This is the Diagonalization Gateway test. Passing on this test is ALL FIVE of the five problems on the test.

Problem 1. **(1 point)**

The matrix $A = \begin{bmatrix} 0 & 3 \\ 5 & 2 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues satisfy

$$\begin{aligned} \det(\begin{bmatrix} 0-\lambda & 3 \\ 5 & 2-\lambda \end{bmatrix}) &= (0-\lambda)(2-\lambda)-(3)(5) \\ &= \lambda^2 - 2\lambda - 15 \\ &= (\lambda+3)(\lambda-5) = 0. \end{aligned}$$

Thus $\lambda = -3$ or $\lambda = 5$.

If $\lambda = -3$, the eigenvector satisfies

$$\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

Similarly, if $\lambda = 5$, the eigenvector satisfies

$$\begin{bmatrix} -5 & 3 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 3 & 3 \\ -3 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

Answer(s) submitted:

- $\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}; \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Correct Answers:

- $\begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}; \begin{bmatrix} 3 & 3 \\ -3 & 5 \end{bmatrix}$

Problem 2. **(1 point)**

The matrix $A = \begin{bmatrix} -8 & 154 \\ -1 & 17 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues of A satisfy

$$\begin{aligned} \det(\begin{bmatrix} -8-\lambda & 154 \\ -1 & 17-\lambda \end{bmatrix}) &= (-8-\lambda)(17-\lambda)-(154)(-1) \\ &= \lambda^2 - 9\lambda + 18 \\ &= (\lambda-6)(\lambda-3) = 0. \end{aligned}$$

Thus $\lambda = 6$ or $\lambda = 3$.

If $\lambda = 6$, the eigenvector satisfies

$$\begin{bmatrix} -14 & 154 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 154 \\ 14 \end{bmatrix}$.

Similarly, if $\lambda = 3$, the eigenvector satisfies

$$\begin{bmatrix} -11 & 154 \\ -1 & 14 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 154 \\ 11 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 154 & 154 \\ 14 & 11 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

Answer(s) submitted:

- $\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}; \begin{bmatrix} 14 & 11 \\ 1 & 1 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Correct Answers:

- $\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}; \begin{bmatrix} 154 & 154 \\ 14 & 11 \end{bmatrix}$

Problem 3. **(1 point)**

Identify the diagonalizability of each of the following matrices.

1. $A = \begin{bmatrix} 5 & -1 \\ 0 & 7 \end{bmatrix}$
   A is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

2. $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$
   B is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

3. $C = \begin{bmatrix} -2 & 2 \\ -2 & -4 \end{bmatrix}$
   C is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

Solution: For each matrix, we determine diagonalizability from the existence of a complete eigenspace. For A, we see that eigenvalues are given by

$$(-\lambda)(7-\lambda)+1\cdot 0=\lambda^2-12\lambda+35=0.$$

Completing the square, we have

$$(\lambda-6)^2-1=0,$$

so that there are two distinct real eigenvalues, so that A is Diagonalizable over real and complex numbers.

Similarly, for B, we have

$$(\lambda-1)^2+0=0,$$

so that there are two repeated real eigenvalues, and because the matrix (after subtracting λI) will have nonzero entries and so cannot have two eigenvectors, so that B is Not diagonalizable over either real or complex numbers.

Similarly, for C, we have

$$(\lambda+3)^2+3=0,$$

so that there are two complex conjugate eigenvalues, so that C is Diagonalizable over complex numbers but not reals.

Answer(s) submitted:

- Diagonalizable over real and complex numbers
- Not diagonalizable over either real or complex numbers
- Diagonalizable over complex numbers but not reals

submitted: (correct)
recorded: (correct)

Correct Answers:

- Diagonalizable over real and complex numbers
- Not diagonalizable over either real or complex numbers
- Diagonalizable over complex numbers but not reals

Problem 4. **(1 point)**

Identify the diagonalizability of each of the following matrices.

1. $A = \begin{bmatrix} -2 & -1 \\ 3 & -4 \end{bmatrix}$
   A is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

2. $B = \begin{bmatrix} 1 & 0 \\ -5 & 3 \end{bmatrix}$
   B is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

3. $C = \begin{bmatrix} -2 & 1 \\ -2 & -2 \end{bmatrix}$
   C is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

Solution: For each matrix, we determine diagonalizability from the existence of a complete eigenspace. For A, we see that eigenvalues are given by

$$(-\lambda)(-4-\lambda)+1\cdot 3 = \lambda^2+6\lambda+11 = 0.$$

Completing the square, we have

$$(\lambda+3)^2+2 = 0,$$

so that there are two complex conjugate eigenvalues, so that A is Diagonalizable over complex numbers but not reals.

Similarly, for B, we have

$$(\lambda-2)^2-1 = 0,$$

so that there are two distinct real eigenvalues, so that B is Diagonalizable over real and complex numbers.

Similarly, for C, we have

$$(\lambda+2)^2+2 = 0,$$

so that there are two complex conjugate eigenvalues, so that C is Diagonalizable over complex numbers but not reals.

Answer(s) submitted:

- Diagonalizable over complex numbers but not reals
- Diagonalizable over real and complex numbers
- Diagonalizable over complex numbers but not reals

submitted: (correct)
recorded: (correct)
Correct Answers:

- Diagonalizable over complex numbers but not reals
- Diagonalizable over real and complex numbers
- Diagonalizable over complex numbers but not reals

Problem 5. **(1 point)**

The matrix $A = \begin{bmatrix} 1 & 5 \\ 0 & -3 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues satisfy

$$\begin{aligned} \det\left(\begin{bmatrix} 1-\lambda & 5 \\ 0 & -3-\lambda \end{bmatrix}\right) &= (1-\lambda)(-3-\lambda)-(5)(0) \\ &= \lambda^2+2\lambda-3 \\ &= (\lambda-1)(\lambda+3) = 0. \end{aligned}$$

Thus $\lambda = 1$ or $\lambda = -3$.

If $\lambda = 1$, the eigenvector satisfies

$$\begin{bmatrix} 0 & 5 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Similarly, if $\lambda = -3$, the eigenvector satisfies

$$\begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

Answer(s) submitted:

- $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}; \begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix}$

submitted: (correct)
recorded: (correct)
Correct Answers:

- $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}; \begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix}$