Math 217: The Power of the Rank Nullity Theorem

The Rank-Nullity Theorem Let $T : V \to W$ be a linear transformation between finitely dimensional vector spaces. Then

$$\dim \ker T + \dim \operatorname{im} T = \dim \operatorname{Source} T.$$

Problem 1.

(a) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation with standard matrix A. We can make a number of statements about T and A, including all of the following:

(i) $\ker(A) = \{\vec{0}\}$.	(iv) The system $A\vec{x} = \vec{0}$ has a unique solution $\vec{0}$.
(ii) $\operatorname{rank}(A) = m$.	(v) The linear system $A\vec{x} = \vec{0}$ has no free variables.
(iii) T is injective.	(vi) For any $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has at most one solution.

Which of these statements are equivalent, and why?

(b) Explain why, if any one of the above conditions holds, we must have $m \leq n$. What can be said about T if $m = n$ and any one of the above conditions holds?

Solution: They are all equivalent:
(i) $\Leftrightarrow$ (iv) by definition of kernel.
(i) $\Leftrightarrow$ (iii) by the theorem that a linear transformation is injective if and only if its kernel is trivial.
(i) $\Leftrightarrow$ (ii) by rank nullity.
So (i), (ii), (iii) and (iv) are all equivalent. Also, (ii) and (v) are the same by the definition of rank. And, finally, (vi) $\Leftrightarrow$ (iii) by the definition of injective. So all are equivalent.

For (b), note that since the rank is the dimension of the image, and then image is a subspace of $\mathbb{R}^n$, we know $\dim \operatorname{im} T \subset \mathbb{R}^n$ so $m \leq n$. If equality holds, then T is invertible if and only if any one of the conditions (and hence all) hold.

Problem 2. TRUE or FALSE? Justify your answer.

(a) There exists a 4×5 matrix of rank 3 and such that the dimension of the space spanned by its columns is 4.

Solution: False. The dimension of the image is the rank of A.

(b) There exists a surjective linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^4$ given by multiplication by a rank 3 matrix.

Solution: False. Surjective means the image is all of $\mathbb{R}^4$, which has dimension 4. The dimension of the image is the rank of the matrix, which is 3.

(c) If A is 7×6 and $rref(A)$ has 6 pivots, then the map given by multiplication by A is injective.

Solution: True. The rank of A is 6 and the source has dimension 6. By rank-nullity, the kernel has dimension 0. This means the map is injective.

(d) If A is a 4×5 matrix, then it is possible for $\text{rank}(A)$ to be 3 and $\dim(\ker(A))$ to be 3.

Solution: No. $\dim \ker + \dim \text{im} = \dim \textit{source} = 5$. So we would have $3 + 3 = 5$, which is impossible.

(e) Every injective linear transformation $\mathbb{R}^5 \to \mathcal{P}_4$ is an isomorphism.

Solution: True. $\dim R^5 = \dim \mathcal{P}_4 = 5$, so if the kernel of a LT $\mathbb{R}^5 \to \mathcal{P}_4$ has dimension zero, then its image is dimension 5 by Rank-Nullity, and must be all of $\mathcal{P}_4$.

(f) Every surjective linear transformation $\mathbb{R}^5 \to \mathcal{P}_4$ is an isomorphism.

Solution: True. $\dim R^5 = \dim \mathcal{P}_4 = 5$, so if a LT $\mathbb{R}^5 \to \mathcal{P}_4$ is surjective, then its image is dimension 5, so by rank-nulity, its kernel has dimension zero, so the map is injective.

(g) There exists a 4×5 matrix A of rank 3 such that $\dim(\ker(A))$ is 2.

Solution: True. It is possible and in fact always true by Rank Nullity.

(h) If A is a 4×5 matrix and B is a 5×3 matrix, then $\text{rank}(A) \leq \text{rank}(B)$.

Solution: False! B could be the zero matrix, which has rank 0. But A will have rank more than 0 if it has even one non-zero entry.

(i) If matrix A has columns $C_1, \ldots, C_4$ which admit the relation $C_1 + C_2 - C_3 - 3C_4 = 0$, then $[1 \;\; 1 \quad -1 \quad -3]^T$ is a solution to $A\vec{x} = 0$.

Solution: True. A relation on the columns is the same as a solution to the corresponding homogenerous system of linear equations.

(j) If A is a 4×5 matrix and B is a 5×3 matrix, then $\text{rank}(A) \leq \text{rank}(AB)$.

Solution: False. If B is the zero matrix, so is AB. So its rank is zero, regardless of A.

(k) There does not exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker(T)$ and $\text{im}(T)$ are both lines in $\mathbb{R}^3$.

Solution: True. By rank-nullity, in that case, we would have $3 = 1 + 1$.

(l) If the linear system $A\vec{x} = \vec{b}$ has at least 5 solutions for some choice of $\vec{b}$, then it must have at least 5 solutions for any other choice of $\vec{b}$.

Solution: False. If it has at least 5 solutions, it must have infinitely many. But maybe it has no solutions for some values of $\vec{b}$. A counterexample is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions, which is at least 5. But $A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solutions (is inconsistent).

(m) If A is an $n \times n$ matrix such that $\operatorname{rank}(A^2) < n$, then $\operatorname{rank}(A) < n$ as well.

Solution: True. Suppose A has rank n. Since A is $n \times n$, this means it is invertible. So also A^2 is invertible. So its rank would be n.

(n) If A is a 4×5 matrix and B is a 5×3 matrix, then $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

Solution: True. Think about the composition linear transformation

$$\mathbb{R}^3 \xrightarrow{B} \mathbb{R}^5 \xrightarrow{A} \mathbb{R}^4.$$

The image of AB is the same as the image of $\operatorname{im} B$ under A. The dimension of a vector space can only go down under a transformation, so the dimension of $\operatorname{im} AB$ is at most $\dim \operatorname{im} B$. So rank $AB \leq$ rank B.

(o) If A is a 4×5 matrix and B is a 5×3 matrix, then $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

Solution: True. Think about the composition linear transformation

$$\mathbb{R}^3 \xrightarrow{B} \mathbb{R}^5 \xrightarrow{A} \mathbb{R}^4.$$

The image of AB is contained in the image of A, so dimension $\operatorname{im} AB \leq \dim \operatorname{im} A$.

(p) For each natural number $n \geq 2$, there is a non-zero square matrix A such that A^n is the zero matrix.

Solution: True. For any $n \geq$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(q) For each natural number $n \geq 2$, there is a non-identity square matrix A such that A^n is the identity matrix.

Solution: TRUE. Think about 2×2 matrices as giving linear transformations of $\mathbb{R}^2$ to itself. The question can be rephrased as asking whether or not there is a transformation that, when composed with itself n times, gives us the identity transformation. Rotation through an angle of $\frac{2\pi}{n}$ will work. Explicitly, the matrix $A = \begin{bmatrix} \cos\frac{2\pi}{n} & -sin\frac{2\pi}{n} \\ sin\frac{2\pi}{n} & cos\frac{2\pi}{n} \end{bmatrix}$ is a square matrix such that A^n is the identity matrix.

(r) If A is a 5×5 matrix such that $A^{20} = I_5$, then the rank of A is 5.

Solution: TRUE. If $A^{20} = I_5$, then $A^{19}A = AA^{19} = I_5$. This means that A is invertible, so has full rank 5.

(s) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ sending $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is given by multiplication by a diagonal matrix.

Solution: FALSE. We can compute the matrix of this map. We just need to see where it sends $\vec{e}_1$ and $\vec{e}_2$. Since $\vec{e}_1 = \frac{1}{2}(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix})$, we see that $T(\vec{e}_1) = \frac{1}{2}(T(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})) = \frac{1}{2}(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So the first column is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Since $\vec{e}_2 = \frac{1}{2}(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix})$, we see that $T(\vec{e}_2) = \frac{1}{2}(T(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})) = \frac{1}{2}(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. So the matrix is $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, which is NOT DIAGONAL.

(t) The linear transformation $\mathcal{P}_7 \to \mathbb{R}$ sending a polynomial f to $f'(0)$ has a six-dimensional kernel.

Solution: False. The map is surjective (since $f(x) = cx$ is sent to $c \in \mathbb{R}$ for any c). By rank nullity, the kernel is 7 dimensional, since the image is 1 dimensional.

(u) There is exactly one linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ sending $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution: TRUE. It is easy to compute the matrix of this map, so map is completely determined by the given information. We need to find $T(\vec{e}_1)$ and $T(\vec{e}_2)$. Compute $T(\vec{e}_1) = T(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix} 0 \\ 2 \end{bmatrix}) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -1 \end{bmatrix}$. This is the first column. Also, since T sends $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we know that $\vec{e}_2$ is sent to $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. So the second column of the matrix is $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. Thus the map T is multiplication by $\begin{bmatrix} 3.5 & .5 \\ -1 & 1 \end{bmatrix}$. This is the only map with the stated properties.

(v) If we know the output of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ on some basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathbb{R}^3$, then we can find the standard matrix of T.

Solution: TRUE. To find the matrix, we would need to find where $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$ go. But each of these is a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and T is linear. So, for example, we can find $\vec{e}_1$ by writing $\vec{e}_1 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$. Then $T(\vec{e}_1) = aT(\vec{v}_1) + bT(\vec{v}_2) + cT(\vec{v}_3)$. This is a column vector in$\mathbb{R}^3$ and the first column of the matrix of T. Similarly, we can find $T(\vec{e}_2)$ and $T(\vec{e}_3)$ which are the second and third columns of the matrix.

(w) The standard matrix of the projection of $\mathbb{R}^3$ onto to some plane through the origin in $\mathbb{R}^3$ has rank 2.

Solution: TRUE. The rank of the matrix is the dimension of the image. Clearly, when we project onto a plane, the image of that mappign is exactly the plane, which is dimension 2.