MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 11, SOLUTIONS

Part A

Solve the following problems from the book: Solve the following problems from the book:
Section 7.2: 16;
Section 7.3: 16, 22, 24;
Section 7.4: 48, 64;
Section 7.5: 30;
Section 8.1: 14.

Solution.

7.2.16: The characteristic polynomial of A is

$$f_A(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2.$$

Its zeros are $\lambda = \dfrac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. Therefore we get two distinct eigenvalues if and only if $(a-c)^2 + 4b^2 > 0$. Since $b \neq 0$, this always holds.

7.3.16: Let A be the given matrix. The characteristic polynomial is

$$f_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix} = -\lambda^3 - \lambda = -\lambda(\lambda^2+1).$$

Therefore the only real eigenvalue of A is $\lambda = 0$. The corresponding eigenspace is

$$E_0 = \ker(A) = \operatorname{span} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

which has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Since A has the nonreal eigenvalues $\pm i$, it is not diagonalizable over $\mathbb{R}$.

7.3.22: Suppose that A is a 2×2 matrix such that $E_7 = \mathbb{R}^2$. Then for all $\vec{v} \in \mathbb{R}^2$, we have $\vec{v} \in E_7$, so $A\vec{v} = 7\vec{v} = 7I_2\vec{v}$. Therefore $A = 7I_2$. Conversely, if $A = 7I_2$, then $E_7 = \mathbb{R}^2$.

7.3.24: We must select A so that 1 is an eigenvalue whose algebraic multiplicity is 2 but whose geometric multiplicity is only 1. Equivalently, we must select $A - I_2$ so that 0 is an eigenvalue whose algebraic multiplicity is 2 but whose geometric multiplicity is only 1. For example, take $A - I_2 = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & -1 \end{bmatrix}$, so that $A = \begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 0 \end{bmatrix}$.

7.4.48: Let us work with respect to the basis $\mathcal{B} = (1, x, x^2)$ of $\mathcal{P}_2$. We have

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Therefore the eigenvalues of T are 1, 2, and 4, with corresponding eigenvectors 1, x, and x^2 (and their nonzero scalar multiples).

7.4.64: Given $S \in \mathbb{R}^{2\times 2}$, let us write $S = \begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix}$. Then

$$AS = \begin{bmatrix} | & | \\ A\vec{v} & A\vec{w} \\ | & | \end{bmatrix} \quad \text{and} \quad SB = \begin{bmatrix} | & | \\ \vec{v} & 3\vec{w} \\ | & | \end{bmatrix}$$

Therefore $AS = SV$ if and only if $\vec{v} \in E_1$ and $\vec{w} \in E_3$. We can compute that E_1 has the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and E_2 has the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Therefore

$$V = \left\{ \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Since

$$\begin{bmatrix} a & b \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

we see that $\mathcal{B} := \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$ spans V. Also, $\mathcal{B}$ is linearly independent since it consists of two nonzero vectors, neither of which is a scalar multiple of the other. Therefore $\mathcal{B}$ is a basis of V, and $\dim(V) = 2$.

7.5.30: (a) Let $A \in \mathbb{R}^{2\times 2}$. The nonreal eigenvalues of A come in complex conjugate pairs, so if $2i$ is an eigenvalue of A, so is $-2i$. Since A has distinct complex eigenvalues, it is diagonalizable over $\mathbb{C}$, i.e. there exists an invertible $S \in \mathbb{C}^{2\times 2}$ such that

$$S^{-1}AS = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}.$$

Then

$$A^2 = \left(S \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} S^{-1} \right)^2 = S \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}^2 S^{-1} = S(-4I_2)S^{-1} = -4SS^{-1} = -4I_2.$$

(b) Let us take $A := \begin{bmatrix} -4\sqrt{2} & -6 \\ 6 & 4\sqrt{2} \end{bmatrix}$. Then the characteristic polynomial of A is

$$f_A(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} -4\sqrt{2} - \lambda & -6 \\ 6 & 4\sqrt{2} - \lambda \end{bmatrix} = \lambda^2 + 4,$$

so indeed the eigenvalues of A are $\pm 2i$. We have $A^2 = -4I_2$, confirming part (a).

8.1.14: From Example 3, we have that $S^{-1}AS = D$, where

$$S := \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad D := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

It turns out that each given matrix is a linear combination of A and I_3, and is therefore also orthogonally diagonalized by S.

(a) The given matrix is $2A$. We have

$$S^{-1}(2A)S = 2(S^{-1}AS) = 2D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

(b) The given matrix is $A - 3I_3$. We have

$$S^{-1}(A - 3I_3)S = S^{-1}AS - 3S^{-1}I_3S = D - 3I_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The given matrix is $\frac{1}{2}A - \frac{1}{2}I_3$. We have

$$S^{-1}(\tfrac{1}{2}A - \tfrac{1}{2}I_3)S = \tfrac{1}{2}S^{-1}AS - \tfrac{1}{2}S^{-1}I_3S = \tfrac{1}{2}D - \tfrac{1}{2}I_3 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Part B

Problem 1. Let V be the infinite-dimensional vector space of all infinite sequences $(x_1, x_2, x_3, \dots)$ of real numbers indexed by $\mathbb{N}$. Consider the linear transformation $T : V \to V$ which deletes all the components with an odd index, i.e.,

$$T(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (x_2, x_4, x_6, \dots) \quad \text{for all } (x_1, x_2, x_3, \dots) \in V.$$

(a) Let E_0 denote the 0-eigenspace of T. Explicitly describe E_0 (as a set).

(b) Prove that every real number λ is an eigenvalue of T. (Hint: explicitly construct an eigenvector $(x_1, x_2, x_3, \dots) \in V$. First consider x_i when i is a power of 2.)

Solution.

(a) We have $E_0 = \ker(T) = \{(x_1, x_2, x_3, \dots) \in V : x_i = 0 \text{ for all even } i \in \mathbb{N}\}$.

(b) Given $\lambda \in \mathbb{R}$, define $\vec{v} = (x_1, x_2, x_3, \dots) \in V$ by

$$x_i := \begin{cases} \lambda^m, & \text{if } i = 2^m \text{ for some integer } m \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{N}$, so that $\vec{v} = (1, \lambda, 0, \lambda^2, 0, 0, 0, \lambda^3, 0, \dots)$. We claim that $\vec{v}$ is an eigenvector of T with eigenvalue λ. First note that $\vec{v}$ is nonzero, since its first component x_1 is equal to 1. Now we show that $T(\vec{v}) = \lambda\vec{v}$. Since $T(\vec{v}) = (x_2, x_4, x_6, \dots)$, we must show that

$$x_{2i} = \lambda x_i \quad \text{for all } i \in \mathbb{N}. \qquad (*)$$

If $i = 2^m$ for some integer $m \geq 0$, then $x_{2i} = \lambda^{m+1}$ and $x_i = \lambda^m$, so $(*)$ holds. Otherwise, we have $x_{2i} = x_i = 0$, and $(*)$ still holds.

Problem 2. If A is an $n \times n$ matrix, define

$$\mathscr{C}(A) = \{B \in \mathbb{R}^{n\times n} \mid AB = BA\}.$$

(a) Let D be a diagonal $n \times n$ matrix with distinct entries along the diagonal, and let $\mathscr{D}$ be the subset of $\mathbb{R}^{n\times n}$ consisting of diagonal matrices. Prove $\mathscr{C}(D) = \mathscr{D}$.

Two $n \times n$ matrices in A and B are said to be *simultaneously diagonalizable* if there exists an invertible $n \times n$ matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

(b) Prove that if A and B are simultaneously diagonalizable $n \times n$ matrices, then $B \in \mathscr{C}(A)$.
(c) Prove that if A and B are $n \times n$ matrices such that A has n distinct eigenvalues and $B \in \mathscr{C}(A)$, then A and B are simultaneously diagonalizable.

Solution.

(a) If D is diagonal with distinct entries, we claim that $\mathscr{C}(D)$ consists of the set of diagonal matrices. If D' is diagonal, then $DD' = D'D$ since each product will be the diagonal matrices whose entries are the products of the corresponding entries in D and D'. On the other hand, if B is a matrix which is not diagonal, we claim that $B \notin \mathscr{C}(D)$. Since B is not diagonal, it has at least one nonzero entry off of the diagonal, say $b_{ij} \neq 0$ where $i \neq j$. Write d_{ii} for the (i, i)-entry of D. Then the (i, j)-entry of DB is $d_{ii}b_{ij}$, but the (i, j)-entry of BD is $d_{jj}b_{ij}$. By assumption D has distinct entries along the diagonal, so $d_{ii} \neq d_{jj}$ since $i \neq j$. Hence $DB \neq BD$.

(b) Suppose A and B are simultaneously diagonalizable, so $S^{-1}AS = D_1$ and $S^{-1}BS = D_2$ are both diagonal for some invertible matrix S. Then

$$AB = (SD_1S^{-1})(SD_2S^{-1}) = SD_1S^{-1}SD_2S^{-1} = SD_1D_2S^{-1}.$$

Since D_1 and D_2 are diagonal, they commute with one another, and therefore

$$AB = SD_2D_1S^{-1} = (SD_2S^{-1})(SD_1S^{-1}) = BA.$$

That is, $B \in \mathscr{C}(A)$.

(c) If A has n distinct eigenvalues, it must be diagonalizable, so there exists some invertible S such that $S^{-1}AS = D$ is diagonal. Moreover, since the diagonal entries of D are the eigenvalues of A, it follows that D has distinct entries along its diagonal. But since $B \in \mathscr{C}(A)$, we also have $S^{-1}BS \in \mathscr{C}(D)$. Indeed,

$$(S^{-1}BS)D = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS$$

But because $B \in \mathscr{C}(A)$, $AB = BA$, and

$$(S^{-1}BS)D = S^{-1}ABS = (S^{-1}AS)(S^{-1}BS) = D(S^{-1}BS).$$

Then by our answer to (a), since $S^{-1}BS \in \mathscr{C}(D)$ it must be the case that $S^{-1}BS$ is diagonal. Hence A and B are simultaneously diagonalizable (by S).

Problem 3. *(Classifying non-diagonalizable[1] 2×2 matrices.)* Let $A \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix.

(a) Suppose that A has eigenvalue 0 but is not diagonalizable. Prove that[2] $\operatorname{im}(A) = E_0$, and conclude from this that $A^2 = 0$.

(b) Let $\lambda \in \mathbb{R}$ and suppose that A has eigenvalue λ but is not diagonalizable. Prove that we have $(A - \lambda I_2)^2 = 0$, and deduce from this that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for every $\vec{v} \in \mathbb{R}^2$.

[Hint: apply part (a) to the matrix $A - \lambda I_2$.]

(c) Prove that if A has eigenvalue λ but is not diagonalizable, then A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

[Hint: consider the basis $\mathcal{B} = (A\vec{v} - \lambda\vec{v}, \vec{v})$ where $\vec{v} \notin E_\lambda$.]

(d) Prove that if A does not have any real eigenvalues, then A is similar to a matrix of the form λQ where Q is an orthogonal matrix and $\lambda > 0$.

Solution.

(a) Since A has an eigenvalue 0, A is not invertible. On the other hand, as A is not diagonalizable, the only eigenvalue of A is 0 and A is not the zero matrix. Therefore, $\operatorname{im}(A)$ has 1 dimension. For every nonzero vector $\vec{v}$ in $\operatorname{im}(A)$, the vector Av is in $\operatorname{im}(A)$, thus a multiple of v. Hence v must be an eigenvector. Combining with 0 is the only eigenvalue, it follows that $\operatorname{im}(A) = E_0$. Hence, $A^2\vec{x} = A(A\vec{x}) = 0$ for every $x \in \mathbb{R}^2$. By picking x to be $\vec{e}_1$ and $\vec{e}_2$, we obtain that A^2 is the zero matrix.

(b) We first prove the following claim:

Claim: A is diagonalizable with eigenvalue λ_1, λ_2 (not necessarily to be different) if and only if $A - \lambda I_2$ is diagonalizable with eigenvalues $\lambda_1 - \lambda, \lambda_2 - \lambda$.

Proof of Claim: A is diagonalizable with eigenvalue λ_1, λ_2 iff there exists an invertible matrix S such that $A = S\begin{bmatrix}\lambda_1\vec{e}_1 & \lambda_2\vec{e}_2\end{bmatrix}S^{-1}$. This is equivalent with

$$A - \lambda I_2 = S\begin{bmatrix}\lambda_1\vec{e}_1 & \lambda_2\vec{e}_2\end{bmatrix}S^{-1} - S\begin{bmatrix}\lambda\vec{e}_1 & \lambda\vec{e}_2\end{bmatrix}S^{-1} = S\begin{bmatrix}\lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda\end{bmatrix}S^{-1}.$$

Thus the claim follows.

Back to the problem, by Claim, the matrix $A - \lambda I_2$ has eigenvalue 0 but is not diagonalizable. Hence, by part (a), $(A - \lambda I)^2 = 0$. Therefore, for every $\vec{v} \in \mathbb{R}^2$,

$$(A - \lambda I_2)(A\vec{v} - \lambda\vec{v}) = (A - \lambda I_2)^2\vec{v} = 0.$$

This implies that $A\vec{v} - \lambda\vec{v} \in \ker(A - \lambda I_2) = E_\lambda$.

(c) Consider $\mathcal{B} = (A\vec{v} - \lambda\vec{v}, \vec{v})$, where $\vec{v} \notin E_\lambda$. We note that $A\vec{v} - \lambda\vec{v} \neq 0$ and belongs to E_λ by part (b), while $\vec{v}$ is a nonzero vector not in E_λ. Therefore they are not multiple of each other, equivalently, they are linearly independent. Thus $\mathcal{B}$ is a basis of $\mathbb{R}^2$. By part (b), $A(A\vec{v} - \lambda\vec{v}) = \lambda(A\vec{v} - \lambda\vec{v})$. It follows that $[A(A\vec{v} - \lambda\vec{v})]_{\mathcal{B}} = \begin{bmatrix}\lambda \\ 0\end{bmatrix}$. On the other hand, $[A\vec{v}]_{\mathcal{B}} = \begin{bmatrix}1 \\ \lambda\end{bmatrix}$. Therefore,

$$[T_A]_{\mathcal{B}} = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix},$$

[1] We work over $\mathbb{R}$ throughout this problem. So "eigenvalue" means *real eigenvalue*, "diagonalizable" means *diagonalizable over* $\mathbb{R}$, and "similar" means *similar over* $\mathbb{R}$.

[2] Recall that for each $\lambda \in \mathbb{R}$, $E_\lambda = \{\vec{v} \in \mathbb{R}^2 \,:\, A\vec{v} = \lambda\vec{v}\}$.

where T_A is the linear map with A as its standard matrix. Now the claim follows as $A = [T_A]_{\mathcal{E}}$ is similar to $[T_A]_{\mathcal{B}}$.

(d) Let $p(x)$ be the characteristic polynomial of A. Suppose that $a + bi$ is a root of p. Then $p(a - bi) = \overline{p(a + bi)} = 0$, where the first equality follows from the fact that all coefficients of p are real. Thus, if A does not have real eigenvalue, we can assume that A has two distinct complex eigenvalues $a + bi$ and $a - bi$.

Suppose $A(\vec{v} + i\vec{w}) = (a + bi)(\vec{v} + i\vec{w})$, so also $A(\vec{v} - i\vec{w}) = (a - bi)(\vec{v} - i\vec{w})$. Then, diagonalizing A over $\mathbb{C}$, we have

$$\begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}^{-1} A \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} = \begin{bmatrix} a+bi & \\ & a-bi \end{bmatrix},$$

Multiply both sides by $\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ (on left) and $\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$ (on right), we get

$$\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}^{-1} A \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

By direct computation we have $\begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$, so

$$\begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}^{-1} A \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The claim now follows from the fact that $\frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is an orthogonal matrix.

Problem 4. Consider the sequence of real numbers defined by the recursive formula

$$x_0 = 0, \quad x_1 = 2, \quad x_{n+2} = 4x_{n+1} - 13x_n \text{ for } n \geq 0.$$

Thus, the sequence starts like this: $0, 2, 8, 6, -80, \ldots$
In this problem we will use linear algebra to find an explicit formula for x_n.

(a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$ for every integer $n \geq 0$.

Solution.

$$A = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}$$

(b) Use part (a) to prove by induction that your matrix A satisfies $A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for every $n \geq 0$.

Solution. The base case is $n = 0$, which asserts that $A^0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, which is true because $A^0 = 1$.

Now for the induction step, assume $A^k \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ for some k. Then

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = A \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A \left(A^k \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) = A^{k+1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

as required.

(c) Find all (real or complex) eigenvalues and corresponding eigenvectors for A.

Solution. The characteristic polynomial of A is $p(\lambda) = \lambda(\lambda - 4) + 13 = \lambda^2 - 4\lambda + 13$. The eigenvalues are the zeroes of this polynomial,

$$\lambda = \frac{4 \pm \sqrt{16 - 4(13)}}{2} = 2 \pm 3i$$

To find an eigenvector corresponding to $\lambda = 2 + 3i$, we need a basis of the kernel of $\begin{bmatrix} 2+3i & -1 \\ 13 & -2+3i \end{bmatrix}$. This kernel is spanned by the complex vector $\begin{bmatrix} 1 \\ 2+3i \end{bmatrix}$. Likewise, to find an eigenvector corresponding to $\lambda = 2 - 3i$, we need a basis of the kernel of $\begin{bmatrix} 2-3i & -1 \\ 13 & -2-3i \end{bmatrix}$. This kernel is spanned by the complex vector $\begin{bmatrix} 1 \\ 2-3i \end{bmatrix}$.

(d) Find an invertible (real or complex) matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Solution.

$$P = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}, D = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}.$$

(e) First give an explicit formula for D^n, and then use this to give an explicit formula for A^n.

Solution.

$$D^n = \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix}$$

so

$$A^n = (PDP^{-1})^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \left(\frac{-1}{6i} \begin{bmatrix} 2-3i & -1 \\ -2-3i & 1 \end{bmatrix} \right)$$

where in the last step we have used the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(f) Using parts (b) and (e), give an explicit formula for x_n, the nth term in the sequence. (Your formula may involve complex numbers, and need not be fully simplified.)

Solution. We have

$$\begin{aligned}
\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} &= A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \left(\frac{-1}{6i} \begin{bmatrix} 2-3i & -1 \\ -2-3i & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \frac{-1}{6i} \begin{bmatrix} (2+3i)^n & (2-3i)^n \\ (2+3i)^{n+1} & (2-3i)^{n+1} \end{bmatrix} \begin{bmatrix} 2-3i & -1 \\ -2-3i & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \frac{-1}{6i} \begin{bmatrix} (2+3i)^n & (2-3i)^n \\ (2+3i)^{n+1} & (2-3i)^{n+1} \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
&= \frac{-1}{6i} \begin{bmatrix} -2(2+3i)^n + 2(2-3i)^n \\ -2(2+3i)^{n+1} + 2(2-3i)^{n+1} \end{bmatrix}
\end{aligned}$$

from which we conclude, finally, that

$$x_n = \frac{-1}{6i} \left(-2(2+3i)^n + 2(2-3i)^n \right)$$

or, in a slightly simpler form,

$$x_n = \frac{1}{3i} \left((2+3i)^n - (2-3i)^n \right)$$

It is perhaps worth noting that this expression, although written in a form that requires complex numbers, nevertheless produces real solutions for all integer values of n.

Problem 5. Let $V = C^\infty(\mathbb{R})$, let $\mathcal{A} = (e^{3x}, \cos 2x, \sin 2x)$, and let $W = \operatorname{span} \mathcal{A}$. Let $T : W \to W$ be the linear transformation defined by $T(f) = f'$.

(a) Find $[T]_\mathcal{A}$.

Solution.

$$[T]_\mathcal{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

(b) Find all (real or complex) eigenvalues of the matrix $[T]_\mathcal{A}$.

Solution. The characteristic polynomial is $(\lambda - 3)(\lambda^2 + 4)$, which has three distinct complex solutions: $\lambda = 3, \lambda = 2i, \lambda = -2i$.

(c) Viewing the matrix $[T]_\mathcal{A}$ as a linear transformation of the complex vector space $\mathbb{C}^3$, find a complex eigenvector for $[T]_\mathcal{A}$ for each of the eigenvalues you found in (b).

Solution.

For $\lambda = 3$ it is clear that $E_3 = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$. For $\lambda = \pm 2i$ we find

$$E_{\pm 2i} = \ker \begin{bmatrix} \pm 2i - 3 & 0 & 0 \\ 0 & \pm 2i & -2 \\ 0 & 2 & \pm 2i \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix}0\\1\\ \pm i\end{bmatrix}\right)$$

(d) Interpret the eigenvectors you found in (c) as a set of three complex-valued functions

$$\mathcal{B} = (f_1(x), f_2(x), f_3(x))$$

with the property that any complex linear combination of the vectors in $\mathcal{A}$ (that is, a linear combination with coefficients in $\mathbb{C}$) can be written as a complex linear combination of the vectors in $\mathcal{B}$, and vice versa.

Solution. The function corresponding to $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is $f_1(x) = e^{3x}$.

Corresponding to $\begin{bmatrix}0\\1\\i\end{bmatrix}$ we have $f_2(x) = \cos(2x) + i\sin(2x)$, and corresponding to $\begin{bmatrix}0\\1\\-i\end{bmatrix}$ we have $f_3(x) = \cos(2x) - i\sin(2x)$. So we set $\mathcal{B} = (e^{3x}, \cos(2x) + i\sin(2x), \cos(2x) - i\sin(2x))$.

We observe that each of the functions in $\mathcal{B}$ is already expressed as a (complex) linear combination of the functions in $\mathcal{A}$. The reverse is also true:

$$\begin{aligned} e^{3x} &= f_1(x) \\ \cos 2x &= \frac{1}{2} f_2(x) + \frac{1}{2} f_3(x) \\ \sin 2x &= \frac{1}{2i} f_2(x) - \frac{1}{2i} f_3(x) \end{aligned}$$

(e) **(Recreational).** *Euler's formula* allows us to work with complex exponential functions via the definition $e^{i\theta} = \cos\theta + i\sin\theta$. Find three constants $a, b, c \in \mathbb{C}$ such that $\mathcal{C} = (e^{ax}, e^{bx}, e^{cx})$ has the same span over $\mathbb{C}$ as does $\mathcal{B}$, and such that $[T]_{\mathcal{C}}$ is a diagonal matrix.

Solution.
$\mathcal{C} = (e^{3x}, e^{2ix}, e^{-2ix})$ is, using Euler's formula, identical with $\mathcal{B}$, and with respect to this basis we have

$$[T]_{\mathcal{C}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

Problem 6. In this problem we apply some of the theory we have learned to Physics. Consider a solid three-dimensional object with mass density given by a function $\rho(\vec{r})$, where $\vec{r} = \langle r_2, r_2, r_3 \rangle$ is the standard position vector in $\mathbb{R}^3$. When such an object rotates in space, it has a nonzero *angular velocity*, which is represented as a vector $\vec{\omega} \in \mathbb{R}^3$ pointing along the axis of rotation. The rotating object also has an *angular momentum*, which is represented by a vector $\vec{L} \in \mathbb{R}^3$, and which is related to $\vec{\omega}$ by the equation $\vec{L} = I\vec{\omega}$, where I is a fixed 3×3 real matrix called the *moment of inertia tensor* for the solid object. The rotating object will wobble (that is, its axis of rotation will precess) if and only if $\vec{L}$ and $\vec{\omega}$ point along different lines.

(a) Show that if I has a real eigenvalue λ than there exists an axis around which the solid object can rotate without wobbling.

Solution. If $\lambda \in \mathbb{R}$ is an eigenvalue of I then there exists some nonzero vector $\vec{\omega} \in \mathbb{R}^3$ such that $\vec{L} = I\vec{\omega} = \lambda\vec{\omega}$, and hence $\vec{L}$ and $\vec{\omega}$ point along the same line, so the object can rotate around that line without wobbling.

(b) Show that I always has at least one real eigenvalue λ (and hence by (a) there always exists an axis around which a solid object can rotate without wobbling).

Solution. I is a 3×3 matrix, so its characteristic polynomial is cubic, which means it always has at least one real solution.

(c) Show that if $\operatorname{gemu}(\lambda) = 3$ then the solid object can rotate around any axis without wobbling.

Solution. If λ is an eigenvalue with geometric multiplicity 3, then the matrix I is similar to a scalar matrix, and hence I **is** a scalar matrix already. Then for any vector $\vec{\omega}$, we have $\vec{L} = I\vec{\omega} = \lambda\vec{\omega}$, so we can rotate around any axis without wobbling.

(d) Show that if I has three distinct real eigenvalues then there exist three axes around which the solid object can rotate without wobbling.

Solution. If I has three distinct real eigenvalues $\lambda_1, \lambda 2, \lambda_3$ then it is diagonalizable over the reals. This means that there exist three vectors $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$ such that $I\vec{\omega}_k = \lambda_k \omega_k$. Each of these eigenvectors corresponds to an axis of rotation around which the solid object can rotate without wobbling.

(e) It can be shown (although you do not have to worry about the proof of this!) that the (i, j)-component of the moment of inertia tensor is given by a volume integral:

$$I_{ij} = \begin{cases} -\iiint r_i r_j \, \rho(\vec{r}) \, dV, & i \neq j \\ \iiint \|\vec{r} - \operatorname{proj}_{\vec{e}_i} \vec{r}\|^2 \, \rho(\vec{r}) dV, & i = j \end{cases}$$

where $\vec{r} = \langle r_2, r_2, r_3 \rangle$ is the standard position vector in $\mathbb{R}^3$, and $\rho(\vec{r})$ is the mass density of the object at $\vec{r}$. Prove that for any solid object, there exist three **perpendicular** axes of rotation around which the object will not wobble. (These are called the *principal axes* of the object.)

[*Hint:* compare I_{ij} and I_{ji}, and consider the Spectral Theorem.]

Solution. By inspection, we note that $I_{ij} = I_{ji}$, and hence the matrix I is symmetric. By the Spectral Theorem, it is orthogonally diagonalizable, which means that it has three orthogonal eigenvectors $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$. Each of these eigenvectors corresponds to an axis of rotation around which the object can rotate without wobbling.