Math 217 – Midterm 1

Fall 2019

Solutions

Student ID Number: ______________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	12	
4	12	
5	13	
6	12	
7	12	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *injective*

   > **Solution:** The function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

   (b) The list $(\vec{v}_1, \ldots, \vec{v}_m)$ of vectors in the subspace V of $\mathbb{R}^n$ is a *basis* of V

   > **Solution:** The list $(\vec{v}_1, \ldots, \vec{v}_m)$ of vectors in the subspace V of $\mathbb{R}^n$ is a *basis* of V if $(\vec{v}_1, \ldots, \vec{v}_m)$ is linearly independent and spans V.

   (c) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly dependent*

   > **Solution:** The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly dependent* if there exist $c_1, \ldots, c_n \in \mathbb{R}$ that are not all zero such that $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$.

   (d) For vector spaces V and W, the function $T : V \to W$ is an *isomorphism* from V to W

   > **Solution:** The function $T : V \to W$ is an *isomorphism* from V to W if T is a bijective linear transformation.

   > **Solution:** The function $T : V \to W$ is an *isomorphism* from V to W if T is an *invertible* linear transformation, meaning that there is a linear transformation $S : W \to V$ such that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For all 2×2 matrices A and B, if $AB = 0$ then $BA = 0$.

Solution: False. Counterexample: Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad BA = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

(b) (3 points) For every 2×2 matrix A, if $A^2 = I_2$ then $A = I_2$ or $A = -I_2$.

Solution: False. Counterexample: Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $A^2 = I_2$.

(c) (3 points) For all $n \times n$ matrices A and B, we have $\text{rref}(A+B) = \text{rref}(A) + \text{rref}(B)$.

Solution: False. Let $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$\text{rref}(A) = \text{rref}(B) = \text{rref}(A+B) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \text{rref}(A) + \text{rref}(B).$$

(Problem 2, Continued).

(d) (3 points) For all finite-dimensional vectors spaces U, V, and W and for all linear transformations $T : U \to V$ and $S : V \to W$, if T is injective and S is surjective and $\operatorname{im}(T) = \ker(S)$, then $\dim(V) = \dim(U) + \dim(W)$.

Solution: True. Proof:
Since T is injective, $\ker(T) = \{0\}$ and $\dim(\ker T) = 0$. Since S is surjective, $\operatorname{im}(S) = W$ and $\dim(\operatorname{im} S) = \dim(W)$. By Rank-Nullity theorem for T, we have $\dim(U) = \dim(\operatorname{im} T) + \dim(\ker T) = \dim(\operatorname{im} T) = \dim(\ker S)$. By Rank-Nullity theorem for S, we know $\dim(V) = \dim(\operatorname{im} S) + \dim(\ker S) = \dim(W) + \dim(\operatorname{im} T) = \dim(W) + \dim(U)$.

(e) (3 points) For every $m \times n$ matrix A, if the columns of A span $\mathbb{R}^m$ then the rows of A span $\mathbb{R}^n$.

Solution: False. Counterexample: Let A be the 1×2 matrix, $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Then, the columns of A span $\mathbb{R}^1$: $\operatorname{span}(1, 0) = \operatorname{span}(1) = \mathbb{R}$, but the rows of A does not span $\mathbb{R}^2$. This is because $\mathbb{R}^2$ has dimension 2, so it has be spanned by at least two vectors, but A only has a single row.

3. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & 1 \end{bmatrix}$$

with $a, b \in \mathbb{R}$.

(*No justification is required on any part of this problem. Your answers may be written in terms of a and b.*)

(a) (3 points) Assuming that T_A is *not* invertible, what, if anything, can be said about the values of a and b?

Solution: T_A is not invertible if and only if the columns of the matrix A are linearly dependent. By inspection, this means that $a = 1$ and b can be any real number.

(b) (3 points) Assuming that T_A is *not* invertible, find a basis of $\ker(T_A)$.

Solution: Notice that the rank of A is 2. Therefore, by the Rank-Nullity Theorem, the dimension of $\ker(T_A)$ is one. If T_A is not invertible (i.e. $a = 1$), then we have the following nontrivial dependence relation among the columns of A:

$$-b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix} = \vec{0}.$$

Since the dimension of $\ker(T_A)$ is one, this means that $\begin{bmatrix} -b \\ -1 \\ 1 \end{bmatrix}$ spans the kernel of T_A. Therefore, a basis for $\ker(T_A)$ is given by the singleton set: $\left\{ \begin{bmatrix} -b \\ -1 \\ 1 \end{bmatrix} \right\}$.

(c) (3 points) Assuming that T_A is *not* invertible, find a basis of $\operatorname{im}(T_A)$.

Solution: The columns of A always span the image of T_A. Since the rank of A is two and the first two column vectors form a linearly independent set, then they also form a basis for the image of T_A. Therefore, a basis for $\operatorname{im}(T_A)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Of course, any choice of two out of the three columns of A also form a basis.

(d) (3 points) Assuming now that T_A *is* invertible, what is the second column of A^{-1}?

Solution: Observe that $A\vec{e}_1 = \vec{e}_2$. Multiplying both sides of this formula by A^{-1} (which we're assuming exists) yields $\vec{e}_1 = A^{-1}\vec{e}_2$. However, $A^{-1}\vec{e}_2$ is just the second column of A^{-1}. Therefore, the answer is $\vec{e}_1$.

4. Let $a, b, c, d \in \mathbb{R}$, and consider the subset V of $\mathbb{R}^4$ defined by

$$V = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} d \\ 0 \\ c \\ 1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

In each of (a) – (d) below, find values of a, b, c, d which make the given claim true, or else write *none* if no such values exist. (*No justification is required on this problem.*)

(a) (2 points) $V = \mathbb{R}^4$.

Solution: None.

(b) (2 points) V is the kernel of some linear transformation.

Solution: For instance, $a = b = c = d = 0$.

(c) (2 points) V is a subspace of $\mathbb{R}^4$ and $\dim V = 1$.

Solution: For instance, $a = b = c = 0$ and $d = 1$.

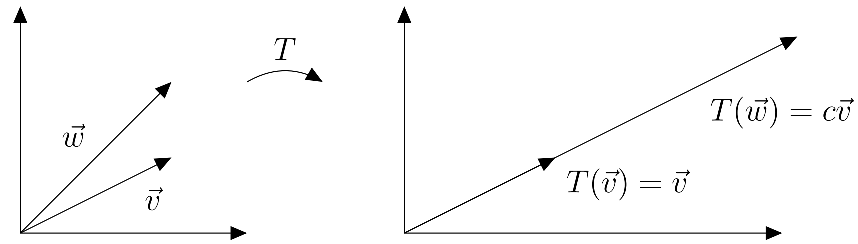
(d) (3 points) V is the solution set of $A\vec{x} = \vec{v}$, where $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution: For instance, $a = 5$, $b = 1$, $c = 0$, and $d = 1$.

(e) (3 points) Assuming that $a = b = c = d = 0$, find a 4×5 matrix such that $V = \operatorname{im}(A)$ or else explain why this is impossible.

Solution: For instance, $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$.

5. Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ be the vectors and $T : \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation shown below, where you may assume that the pictures are drawn accurately and to scale, so in particular c is a positive scalar. Your answers in (b) and (c) may involve c, $\vec{v}$, or $\vec{w}$.



(a) (3 points) Answer each question *yes*, *no*, or *not enough information.*

Is T injective? ____________

Is T surjective? ____________

Is T invertible? ____________

(*No justification needed*)

Solution: T is neither injective nor surjective nor invertible.

(b) (4 points) Find a basis of $\ker(T)$, and justify your answer.

Solution: From the picture we see $\vec{w} - c\vec{v} \neq \vec{0}$, but we have

$$T(\vec{w} - c\vec{v}) \;=\; T(\vec{w}) - cT(\vec{v}) \;=\; c\vec{v} - c\vec{v} \;=\; \vec{0},$$

so $\vec{w} - c\vec{v}$ is a nonzero vector in $\ker(T)$. Since T is not the zero transformation we conclude $\dim(\ker(T)) = 1$, and thus $\{\vec{w} - c\vec{v}\}$ is a basis of $\ker(T)$.

(c) (3 points) Assuming $\vec{w} - \vec{v} = 2\vec{e}_2$, find the second column of the standard matrix of T. Be sure to show your work.

Solution: The second column of the standard matrix of T is

$$T(\vec{e}_2) \;=\; T\big(\tfrac{1}{2}(\vec{w} - \vec{v})\big) \;=\; \tfrac{1}{2}\big(T(\vec{w}) - T(\vec{v})\big) \;=\; \tfrac{1}{2}(c\vec{v} - \vec{v}) \;=\; \big(\tfrac{c-1}{2}\big)\vec{v}.$$

(d) (3 points) Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ and $R : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto ℓ and reflection over ℓ, respectively, where $\ell = \operatorname{Span}(\vec{v})$. Find an expression for $T(R(\vec{w}))$ that does not involve T or R, but can include other parameters used in this problem such as c, $\vec{v}$, $\vec{w}$, or P. Be sure to show some work, and circle your final answer.

Solution: We have $\vec{w} + R(\vec{w}) = 2P(\vec{w})$, so $T(\vec{w}) + T(R(\vec{w})) = 2T(P(\vec{w})) = 2P(\vec{w})$, and therefore

$$T(R(\vec{w})) \;=\; 2P(\vec{w}) - T(\vec{w}) \;=\; 2P(\vec{w}) - c\vec{v}.$$

6. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Let

$$V = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \ : \ a, b \in \mathbb{R} \right\},$$

and let $T : V \to V$ be the map defined by $T(A) = A + A^\top$ for all $A \in V$.

(a) (3 points) Show that V is a subspace of $\mathbb{R}^{2\times 2}$.

Solution: Let $k \in \mathbb{R}$ and $A, B \in V$. Then $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$.

Taking $a = b = 0$ we get $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$.

V is closed under addition since

$$A + B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \in V.$$

V is closed under scalar multiplication since

$$kA = k \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ka & -(kb) \\ kb & ka \end{bmatrix} \in V.$$

(b) (3 points) Find a basis of V. What is $\dim(V)$? (*No justification necessary.*)

Solution: Basis of V : $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, $\dim(V) = 2$.

(c) (3 points) Show that T is a linear transformation.

Solution: Let $A, B \in V$ and $k \in \mathbb{R}$.

- $T(A+B) = A + B + (A+B)^T = A + B + A^T + B^T = A + A^T + B + B^T = T(A) + T(B)$
- $T(kA) = kA + (kA)^T = kA + kA^T = k(A + A^T) = kT(A)$

(d) (3 points) Find a basis of $\operatorname{im}(T)$ and a basis of $\ker(T)$. (*No justification necessary.*)

Solution:
Basis of $\operatorname{im}(T)$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Basis of $\ker(T)$: $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

7. Let $n \in \mathbb{N}$, let V be an n-dimensional vector space, let $T : V \to V$ and $S : V \to V$ be linear transformations, and suppose $S \circ T$ is the identity transformation on V, so $S(T(v)) = v$ for all $v \in V$.

   (a) (4 points) Prove that $\ker(T) = \{0_V\}$.

Solution: (Solution 1:) Let $\vec{x} \in \ker(T)$. Then $T(\vec{x}) = \vec{0}_V$. So:

$$\vec{x} = S \circ T(\vec{x}) = S(\vec{0}_V) = \vec{0}_V,$$

where in the first equality we used our hypothesis about S and T and in the third equality we used the fact that S is linear. Thus, $\vec{x} = \vec{0}_V$. This proves that $\ker(T) \subseteq \{\vec{0}_V\}$. The other set inclusion follows from the linearity of T.
(Solution 2:) It suffices to show that T is injective since injectivity implies that $\ker(T) = \{\vec{0}_V\}$ by linearity of T. Assume that $\vec{x}_1, \vec{x}_2 \in V$ and $T(\vec{x}_1) = T(\vec{x}_2)$. Then:

$$\vec{0}_V = S(\vec{0}_V) = S(T(\vec{x}_1) - T(\vec{x}_2)) = S(T(\vec{x}_1)) - S(T(\vec{x}_2)) = \vec{x}_1 - \vec{x}_2.$$

Therefore, $\vec{x}_1 = \vec{x}_2$ and T is injective. Alternatively, see that $T(\vec{x}_1) = T(\vec{x}_2)$ implies $x_1 = S(T(\vec{x}_1)) = S(T(\vec{x}_2)) = \vec{x}_2$.
(Solution 3:) First, notice that $\ker(T) \subseteq \ker(S \circ T)$. Indeed, if $\vec{x} \in \ker(T)$ then $S \circ T(\vec{x}) = S(\vec{0}_V) = \vec{0}_V$. Since $S \circ T = \mathrm{id}$, then:

$$\{\vec{0}_V\} \subseteq \ker(T) \subseteq \ker(S \circ T) = \{\vec{0}_V\}.$$

Thus, $\ker(T) = \{\vec{0}_V\}$.

   (b) (4 points) Prove that T is surjective.

Solution: By part (a) we know that $\dim(\ker(T)) = 0$. Since V has dimension n, the Rank-Nullity Theorem says:

$$n = \dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(\mathrm{im}(T)).$$

Since $\dim(\mathrm{im}(T))$ is a subspace of V with dimension $n = \dim(V)$, we conclude that $\mathrm{im}(T) = V$ and T is surjective.

Remark. It is **not possible** to prove this claim by definition. The statement requires both that T is linear, and that V is finite dimensional. A proof by definition does not invoke these assumptions. Here are counterexamples with each assumption relaxed:

Consider $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = e^x$. Note that T is not linear, so the assumption that T is not linear has been relaxed. The assumption that $\dim V$ is finite holds for this example. Now if we consider the function $S(x) = \log(x)$, then clearly $S \circ T$ is the identity on $\mathbb{R}$. (Evidently S is only defined on $(0, \infty)$

but one could modify the definition of S so that S is defined piecewise and equal to zero on $(-\infty, 0]$.) Notice that T is not surjective.

Alternatively, we have an example where T is linear but V is not finite dimensional. Let $V = \mathbb{R}^{\mathbb{N}}$, the space of sequences of real numbers, and let $T\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the linear transformation given by $T\big((x_1, x_2, x_3, \ldots)\big) = (0, x_1, x_2, x_3, \ldots)$. One can check that T is linear. Then consider the linear transformation $S\colon R^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ given by $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. One can also check that S is linear. Moreover, $S \circ T$ is the identity on $\mathbb{R}^{\mathbb{N}}$. Notice T is not surjective again.

(c) (4 points) Does T have to be invertible? Justify your answer.

Solution: By part (a) we proved that T is injective. By part(b) we proved that T is surjective. So T is bijective and therefore invertible.

8. Let V be a vector space, and let $X = (\vec{x}_1, \ldots, \vec{x}_n)$ and $Y = (\vec{y}_1, \ldots, \vec{y}_m)$ be linearly independent lists of vectors in V of lengths n and m, respectively.

   (a) (6 points) Prove that if the list $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, then $\operatorname{Span}(X) \cap \operatorname{Span}(Y) = \{\vec{0}\}$.

Solution: Suppose $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, and let $\vec{v} \in \operatorname{Span}(X) \cap \operatorname{Span}(Y)$. Then we may fix $c_1, \ldots, c_n \in \mathbb{R}$ and $d_1, \ldots, d_m \in \mathbb{R}$ such that $\vec{v} = \sum_{i=1}^n c_i\vec{x}_i = \sum_{j=1}^m d_j\vec{y}_j$. Then

$$\vec{0} \;=\; \vec{v} - \vec{v} \;=\; \sum_{i=1}^n c_i\vec{x}_i + \sum_{j=1}^m (-d_j)\vec{y}_j,$$

which implies that each $c_i = 0$ and each $d_j = 0$ since $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, and thus $\vec{v} = \vec{0}$. This shows $\operatorname{Span}(X) \cap \operatorname{Span}(Y) \subseteq \{\vec{0}\}$, and the reverse inclusion is immediate since $\vec{0} = \sum_{i=1}^n 0\vec{x}_i = \sum_{j=1}^m 0\vec{y}_j \in \operatorname{Span}(X) \cap \operatorname{Span}(Y)$.

Solution: To prove the contrapositive, suppose $\vec{z}$ is a nonzero vector in $\operatorname{Span}(X)\cap \operatorname{Span}(Y)$, say $\vec{z} = \sum_{i=1}^n c_i\vec{x}_i = \sum_{j=1}^m d_j\vec{y}_j$. Then

$$\sum_{i=1}^n c_i\vec{x}_i + \sum_{j=1}^m (-d_j)\vec{y}_j \;=\; \vec{0},$$

but $c_i \neq 0$ and $d_j \neq 0$ for some i and j since $\vec{z} \neq \vec{0}$, which shows that $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly dependent.

   (b) (6 points) Prove the converse of what you proved in part (a); that is, prove that if $\operatorname{Span}(X) \cap \operatorname{Span}(Y) = \{\vec{0}\}$, then the list $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent.

Solution: (Solution 1:) Assume $\operatorname{Span}(X)\cap\operatorname{Span}(Y) = \{\vec{0}\}$, let $c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathbb{R}$ be arbitrary, and suppose $\sum_{i=1}^n c_i\vec{x}_i + \sum_{j=1}^m d_j\vec{y}_j = \vec{0}$. Then

$$\sum_{i=1}^n c_i\vec{x}_i \;=\; \sum_{j=1}^m (-d_j)\vec{y}_j \;\in\; \operatorname{Span}(X) \cap \operatorname{Span}(Y) \;=\; \{\vec{0}\},$$

so each $c_i = 0$ since X is linearly independent and each $d_j = 0$ since Y is linearly independent. This shows $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, as desired.

(Solution 2:) By contrapositive, assume that the list $(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}_1, \ldots, \vec{y}_m)$ is linearly dependent. Then there exists a redudant vector in the list. Since the vectors $(\vec{x}_1, \ldots, \vec{x}_n)$ are linearly independent, the redundant vector must be $\vec{y}_i$ for some i. Then there are scalars $a_1, \ldots, a_m$ and, if $i > 1$, $b_1, \ldots, b_{i-1}$ which are not all zero such that

$$\vec{y}_k = \sum_{j=1}^{n} a_j \vec{x}_j + \sum_{k=1}^{i-1} b_k \vec{y}_k.$$

Then, rearranging the above equation,

$$\vec{y}_k - \sum_{k=1}^{i-1} b_k \vec{y}_k = \sum_{j=1}^{n} a_j \vec{x}_j \in \mathrm{Span}(X) \cap \mathrm{Span}(Y).$$

Since $(\vec{y}_1, \ldots, \vec{y}_m)$ is linearly independent, $\vec{y}_k$ cannot be redundant, so $\vec{y} - \sum_{k=1}^{i-1} b_k \vec{y}_k$ is not zero and the proof is complete.