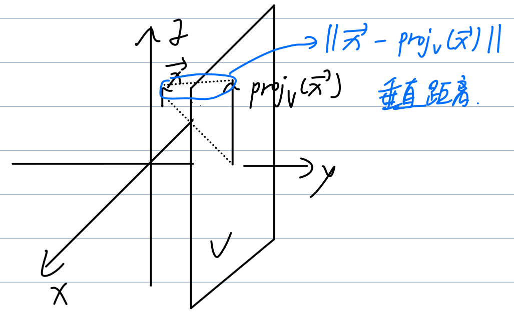
WS 15. Thm ①

对于任意 subspace $V \subseteq \mathbb{R}^n$

$\forall \vec{x} \in \mathbb{R}^n$, $\boxed{proj_V(\vec{x})\text{ 是 } V \text{ 中离 } \vec{x} \text{ 最近的 vector}}$

即: $\forall \vec{v} \in V$, $\|\vec{x} - proj_V(\vec{x})\| \le \|\vec{x} - \vec{v}\|$

P1



P2 (a) $A\vec{x} = \vec{b}$ is consistent iff $\vec{b} \in im A$

显然.
$$\begin{cases} \exists \vec{x} \text{ s.t. } A\vec{x} = \vec{b} \Rightarrow \vec{b} \in im A \\ \vec{b} \in im A \Rightarrow \exists \vec{x} \text{ s.t. } A\vec{x} = \vec{b} \end{cases}$$

(b). 不可能. $im A = span(\text{cols of } A)$

$\vec{b} \notin span(\text{cols of } A) \Rightarrow A\vec{x} = \vec{b}$ not consistent.

(c). least squares solutions:
(d)

$A\vec{x} = \vec{b}$ 如果不 consistent

考虑 $V = im A$, $\vec{b}' = proj_V(\vec{b})$

则 $A\vec{x} = \vec{b}'$ 有解 (因为 $\vec{b}' \in V$)

这个 solution set 中的 solutions 称为 the least squares solution to this system

$$A\vec{x} \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y}$$
$$= (\vec{x} \cdot (A^T \vec{y}))$$

P4 (1) Proof of Thm 2:

$\forall$ m×n matrix A, $\ker A^T = (im A)^\perp$

Pf let $A = \begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_n} \\ | & & | \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{bmatrix}$

① let $\vec{x} \in \mathbb{R}^m \in \ker A^T \Rightarrow \begin{bmatrix} \vec{v_1}^T \\ \vec{v_2}^T \\ \vdots \\ \vec{v_n}^T \end{bmatrix} \vec{x} = \vec{0}$

$$\Rightarrow \begin{bmatrix} \vec{v_1}^T \vec{x} \\ \vec{v_2}^T \vec{x} \\ \vdots \\ \vec{v_n}^T \vec{x} \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} \vec{v_1} \cdot \vec{x} \\ \vec{v_2} \cdot \vec{x} \\ \vdots \\ \vec{v_n} \cdot \vec{x} \end{bmatrix} = \vec{0}$$

$\Rightarrow \forall i, \vec{x} \perp \vec{v_i} \Rightarrow \vec{x} \in (im A)^\perp$

Since $im A = span(\vec{v_1}, \dots, \vec{v_n})$, $\vec{x} \perp im A \Rightarrow \vec{x} \in (im A)^\perp$

② let $\vec{x} \in (im A)^\perp \Rightarrow \forall \vec{v} \in im A, \vec{v} \perp \vec{x}$

Since $im A = span(\vec{v_1}, \dots, \vec{v_n})$, $\vec{v_1}, \dots, \vec{v_n} \in im A$

$\Rightarrow \forall i, \vec{v_i} \cdot \vec{x} = 0 \Rightarrow A^T \vec{x} = \vec{0}$

$\Rightarrow \vec{x} \in \ker A^T$

Pf way 2: use Lemma

Lemma: $\forall \vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n \Rightarrow (A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^T \vec{y})$

Let $\vec{y} \in \ker A^T \Rightarrow A^T \vec{y} = \vec{0} \Rightarrow (A\vec{x}) \cdot \vec{y} = \vec{0}$

So $A\vec{x} \perp \vec{y} \Rightarrow \vec{y} \in (im A)^\perp$

Let $\vec{y} \in (im A)^\perp \Rightarrow y \perp im A \Rightarrow A\vec{x} \cdot \vec{y} = \vec{0}$ ($\forall x$)

$\Rightarrow \forall \vec{x}, \vec{x} \cdot (A^T \vec{y}) = \vec{0} \Rightarrow A^T \vec{y} = \vec{0}$

$\Rightarrow \vec{y} \in \ker A^T$

So $\ker(A^T) = (im A)^\perp$

(2) By Thm 2, also: $(\ker(A^T))^\perp = im A$

$\ker(A)^\perp = im(A^T)$

Since $\ker(A^T) = (im A)^\perp$

$\Rightarrow (\ker(A^T))^\perp = ((im A)^\perp)^\perp = im A$

$\ker((A^T)^T) = (im A^T)^\perp \Rightarrow \ker A = (im A^T)^\perp \Rightarrow (\ker A)^\perp = im A^T$.

(3) By Thm 2: $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

因为 $\operatorname{rank}(A^T) = n - \underline{\dim(\ker A^T)}$

$= \dim((\operatorname{im} A)^\perp)$

$= n - \dim(\operatorname{im} A)$

$\Longrightarrow \operatorname{rank}(A^T) = \dim(\operatorname{im} A) = \operatorname{rank} A$.

The normal equation:

$A\vec{x} = \vec{b}$ 的 least square 解

即 $\underline{A\vec{x}^* = \operatorname{proj}_{\operatorname{im}(A)}(\vec{b})}$ 的解 $\vec{x}^*$

就是 $\underline{A^TA\vec{x} = A^T\vec{b}}$ 的解.

↳ 称为 normal equation

P6. (a) Proof of the normal equation.

Claim (a): $\forall$ $m\times n$ matrix A, $\ker(A) = \ker(A^TA)$

① if $\vec{x} \in \ker(A^TA)$, then $A^TA\vec{x} = 0$

$\Longrightarrow A^T(A\vec{x}) = \vec{0} \Rightarrow A\vec{x} \in \ker A^T$

By thm 2, $\underline{A\vec{x} \in (\operatorname{im} A)^\perp}$

Since $\underline{A\vec{x} \in \operatorname{im} A}$ and $\boxed{\operatorname{im} A \cap (\operatorname{im} A)^\perp = \vec{0}}$ ☆

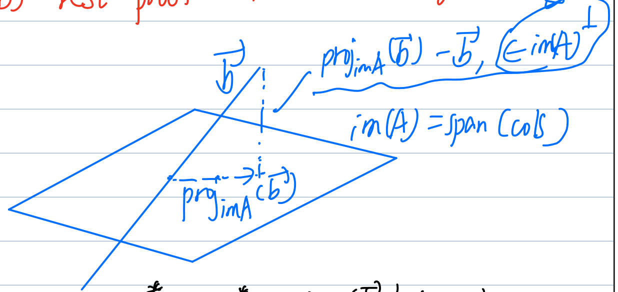
$\Longrightarrow A\vec{x} = \vec{0} \Longrightarrow \underline{\vec{x} \in \ker A}$

② if $\underline{\vec{x} \in \ker(A)}$, then $A\vec{x} = 0$

So $A^TA\vec{x} = 0 \Longrightarrow \underline{\vec{x} \in \ker(A^TA)}$

Therefore $\underline{\ker(A) = \ker(A^TA)}$

(b) rest proof of normal equation



$\vec{x}^*$ 为 $A\vec{x}^* = \operatorname{proj}_{\operatorname{im}A}(\vec{b})$ 的 sol

$\Longleftrightarrow \underbrace{A\vec{x}^* - \vec{b} \left(= \operatorname{proj}_{\operatorname{im}A}(\vec{b}) - \vec{b}\right) \in \underbrace{(\operatorname{im}A)^\perp}_{=\ker(A^T)}}$ ☆

Note that $A\vec{x}^* - \vec{b} \in \ker(A^T)$

即 $A^T(A\vec{x}^* - \vec{b}) = \vec{0}$

即 $\underline{A^TA\vec{x}^* = A^T\vec{b}}$. ☆☆

总结: 可以这么说:

对于 $T_A: \mathbb{R}^n \to \mathbb{R}^m$.

$T_{A^T}: \mathbb{R}^m \to \mathbb{R}^n$ 使得

取任意 $\vec{x} \in \mathbb{R}^n$ 及 $\vec{y} \in \mathbb{R}^m$,

$\underline{(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^T\vec{y})}$ 是 transpose 本质的

性质: 它试图撤销 T_A 对 $\mathbb{R}^n$ 的影响

但不一定能做到.