Math 217 Worksheet 22: Eigenvalues and Eigenvectors (§7.1)

Definition. Let V be a vector space and let $T : V \to V$ be a linear transformation.

A *non-zero* vector $v \in V$ is an **eigenvector** of T if there exists $\lambda \in \mathbb{R}$ such that $T(v) = \lambda v$.
A scalar λ is an **eigenvalue** of T if there exists a *non-zero* vector $v \in V$ such that $T(v) = \lambda v$.

Whenever a pair consisting of a non-zero $v \in V$ and a scalar $\lambda \in \mathbb{R}$ satisfies $T(v) = \lambda v$, we say that v is an eigenvector of the eigenvalue λ, or that v is a **λ-eigenvector.**

Note that, by definition, eigenvectors (but not eigenvalues) are required to be nonzero!

Problem 1. Warm-up.

(a) Find all eigenvectors of the *identity map* $V \to V$. What are the associated eigenvalues?

(b) Find all eigenvectors of the *zero map* $V \to V$. What are the associated eigenvalues?

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map "scale by 2". What are the eigenvectors of T? What are the associated eigenvalues?

(d) The transformation $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ rotating counterclockwise by $\pi/6$ has *no eigenvectors.* Why not?

Solution: Every non-zero vector is an eigenvector of the identity map, with eigenvalue 1.
Every non-zero vector is an eigenvector of the zero map, with eigenvalue 0.
Every non-zero vector is an eigenvector of the "scale by k" map, with eigenvalue k. For (d), note that rotation in $\mathbb{R}^2$ moves every vector in such a way that no vector is taken to a scaling of itself (unless we rotate by π, so vectors go to a their negatives, in which case -1 is an eigenvalue. So there are no eigenvectors (nor eigenvalues) for (d).

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be multiplication by $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. Explain why any non-zero vector in the span of $\vec{e}_1$ is an eigenvector. What are the corresponding eigenvalues? Find all eigenvectors with eigenvalue 4. Are there any eigenvectors with eigenvalue zero? Prove you've found *all* eigenvectors of T. [Hint: Take arbitrary $\vec{v}$ which is *not* one you found before...]

Solution: Note that $T(k\vec{e}_1) = 3k\vec{e}_1$, so $k\vec{e}_1$ is an eigenvector with eigenvalue 3. The eigenvectors with eigenvalue 4 are the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $T(\begin{bmatrix} a \\ b \end{bmatrix}) = 4\begin{bmatrix} a \\ b \end{bmatrix}$. To find them, we solve $\begin{bmatrix} 3a \\ 4b \end{bmatrix} = 4\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a \\ 4b \end{bmatrix}$. We must have $a = 0$, but b can be arbitrary. So the eigenvectors with eigenvalues 4 are exactly the span of $\vec{e}_2$ (except for $\vec{0}$). To see we that there are no further eigenvectors, suppose, on the contrary, that $c \neq 3, 4$ is an eigenvalue. This means there exists a non-zero vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $T(\begin{bmatrix} a \\ b \end{bmatrix}) = c\begin{bmatrix} a \\ b \end{bmatrix}$. In other words, $\begin{bmatrix} 3a \\ 4b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$. Since one of either a or b is non-zero, we conclude that either $c = 3$ or $c = 4$.

Problem 3. Explain why the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line spanned by $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$, $\begin{bmatrix} -\pi \\ -5\pi \end{bmatrix}$, $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1/5 \end{bmatrix}$. What are the eigenvalues of each of these eigenvectors?

Solution: The vectors on the one-dimensional subspace L (a line through the origin) are taken to themselves when we project to L. So the vectors $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$, and $\begin{bmatrix} -\pi \\ -5\pi \end{bmatrix}$ are eigenvectors for the eigenvalue 1, since they are all on the line spanned by $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Everything in $L^{\perp}$ is sent to zero, so zero is also an eigenvalue, and the (non-zero) vectors $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1/5 \end{bmatrix}$, which are in $L^{\perp}$, are eigenvectors with eigenvalue 0.

Problem 4. Let $\mathcal{C}^{\infty}(\mathbb{R})$ be the vector space of all real valued functions that are infinitely differentiable. Consider the differentiation map $d : \mathcal{C}^{\infty}(\mathbb{R}) \to \mathcal{C}^{\infty}(\mathbb{R})$. Why is this linear? For any constant k, show that the function $f(x) = e^{kx}$ is an eigenvector of d. What is its eigenvalue? What are all the possible eigenvalues of differentiation on $\mathcal{C}^{\infty}(\mathbb{R})$?

Solution: Check: $d(f) = \frac{d}{dx}e^{kx} = ke^{kx} = kf$, so f is an eigenvector with eigenvalue k. This shows that every real number is an eigenvalue for d.

Problem 5. Prove or disprove the following statements.

(a) Any non-zero element in the kernel of $V \xrightarrow{T} V$ is an eigenvector of T.

(b) The linear transformation $V \xrightarrow{T} V$ is injective if and only if zero is not an eigenvalue of T.

(c) If v is an an eigenvector of T, then v is an eigenvector of T^2 as well.

(d) If v is an eigenvector for T with eigenvalue λ, then so is kv for any non-zero scalar k.

Solution:

(a) True: $T(v) = 0 = 0\,v$ so the 0-eigenvectors are exactly the (non-zero) elements in the kernel.

(b) True: This basically restates (a), remembering that T is injective if and only if $\ker T$ is zero.

(c) True: If $T(v) = \lambda v$, then $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$. So v is an eigenvector for T also (though the corresponding eigenvalue could be different.)

(d) True. Suppose v is an eigenvector with eigenvalue λ. This means $T(v) = \lambda v$. Compute $T(kv) = kT(v) = k\lambda v = \lambda(kv)$. So kv is also an λ-eigenvector. The first $=$ is coming from the fact T is linear, the second is from the fact that v is an λ-eigenvector, and the third from the axioms of vector spaces.

Definition. Let $T : V \to V$ be a linear transformation. Fix any eigenvalue λ.
The **λ-eigenspace** of T is the subset of V defined by

$$E_\lambda \;=\; \{v \in V :\; T(v) = \lambda v\}.$$

Problem 6. Let $T : V \to V$ be a linear transformation of a vector space V.

(a) Prove that for any scalar λ (eigenvalue or not), the set

$$E_\lambda \;=\; \{v \in V :\; T(v) = \lambda v\}$$

is a subspace of V.

(b) With E_λ as defined in (a), prove that λ is an eigenvalue of T if and only if $E_\lambda \neq \{0_V\}$.

(c) For an eigenvalue λ of T, the subspace E_λ is almost, but not quite, the set of all λ-eigenvectors of T. Explain.

Solution:

(a) Since $T(0_V) = 0_V = \lambda 0_V$, we have $0_V \in E_\lambda$. If $v \in E_\lambda$ and $c \in \mathbb{R}$, then $T(cv) = cT(v) = \lambda(cv)$, so $cv \in E_\lambda$. Finally, if $v, w \in E_\lambda$ then $T(v+w) = T(v)+T(w) = \lambda v + \lambda w = \lambda(v+w)$, so $v + w \in E_\lambda$. This shows that E_λ is a subspace of V. Furthermore, λ is an eigenvalue of T if and only if $E_\lambda \neq \{\vec{0}\}$.

(b) If $E_\lambda \neq \{0_V\}$, then there is a non-zero vector $v \in V$ such that $T(v) = \lambda v$. This means v is an eigenvector, and hence λ is an eigenvalue. On the other hand, if $E_\lambda = \{0_V\}$, then there is no non-zero vector v such that $T(v) = \lambda v$. This means λ is not an eigenvalue.

(c) Since eigenvectors must be non-zero, the set of λ-eigenvectors never includes the zero vector, so can not be a subspace. But the zero is the only missing element to make the set of eigenvectors a subspace.

Problem 7. For the following transformations $T : V \to V$, try to find an eigenvalue using any methods you can think of, including basic geometry, if this is possible. For your eigenvalue, find the corresponding **eigenspace.** If none exist, explain why.

(a) $V = \mathbb{R}^2$, $T =$ orthogonal projection onto the x-axis.

(b) $V = \mathbb{R}^2$, $T =$ reflection over the line $x = y$.

(c) $V = \mathbb{R}^2$, $T =$ rotation by $90°$.

(d) $V = \mathbb{R}^2$, $T =$ left multiplication by $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(e) $V = \mathbb{R}^3$, T is rotation around some axis L through the origin in $\mathbb{R}^3$.

(f) $V = \mathcal{P}_3$ the space of polynomials of degree less than or equal 3 in the variable t,

$$T(f) = f'.$$

(g) $V = \mathbb{R}^{2\times 2}$, T is the zero transformation.

Solution:

(a) The eigenvalues are 1 and 0, with eigenspaces the x-axis and y-axis, respectively.

(b) The eigenvalues are 1 and -1, with eigenspaces the line spanned by $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$, respectively.

(c) no eigenvectors or eigenvalues

(d) Since $T(\vec{e}_1) = 2\vec{e}_2$, the x-axis is the eigenspace of the eigenvalue 2. It's not as easy to find, but also, $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector with eigenvalues 3. It's eigenspace is the line $y = x$.

(e) The line L is the eigenspace for the eigenvalue 1. Rotation in $\mathbb{R}^3$ fixes every vector on the axis of rotation. So each (non-zero) vector in this axis is an eigenvector with eigenvalue 1. This is the only eigenvalue unless we are rotating by π, in which case every vector perpendicular to the axis of rotation is taken to its negative. So if we rotate by π, then -1 is also an eigenvalue.

(f) Any constant function is an eigenvector, with eigenvalue 0. The eigenspace of 0 is the subspace of constant functions.

(g) Any zero transformation has $\lambda = 0$ as its only eigenvalue, with the entire space $\mathbb{R}^{2\times 2}$ the corresponding eigenspace.

Problem 8. Show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation,* then the only possible eigenvalues are 1 or -1. [Hint: Recall that orthogonal transformations *preserve lengths.*]

Solution: Suppose that λ is an eigenvalue of an orthogonal transformation T, and let $\vec{v}$ be the corresponding eigenvector. Then $\|T(\vec{v})\| = \|(\lambda\vec{v})\| = |\lambda|\|\vec{v}\|$. But since T is orthogonal this means $\|\vec{v}\| = |\lambda|\ \|\vec{v}\|$. So $|\lambda| = 1$.

Problem 9. Suppose that λ is an eigenvalue of the linear transformation $T : V \to V$.

(a) Given $n \in \mathbb{N}$, show that λ^n is an eigenvalue of T^n. [Hint: You thought about this already in 5c.]

(b) Supposing that T is invertible, what can you say about the eigenvalues of T^{-1}?

Solution:

(a) If v is an eigenvector of T corresponding to λ, then $T^n(v) = \lambda^n v$, so v is an eigenvector of T^n with corresponding eigenvalue λ^n.

(b) If T is invertible and v is an eigenvector of T with corresponding eigenvalue λ, then $T^{-1}(\lambda v) = v$, so $T^{-1}(v) = \frac{1}{\lambda}v$, which shows that v is an eigenvector of T^{-1} with corresponding eigenvalue λ^{-1}. Indeed, λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Problem 10. Consider the transpose transformation $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $T(A) = A^\top$.

(a) Prove that 1 is an eigenvalue of T, with eigenspace the subspace of all *symmetric* matrices.

(b) Prove that -1 is an eigenvalue of T, with eigenspace the subspace of all *skew-symmetric* matrices.

(c) Are there any other eigenvalues for the transpose map?

Solution:

(a) By definition, $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if $A = A^\top$. So $T(A) = 1\ A^\top$ if and only if A is symmetric, which means 1 is an eigenvalue with eigenspace the set of symmetric matrices.

(b) By definition, $A \in \mathbb{R}^{n\times n}$ is skew-symmetric if and only if $A = -A^\top$. So $T(A) = -1\ A^\top$ if and only if A is skew-symmetric, which means -1 is an eigevnalue with eigenspace the set of skew symmetric matrices.

(c) No. Assume c is an eigenvalue not equal to 1 or -1. This means there is a non-zero matrix $A \in \mathbb{R}^{n\times n}$ such that $T(A) = cA$. In other words, such that $cA = A^\top$. So for all i, j, we have $ca_{ji} = a_{ij}$ and so substituting $a_{ji} = ca_{ij}$, we get $c^2 a_{ij} = a_{ij}$ for all i, j. Now, because A is non-zero, some entry $a_{ij} \neq 0$. Thus we can divide by a_{ij} and see that $c^2 = 1$, so $c = \pm 1$.