P5 (b)

$z=a+bi$
$w=c+di$

$\Rightarrow \underset{\sim}{\overline{z+w}} = (a+c)-(b+d)i$
$= \underset{\sim}{\bar{z}+\bar{w}}.$

$\overline{zw} = \overline{ac-bd+(bc+ad)i}$
$\| \quad = (ac-bd)-(bc+ad)i$
$\bar{z}\bar{w} = (a-bi)(c-di) = ac-bd-(bc+ad)i$

(c) Fact: if $f(x)$的 coeffs $\in \mathbb{R} \Rightarrow$ $\underline{\lambda \text{ 为 } f(x) \text{ 的 root} \Longleftrightarrow \bar{\lambda} \text{ 为 } f(x) \text{ 的 root}}$.

$\lambda$为 root
$\Rightarrow a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$

$\Rightarrow \overline{\sum_k a_k\lambda^k} = \sum_k \overline{a_k\lambda^k} = \sum_k \bar{a}_k \bar{\lambda}^k$
$= \sum_k a_k \bar{\lambda}^k$
$= 0$

P8.

$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathbb{R}^{2\times 2}$

$\chi_A(\lambda) = \underset{\sim}{(a-\lambda)^2 + b^2}$

let $\chi_A(x) = 0 \Rightarrow \underset{\sim}{\lambda = a \pm bi}$

(b) Factor $A$ into 一个 scalar matrix $rI_2$ 和一个 rotation matrix $R_\theta$

$A = \underset{\sim}{rI_2 \cdot R_\theta} = \sqrt{a^2+b^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

where $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$,
$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$
$(\theta = \arctan\frac{b}{a})$

(c) Diagonalization over $\mathbb{C}$.

$D = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$

$\det(A-\lambda_1 I_2) = \begin{bmatrix} a-(a+bi) & -b \\ b & a-(a+bi) \end{bmatrix}$
$= \begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix}$

$\Rightarrow \left(\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$为 basis for $E_{\lambda_1}$

同理 $\left(\begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$为 basis for $E_{\lambda_2}$

$\Rightarrow D = S^{-1}AS = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} A \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \underline{A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}}.$

P9: 任何有一对 $\underset{\sim}{\text{complex eigenvalue}}$ $(a\pm bi)$ 的 $A \in \mathbb{R}^{2\times 2}$ 都 similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (一个旋转scaling matrix).

(因为 $A \xrightarrow{\text{diag}} D = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$
$\Downarrow$
$\sim \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ by P8.)