Math 217 Worksheet 21B: Elementary Matrices and Determinants

Definition: An **elementary matrix** is an $n \times n$ matrix obtained by performing a *single* elementary row operation on an $n \times n$ identity matrix I_n.

Theorem 1: If E is an elementary matrix obtained by applying an elementary row operation on I_n, then for $A \in \mathbb{R}^{n \times d}$, the matrix EA is obtained by applying *the same* elementary row operation to A.

Problem 1. Recall and discuss the three different types of elementary row operations.

(a) Write out three examples of 3×3 elementary matrices, with at least one of each type.

(b) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ be an arbitrary 3×4 matrix. Verify Theorem 1 for each of your three elementary row operations (matrices) in (a).

(c) Do you see why Theorem 1 is true? Without writing out details, discuss a scaffold for its proof.

Solution:

(a) Here is one answer (of many possible):

- $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (swapping rows 2 and 3);
- $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (multiply row 2 by π);
- $E_3 = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (add $\frac{1}{3}$ times row 3 to row 1).

(b)
- $E_1 A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$, which is the same as swapping rows 2 and 3 of A.
- $E_2 A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \pi b_1 & \pi b_2 & \pi b_3 & \pi b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$, which is the same as multiplying row 2 of A times π;
- $E_3 A = \begin{bmatrix} a_1 + \frac{c_1}{3} & a_2 + \frac{c_2}{3} & a_3 + \frac{c_3}{3} & a_4 + \frac{c_4}{3} \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$, which is the same as adding $\frac{1}{3}$ times row 3 to row 1.

In all cases, EA is the same as what we get after performing the corresponding row operation on A.

(c) The scaffold of the proof would have three cases: for each of the three types of elementary row operations, we would need to check that multiplying a matrix A by the corresponding elementary matrix has the same effect as performing that operation on A.

Problem 2. Suppose $i \neq j$. Find the determinant of the elementary matrix:

(a) obtained from I_n by scaling row i by non-zero $a \in \mathbb{R}$; [Hint: Use the linearity of the determinant in row i.]

(b) obtained from I_n by interchanging rows i and j; [Hint: Use the alternating property of determinants.]

(c) obtained from I_n by adding a times row i to row j; [Hint: Use linearity in row j and alternating prop.]

Solution:

(a) Using the linearity property in row i, $\det E = a \det I_n = a$.

(b) Using the alternating property of determinants, $\det E = -\det I_n = -1$.

(c) Here, $\det E$ is 1, as this operation has no effect on the determinant. Adding a times row i to row j changes only row j. Starting with I_n, the new matrix has j-th row the sum of the original row j (which is $\vec{e}_j^{\top}$) plus $a\vec{e}_i^{\top}$. So $\det E = \det I_n + a \det B$, where B is obtained from the identity matrix by replacing row j by row i. But since two rows of B are identical, $\det B = 0$ by the alternating property. So $\det E = \det I_n = 1$.

Alternatively, the matrix E is upper/lower triangular with all 1's on the diagonal. So by Problem 3 on Worksheet 21A, the determinant is 1.

Problem 3.

(a) Elementary matrices are invertible, with inverse also an elementary matrix. Explain.

(b) Prove that an invertible matrix is a product of elementary matrices.
[Hint: Use Theorem 1 repeatedly, performing elementary row ops to get rref(A).]

Solution:

(a) There are three cases. If E is obtained from I_n by scaling row i by $a \neq 0$, then E^{-1} is obtained from I_n by scaling row i by $\frac{1}{a}$. If E is obtained by swapping rows i and j, then E^{-1} also swaps i and j, so $E^{-1} = E$. If E is obtained from I_n by adding a times row i to j, then E^{-1} is obtained from I_n by adding $-a$ times row i to j.

(b) Starting with an invertible matrix A, we can do a sequence of row reductions, say t elementary operations, to get A into row reduced echelon form, which is I_n. Each of these is multiplication by some elementary matrix, so

$$I_n = E_t E_{t-1} \cdots E_2 E_1 A.$$

This means that, multiplying one by one on the left by E_t^{-1}, then E_{t-1}^{-1}, etc,

$$E_1^{-1} E_2^{-1} \cdots E_{t-1}^{-1} E_t^{-1} I_n = A.$$

Now, since the inverses of elementary matrices are also elementary (part (a)), we have expressed A as a product of elementary matrices.

Problem 4. Another way to compute determinants.

(a) For $A \in \mathbb{R}^{n\times n}$, what is the effect on $\det A$ when we apply each type of elementary row operation?

(b) For a matrix $A \in \mathbb{R}^{n \times n}$, the determinant can be computed by row reducing A, and keeping track of how many row swaps were performed, and all the row scalings performed. Explain.

(c) Use row ops to compute the determinant of $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \\ 2 & -4 & -2 & 8 \\ -1 & 3 & 6 & -1 \\ 1 & -3 & -1 & 2 \end{bmatrix}$.

Solution:

(a) Scaling a row by a multiples the determinant by a. Swapping two rows changes the sign of the determinant, and adding a multiple of one row to another leaves the determinant the same. More precisely, if A' is obtained from A by scaling row i by a, then $\det A' = a \det A$. Thinking about trying to understand $\det A$, it might be better to write this as $\det A = \frac{1}{a} \det A'$.

(b) Think about row reducing A to $\text{rref}(A)$ (or even just an upper triangular form with 1's and 0's on the diagonal). So if $a_1, \dots, a_r$ are the non-zero scalars we use in those row ops that simply multiply one row by $\frac{1}{a_i}$, and D is the number of row swaps we perform, we have

$$\det A = (-1)^D \prod_{i=1}^{r} a_i \det \text{rref}(A) = \begin{cases} (-1)^D \prod_{i=1}^{r} a_i & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases}$$

(Remember that A is invertible if and only if $\text{rref}(A) = I_n$, so $\det \text{rref} A = \det I_n = 1$ in the first case above, and otherwise $\text{rref} A$ has a row of zeros, so has determinant zero.) Note that the most commonly used row operation, when we add multiples of one row to another, *has no effect on the determinant!* This is actually a much more efficient way to compute the determinant of a matrix—in fact, this is how computers compute the determinant. For an $n \times n$ matrix, it is interesting to estimate the number of arithmetic steps needed to compute the determinant via a Laplace expansion versus by row reduction.

(c) Scaling the first row of A by 2 and then applying three successive row addition operations reduces A to the upper triangular matrix

$$R = \begin{bmatrix} 1 & -3 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Thus $-30 = \det R = 2 \det A$, so $\det A = -15$.

Problem 5. In this problem, we will prove the multiplicative property of determinants: $\det(AB) = \det A \ \det B$. So answer all parts below *without using* the multiplicative property. Fix $A, B \in \mathbb{R}^{n \times n}$.

(a) Prove that $\det(EA) = \det(E) \det(A)$. [Hint: There are three cases. Use Theorem 1 and Problems 2 and 4.]

(b) If $A = E_1 E_2 \cdots E_t$, where the E_i are elementary matrices, prove $\det A = \prod_{i=1}^{t} \det E_i$. [Hint: Induce!]

(c) Prove that $\det(AB) = \det A \det B$. [Hint: Multiply A by appropriate $E_1 \cdots E_t$ to row reduce; induce on t.]

Solution:

(a) There are three cases.

(i) If E is obtained from I_n by scaling row i by a, then $\det E = a$. Likewise, by linearity of the determinant in row i, $\det(EA) = a \det A$, since EA is A with row i scaled by a. So $\det(EA) = \det E \det A$.

(ii) If E is obtained by swapping two rows, then $\det(EA) = -\det A$ by Theorem 1 and the alternating property of determinants. So again $\det(EA) = \det E \det A$.

(iii) If E is obtained from I_n by adding a times row i to row j, then EA is the same as A except in row j, where it is row j plus $a\times$ the i-th row of A. So by linearity in row j, $\det EA = \det A + \det A'$ where A' is almost A—all rows are the same except for the jth, which is a time row i.

In all three cases, we have $\det(EA) = (\det E)(\det A)$.

(c) Using (b), write $A = E_1 \cdots E_t \mathrm{rref}(A)$. Then there are two cases. If A is invertible, then $\mathrm{rref}(A) = I_n$ and $AB = E_1 \cdots E_t B$. We will prove by induction on t, that $\det AB = \det A \det B$. The base case, where $t = 1$, was proved in (a). Assume, inductively, that we know the result for $t-1$. Write $AB = E_1((E_2 \cdots E_t)B)$, so that again by (a), we have $\det(AB) = \det(E_1(E_2 \cdots E_t B)) = \det E_1 \det(E_2 \cdots E_t B)$. By induction, $\det(E_2 \cdots E_t B) = \det((E_2 E_3 \ldots E_t) \det B)$. So $\det(AB) = \det E_1 \det(E_2 \cdots E_t) \det B$. Finally, applying (a) again, we have $\det E_1 \det(E_2 \cdots E_t) = \det(E_1(E_2 \cdots E_t)) = \det A$, so $\det(AB) = \det(A)\det(B)$.

On the other hand if A is not invertible, then also AB is not invertible. For both A and AB, we can apply elementary row operations to get

$$A = E_1 \cdots E_t \mathrm{rref}(A) \qquad \text{and } AB = E_1' \cdots E_s' \mathrm{rref}(AB)$$

where both $\mathrm{rref}(A)$ and $\mathrm{rref}(AB)$ have bottom row all zeros, and hence determinant zero. By part (a) and induction, we conclude that both A and AB have determinant zero.

Math 217 Worksheet 21C: Determinants and Volume

Definition: The **standard unit n-cube** in $\mathbb{R}^n$ is the set $\{t_1\vec{e}_1 + \cdots + t_n\vec{e}_n \mid 0 \le t_i \le 1\} \subseteq \mathbb{R}^n$.
Theorem 2: *Consider a linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$. Let P be the parallelepiped which is the image of the standard unit n-cube under T. Then the n-volume of P is $|det\, T|$.*

Problem 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $\begin{bmatrix} 7 & 3 \\ 0 & 4 \end{bmatrix}$.

(a) The image $T[\{t_1\vec{e}_1 + t_2\vec{e}_2 \mid 0 \le t_i \le 1\}]$ of the standard unit square* is a parallelogram. Sketch it.

(b) Verify Theorem 2 in this example.

Solution: Using the Key Theorem, the transformation stretches out $\vec{e}_1$ to $7\vec{e}_1$ and sends $\vec{e}_2$ to the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Your sketch should show a parallelogram with one side along the x-axis, connecting the origin to the point $(7, 0)$. Another side connects the origin to the point $(3, 4)$. So the four vertices are $(0, 0), (7, 0), (3, 4)$, and $(10, 4)$. This parallelogram has base of length 7 and height of length 4, so its area is 28, which also is $|\det A|$.

* "Unit square" is another name for "unit 2-cube".

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$, where $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ are *orthogonal.*

(a) The image of the unit square under T is a *rectangle* with sides of lengths $||\vec{v}_1||$ and $||\vec{v}_2||$. Why? Sketch it. What does Theorem 2 tell us about $\det A$?

(b) Verify Theorem 2 for this T. [HINT: One way to find $\det A$ uses QR factorization; Another writes out $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.]

Solution:

(a) The first column of the matrix, $\vec{v}_1$, is the image of $\vec{e}_1$, and the second column, $\vec{v}_2$, of A is the image of $\vec{e}_2$. These are two of the the sides of the image parallelogram. Since $\vec{v}_1$ are perpendicular, the image is a rectangle with side lengths $||\vec{v}_1||$ and $||\vec{v}_2||$. Your sketch should show it neatly squared up along the x and y axis with one corner at the origin. The sides of the rectangle have lengths $||\vec{v}_1||$ and $||\vec{v}_2||$. So the area is $||\vec{v}_1||\,||\vec{v}_2||$. Theorem 2 tells us that $|\det A|$ must be $||\vec{v}_1||\,||\vec{v}_2||$.

(b) Using QR-factorization, write $A = QR$, so $|\det A| = |\det Q||\det R|$ by the multiplicative property of determinants. The matrix R is the change of basis matrix for the Gram-Schmidt process. Since the $\vec{v}_i$ are already perpendicular, all we would do is scale each column of A by its length. This means R is **diagonal:** $R = \begin{bmatrix} ||\vec{v}_1|| & 0 \\ 0 & ||\vec{v}_2|| \end{bmatrix}$. Finally, Q is an orthogonal matrix, so its determinant is ± 1. So $|\det A| = |\det R| = ||\vec{v}_1||\,||\vec{v}_2||$, which is the area of the rectangle. This confirms the theorem in this case.

Another way to compute the determinant of A is to write $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, and then observe that because $\vec{v}_1 \cdot \vec{v}_2 = 0$, we must have $\vec{v}_2 = \begin{bmatrix} -kb \\ ka \end{bmatrix}$ for some $k \in \mathbb{R}$. So $\det A = \det \begin{bmatrix} a & -kb \\ b & ka \end{bmatrix} = k(a^2 + b^2)$. Since $||\vec{v}_1|| = \sqrt{(a^2 + b^2)}$ and $||\vec{v}_2|| = |k|\sqrt{(a^2 + b^2)}$, we again have $|\det A| = ||\vec{v}_1||\,||\vec{v}_2||$ is the area of $T[Q]$, confirming the theorem in this case.

Problem 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix A, where A has linearly *dependent* columns $\vec{v}_1, \vec{v}_2$.

(a) The image $T[\{t_1\vec{e}_1 + t_2\vec{e}_2 \,|\, 0 \le t_i \le 1\}]$ of the standard unit square Q is a *line segment.* Why?

(b) Verify Theorem 2 in this example.

Solution: Since $\vec{v}_1$ and $\vec{v}_2$ are dependent, they span a line L (through the origin) in $\mathbb{R}^2$ (or it could be just the origin if A is the zero matrix, but we'll leave this trivial case to you). The image of T is this line, and in particular, the image $T[Q]$ is contained in this line. The image $T[Q]$ contains $\vec{0}$ and so it is a segment containing $\vec{0}$ on L (it is not so important exactly what the segment is, but you can figure it out...if both $\vec{v}_1$ and $\vec{v}_2$ are in quadrant 1, for example, segment has endpoints $\vec{0}$ and $\vec{v}_1 + \vec{v}_2$; if $\vec{v}_1$ is in quadrant 1 but $\vec{v}_2$ is in quadrant 3, the origin will be in the interior of the segment and the end points will be $\vec{v}_1$ and $\vec{v}_2$.) In any case, the "area" of a line segment is zero, which is also the determinant of the matrix, verifying Theorem 2 in this case.

Problem 4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix A, where A has linearly *independent* columns $\vec{v}_1, \vec{v}_2$.

(a) The image $T[Q]$ of the standard unit square Q is a *parallelogram.* Sketch it, labelling the vectors $\vec{v}_1$ and $\vec{v}_2$ on your sketch. [PROTIP: Placing $\vec{v}_1$ and $\vec{v}_2$ in Quadrant 1 will make the sketch more manageable.]

(b) Suppose we apply the Gram Schmidt process to $\{\vec{v}_1, \vec{v}_2\}$ and get the vectors $\{\vec{u}_1, \vec{u}_2\}$. Add $\vec{u}_1$ to your sketch, clearly showing its relationship to $\vec{v}_1$. Show also $\vec{u}_2$ on your sketch.

(c) Compute that the base length and the height of the parallelogram $T[Q]$ are $\vec{v}_1 \cdot \vec{u}_1$ and $\vec{v}_2 \cdot \vec{u}_2$.

(d) Prove Theorem 2 in dimension two. [Hint: Compute the determinant of A *using its QR factorization.*]

Solution: You sketch should show the parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$; it's vertices are the origin and (the heads of) $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_1 + \vec{v}_2$. Your sketch should show $\vec{u}_1$ a unit vector in the same direction as $\vec{v}_1$, whereas $\vec{u}_2$ is perpendicular to $\vec{v}_1$. The vector $\vec{v}_2^{\perp}$ is an altitude representing its height. We can think of this vector $\vec{v}_2^{\perp}$ as the component of $\vec{v}_2$ in the $\vec{u}_2$ direction, so its length is $\vec{v}_2 \cdot \vec{u}_2$. So the length of base of our parallelogram is $||\vec{v}_1|| = \vec{v}_1 \cdot \vec{u}_1$ and the height is $|\vec{v}_2^{\perp}| = \vec{v}_2 \cdot \vec{u}_2$. So the area is "base times height" or $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)$. The QR factorization is

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \cdot \vec{u}_1 & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & \vec{v}_2 \cdot \vec{u}_2 \end{bmatrix}.$$

So determinant of A is $\det Q \det R = \pm(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2) = \pm$ *height* $\times$ *base* $=$ area of image parallelogram.

Problem 5. Let A be the 3×3 matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be left multiplication by A.

(a) Assuming the columns of A are linearly independent, use the QR-factorization to show that

$$|\det A| = (\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_3 \cdot \vec{u}_3),$$

where $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is obtained from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ by the Gram-Schmidt process.

(b) The image of the standard unit cube Q_3 under T is a parallelepiped with one vertex at $\vec{0}$ and $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ as three of its edges. Why?

(c) The image parallelepiped $T[Q_3]$ has sides that are parallelograms. Explain why one of these sides (let's call it the "base") has area $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)$. Explain why the height of the parallelepiped is $(\vec{v}_3 \cdot \vec{u}_3)$.

(d) Prove Theorem 2 in dimension 3. Do you see how one might construct an inductive proof for Theorem 2 in arbitrary dimension?

Solution:

(a) Writing $A = QR$, we see that $\det A = \det Q \det R$. Because Q is orthogonal, its determinant is ± 1. So $|\det A| = \det R$, which is positive because it is an upper triangular matrix with positive entries on the diagonal. The determinant of R is the product of the diagonal entries, or $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_3 \cdot \vec{u}_3)$.

(b) The image of the unit cube is a parallelepiped whose edges are $\vec{v}_1, \vec{v}_2, \vec{v}_3$. This a prism, with base a parallelogram formed by $\vec{v}_1, \vec{v}_2$. The length of this parallelogram is $||\vec{v}_1||$ or $\vec{v}_1 \cdot \vec{u}_1$ and the height of the parallelogram is $||\vec{v}_2^{\perp}||$, or $\vec{v}_2 \cdot \vec{u}_2$. The solid parallelepiped has height which is $||\vec{v}_3^{\perp}||$, or $\vec{v}_3 \cdot \vec{u}_3$.

(c) The area is base times height or $\vec{v}_1 \cdot \vec{u}_1$ times $\vec{v}_2 \cdot \vec{u}_2$. The height is $(\vec{v}_3 \cdot \vec{u}_3)$, so the volume of the parallelepiped is $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_3 \cdot \vec{u}_3)$.

(d) From (a) and (c), we have the same result: the volume is $|detA| = \det R = (\vec{v}_1 \cdot \vec{u}_1)\,(\vec{v}_2 \cdot \vec{u}_2)\,(\vec{v}_3 \cdot \vec{u}_3)$. On the other hand, if the columns of A are linearly dependent, then $\text{im}T$ has dimension two or less, which means that $T[Q_3]$, which lives inside $\text{im}T$, can not be 3-dimensional and must have zero 3-volume (which is the same as the tradition volume in 3D).

For the n-dimensional case, we can argue by induction. If the matrix has rank n, then the image is an n-dimensional parallelepiped with a "side" that is a parallelepiped of dimension $n-1$. If that "side" is constructed from the vectors $\vec{v}_1, \ldots, \vec{v}_{n-1}$, then its $n-1$-volume is the product $(\vec{v}_1 \cdot \vec{u}_1) \cdots (\vec{v}_{n-1} \cdot \vec{u}_{n-1})$. The height of $T[Q_n]$ is then $(\vec{v}_n \cdot \vec{u}_n)$, so the n-volume is $(\vec{v}_1 \cdot \vec{u}_1) \cdots (\vec{v}_n \cdot \vec{u}_n)$. Thinking about the QR factorization of A, we see that this is also $|\det A|$.

Problem 6. THE SIGN OF THE DETERMINANT. Let A be a 2×2 matrix representing a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ in standard coordinates. Investigate the geometric meaning of the *sign of the deteminant* by sketching the images $\vec{v}_1$ and $\vec{v}_2$ of $\vec{e}_1$ and $\vec{e}_2$ in several different cases, some where the deteminant of A is negative and some where it is positive. What happens for 3×3 matrices? What general observation can you make?

Solution: The sign is positive if the orientation of $\{T(\vec{e}_1), T(\vec{e}_2)\}$ is the same as $\{\vec{e}_1, \vec{e}_2\}$. This means the acute angle between them has $T(\vec{e}_1)$ as the right edge and $T(\vec{e}_2)$ as the left edge (just like the acute angle between $\vec{e}_1$ and $\vec{e}_2$). The sign is negative if the orientation is swapped. The same is true for 3×3 matrices. Remember the "right hand rule" from Calc 3? The orientation $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is righthanded: this means if you put your right hand on $\vec{e}_1$ and curl your fingers towards $\vec{e}_2$, your thumb points in the (positive!) $\vec{e}_3$ direction. The tranformation T PRESERVES ORIENTATION if the orientation of $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$ is righthanded: putting your right hand on $T(\vec{e}_1)$ and curling your fingers towards $T(\vec{e}_2)$, your thumb points in the same direction as $T(\vec{e}_3)$. The sign of the determinant is postive if T is orientation preserving and negative if its orientation reversing.