Mathematical Hygiene

These notes are designed to expose you to elementary logic, the grammar of mathematical communication. Once internalized, this material will help keep your mathematics "healthy and strong."[1] For a rigorous treatment of logic, you may wish to take Math 481, *Introduction to Mathematical Logic*.

[1] "Logic is the hygiene that the mathematician practices to keep his ideas healthy and strong." (Hermann Weyl, quoted in *American Mathematical Monthly*, November 1992)

Statements

Ambiguity is accepted, maybe even welcomed, in certain methods of discourse. As Definition 1 suggests, it is generally avoided in math.

Definition 1. *A* statement*, also called a* proposition*, is a sentence that is either true or false, but not both.*

For example, the sentences $\boxed{1+2+3 = 1 \cdot 2 \cdot 3.}$, $\boxed{5+4=8.}$, and $\boxed{\text{In terms of future career satisfaction, math is a top-ranked degree.}}$ are all statements. However, the sentences $\boxed{\text{This sentence is false.}}$, $\boxed{x = \pi + 34.}$, $\boxed{\text{When does Michigan play today?}}$, and $\boxed{\text{Go Blue!}}$ are all NOT statements.

See www.math.lsa.umich.edu/career/ for information about careers for students of math.

For a fixed value of x the sentence $\boxed{x = \pi + 34.}$ is either true or false. However, as a value for x has not been specified, the sentence is neither true nor false.

To help distinguish between examples and the running text, statements will often be placed in parentheses. For example, for a fixed object x and a fixed set S both ($x \in S$.) and ($x \notin S$.) are statements.

In math, the symbols P and Q are often used as short hand for statements. If P is a statement, then its *truth value* is T if P is true and F if P is false. For example, the truth value of the statement ($3 \cdot 4 = 13$.) is F, while the the truth value of both ($1001 = 7 \cdot 11 \cdot 13$.) and (The Michigan Math Club meets on Thursdays at 4PM in the Nesbitt Commons Room, East Hall.) is T.

The phrase "mind your P's and Q's" becomes especially relevant in this part of mathematics.

Math Club events feature an engaging math talk and free pizza and pop. See www.math.lsa.umich.edu/mathclub.

Negation and truth tables

The *negation* of a statement P is written $\neg P$ and read "not P." The negation can usually be formed by inserting the word *not* into the original statement. For example, the negation of (1000009 is prime.) is (1000009 is not prime.). We require that $\neg P$ have the opposite truth value of P, and so, for example, $\neg$(All mathematicians are left-handed.) is (Not all mathematicians are left-handed.) rather than (All mathematicians are not left-handed.).

By writing it as a sum of two squares in two different ways, Euler deduced that $1000009 = 293 \cdot 3413$.

A *truth table* is a tabulation of the possible truth values of a logical operation. For example, the truth table for negation appears in Table 1. For each possible input (the truth value of P is either T or F) the table records the output of the negation operation.

P	$\neg P$
T	F
F	T

Table 1: The truth table for negation.

Equivalent statements

Suppose the edges of a triangle T have lengths a, b, and c with $a \leq b \leq c$. Thanks to Pythagoras and others we know[2] that the statement $(a^2 + b^2 = c^2.)$ is *equivalent* to the statement (T is a right triangle.). Similarly, (Not all mathematicians are left-handed.) is equivalent to (Some mathematicians are not left-handed.).

[2] *Euclid's Elements*, Book I, Propositions 47 and 48.

When statements P and Q are equivalent, we write $P \Leftrightarrow Q$. We remark that equivalent statements have the same truth values.

In the standard interpretation of English, two negatives make a positive. The same is true in logic: for all statements P we have $\neg(\neg P) \Leftrightarrow P$. As expected, $\neg(\neg P)$ and P have the same truth values:

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

"The English linguistics professor J.L. Austin was lecturing one day. 'In English,' he said, 'a double negative forms a positive. In some languages though, such as Russian, a double negative is still a negative. However,' he pointed out, 'there is no language wherein a double positive can form a negative.' From the back of the room, the voice of philosopher Sydney Morgenbesser piped up, 'Yeah, right.' " (*The Times*, September 8, 2004)

Compound statements: Conjunctions and Disjunctions

Mathematics and English agree about the meaning of "and." The *conjunction* of statements P and Q is the statement (P and Q), often written $(P \wedge Q)$. Note that the statement $(P \wedge Q)$ is true when both P and Q are true and is false otherwise.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

However, Mathematics and English disagree when it comes to the meaning of the word "or." For example, if your mathematics instructor says

"As a prize, you may have a t-shirt or a keychain,"

then the standard interpretation of this statement is "As a prize, you may have a t-shirt or a keychain, but not both." THIS IS NOT THE MATHEMATICAL MEANING OF THE STATEMENT. The mathematical meaning is "As a prize, you may have a t-shirt, a keychain, or both." The *disjunction* of statements P and Q is the statement (P or Q), often written $(P \vee Q)$. Note that the statement $(P \vee Q)$ is false when both P and Q are false and is true otherwise.

Michigan Math t-shirts are available for purchase in the Undergraduate Office, 2082 East Hall.

In the table below, the first row of truth values reflects the difference between mathematics and English.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The operations of negation, conjunction, and disjunction correspond[3] to the set operations of complement, intersection, and union, respectively. It is therefore not surprising that relations among negation, conjunction, and disjunction are encapsulated in DeMorgan's Laws:

$$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q) \quad \text{and} \quad \neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q).$$

[3]
$$\begin{aligned} A^{\complement} &= \{x \mid \neg(x \in A)\} \\ A \cap B &= \{☺ \mid (☺ \in A) \wedge (☺ \in B)\} \\ A \cup B &= \{☺ \mid (☺ \in A) \vee (☺ \in B)\} \end{aligned}$$

Conditional Statements

When Bruce Willis' character in *Die Hard* expounds "If you're not part of the solution, [then] you're part of the problem," he has com-

PRACTICE: To gain familiarity with compound statements and conditionals, use Doug Ensley's materials at `math.lsa.umich.edu/courses/101/imply.html`, `math.lsa.umich.edu/courses/101/tt1.html`, and `math.lsa.umich.edu/courses/101/tt2.html`.

bined the statements r = (You're not part of the solution.) and s = (You're part of the problem.) to form the *conditional statement* (If r, then s.).

For statements P and Q the conditional statement (If P, then Q.) is often written ($P \Rightarrow Q$) and read "P implies Q." The statement P is called the *hypothesis* (or *antecedent* or *premise*) and the statement Q is called the *conclusion* (or *consequent*). Mathematically, the statement ($P \Rightarrow Q$) is false when P is true and Q is false and is true otherwise.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that $P \Rightarrow Q$ is false exactly once: when a true hypothesis implies a false conclusion. Does this agree with our ordinary understanding of implication? Consider Almira Gulch's threat to Dorothy:

When $P \Rightarrow Q$ is true, we say that P is a *sufficient* condition for Q. For example, a sufficient condition for a function on the real numbers to be continuous at π is that the function be differentiable at π.

> "If you don't hand over that dog, then I'll bring a damage suit that'll take your whole farm." *The Wizard of Oz*, 1939

Suppose that Dorothy hands over that dog, Toto, thus FAILING to carry out the hypothesis. In this case, Ms. Gulch's statement is true independent of whether or not she fulfills the conclusion by bringing a damage suit. Should Dorothy choose to fulfill the hypothesis by not handing over the dog, then Ms. Gulch's statement is false unless she files suit. So, it appears mathematics and English agree for this example. On the other hand, the mathematically correct statement (If $3 = 7$, then $8 = 4 + 4$.) sounds bizarre, even to a mathematician.

When $P \Rightarrow Q$ is true, we say that Q is a *necessary* condition for P. For example, $\lim_{n\to\infty} a_n = 0$ is a necessary condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

As with all statements, the statement $P \Rightarrow Q$ may be negated. Since the negation of $(P \Rightarrow Q)$ is required to be true when P is true and Q is false, and false otherwise, we must have $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$. Thus, the negation of (If you're not part of the solution, then you're part of the problem.) is (You are not part of the solution and yet you are not part of the problem.), and for a function f on the real numbers, the negation of (If f is differentiable at π, then f is continuous at π.) is (f is differentiable at π, and f is not continuous at π.).

P	Q	$P \wedge \neg Q$	$\neg(P \Rightarrow Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Predicates

The sentence $\boxed{y > 4.}$ is not a statement because, depending on the value of the variable y, the sentence may be either true or false. Since sentences such as $\boxed{y > 4.}$ arise very often, we give them their own name, *predicate*. We often use notation like $P(x)$ to denote a predicate that depends on a variable x. So, for example, $P(x)$ might denote the predicate $\boxed{2 < e^x < e}$ and $Q(☺, ☹)$ might denote the predicate $\boxed{☺^2 + ☹^2 = 34.}$.

PRACTICE: To gain familiarity with predicates, use Doug Ensley's material at `math.lsa.umich.edu/courses/101/predicate.html`.

As with statements, a predicate can be negated. For example, suppose $Q(☺, ☹) = \boxed{☺^2 + ☹^2 = 34.}$ and $r(y) = \boxed{y > 4.}$, then $\neg Q(☺, ☹)$ is $\boxed{☺^2 + ☹^2 \neq 34.}$ and $\neg r(y)$ is $\boxed{y \leq 4.}$.

PRACTICE: To gain familiarity with negating predicates, use Doug Ensley's material at `math.lsa.umich.edu/courses/101/np1.html` and `math.lsa.umich.edu/courses/101/np2.html`.

Quantifiers

By quantifying the variable that occurs in a predicate, we can create statements. For example,

$$\text{(There exists a real number } y \text{ such that } y > 4.) \tag{1}$$

is true (and, since it is not false, is therefore a statement), and

$$\text{(For all real numbers } y\text{, we have } y > 4.) \tag{2}$$

is false (and, since it is not true, is therefore a statement). The words *there exists* and *for all* in statements (1) and (2) are called *quantifiers*. While the words "for all" and "there exists ... such that" don't take long to write out, they appear so frequently that the following shorthand has been adopted: the symbol $\forall$ translates as "for all" and the symbol $\exists$ translates as "there exists ... such that." Thus, statement (1) is equivalent to ($\exists y \in \mathbb{R}\ r(y)$.), and statement (2) is equivalent to ($\forall y \in \mathbb{R}, r(y)$.).

Often, quantifiers are hidden. For example, the statement (Every integer is even.) can be written ($\forall n \in \mathbb{Z}$, n is even.) and the statement (Some integers are even.) is equivalent to ($\exists m \in \mathbb{Z}$ m is even.). Ferreting out hidden quantifiers can be more than half the battle.

Here are two final examples that may be familiar to you. Fermat's Last Theorem says

$$\forall n \in \mathbb{N}, ((\exists a, b, c \in \mathbb{N}\ a^n + b^n = c^n) \Rightarrow (n \leq 2))$$

and, for a predicate S, the Principle of Mathematical Induction states

$$[S(1) \wedge (\forall n \in \mathbb{N}, S(n) \Rightarrow S(n+1))] \Rightarrow (\forall m \in \mathbb{N}, S(m)).$$

Negation and quantifiers

Recall that if P is a statement, then the symbol $\neg P$ denotes the negation of P. With the addition of quantifiers to the mix, negation can be more challenging. For example, ¬(Everyone remembers how to negate statements.) is (Somebody does not remember how to negate statements.) and the negation of (Some integers are even.) is (Every integer is odd.). The negation of statement (1) is (For all real numbers w, $w \leq 4$.), and the negation of statement (2) is (There exists a real number z such that $z \leq 4$.). Do you see the pattern? For a predicate $P(x)$ we have

$$\neg(\forall x, P(x)) \text{ is } \exists z\ \neg P(z) \quad \text{and} \quad \neg(\exists w\ P(w)) \text{ is } \forall v, \neg P(v).$$

Thus, the negation of (Every triangle is isosceles.) is (Some triangle is not isosceles.) and ¬(There is a positive real number that is greater than its square.) is (Every positive real number is less than or equal to its square.).

PRACTICE: To gain familiarity with quantifiers, use Doug Ensley's material at `math.lsa.umich.edu/courses/101/quantifiers.html`.

We have used the phrase "such that" rather than the incorrect "so that." See the comments of former Michigan mathematics professor J.S. Milne at `www.jmilne.org/math/words.html`.

"For all" is called a *universal* quantifier and "there exists" is called an *existential* quantifier.

... cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

In Calculus, a function f is said to be continuous at a provided that

$$\forall \varepsilon > 0, \exists \delta > 0\ \forall x \in \mathbb{R}, \\ (|x-a| < \delta) \Rightarrow (|f(x) - f(a)| < \varepsilon).$$

Thus, as you should verify, a function f is NOT continuous at a provided that

$$\exists \varepsilon > 0\ \forall \delta > 0, \exists x \in \mathbb{R} \\ (|x-a| < \delta) \wedge (|f(x) - f(a)| \geq \varepsilon).$$

"Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs." (Paul Halmos, *I Want to be a Mathematician*, 1985)