The vector $\vec{x}^*$ is a *least-squares solution* of the linear system $A\vec{x} = \vec{b}$

   Solution: The vector $\vec{x}^*$ is a *least-squares solution* of the linear system $A\vec{x} = \vec{b}$ if $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{m \times n}$.

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the inner product space V is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ on V

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ if for all $1 \leq i, j \leq n$ we have $\langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$

   (d) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B

   Solution: The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B if there is an invertible matrix S such that $A = P^{-1}BP$.

2. State whether each statement is True or False and provide a short proof of your claim.

   (a) (3 points) There exists an orthonormal list of 3 vectors in $\mathbb{R}^3$ that does not span $\mathbb{R}^3$.

   Solution: FALSE. Let $\mathcal{U} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ be an orthonormal list of vectors in $\mathbb{R}^3$. Then $\mathcal{U}$ is linearly independent. Since $\dim \mathbb{R}^3 = 3$ and $\mathcal{U}$ has 3 vectors in it, it follows that $\mathcal{U}$ is a basis of $\mathbb{R}^3$ and thus $\mathcal{U}$ spans $\mathbb{R}^3$.

   (b) (3 points) For every inner product space V (with inner product $\langle \cdot, \cdot \rangle$) and for all vectors $\vec{x}, \vec{y} \in V$, if $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$ then $\vec{x}$ and $\vec{y}$ are orthogonal.

   Solution: TRUE. Let $\vec{x}, \vec{y} \in V$, and suppose $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2$. Expanding, we have

   $$\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle = \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2.$$

   Thus $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2$, so $\langle \vec{x}, \vec{y} \rangle = 0$ and thus $\vec{x}$ and $\vec{y}$ are orthogonal as claimed.

   (c) (3 points) For all vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{n \times n}$, if $A^\top B = I_n$ then $A\vec{x} \cdot B\vec{y} = \vec{x} \cdot \vec{y}$.

   Solution: TRUE. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$, and suppose $A^\top B = I_n$. Then

   $$A\vec{x} \cdot B\vec{y} = (A\vec{x})^\top B\vec{y} = \vec{x}^\top A^\top B\vec{y} = \vec{x}^\top I_n \vec{y} = \vec{x}^\top \vec{y} = \vec{x} \cdot \vec{y}.$$

(Problem 2, Continued).

(d) (3 points) For every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and for every pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^n$, if $[T]_\mathcal{B} = [T]_\mathcal{C}$ then $\mathcal{B} = \mathcal{C}$.

Solution: FALSE. For a counterexample, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation. Then

$$[T]_\mathcal{B} \;=\; \begin{bmatrix} | & & | \\ [T(\vec{b}_1)]_\mathcal{B} & \cdots & [T(\vec{b}_n)]_\mathcal{B} \\ | & & | \end{bmatrix} \;=\; \begin{bmatrix} | & & | \\ [\vec{b}_1]_\mathcal{B} & \cdots & [\vec{b}_n]_\mathcal{B} \\ | & & | \end{bmatrix} \;=\; I_n$$

for *every* basis $\mathcal{B}$ of $\mathbb{R}^n$, so as long as $\mathcal{B}$ and $\mathcal{C}$ are distinct bases of $\mathbb{R}^n$ we will have $[T]_\mathcal{B} = [T]_\mathcal{C}$ even though $\mathcal{B} \neq \mathcal{C}$.

(e) (3 points) For every symmetric matrix A, $\ker(A^2) = \operatorname{im}(A)^\perp$.

Solution: TRUE. Let A be a symmetric matrix, so $A = A^\top$. Then, using the fact that $\ker(B) = \ker(B^\top B)$ for every matrix B and the fact that $\ker(B^\top) = \operatorname{im}(B)^\perp$ for every matrix B, we have

$$\ker(A^2) \;=\; \ker(AA^\top) \;=\; \ker(A^\top) \;=\; \operatorname{im}(A)^\perp.$$

3. Let V be the subspace of $\mathbb{R}^3$ consisting of solutions of the linear equation $x+2y-z=0$, and let $\mathcal{B}$ be the orthonormal basis of V given by

$$\mathcal{B} \;=\; (\vec{b}_1, \vec{b}_2) \;=\; \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

(a) (3 points) Find the $\mathcal{B}$-coordinate vector of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in V. *(No justification required).*

Solution: $\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

(b) (3 points) Find a vector $\vec{v} \in V$ such that $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *(No justification required).*

Solution: $\frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(c) (3 points) Let $L_{\mathcal{B}} : V \to \mathbb{R}^2$ be the coordinate isomorphism defined by $L_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ for each $\vec{v} \in V$, and define the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = L_{\mathcal{B}}^{-1}(\vec{x})$ for each $\vec{x} \in \mathbb{R}^2$. Find the standard matrix of T.

Solution: $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(d) (3 points) Find a vector $\vec{b}_3 \in \mathbb{R}^3$ such that the matrix $B = \begin{bmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{bmatrix}$ is orthogonal, or else explain why this is impossible.

Solution: $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

4. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable x.

   (a) (6 points) In each of (i) – (iii) below, determine whether the given rule defines an inner product on $\mathcal{P}_2$. Write YES or NO. *(No justification necessary.)*

   (i) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = p'q' \in \mathcal{P}_2$ NO

   (ii) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = p(1)q(1)$ NO

   (iii) for all $p, q \in \mathcal{P}_2$, $\langle p, q\rangle = \int_0^1 \big[p(x)q(x)\big]^2 dx$ NO

For the rest of this problem, let p, q, and r be the polynomials in $\mathcal{P}_2$ defined for all $x \in \mathbb{R}$ by the rules $p(x) = 1$, $q(x) = x$, and $r(x) = x^2$. Also let $\langle \cdot, \cdot\rangle$ be the inner product on $\mathcal{P}_2$ given by

$$\langle f, g\rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

for all $f, g \in \mathcal{P}_2$. (You do not have to prove that this is an inner product.)

   (b) (3 points) Fill in the missing values in the following table of inner products, where, for instance, the inner product $\langle q, p\rangle = \frac{1}{2}$ is one of those given.

	p	q	r
p	1	$\mathbf{\frac{1}{2}}$	$\frac{1}{3}$
q	$\frac{1}{2}$	$\frac{1}{3}$	$\mathbf{\frac{1}{4}}$
r	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{1}{5}}$

   (c) (2 points) Find an orthonormal basis of Span(q).

Solution: $(\sqrt{3}q)$

   (d) (3 points) Find an orthonormal basis of Span(p, q).

Solution: Applying Gram-Schmidt to (q, p), we have

$$p - \frac{\langle p, q\rangle}{\langle q, q\rangle} \;=\; p - \frac{3}{2}q,$$

and

$$\langle p - \frac{3}{2}q, p - \frac{3}{2}q\rangle \;=\; \langle p, p\rangle - 3\langle p, q\rangle + \frac{9}{4}\langle q, q\rangle \;=\; \frac{1}{4}.$$

Thus we get orthonormal basis $\big(\sqrt{3}q, 2p - 3q\big)$.

   (e) (3 points) Find the vector in Span(q) that is closest to r.

Solution: $\mathrm{proj}_q(r) \;=\; \frac{\langle r,q\rangle}{\langle q,q\rangle}q \;=\; \frac{3}{4}q.$

5. Let $a, b, c \in \mathbb{R}$, and suppose that

$$\mathcal{B} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ b \end{bmatrix} \right)$$

is an ordered basis of $\mathbb{R}^3$.

(a) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{E}}$, where $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is the standard basis of $\mathbb{R}^3$. (Your answer may depend on a, b, and c.)

Solution: $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 0 & a & 1 \\ 0 & 1 & c \\ 1 & 0 & b \end{bmatrix}$

(b) (4 points) Suppose $c = 0$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection over the xy-plane. Find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T. (Your answer may depend on a and b.)

Solution: $[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & -2b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) (4 points) Suppose $a = 1$, and let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection onto the plane in $\mathbb{R}^3$ defined by the equation $x + py + qz = 0$. Find values of b, c, p, q such that $[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Circle your answers.

Solution: The expression for $[P]_{\mathcal{B}}$ tells us that the first and second basis vectors of $\mathcal{B}$ have to be on the plane $x + py + qz = 0$, and the third one has to be perpendicular to the plane. Using that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is on $x + py + qz = 0$, we get that $q = 0$. Similarly, using that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is on $x + py + qz = 0$, we get that $p = -1$. Since $\begin{bmatrix} 1 \\ p \\ q \end{bmatrix}$ is a normal vector to the plane $x + py + qz = 0$, we get that $\begin{bmatrix} 1 \\ p \\ q \end{bmatrix}$ and $\begin{bmatrix} 1 \\ c \\ b \end{bmatrix}$ must be parallel, and that can only happen if $p = c$ and $b = q$.

Hence, the only possible answer is $b = q = 0$ and $c = p = -1$.

6. Let $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{4\times 3}$ and $B = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{4\times 2}$, where $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a linearly independent list of vectors in $\mathbb{R}^4$. Suppose A has QR-factorization

$$A = \underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}}_{R} \in \mathbb{R}^{4\times 3}.$$

(a) (3 points) Compute $A^\top A$.

Solution: $A^\top A = (QR)^\top (QR) = R^\top (Q^\top Q) R = R^\top R = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & -4 \\ 0 & -4 & 13 \end{bmatrix}$.

(b) (3 points) Compute $\vec{v}_1 \cdot (\vec{v}_2 + 2\vec{v}_3)$.

Solution: Since the (i,j)-entry of $A^\top A$ is $\vec{v}_i \cdot \vec{v}_j$, we have $\vec{v}_1 \cdot (\vec{v}_2 + 2\vec{v}_3) = (\vec{v}_1 \cdot \vec{v}_2) + 2(\vec{v}_1 \cdot \vec{v}_3) = 2 + 2(0) = 2$.

(c) (3 points) Find a vector $\vec{x} \in \mathbb{R}^2$ such that $B\vec{x} = \vec{u}_2$.

Solution: Since $R(2,2) = 2$, we have $2\vec{u}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = -\vec{u}_1 + \vec{v}_2 = -\frac{1}{\|\vec{v}_1\|}\vec{v}_1 + \vec{v}_2 = -\frac{1}{2}\vec{v}_1 + \vec{v}_2$. Thus $\vec{x} = \begin{bmatrix} -1/4 \\ 1/2 \end{bmatrix}$.

(d) (4 points) Find a least-squares solution $\vec{x}^*$ of the linear system $B\vec{x} = \vec{v}_3$.

Solution: We have $B^\top \vec{v}_3 = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_3 \\ \vec{v}_2 \cdot \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, and from (a) we see that $B^\top B = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$. Thus the normal equation $B^\top B\vec{x} = B^\top \vec{v}_3$ becomes $\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, and solving this gives least-squares solution $\vec{x}^* = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$.

Solution: We have that $\vec{x}^*$ is a least-squares solution of $B\vec{x} = \vec{v}_3$ iff $B\vec{x}^* = \operatorname{proj}_{\operatorname{im} B}(\vec{v}_3)$. The third column of R shows that $\vec{v}_3 = -2\vec{u}_2 + 3\vec{u}_3$, so $\operatorname{proj}_{\operatorname{im} B}(\vec{v}_3) = -2\vec{u}_2$ since $\operatorname{im}(B) = \operatorname{Span}(\vec{u}_1, \vec{u}_2)$. So we must find $\vec{x}^*$ such that $B\vec{x}^* = -2\vec{u}_2$, which by part (c) gives us $\vec{x}^* = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$.

7. Let $n \in \mathbb{N}$, and let $\mathcal{B} = (\vec{u}_1, \dots, \vec{u}_n)$ be an orthonormal basis of $\mathbb{R}^n$.

   (a) (4 points) Prove that for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, $[\vec{x}]_\mathcal{B} \cdot [\vec{y}]_\mathcal{B} = \vec{x} \cdot \vec{y}$.

Solution: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, and write $\vec{x} = \sum_{k=1}^n c_k \vec{u}_k$ and $\vec{y} = \sum_{\ell=1}^n d_\ell \vec{u}_\ell$. Then

$$\vec{x} \cdot \vec{y} \;=\; \left(\sum_{k=1}^n c_k \vec{u}_k\right) \cdot \left(\sum_{\ell=1}^n d_\ell \vec{u}_\ell\right) \;=\; \sum_{k=1}^n \sum_{\ell=1}^n c_k d_\ell (\vec{u}_k \cdot \vec{u}_\ell) \;=\; \sum_{k=1}^n c_k d_k \;=\; [\vec{x}]_\mathcal{B} \cdot [\vec{y}]_\mathcal{B}.$$

Solution: Let S be the $n \times n$ matrix whose ith column is $\vec{u}_i$. Then $S = S_{\mathcal{B} \to \mathcal{E}}$ is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{E}$, and S is orthogonal since $\mathcal{B}$ is orthonormal. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then $S[\vec{x}]_\mathcal{B} = \vec{x}$ and $S[\vec{y}]_\mathcal{B} = \vec{y}$, so

$$\vec{x} \cdot \vec{y} \;=\; S[\vec{x}]_\mathcal{B} \cdot S[\vec{y}]_\mathcal{B} \;=\; (S[\vec{x}]_\mathcal{B})^\top S[\vec{y}]_\mathcal{B} \;=\; [\vec{x}]_\mathcal{B}^\top S^\top S [\vec{y}]_\mathcal{B} \;=\; [\vec{x}]_\mathcal{B}^\top I_n [\vec{y}]_\mathcal{B} \;=\; [\vec{x}]_\mathcal{B} \cdot [\vec{y}]_\mathcal{B}.$$

   (b) (6 points) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and let k be an integer such that $1 \leq k < n$. Let $V = \operatorname{Span}(\vec{u}_1, \dots, \vec{u}_k)$, and let $W = \operatorname{Span}(\vec{u}_{k+1}, \dots, \vec{u}_n)$. Prove that if $\ker(A) = V$ then $\operatorname{im}(A) = W$.

Solution: Assume the hypotheses, and suppose $\ker(A) = V$. Using the identity $\ker(A^\top) = \operatorname{im}(A)^\perp$ and the fact that A is symmetric, we have $V = \ker(A) = \ker(A^\top) = \operatorname{im}(A)^\perp$. Taking orthogonal complements, this implies $V^\perp = \operatorname{im}(A)$. So we just need to show that $W = V^\perp$. We know $\dim(V) = k$ and $\dim(V^\perp) = n - k = \dim(W)$, so in fact it will suffice to show that $W \subseteq V^\perp$ since then W would be a subspace of $V^\perp$ of the same dimension as $V^\perp$.

To see that $W \subseteq V^\perp$, let $\vec{w} \in W$ and write $\vec{w} = \sum_{i=k+1}^n c_i \vec{u}_i$. Then for each $1 \leq j \leq k$ we have

$$\vec{u}_j \cdot \vec{w} \;=\; \vec{u}_j \cdot \sum_{i=k+1}^n c_i \vec{u}_i \;=\; \sum_{i=k+1}^n c_i (\vec{u}_j \cdot \vec{u}_i) \;=\; 0$$

since $\mathcal{B}$ is orthonormal. Thus $\vec{w}$ is orthogonal to every vector in the basis $(\vec{u}_1, \dots, \vec{u}_k)$ of V, so $\vec{w} \in V^\perp$ as desired.

8. Let $n \in \mathbb{N}$, let V be an n-dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$, let $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ be an ordered basis of V, and let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ be a list of vectors in V. Let S be the $n \times n$ matrix whose jth column is $[\vec{b}_j]_{\mathcal{U}}$.

   (a) (5 points) Prove that if S is invertible, then $\mathcal{B}$ is a basis of V.

Solution: Suppose S is invertible. Let $c_1, \ldots, c_n \in \mathbb{R}$ and suppose $\sum_{i=1}^n c_i \vec{b}_i = \vec{0}$. Let $L_{\mathcal{U}} : V \to \mathbb{R}^n$ be the $\mathcal{U}$-coordinate isomorphism defined by $L_{\mathcal{U}}(\vec{v}) = [\vec{v}]_{\mathcal{U}}$ for all $\vec{v} \in V$. Then

$$S\vec{c} \;=\; \sum_{i=1}^n c_i L_{\mathcal{U}}(\vec{b}_i) \;=\; L_{\mathcal{U}}\left(\sum_{i=1}^n c_i \vec{b}_i\right) \;=\; L_{\mathcal{U}}(\vec{0}) \;=\; \vec{0}.$$

Since S is invertible, this implies $\vec{c} = S^{-1}\vec{0} = \vec{0}$, showing that $\mathcal{B}$ is linearly independent. Since $\dim V = n$ and $\mathcal{B}$ consists of n vectors, it follows that $\mathcal{B}$ is a basis of V.

Solution: Let us define matrices $U = [\vec{u}_1, \ldots, \vec{u}_n]$ and $B = [\vec{b}_1, \ldots, \vec{b}_n]$. (Notice that since $\vec{u}_k, \vec{b}_k \in V$, these two matrices U and B are not in $\mathbb{R}^{n\times n}$!) Then, they are related by $B = US$. Consider the linear relation: $\sum_{k=1}^n c_k \vec{b}_k = \vec{0}$, which can be rewritten as $B\vec{c} = \vec{0}$ with $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. So, we obtain the equation $US\vec{c} = \vec{0}$.
Since the columns of U form a basis, they are linearly independent. So, $S\vec{c} = \vec{0}$. Since S is invertible, we have $\vec{c} = \vec{0}$. Thus, $\mathcal{B}$ is linearly independent. Moreover, since $\mathcal{B}$ consists of n vectors and $\dim V = n$, $\mathcal{B}$ is a basis.

Solution: Proof by contrapositive. We prove "If $\mathcal{B}$ is not a basis of V, then S is not invertible".

Since $\mathcal{B}$ has n vectors, it is not a basis of V implies that it is linearly dependent. There exist not all zero constants $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^n c_i \vec{b}_i = \vec{0}$. Then, the $\mathcal{U}$-coordinates satisfy $[\sum_{i=1}^n c_i \vec{b}_i]_{\mathcal{U}} = [\vec{0}]_{\mathcal{U}}$. In other words, $\sum_{i=1}^n c_i [\vec{b}_i]_{\mathcal{U}} = \vec{0}$ with some constant c_i nonzero. The columns of S are linearly dependent, so S is not invertible.

   (b) (5 points) Prove that if $\mathcal{U}$ is an orthonormal basis of V and S is orthogonal, then $\mathcal{B}$ is an orthonormal basis of V.

Solution: Suppose $\mathcal{U}$ is orthonormal and S is orthogonal, so $[\vec{b}_i]_{\mathcal{U}} \cdot [\vec{b}_j]_{\mathcal{U}} = \delta_{ij}$ for all $1 \le i, j \le n$. By (a), $\mathcal{B}$ is a basis of V. To show that $\mathcal{B}$ is orthonormal, let

$1 \le i, j \le n$ be arbitrary and write $\vec{b}_i = \sum_{k=1}^n c_k \vec{u}_k$ and $\vec{b}_j = \sum_{\ell=1}^n d_\ell \vec{u}_\ell$. Then

$$\begin{aligned} \langle \vec{b}_i, \vec{b}_j \rangle \ &= \ \left\langle \sum_{k=1}^n c_k \vec{u}_k,\ \sum_{\ell=1}^n d_\ell \vec{u}_\ell \right\rangle \ = \ \sum_{k=1}^n \sum_{\ell=1}^n c_k d_\ell \langle \vec{u}_k, \vec{u}_\ell \rangle \\ &= \ \sum_{k=1}^n c_k d_k \ = \ [\vec{b}_i]_{\mathcal{U}} \cdot [\vec{b}_j]_{\mathcal{U}} \ = \ \delta_{ij} \end{aligned}$$

as desired. (As an alternative to showing the full calculation above, we can observe directly that $\langle \vec{b}_i, \vec{b}_j \rangle = [\vec{b}_i]_{\mathcal{U}} \cdot [\vec{b}_j]_{\mathcal{U}}$ by the same argument used in 7(a).)