Math 217 – Midterm 1

Winter 2019

Solutions

Name: ____________________ Section: __________

Question	Points	Score
1	12	
2	15	
3	14	
4	11	
5	12	
6	14	
7	11	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) (3 points) The function $f : X \to Y$ is *surjective*

   Solution: The function $f : X \to Y$ is *surjective* if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

   Solution: The function $f : X \to Y$ is *surjective* if the image of f is equal to Y.

   (b) (2 points) Given vector spaces V and W, the *kernel* of the linear transformation $T : V \to W$

   Solution: The *kernel* of the linear transformation $T : V \to W$ is the set $\ker(T) = \{\vec{v} \in V \ : \ T(\vec{v}) = \vec{0}\}$.

   (c) (4 points) The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent*

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent* if <u>for all</u> $c_1, \ldots, c_n \in \mathbb{R}$, <u>if</u> $\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$, <u>then</u> $c_i = 0$ for all $i = 1, \ldots, n$.

   Solution: The list of vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ in the vector space V is *linearly independent* if <u>for any</u> linear equation $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ where $c_1, \ldots, c_n \in \mathbb{R}$, the <u>only</u> solution is the trivial solution, $c_1 = \cdots = c_n = 0$.

   (d) The subset V of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$

   Solution: The subset V of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$ if $\vec{0} \in V$ and for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{R}$, the vectors $\vec{x} + \vec{y}$ and $c\vec{x}$ belong to V.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For every 3×3 matrix A, if $A^2 = A$ then $A^3 = A$.

Solution: TRUE. Let $A \in \mathbb{R}^{3\times 3}$, and suppose $A^2 = A$. Then $A^3 = A(A^2) = AA = A^2 = A$.

(b) (3 points) For all vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{v}$ in $\mathbb{R}^3$, if the list $(\vec{x}, \vec{y}, \vec{z})$ is linearly independent, then the list $(\vec{x}+\vec{v}, \vec{y}, \vec{z})$ is also linearly independent.

Solution: FALSE. For a counterexample, let $(\vec{x}, \vec{y}, \vec{z}) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$, so that $(\vec{x}, \vec{y}, \vec{z})$ is linearly independent, and let $\vec{v} = -\vec{e}_1$. Then $(\vec{x}+\vec{v}, \vec{y}, \vec{z}) = (\vec{0}, \vec{e}_2, \vec{e}_3)$ is linearly dependent, since it contains $\vec{0}$.

(c) (3 points) For every $n \in \mathbb{N}$, the set of non-invertible $n \times n$ matrices is a subspace of $\mathbb{R}^{n\times n}$.

Solution: FALSE. For a counterexample, let $n = 2$, and consider the non-invertible matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A + B = I_2$ is invertible, so the set of non-invertible 2×2 matrices is not closed under addition and therefore is not a subspace of $\mathbb{R}^{2\times 2}$.

(Problem 2, Continued).

(d) (3 points) For every $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, if there exists a vector $\vec{b} \in \mathbb{R}^n$ such that the linear system $A\vec{x} = \vec{b}$ is inconsistent, then $\text{rank}(A) < n$.

Solution: TRUE. Let $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, and suppose there is $\vec{b} \in \mathbb{R}^n$ such that the linear system $A\vec{x} = \vec{b}$ is inconsistent. Then there is $\vec{b} \in \mathbb{R}^n$ such that $\vec{b} \notin \text{im}(A)$, so $\text{im}(A) \neq \mathbb{R}^n$. Thus $\text{im}(A)$ is a proper subspace of $\mathbb{R}^n$, so $\text{rank}(A) = \dim(\text{im}(A)) < n$.

Solution: TRUE. The linear system $A\vec{x} = \vec{b}$ is inconsistent if and only if the RREF of $A|\vec{b}$ has a zero row augmented by a non-zero entry. This implies that the RREF of A has fewer leading ones than rows. It follows that $\text{rank}(A) < n$.

(e) (3 points) There exists a linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ such that $\text{im}(T) = \ker(T)$, where $\mathcal{P}_2$ is the vector space of all polynomial functions in the variable x with real coefficients of degree at most 2.

Solution: FALSE. Suppose for contradiction that $T : \mathcal{P}_2 \to \mathcal{P}_2$ is a linear transformation such that $\text{im}(T) = \ker(T)$. Then $\dim(\text{im}(T)) = \dim(\ker(T))$. Using Rank-Nullity, it follows that

$$\dim(\mathcal{P}_2) \;=\; \dim(\text{im}(T)) + \dim(\ker(T)) \;=\; 2\dim(\text{im}(T)).$$

Since $\dim(\text{im}(T))$ is an integer, this implies that $\dim(\mathcal{P}_2)$ is an even integer, contradicting the fact that $\dim(\mathcal{P}_2) = 3$.

3. Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of $\mathbb{R}^3$, and suppose that $\vec{z} = \vec{u} + \vec{v} + \vec{w}$. Let A be the 3×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{u} & \vec{v} & \vec{w} & \vec{z} \\ | & | & | & | \end{bmatrix},$$

and let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be the map with standard matrix A, so $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^4$. *(No justification is required on any part of this problem.)*

(a) (3 points) Find the reduced row echelon form of A.

Solution: $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(b) (2 points) Find $\text{rank}(A)$ and $\text{nullity}(A)$.

Solution: $\text{rank}(A) = 3$ and $\text{nullity}(A) = 1$.

(c) (3 points) Find a basis of $\text{im}(A)$.

Solution: $(\vec{u}, \vec{v}, \vec{w})$ is a basis of $\text{im}(A)$. (So is $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$).

(d) (3 points) Find a basis of $\ker(A)$.

Solution: $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$ is a basis of $\ker(A)$.

(e) (3 points) Assuming that $\vec{z} = \vec{e}_1$, find the first column of the inverse of the 3×3 matrix $\begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$.

Solution: If $\vec{z} = \vec{e}_1$, then the first column of $\begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}^{-1}$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^m$, $S : \mathbb{R}^n \to \mathbb{R}^d$, and $R : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations.

(a) (2 points) State exact conditions on m, n, and d for which the composition $R \circ S \circ T$ is defined. *(No justification necessary).*

Solution: $m = n$ and $d = 2$.

(b) (2 points) Let A, B, and C be the standard matrices of R, S, and T, respectively. Assuming that $R \circ S \circ T$ is defined, find its standard matrix in terms of A, B, and C. *(No justification necessary).*

Solution: The standard matrix of $R \circ S \circ T$ is ABC.

(c) (3 points) Again assume that $R \circ S \circ T$ is defined, and suppose $T(\vec{e}_1) = \vec{v}_1$, $T(\vec{e}_2) = \vec{v}_2$, $S(\vec{v}_1) = \vec{w}_1$, $S(\vec{v}_2) = \vec{w}_2$, $R(\vec{w}_1) = \vec{e}_1 + 2\vec{e}_2$, and $R(\vec{w}_2) = 2\vec{e}_1 + 5\vec{e}_2$. Find the standard matrix of $R \circ S \circ T$.

Solution: The standard matrix of $R \circ S \circ T$ is

$$\begin{bmatrix} | & | \\ R(S(T(\vec{e}_1))) & R(S(T(\vec{e}_2))) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ R(S(\vec{v}_1)) & R(S(\vec{v}_2)) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ R(\vec{w}_1) & R(\vec{w}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

(d) (4 points) Assume $m = n = d = 2$, and suppose R is reflection over the x-axis, S is projection onto the y-axis, and T is reflection over the line $y = -x$. Find the standard matrix of $R \circ S \circ T$.

Solution: We have

$$R(S(T(\vec{e}_1))) = R(S(-\vec{e}_2)) = R(-\vec{e}_2) = \vec{e}_2$$

and

$$R(S(T(\vec{e}_2))) = R(S(-\vec{e}_1)) = R(\vec{0}) = \vec{0},$$

so the standard matrix of $R \circ S \circ T$ is

$$\begin{bmatrix} | & | \\ R(S(T(\vec{e}_1))) & R(S(T(\vec{e}_2))) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

5. Consider the system of linear equations $A\vec{x} = \vec{b}$, where $A \in \mathbb{R}^{3\times 4}$ and $\vec{b} \in \mathbb{R}^3$. Throughout this problem, suppose the reduced row-echelon form of the augmented matrix $[\, A \mid \vec{b}\,]$ is

$$\left[\begin{array}{cccc|c} p & q & 3 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & r & s \end{array}\right], \quad \text{where } p, q, r, s \in \mathbb{R}.$$

(No justification is required on any part of this problem.)

(a) (3 points) Find all values of p and q that are consistent with the given information, or else write *none* if there are no such values.

Solution: $p = 1$ and $q = 0$.

(b) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that the linear system $A\vec{x} = \vec{b}$ has no solutions, or else write *none* if there are no such values.

Solution: $r = 0$ and $s = 1$.

(c) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that the linear system $A\vec{x} = \vec{b}$ has a unique solution, or else write *none* if there are no such values.

Solution: None.

(d) (3 points) Find all values of r and s that are consistent with both the given information and the assumption that $\operatorname{rank}(A) = 2$, or else write *none* if there are no such values.

Solution: $r = 0$ and ($s = 0$ or $s = 1$).

6. Given $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the map given by $T_\alpha(A) = SA$, where $S = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

   (a) (4 points) Show that for all $\alpha \in \mathbb{R}$, T_α is a linear transformation.

Solution: Let $\alpha \in \mathbb{R}$. Let $A, B \in \mathbb{R}^{2\times 2}$ and let $c \in \mathbb{R}$. Then

$$T_\alpha(A+B) \;=\; S(A+B) \;=\; SA+SB \;=\; T_\alpha(A)+T_\alpha(B)$$

and

$$T_\alpha(cA) \;=\; S(cA) \;=\; c(SA) \;=\; cT_\alpha(A),$$

which shows that T_α is linear.

   (b) (4 points) Show that if S is an invertible matrix, then T_α is an isomorphism.

Solution: Suppose that S is invertible. Let $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by $L(A) = S^{-1}A$ for all $A \in \mathbb{R}^{2\times 2}$. Then for all $A \in \mathbb{R}^{2\times 2}$, we have

$$L(T_\alpha(A)) \;=\; S^{-1}(SA) \;=\; (S^{-1}S)A \;=\; I_2A \;=\; A$$

and

$$T_\alpha(L(A)) \;=\; S(S^{-1}A) \;=\; (SS^{-1})A \;=\; I_2A \;=\; A.$$

This shows that L is the inverse of T_α, and thus T_α is an invertible linear transformation; that is, T_α is an isomorphism.

(Problem 6, Continued). *Recall the instructions for Problem 6:*

Given $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the map given by $T_\alpha(A) = SA$, where $S = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

(c) (3 points) Letting $\alpha = 1$, find a basis of $\ker(T_1)$. *You should indicate how you obtain your answer, but you do not need to prove that your answer is actually a basis of* $\ker(T_1)$.

Solution: Letting $\alpha = 1$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$T_\alpha(A) \;=\; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;=\; \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}.$$

Since a, b, c, d can be arbitrary, we see from this calculation that $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ belong to $\mathrm{im}(T_1)$ while $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ belong to $\ker(T_1)$. Since these four matrices form a linearly independent set in $\mathbb{R}^{2\times 2}$ and

$$4 \;=\; \dim(\mathbb{R}^{2\times 2}) \;=\; \dim(\ker(T_1)) + \dim(\mathrm{im}(T_1))$$

by Rank-Nullity, we conclude that $\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right)$ is a basis of $\ker(T_1)$ while $\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$ is a basis of $\mathrm{im}(T_1)$.

(d) (3 points) Again letting $\alpha = 1$, find a basis of $\mathrm{im}(T_1)$. *You should indicate how you obtain your answer, but you do not need to prove that your answer is actually a basis of* $\mathrm{im}(T_1)$.

Solution: As shown above, $\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$ is a basis of $\mathrm{im}(T_1)$.

7. Let V and W be vector spaces, and suppose that $T : V \to W$ is a linear transformation.

(a) (2 points) Prove that $T(\vec{0}_V) = \vec{0}_W$.

Solution: Using the fact that T preserves scalar multiplication, we have

$$T(\vec{0}_V) \;=\; T(0\vec{0}_V) \;=\; 0T(\vec{0}_V) \;=\; \vec{0}_W.$$

(b) (4 points) Prove that if T is injective, then $\ker(T) = \{\vec{0}_V\}$.

Solution: By part (a), we know $T(\vec{0}_V) = \vec{0}_W$, so $\vec{0}_V \in \ker(T)$. Suppose that T is injective, and let $\vec{v} \in \ker(T)$. Then $T(\vec{v}) = \vec{0}_W = T(\vec{0}_V)$, so by injectivity of T, we have $\vec{v} = \vec{0}_V$. Thus $\ker(T) \subseteq \{\vec{0}_V\}$. It follows that $\ker(T) = \{\vec{0}_V\}$.

(c) (5 points) Prove that if $\ker(T) = \{\vec{0}_V\}$, then T is injective.

Solution: Suppose that $\ker(T) = \{\vec{0}_V\}$. Let $\vec{v}_1, \vec{v}_2 \in V$, and suppose $T(\vec{v}_1) = T(\vec{v}_2)$. Then

$$\vec{0}_V \;=\; T(\vec{v}_1) - T(\vec{v}_2) \;=\; T(\vec{v}_1 - \vec{v}_2),$$

which shows that $\vec{v}_1 - \vec{v}_2 \in \ker(T)$. Thus $\vec{v}_1 - \vec{v}_2 = \vec{0}_V$, so $\vec{v}_1 = \vec{v}_2$. This shows that T is injective.

8. Let $n \geq 2$, let A and B be $n \times n$ matrices, and write O for the $n \times n$ zero matrix.

(a) (6 points) Prove that if $AB = O$, then $\operatorname{rank}(A) + \operatorname{rank}(B) \leq n$.

Solution: Assume $AB = O$. Let $\vec{y} \in \operatorname{im}(B)$, say $\vec{y} = B\vec{x}$ where $\vec{x} \in \mathbb{R}^n$. Then $A\vec{y} = A(B\vec{x}) = (AB)\vec{x} = O\vec{x} = \vec{0}$, so $\vec{y} \in \ker(A)$. This shows that $\operatorname{im}(B) \subseteq \ker(A)$, and therefore $\dim(\operatorname{im}(B)) \leq \dim(\ker(A))$. By Rank-Nullity, $\operatorname{rank}(A) + \dim(\ker(A)) = n$. Thus

$$\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(A) + \dim(\operatorname{im}(B)) \leq \operatorname{rank}(A) + \dim(\ker(A)) = n.$$

(b) (2 points) State the converse of the statement you were asked to prove in part (a). *(No justification required).*

Solution: If $\operatorname{rank}(A) + \operatorname{rank}(B) \leq n$, then $AB = O$.

(c) (3 points) Prove the converse of the statement in part (a) if it is true for all $n \geq 2$, or else give a counterexample to show that it can fail for some $n \geq 2$.

Solution: The converse statement is false. To see this, let $n = 2$, and let $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\operatorname{rank}(A) = \operatorname{rank}(B) = 1$, so $\operatorname{rank}(A) + \operatorname{rank}(B) = 2 \leq n$, but $AB = A \neq O$.