Math 217 Worksheet 21B: Elementary Matrices and Determinants

Definition: An **elementary matrix** is an $n \times n$ matrix obtained by performing a *single* elementary row operation on an $n \times n$ identity matrix I_n.

Theorem 1: If E is an elementary matrix obtained by applying an elementary row operation on I_n, then for $A \in \mathbb{R}^{n \times d}$, the matrix EA is obtained by applying *the same* elementary row operation to A.

Problem 1. Recall and discuss the three different types of elementary row operations.

(a) Write out three examples of 3×3 elementary matrices, with at least one of each type.

(b) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ be an arbitrary 3×4 matrix. Verify Theorem 1 for each of your three elementary row operations (matrices) in (a).

(c) Do you see why Theorem 1 is true? Without writing out details, discuss a scaffold for its proof.

Problem 2. Suppose $i \neq j$. Find the determinant of the elementary matrix:

(a) obtained from I_n by scaling row i by non-zero $a \in \mathbb{R}$; [Hint: Use the linearity of the determinant in row i.]

(b) obtained from I_n by interchanging rows i and j; [Hint: Use the alternating property of determinants.]

(c) obtained from I_n by adding a times row i to row j; [Hint: Use linearity in row j and alternating prop.]

Problem 3.

(a) Elementary matrices are invertible, with inverse also an elementary matrix. Explain.

(b) Prove that an invertible matrix is a product of elementary matrices.
[Hint: Use Theorem 1 repeatedly, performing elementary row ops to get rref(A).]

Problem 4. Another way to compute determinants.

(a) For $A \in \mathbb{R}^{n \times n}$, what is the effect on $\det A$ when we apply each type of elementary row operation?

(b) For a matrix $A \in \mathbb{R}^{n \times n}$, the determinant can be computed by row reducing A, and keeping track of how many row swaps were performed, and all the row scalings performed. Explain.

(c) Use row ops to compute the determinant of $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ 2 & -4 & -2 & 8 \\ -1 & 3 & 6 & -1 \\ 1 & -3 & -1 & 2 \end{bmatrix}$.

Problem 5. In this problem, we will prove the multiplicative property of determinants: $\det(AB) = \det A \ \det B$. So answer all parts below *without using* the multiplicative property. Fix $A, B \in \mathbb{R}^{n \times n}$.

(a) Prove that $\det(EA) = \det(E)\det(A)$. [Hint: There are three cases. Use Theorem 1 and Problems 2 and 4.]

(b) If $A = E_1 E_2 \cdots E_t$, where the E_i are elementary matrices, prove $\det A = \prod_{i=1}^{t} \det E_i$. [Hint: Induce!]

(c) Prove that $\det(AB) = \det A \det B$. [Hint: Multiply A by appropriate $E_1 \cdots E_t$ to row reduce; induce on t.]

Math 217 Worksheet 21C: Determinants and Volume

Definition: The **standard unit n-cube** in $\mathbb{R}^n$ is the set $\{t_1\vec{e}_1 + \cdots + t_n\vec{e}_n \,|\, 0 \le t_i \le 1\} \subseteq \mathbb{R}^n$.
Theorem 2: *Consider a linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$. Let P be the parallelepiped which is the image of the standard unit n-cube under T. Then the n-volume of P is $|\det T|$.*

Problem 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $\begin{bmatrix} 7 & 3 \\ 0 & 4 \end{bmatrix}$.

(a) The image $T[\{t_1\vec{e}_1 + t_2\vec{e}_2 \,|\, 0 \le t_i \le 1\}]$ of the standard unit square* is a parallelogram. Sketch it.

(b) Verify Theorem 2 in this example.

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$, where $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ are *orthogonal.*

(a) The image of the unit square under T is a *rectangle* with sides of lengths $||\vec{v}_1||$ and $||\vec{v}_2||$. Why? Sketch it. What does Theorem 2 tell us about $\det A$?

(b) Verify Theorem 2 for this T. [Hint: One way to find $\det A$ uses QR factorization; Another writes out $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.]

Problem 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix A, where A has linearly *dependent* columns $\vec{v}_1, \vec{v}_2$.

(a) The image $T[\{t_1\vec{e}_1 + t_2\vec{e}_2 \,|\, 0 \le t_i \le 1\}]$ of the standard unit square Q is a *line segment.* Why?

(b) Verify Theorem 2 in this example.

Problem 4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix A, where A has linearly *independent* columns $\vec{v}_1, \vec{v}_2$.

(a) The image $T[Q]$ of the standard unit square Q is a *parallelogram.* Sketch it, labelling the vectors $\vec{v}_1$ and $\vec{v}_2$ on your sketch. [Protip: Placing $\vec{v}_1$ and $\vec{v}_2$ in Quadrant 1 will make the sketch more manageable.]

(b) Suppose we apply the Gram Schmidt process to $\{\vec{v}_1, \vec{v}_2\}$ and get the vectors $\{\vec{u}_1, \vec{u}_2\}$. Add $\vec{u}_1$ to your sketch, clearly showing its relationship to $\vec{v}_1$. Show also $\vec{u}_2$ on your sketch.

(c) Compute that the base length and the height of the parallelogram $T[Q]$ are $\vec{v}_1 \cdot \vec{u}_1$ and $\vec{v}_2 \cdot \vec{u}_2$.

(d) Prove Theorem 2 in dimension two. [Hint: Compute the determinant of A *using its QR factorization.*]

Problem 5. Let A be the 3×3 matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be left multiplication by A.

(a) Assuming the columns of A are linearly independent, use the QR-factorization to show that

$$|\det A| = (\vec{v}_1 \cdot \vec{u}_1)\,(\vec{v}_2 \cdot \vec{u}_2)\,(\vec{v}_3 \cdot \vec{u}_3),$$

where $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is obtained from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ by the Gram-Schmidt process.

(b) The image of the standard unit cube Q_3 under T is a parallelepiped with one vertex at $\vec{0}$ and $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ as three of its edges. Why?

(c) The image parallelepiped $T[Q_3]$ has sides that are parallelograms. Explain why one of these sides (let's call it the "base") has area $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)$. Explain why the height of the parallelepiped is $(\vec{v}_3 \cdot \vec{u}_3)$.

(d) Prove Theorem 2 in dimension 3. Do you see how one might construct an inductive proof for Theorem 2 in arbitrary dimension?

* "Unit square" is another name for "unit 2-cube".

Problem 6. THE SIGN OF THE DETERMINANT. Let A be a 2×2 matrix representing a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ in standard coordinates. Investigate the geometric meaning of the *sign of the deteminant* by sketching the images $\vec{v}_1$ and $\vec{v}_2$ of $\vec{e}_1$ and $\vec{e}_2$ in several different cases, some where the deteminant of A is negative and some where it is positive. What happens for 3×3 matrices? What general observation can you make?