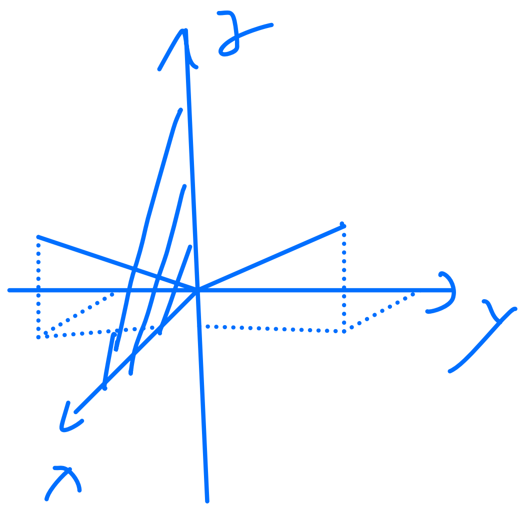
2-2

Find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ given in Exercises 19 through 23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

20. The reflection about the x–z-plane.

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$



The reflection of $\vec{v}$ about the x-z plane is $\begin{bmatrix} v_1 \\ -v_2 \\ v_3 \end{bmatrix}$ since x, z-coordinate is unchanged and y-coordinate is reversed

So $T(\vec{v}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{v}$

38. The *determinant* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$ (we have seen this quantity in Exercise 2.1.13 already). Find the determinant of a matrix that represents a(n)

a. orthogonal projection **b.** reflection about a line

c. rotation **d.** (horizontal or vertical) shear.

What do your answers tell you about the invertibility of these matrices?

a. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be an arbitrary unit vector

By def 2.2.1, it represents any matrix of orthogonal projection and the matrix is $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$

its determinant is $u_1^2 u_2^2 - (u_1 u_2 \cdot u_1 u_2) = \boxed{0.}$

By theorem 2.4.9 the matrix is not invertible.

b. By def 2.2.2, matrix A representing a reflection about a line is $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2+b^2=1$

So $\det(A) = a(-a) - b^2 = -(a^2+b^2) = \boxed{-1}$

By theorem 2.4.9 the matrix is invertible.

c. By theorem 2.2.4, matrix A representing a rotation is $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a^2+b^2=1$

So $\det(A) = a^2 - b(-b) = a^2+b^2 = 1$

By theorem 2.4.9 the matrix is invertible.

d. By theorem 2.2.5

the matrix of a shear is $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for horizontal and $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ for vertical

No matter in which case, $\det A = 1$ and this shows that the matrix is invertible.

2-3

In the Exercises 17 through 26, find all matrices that commute with the given matrix A.

18. $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

Suppose $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with A

Then $AB = BA$ by definition

$$AB = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+3c & 2b+3d \\ -3a+2c & -3b+2d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2a-3b & 3a+2b \\ 2c-3d & 3c+2d \end{bmatrix}$$

So we have the linear system:

$$\begin{cases} 2a+3c = 2a-3b \\ 2b+3d = 2b+3a \\ 2c-3d = 2c-3a \\ 2d-3b = 2d+3c \end{cases} \Rightarrow \begin{matrix} b = -c \\ a = d \end{matrix}$$

Therefore B is any matrix of form: $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

For the matrices A in Exercises 33 through 42, compute $A^2 = AA$, $A^3 = AAA$, and A^4. Describe the pattern that emerges, and use this pattern to find A^{1001}. Interpret your answers geometrically, in terms of rotations, reflections, shears, and orthogonal projections.

34. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^3A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

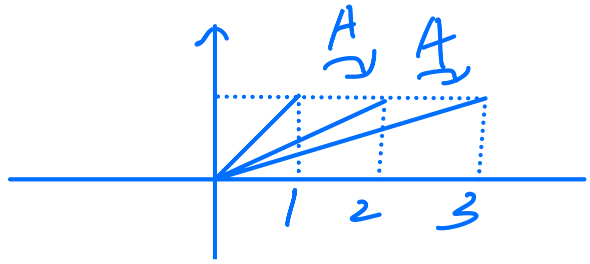
By this pattern, $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$

(Can be induced: Base case $A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Inductive step: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \Rightarrow A^{k+1} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$

So for all $k \in \mathbb{Z}^+$, $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$)

Geometrical interpretation:



The horizontal shear shears a vector's x-axis by 1 unit every time.

2-4

Decide whether the matrices in Exercises 1 through 15 are invertible. If they are, find the inverse. Do the computations with paper and pencil. Show all your work.

12. $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

Denote the matrix by A

By $\boxed{\text{Theorem 2.4.5}}$, we compute $\operatorname{rref}(A \vdots I_4)$ to determine whether A is invertible and to get A^{-1} if A is invertible.

$$A \vdots I_4 = \left[\begin{array}{cccc|cccc} 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2\times I \\ \\ -2\times III \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} +3\times II \\ \times(-1) \\ -2\times IV \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right].$$

So by $\boxed{\text{Theorem 2.4.5}}$, A is invertible

and $A^{-1} = \begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

34. Consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

a. For which values of a, b, and c is A invertible? If it is invertible, what is A^{-1}?

b. For which values of the diagonal elements is a diagonal matrix (of arbitrary size) invertible?

a. By elementary transformation, (divide every row vector by a, b, c respectively), we can get $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ whenever $a, b, c \neq 0$

So A is inverible if $a, b, c \neq 0$. by theorem 2.4.3.

By theorem 2.4.5, since $\text{rref}\left(\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{c} \end{array}\right]$,

$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

b. By a., a diagonal matrix is invertible if all its diagonal elements are not 0.

Part B (25 points)

The definitions of *trace*, *determinant* and *transpose* will be needed in this part.

Definition 1. Given a square $n \times n$ matrix $C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$, we define the **trace** of C to be the sum of the diagonal elements $c_{11} + \cdots + c_{nn} = \sum_{i=1}^{n} c_{ii}$, denoted $\mathrm{tr}(C)$.

Definition 2. The **determinant** of a square matrix C will be denoted $\det(C)$. We define the determinant of a 1×1 matrix by $\det[a] = a$, and the determinant of a 2×2 matrix by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ (We will wait until Chapter 6 to define determinants of larger square matrices).

Definition 3. Consider an $m \times n$ matrix A. The **transpose** $A^\top$ of A is the $n \times m$ matrix obtained from A by rewriting all of the columns of A as rows, and vice versa, so that the (i, j)-entry of $A^\top$ is the (j, i)-entry of A. Further, we say that the square matrix A is **symmetric** if $A^\top = A$.

Problem 1. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) For all 2×2 matrices A and B, $(AB)^\top = A^\top B^\top$.
(b) For all 2×2 matrices A and B, $(AB)^\top \neq A^\top B^\top$.
(c) For all matrices A and B such that the matrix product AB exists, $(AB)^\top = B^\top A^\top$.
(d) If A is a symmetric matrix, then for all $n \in \mathbb{N}$, A^n is also symmetric.
(e) If A is a square matrix and A^2 is symmetric, then so is A.

(a) False.

Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}$

$\Rightarrow B^T = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A^T B^T = \begin{bmatrix} 5 & 5 \\ 4 & 4 \end{bmatrix}$

(b) False.

Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow AB = A^T B^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

(c) True.

Pf. Since AB exists, by $\boxed{\text{Definition 2.3.1}}$,

Let A be a $n \times p$ matrix
B be a $p \times m$ matrix
where n, p, m are arbitrary integers

So A, B can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2m} \\ \vdots & & & & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pm} \end{bmatrix}$$

$$B^T = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{p1} \\ \vdots & & & \vdots \\ b_{1j} & b_{2j} & \cdots & b_{pj} \\ \vdots & & & \vdots \\ b_{1m} & b_{2m} & \cdots & b_{pm} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{i1} & \cdots & a_{n1} \\ a_{12} & \cdots & a_{i2} & \cdots & a_{n2} \\ \vdots & & \vdots & & \vdots \\ a_{1p} & \cdots & a_{ip} & \cdots & a_{np} \end{bmatrix}$$

First, we claim that $(B^T \cdot A^T)$ exists by $\boxed{\text{def 2.3.1}}$, and the shape is $m \times n$ which is the same with $(AB)^T$.

By $\boxed{\text{Theorem 2.3.4}}$

The ij^{th} entry of $A \cdot B$ is $\sum_{k=1}^{p} a_{ik} b_{kj}$ $(1 \le i \le n, 1 \le j \le m)$

So by definition of transpose,

the ji^{th} entry of $(A \cdot B)^T$ is $\sum_{k=1}^{p} a_{ik} b_{kj}$

Similarly, the ji^{th} entry of $(B^T \cdot A^T)$ is $\sum_{k=1}^{p} b_{kj} \cdot a_{ik}$

$$= \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= \text{the } ji^{th} \text{ entry of } (AB)^T$$

Since i, j are arbitrary we know every corresponding entry of $B^T A^T$ and $(AB)^T$ are the same

So $(AB)^T = B^T A^T$.

Since A, B are arbitrary matrix,

We have proved the general case.

(d) $\boxed{\text{True.}}$ We will prove it by induction.

Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ be a symmetric matrix by definition

where a, b are arbitrary entries

Base case: $A^1 = A$ is symmetric by definition

Inductive step: We want to show

A^{n+1} is symmetric whenever A^n is symmetric, where $n \in \mathbb{N}$.

So assume that A^n is symmetric,

so $A^n = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ for some c, d (of same type as a, b).

Then $A^{n+1} = A^n \cdot A = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$$= \begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix}$$

By definition it is symmetric.

Since we have base case and inductive step as true statement we have proved that for all $n \in \mathbb{N}$, if A is symmetric, then A^n is also symmetric

(e) $\boxed{\text{False}}$

Counterexample: $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, not symmetric

$A^2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, symmetric.

Problem 2. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) Every 3-by-3 matrix that has a row of zeros is not invertible.
(b) Every square matrix with 1's down the main diagonal is invertible.
(c) For any matrix A, if A is invertible, then so is A^{-1}.
(d) For any matrix A, if A is invertible, then A^n is invertible.

(a) True.

Pf Denote the matrix by A.
Since A has a row of zeros, there are at most $(3-1)=2$ leading 1's in rref(A)

So by $\boxed{\text{definition 1.3.2}}$, $\operatorname{rank}(A) \leq 2$

Therefore by $\boxed{\text{Theorem 2.4.3}}$, A is not invertible

(b) False

Counterexample : $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$$

So by $\boxed{\text{Theorem 2.4.3}}$, A is not invertible

(c) True.

Pf Let A be a $n \times n$ matrix

Assume A is invertible

By Theorem 2.4.3

$$A^{-1}A = AA^{-1} = I_n$$

So for A^{-1}, We have $AA^{-1} = A^{-1}A = I_n$

By definition of invertible matrix on worksheet 7, A^{-1} is invertible

(d) <u>True</u>

Assume A is an invertible $m \times m$ square matrix, where m is an arbitrary positive integer

Let n be an arbitrary positive integer.

Since A is invertible, there exist $m \times m$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_m$

So Consider $(A^{-1})^n$

By $\boxed{\text{Theorem 2.3.6}}$ (matrix multiplication is associative),

$$(A^{-1})^n \cdot A^n = (A^{-1} \cdot (A^{-1} \cdot \ldots (A^{-1}(A^{-1} \cdot A) A) \ldots \cdot A) \cdot A)$$
$$= (A^{-1} \ldots (A^{-1} I_n \cdot A) \ldots \cdot A)$$
$$= (A^{-1} \ldots A^{-1}(A^{-1}A) A \ldots A)$$
$$= (A^{-1} \ldots (A^{-1} I_n \cdot A) \ldots A)$$
$$= A^{-1}A = I_m$$

Similarly, $A^n \cdot (A^{-1})^n = I_m$

So By definition of invertible matrix on worksheet A^n is invertible, with $(A^{-1})^n$ as its inverse matrix?

Problem 3. Let A be an $m \times n$ matrix.

(a) Prove that if there exists an $n \times m$ matrix B such that $BA = I_n$, then the system of linear equations $A\vec{x} = \vec{0}$ has a unique solution. (Note: a matrix B with this property is called a *left-inverse* for A. Can you guess why?)

(b) (**Recreational**) State and prove the converse of the statement in (a).

(a) Proof. Assume $\vec{x_1}, \vec{x_2}$ are two solutions to $A\vec{x} = \vec{0}$, that is, $A\vec{x_1} = \vec{0}$, $A\vec{x_2} = \vec{0}$

So $A\vec{x_1} - A\vec{x_2} = \vec{0} - \vec{0} = \vec{0}$

$A(\vec{x_1} - \vec{x_2}) = \vec{0}$ by $\boxed{\text{Theorem 1.3.10}}$ (Algebraic rules)

Since BA exists and since $A\vec{x}$ and $\vec{0}$ has m components by definition of matrix times vector so $\underset{n\times m}{B}\underset{m}{\vec{0}}$ exists,

We can multiply B to the left of both sides.

Then we have $BA(\vec{x_1} - \vec{x_2}) = B\vec{0} = \vec{0}$

Since $BA = I_n$, $\vec{x_1} - \vec{x_2} = \vec{0}$, So $\vec{x_1} = \vec{x_2}$

Therefore we have proved any solutions of $A\vec{x} = \vec{0}$ are equal, which means that $A\vec{x} = \vec{0}$ has unique solution.

Problem 4. Given two matrices A and B such that the product AB is defined (say, A is $n \times m$ and B is $m \times k$), exactly one of the following two statements is true:

(a) Every column of AB is a linear combination of columns of A,
(b) Every column of AB is a linear combination of columns of B.

Prove the one that is true, and provide a counterexample for the one that is false.

(a) $\boxed{\text{True.}}$

Denote A by
$$\begin{array}{cc} \overleftrightarrow{\quad m \quad} & \\ \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \\ | & | & & | \end{bmatrix} & \updownarrow n \end{array}$$

B by
$$\begin{array}{cc} \overleftrightarrow{\quad n \quad} & \\ \begin{bmatrix} | & | & & | \\ \vec{w_1} & \vec{w_2} & \cdots & \vec{w_n} \\ | & | & & | \end{bmatrix} & \updownarrow m \end{array}$$

for some column vectors $\vec{v_1}, \dots, \vec{v_m}$, $\vec{w_1}, \dots, \vec{w_n}$,

By $\boxed{\text{theorem 2.3.2}}$,
$$AB = A \begin{bmatrix} | & | & & | \\ \vec{w_1} & \vec{w_2} & \cdots & \vec{w_n} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{w_1} & A\vec{w_2} & \cdots & A\vec{w_n} \\ | & | & & | \end{bmatrix}$$

Let i be an arbitrary integer with $1 \le i \le n$

take $A\vec{w_i}$, that is, any column vector of AB,

$$A\vec{w_i} = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \\ | & | & & | \end{bmatrix} \vec{w_i}$$

Denote $\vec{w_i}$ by $\begin{bmatrix} w_{i1} \\ \vdots \\ w_{im} \end{bmatrix}$ for its components.

By $\boxed{\text{Theorem 1.3.8}}$, $A\vec{w_i} = w_{i1}\vec{v_1} + w_{i2}\vec{v_2} + \dots + w_{im}\vec{v_m}$

Therefore, every column of AB is a linear combination of columns of A.

(b) False

Counterexample:

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since $\text{rref}\left(\left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right]\right) = \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

Which shows that the linear system

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is inconsistent

So by worksheet 4 Problem 5,

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem 5. Let $f\colon X \to X$ be a function. We let f^n denote the function $f^n\colon X \to X$ given by composing f iteratively, n many times. In other words, $f^n(x) = \underbrace{(f \circ \cdots \circ f)}_{n \text{ times}}(x)$. Also, we define f^0 to be the identity function, i.e. $\forall x \in X$, $f^0(x) = x$.

(a) Assume that $X = \mathbb{R}^d$. Prove by induction that if f is a linear transformation, then the nth iterate f^n is also a linear transformation.

(b) Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is **not** a linear transformation, but for which there exists an n such that the nth iterate f^n is a linear transformation.

(c) Prove that for $X = \mathbb{R}^d$ and f linear, if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.

(a) Assume that f is a linear transformation.
Prove by Induction on $m \in \mathbb{N}$.

Base case: $m = 1$, $f^m(x) = f(f^0(x)) = f(x)$ is a linear transformation by assumption.

Inductive step: We want to show: $f^{m+1}(x)$ is a linear transformation whenever $f^m(x)$ is a linear transformation.

Assume for $m \in \mathbb{N}$, $f^m(x)$ is a linear transformation.

Since f is a linear transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$

By key theorem on worksheet 4, (definition 2.1.1 on book)

there exist a $d \times d$ matrix A such that $f(x) = Ax$ $(x \in \mathbb{R}^d)$

Since by composing a function from $\mathbb{R}^d$ to $\mathbb{R}^d$ m times, source and target is still $\mathbb{R}^d$

So the linear tranformation $f^m(x)$ is also from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Then by key theorem on worksheet 4. (definition 2.1.1 on book)

there exist a $d \times d$ matrix B such that $f^m(x) = Bx$ $(x \in \mathbb{R}^d)$

Then $f^{m+1}(x) = f \circ f^m(x) = f(f^m(x))$

$= f(Bx) = A(Bx) = (AB)x$

by theorem 2.3.6

By definition 2.3.1, (AB) is a $d \times d$ matrix.

So $f^{m+1}(x) = (AB)x$.

By definition 2.1.1, $f(x)$ is a linear transformation.

Therefore we have proved $f^m(x)$ is linear transformation implies $f^{m+1}(x)$ is linear transformation

Since we have proved the base case and inductive step. we have proved that if $f(x)$ is a linear transformation, then $f^n(x)$ is a linear transformation for all $n \in \mathbb{N}$.

(b) Def: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x) = \begin{cases} -x, & \text{if } \|x\| = 2\pi \\ x, & \text{otherwise.} \end{cases}$$

$f(x)$ is not linear since

if we choose x_1, x_2 where $\|x_1\| = 2\pi$, $\|x_2\| \neq 2\pi$

Then $f(x_1 + x_2) = x_1 + x_2$

$f(x_1) + f(x_2) = -x_1 + x_2 \neq f(x_1 + x_2)$.

But $f^2(x)$ is linear since for any $\|x\| = 2\pi$,

$f^2(x) = f(f(x)) = f(-x) = x$

So $f^2(x) = x$ for all $x \in \mathbb{R}^2$, which is the identity function, which is linear transformation.

(c) Since $f: \mathbb{R}^d \to \mathbb{R}^d$ is a linear transformation, by key theorem on worksheet 4, there exist a $d \times d$ matrix A such that $f(x) = Ax$ for all $x \in \mathbb{R}^d$.

Assume $f(x)=0$ has a unique solution

this means system $Ax=0$

has a unique solution

By theorem 1.3.4, $\text{rref}(A) = I_d$,

that is, $\text{rank}(A) = d$.

So by theorem 2.4.3, A is invertible

In problem 3 we have proved that

if A is invertible, then for any $n \in \mathbb{N}$,

A^n is invertible.

Therefore A^n is invertible for any $n \in \mathbb{N}$.

So $f^n(x) = A\cdot(A\cdot(A\cdots(Ax)\cdots)A)$

$= \underbrace{(A\cdot A \cdots \cdot A)}_{n \text{ times}} x$ by theorem 2.3.7

So $f^n(x) = A^n x$

Since A^n is invertible, $\text{rank}(A^n) = n$

So by theorem 1.3.4, $A^n x = 0$ has a unique solution.

Then we have proved the statement.