MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 3, SOLUTIONS

Part A (10 points)

Solve the following problems from the book:
Section 2.2: 20, 38;
Section 2.3: 18, 34;
Section 2.4: 12, 34.

Solution.

2.2.20

Denote by T this transformation and

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

then

$$T(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad T(\vec{e}_2) = \begin{bmatrix}0\\-1\\0\end{bmatrix}, \qquad T(\vec{e}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

By Theorem 2.1.2, the matrix of T is $\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}$.

2.2.38

a. Orthogonal projection: $\det\begin{bmatrix}\cos(\theta)^2 & \cos(\theta)\sin(\theta)\\ \cos(\theta)\sin(\theta) & \sin(\theta)^2\end{bmatrix} = 0$.

b. Reflection: $\det\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix} = -1$.

c. Rotation: $\det\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} = 1$.

d. Shear: $\det\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix} = \det\begin{bmatrix}1 & 0\\ k & 1\end{bmatrix} = 1$.

2.3.18

If $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ commutes with $\begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix}$, then

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Rightarrow \begin{cases}2a-3b = 2a+3c\\ 3a+2b = 2b+3d\\ 2c-3d = -3a+2c\\ 3c+2d = -3b+2d\end{cases} \quad \Rightarrow \begin{cases}a = s\\ b = t\\ c = -t\\ d = s,\end{cases}$$

hence the set of all matrices which commute with A is equal to $\left\{\begin{bmatrix}s & t\\ -t & s\end{bmatrix} \,\middle|\, s, t \in \mathbb{R}\right\}$.

2.3.34:
$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.
By induction, $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$.

2.4.12

$$\left[\begin{array}{cccc|cccc} 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow []\text{RREF} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right],$$

Then the matrix is invertible with the inverse $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

2.4.34

(a) A is invertible if $a \neq 0, b \neq 0, c \neq 0$ and $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

(b) If any of the diagonal elements are zero, then the diagonal matrix is not invertible. Then a diagonal matrix is invertible if all the diagonal entries ara non zero.

Part B (25 points)

The definitions of *trace*, *determinant* and *transpose* will be needed in this part.

Definition 1. Given a square $n \times n$ matrix $C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$, we define the **trace** of C to be the sum of the diagonal elements $c_{11} + \cdots + c_{nn} = \sum_{i=1}^{n} c_{ii}$, denoted $\mathrm{tr}(C)$.

Definition 2. The **determinant** of a square matrix C will be denoted $\det(C)$. We define the determinant of a 1×1 matrix by $\det[a] = a$, and the determinant of a 2×2 matrix by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. (We will wait until Chapter 6 to define determinants of larger square matrices).

Definition 3. Consider an $m \times n$ matrix A. The **transpose** $A^\top$ of A is the $n \times m$ matrix obtained from A by rewriting all of the columns of A as rows, and vice versa, so that the (i, j)-entry of $A^\top$ is the (j, i)-entry of A. Further, we say that the square matrix A is **symmetric** if $A^\top = A$.

Problem 1. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) For all 2×2 matrices A and B, $(AB)^\top = A^\top B^\top$.
(b) For all 2×2 matrices A and B, $(AB)^\top \neq A^\top B^\top$.

(c) For all matrices A and B such that the matrix product AB exists, $(AB)^\top = B^\top A^\top$.
(d) If A is a symmetric matrix, then for all $n \in \mathbb{N}$, A^n is also symmetric.
(e) If A is a square matrix and A^2 is symmetric, then so is A.

Solution.

(a) Not true. The following pair for example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

(Student solutions should include explicitly working out the products $(AB)^\top$ and $A^\top B^\top$ and verifying that they are unequal.)

(b) False. Any pair of diagonal matrices is a counterexample.

(c) True. $(AB)_{i,j} = \vec{a_i}\vec{b_j}$ where $\vec{a_i}$ is the i-th row of A and $\vec{b_j}$ is the j-th column of B. Notice that i-th row of $A^\top$ is the i-th column of A.

$$(AB)^\top_{i,j} = (AB)_{j,i} = \vec{a_j}\vec{b_i} = \vec{b_i}\vec{a_j}$$
$$(B^\top A^\top)_{i,j} = \vec{b_i}\vec{a_j}$$

Then $(AB)^\top_{i,j} = (B^\top A^\top)_{i,j}$ for each (i,j)-entry. Therefore, $(AB)^\top = B^\top A^\top$.

(d) True. A is symmetric if $A = A^\top$. The statement is true for $n = 1$.

Assume that A^n is symmetric for some $n \geq 1$. Then $A^{n+1} = A^n A$. Then

$$\begin{aligned}(A^{n+1})^\top &= (A^n A)^\top \\ &= A^\top (A^n)^\top \text{ (by (a))} \\ &= AA^n \text{ (by induction hypothesis and the assumption that } A \text{ is symmetric)} \\ &= A^{n+1}.\end{aligned}$$

Then A^{n+1} is symmetric. Therefore, A^n is symmetric for all $n \in \mathbb{N}$.

(e) False. Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a nonsymmetric matrix, but A^2 is the zero matrix and hence symmetric.

Problem 2. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) Every 3-by-3 matrix that has a row of zeros is not invertible.
(b) Every square matrix with 1's down the main diagonal is invertible.
(c) For any matrix A, if A is invertible, then so is A^{-1}.
(d) For any matrix A, if A is invertible, then A^n is invertible.

Solution.

(a) True. If it has a row of zeros, then rank is less than 3. Therefore, it is not invertible.

(b) False. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

(c) True, A^{-1} is invertible and its inverse matrix is A. This follows from the fact that $AA^{-1} = A^{-1}A = I_n$.

(d) True. In fact the inverse of A^n is $\left(A^{-1}\right)^n$.

Problem 3. Let A be an $m \times n$ matrix.

(a) Prove that if there exists an $n \times m$ matrix B such that $BA = I_n$, then the system of linear equations $A\vec{x} = \vec{0}$ has a unique solution. (Note: a matrix B with this property is called a *left-inverse* for A. Can you guess why?)

(b) (**Recreational**) State and prove the converse of the statement in (a).

Solution. Clearly $\vec{x} = \vec{0}$ is one solution for the desired system. Now if $\vec{x}$ is some other solution, it means that the equation $A\vec{x} = \vec{0}$ is satisfied. Multiplying both sides of this equation by B we obtain $\vec{x} = I_n\vec{x} = (BA)\vec{x} = B(A\vec{x}) = B\vec{0} = \vec{0}$, showing that the only possibility for an $\vec{x}$ that satisfies this equation would be $\vec{x} = \vec{0}$.

Problem 4. Given two matrices A and B such that the product AB is defined (say, A is $n \times m$ and B is $m \times k$), exactly one of the following two statements is true:

(a) Every column of AB is a linear combination of columns of A,

(b) Every column of AB is a linear combination of columns of B.

Prove the one that is true, and provide a counterexample for the one that is false.

Solution. It turns out that (a) is true and b is false.

(a) Let's let $\vec{a}_i$ denote the ith column of matrix A and $\vec{b}_j$ denote the jth column of matrix B. By definition, AB is the matrix whose jth column is $A\vec{b}_j$. Also by definition, $A\vec{b}_j$ is the linear combination

$$b_{j,1}\vec{a}_1 + b_{j,2}\vec{a}_2 + \cdots + b_{j,m}\vec{a}_m$$

so each column of AB is a linear combination of the columns of A.

(b) **Counterexample**: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, so that $AB = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$. We can observe directly that the columns of AB are obviously not linear combinations of the columns of B: any linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ would be of the form $\begin{bmatrix} r \\ r \end{bmatrix}$ for some $r \in \mathbb{R}$, and neither of the columns of AB has that form.

Problem 5. Let $f \colon X \to X$ be a function. We let f^n denote the function $f^n \colon X \to X$ given by composing f iteratively, n many times. In other words, $f^n(x) = \underbrace{(f \circ \cdots \circ f)}_{n \text{ times}}(x)$. Also, we define f^0 to be the identity function, i.e. $\forall x \in X, f^0(x) = x$.

(a) Assume that $X = \mathbb{R}^d$. Prove by induction that if f is a linear transformation, then the nth iterate f^n is also a linear transformation.

(b) Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is **not** a linear transformation, but for which there exists an n such that the nth iterate f^n is a linear transformation.

(c) Prove that for $X = \mathbb{R}^d$ and f linear, if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.

Solution.

(a) Prove by induction. The base is just the definition of linear function.

For the induction step, we assume f^n is linear for some $n \geq 1$. We want to prove that f^{n+1} is linear as well. We observe that

$$\begin{aligned} f^{n+1}(u+v) &= f(f^n(u+v)) \\ &= f(f^n(u) + f^n(v)) \text{ (using the induction hypothesis)} \\ &= f(f^n(u)) + f(f^n(v)) \text{ (using the linearity of } f\text{)} \\ &= f^{n+1}(u) + f^{n+1}(v) \end{aligned}$$

Similarly, for any constant $c \in \mathbb{R}$, we have

$$\begin{aligned} f^{n+1}(cv) &= f(f^n(cv)) \\ &= f(cf^n(v)) \text{ (using the induction hypothesis)} \\ &= cf(f^n(v)) \text{ (using the linearity of } f\text{)} \\ &= cf^{n+1}(v) \end{aligned}$$

(b) **Example**: Note that f does not need to be continuous. Then we define f as follows:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{cases} \begin{bmatrix} x \\ y \end{bmatrix}, & x > 0 \\ \\ \begin{bmatrix} x \\ -y \end{bmatrix}, & x \leq 0 \end{cases}$$

This function is not linear, as can be seen from the fact that

$$f\left(-1\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = f\left(\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

but

$$-1f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = -1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

However,

$$f^2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}2$, as can be verified by a case-by-case analysis, and this is a linear transformation.

(There are many other solutions possible.)

(c) Suppose that $f(x) = 0$ has a unique solution. We will prove by induction on n that $f^n(x) = 0$ also has a unique solution. The base case ($n = 1$) is just that $f(x) = 0$ has a unique solution, which is our assumption on f. Now for the induction step, assume that for some n, $f^n(x) = 0$ has a unique solution. Since f^n is linear (by (a)), we know that $f^n(0) = 0$, and so the unique solution must be the zero vector.

Now we prove that $f^{n+1}(x) = 0$ also has a unique solution, and that this solution is also the zero vector. Suppose that $\vec{a} \in \mathbb{R}^d$ is a vector such that $f^{n+1}(\vec{a}) = 0$. This is equivalent to $f(f^n(\vec{a})) = 0$, which means that $f^n(\vec{a})$ is a solution to the equation $f(x) = 0$. But by hypothesis, $f(x) = 0$ only has one solution, namely the zero vector. We conclude that $f^n(\vec{a}) = 0$ as well. However, by the induction hypothesis $f^n(x) = 0$ also has only one solution, which is also the zero vector. We conclude from this that $\vec{a} = 0$. This shows that the only solution to the equation $f^{n+1}(x) = 0$ is $x = 0$.

(Alternate proof: Note that the coefficient matrix of the system $Ax = 0$ is a square matrix. By textbook theorem 2.4, such a linear system has a unique solution if and only if its coefficient matrix is invertible. Then by 2 d), A is invertible implies A^n is invertible. This means if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.)