Worksheet 14: Modeling Linear Transformations with Matrices (§3.4, §4.3)

IMPORTANT CONCEPT: In Chapter 2, we learned that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ can be conveniently represented by matrix multiplication. (The Key Theorem)

On Worksheet 13, we saw that fixing a basis $\mathcal{B}$ for a finite dimensional vector space V gives us a **coordinate system** V, allowing us to *identify* V with $\mathbb{R}^n$. (The Coordinate Isomorphism)

On Worksheet 14, we will discover that fixing a basis $\mathcal{B}$ for V also allows us to identify a linear transformation $T : V \to V$ with a matrix multiplication $\mathbb{R}^n \to \mathbb{R}^n$ for some suitable matrix.

Problem 1. The Coordinate Isomorphism: Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis for a vector space V. Recall the definition of the **coordinate isomorphism**

$$\begin{aligned} L_{\mathcal{B}} : V &\longrightarrow \mathbb{R}^n \\ v &\mapsto [v]_{\mathcal{B}}. \end{aligned}$$

(a) The isomorphism $L_{\mathcal{B}}$ gives an *identification* of V with $\mathbb{R}^n$ *as vector spaces*. What does this mean? How, exactly, does it work? What more does the phrase "*as vector spaces*" add to the word "*identification*"?

(b) The target of $L_{\mathcal{B}}$ should be thought of as the *$\mathcal{B}$-coordinate space* $\mathbb{R}^n$ *of* V. Explain what this adds over just calling it $\mathbb{R}^n$. If v, w, y are vectors in V:

  (i) What and *where* are $[v]_{\mathcal{B}}, [w]_{\mathcal{B}}, [y]_{\mathcal{B}}$?

  (ii) What column vector is $[3v_1]_{\mathcal{B}}$? $[\frac{\pi}{2}v_2]_{\mathcal{B}}$? $[3v_1 + \frac{\pi}{2}v_2]_{\mathcal{B}}$?

  (iii) Is $[\pi v + \sqrt{17}w - 4y]_{\mathcal{B}}$ the same as $\pi[v]_{\mathcal{B}} + \sqrt{17}[w]_{\mathcal{B}} - 4[y]_{\mathcal{B}}$?

(c) Under identification $L_{\mathcal{B}}$, what does the basis $(v_1, \ldots, v_n)$ of V correspond to in $\mathbb{R}^n$?

Solution: For (a) and (b): Saying the map $L_{\mathcal{B}}$ is an isomorphism says two things:

- It is bijective: every element of V is matched up with exactly one element in $\mathbb{R}^n$—we can think of it as a "renaming" of elements of V, giving them unique names/tags in $\mathbb{R}^n$. So we can identify each element of V with its column vector of coordinates.
- This bijection is a linear transformation, so it preserves the vector space structure: adding vectors (or scalar multiplying) in V and then looking at the coordinates of the result the "same" as adding (or scalar multipliying) their coordinates straight away. We can add (scalar multiply) before or after making the identification and the result is the same. For example,

  (i) $[v + w]_{\mathcal{B}} = [v]_{\mathcal{B}} + [w]_{\mathcal{B}}$.

  (ii) $[cv]_{\mathcal{B}} = c[v]_{\mathcal{B}}$ for all scalars c.

  (iii) $[\pi v + \sqrt{17}w - 4y]_{\mathcal{B}} = \pi[v]_{\mathcal{B}} + \sqrt{17}[w]_{\mathcal{B}} - 4[y]_{\mathcal{B}}$, and similarly for any linear combination, since linear transformations respect linear combinations.

For example, $[3v_1]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $[\frac{\pi}{2}v_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \vdots \\ 0 \end{bmatrix}$, and $[3v_1 + \frac{\pi}{2}v_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ \pi/2 \\ \vdots \\ 0 \end{bmatrix}$. Calling the target the $\mathcal{B}$-coordinate space reminds us that it is a *coordinate space* (a vector space of the form $\mathbb{R}^n$) but even more: the specific coordinates are the $\mathcal{B}$-coordinates. So given a column vector in the $\mathcal{B}$-coordinate space, we know it corresponds to a specific vector in V and how to figure out which one. Using a different basis to represent the elements of V, we would get different coordinates.

For (c), note that $[v_i]_{\mathcal{B}} = \vec{e}_i$. So the basis $\mathcal{B} = (v_1, \ldots, v_n)$ is taken to the standard basis $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ of $\mathbb{R}^n$. We immediately recognize a vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ in the $\mathcal{B}$-coordinate space $\mathbb{R}^n$ as representing the element $a_1v_1 + \cdots + a_nv_n$ in V.

Problem 2. Recall that $\mathbb{C}$ is a 2-dimensional vector space, and that $\mathcal{E} = (1, i)$ is a basis.

(a) Describe the map $L_{\mathcal{E}}$ explicitly—its source, target and effect on an arbitrary $z = x + iy \in \mathbb{C}$.

(b) If $z, w \in \mathbb{C}$, what is the relationship between $[z + w]_{\mathcal{E}}$ and $[z]_{\mathcal{E}} + [w]_{\mathcal{E}}$? If $c \in \mathbb{R}$, what is the relationship between $[cz]_{\mathcal{E}}$ and $c[z]_{\mathcal{E}}$? How is this expressed using the notation $L_{\mathcal{E}}$?

(c) Prove, using the definition, that $\mu : \mathbb{C} \to \mathbb{C}$ defined by $\mu(z) = iz$ is a linear transformation.

(d) Write a formula for $\mu(x + iy)$. Write a formula for $[\mu(x + iy)]_{\mathcal{E}}$.

(e) Writing elements $x + iy \in \mathbb{C}$ as column vectors $[x + iy]_{\mathcal{E}} = \begin{bmatrix} x \\ y \end{bmatrix}$, can you find a matrix B that describes the map μ? That is, find B such that $[\mu(z)]_{\mathcal{E}} = B[z]_{\mathcal{E}}$ for arbitrary $z = x + iy$. Compare/contrast with the Key Theorem.

Solution: For (a) and (b), $L_{\mathcal{E}} : \mathbb{C} \to \mathbb{R}^2$ is defined by $L_{\mathcal{E}}(x + iy) = \begin{bmatrix} x \\ y \end{bmatrix}$. Easily we see that if $z = x + iy$ and $w = a + bi$, then

$$[(x + iy) + (a + bi)]_{\mathcal{E}} = [(x + a) + (y + bi)]_{\mathcal{E}} = \begin{bmatrix} x + a \\ y + b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = [x + yi]_{\mathcal{E}} + [a + bi]_{\mathcal{E}}.$$

Similarly, $[cz]_{\mathcal{E}} = \begin{bmatrix} cx \\ cy \end{bmatrix} = c\begin{bmatrix} x \\ y \end{bmatrix} = c[z]_{\mathcal{E}}$.

For (c): Take arbitrary $z, w \in \mathbb{C}$. Then $\mu(z + w) = i(z + w) = iz + iw = \mu(z) + \mu(w)$ by the distributive property in $\mathbb{C}$. Also for $c \in \mathbb{R}$, $\mu(cz) = i(cz) = c(iz) = c\mu(z)$ by the commutativity of multiplication in $\mathbb{C}$.

For (d): $\mu(x + iy) = i(x + iy) = -y + xi$, so $[\mu(x + iy)]_{\mathcal{E}} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

For (e): In $\mathcal{E}$-coordinates μ sends $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$. So take $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We have $[\mu(z)]_{\mathcal{E}} = B[z]_{\mathcal{E}}$ for all $z \in \mathbb{C}$. So like in the Key theorem, this map can be represented by matrix multiplication if we work in $\mathcal{E}$-coordinates.

Definition: Let $T : V \to V$ be a linear transformation of a vector space V.
Let $\mathfrak{B} = (v_1, \ldots, v_n)$ be an ordered basis of V.
The **matrix of T with respect to $\mathfrak{B}$** is the $n \times n$ matrix whose j-th column is $[T(v_j)]_{\mathfrak{B}}$.
That is, matrix of T with respect to $\mathfrak{B}$ is

$$\begin{bmatrix}[T(v_1)]_{\mathfrak{B}} & [T(v_2)]_{\mathfrak{B}} & \ldots & [T(v_n)]_{\mathfrak{B}}\end{bmatrix}.$$

For short, we call the $n \times n$ matrix $[T]_{\mathfrak{B}}$ the $\mathfrak{B}$-matrix of T.

Generalized Key Theorem. Let $T : V \to V$ be a linear transformation of a vector space V.
Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis for V. Then there exists a unique matrix B such that for all $v \in V$, we have $B[\vec{v}]_{\mathcal{B}} = [T(v)]_{\mathcal{B}}$.
Explicitly, B is $[T]_{\mathcal{B}}$, the matrix of T with respect to $\mathcal{B}$, whose j-th column is $[T(\vec{v}_j)]_{\mathcal{B}}$.

Problem 3. With notation as in Problem 2, find the matrix of the linear transformation $\mu : \mathbb{C} \to \mathbb{C}$ with respect to the basis $\mathcal{E}$. Verify that the Generalized Key Theorem holds in this case.
[Hint: You did most of the work already in Problem 2!]

Problem 4. Let $\mathcal{S}$ be the basis $\{1, x, x^2, x^3\}$ of the vector space $\mathcal{P}_3$.

(a) Prove that the map $D : \mathcal{P}_3 \to \mathcal{P}_3$ sending $f \mapsto f - f'$ is a linear transformation.

(b) For arbitrary $p = a + bx + cx^2 + dx^3$, find $D(p)$.

(c) For arbitrary $p = a + bx + cx^2 + dx^3$, compute $[p]_{\mathcal{S}}$ and $[D(p)]_{\mathcal{S}}$.

(d) Find the matrix $[D]_{\mathcal{S}}$ of D with respect to the basis $\mathcal{S}$, that is, the $\mathcal{S}$-matrix of D.

(e) Verify the Generalized Key Theorem for D by comparing the matrix product $[D]_{\mathcal{S}}[p]_{\mathcal{S}}$ to the column vector $[D(p)]_{\mathcal{S}}$.

Solution:

(a) For arbitrary $f, g \in \mathcal{P}_4$, $D(f+g) = (f+g) - (f+g)' = (f - f') + (g - g') = D(f) + D(g)$. Also $D(cf) = (cf) - (cf)' = cf - cf' = c(f - f') = cD(f)$ for any scalar c.

(b) $D(p) = (a - b) + (b - 2c)x + (c - 3d)x^2 + dx^3$.

(c) $[p]_{\mathcal{S}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. So $[D(p)]_{\mathcal{S}} = \begin{bmatrix} a-b \\ b-2c \\ c-3d \\ d \end{bmatrix}$.

(d) $[D]_{\mathcal{S}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $[D]_{\mathcal{S}}[p]_{\mathcal{S}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ b-2c \\ c-3d \\ d \end{bmatrix}$. This is the same as $[D(p)]_{\mathcal{S}}$!

Problem 5. Standard Coordinates. Let $\mathcal{E} = (\vec{e}_1, \ldots, \vec{e}_n)$ be the **standard basis** for $\mathbb{R}^n$.

(a) For arbitrary $\vec{v} \in \mathbb{R}^n$, explain why $[\vec{v}]_\mathcal{E} = \vec{v}$.

(b) Find the matrix $[T]_\mathcal{E}$ of T with respect to $\mathcal{E}$, when T is the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{defined by} \qquad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 2x + 3y \end{bmatrix}.$$

How does $[T]_\mathcal{E}$ compare to the matrix guaranteed by the Key Theorem in this case?

(c) Let A be any $n \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T(\vec{x}) = A\vec{x}$. Prove that $[T]_\mathcal{E} = A$. That is, show that for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, the matrix $[T]_\mathcal{E}$ of T with respect to the *standard basis* $\mathcal{E}$ is just **standard matrix** of T.

(d) Now, carefully unravel the notation above to deduce the following:

Fact: For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, when we work with the *standard basis*, the Generalized Key Theorem specializes to the Key theorem.

Solution: For (a), write $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and observe this is just the same as $\vec{v} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$. By definition, $[\vec{v}]_\mathcal{E} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ which gives back $\vec{v}$.

For (b), the definition of $[T]_\mathcal{E}$ says that the first column is $[T(\vec{e}_1)]_\mathcal{E}$, which is just $T(\vec{e}_1)$ by (a), and likewise the second column is $[T(\vec{e}_2)]_\mathcal{E}$, which is just $T(\vec{e}_2)$. This is exactly how we compute the matrix A of the key theorem! We see where the standard unit column vectors go, and make them the columns of A. So $[T]_\mathcal{E} = A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

For (c), this all follows from (a). We don't need to write the subscript $\mathcal{E}$ on the vectors $\vec{v} \in \mathbb{R}^n$ since the $\mathcal{E}$-coordinates are just the usual (standard) coordinates. Also, the definition of $[T]_\mathcal{E}$ in this case is what we've always done to find the matrix of a transformation: plug each $\vec{e}_j$ into T to find the j-th column!

Problem 6. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Given $P \in \mathbb{R}^{2\times 2}$, define the function $T_P : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T_P(A) = PA$ for all $A \in \mathbb{R}^{2\times 2}$.

(a) Is T_P always linear? If so, is T_P ever an isomorphism?

Solution: Yes, T_P is linear for every P: for every pair $A, B \in \mathbb{R}^{2\times 2}$, $T_P(A + B) = P(A+B) = PA+PB = T_P(A)+T_P(B)$ (by the distributive law of matrix multiplication); and $T_P(kA) = P(kA) = P(kI_2)A = (kI_2)PA = kT_P(A)$ for all $k \in \mathbb{R}$. Furthermore, T_P is an isomorphism if and only if P is invertible. Indeed, if P has inverse P^{-1}, then $T_P \circ T_{P^{-1}} = T_{P^{-1}} \circ T_P$ is the identity map, so T_P is an isomorphism. Conversely, if P is not invertible, then (because it is square) its kernel in non-zero. Say $\begin{bmatrix} a \\ b \end{bmatrix} \in \ker P$. Then $\begin{bmatrix} a & a \\ b & b \end{bmatrix} \in \ker T_P$, so T_P is not an isomorphism.

(b) Find $[T_P]_{\mathcal{E}}$ if $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

Solution: $[T_P]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$.

(c) Is T invertible for P as in (b). What about the matrix $[T]_{\mathcal{E}}$? Is the invertibility of T related to the invertibility of $[T]_{\mathcal{B}}$ in general?

Solution: Yes, the matrix P is invertible, so T is invertible. Row reducing $[T]_{\mathcal{E}}$, we see that so is $[T]_{\mathcal{E}}$. In general T is invertible if and only $[T]_{\mathcal{B}}$ is. Because the source and target have the same dimension, invertibility of T and of $[T]_{\mathcal{B}}$ is equivalent to having a non-zero kernel. Since $[T]_{\mathcal{B}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{B}}$, we see that $v \in \ker T$ if and only $[v]_{\mathcal{B}} \in \ker[T]_{\mathcal{B}}$.

Problem 7. Let V be a vector space with ordered basis (v_1, v_2, v_3). Let $T : V \to V$ be a linear transformation such that $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find a vector $v \in V$ in the kernel of T. Find a vector $v \in V$ that is *fixed* by T (meaning $T(v) = v$). Find a vector v satisfying $T(v) = 2v$. Find a basis for the image of T. [Hint: If you are stuck, go back and unravel the definition of $[T]_{\mathcal{B}}$ in this case.]

Solution: $v_3 \in \ker T$, v_1 is fixed by T and v_2 is sent to $2v_2$. A basis for the image is $\{v_1, v_2\}$.

Problem 8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be counterclockwise rotation about the z-axis through an angle θ. Find $[T]_{\mathcal{E}}$ where $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Solution: $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 9. Reflections, rotations and projections in $\mathbb{R}^3$. Let $V \subseteq \mathbb{R}^3$ be the plane with equation $x + y + z = 0$. Note that $(\vec{v}_1, \vec{v}_2) = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$ spans V and the vector $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is normal to V, so that $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for $\mathbb{R}^3$. Find the $\mathcal{B}$-matrix of the following transformations $\mathbb{R}^3 \to \mathbb{R}^3$:

(i) reflection over V;

(ii) projection onto V;

(iii*) rotation by $2\pi/3$ around the normal line $\mathrm{Span}\{\vec{v}_3\}$. [Note: There are two answers, depending on the direction of rotation.]

Why is using the basis $\mathcal{B}$ to model these transformations is much easier than using the standard basis? If we replace V be some other plane W through the origin in $\mathbb{R}^3$, how should one choose a convenient basis to model reflections, rotations, and projections involving W?

Solution: For (i): $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. For (ii): $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For (iii), observe that the angle θ between $\vec{v}_1$ and $\vec{v}_2$ is $\frac{2\pi}{3}$, since $\vec{v}_1 \cdot \vec{v}_2 = ||\vec{v}_1||\,||\vec{v}_2|| \cos\theta$ gives $\cos\theta = \frac{-1}{2}$. So a little trig in the plane V spanned by the vectors $\vec{v}_1$ and $\vec{v}_2$ shows that the rotation T will take $\vec{v}_1$ to $\vec{v}_2$, and $\vec{v}_2$ to $-\vec{v}_1 - \vec{v}_2$. So $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, or its inverse if we rotate the other way.

Since all these transformation fix V set-wise, it is much easier to think about where the elements of V get sent by the transformation. Furthermore, the matrices modelling these transformations are *sparse*, meaning they have lots of zeros; this makes them easier for humans and faster for computers to compute with. The standard vectors are not so easy to follow in this case, unless V is one of the coordinate planes (such as the xy plane). In general, choosing a basis so that two of the vectors are W and the third is normal to W will produce a similarly convenient model.

Problem 10. Let ℓ be the line in $\mathbb{R}^2$ whose equation is $y = \frac{1}{2}x$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of $\mathbb{R}^2$ over the line ℓ.

(a) Find the matrix $[T]_{\mathcal{E}}$ where $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$. [Hint: Recall that on Worksheet 5, we found that the standard matrix of reflection over a line through the origin in the direction of unit vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is $\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$.]

(b) Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the matrix of T relative to $\mathcal{B}$ is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution: For (a), plugging in the unit vector $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (lying on the line $y = x/2$) into the formula provided in the hint, we get the standard matrix is $\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$.

For (b), use $\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\}$. There are other answers: we can replace either basis element by any scalar multiple. The point is, the first one should be fixed by T (so it should be in the direction of ℓ and the second should be sent to its negative, so it should be perpendicular to ℓ.

Problem 11: Rotations in $\mathbb{R}^2$. For each angle θ, let R_θ be the counterclockwise rotation of $\mathbb{R}^2$ through an angle θ. Let $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ be the standard basis of $\mathbb{R}^2$.

(a) Remind yourself what the standard matrix $[R_\theta]_{\mathcal{E}}$ of R_θ is.

(b) Let $\mathcal{B} = (2\vec{e}_1, 2\vec{e}_2)$. Find $[R_\theta]_{\mathcal{B}}$.

(c) Let $\mathcal{C} = (2\vec{e}_1, 3\vec{e}_2)$. Find $[R_\theta]_{\mathcal{C}}$.

(d) Let $\mathcal{D} = (\vec{e}_1, \vec{e}_1 + \vec{e}_2)$. Find $[R_\theta]_{\mathcal{D}}$.

(e) Can you find all ordered bases $\mathcal{U} = (\vec{u}, \vec{v})$ such that $[R_\theta]_\mathcal{U} = [R_\theta]_\mathcal{E}$ for each θ?

Solution:

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(c) $\begin{bmatrix} \cos\theta & -\frac{3}{2}\sin\theta \\ \frac{2}{3}\sin\theta & \cos\theta \end{bmatrix}$.

(d) $\begin{bmatrix} \cos\theta - \sin\theta & -2\sin\theta \\ \sin\theta & \sin\theta + \cos\theta \end{bmatrix}$.

(e) All bases of the form $(\vec{u}, \vec{v}) = \left(\begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix}, \begin{bmatrix} -a\sin\theta \\ a\cos\theta \end{bmatrix} \right)$ where $0 \neq a \in \mathbb{R}$ and $\theta \in \mathbb{R}$.