Math 217 Worksheet 13: Coordinates (§3.4, §4.2)

Let $\mathfrak{B} = (v_1, v_2, \ldots v_d)$ be an ordered basis for a vector space V.

Definition. Let v be a vector in V. The $\mathfrak{B}$**-coordinates** of v are the *unique scalars* $a_1, \ldots, a_d$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_dv_d$. We arrange the $\mathfrak{B}$-coordinates into the column vector

$$[v]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix},$$

which we call the $\mathfrak{B}$**-coordinate column vector of** v**.**

Problem 1. Practice with Coordinates. Consider the ordered basis $\mathfrak{B} = (x-1, x+1, x^2+x, x^3)$ for the vector space $\mathcal{P}_3$ of polynomials of degree at most 3. Let $g = 2x^3 + 4x^2 + 4$.

(a) Verify that $\mathfrak{B}$ is a basis for $\mathcal{P}_3$.
[Hint: You should be able to do this quickly; don't forget about Theorems learned on Worksheet 11.]

(b) Write g as a linear combination of the elements in $\mathfrak{B}$. Is there more than one way? Find $[g]_{\mathfrak{B}}$.

(c) Suppose we *reorder* the basis to get $\mathfrak{B}' = (x^3, x^2+x, x-1, x+1)$. Find $[g]_{\mathfrak{B}'}$, and compare to $[g]_{\mathfrak{B}}$. Why do we need an *ordered* basis to define the $\mathfrak{B}$-coordinate column vector?

(d) Does $\mathcal{P}_3$ have a "standard" basis? Or at least one more convenient than $\mathfrak{B}$? Write down your choice of convenient basis $\mathcal{S}$ for $\mathcal{P}_3$. Find $[g]_{\mathcal{S}}$.

(e) With $\mathcal{S}$ the basis you found in (d), consider the map

$$\mathcal{P}_3 \longrightarrow \mathbb{R}^4 \qquad f \mapsto [f]_{\mathcal{S}}.$$

Write a formula for an arbitrary $f \in \mathcal{P}_3$. What can you say about this map? Is it injective? surjective? bijective? linear? an isomorphism?

Solution:

(a) We know $\mathcal{P}_3$ has dimension 4. So we can check *either* that they span *or* that they are independent (don't waste time doing both!). Both are pretty easy. For span: note that from these elements, we get $1 = -1/2(x-1) + 1/2(x+1)$, so 1 is in the span. This means also x, and then x^2 are in the span. Since the span contains $\{1, x, x^2, x^3\}$, we know that the span contains every polynomial in $\mathcal{P}_3$.

(b) $2x^3+4x^2+4 = -4(x-1)+0(x+1)+4(x^2+x)+2(x^3)$. This is the only way to write g as a linear combination of these polynomials, since they form a basis (Theorem 1). So $[g]_{\mathfrak{B}} = [-4\ 0\ 4\ 2]^T$.

(c) $[g]_{\mathfrak{B}'} = [2\ 4\ \ -4\ 0]^T$.

(d) Let's use $\mathcal{S} = \{1, x, x^2, x^3\}$. Then we have $[g]_{\mathcal{S}} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 2 \end{bmatrix}$.

(e) The map $L_{\mathcal{S}} : \mathcal{P}_3 \to \mathbb{R}^4$ sends $a + bx + cx^2 + dx^3$ to $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. The fact that $L_{\mathcal{S}}$ preserves addition and scalar multiplication is easy to check: just verify

$L_{\mathcal{S}}\left((a + bx + cx^2 + dx^3) + (a' + b'x + c'x^2 + d'x^3)\right) = L_{\mathcal{S}}(a+bx+cx^2+dx^3)+L_{\mathcal{S}}(a'+b'x+c'x^2+d'x^3)$

and $L_{\mathcal{S}}(\lambda(a + bx + cx^2 + dx^3)) = \lambda L_{\mathcal{S}}(a + bx + cx^2 + dx^3)$. It is a bijection because its easy to check an inverse is given by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto a + bx + cx^2 + dx^3$.

Problem 2. Now let $V = \mathbb{R}^{2\times 2}$ with basis $\mathfrak{B} = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$.

(a) Find the $\mathfrak{B}$-coordinates of $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $A + B = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$. Compare $[A + B]_{\mathfrak{B}}$ with $[A]_{\mathfrak{B}} + [B]_{\mathfrak{B}}$.

(b) Define a map

$$L_{\mathfrak{B}} : \mathbb{R}^{2\times 2} \to \mathbb{R}^4 \qquad A \mapsto [A]_{\mathfrak{B}}$$

What does it do to a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(c) Using your formula in (b), verify that $L_{\mathfrak{B}}$ is an isomorphism.

(d) Can you think of an easier basis to work with? Write out a basis of your choice for $\mathbb{R}^{2\times 2}$, and repeat (a), (b) and (c) for it.

Solution:

(a) $[A]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a-b \\ b-c \\ c-d \\ d \end{bmatrix}$

(c) This map from $\mathbb{R}^{2\times 2} \to \mathbb{R}^4$ respects addition and scalar multiplication, so it is a linear transformation. It's easy to see its kernel is zero, since if $\begin{bmatrix} a-b \\ b-c \\ c-d \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we can check $d = c = b = a = 0$. So the map is injective. Since the source is dimension four, rank nullity tells us the dimension of the image is also four. So the map is surjective, too. (You can also directly proof surjectivity by picking arbitrary $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ in the target, and showing that it is the image of $\begin{bmatrix} w+x+y+z & x+y+z \\ y+z & z \end{bmatrix}$.

(d) A possibly nicer basis is $\mathcal{S} = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$. The map now sends $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, which is much more obviously linear and bijective.

Problem 3. Coordinatization. A basis allows us to *model* an exotic vector space like $\mathcal{P}_n$, a plane (through the origin) in $\mathbb{R}^3$, or $\mathbb{R}^{n\times m}$, so that its vectors can be thought of as column vectors.

(a) Discuss with your group how you saw this important concept in Problems 1 and 2.

(b) Prove the following theorem. [First line: "Any $v \in V$ can be written as a *unique* linear..., so we can define..."]

Theorem. Let $\mathfrak{B} = (v_1, v_2, \ldots v_d)$ be an ordered basis for a vector space V.
The map $V \xrightarrow{L_{\mathfrak{B}}} \mathbb{R}^d$ defined by $v \mapsto [v]_{\mathfrak{B}}$ is an **isomorphism**.

Definition. The map $L_{\mathfrak{B}}$ is called the **coordinate isomorphism** given by $\mathfrak{B}$.

Solution: Consider the map $L_{\mathfrak{B}} : V \to \mathbb{R}^d$ sending each v to the coordinate column $[v]_{\mathfrak{B}}$. We need to show this is both **bijective** and a **linear transformation**.

To see it is bijective, one way is just to describe the inverse. The map $\mathbb{R}^d \to V$ sending each $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix}$ to the vector $b_1v_1 + b_2v_2 + \cdots + b_dv_d \in V$ is obviously the inverse map.

To check it is a linear transformation, we need to check both that it respects addition and scalar multiplication. For addition, we need $[v+w]_{\mathfrak{B}} = [v]_{\mathfrak{B}} + [w]_{\mathfrak{B}}$. Write out $v = a_1v_1 + a_2v_2 + \cdots + a_dv_d$ and $w = b_1v_1 + b_2v_2 + \cdots + b_dv_d$, so that $v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_d + b_d)v_d$. This exactly says that, rearranged into a column, $[v + w]_{\mathfrak{B}} = [v]_{\mathfrak{B}} + [w]_{\mathfrak{B}}$.

Scalar multiplication is similar. I will leave this to you. Do it! Ask if you are not sure.

Problem 4. Standard Coordinates on $\mathbb{R}^n$. Consider the **standard** ordered basis $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ for $\mathbb{R}^n$. Find the $\mathcal{E}$-coordinate column vector $[\vec{v}]_{\mathcal{E}}$ of an arbitrary vector $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{R}^n$. What can you say about the coordinate isomorphism $L_{\mathcal{E}}$?

Solution: $[\vec{v}]_{\mathcal{E}} = \vec{v}$, so $L_{\mathcal{E}}$ is the identity map. Writing an element of $\mathbb{R}^n$ in standard coordinates is what we've been doing all along!

Problem 5. Non-Standard Coordinates. Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(a) Sketch a picture of $\mathbb{R}^2$, indicating at least 5 points of the form $a\vec{v}_1 + b\vec{v}_2$ where a and b are *integers*. What are $\mathfrak{B}$-coordinates of these points?

(b) Indicate on your picture the subspace of $\mathbb{R}^2$ spanned by $\vec{v}_1$. What is $[\vec{v}_1]_{\mathfrak{B}}$? $[\vec{v}_2]_{\mathfrak{B}}$? Indicate the set of all points whose $\mathfrak{B}$-coordinates have the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$ where y is some scalar.

(c) Mark the vector $\vec{w}$ whose $\mathfrak{B}$-coordinates are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What is $[\vec{w}]_{\mathcal{E}}$? Mark the vector $\vec{v}$ such that $[\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. How do we represent $\vec{v}$ in standard coordinates? Find $[\vec{e}_1]_{\mathfrak{B}}$ and $[\vec{e}_2]_{\mathfrak{B}}$.

(d) Shade in the set $\mathfrak{Q} = \{c_1\vec{v}_1 + c_2\vec{v}_2 \mid 0 \leq c_1, c_2 \leq 1\}$.

(e) Consider the map $L_{\mathfrak{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ sending each vector $\vec{v} = [\vec{v}]_{\mathcal{E}}$ to $[\vec{v}]_{\mathfrak{B}}$. Is this map linear? An isomorphism? Explain. Where does it take $\mathfrak{Q}$?

Solution:

(a) Your picture should look like grid rotated (and stretched) from the usual grid of integer coordinate points.

(b) $[\vec{v}_1]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[\vec{v}_2]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The space spanned by $\vec{v}_1$ is a line through the origin and $[1 \;\; 1]^T$, the set of all points whose $\mathfrak{B}$-coordinates have the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$ is the set of all points $y\vec{v}_2$, which is the line spanned by $\vec{v}_2$. So in this coordinate system the line spanned by v_1 plays a role like the "x-axis" in the standard coordinate system, and the the line spanned by v_2 plays a role like the "y-axis" in the standard coordinate system.

(c) The point whose $\mathfrak{B}$-coordinates are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is $2\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; this is $[\vec{w}]_{\mathcal{E}}$. The point whose $\mathfrak{B}$-coordinates are $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is $\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $[\vec{e}_1]_{\mathfrak{B}} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$ and $[\vec{e}_2]_{\mathfrak{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

(d) The set $\mathfrak{Q}$ is a square with vertices $(0,0), (1,1), (-1,1)$ and $(0,2)$; it plays the role of the "unit square."

(e) The map is an isomorphism, so a bijective linear transformation. It takes $\mathfrak{Q}$ to the unit square.

Problem 6. Change of Coordinates. For the vector space $\mathbb{R}^2$, let $\mathcal{E}$ be the standard basis (Problem 4) and let $\mathfrak{B}$ be the basis from Problem 5.

(a) What is $[\vec{v}]_{\mathcal{E}}$ for any vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

(b) Consider the **change of coordinates transformation** $\mathbb{R}^2 \to \mathbb{R}^2$ sending $[\vec{v}]_{\mathcal{E}}$ to $[\vec{v}]_{\mathfrak{B}}$. Explain why this is an isomorphism. Find its matrix. This matrix is called the **change of coordinates matrix from $\mathcal{E}$ to $\mathfrak{B}$** and denoted $S_{\mathcal{E}\to\mathfrak{B}}$. [Hint: You did most of the work for this in Problem 5 (d).]

(c) Consider the **change of coordinates transformation** $\mathbb{R}^2 \to \mathbb{R}^2$ sending each vector $[\vec{v}]_{\mathfrak{B}}$ to $[\vec{v}]_{\mathcal{E}}$. Explain why this is an isomorphism. Find its matrix. This matrix is denoted $S_{\mathfrak{B}\to\mathcal{E}}$.

(d) What is the relationship between the matrices in (b) and (c)?

(e) Notice the columns of $S_{\mathfrak{B}\to\mathcal{E}}$ are simply the elements of the basis $\mathfrak{B}$. Is this a coincidence?

Solution: *Thanks to W23 Math 217 student Kristin Krier for pointing out some errors in an earlier version.*

(a) $[\vec{v}]_{\mathcal{E}}$ for $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is $\begin{bmatrix} a \\ b \end{bmatrix}$.

(b) This take vectors expressed in standard coordinates to vectors expressed in $\mathfrak{B}$ coordinates, which explains the name. By the theorem above (for any vector space) that the map $\vec{v} \mapsto [\vec{v}]_{\mathfrak{B}}$ is an isomorphism in general! This is just a special case. To find the matrix, we need to see where $\vec{e}_1$ and $\vec{e}_2$ go under the map. Write $\vec{e}_1 = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$. This tells us the first column is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Also $\vec{e}_2 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$, so the second column in $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. That is, the matrix is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(c) By the key theorem, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. This is because $\vec{e}_1 = [\vec{v}_1]_{\mathfrak{B}}$ is sent to $[\vec{v}_1]_{\mathcal{E}} = \vec{v}_1$, and similarly for the second column.

(d) They are inverses!

(e) No: in general, the j-th column of the change of basis matrix from $\mathfrak{B}$ coordinates to standard coordinates is simply the j-th element in the basis $\mathfrak{B}$.