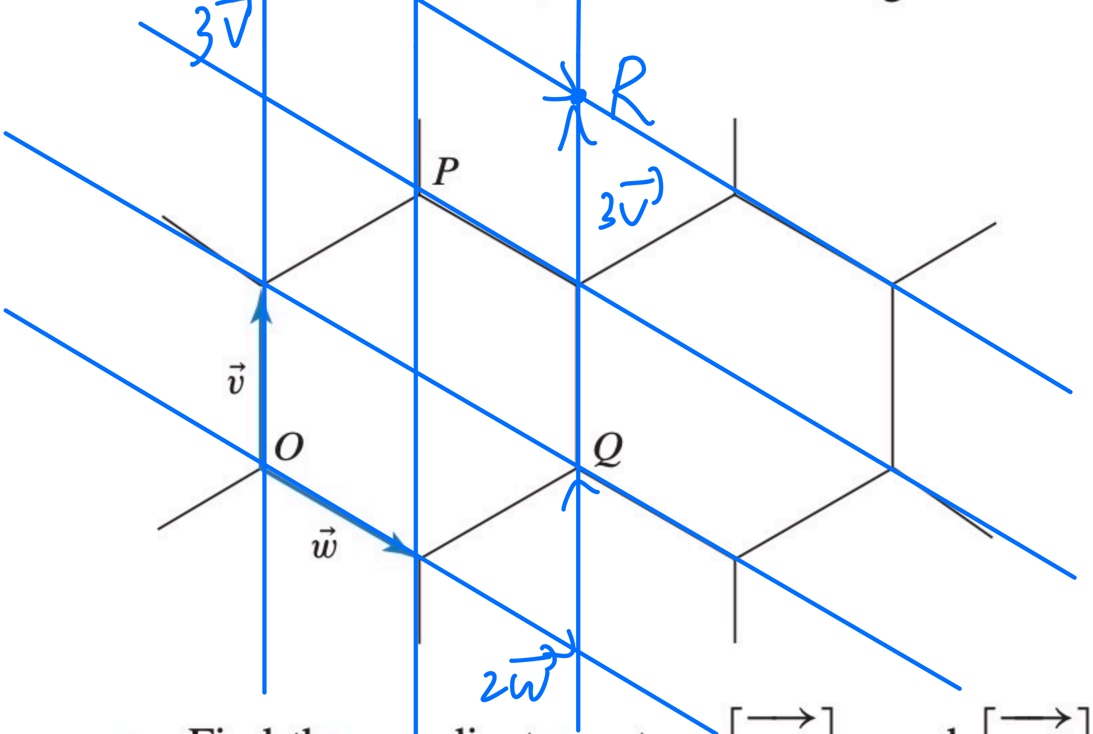
3-4

50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\vec{v}$, $\vec{w}$ in the following sketch:



a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathfrak{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathfrak{B}}$.

Hint: Sketch the coordinate grid defined by the basis $\mathfrak{B} = (\vec{v}, \vec{w})$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R. Is R a vertex or a center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathfrak{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is S a center or a vertex of a tile?

(a) $\left[\overrightarrow{OP}\right]_{\mathfrak{B}} = \left[2\vec{v} + \vec{w}\right]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\left[\overrightarrow{OQ}\right]_{\mathfrak{B}} = \left[\vec{v} + 2\vec{w}\right]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) R is a center of a tile.

(c) $[\vec{OS}]_{\mathfrak{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix} = 8\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Since ① $k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ means "moving parallel to OQ for a $k \times$ (length of OQ)" which does not change whether vertex or center

and ② $k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ means "moving parallel to OP for $k \times$ (length of OP)" which does not change whether vertex or center.

So since $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is a vertex, $\begin{bmatrix} 17 \\ 13 \end{bmatrix}$ is a vertex.

70. Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\mathfrak{B}$-matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular? *Hint*: Think about the first column of B.

Sol $\mathfrak{B} = (\vec{v_1}, \vec{v_2})$ for some $\vec{v_i}$

There does not exist such basis $\mathfrak{B}$.

Assume $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ for contradiction

So by generalized key theorem,

$$[T(\vec{v_1})]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

So $$\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{v_1}\right]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{v_1} = a\vec{v_1}$

Let $\vec{v_1} = \begin{bmatrix} x \\ y \end{bmatrix}$

So $x - y = ax + ay$

$\Rightarrow (a-1)x = (-a-1)y$

$\Rightarrow x = y = 0$ since $a-1$ and $-a-1$ can not both be 0.

This is impossible because $\vec{0}$ can not be in a basis.

4-1

58. In this exercise we will show that the functions $\cos(x)$ and $\sin(x)$ span the solution space V of the differential equation $f''(x) = -f(x)$. See Example 1 of this section.

a. Show that if $g(x)$ is in V, then the function $(g(x))^2 + (g'(x))^2$ is constant. *Hint*: Consider the derivative.

b. Show that if $g(x)$ is in V, with $g(0) = g'(0) = 0$, then $g(x) = 0$ for all x.

c. If $f(x)$ is in V, then $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$ is in V as well (why?). Verify that $g(0) = 0$ and $g'(0) = 0$. We can conclude that $g(x) = 0$ for all x, so that $f(x) = f(0)\cos(x) + f'(0)\sin(x)$. It follows that the functions $\cos(x)$ and $\sin(x)$ span V, as claimed.

(a) Assume $g(x) \in V$

So $g''(x) = -g(x)$

Let $f(x) = (g(x))^2 + (g'(x))^2$

So $f'(x) = 2g(x)g'(x) + 2g'(x)g''(x)$

$= 2g(x)g'(x) - 2g(x)g'(x) = 0$

So $f(x) = (g(x))^2 + (g'(x))^2$ is constant.

(b) Assume $g(x) \in V$ and g

since we have shown $(g(x))^2 + (g'(x))^2 = k$ for some constant k for all x, And $g(0) = g'(0) = 0$

so $k = (g(0))^2 + (g'(0))^2 = 0$

Therefore $(g(x))^2 = (g'(x))^2 = 0$ for all x

Therefore $g(x) = g'(x) = 0$ for all x

(c) V is a vector space since it satisfies all VS1-8.

Since $(\sin x)'' = -\sin x$, $(\cos x)'' = -\cos x$

$\sin x, \cos x \in V$

Therefore if $f(x) \in V$, then

$f(x) - f(0)\cos(x) - f'(0)\sin(x) \in V$

since V is closed under linear combination.

$g(0) = f(0) - f(0)\cdot\cos(0) - f'(0)\sin(0)$

$= f(0) - f(0) - 0 = 0$

and $g'(x) = f'(x) + f(0)\sin(x) - f'(0)\cos x$

so $g'(0) = f'(0) + 0 - f'(0) = 0$

So by (b), $g(x) = 0$ for all x

Therefore $f(x) = f(0)\cos(x) - f'(0)\sin x$ for all x

Therefore $(\cos(x), \sin(x))$ spans V.

4-2

Find out which of the transformations in Exercises 1 through 50 are linear. For those that are linear, determine whether they are isomorphisms.

46. $T\big(f(t)\big) = (t-1)f(t)$ from P to P

Sol. Select arbitrary $f(t), g(t) \in P$

$$T(f(t)+g(t)) = (t-1)(f(t)+g(t))$$
$$=(t-1)f(t)+(t-1)g(t) = T(f(t))+T(g(t))$$

Select arbitrary scalar k

$$T(kf(t)) = (t-1)(kf(t)) = k(t-1)f(t) = kT(f(t))$$

Therefore T is a linear transformation.

Note that T is not isomorphic because T is not surjective. $f(t)=c$ for any constant c except 0 in the target is not mapped by any element in the source by T.

68. For which constants k is the linear transformation

$$T(M) = M\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix} M$$

an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$?

Sol

let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$

So $T(M) = \begin{bmatrix} 5a & b \\ 5c & d \end{bmatrix} - \begin{bmatrix} 2a & 2b \\ kc & kd \end{bmatrix}$

$$= \begin{bmatrix} 3a & -b \\ (5-k)c & (1-k)d \end{bmatrix}$$

Since T is a linear transformation, T is an isomorphism if and only if T is invertible.

And since source and target of T has the same dimension, it suffices that T is injective

So T is an isomorphism if and only if $\ker T = \{\vec{0}\}$ by theorem A on WS 8.

So when $k \neq 1$ and $k \neq 5$, T is an isomorphism (otherwise $\dim(\ker T) = 1$, $\ker T$ is not $\{\vec{0}\}$)

Therefore for $k = 1$ or 5, T is not an isomorphism for other values of k, T is an isomorphism

Part B

Problem 1. Let V be a vector space, and let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a list of vectors in V. Define the function $T : \mathbb{R}^n \to V$ by

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \text{for all} \quad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

(a) Prove that T is a linear transformation.
(b) Prove that T is injective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.
(c) Prove that T is surjective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V.
(d) Prove that T is an isomorphism if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V.

(a) Proof Let $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ be two arbitrary elements in V

① $T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+c_1 \\ \vdots \\ a_n+c_n \end{bmatrix}\right) = (a_1+c_1)\vec{v_1} + (a_2+c_2)\vec{v_2} + \ldots + (a_n+c_n)\vec{v_n} = a_1\vec{v_1} + \ldots + a_n\vec{v_n} + c_1\vec{v_1} + \ldots + c_n\vec{v_n} = T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) + T\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right)$

② Let k be an arbitrary scalar

so $T\left(k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = T\left(\begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix}\right) = ka_1\vec{v_1} + \ldots + ka_n\vec{v_n}$
$= k(a_1\vec{v_1} + \ldots + a_n\vec{v_n})$
$= kT\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right)$

By ① ② we have verified that T is a linear transformation.

(b) ① Proof of T is injective implies $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly independent.

Assume T is injective.

Let $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} = \vec{0}$ be an arbitrary relations on $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$

And consider the trivial relation $0\vec{v_1} + 0\vec{v_2} + \ldots + 0\vec{v_n} = \vec{0}$

So $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} = 0\vec{v_1} + 0\vec{v_2} + \ldots + 0\vec{v_n}$

$$\underline{T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right)} \quad *$$

Since T is injective, $*$ implies $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Therefore only trivial relation exists on $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$

Therefore $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$ is linearly independent.

② Proof of $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly independent implies T is injective.

Assume $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly independent.

Let $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ be arbitrary

Assume $T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = T\left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right)$

Then $a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n} = b_1\vec{v_1} + b_2\vec{v_2} + \ldots + b_n\vec{v_n}$

$(a_1 - b_1)\vec{v_1} + \ldots + (a_n - b_n)\vec{v_n} = 0$

Since $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly independent,

$a_1 - b_1 = a_2 - b_2 = \ldots = a_n - b_n = 0$

So $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Therefore $T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = T\left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right)$ implies $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

So T is injective.

By ①②, we have proved $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly independent if and only if T is injective.

(c) ① Proof of T is surjective implies $(\vec{v_1}, \ldots, \vec{v_n})$ spans V

Assume T is surjective.

Let $\vec{v} \in V$ be an arbitrary element.

Since T is surjective, there exists some $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{R}^n$ such that $T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = \vec{v}$

So $a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n} = \vec{v}$ for some $a_1, a_2, \ldots, a_n$

Therefore every element of V can be written as some linear combination of $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$

So $(\vec{v_1}, \ldots, \vec{v_n})$ spans V by definition.

② Proof of $(\vec{v_1}, \ldots, \vec{v_n})$ spans V implies T is surjective

Assume $(\vec{v_1}, \ldots, \vec{v_n})$ spans V

Let $\vec{v} \in V$ be an arbitrary element.

Since $(\vec{v_1}, \dots, \vec{v_n})$ spans V, $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \dots + a_n\vec{v_n}$ for some $a_1, a_2, \dots, a_n \in \mathbb{R}$

So $T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = \vec{v}$

Since $\vec{v}$ is arbitrary, T is surjective

By ①②, we have proved $(\vec{v_1}, \dots, \vec{v_n})$ spans V if and only if T is surjective

(d) ① Assume T is an isomorphism.

By definition of isomorphism, T is surjective and injective

So by (b) (c), $(\vec{v_1}, \dots, \vec{v_n})$ is linearly independent and spans V

Therefore by definition of basis, $(\vec{v_1}, \dots, \vec{v_n})$ is an ordered basis of V.

② Assume $(\vec{v_1}, \dots, \vec{v_n})$ is an ordered basis of V

So $(\vec{v_1}, \dots, \vec{v_n})$ is linearly independent and spans V by definition of basis

So by (b)(c), T is surjective and injective

Then T is bijection.

Since T is a linear transformation and bijection, T is an isomorphism.

By ①②, we have proved T is an isomorphism if and only if $(\vec{v_1}, \vec{v_2}, \dots, \vec{v_n})$ is an ordered basis of V.

Problem 2. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define the **transpose** of A to be the matrix

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Consider the linear transformation

$$T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad T(A) = \frac{1}{2}(A + A^T).$$

(a) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where

$$\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

is the standard ordered basis of $\mathbb{R}^{2\times 2}$.

(b) Find the $\mathfrak{C}$-matrix of T, where $\mathfrak{C}$ is the ordered basis

$$\mathfrak{C} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

(c) Compute the kernel of $[T]_{\mathcal{E}}$. This will be a subspace of the $\mathcal{E}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.
(d) Find a basis for the corresponding subspace of $\mathbb{R}^{2\times 2}$—that is, for the image of $\ker[T]_{\mathcal{E}}$ under the coordinate isomorphism $L_{\mathcal{E}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.
(e) Compute the kernel of the $\mathfrak{C}$-matrix. This will be a subspace of the $\mathfrak{C}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.
(f) Compute the image of the subspace $\ker[T]_{\mathfrak{C}}$ under the coordinate isomorphism $L_{\mathfrak{C}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.
(g) Compare your answers in (d) and (f). How are they related to $\ker T$?
(h) Find a basis for the image of T using **either** $\mathcal{E}$-coordinates or $\mathfrak{C}$-coordinates (which seems easier?) Don't forget to reinterpret vectors in the coordinate space as elements in $\mathbb{R}^{2\times 2}$!

(a) $T(A) = \frac{1}{2}(A+A^T) = \begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}$, so $[T(A)]_{\mathcal{E}} = \begin{bmatrix} a \\ \frac{b+c}{2} \\ \frac{b+c}{2} \\ d \end{bmatrix}$

Since $[A]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $[T(A)]_{\mathcal{E}} = \begin{bmatrix} a \\ \frac{b+c}{2} \\ \frac{b+c}{2} \\ d \end{bmatrix}$

By generalized key theorem.
the $\mathcal{E}$-matrix $[T]_{\mathcal{E}} = \left[\left[T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} \left[T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} \left[T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right]_{\mathcal{E}} \left[T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]_{\mathcal{E}} \right]$

So the ε-matrix $[T]_\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) Denote $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as V_1, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as V_2, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ as V_3 and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as V_4

So $T(V_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T(V_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T(V_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $T(V_4) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

By generalized key theorem.

the C-matrix $[T]_C = \begin{bmatrix} | & | & | & | \\ [T(V_1)]_C & [T(V_2)]_C & [T(V_3)]_C & [T(V_4)]_C \\ | & | & | & | \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Let $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ be arbitrary.

Assume $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Then $\begin{bmatrix} a \\ \frac{b+c}{2} \\ \frac{b+c}{2} \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, So $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -r \\ r \\ 0 \end{bmatrix}$, $r \in \mathbb{R}$

So $\ker[T]_\mathcal{E} = \left\{ \begin{bmatrix} 0 \\ -r \\ r \\ 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \left\{ r\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$

(d) The corresponding subspace of $\mathbb{R}^{2\times 2}$ is

$$\{ L_\mathcal{E}^{-1}(A) \mid A \in \ker[T]_\mathcal{E} \}$$

Since $L_\mathcal{E} : \mathbb{R}^{2\times2} \to \mathbb{R}^4$ sends $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$L_\mathcal{E}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times2}$ sends $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So

$$\{ L_\mathcal{E}^{-1}(A) \mid A \in \ker[T]_\mathcal{E} \} = \left\{ r\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$$

So a basis of it is $\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$.

(e) Let $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ be arbitrary.

Assume $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Then $\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, So $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \end{bmatrix}$, $r, s, t \in \mathbb{R}$

So $\ker [T]_C = \left\{ r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$

(f) Let $\frac{b+c}{2} = x$, $a = y$, $d = z$, $\frac{b-c}{2} = w$.

So $b = x+w$, $c = x-w$

$L_c : \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^4$ is defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{b+c}{2} \\ a \\ d \\ \frac{b-c}{2} \end{bmatrix}$

So $L_c^{-1} : \mathbb{R}^4 \longrightarrow \mathbb{R}^{2\times2}$ sends $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \longmapsto \begin{bmatrix} y & x+w \\ x-w & z \end{bmatrix}$

Therefore $\{ L_c^{-1}(A) \mid A \in \ker [T]_C \} = \left\{ r \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$

$= \left\{ r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$

(g) The image of the subspace $\ker [T]_\mathcal{E}$ under $L_\mathcal{E}^{-1}$ and the image of the subspace $\ker [T]_C$ under L_C^{-1} are the same.

And they are also the same as $\ker T$.

T sends $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}$

Let $\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

we get $\ker T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{array}{l} a, d = 0, \\ b = -c \\ \in \mathbb{R} \end{array} \right\}$

$= \underbrace{\left\{ r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}}$

which is the same.

ch) We use the c-coordinate

Let $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ be arbitrary.

$$[T(A)]_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$$

So $\operatorname{im}(T) = \left\{ A \in \mathbb{R}^{2\times 2} \middle| [A]_c = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \text{ where } a, b, c \in \mathbb{R} \right\}$

So $\operatorname{im}(T) = \left\{ a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

Therefore a basis of $\operatorname{im}(T)$ is $\underbrace{\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)}$

Problem 3. Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. In other words, every vector $f \in C^\infty(\mathbb{R})$ is a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable k-times for all $k \in \mathbb{N}$. Let $f_1, \dots, f_6$ be the six functions in $C^\infty(\mathbb{R})$ defined by

$$f_1(x) = 1, \quad f_2(x) = \sin(2x), \quad f_3(x) = \cos(2x),$$

$$f_4(x) = \sin^2(x), \quad f_5(x) = \cos^2(x), \quad f_6(x) = \sin x \cos x.$$

Let $V = \operatorname{Span}(f_1, f_2, f_3, f_4, f_5, f_6)$, and let $\mathcal{B} = (f_1, f_2, f_4) = (1, \sin 2x, \sin^2 x)$.

(a) Prove that $\mathcal{B}$ is an ordered basis of V. [*Hint*: For linear independence, write a relation and evaluate it at one or more carefully-chosen values of x. For spanning, remember (or look up) some trig identities.]

(b) For each $i \in \{1, \dots, 6\}$, find $[f_i]_{\mathcal{B}}$.

(c) Show that for all $f \in V$, the derivative of f is also in V.

(d) As a result of (c), we can define the linear transformation $T : V \to V$ by $T(f) = f' + 2f$ for all $f \in V$. Compute the $\mathfrak{B}$-matrix $[T]_{\mathfrak{B}}$ of T.

(e) **Without using Calculus**, find $[T]_{\mathcal{B}}^{-1}$.

(f) Using **matrix methods only** (and **without directly using calculus**), find a function $f(x) \in V$ such that

$$f'(x) + 2f(x) = 4 + 8\sin^2(x)$$

Note: In (e) and (f) you will **not** receive credit for computing integrals using "Calc 2" methods (e.g., u-substitution) or methods from the theory of differential equations.

(a) Let f be an arbitrary element in

so $f = a_1 + a_2 \sin(2x) + a_3 \cos(2x) + a_4 \sin^2(x)$
$+ a_5 \cos^2(x) + a_6 \sin x \cos x$ for some $a_1, \dots, a_6 \in \mathbb{R}$

Then $f = a_1 + a_2 \sin 2x + a_3(1 - 2\sin^2 x) + a_4 \sin^2(x)$
$+ a_5(1 - \sin^2 x) + a_6 \cdot \frac{1}{2} \sin 2x$

$= (a_1 + a_3 + a_5) + (a_2 + \frac{1}{2} a_6) \sin 2x + (a_4 - 2a_3 - a_5) \sin^2 x$
$\in \operatorname{span}(f_1, f_2, f_4)$

So every element of V can be written as some linear combination of f_1, f_2, f_4

So (f_1, f_2, f_4) spans V by definition.

Let $b_1 f_1 + b_2 f_2 + b_4 f_4 = 0$

be an arbitrary relation on (f_1, f_2, f_4)

that is $b_1 + b_2 \sin 2x + b_4 \sin^2 x = 0$

By taking $x = \pi$, we get $b_1 = 0$

Then by taking $x = \frac{\pi}{2}$, we get $b_4 = 0$

Then by taking $x = \frac{\pi}{4}$, we get $b_2 = 0$

So the relation is trivial.

Since the relation is arbitrary, we get (f_1, f_2, f_4) is linearly independent.

Since (f_1, f_2, f_4) spans V and is linearly independent, it is an orderd basis of V by definition

(b) $L_{\mathcal{B}} : V \to \mathbb{R}^3$ sends

$f = a_1 + a_2 \sin(2x) + a_3 \cos(2x) + a_4 \sin^2(x) + a_5 \cos^2(x) + a_6 \sin x \cos x$ for some $a_1, \ldots, a_6 \in \mathbb{R}$

to $[(a_1 + a_3 + a_5), (a_2 + \frac{1}{2} a_6), (a_4 - 2a_3 - a_5)]^T$

So $[f_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[f_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$[f_3]_{\mathcal{B}} = [\cos(2x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$[f_4]_{\mathcal{B}} = [\sin^2 x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[f_5]_{\mathcal{B}} = [\cos^2 x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$[f_6]_{\mathcal{B}} = [\sin x \cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

(c) Let $f = a_1 + a_2 \sin(2x) + a_3 \sin^2(x)$
for some $a_1, a_2, a_3 \in \mathbb{R}$ be an arbitrary element in V

so $f' = 2a_2 \cos 2x + a_3 \cdot (2\sin(x)\cos(x))$

$= 2a_2(1 - 2\sin^2 x) + a_3 \cdot \sin(2x)$

$= 2a_2 + a_3 \sin(2x) - 4a_2 \sin^2 x \in V$

Therefore we have proved for all $f \in V$, f' is also in V.

(d) Take
$f = a_1 + a_2 \sin(2x) + a_3 \sin^2(x) \in V$

So $[f]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$T(f) = f' + 2f = (2a_1 + 2a_2) + (a_3 + 2a_2)\sin 2x + (2a_3 - 4a_2)\sin^2 x.$

So $[T(f)]_{\mathcal{B}} = \begin{bmatrix} 2a_1 + 2a_2 + 0 \\ 0 + 2a_2 + a_3 \\ 0 - 4a_2 + 2a_3 \end{bmatrix}$

By generalized key theorem,

$$[T]_{\mathcal{B}} = \left[[T(1)]_{\mathcal{B}} \; [T(\sin 2x)]_{\mathcal{B}} \; [T(\sin^2 x)]_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & 2 \end{bmatrix}$$

(e) By theorem 2.4.5, we can find the inverse of $[T]_{\mathcal{B}}$ which is $[T]_{\mathcal{B}}^{-1}$

$$\left[\begin{array}{ccc:ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \times\frac{1}{2} \\ \\ \times\frac{1}{2} \end{array} \rightarrow \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} -\frac{1}{2}\times II \\ \times\frac{1}{2} \\ +II \end{array}$$

$$\rightarrow \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ -\frac{1}{4}\times III \\ \times\frac{1}{2} \end{array} \rightarrow \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} \end{array}\right] \begin{array}{l} -II \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} \end{array}\right], \quad \text{So } [T]_{\mathcal{B}}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

So $[T]_{\mathcal{B}}^{-1}$ sends $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto \begin{bmatrix} \frac{1}{2}a - \frac{1}{4}b + \frac{1}{8}c \\ \frac{1}{4}b - \frac{1}{8}c \\ \frac{1}{2}b + \frac{1}{4}c \end{bmatrix}$

(Therefore T^{-1} sends $a_1 + a_2\sin(2x) + a_3\sin^2(x)$ to $(\frac{1}{2}a - \frac{1}{4}b + \frac{1}{8}c) + (\frac{1}{4}b - \frac{1}{8}c)\sin 2x + (\frac{1}{2}b + \frac{1}{4}c)\sin^2 x$)

(f) This is to find $f \in V$ such that

$T(f) = 4+8\sin^2(x)$

So $f = T^{-1}(4+8\sin^2 x) = L_{\mathcal{B}}^{-1}\left([T]_{\mathcal{B}}^{-1}\left(\begin{bmatrix}4\\0\\8\end{bmatrix}\right)\right)$

By (e) we know by $[T]_{\mathcal{B}}^{-1}$,

$\begin{bmatrix}4\\0\\8\end{bmatrix}$ is sent to $\begin{bmatrix}3\\-1\\2\end{bmatrix}$

so $f = L_{\mathcal{B}}^{-1}\left(\begin{bmatrix}3\\-1\\2\end{bmatrix}\right) = \underline{3-\sin 2x+2\sin^2 x}$

Let A and B be sets. Recall from the handout *More Joy of Sets* that we define the *Cartesian product* of A and B to be the set

$$A \times B := \{(a,b) : a \in A \text{ and } b \in B\}.$$

If X and Y are vector spaces, then $X \times Y$ is also a vector space, with addition and scalar multiplication given by

Problem 4. Let X and Y be finite-dimensional vector spaces.

(a) Describe the zero vector of $X \times Y$. (*No justification necessary.*)

(b) Let $\{\vec{x}_1, \ldots, \vec{x}_m\}$ be a basis of X, and let $\{\vec{y}_1, \ldots, \vec{y}_n\}$ be a basis of Y. Prove that

$$\{(\vec{x}_1, \vec{0}_Y), \ldots, (\vec{x}_m, \vec{0}_Y), (\vec{0}_X, \vec{y}_1), \ldots, (\vec{0}_X, \vec{y}_n)\}$$

is a basis of $X \times Y$.

(c) Determine $\dim(X \times Y)$ in terms of $\dim(X)$ and $\dim(Y)$.

(a) Let $0_x, 0_y$ respectively represent the zero vector of X, Y

then the zero vector of $X \times Y$ is $(0_x, 0_y)$.

(b) First we prove $\{(\vec{x_1}, \vec{0_Y}), \ldots, (\vec{x_m}, \vec{0_Y}), (\vec{0_x}, \vec{y_1}), \ldots, (\vec{0_x}, \vec{y_n})\}$

spans $X \times Y$.

Let (a,b) be an arbitrary element of $X \times Y$,
where $a \in X$ and $b \in Y$

So $(a,b) = (a, 0_Y) + (0_X, b)$

Since $(\vec{x_1}, \dots, \vec{x_m})$ is a basis of X and $(\vec{y_1}, \dots, \vec{y_m})$ is a basis of Y,
$(\vec{x_1}, \dots, \vec{x_m})$ spans X and $(\vec{y_1}, \dots, \vec{y_m})$ spans Y by definition.

So $a = a_1\vec{x_1} + a_2\vec{x_2} + \dots + a_m\vec{x_m}$

$b = b_1\vec{y_1} + b_2\vec{x_2} + \dots + b_m\vec{x_m}$ for some scalars $a_1, \dots, a_m$ and $b_1, \dots, b_m$ by definition

So $(a, 0_Y) = a_1(\vec{x_1}, 0_Y) + a_2(\vec{x_2}, 0_Y) + \dots + a_m(\vec{x_m}, 0_Y)$

$(0_X, b) = b_1(0_X, \vec{y_1}) + b_2(0_X, \vec{y_2}) + \dots + b_m(0_X, \vec{y_m})$

So $(a,b) = a_1(\vec{x_1}, 0_Y) + a_2(\vec{x_2}, 0_Y) + \dots + a_m(\vec{x_m}, 0_Y) +$
$b_1(0_X, \vec{y_1}) + b_2(0_X, \vec{y_2}) + \dots + b_m(0_X, \vec{y_m})$

Therefore every element of $X \times Y$ is some linear combination of $\{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y}), (\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})\}$

Therefore $\{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y}), (\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})\}$ spans $X \times Y$ by definition.

Then we prove $\{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y}), (\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})\}$ is linearly independent.

Let $(0_X, 0_Y) = a_1(\vec{x_1}, 0_Y) + a_2(\vec{x_2}, 0_Y) + \dots + a_m(\vec{x_m}, 0_Y) +$
$b_1(0_X, \vec{y_1}) + b_2(0_X, \vec{y_2}) + \dots + b_m(0_X, \vec{y_m})$
be an arbitrary relation.

So $0_X = a_1\vec{x_1} + a_2\vec{x_2} + \dots + a_m\vec{x_m}$,

$0_Y = b_1\vec{y_1} + b_2\vec{y_2} + \dots + b_m\vec{y_m}$

Since $(\vec{x_1}, \dots, \vec{x_m})$ is a basis of X and $(\vec{y_1}, \dots, \vec{y_m})$ is a basis of Y,

$(\vec{x_1}, \dots, \vec{x_m})$ is linearly independent and

$(\vec{y_1}, \dots, \vec{y_m})$ is linearly independent

Therefore $a_1 = a_2 = \dots = a_m = 0$ and $b_1 = b_2 = \dots = b_m = 0$ by definition of linear combination.

Therefore $\underline{\{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y}), (\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})\}}$ is <u>linearly independent.</u>

Since $\{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y}), (\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})\}$ spans $X \times Y$ and is linearly independent, it is an basis of $X \times Y$ by definition

(c) <u>$\dim(X \times Y) = \dim X + \dim Y$</u>

Denote a basis of X by $\mathcal{B}_X = (\vec{x_1}, \vec{x_2}, \dots, \vec{x_m})$
a basis of Y by $\mathcal{B}_Y = (\vec{y_1}, \vec{y_2}, \dots, \vec{y_m})$

Since by (b) we can conclude a basis of $X \times Y$

is $\{\underbrace{(\vec{x_1}, \vec{0_Y}), \dots, (\vec{x_m}, \vec{0_Y})}_{|\mathcal{B}_X| = \dim X}, \underbrace{(\vec{0_X}, \vec{y_1}), \dots, (\vec{0_X}, \vec{y_n})}_{|\mathcal{B}_Y| = \dim Y}\}$ which consists

of exactly all $(\vec{x_i}, \vec{0_Y})$ where $x_i \in \mathcal{B}_X$ and
all $(\vec{0_X}, \vec{y_i})$ where $y_i \in \mathcal{B}_Y$

Since $|\mathcal{B}_X| = \dim X$ and $|\mathcal{B}_Y| = \dim Y$ by definition of basis, $\dim(X \times Y) = |\mathcal{B}_X| + |\mathcal{B}_Y| = \dim X + \dim Y$.

If X and Y are subspaces of some vector space V, we also define the *sum* of X and Y to be the set

$$X + Y \;:=\; \{\vec{x} + \vec{y} : \vec{x} \in X \text{ and } \vec{y} \in Y\} \;\subseteq\; V.$$

It is fairly straightforward to verify that $X + Y$ is a subspace of V, and in fact that $X + Y$ is the smallest subspace containing $X \cup Y$ (in the sense that if Z is any subspace containing $X \cup Y$, then $X + Y \subseteq Z$). (You do not have to prove these facts in what follows, but you may find it good practice to do so!)

Problem 5. Let V be a vector space, and let X and Y be subspaces of V. Define the function $T : X \times Y \to X + Y$ by

$$T(\vec{x}, \vec{y}) := \vec{x} + \vec{y} \quad \text{for all } (\vec{x}, \vec{y}) \in X \times Y.$$

(a) Prove that T is a linear transformation and that T is surjective.
(b) Prove that $\ker(T)$ is isomorphic to $X \cap Y$.
(c) Assuming that X and Y are finite-dimensional, prove that

$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y).$$

(d) Let X and Y be 3-dimensional subspaces of $\mathbb{R}^5$. Is it possible that $X \cap Y = \{\vec{0}\}$? Now instead assume that X and Y are 3-dimensional subspaces of $\mathbb{R}^6$. Is it possible that $X \cap Y = \{\vec{0}\}$? Prove your answers.

(a) Let $(\vec{x_1}, \vec{y_1}), (\vec{x_2}, \vec{y_2})$ be two arbitrary element of $X \times Y$

$T((\vec{x_1}, \vec{y_1}) + (\vec{x_2}, \vec{y_2})) = T(\vec{x_1} + \vec{x_2}, \vec{y_1} + \vec{y_2}) = \vec{x_1} + \vec{x_2} + \vec{y_1} + \vec{y_2}$

$= (\vec{x_1} + \vec{y_1}) + (\vec{x_2} + \vec{y_2}) = T(\vec{x_1}, \vec{y_1}) + T(\vec{x_2}, \vec{y_2})$ ①

Let k be an arbitrary scalar and $(\vec{x}, \vec{y})$ be an arbitrary element of $X \times Y$

So $T(k(\vec{x}, \vec{y})) = k\vec{x} + k\vec{y} = k(\vec{x} + \vec{y}) = k\,T(\vec{x}, \vec{y})$ ②

Since ①②, T is a linear transformation.

Let $\vec{a}$ be an arbitrary element in $X + Y$.

Consider $T(\vec{0_x}, \vec{a}) = \vec{0_x} + \vec{a} = \vec{a}$ since X is a subspace which is guranteed to contain $\vec{0_x}$ by definition of subspace.

Therefore for every element in $X+Y$, there is some element in $X\times Y$ that is mapped to it by T.

So T is surjective by definition.

(b) Proof. Assume $\underbrace{\vec{x}+\vec{y}=\vec{0}_V}$, then $\vec{x}=-\vec{y}$

So $-y\in X$ and $-x\in Y$

then $y\in X$ and $x\in Y$ since vector space preserves scalar multiplication.

So $\underbrace{\ker T}=\{(\vec{a},-\vec{a})\mid \vec{a}\in X \text{ and } \vec{a}\in Y\}$

$=\underbrace{\{(\vec{a},-\vec{a})\mid \vec{a}\in X\cap Y\}}$

Consider the linear transformation

$T_1:\ker T\to X\cap Y$ sending

$(\vec{x},-\vec{x})\mapsto \vec{x}$

This is a linear transformation since

① Take arbitrary element $(\vec{x},-\vec{x})$ and $(\vec{y},-\vec{y})\in\ker T$,

$T_1((\vec{x},-\vec{x})+(\vec{y},-\vec{y}))=\vec{x}+\vec{y}=T_1(\vec{x},-\vec{x})+T_1(\vec{y},-\vec{y})$

② Take arbitrary element $(\vec{a},-\vec{a})\in\ker T$ and arbitrary scalar k

$T_1(k(\vec{a},-\vec{a}))=T_1(k\vec{a},-k\vec{a})=k\vec{a}=kT_1(\vec{a},-\vec{a})$

And it is injective since: assume $T_1(\vec{a},-\vec{a})=T_1(\vec{b},-\vec{b})$

we get $\vec{a}=\vec{b}$

And it is surjective since if we select arbitrary element $\vec{x}\in X\cap Y$

Consider $(\vec{x},-\vec{x})\in\ker T$, we have $T_1(\vec{x},-\vec{x})=\vec{x}$

So T_1 is bijective

Since T_1 is a linear transformation and is bijective, it is an isomorphism by definition.

So $\ker T \cong X \cap Y$.

(c) Since $\ker T \cong X \cap Y$, $\underbrace{\dim(\ker T) = \dim(X \cap Y)}_{\text{by theorem } 4.2.4}$ ①

Since $T: X \times Y \to X+Y$ is surjective

By 4(c) we have proved $\dim(X \times Y) = \dim X + \dim Y$ ②

Since T is surjective, $\dim(\operatorname{im}(T)) = \dim(\text{target})$
$= \dim(X+Y)$ ③

By rank-nullity theorem, $\dim(\operatorname{im}(T)) + \dim(\ker(T)) = \dim(X \times Y)$

So $\dim(X+Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$

(d) if X and Y are 3-dimensional subspaces of $\mathbb{R}^5$, it is not possible that $X \cap Y = \{\vec{0}\}$. but it is possible if it is $\mathbb{R}^6$ instead of $\mathbb{R}^5$

Proof Assume $X \cap Y = \{\vec{0}\}$. then $\dim(X \cap Y) = 0$.

By (c), $\dim(X+Y) = \dim(X) + \dim(Y) = 6$

Assume X, Y are subspaces of $\mathbb{R}^5$.

So $X+Y$ is a subspace of $\mathbb{R}^5$ by definition,

by theorem A on worksheet 12,

$\dim(X+Y) \le \dim(\mathbb{R}^5) = 5$, which contradicts with $\dim(X+Y) = 6$

So it is impossible.

Now we assume X, Y are subspaces of $\mathbb{R}^6$.

Consider $X = \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

$Y = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ x \\ y \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$

So $X + Y = \mathbb{R}^6$, $X \cap Y = \{\vec{0}\}$

Therefore the $\mathbb{R}^6$ case exists.