Math 217 – Midterm 1

Fall 2021

Solutions

Student ID Number: ______________________ Section: ________

Question	Points	Score
1	12	
2	15	
3	13	
4	14	
5	14	
6	13	
7	11	
8	8	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *surjective*

   Solution: The function $f : X \to Y$ is *surjective* if for every $y \in Y$ there is $x \in X$ such that $f(x) = y$.

   (b) The *span* of the finite set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^m$

   Solution: The *span* of the finite set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^m$ is the set of all vectors that can be expressed as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

   Solution: The *span* of the finite set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^m$ is the set

   $$\left\{ \sum_{i=1}^{n} c_i \vec{v}_i \;:\; c_1, \dots, c_n \in \mathbb{R} \right\}.$$

   (c) For vector spaces V and W, the function $T : V \to W$ is a *linear transformation*

   Solution: For vector spaces V and W, the function $T : V \to W$ is a *linear transformation* if for all $\vec{v}_1, \vec{v}_2 \in V$ and $c \in \mathbb{R}$, we have $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ and $T(c\vec{v}_1) = cT(\vec{v}_1)$.

   (d) The list of vectors $(\vec{v}_1, \dots, \vec{v}_n)$ in the vector space V is *linearly dependent*

   Solution: The list of vectors $(\vec{v}_1, \dots, \vec{v}_n)$ in the vector space V is *linearly dependent* if there exist $c_1, \dots, c_n \in \mathbb{R}$ that are not all zero such that $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T)$ contains exactly two elements.

Solution: FALSE. We know that for any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\ker(T)$ is a subspace of $\mathbb{R}^2$, and every nontrivial subspace (i.e., every subspace that is not just $\{\vec{0}\}$) of a vector space must have infinitely many vectors in it.

Solution: FALSE. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and suppose $\ker(T)$ contains at least two elements, say $\vec{v}$ and $\vec{w}$ where $\vec{v} \neq \vec{w}$, so $T(\vec{v}) = \vec{0} = T(\vec{w})$. Then $\vec{v}$ and $\vec{w}$ cannot both be $\vec{0}$, so assume $\vec{v} \neq \vec{0}$. Then for all $c \in \mathbb{R}$ we have $T(c\vec{v}) = cT(\vec{v}) = c\vec{0} = \vec{0}$, so the entire (infinite) line $\{c\vec{v} \ : \ c \in \mathbb{R}\}$ is contained in $\ker(T)$.

(b) (3 points) For every matrix $A \in \mathbb{R}^{4\times 3}$ and $\vec{x}, \vec{y} \in \mathbb{R}^3$, if the columns of A are linearly independent and $A\vec{x} = A\vec{y}$ then $\vec{x} = \vec{y}$.

Solution: TRUE. Let $A = [\,\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3\,] \in \mathbb{R}^{4\times 3}$, let $\vec{x}, \vec{y} \in \mathbb{R}^3$, and suppose $A\vec{x} = A\vec{y}$. Then $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y} = \vec{0}$, so $\vec{x} - \vec{y} \in \ker(A)$. Let us write $\vec{x} - \vec{y} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, so $A(\vec{x} - \vec{y}) = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = \vec{0}$. Since this columns of A are linearly independent, we see that $c_1 = c_2 = c_3 = 0$, so $\vec{x} - \vec{y} = \vec{0}$, and therefore $\vec{x} = \vec{y}$ as desired.

(c) (3 points) For any linear maps $S : V \to U$ and $T : U \to W$ between vector spaces V, U, and W, if $T \circ S = 0$ then $S = 0$ or $T = 0$. (Here, we write 0 for the zero map between the appropriate spaces.)

Solution: FALSE. For a counterexample, let $V = U = W = \mathbb{R}^2$, and let S and T have standard matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

respectively. Then neither S nor T is the zero map (since neither A nor B is the zero matrix), but $T \circ S$ *is* the zero map since its standard matrix is

$$BA \ = \ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ = \ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(Problem 2, Continued).

(d) (3 points) There exists a linear transformation $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$ whose kernel has dimension 3.

Solution: FALSE. Let $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$ be a linear transformation. Note that $\dim \mathbb{R}^{3\times 3} = 9$ and $\dim \mathbb{R}^{2\times 2} = 4$, so $\dim \operatorname{im} T \leq 4$ since $\operatorname{im}(T) \subseteq \mathbb{R}^{2\times 2}$. But

$$\dim \ker T + \dim \operatorname{im} T \; = \; 9$$

by Rank-Nullity, so $\dim \ker T = 9 - \dim \operatorname{im} T \geq 5$. In particular, $\dim \ker T \neq 3$.

(e) (3 points) For every $n \in \mathbb{N}$ and subspace V of $\mathbb{R}^n$, there is $m \in \mathbb{N}$ and a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ such that $V = \operatorname{im}(T)$.

Solution: TRUE. Let $n \in \mathbb{N}$ and let V be a subspace of $\mathbb{R}^n$, so $\dim V \leq n$. Let's say $\dim V = m$, and let $(\vec{v}_1, \ldots, \vec{v}_m)$ be a basis of V. Let

$$A \; = \; \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \; \in \; \mathbb{R}^{n\times m},$$

and let T_A be the linear transformation induced by A, so $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$. Then

$$V = \operatorname{Span}(\vec{v}_1, \ldots, \vec{v}_m) = \left\{ \sum_{i=1}^{m} c_i \vec{v}_i \; : \; c_1, \ldots, c_m \in \mathbb{R} \right\} = \{A\vec{c} : \vec{c} \in \mathbb{R}^m\} = \operatorname{im}(T_A).$$

3. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_5 \\ | & & | \end{bmatrix} \in \mathbb{R}^{4\times 5}$ be a 4×5 matrix with columns $\vec{a}_1, \ldots, \vec{a}_5$, let $\vec{b} \in \mathbb{R}^4$, and suppose the reduced row echelon form of the matrix $\left[A \mid \vec{b}\right]$ is

$$\operatorname{rref}\begin{bmatrix} | & & | & | \\ \vec{a}_1 & \cdots & \vec{a}_5 & \vec{b} \\ | & & | & | \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (2 points) Find all values of c and d for which the linear system $A\vec{x} = \vec{b}$ is inconsistent. *(No justification required.)*

Solution: $c = 0$ and $d = 1$.

(b) (3 points) Write $\vec{a}_4$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$, or else briefly explain why this is impossible.

Solution: $\vec{a}_4 = \vec{a}_1 + 0\vec{a}_2 + 5\vec{a}_3$.

(c) (3 points) Find A^{-1}, or else briefly explain why this is impossible.

Solution: This is impossible since A is not even square, so A cannot be invertible!

(d) (5 points) Assuming that $c = 1$ and $d = 4$, find the solution set of the linear system $A\vec{x} = \vec{b}$. (Please write your answer parametrically, using proper set notation.)

Solution:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} \ : \ x_2, x_4 \in \mathbb{R} \right\}$$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects any vector in $\mathbb{R}^2$ onto the line $y = -x$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects any vector in $\mathbb{R}^2$ across the x-axis.

   No justification is required on any part of this problem.

   (a) (4 points) Find matrices A and B such that $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

   Solution:

   $$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} S(\vec{e}_1) & S(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (b) (3 points) Find a matrix C such that $(S \circ T)(\vec{x}) = C\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

   Solution:

   $$C = BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

   (c) (4 points) Find a basis of $\operatorname{im}(T)$ and a basis of $\ker(T)$.

   Solution: $\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ is a basis of $\operatorname{im}(T)$, and $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is a basis of $\ker(T)$.

   (d) (3 points) For each of the matrices A, B, and C above, either find its inverse or write "DNE" if the matrix has no inverse.

   $A^{-1} =$ $B^{-1} =$ $C^{-1} =$

   Solution: A^{-1} DNE, $B^{-1} = B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and C^{-1} DNE.

5. Let V be the subspace of $C^\infty(\mathbb{R})$ given by $V = \{a\cos x + b\sin x \ : \ a, b \in \mathbb{R}\}$. Let $T : V \to \mathbb{R}^2$ be the linear transformation given by

$$T(f) = \begin{bmatrix} f(0) \\ f''(0) \end{bmatrix} \text{ for each } f \in V.$$

$\Big($*Recall that* $\frac{d}{dx}(\cos x) = -\sin x$ *and* $\frac{d}{dx}(\sin x) = \cos x$, *and that* $\cos(0) = 1$ *and* $\sin(0) = 0.\Big)$

(a) (3 points) Find a basis of $\operatorname{im}(T)$.

Solution: Letting $a, b \in \mathbb{R}$ and writing $f(x) = a\cos x + b\sin x$, we have $f(0) = a$ and $f''(x) = -a\cos x - b\sin x$ so $f''(0) = -a$. Thus $T(f) = \begin{bmatrix} a \\ -a \end{bmatrix}$. This shows that $\operatorname{im}(T) = \left\{\begin{bmatrix} a \\ -a \end{bmatrix} \ : \ a \in \mathbb{R}\right\}$, so $\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ is a basis of $\operatorname{im}(T)$.

(b) (3 points) Find a basis of $\ker(T)$.

Solution: Again letting $f(x) = a\cos x + b\sin x$, the calculation we made in (a) shows that $f \in \ker(T)$ if and only if $a = 0$. Thus $\ker(T) = \{b\sin x \ : \ b \in \mathbb{R}\}$, so $(\sin x)$ is a basis of $\ker(T)$.

(c) (4 points) Is T injective? Is T surjective? Justify your answers.

Solution: No, T is not injective since $\dim(\ker T) = 1$ by our solution to (b), and we know a linear map is injective iff its kernel is trivial. By part (a), $\operatorname{im} T$ is a 1-dimensional subspace of $\mathbb{R}^2$. In particular, $\operatorname{im} T \neq \mathbb{R}^2$ so T is not surjective.

(d) (4 points) Show that $\{\cos x, \sin x\}$ is a linearly independent subset of V.

Solution: Let $a, b \in \mathbb{R}$ and suppose $a\cos x + b\sin x = \vec{0}_V$. Since $\vec{0}_V$ is the constant zero function from $\mathbb{R}$ to $\mathbb{R}$, this means $a\cos x + b\sin x = 0$ for all $x \in \mathbb{R}$. In particular, setting $x = 0$ gives us $a = 0$, and setting $x = \frac{\pi}{2}$ gives us $b = 0$. This shows that $\{\cos x, \sin x\}$ is linearly independent.

Solution: We know from part (a) that $\operatorname{im}(T)$ has dimension 1 and from part (b) that $\ker(T)$ has dimension 1. By the Rank–Nullity Theorem applied to T, the dimension of V must be 2. Since $\{\cos x, \sin x\}$ is a set of size 2 that spans a vector space of dimension 2, it must be linearly independent.

6. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *scaling* if there is $c \in \mathbb{R}$ for which $T(\vec{x}) = c\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) (4 points) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not a scaling but $T \circ T$ is the identity map on $\mathbb{R}^2$? Clearly state *yes* or *no* and justify your answer.

   Solution: Yes, there does. For instance, we know that any reflection of $\mathbb{R}^2$ over a line through the origin is invertible and is equal to its own inverse, so composing with itself will give the identity. For a specific example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the x-axis, so T has standard matrix

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Then $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = -\vec{e}_2$ so T is not a scaling, but $T = T^{-1}$ so $T \circ T = \mathrm{id}_{\mathbb{R}^2}$ (or, equivalently, $A^2 = I_2$).

   (b) (4 points) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is not a scaling but $T \circ T = T$? Clearly state *yes* or *no* and justify your answer.

   Solution: Yes, there does. For instance, any orthogonal projection map onto a line through the origin in $\mathbb{R}^2$ will have this property. For a specific example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the x-axis in $\mathbb{R}^2$, so T has standard matrix

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

   Then $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = \vec{0}$ so T is not a scaling, but for all $\vec{x} \in \mathbb{R}^2$ the vector $T(\vec{x})$ already lies on the x-axis, so $T(T(\vec{x})) = T(\vec{x})$. This shows $T \circ T = T$ (or, equivalently, $A^2 = A$).

   (c) (5 points) Prove that for all $n \in \mathbb{N}$ and matrices $A \in \mathbb{R}^{n \times n}$, if $A \neq I_n$ and $A^2 = A$ then A is not invertible.

   Solution: Let $n \in \mathbb{N}$, let $A \in \mathbb{R}^{n \times n}$, and suppose $A \neq I_n$ and $A^2 = A$. Since $A \neq I_n$, we can fix $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} \neq \vec{v}$. (For instance, we could let $\vec{v} = \vec{e}_k$ where k is such that the kth column of A is not $\vec{e}_k$.) Then $A\vec{v} - \vec{v} \neq \vec{0}$, but $A(A\vec{v} - \vec{v}) = A^2\vec{v} - A\vec{v} = A\vec{v} - A\vec{v} = \vec{0}$, so $A\vec{v} - \vec{v}$ is a nonzero vector in $\ker(A)$. Thus A has nontrivial kernel, which implies that A is not invertible.

   Solution: Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n \times n}$. To prove the contrapositive, assume A *is* invertible; we must show $A = I_n$ or $A^2 \neq A$. In order to prove this, assume

$A^2 = A$. Then

$$A = AI_n = A(AA^{-1}) = A^2A^{-1} = AA^{-1} = I_n,$$

as desired.

7. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Recall that for all $X \subseteq V$ and $Y \subseteq W$, we define

$$T[X] = \{T(\vec{x}) \,:\, \vec{x} \in X\} \quad \text{and} \quad T^{-1}[Y] = \{\vec{x} \in V \,:\, T(\vec{x}) \in Y\}.$$

(a) (2 points) Clearly state what $T[\ker(T)]$ and $T^{-1}[\operatorname{im}(T)]$ are. *(No justification necessary.)*

Solution: $T[\ker(T)] = \{\vec{0}_W\}$ and $T^{-1}[\operatorname{im}(T)] = V$.

(b) (5 points) Prove that for every subspace U of V, the set $T[U]$ is a subspace of W.

Solution: Let U be a subspace of V, so $\vec{0}_V \in U$ and for all $\vec{u}_1, \vec{u}_2 \in U$ and $c \in \mathbb{R}$ we have $\vec{u}_1 + \vec{u}_2 \in U$ and $c\vec{u}_1 \in U$. Since T is linear we have $T(\vec{0}_V) = \vec{0}_W$, so $\vec{0}_W \in T[U]$. Now let $\vec{w}_1$ and $\vec{w}_2$ be arbitrary vectors in $T[U]$, and fix $\vec{u}_1, \vec{u}_2 \in U$ such that $T(\vec{u}_1) = \vec{w}_1$ and $T(\vec{u}_2) = \vec{w}_2$. Then $\vec{u}_1 + \vec{u}_2 \in U$ since U is a subspace of V, and $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2) = \vec{w}_1 + \vec{w}_2$ since T is linear, so $\vec{w}_1 + \vec{w}_2 \in T[U]$. Finally, let $c \in \mathbb{R}$ be arbitrary. Then $c\vec{u}_1 \in U$ since U is a subspace of V, and $T(c\vec{u}_1) = cT(\vec{u}_1) = c\vec{w}_1$ since T is linear, so $c\vec{w}_1 \in T[U]$.

We have shown that $T[U]$ contains $\vec{0}_W$ and is closed under vector addition and scalar multiplication, so we conclude that $T[U]$ is a subspace.

(c) (4 points) Is it true that for every subspace $S \subseteq W$ we have $T[T^{-1}[S]] = S$? Clearly state your answer and prove your claim.

Solution: No, this is not true. For a counterexample, let $V = W = \mathbb{R}^2$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the zero map, and let $S = \mathbb{R}^2$. Then $T[T^{-1}[S]] = T[\mathbb{R}^2] = \{\vec{0}\} \neq S$.

In general, we will have $T[T^{-1}[S]] = S$ for all $S \subseteq W$ if and only if T is surjective. (*This observatin is not required as part of a correct solution.*)

8. (8 points) Let V be a vector space, let X be a subset of V, and let $u, v \in V$. Prove that if $v \in \operatorname{Span}(X \cup \{u\})$ and $v \notin \operatorname{Span}(X)$, then $u \in \operatorname{Span}(X \cup \{v\})$.

Solution: Assume the hypotheses, and suppose $v \in \operatorname{Span}(X \cup \{u\})$ but $v \notin \operatorname{Span}(X)$. Since $v \in \operatorname{Span}(X \cup \{u\})$, we can fix a finite list of vectors $x_1, \ldots, x_n$ in X along with scalars $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$v = c_0 u + \sum_{i=1}^{n} c_i x_i. \tag{1}$$

If $c_0 = 0$ then Equation (1) becomes $v = \sum_{i=1}^{n} c_i x_i$ which would put $v \in \operatorname{Span}(X)$, contrary to our assumption that $v \notin \operatorname{Span}(X)$. Thus $c_0 \neq 0$, so we can solve for u in Equation (1) to obtain

$$u = \frac{1}{c_0}\left(v - \sum_{i=1}^{n} c_i x_i\right) = c_0^{-1} v + \sum_{i=1}^{n} \left(\frac{-c_i}{c_0}\right) x_i,$$

which shows $u \in \operatorname{Span}(X \cup \{v\})$.