Worksheet 1: Systems of Linear Equations (§1.1)

Vocabulary: Linear expression, linear equation, $\mathbb{R}^3$, $\mathbb{R}^2$, line, plane, linear system of equations, inconsistent, consistent.

Problem 1. Consider the set Λ of points in $\mathbb{R}^3$ that satisfy $x - y = 0$ and the set Π of points satisfying $x - 2z = 0$.

(a). What type of geometric objects are the sets Λ and Π?

(b). What type of geometric object is the set $\Lambda \cap \Pi$?

(c). Describe geometrically the set of all points in $\mathbb{R}^3$ that satisfy both of the linear constraints

$$\begin{aligned} x - y \quad\quad &= 0 \\ x \quad\quad - 2z &= 0. \end{aligned}$$

What does this have to do with your answer to (b)?

(d). Find constants a, b, c so that your answer to (c) can be written

$$\{t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid t \in \mathbb{R}\}.$$

Is your answer *unique*? That is, is it the only possible answer?

(e). What do we mean by the "direction vector" of a line in $\mathbb{R}^3$? Is it unique? Relate this to (c) and (d). Does a line in $\mathbb{R}^3$ have a "slope"?

(f). What do we mean by "normal vector" to a plane in $\mathbb{R}^3$? Is it unique? Find normal vectors to Λ and for Π.

Solution: Both Λ and Π are planes. The intersection is a line. Solving the system gives the set of solutions $\{t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R}\}$. Geometrically, the solution set is the line through the origin in the direction of the vector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, which is the same as the intersection of the 2 planes. This direction vector (or slope) is not unique. For example, $\begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$ also works. Planes do not have slopes, but they do have normal vectors. The plane Λ has normal vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and passes through $\begin{bmatrix}0\\0\\0\end{bmatrix}$. These are not unique: for example, we could take $\begin{bmatrix}3\\-3\\0\end{bmatrix}$ as a normal vector, and the point on Λ to be $\begin{bmatrix}1\\1\\0\end{bmatrix}$. A normal vector for Π is $\begin{bmatrix}1\\0\\-2\end{bmatrix}$ and passes through $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Problem 2. Suppose we have a system of 3 linear equations in 3 unknowns:

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + kz &= r, \end{aligned}$$

where $a, b, c, d, e, f, g, h, k, p, q, r$ are all real numbers.

(a). Discuss with your group why we can think of the solution space as an intersection of three planes in $\mathbb{R}^3$.*

(b). Find an explicit non-trivial example (values of the constants) in which the solution set is a plane. Can you find one where a, d, and g are all different numbers?

(c). Find an explicit non-trivial example in which the solution set is a line.

(d). Find an explicit non-trivial example in which the solution set is a point.

(e). Find an explicit non-trivial example so that the system is inconsistent (ie. has no solutions).

(f). Are there values of the constants so that the solution set is a circle? A parabola? A union of two lines? Exactly two points? Place a bet on the shape of the solution space if we randomly pick the constants.

Solution: Each equation represents a plane so the points satisfying all three equations is the set of points on all three planes, or the intersection of the three planes. One way to get a solution space which is a plane is to make sure all the planes are the same. We can do this by making the equations be multiples of each other. For an explicit example:

$$\begin{aligned} x + y + z &= 2 \\ 2x + 2y + 2z &= 4\ . \\ 3x + 3y + 3z &= 6, \end{aligned}$$

To make the solution space a line, we can take two equations that do not define the same plane (so their common solution set is the intersection of two planes, hence a line), and then throw in another which will not give any new constraints. For example:

$$\begin{aligned} 2x - y + z &= 0 \\ x + 3y + 4z &= 0 \\ 3x + 2y + 5z &= 0. \end{aligned}$$

*A useful technique used by professional mathematicians: if stuck on a problem, try a simpler version of it. In this case, you can start by thinking about the solution space for a system of 2 linear equations in 2 unknowns, which will be a subset of $\mathbb{R}^2$. This should be familiar from high school.

Here, the first two equations define *different* planes, since they have different normal vectors, so the first two equations define a line. But the third equation is the sum of the first two, so it doesn't impose new constraints.

The easiest way to get a point is to take, for example, $x + 0y + 0z = 0, 0x + y + 0z = 0$ and $0x + 0y + z = 0$. On the other hand, most choices of the constants will get a point as the solution space, since typically we expect three planes to intersect in a point.

One way to make the system inconsistent is to make two of the plane parallel: for example

$$\begin{aligned} x + y + z &= 2 \\ x + y + z &= 4 \\ x - y + 3z &= 6, \end{aligned}$$

is consistent since the first two planes don't intersect at all. The solution space can never be a circle, parabola, two points, two lines, or anything other than a point, line or plane (or the empty set).

Problem 3. Consider the *Axiom of Parental Support*: If you get a "B" or better in this course, your parents will buy you a new car. Let us accept this as true (your experience notwithstanding), and take the following definitions:

Definition: An *"A" student* never gets a grade lower than "A−" in a given semester.

Definition: A *"B" student* gets at most one grade lower than a "B" in a given semester.

Definition: A *"C" student* gets no grade higher than "C" in a given semester.

Given these axioms and definitions, decide which of the following statements are THEOREMS.* Justify each of your claims with either an argument or a counterexample.

(a) If I am an "A" student, I will get a new car from my parents at the end of the semester.
(b) If I am a "B" student, I will get a new car from my parents at the end of the semester.
(c) If I am a "C" student, I will not get a new car from my parents at the end of the semester.

Solution: (a) is a theorem, while (b) and (c) are not theorems. To prove (a), suppose you are an "A" student. Then by definition, you never get a grade lower than "A−" in a given semester. In particular, you will not get a grade lower than "A−" in *this* course. This means you will get a "B" or better in this course, which by the Axiom of Parental Support means that your parents will indeed buy you a new car.

For a counterexample to (b), I could be a "B" student who gets a "C" in this class and thus my parents refuse to buy me a car.

For a counterexample to (c), I might be a "C" student who gets a "C" in all my classes this semester but my parents decide to buy me a new car anyway. This does not contradict the Axiom of Parental Support because in math a statement of the form "If P then Q" is taken

*A *theorem* in an axiomatic system is a statement that is logically implied by the axioms, so that it *must be true* provided that the axioms themselves are true. Another way to think of theorems is that they are the statements that can be proved using the axioms.

to be true whenever Q is true or P is false (or both! — remember that "or" is always used inclusively in math).

Problem 4. Solve each of the following systems of equations and describe the solution set geometrically.

(a)

$$\begin{aligned} y &= 2w + 3z - 8 \\ x &= w + z - 4 \\ y &= 6w - 6x + 6z - 24 \\ w + z &= 3 \end{aligned}$$

Solution: The unique solution is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, so the solution set is a single point. An easy way to do this is to notice we can substitute $w + z = 3$ into the second equation to get $x = -1$. Then the third equation as well can be written $y = 6(w + z) - 6x - 24 = 6 \times 3 - 6 \times (-1) - 24 = 0$, and so on.

(b)

$$3w + 3x - 5z \;=\; 3w + 3x - 3y \;=\; 6w + 6x - 6y - 5z \;=\; w + x \;=\; 0$$

Solution: Since $w + x = 0$, the first equation tells us $z = 0$ since it says $3(w + x) = 5z$. Similarly, the second equation tells us $y = 0$. The third is always satisfied when $y = z = x + w = 0$, so it imposes no additional constraint. So the solution set of is the line in 4-space given by $w = -x$, $y = z = 0$. This can also be described as the line through the origin with slope vector $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Alternatively, we can write down the augmented matrix, and row reduce it to: $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so the solution set is the line in 4-space given by $w = -x$, $y = z = 0$.

(c)

$$\begin{array}{rcrcrcr} -5x & + & 3y & + & 3z & = & -5 \\ -7x & + & 4y & + & 4z & = & -5 \\ -2x & + & y & + & z & = & 5 \end{array}$$

Solution: If we add the first and third equations we get $-7x + 4y + 4z = 0$, which is clearly inconsistent with the second equation. So there are no solutions to this system. Alternatively, the augmented matrix row reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is inconsistent (i.e., the solution set is $\emptyset$).