Math 217 – Midterm 2

Fall 2018

Solutions

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	12	
4	11	
5	15	
6	12	
7	12	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The vector space V is *isomorphic* to the vector space W

   Solution: The vector space V is *isomorphic* to the vector space W if there exists an isomorphism from V to W.

   (b) The *coordinates* of the vector $\vec{v}$ in the vector space V relative to the ordered basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ of V

   Solution: The *coordinates* of the vector $\vec{v}$ in the vector space V relative to the ordered basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ of V are the unique scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that $\vec{v} = \sum_{i=1}^{n} c_i \vec{b}_i$.

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the inner product space V is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ on V

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the inner product space V is *orthonormal* relative to the inner product $\langle \cdot, \cdot \rangle$ on V if for all integers $1 \leq i, j \leq n$, we have $\langle \vec{v}_i, \vec{v}_j \rangle = 1$ if $i = j$ and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if $i \neq j$.

   (d) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B

   Solution: The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B if there exists an invertible $n \times n$ matrix S such that $A = S^{-1}BS$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For all square matrices A and B, if A is similar to B and A is invertible, then B is also invertible.

Solution: TRUE. Let A and B be square matrices, and suppose A is similar to B and A is invertible. Let S be an invertible matrix such that $B = S^{-1}AS$. Note that A and B (and S) are the same size, say $n \times n$. Then

$$BS^{-1}A^{-1}S \;=\; S^{-1}ASS^{-1}A^{-1}S \;=\; S^{-1}AA^{-1}S \;=\; SS^{-1} \;=\; I_n,$$

which shows that B is invertible with inverse $B^{-1} = S^{-1}A^{-1}S$.

Solution: TRUE. If A is similar to B, then B is similar to A. Let S be the invertible matrix such that $B = SAS^{-1}$. Then S, A, and S^{-1} are invertible, so B is a product of invertible matrices which must be invertible.

(b) (3 points) For every subspace V of $\mathbb{R}^n$, the orthogonal projection $\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ onto V is an orthogonal transformation.

Solution: FALSE. For a counterexample, let $V = \text{span}(\vec{e}_1) \subseteq \mathbb{R}^2$, so V is a subspace of $\mathbb{R}^2$, and consider orthogonal projection $\text{proj}_V \colon \mathbb{R}^2 \to \mathbb{R}^2$ onto V. Then $\text{im}(\text{proj}_V) = V \neq \mathbb{R}^2$, so proj_V is not surjective, hence not invertible, which means proj_V is not an orthogonal transformation. (In fact, any orthogonal projection which is not the identity transformation will fail to be invertible, and thus will not be an orthogonal transformation.)

(c) (3 points) For every inner product space V, if $\vec{x}$ is a vector in V such that $\langle \vec{x}, \vec{y} \rangle = 0$ for every $\vec{y} \in V$, then $\vec{x} = \vec{0}$.

Solution: TRUE. Let V be an inner product space, and let $\vec{x}$ be a vector in V such that $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{y} \in V$. Then in particular $\langle \vec{x}, \vec{x} \rangle = 0$, which implies $\vec{x} = \vec{0}$ since inner products are positive definite.

Solution: TRUE. Note that V is a subspace of itself. If $\vec{x} \in V$ is such that $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{y} \in V$, then $\vec{x} \in V^\perp$ by definition of the orthogonal complement. Then $\vec{x} \in V^\perp \cap V = \{\vec{0}\}$ implies $\vec{x} = \vec{0}$.

(Problem 2, Continued).

(d) (3 points) The polynomial functions $p(t) = 2t + 1$ and $q(t) = 2t - 1$ are orthogonal in the inner product space $\mathcal{P}_2$ of polynomials of degree at most 2 with inner product given by $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$ for all $f, g \in \mathcal{P}_2$.

Solution: FALSE. By definition, p and q are orthogonal if $\langle p, q\rangle = 0$. But

$$\langle p, q\rangle = \int_0^1 (2t+1)(2t-1)\,dt \;=\; \int_0^1 (4t^2 - 1)\,dt \;=\; \left(\frac{4}{3}t^3 - t\right)\Big]_0^1 \;=\; \frac{1}{3} \;\neq\; 0,$$

so p and q are not orthogonal.

(e) (3 points) For every $n \times k$ matrix A and vector $\vec{b} \in \mathbb{R}^n$, if the columns of A form an orthonormal list of vectors, then $A^\top \vec{b}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$.

Solution: TRUE. Let $A \in \mathbb{R}^{n\times k}$ and $\vec{b} \in \mathbb{R}^n$, and suppose the columns of A are orthonormal, so $A^\top A = I_k$. Then

$$A^\top A(A^\top \vec{b}) \;=\; I_k A^\top \vec{b} \;=\; A^\top \vec{b},$$

so $A^\top \vec{b}$ is a solution of the normal equation $A^\top A\vec{x} = A^\top \vec{b}$ of the linear system $A\vec{x} = \vec{b}$, and is thus a least-squares solution of $A\vec{x} = \vec{b}$.

Solution: TRUE. If $A \in \mathbb{R}^{n\times k}$ has orthonormal columns, then the standard matrix of $\mathrm{proj}_{\mathrm{im}(A)} \colon \mathbb{R}^n \to \mathbb{R}^n$ is $AA^\top$. Then

$$A(A^\top \vec{b}) = (AA^\top)\vec{b} = \mathrm{proj}_{\mathrm{im}(A)}(\vec{b})$$

implies $A^\top \vec{b}$ is a least squares solution to $A\vec{x} = \vec{b}$.

3. Let W be the subspace of $\mathbb{R}^4$ consisting of all solutions of the linear system

$$\begin{aligned} x_1 - x_2 &= 0, \\ x_1 + 2x_3 - x_4 &= 0. \end{aligned}$$

(a) (4 points) Find a 4×4 matrix A such that $\ker(A) = W$. *(No justification required).*

Solution: For instance, we could let $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) (4 points) Find a basis of $W^\perp$. *(No justification required).*

Solution: Since $W = \ker(A)$, we have $W^\perp = \ker(A)^\perp = \operatorname{im}(A^\top)$. One basis is

$$\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right).$$

(c) (4 points) Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^4$ such that the $\mathcal{B}$-matrix of the orthogonal projection onto W in $\mathbb{R}^4$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution: In order for the matrix of proj_W relative to $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4)$ to be as given, we need $(\vec{b}_1, \vec{b}_2)$ to be a basis of W, and $(\vec{b}_3, \vec{b}_4)$ to be a basis of $W^\perp$.

Thus, for instance, we may let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right)$.

4. Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by $T(A) = MA$ for all $A \in \mathbb{R}^{2\times 2}$. Also let

$$\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad \mathcal{B} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right),$$

so that $\mathcal{E}$ and $\mathcal{B}$ are ordered bases of $\mathbb{R}^{2\times 2}$ (you do not have to prove that they are bases).

(a) (3 points) Find the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{E}}$ which changes from $\mathcal{B}$-coordinates to $\mathcal{E}$-coordinates. *(No justification required).*

Solution: Writing $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4)$, we have

$$S_{\mathcal{B}\to\mathcal{E}} \;=\; \begin{bmatrix} | & | & | & | \\ [\vec{b}_1]_{\mathcal{E}} & [\vec{b}_2]_{\mathcal{E}} & [\vec{b}_3]_{\mathcal{E}} & [\vec{b}_4]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(b) (5 points) Find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T.

Solution: Using the fact that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ for all $a, b, c, d \in \mathbb{R}$, and again writing $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4)$, we see that

$$[T]_{\mathcal{B}} \;=\; \begin{bmatrix} | & | & | & | \\ [T(\vec{b}_1)]_{\mathcal{B}} & [T(\vec{b}_2)]_{\mathcal{B}} & [T(\vec{b}_3)]_{\mathcal{B}} & [T(\vec{b}_4)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) (3 points) Find the $\mathcal{B}$-coordinates of $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$.

Solution: Once again writing $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4)$, we have

$$T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \;=\; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \;=\; 1\vec{b}_1 + 1\vec{b}_2 + 0\vec{b}_3 + 0\vec{b}_4,$$

so the $\mathcal{B}$-coordinate vector of $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

5. Let

$$A = \begin{bmatrix} | & | & 1 \\ \vec{a}_1 & \vec{a}_2 & -1 \\ & & 0 \\ | & | & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & | \\ 0 & 0 & \\ 1/2 & 0 & \vec{u}_3 \\ \sqrt{3}/2 & 0 & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & q & r \\ p & 3 & s \\ 0 & 0 & t \end{bmatrix}}_{R}$$

be the QR-factorization of the 4×3 matrix A, where $\vec{a}_1, \vec{a}_2, \vec{u}_3 \in \mathbb{R}^4$ and $p, q, r, s, t \in \mathbb{R}$.

(a) (2 points) What are all the possible values of p and q that are consistent with the given information? *(No justification necessary).*

Solution: $p = 0$, $q \in \mathbb{R}$.

(b) (4 points) Find $\vec{u}_3$.

Solution: Write $\vec{a}_3$ for the third column of A, and $\vec{u}_1$, $\vec{u}_2$ for the first two columns of Q. Then $\vec{u}_3$ is the normalization of

$$\vec{a}_3 - (\vec{a}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{a}_3 \cdot \vec{u}_2)\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \vec{0} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Since $-\vec{e}_2$ already has length 1, we conclude that $\vec{u}_3 = -\vec{e}_2$.

(c) (5 points) Assuming that $\vec{a}_1 \cdot \vec{a}_2 = 0$, find the values of p, q, r, s, and t.

Solution: We already know $p = 0$, since R is upper triangular, and since $\vec{a}_1 \cdot \vec{a}_2 = 0$ we know $q = 0$ as well. Then solving the equation

$$\vec{a}_3 = r\vec{u}_1 + s\vec{u}_2 + t\vec{u}_3$$

using our solution from part (b), we get $r = 0$, $s = 1$, and $t = 1$.

(d) (4 points) Letting $C = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$, find a vector $\vec{b} \in \mathbb{R}^4$ such that the solution set of $C\vec{x} = \vec{b}$ is the set of least-squares solutions of $C\vec{x} = \vec{e}_3$.

Solution: The set of least-squares solutions of $C\vec{x} = \vec{e}_3$ is the solution set of the linear system $C\vec{x} = \operatorname{proj}_{\operatorname{im}(C)}(\vec{e}_3)$. Thus we can let

$$\vec{b} = \operatorname{proj}_{\operatorname{im}(C)}(\vec{e}_3) = (\vec{e}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_3 \cdot \vec{u}_2)\vec{u}_2 = \frac{1}{2}\vec{u}_1 + 0\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/4 \\ \sqrt{3}/4 \end{bmatrix}.$$

6. Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B} = (b_1, b_2, b_3)$ be an ordered basis of V such that for each $1 \le i, j \le 3$, the number in the ith row and jth column of the table below is $\langle b_i, b_j \rangle$.

	b_1	b_2	b_3
b_1	2	-2	0
b_2	-2	6	3
b_3	0	3	9

Also let (u_1, u_2, u_3) be the orthonormal list obtained by applying the Gram-Schmidt process to $\mathcal{B}$, and let $W = \operatorname{span}(b_1, b_3)$ be the subspace of V spanned by b_1 and b_3.

(a) (5 points) Find u_1 and u_2 as linear combinations of b_1 and b_2.

Solution: First, we have $\vec{u}_1 = \frac{b_1}{\|b_1\|} = \frac{b_1}{\sqrt{\langle b_1, b_1 \rangle}} = \frac{1}{\sqrt{2}} b_1$. Then u_2 is the normalization of

$$b_2 - \frac{\langle b_2, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 \;=\; b_2 - \frac{-2}{2} b_1 \;=\; b_1 + b_2.$$

Since $\|b_1 + b_2\|^2 = \langle b_1 + b_2, b_1 + b_2 \rangle = \langle b_1, b_1 \rangle + 2\langle b_1, b_2 \rangle + \langle b_2, b_2 \rangle = 4$, we get

$$u_2 \;=\; \frac{b_1 + b_2}{\|b_1 + b_2\|} \;=\; \frac{1}{2} b_1 + \frac{1}{2} b_2.$$

(b) (4 points) Find the orthogonal projection of b_2 onto W as a linear combination of b_1 and b_3.

Solution: Since $\vec{b}_1$ and $\vec{b}_3$ are orthogonal, the orthogonal projection of $\vec{b}_2$ onto W is

$$\operatorname{proj}_W(\vec{b}_2) \;=\; \frac{\langle \vec{b}_2, \vec{b}_1 \rangle}{\langle \vec{b}_1, \vec{b}_1 \rangle} \vec{b}_1 + \frac{\langle \vec{b}_2, \vec{b}_3 \rangle}{\langle \vec{b}_3, \vec{b}_3 \rangle} \vec{b}_3 \;=\; \frac{-2}{2} \vec{b}_1 + \frac{3}{9} \vec{b}_3 \;=\; -\vec{b}_1 + \frac{1}{3} \vec{b}_3.$$

(c) (3 points) Find the $\mathcal{B}$-coordinates of the vector in W that is closest to b_2.

Solution: The vector in W that is closest to $\vec{b}_2$ is $\operatorname{proj}_W(\vec{b}_2)$. Therefore, using our answer from part (b), we find its $\mathcal{B}$-coordinates are $[\operatorname{proj}_W(\vec{b}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1/3 \end{bmatrix}$.

7. Let B be an invertible $n \times n$ matrix, and let $A = B^\top B$. For each $1 \le i, j \le n$, let a_{ij} denote the (i, j)-entry of A.

(a) (2 points) Show that A is symmetric.

Solution: $A^\top = (B^\top B)^\top = B^\top (B^\top)^\top = B^\top B = A$, so A is symmetric.

(b) (3 points) Show that the diagonal entries of A are positive.

Solution: Since B is invertible and no invertible matrix can have a zero column, we know $B\vec{e}_i \cdot B\vec{e}_i > 0$ for each $1 \le i \le n$ by positive-definiteness of the dot product. But then by the definition of matrix multiplication, for each $1 \le i \le n$ we have that

$$a_{ii} \;=\; \vec{e}_i^\top B^\top B\vec{e}_i \;=\; B\vec{e}_i \cdot B\vec{e}_i \;>\; 0.$$

(c) (5 points) Prove that $a_{ij}^2 \le a_{ii}a_{jj}$ for all integers $i, j \in \{1, \dots, n\}$.

Solution: Note that for each $1 \le i, j \le n$, we have $a_{ij} = B\vec{e}_i \cdot B\vec{e}_j$. Thus for each $1 \le i, j \le n$ we see that

$$a_{ij}^2 \;=\; (B\vec{e}_i \cdot B\vec{e}_j)^2 \;\le\; \|B\vec{e}_i\|^2 \|B\vec{e}_j\|^2 \;=\; (B\vec{e}_i \cdot B\vec{e}_i)(B\vec{e}_j \cdot B\vec{e}_j) \;=\; a_{ii}a_{ij},$$

where the inequality in the line above is just the Cauchy-Schwarz inequality applied to the vectors $B\vec{e}_i$ and $B\vec{e}_j$.

(d) (2 points) Give an example of an invertible symmetric matrix S that is *not* of the form $C^\top C$ for any invertible matrix C.

Solution: In part (b) we proved that every matrix of the form $C^\top C$, where C is an invertible matrix, has positive diagonal entries. So to give an example of an invertible symmetric matrix S that is *not* of the form $C^\top C$ for some invertible matrix C, we simply need to find one that does not have positive diagonal entries. For instance, $-I_2$ is such an example.

8. Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_k)$ be an orthonormal list of vectors in $\mathbb{R}^n$, let $B = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix}$ be the $n \times k$ matrix whose columns are the vectors in $\mathcal{B}$, and let $V = \text{Span}(\mathcal{B})$. Also let M be an orthogonal $k \times k$ matrix, and let $C = BM$.

(a) (7 points) Prove that the list $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_k)$ of column vectors of C is an orthonormal basis of V.

Solution: For each $1 \le i \le k$, we have

$$\vec{c}_i \;=\; C\vec{e}_i \;=\; (BM)\vec{e}_i \;=\; B(M\vec{e}_i) \;\in\; \text{im}(B) \;=\; V,$$

so each $\vec{c}_i \in V$ and thus $\text{Span}(\mathcal{C}) \subseteq V$. Next, we use the fact that for any matrix A, the columns of A are orthonormal if and only if $A^\top A$ is an identity matrix. Using this, since the columns of both B and of M form orthonormal lists,

$$C^\top C \;=\; (BM)^\top BM \;=\; M^\top B^\top BM \;=\; M^\top M \;=\; I_k,$$

which shows that the columns of C are orthonormal as well, and thus in particular are linearly independent. Therefore $\mathcal{C}$ is an orthonormal basis of $\text{Span}(\mathcal{C})$, but since $\dim(\text{Span}(\mathcal{C})) = k = \dim(V)$ and $\text{Span}(\mathcal{C}) \subseteq V$, we have $\text{Span}(\mathcal{C}) = V$.

(b) (4 points) With $\mathcal{B}$ and $\mathcal{C}$ as above, prove that M is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$; that is, prove that $M = S_{\mathcal{C}\to\mathcal{B}}$.

Solution: For each $1 \le i \le k$, we have $B(M\vec{e}_i) = (BM)\vec{e}_i = \vec{c}_i$, which implies that $M\vec{e}_i = [\vec{c}_i]_{\mathcal{B}}$. Since $S_{\mathcal{C}\to\mathcal{B}} = \begin{bmatrix} | & & | \\ [\vec{c}_1]_{\mathcal{B}} & \cdots & [\vec{c}_k]_{\mathcal{B}} \\ | & & | \end{bmatrix}$, it follows that $S_{\mathcal{C}\to\mathcal{B}} = M$.

Solution: We know that for all $\vec{x} \in V$, $B[\vec{x}]_{\mathcal{B}} = C[\vec{x}]_{\mathcal{C}} = BM[\vec{x}]_{\mathcal{C}}$. Therefore, using $B^\top B = I_k$, we get

$$[\vec{x}]_{\mathcal{B}} \;=\; B^\top B[\vec{x}]_{\mathcal{B}} \;=\; B^\top BM[\vec{x}]_{\mathcal{C}} \;=\; M[\vec{x}]_{\mathcal{C}}.$$

Since $S_{\mathcal{C}\to\mathcal{B}}$ is the unique matrix S such that $S[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$ for all $\vec{x} \in V$, we conclude that $M = S_{\mathcal{C}\to\mathcal{B}}$.

Solution: For each $1 \le i \le k$, we have $B[\vec{c}_i]_{\mathcal{B}} = \vec{c}_i = C\vec{e}_i = BM\vec{e}_i$. Therefore, using $B^\top B = I_k$, we get

$$[\vec{c}_i]_{\mathcal{B}} \;=\; B^\top B[\vec{c}_i]_{\mathcal{B}} \;=\; B^\top BM\vec{e}_i \;=\; M\vec{e}_i$$

for each $1 \leq i \leq k$. Since $S_{\mathcal{C}\to\mathcal{B}} = \begin{bmatrix} | & & | \\ [\vec{c}_1]_{\mathcal{B}} & \cdots & [\vec{c}_k]_{\mathcal{B}} \\ | & & | \end{bmatrix}$, it follows that $S_{\mathcal{C}\to\mathcal{B}} = M$.