Exam 1 Review Problems

Math 217

<u>Problem 1</u>: Determine whether the following statements are true or false. (They are all true)

(1) Every vector space has a basis.

(2) Every linearly independent set in a vector space V is contained in a basis of V.

(3) Every spanning set of vectors in a vector space V contains a basis of V.

(4) No set of more than n vectors in $\mathbb{R}^n$ is linearly independent.

(5) No set of fewer than n vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$.

(6) Every basis of $\mathbb{R}^n$ has exactly n vectors in it.

(7) If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$.

(8) Every linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

(9) Every spanning set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

(10) If the vector space V has a finite spanning set, then V has a finite basis, and any two bases of V have the same number of elements.

(11) Any two bases of a vector space V have the same number of elements, even if this number is infinite.

(12) If V is a subspace of the vector space W, then $\dim(V) \leq \dim(W)$.

(13) If V is a subspace of the finite-dimensional vector space W and $\dim(V) = \dim(W)$, then $V = W$.

(14) An infinite-dimensional vector space could have the same dimension as some proper subspace of itself.

<u>Problem 2</u>: Determine whether the following statements are true or false.

(1) If $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$. False

(2) A system of four linear equations in three unknowns is always inconsistent. False

(3) For any $n \times n$ matrices A and B, if $AB = 0$ then $A = 0$ or $B = 0$. False

(4) If $n > m$, then no linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be injective. True

(5) If $n > m$, then no linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be surjective. False

(6) If T is an injective linear transformation from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^9$, then T is an isomorphism. True

(7) For any linear transformation $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^4$, there is an injective linear transformation $S : \mathbb{R}^4 \to \ker(T)$. True

(8) If A is a matrix and there is a matrix B such that AB is an identity matrix, then A is invertible and $A^{-1} = B$. False

(9) If A is a square matrix and there is a matrix B such that AB is an identity matrix, then A is invertible and $A^{-1} = B$. True

(10) For all matrices A and B, if $\text{rref}(A) = \text{rref}(B)$ then $\text{im}(A) = \text{im}(B)$. False

Problem 3: Worry about these rude questions briefly, and then immediately forget about them.

(1) Is $\emptyset$ a vector space? No

(2) What is the dimension of the vector space $\{\vec{0}\}$? 0

(3) What is a basis of $\{\vec{0}\}$? $\emptyset$

(4) What is $\text{Span}(\emptyset)$? $\{\vec{0}\}$

(5) Is $\{\vec{0}\}$ linearly independent? No

(6) Is $\emptyset$ linearly independent? Yes

(7) Does there exist an injective function $f : \mathbb{R}^2 \to \mathbb{R}$, or a surjective function $g : \mathbb{R} \to \mathbb{R}^2$? Yes, and yes, though not linear ones!

(8) Are there infinite-dimensional vectors spaces V and W such that $V \ncong W$?
Yes – but you don't have to worry about this yet.

(9) Is it true even for infinite-dimensional vector spaces V and W that $V \cong W$ if and only if $\dim V = \dim W$? Yes – but again, you don't have to worry about this yet.

(10) If A is a non-invertible square matrix, is it possible for the set of column vectors of A to be linearly independent?
We have to be very careful here. Suppose A is a square matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$. If A is non-invertible, then the list $(\vec{a}_1, \ldots, \vec{a}_n)$ must satisfy some non-trivial linear dependence relation, i.e., there must exist scalars $c_1, \ldots, c_n$, not all zero, such that $c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{0}$. So the *list* $(\vec{a}_1, \ldots, \vec{a}_n)$ is certainly linearly dependent. However, a *set* is not the same as a list, because in a set order doesn't matter (and neither do repeated elements). Remember that as sets, the set $\{x, y, z, x, y, y\}$ is equal to the set $\{z, y, x\}$. So if two columns of A are equal to each other, then the *set* of column vectors of A might very well be linearly independent. For instance, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then A is a non-invertible square matrix but its *set* of column vectors is just $\{\vec{e}_1\}$, which is linearly independent. This is, of course, quite obnoxious, and is one reason why it is often desirable to use lists of vectors instead of sets of vectors when considering linear dependence / independence. As a notational matter, we write $(\vec{a}_1, \ldots, \vec{a}_n)$ for the list, and $\{\vec{a}_1, \ldots, \vec{a}_n\}$ for the set.

Problem 4

(1) Consider the matrix $A = \begin{bmatrix} 0 & 0 & -1 & 2 & -1 & 5 \\ 1 & -3 & 2 & -3 & 2 & 0 \\ -1 & 3 & 1 & -3 & 1 & 4 \\ 2 & -6 & 1 & 0 & 1 & 7 \end{bmatrix}$ with $\mathrm{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Find bases of $\mathrm{im}(A)$ and $\ker(A)$.

$\mathrm{im}(A)$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 4 \\ 7 \end{bmatrix} \right\}$, and $\ker(A)$ has basis $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(2) Let $\vec{v}_1, \ldots, \vec{v}_7$ be a list of 7 vectors in $\mathbb{R}^4$ that span $\mathbb{R}^4$ and have the property that for each *even* index $k \leq 7$, $\vec{v}_k = \sum_{i=1}^{k-1} \vec{v}_i$. Find $\mathrm{rref}\big([\vec{v}_1 \cdots \vec{v}_7]\big)$.

The columns of A and of $\mathrm{rref}(A)$ satisfy the same linear relations, so $\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(3) Prove or disprove: if V and W are subspaces of the vector space U, then $V \cap W$ is a subspace of U.
True. Assume V and W are subspaces of U. Then $\vec{0} \in V$ and $\vec{0} \in W$, so $\vec{0} \in V \cap W$. Suppose $x, y \in V \cap W$. Then x and y belong to V, which implies $x + y \in V$ since V is a subspace and thus is closed under addition, and similarly x and y belong to W which implies $x + y \in W$ since W is closed under addition. Therefore $x + y \in V \cap W$, showing that $V \cap W$ is closed under addition. Finally, let $x \in V \cap W$ and $c \in \mathbb{R}$. Then $x \in V$ and $x \in W$, and since, being subspaces, both are closed under scalar multiplication, this means that $cx \in V$ and $cx \in W$. So $cx \in V \cap W$, showing that $V \cap W$ is closed under scalar multiplication. Thus $V \cap W$ contains $\vec{0}$ and is closed under both vector addition and scalar multiplication, showing that $V \cap W$ is a subspace of U.

(4) Prove or disprove: if V and W are subspaces of the vector space U, then $V \cup W$ is a subspace of U.
False. For instance, let V be the x-axis and W the y-axis in $U = \mathbb{R}^2$. Then $V \cup W$ is the contains $\vec{e}_1$ and $\vec{e}_2$ but not $\vec{e}_1 + \vec{e}_2$, so it is not closed under vector addition and therefore is not a subspace of U.

(5) Suppose A and B are two matrices such that AB is defined. Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$ and $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.
Let A be $m \times n$ and B $n \times p$, so that AB is $m \times p$. Suppose $\vec{y} \in \mathrm{im}(AB)$, say $\vec{y} = (AB)\vec{x}$ where $\vec{x} \in \mathbb{R}^p$. Then $\vec{y} = (AB)\vec{x} = A(B\vec{x})$, showing that $\vec{y} \in \mathrm{im}(A)$. Since $\vec{y}$ was arbitrary, this shows $\mathrm{im}(AB) \subseteq \mathrm{im}(A)$, and therefore

$$\mathrm{rank}(AB) \;=\; \dim(\mathrm{im}(AB)) \;\leq\; \dim(\mathrm{im}(A)) \;=\; \mathrm{rank}(A).$$

For the other claim, let $\vec{x} \in \ker(B)$. Then $B\vec{x} = \vec{0}$, so $(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$, so $\vec{x} \in \ker(AB)$. Since $\vec{x}$ was arbitrary, this shows $\ker(AB) \supseteq \ker(B)$, which implies $\dim(\ker(AB)) \geq \dim(\ker(B))$. Therefore, using Rank-Nullity, we have

$$\mathrm{rank}(AB) \;=\; p - \dim(\ker(AB)) \;\leq\; p - \dim(\ker(B)) \;=\; \mathrm{rank}(B).$$

(6) Let S be the subset of $\mathbb{R}^{3\times 3}$ consisting of all 3×3 matrices A such that $\text{trace}(A) = 0$. Either prove that S is not a subspace of $\mathbb{R}^{3\times 3}$, or else show that S *is* a subspace of $\mathbb{R}^{3\times 3}$ and find its dimension.
Yes, S is a subspace of $\mathbb{R}^{3\times 3}$, since the trace function is a linear transformation from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}$ with kernel S, and the kernel of a linear transformation is always a subspace. Furthermore, since the image of the trace map is all of $\mathbb{R}$ (which is 1-dimensional), the dimension of S must be $9 - 1 = 8$ by the Rank-Nullity Theorem.

(7) Let A be the 4×6 matrix from Problem (1). Is $\{B \subseteq \mathbb{R}^{4\times 6} : \text{rref}(B) = \text{rref}(A)\}$ a subspace of $\mathbb{R}^{4\times 6}$?
No, because it doesn't even contain $\vec{0}$.

(8) Prove that for all $m \times n$ matrices A and B, if there is an invertible matrix P such that $B = PA$, then $\ker(A) = \ker(B)$.
Let A and B be $m \times n$ matrices, and let P be an invertible $m \times m$ matrix such that $B = PA$. In the solution for (5) we already proved that $\ker(PA) \supseteq \ker(A)$. Conversely, if $PA\vec{x} = \vec{0}$, then since P is invertible (so its kernel is trivial) we must have $A\vec{x} = \vec{0}$, so that $\vec{x} \in \ker(A)$. Thus $\ker(B) = \ker(PA) = \ker(A)$.

(9) Prove that if V and W are vector spaces of the same finite dimension, then for every linear transformation $T : V \to W$, T is injective if and only if T is surjective.
Suppose $\dim(V) = \dim(W) = n$. Then using Rank-Nullity and part (13) of Problem 1, we know that T is injective iff $\ker(T) = \{\vec{0}\}$ iff $\dim(\ker(T)) = 0$ iff $\dim(\text{im}(T)) = \dim(V)$ iff $\dim(\text{im}(T)) = \dim(W)$ iff $V = W$ iff T is surjective.

(10) Give examples to show that the statement in (9) fails if either V and W have the same *infinite* dimension or if T is not linear.
For the infinite-dimensional case, let T be differentiation on the space $\mathcal{P}$ of all polynomials; then T is surjective but not injective. For the non-linear case, let $T : \mathbb{R} \to \mathbb{R}$ be the exponential function $T(x) = e^x$ for all $x \in \mathbb{R}$; then T is injective but not surjective.