Pf of QR factorization

$$M = \begin{bmatrix} | & | & & | \\ m_1 & m_2 & \cdots & m_d \\ | & | & & | \end{bmatrix} \updownarrow n$$

$\xrightarrow{\text{Gram Schmidt}}$ $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_d \\ | & | & & | \end{bmatrix} \updownarrow n$

其中 $q_1 = \frac{m_1}{\|m_1\|}$

$q_i^{\parallel} = \text{proj}_{\text{span}(m_1, \ldots, m_{i-1})} m_i$

$\to q_i^{\perp} \in \text{span}(m_1, \ldots, m_{i-1})^{\perp}$

$$q_i = \frac{m_i - \sum_{k=1}^{i-1} (m_k \cdot q_k) q_k}{\| m_i - \sum_{k=1}^{i-1} (m_k \cdot q_k) q_k \|}$$

$\Rightarrow (q_1, \ldots, q_d)$ orthonormal

因而 Q 为一个 orthogonal matrix

$$\Rightarrow S_{M\to Q} = \begin{bmatrix} | & | & & | \\ [m_1]_Q & [m_2]_Q & \cdots & [m_d]_Q \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} m_1\cdot q_1 & m_2\cdot q_1 & \cdots & m_d\cdot q_1 \\ m_1\cdot q_2 & \ddots & & \\ m_1\cdot q_3 & & \ddots & \\ \vdots & & & \\ m_1\cdot q_d & & & m_d\cdot q_d \end{bmatrix}$$

0

因为 $q_i \perp \text{span}(m_1, \ldots, m_{i-1})$

$\Rightarrow$ 对于 $q_i \cdot m_j$，当 $i > j$ 时为0.

Pf of matrix product Thm:

对于 $\forall$ 两个 order basis $(\vec{b_1}, \ldots, \vec{b_d})$ 与 $(\vec{a_1}, \ldots, \vec{a_d})$ of $W \subseteq \mathbb{R}^n$

$$\begin{bmatrix} | & & | \\ b_1 & \cdots & b_d \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_d \\ | & & | \end{bmatrix} S_{B\to A}$$

$$S_{B\to A} = \begin{bmatrix} [b_1]_A & \cdots & [b_d]_A \end{bmatrix}$$

Note: $b_i = c_1 a_1 + \cdots + c_d a_d$

$\Rightarrow [b_i]_A = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix}$

$$A\, S_{B\to A} = \begin{bmatrix} A[b_1]_A & \cdots & A[b_d]_A \end{bmatrix}$$

$$A[b_i]_A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_d \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$$

$$= c_1 a_1 + \cdots + c_d a_d = b_i$$

So $A\, S_{B\to A} = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_d \\ | & & | \end{bmatrix} = B.$