Worksheet 15: Comparing Coordinate Systems for different Bases (§3.4, 4.3)

Let V be a d-dimensional vector space, with two bases $\mathcal{B} = (b_1, \ldots, b_d)$ and $\mathcal{A} = (a_1, \ldots, a_d)$.

Definition: The **Change of Basis** matrix from $\mathcal{B}$ to $\mathcal{A}$ is the $d \times d$ matrix

$$S_{\mathcal{B}\to\mathcal{A}} = [[b_1]_{\mathcal{A}} \quad [b_2]_{\mathcal{A}} \quad \cdots \quad [b_d]_{\mathcal{A}}].$$

Change of Basis Theorem for Coordinates: For all $v \in V$, $S_{\mathcal{B}\to\mathcal{A}}[v]_{\mathcal{B}} = [v]_{\mathcal{A}}$.

The theorem says that the matrix $S_{\mathcal{B}\to\mathcal{A}}$ transforms $\mathcal{B}$-coordinate vectors into $\mathcal{A}$-coordinate vectors.

Problem 1: Two Bases of $\mathbb{R}^2$. Let $\vec{b}_1 = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix}5\\2\end{bmatrix}$. Consider the ordered bases $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$ of $\mathbb{R}^2$.

(a) Find the following coordinate vectors:

(i) $[\vec{b}_1]_{\mathcal{B}}$ (iii) $[\vec{e}_1]_{\mathcal{B}}$ (v) $[\vec{e}_1]_{\mathcal{E}}$ (vii) $[\vec{b}_1]_{\mathcal{E}}$ (ix) $[2\vec{b}_1 - 1\vec{b}_2]_{\mathcal{B}}$

(ii) $[\vec{b}_2]_{\mathcal{B}}$ (iv) $[\vec{e}_2]_{\mathcal{B}}$ (vi) $[\vec{e}_2]_{\mathcal{E}}$ (viii) $[\vec{b}_2]_{\mathcal{E}}$ (x) $[2\vec{e}_1 - 1\vec{e}_2]_{\mathcal{B}}$

(b) Find the change of basis matrices $S_{\mathcal{B}\to\mathcal{E}}$ and $S_{\mathcal{E}\to\mathcal{B}}$. One is easier than the other—why?
[Hint: For the harder one, you already did most of the work in (a)!]

(c) Verify the Change of Basis Theorem for Coordinates by checking the following:

(i) $S_{\mathcal{B}\to\mathcal{E}}[\vec{b}_1]_{\mathcal{B}} = [\vec{b}_1]_{\mathcal{E}}$ (iii) $S_{\mathcal{E}\to\mathcal{B}}[\vec{b}_1]_{\mathcal{E}} = [\vec{b}_1]_{\mathcal{B}}$

(ii) $S_{\mathcal{B}\to\mathcal{E}}[\vec{b}_2]_{\mathcal{B}} = [\vec{b}_2]_{\mathcal{E}}$. (iv) $S_{\mathcal{E}\to\mathcal{B}}[\vec{b}_2]_{\mathcal{E}} = [\vec{b}_2]_{\mathcal{B}}$.

(d) Verify, by multiplying matrices, that $S_{\mathcal{E}\to\mathcal{B}}$ and $S_{\mathcal{B}\to\mathcal{E}}$ are inverse matrices.

(e) Now give another reason why $S^{-1}_{\mathcal{E}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{E}}$, using the Change of Basis theorem above.

Solution: For (a):

(i) $\begin{bmatrix}1\\0\end{bmatrix}$ (iii) $\begin{bmatrix}2\\-1\end{bmatrix}$ (v) $\begin{bmatrix}1\\0\end{bmatrix}$ (vii) $\begin{bmatrix}3\\1\end{bmatrix}$ (ix) $\begin{bmatrix}2\\-1\end{bmatrix}$

(ii) $\begin{bmatrix}0\\1\end{bmatrix}$ (iv) $\begin{bmatrix}-5\\3\end{bmatrix}$ (vi) $\begin{bmatrix}0\\1\end{bmatrix}$ (viii) $\begin{bmatrix}5\\2\end{bmatrix}$ (x) $\begin{bmatrix}9\\-5\end{bmatrix}$

For (b), $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}$ and $S_{\mathcal{E}\to\mathcal{B}} = \begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}$. For (e), note that by the **change of basis theorem** (above), $S_{\mathcal{B}\to\mathcal{E}}\,(S_{\mathcal{E}\to\mathcal{B}}[\vec{v}]_{\mathcal{E}}) = S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{E}}$ for all $\vec{v} \in V$. This says that the composition $S_{\mathcal{B}\to\mathcal{E}} \circ S_{\mathcal{E}\to\mathcal{B}}$ is the identity on $\mathbb{R}^2$, so the matrix product is $S_{\mathcal{B}\to\mathcal{E}}S_{\mathcal{E}\to\mathcal{B}} = I_2$. Since both are 2×2, this implies the matrices are inverses of eachother.

Problem 2. Prove the following corollary of the Change of Basis theorem for Coordinates:

Corollary: Let V be a vector space, with two bases $\mathcal{B} = (b_1, \dots, b_d)$ and $\mathcal{A} = (a_1, \dots, a_d)$. Then the change of basis matrices $S_{\mathcal{A}\to\mathcal{B}}$ and $S_{\mathcal{B}\to\mathcal{A}}$ are inverse to each other.

Solution: For all $v \in V$, we have by the **change of basis theorem for coordinates**, $S_{\mathcal{B}\to\mathcal{A}}(S_{\mathcal{A}\to\mathcal{B}}[\vec{v}]_{\mathcal{A}}) = S_{\mathcal{B}\to\mathcal{A}}[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{A}}$. This says that the composition $S_{\mathcal{B}\to\mathcal{A}} \circ S_{\mathcal{A}\to\mathcal{B}}$ is the identity on $\mathbb{R}^d$, so the matrix product is $S_{\mathcal{B}\to\mathcal{A}}S_{\mathcal{A}\to\mathcal{B}} = I_d$. Since both matrices are $d \times d$, this implies they are inverses of eachother.

Problem 3. Comparing Matrices of a transformation in different coordinate systems. Consider the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 8x - 15y \\ 2x - 3y \end{bmatrix}$.

(a) Find $[T]_{\mathcal{E}}$ and $[T]_{\mathcal{B}}$ where $\mathcal{E}$ and $\mathcal{B}$ are the bases in Problem 1.

(b) By looking at the $\mathcal{B}$-matrix for T, describe what T is doing geometrically.

(c) Verify that $[T]_{\mathcal{E}} = S_{\mathcal{B}\to\mathcal{E}}[T]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}$ by multiplying the matrices.

(d) Draw a diagram illustrating how the matrix product $S_{\mathcal{B}\to\mathcal{E}}[T]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}$ represents the composition of three transformations. Your diagram should have four copies of $\mathbb{R}^2$ and three labelled arrows.

(e) The source and target of each map in (d) is $\mathbb{R}^2$, but it is good to think of each $\mathbb{R}^2$ more concretely as either an $\mathcal{E}$-coordinate space or a $\mathcal{B}$-coordinate space. Explain.

(f) Give a different proof that $[T]_{\mathcal{E}} = S_{\mathcal{B}\to\mathcal{E}}[T]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}$ by building on your answers to (d) and (e).

Solution:

(a) $[T]_{\mathcal{E}} = \begin{bmatrix} 8 & -15 \\ 2 & -3 \end{bmatrix}$ and $[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) T stretches by 3 in the $\vec{b}_1$ direction and by 2 in the $\vec{b}_2$ direction.

(c) $[T]_{\mathcal{E}} = \begin{bmatrix} 8 & -15 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = S_{\mathcal{B}\to\mathcal{E}}[T]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}$.

(d) $\mathbb{R}^2 \xrightarrow{S_{\mathcal{E}\to\mathcal{B}}} \mathbb{R}^2 \xrightarrow{[T]_{\mathcal{B}}} \mathbb{R}^2 \xrightarrow{S_{\mathcal{B}\to\mathcal{E}}} \mathbb{R}^2$.

(e) The first and last copy of $\mathbb{R}^2$ are $\mathcal{E}$-coordinates, and the middle two copies are $\mathcal{B}$-coordinates.

(f) The composition map $S_{\mathcal{B}\to\mathcal{E}}[T]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}$ does the following to an arbitrary element of $\mathbb{R}^2$:

$$[v]_{\mathcal{E}} \mapsto [v]_{\mathcal{B}} \mapsto [T(v)]_{\mathcal{B}} \mapsto [T(v)]_{\mathcal{E}}$$

which is exactly describing $[T]_{\mathcal{E}}$.

Problem 4. Discuss and illustrate the following theorem by drawing a diagram of maps. You diagram should have six vector spaces, two of which are V and four of which are $\mathbb{R}^n$. Think carefully about *what* coordinate space each copy of $\mathbb{R}^n$ is. You should have 9 arrows, with labels including T, $L_{\mathcal{A}}, L_{\mathcal{B}}, [T]_{\mathcal{A}}, [T]_{\mathcal{B}}, S_{\mathcal{A}\to\mathcal{B}}$ (or the inverses of these). [Caution: You will need plenty of space!]

Change of Basis Theorem for Transformations:
Let V be a n-dimensional vector space, with bases $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$.
Let $T : V \to V$ be a linear transformation. Then the matrices of T with respect to $\mathcal{A}$ and $\mathcal{B}$ are related as follows:

$$[T]_{\mathcal{A}} = S^{-1}[T]_{\mathcal{B}}S$$

where S is the change of basis matrix $S_{\mathcal{A}\to\mathcal{B}}$.

Solution: *Diagram courtesy of Matthew Anderson, Math 217 student, F22.*

$\mathcal{A}$-coordinates

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "{[T]_{\mathcal{A}}}"] \arrow[dd, "S_{\mathcal{A}\to\mathcal{B}}"'] & & & \mathbb{R}^n \\
& V \arrow[ul, "L_{\mathcal{A}}"'] \arrow[dl, "L_{\mathcal{B}}"] \arrow[r, "T"] & V \arrow[ur, "L_{\mathcal{A}}"] \arrow[dr, "L_{\mathcal{B}}"'] & \\
\mathbb{R}^n \arrow[rrr, "{[T]_{\mathcal{B}}}"] & & & \mathbb{R}^n \arrow[uu, "S_{\mathcal{B}\to\mathcal{A}}"']
\end{tikzcd}$$

$\mathcal{B}$-coordinates

Here, the two diagonal arrows pointing upwardward convert transformation $T : V \to V$ into $\mathcal{A}$-coordinates—the top horizontal arrow is the $\mathcal{A}$-coordinate "model" of T. Likewise, the two diagonal arrows pointing downward convert transformation T into $\mathcal{B}$-coordinates. The vertical arrows go between $\mathcal{A}$-coordinates and $\mathcal{B}$-coordinates.

Problem 5. Let $\mathcal{S} = (1, x, x^2)$ and $\mathcal{A} = (x^2, x, x - 1)$ be two bases for $\mathcal{P}_2$.

(a) Find $[g]_{\mathcal{S}}$ and $[g]_{\mathcal{A}}$ for arbitrary $g = a + bx + cx^2$.

(b) Using the definition, find the **change of basis matrices** $S_{\mathcal{A}\to\mathcal{S}}$ and $S_{\mathcal{S}\to\mathcal{A}}$. How are they related?

(c) Verify the **Change of Basis Theorem for coordinates** in this case by checking that, for arbitrary $g \in \mathcal{P}_2$, $\quad S_{\mathcal{A}\to\mathcal{S}}[g]_{\mathcal{A}} = [g]_{\mathcal{S}}$, by multiplying out the matrices you found in (a) & (b).

(d) Compute the product $S_{\mathcal{S}\to\mathcal{A}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ another way, without multiplying the matrices.

Solution:

(a) Find $[g]_{\mathcal{S}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $[g]_{\mathcal{A}} = \begin{bmatrix} c \\ b + a \\ -a \end{bmatrix}$.

(b) $S_{\mathcal{A}\to\mathcal{S}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $S_{\mathcal{S}\to\mathcal{A}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ They are inverses.

(c) $S_{\mathcal{A}\to\mathcal{S}}[g]_{\mathcal{A}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b+a \\ -a \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(d) If we view $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as the $\mathcal{S}$-coordinates of the polynomial $-1 + x + x^2$, then $S_{\mathcal{S}\to\mathcal{A}}$ will transform $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ into the $\mathcal{A}$ coordinate vector of $-1 + x + x^2$, which is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 6. Consider the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(g) = g(1) + g'$. Let $\mathcal{S}$ and $\mathcal{A}$ be the two bases for $\mathcal{P}_2$ from Problem 5.

(a) Find the matrices $[T]_{\mathcal{S}}$ and $[T]_{\mathcal{A}}$ of T with respect to the basis $\mathcal{S}$ and $\mathcal{A}$, respectively.
[Caution: Remember to rewrite each in the correct coordinates after applying T.]

(b) Explain how the linear transformation $\mathbb{R}^3 \xrightarrow{[T]_{\mathcal{S}}} \mathbb{R}^3$ models T. For example, how should we think of the column vectors in $\mathbb{R}^3$ here? How can we use $[T]_{\mathcal{S}}$ to compute $T(x^2 + 5x + 7)$?

(c) For arbitrary $g = a + bx + cx^2$, compute $T(g)$, $[T(g)]_{\mathcal{S}}$, and $[T(g)]_{\mathcal{A}}$.

(d) Consider the composition of linear transformations:

$$\mathbb{R}^3 \xrightarrow{S_{\mathcal{A}\to\mathcal{S}}} \mathbb{R}^3 \xrightarrow{[T]_{\mathcal{S}}} \mathbb{R}^3 \xrightarrow{S_{\mathcal{S}\to\mathcal{A}}} \mathbb{R}^3.$$

Taking a clue from the names above each arrow, we see that for the first map, the source is the $\mathcal{A}$-coordinate space, while the target is the $\mathcal{S}$-coordinate space. Explain. What are the sources and targets of the other arrows? Make sure your answer comports with the fact that in order for a composition to be defined, the target of the incoming map must be the same as the outgoing map.

(e) Show that for arbitrary $g \in \mathcal{P}_2$, the composition in (d) sends $[g]_{\mathcal{A}}$ to $[T(g)]_{\mathcal{A}}$. Use the generalized Key Theorem, then, to deduce that the matrix of the composition is $[T]_{\mathcal{A}}$.

(f) Verify the matrix equation $[T]_{\mathcal{A}} = S_{\mathcal{S}\to\mathcal{A}}[T]_{\mathcal{S}}S_{\mathcal{A}\to\mathcal{S}}$ in this case.

Solution:

(a) $[T]_{\mathcal{S}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $[T]_{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$

(b) To compute $T(x^2+5x+7)$ using $\mathcal{S}$-coordinates, convert the polynomial to $\begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}$ and multiply by the $\mathcal{S}$-matrix. The result is $[T]_{\mathcal{S}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 2 \\ 0 \end{bmatrix}$ which is the $\mathcal{S}$-coordinate vector of $T(x^2 + 5x + 7)$. So $T(x^2 + 5x + 7) = 18 + 2x$.

(c) $T(g) = (a+2b+c) + 2c\,x$. $[T(g)]_{\mathcal{S}} = \begin{bmatrix} a+2b+c \\ 2c \\ 0 \end{bmatrix}$. $[T(g)]_{\mathcal{A}} = \begin{bmatrix} 0 \\ a+2b+3c \\ -(a+2b+c) \end{bmatrix}$.

(d) The first and last copy of $\mathbb{R}^3$ are $\mathcal{A}$-coordinate spaces for $\mathcal{P}_2$. The middle copies are of $\mathbb{R}^3$ are $\mathcal{S}$-coordinate spaces for $\mathcal{P}_2$.

(e) For $g = a + bx + cx^2$, we follow the path of $[g]_{\mathcal{A}} = \begin{bmatrix} c \\ b+a \\ -a \end{bmatrix}$ through the compositions:

$$\begin{bmatrix} c \\ b+a \\ -a \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b+c \\ 2c \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a+2b+c \\ 2c \\ 0 \end{bmatrix}$$

which is $\begin{bmatrix} 0 \\ a+2b+3c \\ -a-2b-c \end{bmatrix}$. This recovers $[T(g)]_{\mathcal{A}}$.

(f) This is just a matter of multiplying out the matrices $S_{\mathcal{S}\to\mathcal{A}}[T]_{\mathcal{S}}S_{\mathcal{A}\to\mathcal{S}}$ and checking that we get $[T]_{\mathcal{B}}$. Check: $S_{\mathcal{S}\to\mathcal{A}}[T]_{\mathcal{S}}S_{\mathcal{A}\to\mathcal{S}} =$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}.$$

It works! This is $[T]_{\mathcal{A}}$.

Definition: Two square matrices $A, B \in \mathbb{R}^{n\times n}$ are **similar** if there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $A = S^{-1}BS$.

Problem 7. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space V. Prove that, if $\mathcal{A}$ and $\mathcal{A}'$ are two bases for V, then $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{A}'}$ are *similar* matrices. CHALLENGE: If B is some third matrix similar to $[T]_{\mathcal{A}}$, must it be the $\mathcal{B}$-matrix of T for some for basis $\mathcal{B}$ of V?

Solution: This follows immediately from the change of basis theorem for transformations. For the second question, let $\mathcal{A} = (a_1, \ldots, a_n)$ and say $B = S^{-1}[T]_{\mathcal{A}}S$. Let $\begin{bmatrix} s_{1j} \\ \vdots \\ s_{nj} \end{bmatrix}$ be the j-th column of S. Set $b_j = \sum_{i=1}^n s_{ij}a_i$. Then, setting $\mathcal{B} = (b_1, \ldots, b_n)$, we claim that B is the $\mathcal{B}$-matrix of T. Indeed, note that $\mathcal{B}$ is a basis and S is the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$, so $[T]_{\mathcal{B}} = S^{-1}[T]_{\mathcal{A}}S$ by the change of basis theorem for transformations.

Problem 8. Proof of the Change of Basis Theorem for Coordinates. Fix a vector space V with two ordered bases $\mathcal{B} = (b_1, \ldots, b_d)$ and $\mathcal{A} = (a_1, \ldots, a_d)$. We thus have two *different* coordinatizations of V. These are the coordinate isomorphisms

$$V \xrightarrow{L_{\mathcal{B}}} \mathbb{R}^d \qquad \text{and} \qquad V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^d.$$

(a) Consider the natural **change of coordinates map**

$$\mathbb{R}^d \longrightarrow \mathbb{R}^d \qquad [\vec{v}]_{\mathcal{B}} \mapsto [\vec{v}]_{\mathcal{A}}.$$

Discuss this map with your group. Explain why we should think of the source, not just as $\mathbb{R}^d$, but as *the* $\mathcal{B}$*-coordinate space of* V. What is the target? Write this map as a composition, using the notation for the coordinate isomorphisms. Is it bijective? Linear ? An isomorphism?

(b) The Key Theorem tells us that there is a matrix A (the standard matrix) such that the change of coordinates map in (a) is given by multiplication by A. Find this matrix. Compare to $S_{\mathcal{B}\to\mathcal{A}}$. [HINT: Remember that the source is the $\mathcal{B}$-coordinate space of V. What vector in V corresponds to $\vec{e}_j$?]

(c) Prove the Change of Basis Theorem for Coordinates.

Solution:

(a) The map is the composition $\mathbb{R}^d \xrightarrow{L_{\mathcal{B}}^{-1}} V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^d$. Since a composition of isomorphisms is an isomorphism, this change of coordinates map is an isomorphism.

(b) To find the matrix, we do it column by column. The standard column $\vec{e}_j$, when viewed in the source, is $\vec{e}_j = [b_j]_{\mathcal{B}}$. This is taken to $[b_j]_{\mathcal{A}}$ under the map. So the matrix given by the Key Theorem is $[[b_1]_{\mathcal{A}} \quad \cdots \quad [b_d]_{\mathcal{A}}]$. This is the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$.

(c) Now the change of basis theorem for coordinates follows immediately from the Key Theorem! Since the standard matrix of a linear map between coordinates is unique, and $[[b_1]_{\mathcal{A}} \quad \cdots \quad [b_d]_{\mathcal{A}}]$ does the job, we are done!

Problem 9. Let $V \subseteq \mathbb{R}^3$ be the plane defined by $x+y+z=0$, with bases $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$ and $\mathcal{A} = (\vec{a}_1, \vec{a}_2)$ where $\vec{b}_1 = \vec{a}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$.

(a) Discuss and try to visualize the coordinate system on V determined by $\mathcal{B}$. Which points in V have $\mathcal{B}$-coordinates $\vec{e}_1$ and $\vec{e}_2$? Where are the points in V whose $\mathcal{B}$-coordinates are of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$? Can you visualize the set of all *integer* linear combinations of your basis elements?

(b) Find the following coordinate vectors:

(i) $[\vec{b}_1]_{\mathcal{B}}$ (iii) $[\vec{a}_1]_{\mathcal{B}}$ (v) $[\vec{a}_2]_{\mathcal{A}}$ (vii) $[\vec{0}]_{\mathcal{B}}$ (ix) $\begin{bmatrix} 10 \\ -5 \\ -5 \end{bmatrix}_{\mathcal{A}}$

(ii) $[\vec{b}_2]_{\mathcal{B}}$ (iv) $[\vec{a}_1]_{\mathcal{A}}$ (vi) $[\vec{b}_2]_{\mathcal{A}}$ (viii) $[\vec{0}]_{\mathcal{A}}$

(c) Find one of the change of basis matrices $S_{\mathcal{B}\to\mathcal{A}}$ or $S_{\mathcal{A}\to\mathcal{B}}$ using the definition; note that one may be easier than the other. Find the other by computing the inverse.

(d) Does V have a "standard basis"?

Solution:

(a) You should see a grid imposed on the plane V. The points in V that have $\mathcal{B}$-coordinates $\vec{e}_1$ and $\vec{e}_2$ are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The points in V whose $\mathfrak{B}$-coordinates are of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$ are the lines spanned by these two vectors—these are the axes of your grid. The integer points form a grid of evenly spaced dots.

(b) (i) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (vi) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (viii) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (ix) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

(c) $S_{\mathcal{A}\to\mathcal{B}} = [[\vec{a}_1]_{\mathcal{B}} \ [\vec{a}_2]_{\mathcal{B}}] = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ and $S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$, its inverse.

(d) No standard basis! we have to pick some coordinates and there are many nice choices!