A. 1-1

20. Consider the linear system

$$\left| \begin{array}{l} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{array} \right|,$$

where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & k^2-5 & k \end{array}\right]$$

$$\begin{array}{c} \\ -I \\ -I \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & k^2-5 & k \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right] \begin{array}{c} -II \\ \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc} 1 & 0 & -3 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right]$$

Case 1. System has infinitely many solutions

$\Rightarrow k^2-4=0$ and $k-2=0$

$(k-2)(k+2)=0$

$\Rightarrow k=2$

Case 2, System has no solution.

$\Rightarrow k^2-4=0$ but $k-2\neq 0$

$\Rightarrow k=-2$

Case 3. System has one solution

equivalent to negation of: (System has no solution or infinitely many solutions)

Hence $k\neq\pm 2$

(k be any value in $(-\infty,-2)\cup(-2,2)\cup(2,+\infty)$)

32. Find a polynomial of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, p)$, $(2, q)$, $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

By the question's meaning

To ensure the polynomial exists

$$\begin{cases} t=1: & a+b+c=p \\ t=2: & a+2b+4c=q \\ t=3: & a+3b+9c=r \end{cases}$$

system need to have solution.

that is, the system is consistent.

the augmented matrix of the system is

$$A = \left[\begin{array}{ccc:c} 1 & 1 & 1 & p \\ 1 & 2 & 4 & q \\ 1 & 3 & 9 & r \end{array}\right] \begin{array}{l} \\ -\mathrm{I} \\ -\mathrm{I} \end{array}$$

Now we calculate rref(A)

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & p \\ 0 & 1 & 3 & q-p \\ 0 & 2 & 8 & r-p \end{array}\right] \begin{array}{l} -\mathrm{II} \\ \\ -2\times\mathrm{II} \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & p-q+p \\ 0 & 1 & 3 & q-p \\ 0 & 0 & 2 & r-p-2q+2p \end{array}\right] \begin{array}{l} \\ \\ \div 2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & p-q+p \\ 0 & 1 & 3 & q-p \\ 0 & 0 & 1 & \frac{r+p-2q}{2} \end{array}\right] \begin{array}{l} +2\times\mathrm{III} \\ -3\times\mathrm{III} \\ \\ \end{array}$$

$$\Rightarrow \mathrm{rref}(A) = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 3p-3q+r \\ 0 & 1 & 0 & -\frac{5}{2}p+4q-\frac{3}{2}r \\ 0 & 0 & 1 & \frac{1}{2}p-q+\frac{1}{2}r \end{array}\right]$$

Therefore A is always consistent

with solution $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3p-3q+r \\ -\frac{5}{2}p+4q-\frac{3}{2}r \\ \frac{1}{2}p-q+\frac{1}{2}r \end{bmatrix}$

Hence such polynomial always exists for all value of p, q, r.

34. Find all the polynomials $f(t)$ of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graphs run through the points (1, 1) and (2, 0), such that $\int_1^2 f(t)\,dt = -1$.

Since graphs run through (1,1) and (2,0)

We have $\begin{cases} a+b+c=1 \\ a+2b+4c=0 \end{cases}$

And by $\int_1^2 f(t)\,dt = -1$

$$\int_1^2 a+bt+ct^2 = -1$$

$$\left[at+\tfrac{1}{2}bt^2+\tfrac{1}{3}ct^3\right]_1^2 = -1$$

$$2a+2b+\tfrac{8}{3}c-a-\tfrac{1}{2}b-\tfrac{1}{3}c=-1$$

$$a+\tfrac{3}{2}b+\tfrac{7}{3}c=-1$$

Therefore we have the linear system:

$$\begin{cases} a+b+c=1 \\ a+2b+4c=0 \\ a+\frac{3}{2}b+\frac{7}{3}c=-1 \end{cases}$$

The polynomial we are looking for are the ones with a,b,c as coefficients such that the linear system is consistent.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 0 \\ 1 & \frac{3}{2} & \frac{7}{3} & -1 \end{array}\right] \begin{array}{c} \\ -I \\ -I \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & \frac{1}{2} & \frac{4}{3} & -2 \end{array}\right] \begin{array}{l} -\mathrm{II} \\ \\ \times 2 - \mathrm{II} \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -\frac{1}{3} & -3 \end{array}\right] \begin{array}{l} \\ \\ \times (-3) \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 9 \end{array}\right] \begin{array}{l} +2\times \mathrm{III} \\ -3\times \mathrm{III} \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -28 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

Therefore the only polynomial satisfying the requirement is $f(t) = 20 - 28t + 9t^2$

44. Find a system of linear equations with three unknowns whose solutions are the points on the line through (1, 1, 1) and (3, 5, 0).

Let A denote $(1,1,1)$, B denote $(3,5,0)$

then $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ is the direction vector of the line.

And since $(1,1,1)$ is a point of the line, Let $t \in \mathbb{R}$ be arbitrary.

$(1+2t, 1+4t, 1-t)$ is a point of the line, determined by the direction vector and the point.

Therefore the line can be described as $\begin{cases} x = 1+2t \\ y = 1+4t \\ z = 1-t \end{cases}$,

so $t = \frac{x-1}{2} = \frac{y-1}{4} = 1-z$

so $2x-2 = y-1$, $x-1 = 2-2z$.

Hence the linear system that can be used to describe the line is:

$$\begin{cases} 2x - y \quad = 3 \\ x \quad + 2z = 3 \end{cases}$$

1-2

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work.

12. $$\left| \begin{array}{l} 2x_1 - 3x_3 + 7x_5 + 7x_6 = 0 \\ -2x_1 + x_2 + 6x_3 - 6x_5 - 12x_6 = 0 \\ x_2 - 3x_3 + x_5 + 5x_6 = 0 \\ -2x_2 + x_4 + x_5 + x_6 = 0 \\ 2x_1 + x_2 - 3x_3 + 8x_5 + 7x_6 = 0 \end{array} \right|$$

the augmented matrix is :

$$A = \begin{array}{c} \left[\begin{array}{cccccc|c} 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ -2 & 1 & 6 & 0 & -6 & -12 & 0 \\ 0 & 1 & -3 & 0 & 1 & 5 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & -3 & 0 & 8 & 7 & 0 \end{array}\right] \end{array} \begin{array}{l} \times(-\frac{1}{2}) \\ \\ \\ \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 1 & -\frac{1}{2} & -3 & 0 & 3 & 6 & 0 \\ 2 & 0 & -3 & 0 & 7 & 7 & 0 \\ 0 & 1 & -3 & 0 & 1 & 5 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & -3 & 0 & 8 & 7 & 0 \end{array}\right] \begin{array}{l} \\ -2\times I \\ \\ \\ -2\times I \end{array}$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 1 & -\frac{1}{2} & -3 & 0 & 3 & 6 & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 1 & -3 & 0 & 1 & 5 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 & 2 & -5 & 0 \end{array}\right] \begin{array}{l} +\frac{1}{2}\times II \\ \\ -II \\ +2\times II \\ -2\times II \end{array}$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{3}{2} & 0 & \frac{7}{2} & \frac{7}{2} & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 0 & -6 & 0 & 0 & 10 & 0 \\ 0 & 0 & 6 & 1 & 3 & -9 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \end{array}\right] \begin{array}{l} \\ \\ \times -\frac{1}{6} \\ \\ +\frac{1}{2}\times III \end{array}$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{3}{2} & 0 & \frac{7}{2} & \frac{7}{2} & 0 \\ 0 & 1 & 3 & 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 6 & 1 & 3 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +\frac{3}{2}\times III \\ -3\times III \\ \\ -6\times III \\ \\ \end{array}$$

$$\Rightarrow \operatorname{rref}(A) = \left[\begin{array}{cccccc:c} 1 & 0 & 0 & 0 & \frac{7}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_5 = t$, $x_6 = r$ where t, r is any real number.

Therefore the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2}t - r \\ -t \\ \frac{5}{3}r \\ -3t - r \\ t \\ r \end{bmatrix}$$

$$= t\begin{bmatrix} -\frac{7}{2} \\ -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ \frac{5}{3} \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

36. The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

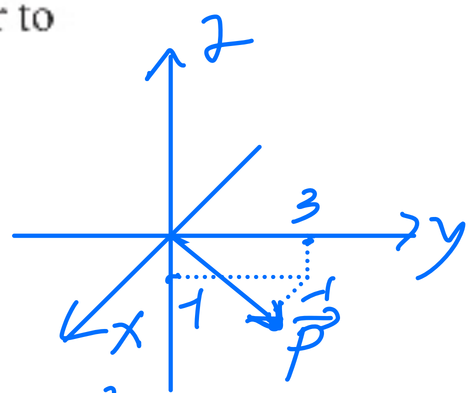
$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$.

Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.



Let $\vec{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

By definition of perpendicular,

$$\vec{p} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 0$$

By definition of dot product,

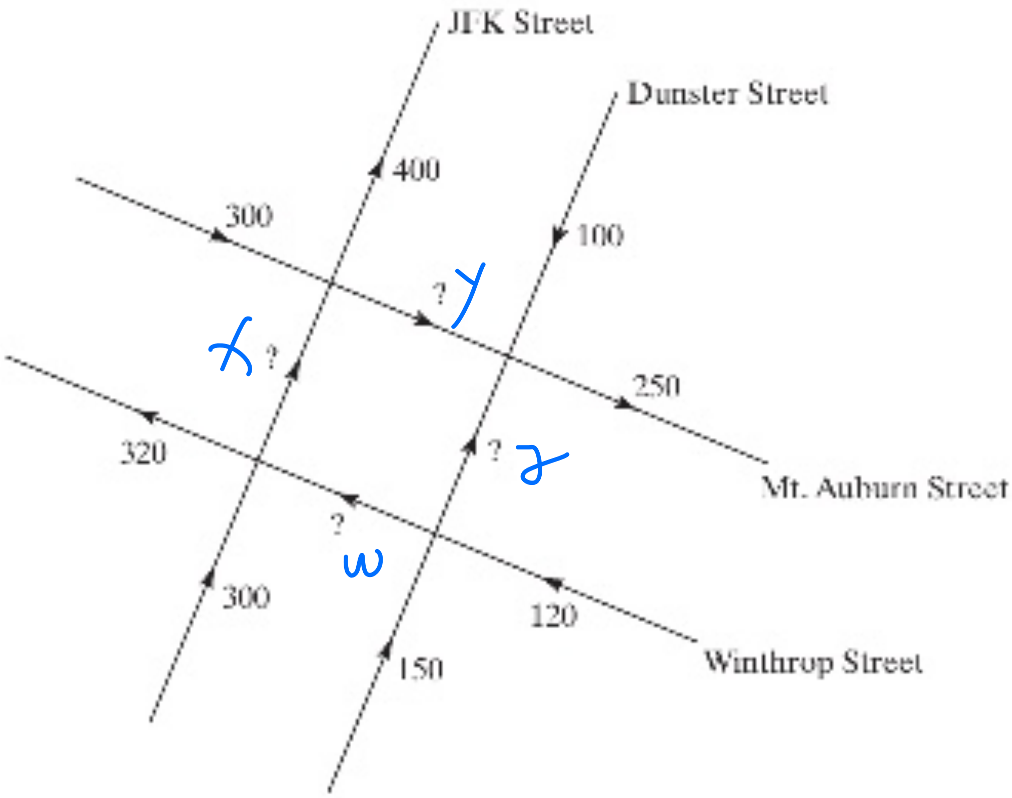
$$a + 3b - c = 0$$

Therefore All $\vec{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying this linear equation is perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

Let $b = t$, $c = r \in \mathbb{R}$

Hence Any vector $\vec{p} = \begin{bmatrix} -3t+r \\ t \\ r \end{bmatrix}$ where t, r are arbitrary real numbers is perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

44. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

Label traffic volume of the four location as w, x, y, z as shown in picture, respectively.

Then we have

$$\begin{cases} 300 + x = 400 + y \\ 300 + w = 320 + x \\ 150 + 120 = w + z \\ y + z + 100 = 250 \end{cases}$$

Rearrange the linear system in order of variables then we have

$$\begin{cases} x - y = 100 \\ w - x = 20 \\ w + z = 270 \\ y + z = 150 \end{cases}$$

So the augmented matrix is

$$\left[\begin{array}{cccc:c} 0 & 1 & -1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 1 & 270 \\ 0 & 0 & 1 & 1 & 150 \end{array}\right] \quad \text{reorder}$$

$$\Rightarrow \begin{array}{c} \left[\begin{array}{cccc:c} 1 & -1 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & 0 & 1 & 270 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ -I \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc:c} 1 & -1 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 1 & 0 & 1 & 250 \end{array}\right] \begin{array}{l} +II \\ \\ \\ -II \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc:c} 1 & 0 & -1 & 0 & 120 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 1 & 1 & 150 \end{array}\right] \begin{array}{l} +III \\ +III \\ \\ -III \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc:c} 1 & 0 & 0 & 1 & 270 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = t \in \mathbb{R}$

By the question's scene

$t \geq 0$ and $t \leq 250$.

The solution to this linear system is

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 270-t \\ 250-t \\ 150-t \\ t \end{bmatrix}$$

Notice that w, x, y, z are all greater than or equal to 0, so $t \leq 150$

together with $0 \leq t \leq 250$ we then know $0 \leq t \leq 150$

Hence the highest and lowest traffic volume for location w are 270 and 120 respectively;

for location x are 250 and 100 respectively;

for location y and z are both 150 and 0 respectively.

Problem 1. Decide whether the following statements are true or false. *Briefly* justify your answers.[1]

(a) 2 is even or 3 is odd.
(b) If the Riemann Hypothesis is true, then 217 is not a prime number.
(c) $\frac{d}{dx}(x^2) = 2x$ if and only if $\tan(\pi/6) = \sqrt{3}$.

---

[1] Don't work too hard here. For instance, sufficient justification for claiming that "If 13 is prime then $\sqrt{2}$ is rational" is false could be: "FALSE by the meaning of 'if…then', since '13 is prime' is true, but '$\sqrt{2}$ is rational' is false." (In particular, you would *not* need to prove that 13 is prime or that $\sqrt{2}$ is irrational!)

(d) If the set of even prime numbers is infinite, then 10 is even and 10^{10} is odd.[2]
(e) If every right triangle in $\mathbb{R}^2$ has two acute angles, then every real number has a positive cube root.

(a) true.
"2 is even" is true
"3 is odd" is true.
Since "2 is even or 3 is odd" is true whenever any one of ①, ② is true, it is true.

(b) true.
It is true that 217 is not a prime number. ($217 = 7 \times 31$), so by the meaning of "if then", no matter whether the Riemann Hypothesis is true, the proposition is true.

(c) false.
"$\frac{d}{dx}(x^2) = 2x$" is true, "$\tan(\frac{\pi}{6}) = \sqrt{3}$" is false ($= \frac{\sqrt{3}}{3}$)
By the meaning of "if and only if", the proposition is true only when both sides are true. Since "$\tan(\frac{\pi}{6}) = \sqrt{3}$" is false, the proposition is false.

(d) true.

"The set of even prime number is infinite" is false since the only even prime number is 2 (all other even number except 0 has at least 2 as a factor).

By meaning of "if - then" the proposition is always true since the premise is false.

(e) true.

"Every right triangle in $\mathbb{R}^2$ has 2 acute angles" is true, and "every real number has a cube root" is also true. By the meaning of "if-then", the proposition is true

Problem 2.

(a) Let $P(x)$ be a statement whose truth value depends on x. An *example* is a value of x that makes $P(x)$ true, and a *counterexample* is a value of x that makes $P(x)$ false. Fill in the blank spaces with "is true", "is false", or "nothing" as appropriate:

	"$\forall x, P(x)$"	"$\exists x$ s.t. $P(x)$"
An example proves	nothing	is true
A counterexample proves	is false	nothing

Determine whether each of the given statements is true or false, and briefly justify your answer (as you did for Problem 1).

(b) Every prime number is even or odd.
(c) Every prime number is even or every prime number is odd.
(d) There exists $n \in \mathbb{Z}$ such that for every $x \in \mathbb{R}$, $n < x$.
(e) For every $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n < x$.
(f) Some squares are rectangles.
(g) For every nonnegative real number a, there exists a unique[3] real number x such that $x^2 = a$.

[2] By convention, the connectives "and" and "or" bind more strongly than do the "if … then" and "if and only if" connectives. This means that you should read the statement in Problem 1(d) as "If (the set of prime numbers is finite), then (10 is even and 10^{10} is odd)" rather than "(If the set of prime numbers is finite, then 10 is even) and (10^{10} is odd)". *Negation*, which is signified by the word "not," binds even more strongly than "and" and "or" do.

[3] The statement "there exists unique $x \in X$ such that $P(x)$" means that there is one and only one element in the set X having property P. For those who like fancy symbolisms, this is sometimes abbreviated "$\exists! x \in X$ s.t. $P(x)$."

(b) True.

Every integer is either even or odd.
And every prime is an integer.
Then we know every prime is either even or odd.

(c) False.

By counterexample of 2 as a prime but is even, "every prime number is odd" is false.
By counterexample of 3 as a prime but is odd, "every prime number is even" is false.
Since all two propositions of the "or" expression is false, the overall proposition is false.

(d) False.

Pf. Let n be an integer such that for all $x \in \mathbb{R}$, $n < x$, seeking contradiction,

Consider $x = n-1$, then since $0 > 1$, $n > n-1$, $n > x$, contradicts with the assumption.

Therefore there does not exist such n.

(e) True.

Pf. Let x be an arbitrary real number.
Want to show there exists $n \in \mathbb{Z}$ such that $n < x$

Consider $\lfloor x \rfloor + 1$.

Since $\lceil x \rceil - \lfloor x \rfloor = 1$ and $\lfloor x \rfloor < x < \lceil x \rceil$,
We have $\lfloor x \rfloor < x < \lfloor x \rfloor + 1$.
Therefore $\lfloor x \rfloor + 1 > x$
Since x is arbitrary, we have proved the proposition.

(f) True

By definition all squares are rectangles,
hence some squares are rectangles.

(g) False

Consider $4 \in \mathbb{R}^+$
We have $2^2 = 4$ and $(-2)^2 = 4$
There are two square roots, contradicts with uniqueness
Therefore the proposition is false.

Problem 3. Formulate the negation of each of the statements below in a meaningful way (these statements have been recycled from Problems 1 and 2). Note: just writing "It is not the case that ..." before each statement will not receive credit, as that does not help the reader understand the meaning of the negation. (No justification is needed – you may just write the negation).

(a) 2 is even or 3 is odd.
(b) If the Riemann Hypothesis is true, then 217 is not a prime number.
(c) $\frac{d}{dx}(x^2) = 2x$ if and only if $\tan(\pi/6) = \sqrt{3}$.
(d) If the set of even prime numbers is infinite, then 10 is even and 10^{10} is odd.
(e) If every right triangle in $\mathbb{R}^2$ has two acute angles, then every real number has a positive cube root.
(f) There exists $n \in \mathbb{N}$ such that for every $x \in \mathbb{R}$, $x < n$.
(g) Some squares are rectangles.

(a) 2 is not even and 3 is not odd.

(b) The Riemann Hypothesis is true and 217 is a prime number.

(c) Either $\frac{d}{dx}(x^2) \neq 2x$ or $\tan(\frac{\pi}{6}) = \sqrt{3}$ (or both).

(d) The set of even prime numbers is infinite, and either 10 is not even or 10^{10} is not odd or both.

(e) Every right triangle in $\mathbb{R}^2$ has two acute angles, and there exists some real number that does not have a positive cube root.

(f) For all $n \in \mathbb{N}$ there exists $x \in \mathbb{R}$ such that $x \geq n$.

(g) All squares are not rectangles.

Problem 4. Write both the converse and the contrapositive of the following "if-then" statements.

(a) If something can think, then it exists[4].

(b) If p is an irrational number, then p^2 is an irrational number.

(c) If $n > 2$ is a natural number such that the Collatz sequence beginning with n does not eventually reach 1, then $n^2 + 1$ is prime.

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[4]You may recognize this as a paraphrase of "I think, therefore I am", which is itself a translation of "*Cogito, ergo sum*", an axiom used by the 17th century mathematician in his work of philosophy *Discourse on the Method.*

(a) converse : If something exists then it can think

contrapositive : If something does not exist, then it cannot think.

(b) converse : If p^2 is an irrational number, then p is an irrational number.

contrapositive : If p^2 is not an rational number, then p is not an rational number.

(c) converse: If n^2+1 is prime then n is a natural number that is greater than 2 such that the Collatz sequence beginning with n does not eventually reach 1.

contrapositive ; If n^2+1 is not prime number then n is not greater than 2, or it is not a natural number, or it is a natural number such that the Collatz sequence beginning with it eventually reaches 1.

(or any combination of these three propositions)

A *set* is a container with no distinguishing feature other than its contents. The objects contained in a set are called the *elements* of the set. We write $a \in S$ to signify that the object a is an element of the set S. The number of elements in a set S is called the *cardinality* of the set, and is denoted by $|S|$.

Since a set has no distinguishing feature other than its contents, there is a unique set containing no elements which is called the *empty set* and is denoted $\emptyset$. Some other very common sets are the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}$ of all integers, the set $\mathbb{Q}$ of all rational numbers, the set $\mathbb{R}$ of all real numbers, and the set $\mathbb{C}$ of all complex numbers.

There are two important ways to specify a set.

- *Enumeration.* One can list the contents of the set, in which case the set is denoted by enclosing the list in curly braces. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- *Comprehension.* One can describe the contents of the set by a property of its elements. If $P(a)$ is a property of the object a, then the set of all objects a such that $P(a)$ is true is denoted by $\{a \mid P(a)\}$, or equivalently $\{a : P(a)\}$. For example,

$$\mathbb{Q} = \{x \mid x = \tfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

  Comprehension can also be used together with functions. For instance, $\{n^2 : n \in \mathbb{N}\}$ is the set of all perfect squares, and $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is the set of all reciprocals of natural numbers.

Let X and S be sets. We say that S is a *subset* of X if $a \in S \implies a \in X$ holds for all objects a. We write $S \subseteq X$ to signify that S is a subset of X. This means that S is a set each of whose elements also belongs to X. The subset of X consisting of all elements a of X such that property $P(a)$ holds true is denoted $\{a \in X \mid P(a)\}$ or $\{a \in X : P(a)\}$.

Problem 5.

(a) Give common English descriptions of the following sets:
  (1) $\{n \in \mathbb{N} \mid \text{there exist } a \in \mathbb{N} \text{ such that } n = 2a - 1\}$.
  (2) $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1 \text{ and } a \geq 0\}$.

(b) Use set comprehension notation to give a description of each of the following sets:
  (i) The unit sphere in $\mathbb{R}^3$.
  (ii) The set of all integer multiples of $\sqrt{2}$.

(c) Determine whether each of the following statements is true or false (no justification necessary):

(c)

(i) $\sqrt{2} \in \mathbb{R}$ true
(ii) $\sqrt{2} \subseteq \mathbb{R}$ false
(iii) $\{\sqrt{2}\} \in \mathbb{R}$ false
(iv) $\{\sqrt{2}\} \subseteq \mathbb{R}$ true
(v) $\emptyset \in \mathbb{R}$ false
(vi) $\emptyset \subseteq \mathbb{R}$ true
(vii) $\emptyset \in \emptyset$ false
(viii) $\emptyset \subseteq \emptyset$ true

(a) (1) The set of all odd natural numbers.
(2) All points contained within the right half of the circle in $\mathbb{R}^2$ centered at $(0,0)$ with radius $= 1$, containing boundary.

(b) (i) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$
(ii) $\{\sqrt{2}n \mid n \in \mathbb{Z}\}$

6. Set Operations.

Starting from given sets, we can use set operations to form new sets.

- Given sets X and Y, the *intersection* of X and Y is defined as
$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$
- Given sets X and Y, the *union* of X and Y is defined as
$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}.$$
- Given sets X and Y, the *difference* of X and Y, denoted $X \setminus Y$ or $X - Y$, is the set
$$\{x \in X \mid x \notin Y\}.$$
- Given a set Y inside some larger set X, the *complement* of Y with respect to X, denoted Y^C, is $X \setminus Y$. (The larger set X, sometimes referred to as the *universe*, is often suppressed in the notation).

Problem 6. For each rational number q, let $q\mathbb{N} = \{qm \mid m \in \mathbb{N}\}$, so that we have $q\mathbb{N} \subseteq \mathbb{Q}$.

(a) Use enumeration to describe each of the following sets (listing at least the first six elements of each set, in order from smallest to largest): $\frac{1}{2}\mathbb{N}$, $\frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \cap \frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$, $\frac{1}{2}\mathbb{N} \setminus \frac{1}{3}\mathbb{N}$, and $(3\mathbb{N})^C$ (where the complement is taken inside $\mathbb{N}$).

(b) What is the smallest natural number n such that every set from part (a) is contained in $\frac{1}{n}\mathbb{N}$? (Alternatively, if you think no such n exists, explain why.)

(a) Since $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$

We have $\frac{1}{2}\mathbb{N} = \{\frac{1}{2}m \mid m \in \mathbb{N}\} = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$

Similary $\frac{1}{3}\mathbb{N} = \{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \ldots\}$

$3\mathbb{N} = \{3, 6, 9, 12, 15, 18, \ldots\}$

So $\frac{1}{2}\mathbb{N} \cap \frac{1}{3}\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$

$\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N} = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, \ldots\}$

$\frac{1}{2}\mathbb{N} \setminus \frac{1}{3}\mathbb{N} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \ldots\}$

$(3\mathbb{N})^C = \mathbb{N} \setminus 3\mathbb{N} = \{1, 2, 4, 5, 7, 8, 10, \ldots\}$

(b) It is easy to show that every set in (a) is a subset of $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$, so first we want to show that if such n exists, $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N} \subseteq \frac{1}{n}\mathbb{N}$.

Step ① Let x be an element of $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$,
then $x = \frac{1}{2}m$ for some integer m,
or $x = \frac{1}{3}p$ for some integer p,
so $x = \frac{1}{2}m + \frac{1}{3}p$ for some nonnegative integer m, p
by always picking 0 for one of m, p as its general form.

Step ② Hence $x = \dfrac{3n+2p}{6}$ for some integer m, p.
Since one of m, p must not be 0 (≥ 1), and the other must be 0, we have $3n+2m \in \mathbb{N}$, so it is sure that $x \in \frac{1}{6}\mathbb{N}$

Step ③ Hence $\frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N} \subseteq \frac{1}{6}\mathbb{N}$,
Therefore $n = 6$ satisfies the requirement.

Step ④ Now we show that 6 is smallest by listing a counterexample for every natural number less than 6. For $n = 1, 3, 5$, we have $\frac{1}{2} \in \frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$ not in $\frac{1}{n}\mathbb{N}$. For $n = 2, 4$, we have $\frac{1}{3} \in \frac{1}{2}\mathbb{N} \cup \frac{1}{3}\mathbb{N}$ not in $\frac{1}{n}\mathbb{N}$.

Therefore $\boxed{n = 6}$ is the answer.

Problem 7 (Recreational Problem). [5] According to legend, Abraham Lincoln once said:

> "You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time."

Form an intelligible negation of this statement.

Hint: Sometimes when you are dealing with a complex or potentially ambiguous statement in natural language, you can use logic to diagram the statement and remove any ambiguity. In this case, it will help to use a two-variable predicate, say $F(x,t)$, which intuitively says "you can fool person x at time t." Then Honest Abe's statement can be translated into logic as follows:

$$\exists t \forall x F(x,t) \wedge \exists x \forall t F(x,t) \wedge \neg \forall t \forall x F(x,t).$$

Much better, no?[6] *Now* form the negation.

[5] Recreational problems may come up from time to time and exist for your amusement and edification, but they are optional and will not be graded. Handle with care. These problems are appropriate if (and only if) you need an additional challenge after finishing all of the other problems.

[6] You can probably guess from this that $\wedge$ is the logical symbol for "and" and $\neg$ is the logical symbol for "not". Symbols like this, and also $\forall$ and $\exists$, can safely appear in your scratch work but should never appear in your written proofs, which need to be intelligible to human beings!

Let P denote the set of allpeopel
T denote the set of all time.

and let $F(x,t)$ be a predicate denoting:
you can fool $x \in P$ at $t \in T$.

Then we can translate the first part as:
There exists $t \in T$ that for all $x \in P$, $F(x,t)$.
Use logical symbols for logical operation:
$\exists t \forall x F(x,t)$

Similarly the second part as; $\exists x \forall t F(x,t)$
the third part as: $\neg \forall t \forall x F(x,t)$

And "and", "but" can be translated into "$\wedge$"

Then the statement becomes:

$$(\exists t \forall x\, F(x,t)) \wedge (\exists x \forall t\, F(x,t)) \wedge (\neg \forall t \forall x\, F(x,t)) \quad (1)$$

To negate it:

$$\neg\big((\exists t \forall x\, F(x,t)) \wedge (\exists x \forall t\, F(x,t)) \wedge (\neg \forall t \forall x\, F(x,t))\big) \quad (2)$$

And it is logically equivalent to:

$$\neg(\exists t \forall x\, F(x,t)) \vee \neg\big((\exists x \forall t\, F(x,t)) \wedge (\neg \forall t \forall x\, F(x,t))\big) \quad (3)$$

and then

$$\neg(\exists t \forall x\, F(x,t)) \vee \neg(\exists x \forall t\, F(x,t)) \vee (\forall t \forall x\, F(x,t)) \quad (4)$$

by Demorgan's Law

which is logically equivalent to:

$$(\forall t \exists x\, \neg F(x,t)) \vee (\forall x \exists t\, \neg F(x,t)) \vee (\forall t \forall x\, F(x,t))$$

This is a form which is easy to understand in plain English.

The negation is:

For all the time there are some people you cannot fool, or for some time you can fool no people in the world, or for all time you can fool all people. (You can at least achieve one of the three things).