Math 217 – Midterm 1

Winter 2022

Solutions

Student ID Number: ______________________ Section: ________

Question	Points	Score
1	12	
2	12	
3	12	
4	14	
5	15	
6	12	
7	11	
8	12	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) The function $f : X \to Y$ is *injective*

   Solution: The function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

   Solution: The function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

   (b) The *image* of the linear transformation $T : V \to W$ from the vector space V to the vector space W

   Solution: The *image* of the linear transformation $T : V \to W$ is the set $\operatorname{im} T = \{T(\vec{v}) \in W \ : \ \vec{v} \in V\}$.

   (c) The *span* of the finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V

   Solution: The *span* of the finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is the set

$$\operatorname{Span}(\vec{v}_1, \ldots, \vec{v}_n) \ = \ \left\{ \sum_{i=1}^{n} c_i \vec{v}_i \ : \ c_1, \ldots, c_n \in \mathbb{R} \right\}.$$

   Solution: The *span* of the finite list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is the set of all vectors in V that can be expressed as a linear combination of the vectors $\vec{v}_1, \ldots \vec{v}_n$.

   (d) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in $\mathbb{R}^m$ is *linearly independent*

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in $\mathbb{R}^m$ is *linearly independent* if for all scalars $c_1, \ldots, c_n \in \mathbb{R}$, if $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$ then $c_i = 0$ for all $i \in \{1, \ldots, n\}$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (4 points) If the linear system $A\vec{x} = \vec{b}$ has more variables than equations, then it has infinitely many solutions.

Solution: FALSE, since the linear system could have *no* solutions. For instance, if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then the linear system $A\vec{x} = \vec{b}$ has two equations and three variables but no solutions!

(b) (4 points) There is a linear transformation from $\mathcal{P}_3$ to $\mathbb{R}$ whose kernel is the set of constant functions in $\mathcal{P}_3$. (Here $\mathcal{P}_3$ is the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree at most 3.)

Solution: FALSE. Suppose $T : \mathcal{P}_3 \to \mathbb{R}$ is linear. Then since $\dim \mathcal{P}_3 = 4$, by Rank-Nullity we have

$$\dim \ker T \;=\; \dim \mathcal{P}_3 - \dim \operatorname{im} T \;=\; 4 - \dim \operatorname{im} T.$$

Since $\operatorname{im} T \subseteq \mathbb{R}$ and $\dim \mathbb{R} = 1$, this implies $\dim \ker T \geq 3$. But the set of constant functions in $\mathcal{P}_3$ is a subspace of dimension 1, so it cannot be $\ker T$.

(c) (4 points) For every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and subspace V of $\mathbb{R}^n$, the set $S = \{T(\vec{v}) \;:\; \vec{v} \in V\}$ is a subspace of $\mathbb{R}^m$.

Solution: TRUE. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, let V be a subspace of $\mathbb{R}^n$, and let $S = \{T(\vec{v}) \;:\; \vec{v} \in V\}$. Then $\vec{0} \in V$ since V is a subspace, and $T(\vec{0}) = \vec{0}$ since T is linear, so $\vec{0} \in S$. For closure under addition and scalar multiplication, let $\vec{y}_1, \vec{y}_2 \in S$ and $c \in \mathbb{R}$. Fix $\vec{x}_1, \vec{x}_2 \in V$ such that $T(\vec{x}_1) = \vec{y}_1$ and $T(\vec{v}_2) = \vec{y}_2$. Then $\vec{x}_1 + \vec{x}_2 \in V$ and $c\vec{v}_1 \in V$ since V is a subspace, so by linearity of T we have

$$\vec{y}_1 + \vec{y}_2 \;=\; T(\vec{x}_1) + T(\vec{x}_2) \;=\; T(\vec{x}_1 + \vec{x}_2) \;\in\; S$$

and

$$c\vec{y}_1 \;=\; cT(\vec{x}_1) \;=\; T(c\vec{x}_1) \;\in\; S.$$

Thus S contains $\vec{0}$ and is closed under vector addition and scalar multiplication, so it is indeed a subspace of $\mathbb{R}^m$.

3. Consider the 3×5 matrices

$$A = \begin{bmatrix} | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \\ | & | & | & | & | \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where we are given that B is the reduced row echelon form of A. (*No justification is required in parts (a) or (b), but you should show your work in part (c).*)

(a) (4 points) Find integers m and n such that the solution set of the linear system $A\vec{x} = \vec{0}$ is an m-dimensional subspace of $\mathbb{R}^n$.

Solution: Since A has 5 columns, the linear system $A\vec{x} = \vec{0}$ has 5 variables, so its solution set is a subset of $\mathbb{R}^5$. By examining B we see that there are 3 *free* variables (i.e., 3 non-pivot columns), so the solution set of the system $A\vec{x} = \vec{0}$ is 3-dimensional. Thus $n = 5$ and $m = 3$.

(b) (4 points) Assuming that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, find A explicitly as a matrix with numerical entries.

Solution: Since the columns of A and B satisfy the same linear relations, we have

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(c) (4 points) Find the solution set of the linear system with *augmented* matrix A; that is, find the solution set of the linear system $C\vec{x} = \vec{b}$ where $\begin{bmatrix} C & \vec{b} \end{bmatrix} = A$.

Solution: The solution set is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \; : \; x_2, x_3 \in \mathbb{R} \right\}.$$

4. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} b \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, where a and b are real numbers.

   (a) (3 points) Without using determinants, find all values of a and b for which $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.

   Solution: The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis of $\mathbb{R}^3$ if and only if the matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ has rank 3. Row reducing, we have

$$\begin{bmatrix} 1 & b & 2 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & b & 2 \\ 0 & 1-ab & -2a \\ 0 & 0 & 1 \end{bmatrix},$$

   so we see that A has rank 3 if and only if $ab \neq 1$. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$ if and only if $ab \neq 1$.

   (b) (3 points) Assuming $a = 2$ and $b = 1$, write $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ explicitly as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

   Solution: Assuming $a = 2$ and $b = 1$, we have

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 12 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

(Problem 4, Continued).

Recall that $\vec{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} b \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, where a and b are real numbers.

(c) (4 points) Assuming $a = 1$ and $b = 0$, find the inverse of the matrix $B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ with column vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Solution: If we row reduce $\left[\, B \mid I_3 \,\right]$ to $\left[\, I_3 \mid R \,\right]$, we will have $R = B^{-1}$. Row reducing, we obtain

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

so $B^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) (4 points) Assuming $a = b = 1$, find a matrix A whose kernel is the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Solution: Assuming $a = b = 1$, we have

$$V \;=\; \mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \;=\; \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) \;=\; \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right),$$

so V is a plane in $\mathbb{R}^3$. If $\vec{w}$ is a vector in $\mathbb{R}^3$ that is perpendicular to this plane, then the row vector $\vec{w}^\top$ will be a (1×3) matrix whose kernel is V. Solving for $\vec{w}$ either by taking a cross-product, by solving the linear system $\vec{w} \cdot \vec{v}_1 = \vec{w} \cdot \vec{v}_3 = 0$, or by inspection, we obtain the matrix

$$\vec{w}^\top \;=\; \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}.$$

If we want a 3×3 matrix whose kernel is V, we could repeat this row to obtain

$$A \;=\; \begin{bmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ 1 & -1 & -2 \end{bmatrix}.$$

5. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation about the origin by an angle of $\frac{\pi}{3}$ radians, or $60°$, so the standard matrix A of S is the matrix

$$A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Also let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be orthogonal projection onto the x-axis in $\mathbb{R}^2$. *(No justification is necessary on any part of this problem.)*

(a) (3 points) Find the standard matrix of $P \circ S$.

Solution: Since

$$P(S(\vec{e}_1)) = P\left(\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \text{ and } P(S(\vec{e}_2)) = P\left(\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}\right) = \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ 0 \end{bmatrix},$$

the standard matrix of $P \circ S$ is $\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix}$.

(b) (4 points) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \circ T = S$. (It is enough to give a geometric description of such T.)

Solution: For instance, we could take T to be counterclockwise rotation about the origin by an angle of $\frac{\pi}{6}$ radians.

(c) (4 points) Find a basis of $\operatorname{im}(S \circ P)$ and a basis of $\ker(S \circ P)$.

Solution: A basis of $\operatorname{im}(S \circ P)$ is $\left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}\right)$ and a basis of $\ker(S \circ P)$ is $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(d) (4 points) Letting B be the standard matrix of P, circle all matrices listed below that are invertible, and cross out those that are not:

$A \qquad B \qquad BA \qquad AB \qquad A^2 \qquad B^2 \qquad 2B - I_2 \qquad A^3 + I_2$

Solution: The matrices A, A^2, and $2B - I_2$ are invertible, while the rest are not. (Note that $2B - I_2$ is a reflection, and $A^3 + I_2$ is the zero matrix.)

6. Let $\mathcal{P}_3$ be the vector space of polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree at most 3, and consider the function $T : \mathcal{P}_3 \to \mathbb{R}$ defined by $T(p) = p(1)$ for each $p \in \mathcal{P}_3$.

   (a) (4 points) Show that T is a linear transformation.

   Solution: Let $p, q \in \mathcal{P}_3$ and $c \in \mathbb{R}$. Then

   $$T(p+q) \;=\; (p+q)(1) \;=\; p(1)+q(1) \;=\; T(p)+T(q)$$

   and

   $$T(cp) \;=\; (cp)(1) \;=\; cp(1) \;=\; cT(p).$$

   (b) (3 points) Is T surjective? Briefly justify your answer.

   Solution: Yes, T is surjective, since given $c \in \mathbb{R}$, the constant function $p(x) = c$ gets mapped to c by T.

   (c) (5 points) Find a basis of $\ker(T)$, and briefly justify your answer.

   Solution: We know $\dim \mathcal{P}_3 = 4$, and $\dim \operatorname{im} T = 1$ by part (b), so by Rank-Nullity we have $\dim \ker T = 3$. But it is easy to check that the functions $x-1$, $(x-1)^2$, and $(x-1)^3$ all belong to $\ker(T)$, and they are linearly independent since they all have different degrees, so

   $$\Big(x-1,\, (x-1)^2,\, (x-1)^3\Big)$$

   is a basis of $\ker(T)$.

7. Let A and B be $n \times n$ matrices, and let T_A and T_B be the linear transformations induced by A and B, respectively, so that $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) (3 points) Prove that $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$.

   Solution: Let $\vec{y} \in \operatorname{im}(AB)$, and fix $\vec{x} \in \mathbb{R}^n$ such that $(AB)\vec{x} = \vec{y}$. Then

   $$\vec{y} \;=\; (AB)\vec{x} \;=\; A(B\vec{x}),$$

   so $\vec{y} \in \operatorname{im}(A)$.

   (b) (3 points) Prove that $\ker(B) \subseteq \ker(AB)$.

   Solution: Let $\vec{x} \in \ker(B)$, so $B\vec{x} = \vec{0}$. Then

   $$(AB)\vec{x} \;=\; A(B\vec{x}) \;=\; A\vec{0} \;=\; \vec{0},$$

   so $\vec{x} \in \ker(AB)$.

   (c) (5 points) Prove that if $\ker(AB) \subseteq \ker(B)$ and T_B is surjective, then T_A is injective.

   Solution: Suppose $\ker(AB) \subseteq \ker(B)$ and that T_B is surjective. Let $\vec{x} \in \ker(T_A)$, so $A\vec{x} = \vec{0}$. Using the fact that T_B is surjective, fix $\vec{y} \in \mathbb{R}^n$ such that $B\vec{y} = \vec{x}$. Then $\vec{0} = A\vec{x} = A(B\vec{y}) = (AB)\vec{y}$, so $\vec{y} \in \ker(AB)$ and thus $\vec{y} \in \ker(B)$. But then $\vec{x} = B\vec{y} = \vec{0}$. This shows $\ker(T_A) = \{\vec{0}\}$, so T_A is injective as desired.

8. Let U and V be finite-dimensional vector spaces, and let $T : U \to V$ and $S : V \to U$ be linear transformations.

   (a) (6 points) Let $k \in \mathbb{N}$, and let $(\vec{u}_1, \ldots, \vec{u}_k)$ be a list of vectors in U. Prove that if $(\vec{u}_1, \ldots, \vec{u}_k)$ spans U and $(T(\vec{u}_1), \ldots, T(\vec{u}_k))$ is linearly independent, then T is injective.

   Solution: Let $\vec{x} \in \ker(T)$, so $\vec{x} \in U$ and $T(\vec{x}) = \vec{0}$. Using the fact that $(\vec{u}_1, \ldots, \vec{u}_k)$ spans U, fix $c_1, \ldots, c_k \in \mathbb{R}$ such that $\vec{x} = \sum_{i=1}^{k} c_i\vec{u}_i$. Then

   $$\vec{0} \;=\; T(\vec{x}) \;=\; T\left(\sum_{i=1}^{k} c_i\vec{u}_i\right) \;=\; \sum_{i=1}^{k} T(\vec{u}_i),$$

   which implies that $c_i = 0$ for each i since $(T(\vec{u}_1), \ldots, T(\vec{u}_k))$ is linearly independent. Thus $\vec{x} = \vec{0}$, so $\ker(T) = \{\vec{0}\}$, which means T is injective.

   (b) (6 points) Prove that if $S \circ T$ is surjective, then $\dim(U) \leq \dim(V)$.

   Solution: Suppose $S \circ T$ is surjective. Then $U = \operatorname{im}(S \circ T) \subseteq \operatorname{im} S$, so in fact $\operatorname{im}(S) = U$ and thus $\dim \operatorname{im}(S) = \dim(U)$. But by Rank-Nullity we have $\dim \operatorname{im}(S) + \dim \ker(S) = \dim V$. Thus $\dim(U) = \dim V - \dim \ker(S)$, and since $\dim \ker(S) \geq 0$, this implies $\dim(U) \leq \dim(V)$ as desired.