Some More Practice Problems for Midterm 2

1. Suppose A is an $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$ be a basis of $\ker(A)$.

   (a) Find all least-squares solutions of $A\vec{x} = \vec{a}_1$.

   Note that $A\vec{e}_1 = \vec{a}_1$. So the system $A\vec{x} = \vec{a}_1$ is consistent, and its least-squares solutions are precisely its solutions. To solve the system $A\vec{x} = \vec{a}_1$, we rewrite it as $A\vec{x} = A\vec{e}_1$ and $A(\vec{x} - \vec{e}_1) = \vec{0}$, i.e., $\vec{x} - \vec{e}_1 \in \ker(A)$. Since $\mathcal{B}$ is a basis of $\ker(A)$, it follows that the solutions can be written as $\vec{x} = \vec{e}_1 + c_1\vec{b}_1 + \cdots + c_k\vec{b}_k$ where $c_1, \ldots, c_k \in \mathbb{R}$.

   (b) If $\vec{w} \in \operatorname{im}(A)^\perp$, find all least-squares solutions of $A\vec{x} = \vec{w}$

   Since $\vec{w} \in \operatorname{im}(A)^\perp = \ker\left(A^\top\right)$, we have $A^\top\vec{w} = \vec{0}$. So the normal equation of the system $A\vec{x} = \vec{w}$ is $A^\top A\vec{x} = \vec{0}$. So the least-squares solutions $\vec{x}^*$ of $A\vec{x} = \vec{w}$ satisfy $A^\top A\vec{x}^* = \vec{0}$, i.e., $\vec{x}^* \in \ker\left(A^\top A\right) = \ker(A)$. So $\vec{x}^* = c_1\vec{b}_1 + \cdots + c_k\vec{b}_k$ where $c_1, \ldots, c_k \in \mathbb{R}$.

2. Let $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, let $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, and let $V = \operatorname{Span}(\vec{v}, \vec{w})$.

   (a) Find an orthonormal basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ of $\mathbb{R}^3$ such that $\operatorname{Span}(\vec{u}_1, \vec{u}_2) = V$.

   Note that $\vec{v} \cdot \vec{w} = 0$. So we can set

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}.$$

   To find $\vec{u}_3$, pick a vector in $\mathbb{R}^3$, say $\vec{e}_1$. Then let $\vec{w}_3$ be the part of $\vec{e}_1$ that is orthogonal to V:

$$\begin{aligned} \vec{w}_3 = \vec{e}_1 - \operatorname{proj}_V(\vec{e}_1) &= \vec{e}_1 - (\vec{e}_1 \cdot \vec{u}_1)\,\vec{u}_1 - (\vec{e}_1 \cdot \vec{u}_2)\,\vec{u}_2 \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 2/6 \end{bmatrix} - \begin{bmatrix} 9/11 \\ 3/11 \\ -3/11 \end{bmatrix} = \begin{bmatrix} 1/66 \\ -7/66 \\ -4/66 \end{bmatrix}. \end{aligned}$$

   Finally, let

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \begin{bmatrix} 1/\sqrt{66} \\ -7/\sqrt{66} \\ -4/\sqrt{66} \end{bmatrix}.$$

(b) Find the $\mathcal{U}$-matrix of the orthogonal projection onto V.

Note that $\operatorname{proj}_V(\vec{u}_1) = \vec{u}_1$, $\operatorname{proj}_V(\vec{u}_2) = \vec{u}_2$, and $\operatorname{proj}_V(\vec{u}_3) = \vec{0}$, because $\vec{u}_1, \vec{u}_2 \in V$ and $\vec{u}_3 \in V^\perp$ respectively. So

$$[\operatorname{proj}_V]_{\mathcal{U}} = \begin{bmatrix} | & | & | \\ [\operatorname{proj}_V(\vec{u}_1)]_{\mathcal{U}} & [\operatorname{proj}_V(\vec{u}_2)]_{\mathcal{U}} & [\operatorname{proj}_V(\vec{u}_3)]_{\mathcal{U}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Find the $\mathcal{U}$-matrix of the reflection through the plane V.

Similar to (b), we find $\operatorname{ref}_V(\vec{u}_1) = \vec{u}_1$, $\operatorname{ref}_V(\vec{u}_2) = \vec{u}_2$, $\operatorname{ref}_V(\vec{u}_3) = -\vec{u}_3$, and

$$[\operatorname{ref}_V]_{\mathcal{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(d) Find the $\mathcal{U}$-matrix of the $180°$ rotation of $\mathbb{R}^3$ about the axis $\operatorname{Span}(\vec{u}_3)$.

Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ denote the $180°$ rotation about the axis $\operatorname{Span}(\vec{u}_3)$. Then $R(\vec{u}_1) = -\vec{u}_1$, $R(\vec{u}_2) = -\vec{u}_2$, and $R(\vec{u}_3) = \vec{u}_3$. So

$$[R]_{\mathcal{U}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(e) Find the $\mathcal{E}$-matrix of one of the three transformations just given.

Let Q be a matrix whose column vectors form an orthonormal basis of V:

$$Q = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 3/\sqrt{11} \\ -1/\sqrt{6} & 1/\sqrt{11} \\ 2/\sqrt{6} & -1/\sqrt{11} \end{bmatrix}.$$

By Theorem 5.3.10, the standard matrix of the orthogonal projection onto V is

$$[\operatorname{proj}_V]_{\mathcal{E}} = QQ^\top = \begin{bmatrix} 65/66 & 7/66 & 4/66 \\ 7/66 & 17/66 & -28/66 \\ 4/66 & -28/66 & 50/66 \end{bmatrix}.$$

Since $\operatorname{ref}_V = 2\operatorname{proj}_V - \mathrm{id}_3$ where $\mathrm{id}_3 : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity transformation, it follows that

$$[\operatorname{ref}_V]_{\mathcal{E}} = 2[\operatorname{proj}_V]_{\mathcal{E}} - I_3 = \begin{bmatrix} 32/33 & 7/33 & 4/33 \\ 7/33 & -16/33 & -28/33 \\ 4/33 & -28/33 & 17/33 \end{bmatrix}.$$

From (c) and (d), we see that $R = -\operatorname{ref}_V$. So

$$[R]_{\mathcal{E}} = -[\operatorname{ref}_V]_{\mathcal{E}} = \begin{bmatrix} -32/33 & -7/33 & -4/33 \\ -7/33 & 16/33 & 28/33 \\ -4/33 & 28/33 & -17/33 \end{bmatrix}.$$

Alternatively, we can use the change of basis theorem for transformations $[T]_{\mathcal{E}} = S^{-1}[T]_{\mathcal{U}}S$ to find the $\mathcal{E}$-matrix of a transformation T from its $\mathcal{U}$-matrix. Here S is the change of basis matrix $S_{\mathcal{E}\to\mathcal{U}}$ which is the inverse of the orthogonal matrix

$$S_{\mathcal{U}\to\mathcal{E}} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}.$$

So $S^{-1} = S_{\mathcal{U}\to\mathcal{E}}$ and $S = (S_{\mathcal{U}\to\mathcal{E}})^\top$.

3. Let S be an $n \times n$ matrix such that every row and every column of A has exactly one nonzero entry.

   (a) Prove that S is invertible.

   The column vectors of S are $s_1\vec{e}_1, \ldots, s_n\vec{e}_n$ in some order, where $s_1, \ldots, s_n \in \mathbb{R}$ are nonzero. So $\operatorname{im}(S) = \operatorname{Span}(s_1\vec{e}_1, \ldots, s_n\vec{e}_n) = \operatorname{Span}(\vec{e}_1, \ldots, \vec{e}_n) = \mathbb{R}^n$. So S is invertible.

   (b) If S is the change-of-coordinates matrix $S_{\mathcal{B}\to\mathcal{C}}$, what can you say about $\mathcal{B}$ and $\mathcal{C}$?

   Each basis element of $\mathcal{B}$ is parallel to a basis element of $\mathcal{C}$.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}$ be a linear transformation. Prove that there is a vector $\vec{w} \in \mathbb{R}^4$ such that $T(\vec{v}) = \vec{w} \cdot \vec{v}$ for all $\vec{v} \in \mathbb{R}^4$.

   Since $T : \mathbb{R}^4 \to \mathbb{R}$ is a linear transformation, it has a standard matrix $A \in \mathbb{R}^{1\times 4}$ such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^4$. Since $A \in \mathbb{R}^{1\times 4}$, it can be written as $A = \vec{w}^\top$ where $\vec{w} \in \mathbb{R}^4$ is a column vector. So $T(\vec{v}) = \vec{w}^\top\vec{v} = \vec{w} \cdot \vec{v}$ for all $\vec{v} \in \mathbb{R}^4$.

5. Show that if $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are inner products on the vector space V, then so is the map $\langle\ ,\ \rangle$ defined by $\langle x, y\rangle = \langle x, y\rangle_1 + \langle x, y\rangle_2$. For which scalars $c \in \mathbb{R}$ is the map $\langle\ ,\ \rangle_c$ defined by $\langle x, y\rangle_c = c\langle x, y\rangle_1$ an inner product on V?

   The map $\langle\ ,\ \rangle$ inherits inner product properties from $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$:
   (Symmetry) $\langle x, y\rangle = \langle x, y\rangle_1 + \langle x, y\rangle_2 = \langle y, x\rangle_1 + \langle y, x\rangle_2 = \langle y, x\rangle$ for all $x, y \in V$.

(Linearity in the First Argument) If $a, b \in \mathbb{R}$ and $x, y, z \in V$, then

$$\begin{aligned}\langle ax + by, z\rangle &= \langle ax + by, z\rangle_1 + \langle ax + by, z\rangle_2 \\ &= (a\langle x, z\rangle_1 + b\langle y, z\rangle_1) + (a\langle x, z\rangle_2 + b\langle y, z\rangle_2) \\ &= a\left(\langle x, z\rangle_1 + \langle x, z\rangle_2\right) + b\left(\langle y, z\rangle_1 + \langle y, z\rangle_2\right) \\ &= a\langle x, z\rangle + b\langle y, z\rangle.\end{aligned}$$

(Linearity in the Second Argument) follows from symmetry and linearity in the first argument:

$$\begin{aligned}\langle z, ax + by\rangle &= \langle ax + by, z\rangle \\ &= a\langle x, z\rangle + b\langle y, z\rangle \\ &= a\langle z, x\rangle + b\langle z, y\rangle.\end{aligned}$$

(Positive Definiteness) $\langle x, x\rangle = \langle x, x\rangle_1 + \langle x, x\rangle_2 \geq 0$ for all $x \in V$. Moreover, if $\langle x, x\rangle = 0$, then $\langle x, x\rangle_1 = \langle x, x\rangle_2 = 0$ which implies $x = 0_V$.

The map $\langle\ ,\ \rangle_c$ always inherits symmetry and linearity in each argument from $\langle\ ,\ \rangle_1$ for any $c \in \mathbb{R}$. In order for $\langle\ ,\ \rangle_c$ to inherit positive definiteness from $\langle\ ,\ \rangle_1$, the scalar c must be positive. For if v is a nonzero element of V, then $\langle v, v\rangle_c = c\langle v, v\rangle_1$ where $\langle v, v\rangle_c$ and $\langle v, v\rangle_1$ are positive; so c must be positive.

6. Let V be an inner product space and suppose that $T : V \to V$ is a linear transformation. Prove that if $\|T(v)\| = \|v\|$ for all $v \in V$, then T is injective.

   Suppose $u, w \in V$ are such that $T(u) = T(w)$. Then

$$\|u - w\| = \|T(u - w)\| = \|T(u) - T(w)\| = \|0_V\| = 0.$$

   So $u - w = 0_V$ and $u = w$.

7. Let V be an inner product space. Prove that $\langle x, y\rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ for all $x, y \in V$.

   Let $x, y \in V$. Then

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x + y\rangle + \langle y, x + y\rangle \\ &= (\langle x, x\rangle + \langle x, y\rangle) + (\langle y, x\rangle + \langle y, y\rangle) \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle,\end{aligned}$$

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= (\langle x,x\rangle - \langle x,y\rangle) - (\langle y,x\rangle - \langle y,y\rangle) \\ &= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle.\end{aligned}$$

So

$$\|x+y\|^2 - \|x-y\|^2 = (\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle) - (\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle) = 4\langle x,y\rangle.$$

8. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line $4x - 3y = 0$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection through the plane $x - 2y + z = 0$.

(a) Find the matrix $[S]_{\mathcal{B}}$ of S relative to the ordered basis $\mathcal{B} = \left(\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}4\\-3\end{bmatrix}\right)$.

$$[S]_{\mathcal{B}} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

(b) Find the matrix $[S]_{\mathcal{B}'}$ of S relative to the ordered basis $\mathcal{B}' = \left(\begin{bmatrix}-4\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right)$.

$$[S]_{\mathcal{B}'} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

(c) Find an ordered basis $\mathcal{C}$ of $\mathbb{R}^3$ relative to which the matrix of T is $[T]_{\mathcal{C}} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$.

Let $\mathcal{C} = (\vec{c}_1, \vec{c}_2, \vec{c}_3)$. Then

$$[T(\vec{c}_1)]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [T(\vec{c}_2)]_{\mathcal{C}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \text{and} \quad [T(\vec{c}_3)]_{\mathcal{C}} = \begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

That is, $T(\vec{c}_1) = \vec{c}_1$, $T(\vec{c}_2) = \vec{c}_2$, and $T(\vec{c}_3) = -\vec{c}_3$. So $\vec{c}_1$ and $\vec{c}_2$ must be in the plane $x - 2y + z = 0$, and $\vec{c}_3$ must be orthogonal to the plane. So an ordered basis of $\mathbb{R}^3$ with the desired property is

$$\mathcal{C} = \left(\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}\right).$$

(d) Find the standard matrix of T. (You may leave your answer as a product of matrices).

By the change of basis theorem for transformations, the standard matrix of T is

$$[T]_{\mathcal{E}} = S_{\mathcal{C}\to\mathcal{E}}[T]_{\mathcal{C}}S_{\mathcal{E}\to\mathcal{C}}$$
$$= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}.$$

9. Given the matrix $M = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, consider the linear transformation $T_M : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ defined as:

$$T_M(A) = AM - MA, \quad \text{for } A \in \mathbb{R}^{2\times2},$$

and the inner product in $\mathbb{R}^{2\times2}$:

$$\langle A, B\rangle = \operatorname{tr}(A^T B), \quad \text{for any } A, B \in \mathbb{R}^{2\times2}.$$

(a) Find an orthonormal basis of $\ker(T_M)$.

An orthonormal basis of $\ker(T_M)$ is $\left(\frac{1}{\sqrt{2}}I_2, \frac{1}{\sqrt{6}}M\right)$.

(b) Find the matrix U of T_M with respect to the standard ordered basis of $\mathbb{R}^{2\times2}$,

$$\mathcal{U} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

$$U = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & -2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

(c) Consider the ordered basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

of $\mathbb{R}^{2\times2}$. If B is the $\mathcal{B}$-matrix of T_M, find a matrix S such that $BS = SU$, where U is the $\mathcal{U}$-matrix of T_M from part (b).

We can take S to be the zero matrix in $\mathbb{R}^{4\times4}$. A less trivial S with the desired property is the change of basis matrix $S_{\mathcal{U}\to\mathcal{B}}$.