Math 217 Worksheet 21 A: Determinants (Chapter 6)

Computing the Determinant. Determinants are defined only for square matrices. The determinant of a 1×1 matrix is simply its unique scalar entry. The determinant for a larger size matrix is a scalar defined recursively using a technique called **Laplace expansion.**
The determinant of A is denoted $\det A$ or $|A|$.

Definition: Let A be an $n \times n$ matrix. The **Laplace expansion along row i of A** is the expression

$$\sum_{j=1}^{n}(-1)^{i+j} a_{ij} \det A_{ij},$$

where A_{ij} is the submatrix of A obtained by deleting row i and column j.

Likewise the **Laplace expansion along column** j is $\sum_{i=1}^{n}(-1)^{i+j} a_{ij} \det A_{ij}$.

Theorem/Definition: Let A be an $n \times n$ matrix. The Laplace expansion along any row or column gives the same scalar, denoted $\det A$, and called the **determinant** of A.

Problem 1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Prove that the determinant of A is well-defined, by checking that all four possible Laplace expansions produce the same result.

Solution: The Laplace expansion along the first row is $a \det[d] - b \det c = ad - bc$.
The Laplace expansion along the second row is $-c \det[b] + d \det[a] = ad - bc$.
The Laplace expansion along the first column is $a \det[d] - c \det b = ad - bc$.
The Laplace expansion along the second column is $-b \det[c] + d \det[a] = ad - bc$.
Since we get the same result no matter how we compute it, the determinant of a 2×2 is well-defined!

Problem 2. Compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

using conveniently chosen Laplace expansions along various rows and columns. Compute using different row/column to check your work.

Solution: Using a Laplace expansion along the second column, we have

$$\det \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix} = -2 \det \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = (-2)(3) = -6.$$

Alternatively: along the top row, we have

$$\begin{aligned}\det\begin{bmatrix}1&0&0&-1\\3&1&2&2\\1&0&-2&1\\2&0&0&1\end{bmatrix} &= \det\begin{bmatrix}1&2&2\\0&-2&1\\0&0&1\end{bmatrix} - (-1)\det\begin{bmatrix}3&1&2\\1&0&-2\\2&0&0\end{bmatrix}\\ &= \det\begin{bmatrix}-2&1\\0&1\end{bmatrix} + (-1)\det\begin{bmatrix}1&-2\\2&0\end{bmatrix}\\ &= -2-4=-6.\end{aligned}$$

Problem 3.

(a) Compute the determinant of the matrix $\begin{bmatrix}0&0&0&-1\\0&0&2&\pi\\0&6&-2&10\\2&5&\sqrt{3}&1\end{bmatrix}$.

(b) Prove that the determinant of an $n\times n$ upper triangular matrix is the product of the diagonal elements $\prod_{i=1}^n a_{ii}$. Is the same true for lower triangular matrices?
[HINT: Induce on n. Recall that an $n\times n$ matrix $A=[a_{ij}]$ is **upper triangular*** if $a_{ij}=0$ for $j<i$.]

Solution:

(a) Computing along the top row: $-(-1)\det\begin{bmatrix}0&0&2\\0&6&-2\\2&5&\sqrt{3}\end{bmatrix} = 2\det\begin{bmatrix}0&6\\2&5\end{bmatrix} = 2\times -12 = -24.$

(b) An upper triangular matrix is one of the form $\begin{bmatrix}a_{11}&a_{12}&a_{13}&\cdots&a_{1n}\\0&a_{22}&a_{23}&\cdots&a_{2n}\\0&0&a_{33}&\cdots&a_{3n}\\\vdots&\vdots&&\ddots&\vdots\\0&0&\cdots&0&a_{nn}\end{bmatrix}$, so that below the main diagonal we have only zero entries. We prove the statement by induction on n. The statement is clear if $n=1$. Assume it holds for $n-1$, and let A be an $n\times n$ upper triangular matrix. This means the first column is $a_{11}\vec{e}_1$, and the matrix A_{11} obtained by deleting the first row and column is upper triangular of smaller size. By induction $\det A_{11}$ is the product of the diagonal elements, which is $a_{22}\cdots a_{nn}$ in this case. So expanding along the first column, we have $\det A = a_{11}\det A_{11} = a_{11}\prod_{i=2}^n a_{ii} = \prod_{i=1}^n a_{ii}$.

Multiplicative Property of Determinants.

Theorem 1: For any $n\times n$ matrices A and B, we have $\det(AB) = \det A \det B$.

Corollary 1a: If A is an invertible matrix, then $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$.

*Caution: The matrix in (a) is neither upper nor lower triangular, though one might still say item is "triangular".

Corollary 1b: Similar matrices have the same determinant.

Definition: The **determinant** of a linear transformation $T : V \to V$ (where V is finite dimensional) is the determinant of the $\mathcal{B}$-matrix $\det[T]_{\mathcal{B}}$, where $\mathcal{B}$ is any basis for V.

Problem 4.

(a) Verify the Multiplicative Property of Determinants for $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$.

(b) Show that Corollary 1a follows from Theorem 1. (The converse is also true; see Theorem 3.)

(c) Show that Corollary 1b follows from Theorem 1.

(d) Check that the determinant of a linear transformation is *well-defined*—that is, it does not depend on the choice of basis $\mathcal{B}$.

Solution:

(a) $\det A \times \det B = 12 \times 2 = 24$. And $AB = \begin{bmatrix} 3 & 14 \\ 0 & 8 \end{bmatrix}$, so $\det(AB) = 24$ as well.

(b) Say A is invertible. So there exists B such that $AB = I_n$. By the multiplicative property, $\det A \ \det B = \det I_n = 1$. So $\det A$ can not be zero. For the second statement, We have $A\ A^{-1} = I_n$. So $\det A \ \det A^{-1} = \det I_n = 1$. This means $\det A^{-1} = \frac{1}{\det A^{-1}}$.

(c) If A and B are similar $n \times n$ matrices, then $A = S^{-1}BS$ for some invertible S. So $\det A = \det(S^{-1}B) \det S = \det(S^{-1}) \det B \det S$; this is just a product of real numbers, so we can change the order (commutative law of multiplication for $\mathbb{R}$). So $\det A = \det B \det(S^{-1}) \det S = \det B$, where the last equality is because $\det S^{-1}$ and $\det S$ are reciprocals.

(d) If $\mathcal{A}$ and $\mathcal{B}$ are two different bases for V, then $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{B}}$ are similar matrices, since $[T]_{\mathcal{A}} = S^{-1}[T]_{\mathcal{B}}S$ where S is the change of basis matrix $S_{\mathcal{A}\to\mathcal{B}}$. So by Corollary 1b, $\det[T]_{\mathcal{A}} = \det[T]_{\mathcal{B}}$.

Orthogonal Matrices.

Theorem 2: If A is an $n \times n$ matrix, then $\det A = \det A^\top$.

Corollary: The determinant of an orthogonal matrix is ± 1. Likewise, the determinant of an orthogonal transformation of $\mathbb{R}^n$ is ± 1.

Problem 5.

(a) Verify Theorem 2 for the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}$ by computing both $\det A$ and $\det A^\top$.

(b) Explain why Theorem 2 holds.
[Hint: Compare the expansion along a row of A with expansion along the corresponding column of $A^\top$.]

(c) Show the corollary to Theorem 2 follows from Theorem 2. [Hint: ... $AA^\top = I_n$ ...]

(d) Consider the transformation T of $\mathbb{R}^2$ given by rotation counterclockwise through θ. Is this an orthogonal transformation? What is its standard matrix? What is the determinant of T?

(e) Consider the transformation T of $\mathbb{R}^2$ given by reflection over a line L through the origin. Is this an orthogonal transformation? What is the determinant of T? [Hint: Choose a basis wisely.]

Solution:

(a) Using a Laplace expansion along the first row, we have

$$\begin{aligned}\det A = \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix} &= 1 \det \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix} \\ &= 1(3 \cdot 0 - 2 \cdot 1) - 0(0 \cdot 0 - 1 \cdot (-1)) + 2(0 \cdot 2 - 3 \cdot (-1)) \\ &= -2 + 2(3) = 4.\end{aligned}$$

This is exactly the same computation we do to compute $\det A^\top$ along the first column:

$$\begin{aligned}\det A^\top = \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix} &= 1 \det \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \\ &= 1(3 \cdot 0 - 2 \cdot 1) - 0(0 \cdot 0 - 1 \cdot (-1)) + 2(0 \cdot 2 - 3 \cdot (-1)) \\ &= -2 + 2(3) = 4.\end{aligned}$$

(b) Expanding along row 1 of A gives the exact same sequence of computations as expanding $A^\top$ along column 1.

(c) If A is orthogonal, $\det A \det A^\top = \det I_n = 1$. Since $\det A^\top = \det A$, this says $(\det A)^2 = 1$. So $\det A = \pm 1$. Now suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation. Then the standard matrix of T is orthogonal, so its determinant is ± 1. In *any coordinate system*, say given by basis $\mathcal{B}$, the determinant of the $\mathcal{B}$-matrix for T is ± 1, since all these matrices are similar to the orthogonal matrix $[T]_{\mathcal{E}}$.

(d) Yes. The standard matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so has determinant $\sin^2\theta + \cos^2\theta = 1$. In *any coordinate system*, say given by basis $\mathcal{B}$, the determinant of the $\mathcal{B}$-matrix for rotation through θ is 1. This is because the $\mathcal{B}$-matrix and the standard matrix (of any linear transformation) are similar, so it follows from Corollary 1b above.

(e) Yes. Here, a convenient basis is $(\vec{v}, \vec{w})$ where $L = \text{Span}(\vec{v})$ and $\vec{w}$ spans $L^\perp$. The matrix of T in this basis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which has determinant -1. Since we can compute the determinant from *any* basis, $\det T = -1$.

Determinant and Invertibility.

Theorem 3: A square matrix is invertible if and only if its determinant is non-zero.

Problem 6. Is the matrix $M = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ \pi & -1 & 2 & 0 \end{bmatrix}$ invertible? Explain.

Solution: No! Expanding along row 3, we easily see the determinant is zero.

Problem 7. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$ be an $n \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$.

(a) For $c \in \mathbb{R}$, what is $\det \begin{bmatrix} | & | & & | \\ c\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$? [Hint: Choose carefully a row/column to expand along.]

Solution: Let $B = [c\vec{a}_1 \ \cdots \ \vec{a}_n]$. Then expanding along column 1, we have

$$\det B = \sum_{i=1}^{n} (-1)^{i+1} c a_{i1} \det A_{ij} = c \left(\sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{ij} \right) = c \det A.$$

(b) For any $\vec{b} \in \mathbb{R}^n$, what is $\det \begin{bmatrix} | & | & & | \\ \vec{a}_1 + \vec{b} & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$?

Solution: Let $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$, let $B = [\vec{b} \ \ \vec{a}_2 \ \cdots \ \vec{a}_n]$ and let $C = [\vec{a}_1 + \vec{b} \ \ \vec{a}_2 \ \cdots \ \vec{a}_n]$. We claim that $\det C = \det A + \det B$. To see this, expand along column one.

$$\det C = \sum_{i=1}^{n} (-1)^{1+i} ([\vec{a}_1]_i + b_i) \det A_{i1}$$

where $[\vec{a}_1]_i, b_i$ are the i-components of the vectors $\vec{a}_1$ and $\vec{b}$, respectively. This breaks up as $\sum_{i=1}^{n} (-1)^{1+i} ([\vec{a}_1]_i) \det A_{i1} + \sum_{i=1}^{n} (-1)^{1+i} b_i \det A_{i1}$, which is $\det \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} + \begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$.

(c) Do your claims in (a) and (b) above carry over to any column of A, and not just its first? How about to any row?

Solution: Yes, the claims in (a) and (b) hold also for any column, with similar proofs. Similarly, it holds for rows.

Alternating Property of Determinants.

Theorem 4. If we swap two rows of a square matrix, then its determinant changes sign. A similar statement holds for columns.

Problem 8.

(a) Verify Alternative Property of Determinants (Theorem 4) holds for an arbitrary 2×2 matrix.

(b) Use Theorem 4 to show that if two columns of A are the same, then its determinant is zero.

(c) Discuss how to prove Theorem 4; you do not need to write out details. [HINT: Induction works well here. The base case is (a). For the inductive step, expand along a column *not* involved in the swap.]

Solution:

(a) The 2×2 case is easy by direct calculation: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and $\det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = bc - ad = -\det A$. The calculation is similar for rows.

(b) Say columns i and j of square matrix A are equal, then swapping them, the new matrix is equal to A. By Theorem 4, $\det A = -\det A$. The only real number equal to its negative is 0, so $\det A = 0$.

(c) Use induction on n, the size of the square matrix. The base case is where $n = 2$; we handled this in (a). For the inductive step, suppose A is 3×3 or bigger. Suppose A' is obtained by swapping columns a and b. Expanding along column j, one of the columns that is *not* swapped (so $j \neq a, b$), we have Then

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

and

$$\det A' = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A'_{ij}.$$

Note that each of the submatrices A_{ij} are size $n-1 \times n-1$, and that A'_{ij} is obtained from A_{ij} by swapping two columns (indexed by a and b). By induction, $\det A_{ij} = -\det A_{ij}$, so

$$\det A' = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A'_{ij} = -\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \det A.$$

Multi-linearity of Determinant.

Theorem 5: Fix $n-1$ column vectors $\vec{v}_1, \dots, \vec{v}_{n-1}$ in $\mathbb{R}^n$. Then the function

$$\mathbb{R}^n \to \mathbb{R}$$

$$\vec{x} \mapsto \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_{n-1} & \vec{x} \end{bmatrix}$$

is a linear transformation. The same is true if $\vec{x}$ is inserted as the j-th column (instead of the n-th) for any j. A similar statement holds for the rows.

Problem 9.

(a) Consider the map $\mathbb{R}^3 \to \mathbb{R}$ sending $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \det \begin{bmatrix} x & 1 & 2 \\ y & -1 & 0 \\ z & 0 & 1 \end{bmatrix}$. Is it linear? Why? If so, find its standard matrix. [HINT: Use Theorem 5 and/or Problem 7.]

(b) Suppose $\vec{v}_1, \ldots, \vec{v}_{n-1}$ are vectors in $\mathbb{R}^n$ and that

$$\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{a}] = 5, \quad \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{b}] = 7, \quad \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{c}] = -3.$$

Find $\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ (2\vec{a} + 4\vec{b} - 6\vec{c} + 17\vec{v}_1)]$.

(c) Prove that if A is an $n \times n$ matrix, then $\det(kA) = k^n \det A$ for any scalar k.

Solution: Let $\vec{v}_1, \ldots, \vec{v}_n$ be the rows of A. Then $\det kA = \det[k\vec{v}_1 \ k\vec{v}_2 \ \ldots \ k\vec{v}_n] = k \det[\vec{v}_1 \ k\vec{v}_2 \ \ldots \ k\vec{v}_n] = k^2 \det[\vec{v}_1 \ \vec{v}_2 \ \ldots k\vec{v}_n] = k^n \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]$. Each time we are pulling out one k, using linearity in the first, then second, then third, etc, column.

(d) Discuss the proof of Theorem 5, and make sure you see why it follows from Problem 7. You do not have to write down all the details.

Problem 10. Prove or disprove: The determinant map $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is linear.
[Remember! Usually, a simple counterexample is the best way to disprove something.]

Solution: No, if $n > 1$ then the determinant map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ is *not* a linear transformation. In fact, if A is $n \times n$ then for any $c \in \mathbb{R}$ we have

$$\det(cA) \ = \ c^n \cdot \det A.$$

Problem 11. Prove Theorem 3: If $\det A \neq 0$, then A is invertible.
[Hint: One proof uses Theorem 5 and the fact that any noninvertible matrix has a column that is a linear combination of the other columns; we will give another proof using elementary matrices on WS 21B.]

Solution: See Worksheet 21B.

Problem 12*. Prove Theorem 1: $\det(A\ B) = \det A \det B$.
[Hint: It helps if you know that an invertible matrix is a product of elementary matrices; see the "Extra worksheet" on *Elementary Matrices* after Worksheet 7. We will cover this on WS 21B.]

Solution: See Worksheet 21B.