Worksheet 1: Systems of Linear Equations (§1.1)

Vocabulary: Linear expression, linear equation, $\mathbb{R}^3$, $\mathbb{R}^2$, line, plane, linear system of equations, inconsistent, consistent.

Problem 1. Consider the set Λ of points in $\mathbb{R}^3$ that satisfy $x - y = 0$ and the set Π of points satisfying $x - 2z = 0$.

(a). What type of geometric objects are the sets Λ and Π?

(b). What type of geometric object is the set $\Lambda \cap \Pi$?

(c). Describe geometrically the set of all points in $\mathbb{R}^3$ that satisfy both of the linear constraints

$$\begin{aligned} x - y \phantom{{}- 2z} &= 0 \\ x \phantom{{}- y} - 2z &= 0. \end{aligned}$$

What does this have to do with your answer to (b)?

(d). Find constants a, b, c so that your answer to (c) can be written

$$\{t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid t \in \mathbb{R}\}.$$

Is your answer *unique*? That is, is it the only possible answer?

(e). What do we mean by the "direction vector" of a line in $\mathbb{R}^3$? Is it unique? Relate this to (c) and (d). Does a line in $\mathbb{R}^3$ have a "slope"?

(f). What do we mean by "normal vector" to a plane in $\mathbb{R}^3$? Is it unique? Find normal vectors to Λ and for Π.

Problem 2. Suppose we have a system of 3 linear equations in 3 unknowns:

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + kz &= r, \end{aligned}$$

where $a, b, c, d, e, f, g, h, k, p, q, r$ are all real numbers.

(a). Discuss with your group why we can think of the solution space as an intersection of three planes in $\mathbb{R}^3$.*

(b). Find an explicit non-trivial example (values of the constants) in which the solution set is a plane. Can you find one where a, d, and g are all different numbers?

*A useful technique used by professional mathematicians: if stuck on a problem, try a simpler version of it. In this case, you can start by thinking about the solution space for a system of 2 linear equations in 2 unknowns, which will be a subset of $\mathbb{R}^2$. This should be familiar from high school.

(c). Find an explicit non-trivial example in which the solution set is a line.

(d). Find an explicit non-trivial example in which the solution set is a point.

(e). Find an explicit non-trivial example so that the system is inconsistent (ie. has no solutions).

(f). Are there values of the constants so that the solution set is a circle? A parabola? A union of two lines? Exactly two points? Place a bet on the shape of the solution space if we randomly pick the constants.

Problem 3. Consider the *Axiom of Parental Support*: If you get a "B" or better in this course, your parents will buy you a new car. Let us accept this as true (your experience notwithstanding), and take the following definitions:

> Definition: An *"A" student* never gets a grade lower than "A−" in a given semester.
>
> Definition: A *"B" student* gets at most one grade lower than a "B" in a given semester.
>
> Definition: A *"C" student* gets no grade higher than "C" in a given semester.

Given these axioms and definitions, decide which of the following statements are THEOREMS.* Justify each of your claims with either an argument or a counterexample.

(a) If I am an "A" student, I will get a new car from my parents at the end of the semester.
(b) If I am a "B" student, I will get a new car from my parents at the end of the semester.
(c) If I am a "C" student, I will not get a new car from my parents at the end of the semester.

Problem 4. Solve each of the following systems of equations and describe the solution set geometrically.

(a)

$$\begin{aligned} y &= 2w + 3z - 8 \\ x &= w + z - 4 \\ y &= 6w - 6x + 6z - 24 \\ w + z &= 3 \end{aligned}$$

(b)

$$3w + 3x - 5z \;=\; 3w + 3x - 3y \;=\; 6w + 6x - 6y - 5z \;=\; w + x \;=\; 0$$

(c)

$$\begin{array}{rcrcrcr} -5x &+& 3y &+& 3z &=& -5 \\ -7x &+& 4y &+& 4z &=& -5 \\ -2x &+& y &+& z &=& 5 \end{array}$$

*A *theorem* in an axiomatic system is a statement that is logically implied by the axioms, so that it *must be true* provided that the axioms themselves are true. Another way to think of theorems is that they are the statements that can be proved using the axioms.