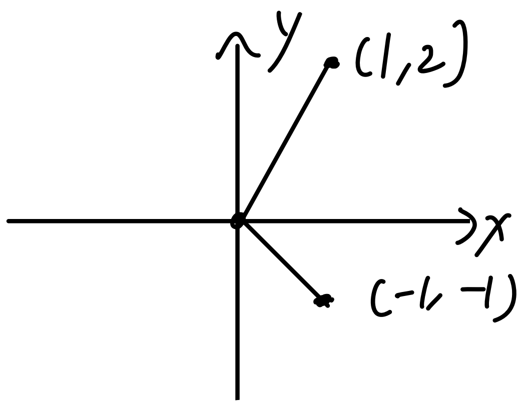
在 2-1 我们知道了 $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ 是 $\mathbb{R}^2$ 中 counterclockwise 转 90° 的 linear transformation

现在再看几个:

$$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\quad B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

$$C=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

$$E=\begin{bmatrix}1 & 0.2\\ 0 & 1\end{bmatrix},\quad F=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

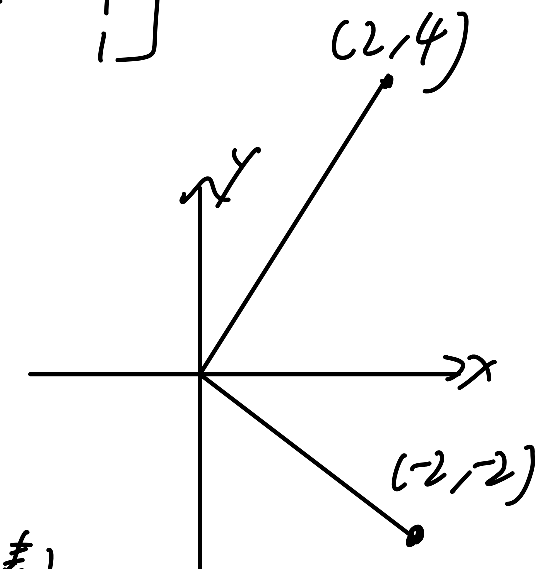


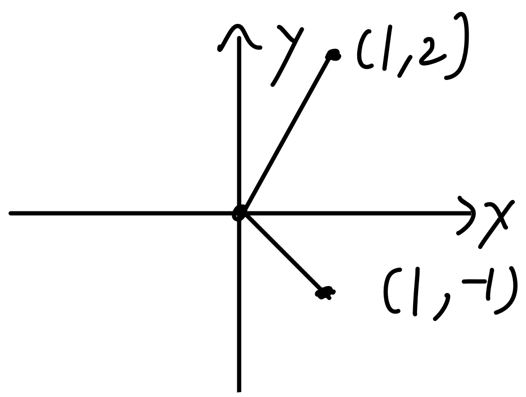
(主diag)

$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

$\Longrightarrow$

放大一倍



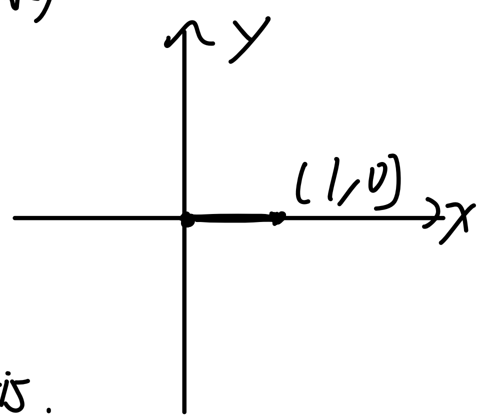


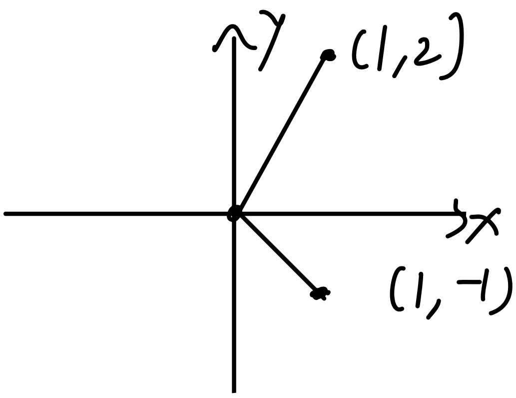
(主diag 首元素)

$B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$\Longrightarrow$

orthogonal projection onto x-axis.



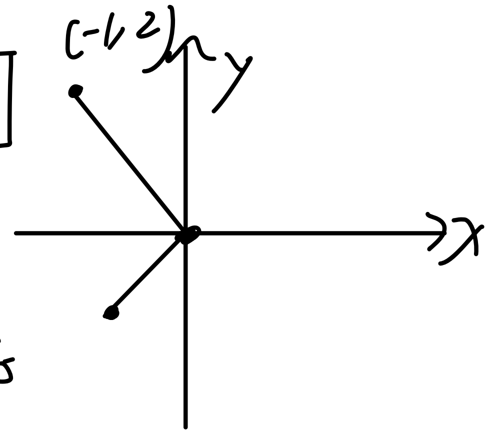


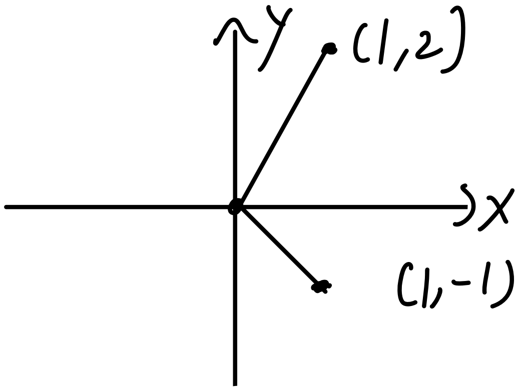
(主diag)

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$

reflection about y-axis



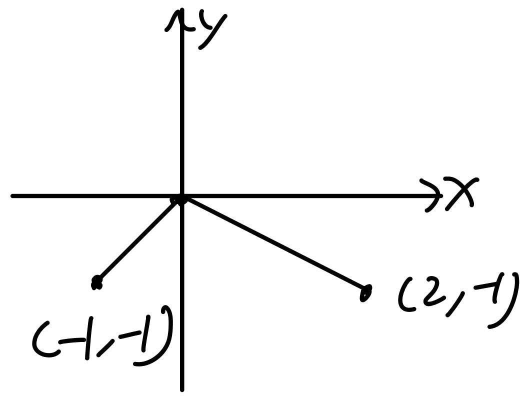


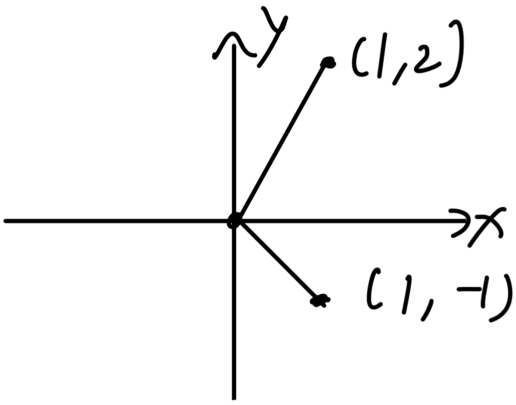
(副diag)

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\Rightarrow$

clockwise 转90°

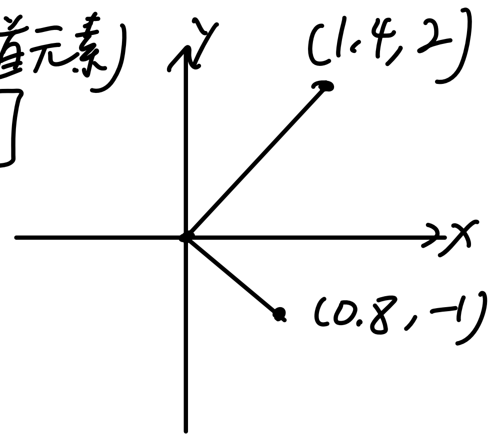


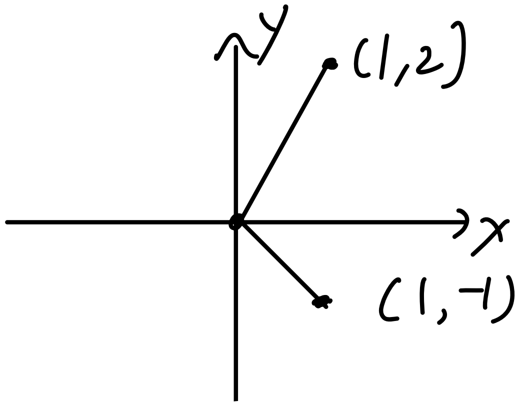


(主diag+副首元素)

$$E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$



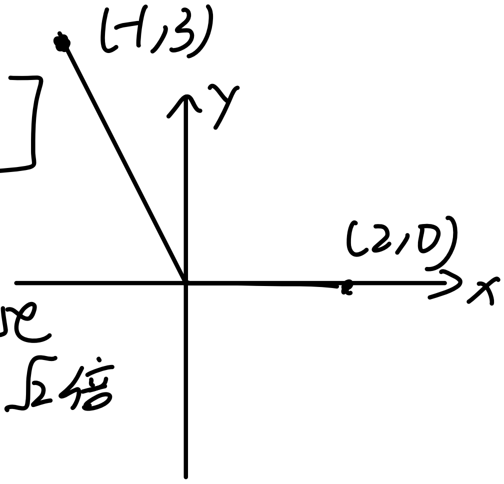


(主,副diag)

$$F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow$

counterclockwise shift + 放大$\sqrt{2}$倍



1. Scaling

[Thm③] Scalings.

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k.

since $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= k\vec{x}$

2. Orthogonal Projection

Def④ 2.2.1 Orthogonal Projection

Consider a line L in coordinate plane, 穿过 $(0,0)$.

任何 $\vec{x} \in \mathbb{R}^2$ 都可以写为 $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

其中 $\vec{x}^{\parallel} \parallel L$, $\vec{x}^{\perp} \perp L$.

而这个 $T(\vec{x}) = \vec{x}^{\parallel}$ 的 transformation 叫做 orthogonal projection of $\vec{x}$ onto L.

denoted : $\boxed{proj_L(\vec{x})}$

随意取 $\vec{w} \parallel L$, $\boxed{proj_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}}$

特别地，如果取 $\vec{w}$ 为 unit vector

$\boxed{\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \parallel L}$

$(\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel})$

可得 $\boxed{proj_L(\vec{x}) = (\vec{x}\cdot\vec{u})\cdot\vec{u}}$

这一 transformation 是 linear 的，with matrix

$$P = \boxed{\frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}} = \boxed{\begin{bmatrix} u^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}}$$

---

3. Reflection

Def ⑤ 2.2.2 Reflection

Consider a line L in coordinate plane, 过 $(0,0)$.

Let $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel}$

$\Rightarrow$ $\boxed{T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}}$ 为 reflection of $\vec{x}$ about L

denoted : $\boxed{ref_L(\vec{x})} = \vec{x}^{\parallel} - \vec{x}^{\perp}$

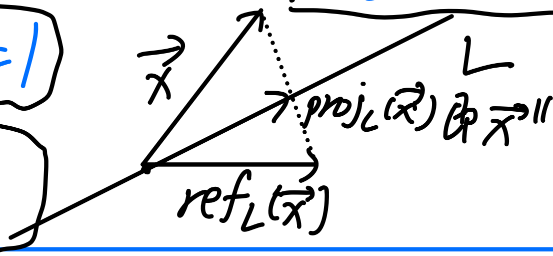
由 Def ④ 2.2.1 可得 $ref_L(\vec{x}) = \vec{x}^{\parallel} - (\vec{x} - \vec{x}^{\parallel})$

$= 2proj_L(\vec{x}) - \vec{x} = \boxed{2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}}$

其 matrix of T 为 form: $\boxed{\begin{bmatrix} a & b \\ b & -a \end{bmatrix}}$,

where $\boxed{a^2+b^2=1}$

$\boxed{\left(\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}\right)}$



$$\text{ref}_L(\vec{x}) = 2\,\text{proj}_L(\vec{x}) - \vec{x}$$
$$= 2P\vec{x} - \vec{x} = (2P - I_2)\vec{x}$$

$$\Rightarrow S = 2P - I_2 = \begin{bmatrix} 2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$$

4. Rotation

Thm 2.2.4 Rotations

$T(\vec{x}) = A\vec{x}$ 表示一个 $\boxed{\text{counterclockwise}}$ rotation in $\mathbb{R}^2$.

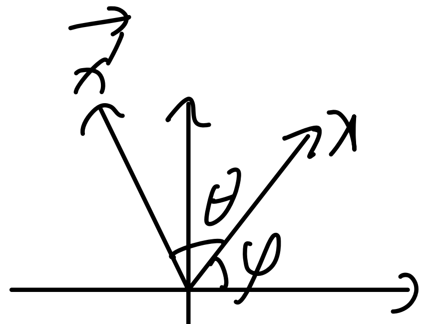
$$A = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$ (rotate by θ)

form: $\boxed{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}$, where $\boxed{a^2+b^2=1}$

Pf 对于原始 vector $\vec{x}$，

将其 coordinate 以 polar

形式表示为 $(r\cos\varphi, r\sin\varphi)$



$\Rightarrow$ rotation 后 $\vec{x}' = (r\cos(\varphi+\theta), r\sin(\varphi+\theta))$

$$\Rightarrow x_1' = r\cos(\varphi+\theta) = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta$$
$$= x\cos\theta - y\sin\theta$$

$$y_1' = r\sin(\varphi+\theta) = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta$$
$$= x\sin\theta + y\cos\theta$$

$$\Rightarrow \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\boxed{\text{Thm ⑤ 2.2.4}}$ $\boxed{\text{Rotations combined with a scaling}}$

将 $\vec{v} \in \mathbb{R}^2$ counterwise rotate θ

并放大至 r 倍:

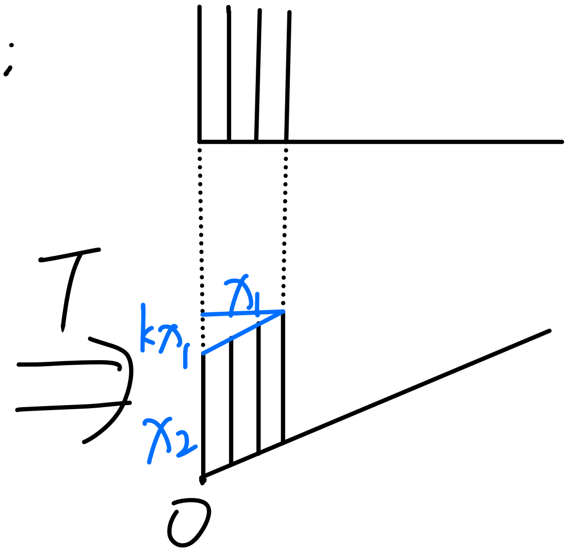
Let $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$$\Rightarrow T(\vec{x}) = A\vec{x},\ A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

---

5. Shearing

vertical:

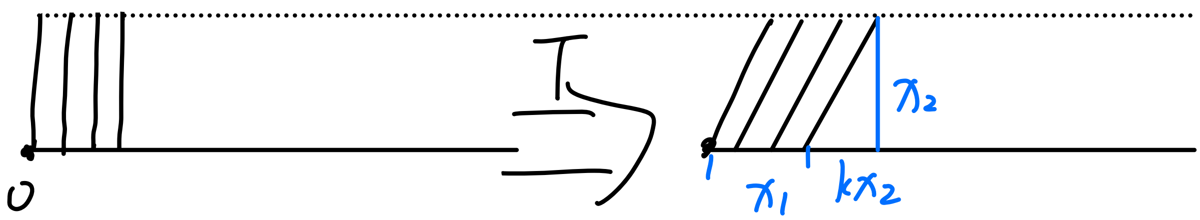


$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x \\ kx_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}} \vec{x}$$

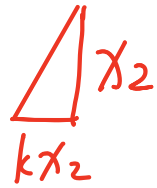
horizontal:



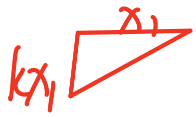
$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\boxed{\text{Thm ⑤ 2.2.5}}$ $\boxed{\text{Horizontal and vertical shearing}}$

horizontal shearing of slope 

: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

vertical shearing of slope 

: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

总结

Transformation	Matrix
Scaling by k	$kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Orthogonal proj onto line L	$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ $\left(\vec{u} \parallel L \;\&\; \lVert \vec{u} \rVert = 1 \right)$
Reflection about line L	$\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$ $\left(\begin{bmatrix} a & b \\ b & -a \end{bmatrix}, a^2+b^2=1 \right)$
Rotation through angle θ (逆时针)	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a^2+b^2=1 \right)$
Rotation through θ with scaling by r	$r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\left(a^2+b^2=r^2 \right)$
Shear	horizontal: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$; vertical: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$