Worksheet 20: Inner Product Spaces (§5.5)

An **inner product** on a vector space V is a function that assigns a real number $\langle v, w\rangle \in \mathbb{R}$ to pairs of vectors $v, w \in V$ and has all the important algebraic properties of the dot product. Specifically, inner products are:

- *symmetric*: $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.
- *Linear in the first argument*: $\langle au+bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$ for all $a, b \in \mathbb{R}$ and $u, v, w \in V$.
- *Linear in the second argument*: $\langle w, au+bv\rangle = a\langle w, u\rangle + b\langle w, v\rangle$ for all $a, b \in \mathbb{R}$ and $u, v, w \in V$.
- *positive definite*: $\langle v, v\rangle > 0$ for all nonzero $v \in V$.

Problem 1. Show that the rule

$$\langle \vec{x}, \vec{y}\rangle \;=\; \vec{x}^{\top}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\vec{y}$$

is an inner product on $\mathbb{R}^2$ which is different from the dot product. Find an explicit formula for $\langle \begin{bmatrix}x_1\\ x_2\end{bmatrix}, \begin{bmatrix}y_1\\ y_2\end{bmatrix}\rangle$. Does $\langle \vec{x}, \vec{y}\rangle = \vec{x}^{\top}\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\vec{y}$ define an inner product on $\mathbb{R}^2$?

Solution: By multiplying out the product $\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$, we see that

$$\langle \vec{x}, \vec{y}\rangle \;=\; x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2.$$

Symmetry follows easily from the above formula, linearity in both arguments follows from the rules of matrix multiplication, and positive definiteness follows from the fact that the square of a nonzero number is always positive. This inner product is different from the dot product because, for instance, $\langle \vec{e}_2, \vec{e}_2\rangle = 2 \neq 1 = \vec{e}_2 \cdot \vec{e}_2$. The second formula does not define an inner product because the formula $\langle \vec{x}, \vec{y}\rangle = x_1y_2 + 2x_2y_2$ is not symmetric: $\langle \vec{e_1}, \vec{e_2}\rangle = 1$ but $\langle \vec{e_2}, \vec{e_1}\rangle = 0$.

Definition: An **inner product space** is a vector space V, together with a *choice* of an inner product on V.

Problem 2. Three of the five in (a) through (e) below define inner product spaces. Which ones?
[Time-saving Tip: Do you actually need to check linearity in both arguments?]

(a) $V =$ the space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$,

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt.$$

(b) $V =$ the space of polynomials in the variable t of degree at most 2,

$$\langle f, g\rangle = \int_{0}^{1} f(t)g(t)dt.$$

(c) $V =$ the space of infinite sequences $a_1, a_2, \ldots$,

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n b_n$$

(d) $V = \mathbb{R}^{n \times n} =$ the space of all $n \times n$ matrices,

$$\langle A, B \rangle = \operatorname{tr}(A + B).$$

(e) $V = \mathbb{R}^{m \times n} =$ the space of all $m \times n$ matrices,

$$\langle A, B \rangle = \operatorname{tr}(A^\top B).$$

Solution: (a), (b), and (e) are inner products. (c) is not because $\langle f, g \rangle$ may not be a real number (it may be ∞). (d) is not because it is not positive definite.

Problem 3. Let $(V, \langle -, - \rangle)$ be an inner product space.* For any vectors $v_1, v_2, w_1, w_2 \in V$, show that $\langle v_1 + w_1, v_2 + w_2 \rangle = \langle v_1, v_2 \rangle + \langle v_1, w_2 \rangle + \langle w_1, v_2 \rangle + \langle w_1, w_2 \rangle$.

Solution: Using the bilinearity, we have

$$\langle v_1 + w_1, v_2 + w_2 \rangle = \langle v_1 + w_1, v_2 \rangle + \langle v_1 + w_1, w_2 \rangle = \langle v_1, v_2 \rangle + \langle w_1, v_2 \rangle + \langle v_1, w_2 \rangle + \langle w_1, w_2 \rangle$$

Definition: A set of vectors $\{v_1, \ldots, v_n\}$ in an inner product space is **orthonormal** if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$ and 1 if $i = j$.

Problem 4. Find two different inner products on $\mathbb{R}^2$, one such that $\{\vec{e}_1, \vec{e}_2\}$ is an orthonormal set and one such that it's not.

Solution: The dot product is an inner product on $\mathbb{R}^2$ for which the vectors $\{\vec{e}_1, \vec{e}_2\}$ are orthonormal. But for the inner product in Problem 1, $\{\vec{e}_1, \vec{e}_2\}$ are not orthonormal, since $\langle \vec{e}_1, \vec{e}_2 \rangle = 1$.

Problem 5. Explain why every finite dimensional inner product space $(V, \langle -, - \rangle)$ has an orthonormal basis and how to find one. [Hint: Gram-Schmidt.]

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*Let's explain this strange notation "$(V, \langle -, - \rangle)$." Remember that an inner product space is a vector space V on which we have defined a function $f : V \times V \to \mathbb{R}$ called an *inner product* that is bilinear, symmetric, and positive-definite. But we don't usually call inner products "f"; rather, we use the notation $\langle \vec{x}, \vec{y} \rangle$ for the inner product of $\vec{x}$ and $\vec{y}$. So $\langle -, - \rangle$ is just a name for our inner product, where the "-" are placeholders indicating that the (two) arguments of the inner product go there. So, taken as a whole, the notation "$(V, \langle -, - \rangle)$" refers to our inner product space and indicates that its underlying vector space is V and that $\langle \vec{x}, \vec{y} \rangle$ is the notation we will be using for the inner product of $\vec{x}, \vec{y} \in V$.

Solution: Since V is a finite dimensional vector space, we know that V has a (finite) basis, say $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$. But now if we apply the Gram-Schmidt process to $\mathcal{B}$, we will obtain an orthonormal set $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ of n vectors in V. Since orthonormal sets of vectors are linearly independent, $\mathcal{U}$ must be a basis of V.

Problem 6. Find an orthonormal basis for the inner product spaces in Problems 1 and 2(b).

Solution: Relative to the inner product defined in Problem 2, $\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right)$ is an orthonormal basis of $\mathbb{R}^2$. Relative to the inner product defined on V in Problem 2b,

$$\left(1, \sqrt{12}(t - \tfrac{1}{2}), \sqrt{180}(t^2 - t + \tfrac{1}{6})\right)$$

is an orthonormal basis of V.

Problem 7. Let $(V, \langle -, - \rangle)$ be the inner product space from Problem 2(a) above.

(a) Are the vectors 1, $\sin t$, and $\cos t$ are orthonormal? How we can orthonormalize them?

(b) Find the orthogonal projection of the function $y = e^t$ onto the subspace $T_1 = \text{Span}\,(1, \sin t, \cos t)$.
[HINT: Recall that $\int e^t \sin t\, dt = \frac{1}{2}e^t(\sin t - \cos t) + C$ and $\int e^t \cos t\, dt = \frac{1}{2}e^t(\sin t + \cos t) + C$.]

Solution: For (a), the vectors are not orthonormal, though they are orthogonal. We can scale each by their length to get an orthonormal basis $\left(\frac{1}{\sqrt{2}}, \sin t, \cos t\right)$ for T_1.

Then

$$\begin{aligned}
\langle e^t, \tfrac{1}{\sqrt{2}} \rangle &= \frac{e^{\pi} - e^{-\pi}}{\sqrt{2}\pi} \\
\langle e^t, \sin t \rangle &= \frac{e^{\pi} - e^{-\pi}}{2\pi} \\
\langle e^t, \sin t \rangle &= -\frac{e^{\pi} - e^{-\pi}}{2\pi},
\end{aligned}$$

so the projection of e^t onto T_1 is

$$\frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{e^{\pi} - e^{-\pi}}{2\pi}\sin t - \frac{e^{\pi} - e^{-\pi}}{2\pi}\cos t.$$

Problem 8. Prove that if $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal basis of the inner product space V, then

$$\langle \vec{x}, \vec{y} \rangle = [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}$$

for all $\vec{x}, \vec{y} \in V$. Is this still true if $\mathcal{U}$ is any basis of V, not necessarily orthonormal?

Solution: Let $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ be an orthonormal basis of V, let $\vec{x}, \vec{y} \in V$, and write $\vec{x} = \sum_{i=1}^n a_i \vec{u}_i$ and $\vec{y} = \sum_{i=1}^n b_i \vec{u}_i$. Then, using the fact that $\mathcal{U}$ is orthonormal, we get

$$\langle \vec{x}, \vec{y} \rangle \;=\; \left\langle \sum_{i=1}^n a_i \vec{u}_i, \sum_{i=1}^n b_i \vec{u}_i \right\rangle \;=\; \sum_{i=1}^n \sum_{j=1}^n a_i b_j \langle \vec{u}_i, \vec{u}_j \rangle \;=\; \sum_{i=1}^n a_i b_i \;=\; [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}.$$

If $\mathcal{U}$ is not orthonormal, then this could fail. For instance, letting V be the inner product space in Problem 1 with (non-orthonormal) basis $\mathcal{E}$, we see that $\langle \vec{e}_1, \vec{e}_2 \rangle = 1 \neq 0 = [\vec{e}_1]_{\mathcal{E}} \cdot [\vec{e}_2]_{\mathcal{E}}$.

Problem 9. Let $\langle -, - \rangle$ be an inner product on $\mathbb{R}^n$. Show that there exists an $n \times n$ matrix A such that

$$\langle \vec{x}, \vec{y} \rangle \;=\; \vec{x}^T A \, \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Show that A is symmetric. [Hint: How can you recover A from the inner products $\langle \vec{e}_i, \vec{e}_j \rangle$?]

Solution: Given an $n \times n$ matrix A and $1 \leq i, j \leq n$, the (i,j)-entry of A is just $\vec{e}_i^T A \vec{e}_j$. Thus we let A be the $n \times n$ matrix whose (i,j)-entry is $a_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have

$$\langle \vec{x}, \vec{y} \rangle \;=\; \left\langle \sum_{i=1}^n x_i \vec{e}_i, \; \sum_{j=1}^n y_j \vec{e}_j \right\rangle \;=\; \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \vec{e}_i, \vec{e}_j \rangle \;=\; \sum_{i=1}^n x_i \left(\sum_{j=1}^n a_{ij} y_j \right) \;=\; \vec{x}^T A \vec{y}.$$

Note that A is symmetric, since for all $1 \leq i, j \leq n$, $A_{i\,j} = \langle \vec{e}_i, \vec{e}_j \rangle = \langle \vec{e}_j, \vec{e}_i \rangle = A_{j\,i}$.

Problem 10. Describe/classify *all* inner products on $\mathbb{R}^2$.

Solution: For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the function $\beta_A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\beta_A(\vec{x}, \vec{y}) \;=\; \vec{x}^T A \vec{y}$$

will be linear in both arguments by the properties of matrix multiplication. Furthermore, β_A will be symmetric precisely when A is symmetric (i.e., when $b = c$). What is a bit harder to see is that β_A will be positive definite if and only if both $a > 0$ and $\det(A) > 0$. Thus by Problem (7), the inner products on $\mathbb{R}^2$ are just those functions of the form β_A where A is a symmetric matrix whose determinant and upper-left entry are both positive.

For the claim about positive definiteness, first note that a simple example such as $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ shows that the conditions $a > 0$ and $\det A > 0$ are necessary. To see that these conditions are sufficient, assume them, and let $(x, y) \neq \vec{0}$; we must show

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; ax^2 + 2bxy + dy^2 \;>\; 0. \tag{$*$}$$

Since $a > 0$, $(*)$ holds if $y = 0$, so we may assume $y \neq 0$. Then by linearity, we may further assume after scaling by a positive number that $y = \pm 1$. But then, using the discriminant, $(*)$ reduces to the condition $(2b)^2 - 4ad = 4(b^2 - ad) < 0$, which holds whenever $\det(A) > 0$.