Math 217: What is Linear about Linear Transformations?

Definition: A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is a mapping (i.e., a function) from $\mathbf{R}^n$ to $\mathbf{R}^m$ satisfying the following:

- $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}$, $\vec{y} \in \mathbf{R}^n$ (that is, "*T respects addition*").
- $T(a\vec{x}) = aT(\vec{x})$ for all $a \in \mathbf{R}$ and $\vec{x} \in \mathbf{R}^n$ (that is, "*T respects scalar multiplication*").

A. A **line** in $\mathbb{R}^n$ is determined by giving any point $\vec{b}$ on it and its "direction vector" $\vec{m}$. Precisely,

Definition: A **line** in $\mathbb{R}^n$ is any set of the form

$$L = \{t\vec{m} + \vec{b} \,|\, t \in \mathbb{R}\},$$

where $\vec{m}$ and $\vec{b}$ are fixed vectors in $\mathbb{R}^n$.

1. In $\mathbb{R}^2$, draw the lines $L = \{t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad | \; t \in \mathbb{R}\}$ and $L' = \{t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad | \; t \in \mathbb{R}\}$.
2. Show that the line L'' in which $\vec{m} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is the same as L from (1).
3. For a line $M = \{t \begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \quad | \; t \in \mathbb{R}\}$ in $\mathbb{R}^2$, in which $a \neq 0$, find the "slope intercept" (or $y = mx + b$) form of the line. What happens when $a = 0$?
4. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be any linear transformation and L any line in the source $\mathbf{R}^n$. Show that T maps the line L to another line in the target $\mathbb{R}^m$ **OR** to a point. How can you distinguish the two cases? When $T(L)$ is a line, what its its direction vector in terms of the direction vector of L?

B. A **plane** in $\mathbb{R}^n$ is any set of the form

$$\Lambda = \{t\vec{m} + s\vec{n} + \vec{b} \,|\, t, s \in \mathbb{R}\},$$

where $\vec{m}$, $\vec{n}$ and $\vec{b}$ are fixed vectors in $\mathbb{R}^n$.

1. Describe/sketch the plane

$$\Lambda = \{t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad | \; s, t \in \mathbb{R}\}$$

   in $\mathbb{R}^3$.
2. What do linear maps do to planes? That is, if $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation and if Λ is a plane in the source, what kind of geometric object is the image $T(\Lambda)$ of Λ?

Challenge Problem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping that sends lines to lines, must it be linear? What if it sends lines to lines and fixes the origin?