本章: ① examine how many sols a system of linear equations can possibly can.
② Then we will present some definitions and rules of matrix algebra.

Number of solutions of a linear system

A system of equa. is said to be

consistent: ≥ 1 sol.

inconsistent: no sol. (iff rref of its augmented matrix contains $[0\ 0 \dots 0 \mid 1]$)

If consistent $\Rightarrow$ either
- infinitely many sols, if ≥ 1 free variable
- exactly one sol, if all variables are leading.

Def The rank of a matrix

The number of leading 1's in rref(A), denoted $\boxed{\text{rank}(A)}$.

ex
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\Rightarrow \text{rank}(A) = 2$$

ex3 Show that for $\boxed{A^{n\times m}}$ (i.e. n equa, m variables) as coeff matrix and $\boxed{A^{n\times(m+1)}}$ as augmented matrix

(a). $\text{rank}(A) \leq m$ AND $\text{rank}(A) \leq n$

(b). If system: consistent $\Rightarrow \text{rank}(A) < n$

(c) If system: 1 sol $\Rightarrow \text{rank}(A) = m$

(d) If system infin sols $\Rightarrow \text{rank}(A) < m$

(a) By def of rref.

(b) If system inconsistent $\Rightarrow$ rref of augumented contains a row of $[0 \dots 0 \mid 1] \Rightarrow$ no leading 1 in that row for coeff $\Rightarrow \text{rank} \leq n-1 < n$

(c) it's worth noting that

(number of variables) = (total number of variables) − (number of leading variables) = $m - \text{rank}(A)$

If system has exactly one sol $\Rightarrow$ no free variables $\Rightarrow m - \text{rank}(A) = 0 \Rightarrow \text{rank}(A) = m$

(d) same as c. infin sol $\Rightarrow \geq 1$ free variable $\Rightarrow m - \text{rank}(A) > 0 \Rightarrow \text{rank}(A) < m$

As contrapositive of ex3 we know:

① If $\text{rank}(A) = n \Rightarrow$ system consistent (has sol)

② If $\text{rank}(A) < m \Rightarrow$ system no sol or infin sols

③ If $\text{rank}(A) = m \Rightarrow$ system no sol or exactly 1 sol.

Thm 1.3.3 Number of Equations vs Number of unknowns

If a linear system has exactly one sol $\Rightarrow$ at least as many equations as variables (i.e. $\boxed{m \leq n}$ for coeff matrix $A^{m\times n}$)

(And its contrapositive: a linear system with fewer equations than variables $\boxed{(n < m)}$ has either no sol or infin sols.)

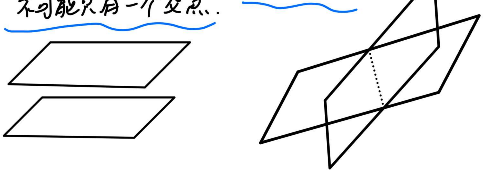
(Pf. By ex3 (a)(c), $m = \text{rank}(A) \leq n$)

To illustrate it: consider 2 linear equations in 3 variables

每个 $ax+by+cz=d$ 都表示一个 plane

而两个 plane 要么平行无交点 (no sol), 要么交于一直线 (infin sols)

不可能只有一个交点.



我们看一个 Thm 1.3.3 的特殊情况.

Thm 1.3.4

A linear system with n equa. and n variables.

By ex3(c) we know,

if this system has $\boxed{\text{exactly 1 sol}}$ (iff) $\Leftrightarrow \boxed{\text{rank}(A) = n}$, and in this case

$$\text{rref} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix Algebra

Def 1.3.5 Sum and Scalar multiples of matrices

① $\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$

② $\forall k \in \mathbb{R}$, $k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$

Def 1.3.6 Dot products of vectors

Consider $\vec{v} = \langle v_1, \ldots, v_n \rangle$, $\vec{w} = \langle w_1, \ldots, w_n \rangle$

$\Rightarrow \boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n}$

Note that the def is not row-column-sensitive
It does not distinguish between row and col vectors.

Note that: the product $A\vec{x}$ is defined only when the $\boxed{\text{num of col}}$ of A = $\boxed{\text{num of components}}$ in $\vec{x}$

Thm 1.3.8 $A\vec{x}$ in terms of cols of A

之前我们用 row vector 组 $\vec{w}$ 来表示 A,
现在我们用 col vector 组 $\vec{v}$ 来表示 A.

即 $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$

$\Rightarrow A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_m\vec{v}_m$

Pf Denote the rows of A by $\vec{w}_1, \ldots, \vec{w}_n$

By def 1.3.7, $A\vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$,

Since $\vec{w}_i = \langle v_{1i}, v_{2i}, \ldots, v_{mi} \rangle$

Def 1.3.7 The product of $A\vec{x}$

Let A be $\boxed{n \times m \text{ matrix}}$ with $\boxed{\text{row vectors}}$ $\boxed{\vec{w}_1, \ldots, \vec{w}_n}$

and $\vec{x}$ be a vector in $\mathbb{R}^m$ $\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right)$

then

$$\boxed{A\vec{x} = \begin{bmatrix} - & \overset{(w_{i1}, w_{i2}, \ldots, w_{im})}{\vec{w}_1} & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}}$$

In words, the $\boxed{i^{th} \text{ component}}$ of $A\vec{x}$ is the $\boxed{\text{dot product}}$ of the $\boxed{i^{th} \text{ row}}$ of A with $\boxed{\vec{x}}$.

ex 9 $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+2+6 \\ 3+0-2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

ex 10 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\left(\forall \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \right)$

the i^{th} component of $A\vec{x} = \vec{w}_i \cdot \vec{x} = x_1 v_{1i} + x_2 v_{2i} + \ldots + x_m v_{mi}$

By def of vector addition,

$A\vec{x} = \begin{bmatrix} x_1 \vec{v}_{11} + x_2 \vec{v}_{21} + \ldots + x_n \vec{v}_{m1} \\ \cdots \\ x_1 \vec{v}_{m1} + x_2 \vec{v}_{m2} + \ldots + x_m \vec{v}_{mn} \end{bmatrix}$

$= x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_m \vec{v}_m$

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$

$A\vec{x} = 2\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(This is remarkable: 可以有 $A \neq 0$, $\vec{x} \neq 0$, 但 $A\vec{x} = 0$)

Def 1.3.9 Linear combination

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if $\boxed{\exists \text{ scalars } x_1, \ldots, x_m}$ s.t.

$\boxed{\vec{b} = x_1 \vec{v}_1 + \ldots + x_m \vec{v}_m}$

ex is $\vec{b} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ a linear combination of the vectors $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}4\\5\\6\end{bmatrix}$

a.c.c. Def 1.3.9, we need to find scalars x, y s.t. $\begin{bmatrix}1\\1\\1\end{bmatrix} = x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}4\\5\\6\end{bmatrix}$

$$= \begin{bmatrix}x+4y\\2x+5y\\3x+6y\end{bmatrix}$$

$\Rightarrow$ solve $\left|\begin{array}{l}x+4y=1\\2x+5y=1\\3x+6y=1\end{array}\right|$

$\Rightarrow \begin{bmatrix}1&0&-\frac{1}{3}\\0&1&\frac{1}{3}\\0&0&0\end{bmatrix} \Rightarrow x=-\frac{1}{3}, y=\frac{1}{3}$

$\Rightarrow \vec{b} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}$

Thm 1.3.10 Algebraic rule for $A\vec{x}$

If A is an $\boxed{n\times m}$ matrix, $\vec{x}, \vec{y} \in \boxed{\mathbb{R}^m}$, k is scalar

$\Rightarrow$ a. $\boxed{A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}}$

b. $\boxed{A(k\vec{x}) = k(A\vec{x})}$

Thm 1.3.11 Matrix form of a linear system

We can write the linear system with augment matrix $[A \,\vdots\, \vec{b}]$ in matrix form

$$A\vec{x} = \vec{b}$$

$$\left(n\left[\overset{m+1}{\vdots}\right]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}\right)$$

$\Rightarrow$ the i^{th} component:

$$\boxed{a_{i1}x_1 + \dots + a_{im}x_m = b_i}$$

same as the i^{th} equation.

ex14 $\left|\begin{array}{l}3x_1+x_2=7\\x_1+2x_2=4\end{array}\right| \Rightarrow \boxed{\left[\begin{array}{cc:c}3&1&7\\1&2&4\end{array}\right]}$

vector form: $x_1\begin{bmatrix}3\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}7\\4\end{bmatrix}$

$\Rightarrow \boxed{\begin{bmatrix}3&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}7\\4\end{bmatrix}}$

ex15 $\left|\begin{array}{l}2x_1-3x_2+5x_3=7\\9x_1+4x_2-6x_3=8\end{array}\right|$

$$\Rightarrow \begin{bmatrix}2&-3&5\\9&4&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}$$

$(A\vec{x} = \vec{b})$

如果我们可以 divide by matrix,

$\vec{x} = \frac{\vec{b}}{A}$

就能直接解出 $\vec{x}$.

换言之我们是否能定义并找 A^{-1} 使 $\vec{x} = A^{-1}\vec{b}$

$\Rightarrow$ Chapter 2