# Ch3 - Def - Thms

## Def ① 3.1.1 Image of a function (像)

对于 $f: X \to Y$

$im(f) = \{f(x) \mid x \in X\}$

note: $im(f) \subseteq$ target.

## Def ② 3.1.2 Span (张成)

(* Def on WS 10: 称 $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_d}\}$ spans vector space $V$ 若 $\forall \vec{v} \in V$ is a linear comb of $\vec{v_1}, \dots, \vec{v_d}$.)

对于 $\vec{v_1}, \dots, \vec{v_m} \in \mathbb{R}^n$

$span(\vec{v_1}, \dots, \vec{v_m}) = \{c_1\vec{v_1} + \dots + c_m\vec{v_m} \mid c_1, \dots, c_m \in \mathbb{R}\}$

即: $\vec{v_1}$ through $\vec{v_m}$ 所有的线性组合的集合

## Thm ① 3.1.3 Image of a linear trans

$n\{\overbrace{[A\vec{e_1}\ A\vec{e_2} \cdots A\vec{e_m}]}^{m}$

对于 linear trans $T(\vec{x}) = A\vec{x}$, $A \in \mathbb{R}^{n\times m}$ $(T: \mathbb{R}^m \to \mathbb{R}^n)$

$im(T) = span(A\vec{e_1}, A\vec{e_2}, \dots, A\vec{e_m})$

也可denote为 $im(A)$ 即: T的 im 为 A的所有 col vec 的 span

## Thm ② 3.1.4 Linear trans 的 Image 的一些性质

(a) $\boxed{\vec{0} \in im(T)}$

(b) $im(T)$ is closed under $+$:
if $\vec{v_1}, \vec{v_2} \in im(T)$
$\Rightarrow \vec{v_1} + \vec{v_2} \in im(T)$

(c) $im(T)$ is closed under scalar $\times$:
if $\vec{v} \in im(T)$, $k$ 是arbitrary scalar.
$\Rightarrow k\vec{v} \in im(T)$

## Def ③ 3.1.5 kernel of linear trans $T: \mathbb{R}^m \to \mathbb{R}^n$

$ker(T) = \{\vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0}\}$

$(ker(A))$ 即: source 中所有映射到 $\vec{0}$ in target 的元素的集合

note: $ker(A) \subseteq$ source

## Thm ③ 3.1.6 Linear trans 的 kernel 的一些性质

$T: \mathbb{R}^m \to \mathbb{R}^n$

(a) $\mathbb{R}^m$ 中的 $\boxed{\vec{0} \in ker(T)}$

(b) kernel is closed under $+$

(c) kernel is closed under scalar $\times$

## Thm ④ 3.1.7 何时 $ker(A) = \vec{0}$.

(1) for square matrix $A$, $ker(A) = \vec{0}$ iff $A$ invertible

(2) For 任意 $n \times m$ matrix $A$,

若 $ker(A) = \{\vec{0}\} \Rightarrow m \le n$

那么若 $m > n$ $ker(A)$ 中一定有非0 vector.

$\boxed{ker(A) = \{\vec{0}\} \text{ iff } rank(A) = m}$

## Summary 3.1

对于一个 square matrix $A \in \mathbb{R}^{n\times n}$,
以下所有条都 equiv

$A\vec{x} = \vec{b}$ 有unique sol $\Leftrightarrow$ $A$ invertible $\Leftrightarrow$ $rref(A) = I_n$ 即 $rank(A) = n$

$A$ invertible $\Leftrightarrow$ $ker(A) = \{\vec{0}\}$

$rref(A) = I_n$ 即 $rank(A) = n$ $\Leftrightarrow$ $im(A) = \mathbb{R}^n$ (surjective 又 linear trans 一定 injective $\Rightarrow$ bijective)

$ker(A) = \{\vec{0}\} \Leftrightarrow im(A) = \mathbb{R}^n$

## Def ④ on WS 9 Subspaces

(对应书subspaces of $\mathbb{R}^n$)

$V$ 是一个vector space. $W \subseteq V$

if
① $\vec{0}_V \in W$
② $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$ (closed under $+$)
③ $\forall \lambda \in \mathbb{R}, w \in W \Rightarrow \lambda w \in W$ (closed under scalar $k$)

$\Rightarrow$ $W$ 是 $V$ 的 subspace

Thm ⑤ (WS) $\boxed{\text{Images and kernel are subspaces}}$

$T: V \to W$ be linear transformation

$\ker(T)$ is a subspace of $V$

$\operatorname{im}(T)$ is a subspace of $W$

| | Subspaces of $\mathbb{R}^2$ | Subspaces of $\mathbb{R}^3$ |
|---|---|---|
| dimension 3 | | $\mathbb{R}^3$(本身) |
| dimension 2 | $\mathbb{R}^2$(本身) | plane |
| dimension 1 | line | line |
| dimension 0 | $\{\vec{0}\}$ | $\{\vec{0}\}$ |

Def ⑤ on WS9 $\boxed{\text{relation}}$ (trivial, non-trivial)

A $\boxed{\text{relation}}$ on a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_d}\}$

是指一个值为 $\vec{0}$ 的 linear comb

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_d\vec{v_d} = \vec{0}$$

特别地, if 所有 $c_i$ 都为 0 $\Longrightarrow$ 称这个 relation 为 $\boxed{\text{trivial}}$ 的

Def ⑥ on WS9 $\boxed{\text{Linearly independent}}$

一个 set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_d}\}$ in Vector space $V$

为 linearly independent 的 if whenever

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_d\vec{v_d} = \vec{0}$$

$$\Longrightarrow c_1 = c_2 = \ldots = c_d = 0,$$

即: 这个 vector set $\boxed{\text{仅存在 trivial relation}}$

不存在其它 relation.

(linearly dependent: 指存在非 trivial 的 relation.)

iff

(Thm 3.2.7)

注意到: 如果 set 中有 $\vec{0}$,

那么一定 lin dependent.

补充 Def on book (linearly independent 另一个 def)

3.2.3

① Redundant vector: $\boxed{\vec{v_i}}$ in $\vec{v_1}, \ldots, \vec{v_m}$ 为一个

redundant vector if 它是其他一些 vector 的 linear comb

② $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_d}\}$ 为 linearly ind 的, if 没有 redundant

vector. (linearly dependent iff $\boxed{\text{至少有一个 redundant vector}}$)

Def ⑦ 3.2.3 $\boxed{\text{Basis}}$

(Same as WS ⑩)

一个 vector space $V$.

set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_d}\}$ 为 $V$ 的一个 basis

if $\underbrace{\boxed{\text{it spans } V}}_{(1)}$ 且 $\underbrace{\boxed{\text{it is linearly independent}}}_{(2)}$

Def ⑧ (对应 3.3.3) $\boxed{\text{Dimension}}$

$V$ is a vector space.

$\boxed{\dim(V)}$ is the number of elements in a basis

of $V$.

Thm ⑥ 3.2-4 $\boxed{\text{Basis of } \operatorname{im}(A)}$

对于 matrix $A$, the basis of $\operatorname{im}(A)$

is all column vectors of $A$ omitting

redundant ones.

ex: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$\to$ basis of $\operatorname{Im}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

Thm ⑦ 3.2.5

对于 $\vec{v_1}, \ldots, \vec{v_m} \in \mathbb{R}^n$

如果 $\vec{v_i}$ 非 $\vec{0}$, 而每个 $\vec{v_i}$ 都有一个位置上的

entry 为 0 而之前 $\vec{v_1}, \ldots \vec{v_{i-1}}$ 在该位置的 entries

都为 0, 则 $\{\vec{v_1}, \ldots, \vec{v_m}\}$ linearly independent.

Thm ⑧ 3.2.8 kernel and relations for $A \in \mathbb{R}^{n \times m}$

$$A\vec{x} = 0 \iff x_1\vec{v_1} + x_2\vec{v_2} + \ldots + x_m\vec{v_m} = \vec{0}$$

$A$ is linearly independent iff $\ker(A) = \{\vec{0}\}$

$\Updownarrow$

$\operatorname{rank}(A) = m$

Sum 3.2.9 对于 $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$, 以下 6 条 equlu

(1) linearly independent (2) 无 redundant vector

(3) 没有一个 $\vec{v_i}$ 是其他 vector 的 linear comb.

(4) 只存在 trivial relation (5) $\ker\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} = \{\vec{0}\}$

(6) $\operatorname{rank}\left(\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix}\right) = m$

Thm ⑨ 3.2.10 $\{\vec{v}_1, \dots, \vec{v}_m\} \in V \subseteq \mathbb{R}^n$

为 V 的一个 basis iff $\boxed{\text{every } \vec{v}_i}$ can be ~~uniquely~~ expressed as a $\boxed{\text{linear comb}}$ of other vectors.

Thm ⑩ on WS10 $T_A : \mathbb{R}^m \to \mathbb{R}^n$ linear trans. $A \in \mathbb{R}^{n\times m}$

(1) A 中对应 rref(A) 的 pivot columns 的 columns 是 $im(T_A)$ 的一个 basis

(2) $\dim(im(T_A)) = rank(A)$

(3) $\dim(source(T_A)) = \boxed{\dim(\ker(T_A))} + \dim(im(T_A))$

即 m. def nullity of A