Math 217 – Midterm 2

Winter 2018

Solutions

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	12	
4	15	
5	12	
6	12	
7	12	
8	10	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

   (a) An *isomorphism* from the vector space V to the vector space W

   Solution: An *isomorphism* from the vector space V to the vector space W is a bijective linear transformation from V to W.

   (b) The matrix $A \in \mathbb{R}^{n \times n}$ is *orthogonal*

   Solution: The matrix $A \in \mathbb{R}^{n \times n}$ is *orthogonal* if $A^T A = AA^T = I_n$.

   (c) A *least-squares solution* of the system of linear equations $A\vec{x} = \vec{b}$

   Solution: For A an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$, the vector $\vec{x}^* \in \mathbb{R}^n$ is a *least-squares solution* of the system of linear equations $A\vec{x} = \vec{b}$ if $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$ for all $\vec{x} \in \mathbb{R}^n$.

   (d) The *norm* (or *magnitude*, or *length*) of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$

   Solution: The *norm* of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$ is the scalar $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if the columns of A form an orthonormal list of vectors in $\mathbb{R}^m$, then $AA^T = I_m$, where I_m is the $m \times m$ identity matrix.

Solution: False.

Any orthogonal projection onto a lower-dimensional subspace would be a counterexample.

For example let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $AA^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq I_2$.

(b) (3 points) If A is an invertible 2018×2018 matrix, then $\det(-A) = -\det(A)$.

Solution: False.

In fact $\det(-A) = (-1)^{2018}\det(A)$ by multilinearity of determinants.

For example let $A = I_{2018}$, then $\det(-I_{2018}) = 1$, while $-\det(I_{2018}) = -1$.

(c) (3 points) For every orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$, $|\det(S_{\mathcal{E}\to\mathcal{B}})| = 1$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^n$ and $S_{\mathcal{E}\to\mathcal{B}}$ is the change-of-coordinates matrix from $\mathcal{E}$ to $\mathcal{B}$.

Solution: True.

Since $\mathcal{B}$ is an orthonormal basis, the change-of-coordinates matrix $S_{\mathcal{E}\to\mathcal{B}}$ is orthogonal. You may quote this result directly, otherwise you may prove this by observing that ${S_{\mathcal{E}\to\mathcal{B}}}^{-1} = S_{\mathcal{B}\to\mathcal{E}} = \left[\begin{array}{c|c|c} \vec{b}_1 & \dots & \vec{b}_n \end{array}\right]$ which has orthonormal columns, together with the fact that the inverse of an orthogonal matrix is also orthogonal.

Therefore $|\det(S_{\mathcal{E}\to\mathcal{B}})| = 1$ since the determinant of any orthogonal matrix is equal to ± 1.

(Problem 2, Continued).

(d) (3 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if the columns of A are linearly independent, then there is an $n \times m$ matrix B such that $BA = I_n$, where I_n is the $n \times n$ identity matrix.

Solution: True.

Here are two proofs using chapter 5 material:

(i) Since A has linearly independent columns, we may apply the QR factorization to write $A = QR$ where Q has orthonormal columns and R is upper-triangular with positive diagonal entries, in particular R is invertible.

Letting $B = R^{-1}Q^T$, we see that $BA = R^{-1}\underbrace{Q^TQ}_{=I_n}R = R^{-1}R = I_n$.

(ii) Since A has linearly independent columns, $\ker(A) = \ker(A^TA) = \{\vec{0}\}$, but A^TA is a square matrix (of size $n \times n$) with zero kernel, hence A^TA is invertible.

Letting $B = (A^TA)^{-1}A^T$, we see that $BA = (A^TA)^{-1}A^TA = I_n$.

Here is another proof without using chapter 5 material:

(iii) Since A has linearly independent columns, each column of A (or its reduced-row echelon form) contains a pivot, in particular its reduced-row echelon form is equal to $\left[\begin{array}{c} I_{n\times n} \\ \hline O_{(m-n)\times n}\end{array}\right] = EA$, where $I_{n\times n}$ denotes the $n \times n$ identity matrix, $O_{k\times n}$ denotes the $k \times n$ matrix where all the entries are equal to zero, and E is the matrix representing the elementary row operations.

Letting $B = \left[\begin{array}{c|c} I_{n\times n} & O_{(m-n)\times n}\end{array}\right]E$, we see that

$$BA = \left[\begin{array}{c|c} I_{n\times n} & O_{(m-n)\times n}\end{array}\right]EA = \left[\begin{array}{c|c} I_{n\times n} & O_{(m-n)\times n}\end{array}\right]\left[\begin{array}{c} I_{n\times n} \\ \hline O_{(m-n)\times n}\end{array}\right] = I_n.$$

(e) (3 points) For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$,

$$\det\begin{bmatrix} | & | \\ \vec{u}-\vec{v} & \vec{v}-\vec{w} \\ | & | \end{bmatrix} = \det\begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix} - \det\begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix}.$$

Solution: False.

In fact

$$\det\left[\begin{array}{c|c} \vec{u}-\vec{v} & \vec{v}-\vec{w}\end{array}\right] = \det\left[\begin{array}{c|c} \vec{u} & \vec{v}\end{array}\right] - \det\left[\begin{array}{c|c} \vec{u} & \vec{w}\end{array}\right] - \underbrace{\det\left[\begin{array}{c|c} \vec{v} & \vec{v}\end{array}\right]}_{=0} + \det\left[\begin{array}{c|c} \vec{v} & \vec{w}\end{array}\right]$$

by standard properties of determinants (multilinearity and alternating), which is not equal to $\det\left[\ \vec{u}\ \middle|\ \vec{v}\ \right] - \det\left[\ \vec{v}\ \middle|\ \vec{w}\ \right]$ in general.

For example let $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{0}$, $\vec{w} = \vec{e}_2$, then

$$\det\left[\ \vec{u} - \vec{v}\ \middle|\ \vec{v} - \vec{w}\ \right] = \det\left[\ \vec{e}_1\ \middle|\ -\vec{e}_2\ \right] = -1,$$

$$\det\left[\ \vec{u}\ \middle|\ \vec{v}\ \right] = \det\left[\ \vec{e}_1\ \middle|\ \vec{0}\ \right] = 0,$$

$$\det\left[\ \vec{v}\ \middle|\ \vec{w}\ \right] = \det\left[\ \vec{0}\ \middle|\ \vec{e}_2\ \right] = 0.$$

3. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$. The rule

$$\langle \vec{x}, \vec{y} \rangle \;=\; \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \;=\; \vec{x}^\top \begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix} \vec{y} \;=\; 8x_1y_1 - 6(x_1y_2 + x_2y_1) + 5x_2y_2$$

defines an inner product on $\mathbb{R}^2$. (You may assume this without proof).

(a) (4 points) Compute the length of $\vec{v}$ with respect to $\langle \cdot, \cdot \rangle$.

Solution: For $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$,
$\|\vec{v}\|^2 \;=\; \langle \vec{v}, \vec{v} \rangle \;=\; \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{\vec{v}^\top} \begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}} \;=\; \begin{bmatrix} -4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; 4$. Thus its length is

$\|\vec{v}\| = 2$.

(b) (4 points) Find a nonzero vector $\vec{w} \in \mathbb{R}^2$ that is orthogonal (relative to $\langle \cdot, \cdot \rangle$) to $\vec{v}$.

Solution: A nonzero vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ is orthogonal to $\vec{v}$ with respect to the given inner product if and only if $\langle \vec{v}, \vec{x} \rangle = 0$. Then $8x_1 - 6(x_2 + 2x_1) + 10x_2 = 0$ gives $-4x_1 + 4x_2 = 0$, that is $x_1 = x_2$.
Any vector $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$ for $a \neq 0$ is orthogonal to $\vec{v}$ (e.g. let $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.)

(c) (4 points) Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$.

Solution: Using part (b), we already have two vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ that are orthogonal to each other. To get an orthonormal basis we just need to normalize them.
By part (a), we know $\|\vec{v}\| = 2$, therefore $\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.
$\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle}\sqrt{8 - 6(1+1) + 5} = 1$, that is $\vec{u}_2 = \frac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
An orthonormal basis is $\mathcal{B} = \left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

4. Let $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and $\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$, so that $\mathcal{E}$ and $\mathcal{B}$ are ordered bases of the vector space V of 2×2 upper-triangular matrices.

(a) (3 points) Find an ordered basis $\mathcal{C}$ of V such that $S_{\mathcal{C}\to\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution: Let c_1, c_2, c_3 denote the basis elements of $\mathcal{C}$. Then we want $[c_i]_{\mathcal{E}}$ to be the i-th column of $S_{\mathcal{S}\to\mathcal{E}}$. So $c_1 = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Similarly, $c_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $c_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, $\mathcal{C} = \left(\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right.$

(b) (3 points) Find the $\mathcal{B}$-coordinates $[A]_{\mathcal{B}}$ of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $[A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(c) (5 points) Find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of the linear transformation $T : V \to V$, where T is defined so that for each $A \in V$, $T(A) = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} A$.

Solution:

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & -2 \\ 0 & -3 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$.

(d) (4 points) Find $\det(T)$, where T is the linear transformation from part (c).

Solution:

$$\det(T) = \det\left(\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}\right) = -1\det\left(\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\right)$$
$$= -1(1-4) = 3.$$

5. Let A be an $m \times n$ matrix with linearly independent columns, let $A = QR$ be the QR-factorization of A, and let $\vec{b} \in \mathbb{R}^m$.

(a) (3 points) Write the normal equation of the linear system $A\vec{x} = \vec{b}$.

Solution: The normal equation is $A^\top A\vec{x} = A^\top\vec{b}$.

(b) (6 points) Show that the vector $\vec{x}^* = R^{-1}Q^\top\vec{b}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$.

Solution: We show that the vector $\vec{x}^* = R^{-1}Q^\top\vec{b}$ is a solution of the normal equation in part(a):

$$\begin{aligned} A^\top A(R^{-1}Q^\top\vec{b}) &= (QR)^\top(QR)(R^{-1}Q^\top\vec{b}) = R^\top Q^\top Q(RR^{-1})Q^\top\vec{b} = R^\top(Q^\top Q)Q^\top\vec{b} \\ &= R^\top Q^\top\vec{b} = A^\top\vec{b} \end{aligned}$$

where we have used the fact that $Q^\top Q = I_n$ since columns of Q form an orthonormal set of vectors.

Alternatively we can show that $\vec{x}^* = R^{-1}Q^\top\vec{b}$ is a solution of the equation $A\vec{x} = \text{proj}_{\text{im}(A)}\vec{b}$. Since $QQ^\top$ is the matrix of orthogonal projection onto $\text{im}(A)$, $A(R^{-1}Q^\top\vec{b}) = Q(RR^{-1})Q^\top\vec{b} = (QQ^\top)\vec{b} = \text{proj}_{\text{im}(A)}\vec{b}$.

(c) (3 points) Is $R^{-1}Q^\top\vec{b}$ the *unique* least-squares solution of the linear system $A\vec{x} = \vec{b}$? Explain.

Solution: Yes, $R^{-1}Q^\top\vec{b}$ the *unique* least-squares solution of the linear system $A\vec{x} = \vec{b}$. Since A has linearly independent columns, $\ker(A^\top A) = \ker(A) = \{\vec{0}\}$. Therefore, $A^\top A$ is invertible and the normal equation yields a unique least-squares solution.

6. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most 2 in the variable t, and consider $\mathcal{P}_2$ as an inner product space with inner product

$$\langle p, q\rangle \;=\; p(0)q(0) + p'(0)q'(0) + \frac{1}{2}p''(0)q''(0).$$

Also let $f(t) = 1 + t$ and $g(t) = 2 - t^2$ be polynomials in $\mathcal{P}_2$, and let $W = \operatorname{span}(f, g)$ be the subspace of $\mathcal{P}_2$ spanned by f and g.

(a) (6 points) Find a basis of W that is orthonormal with respect to $\langle\cdot,\cdot\rangle$.

Solution: We apply the Gram-Schmidt algorithm to the basis (f, g) of W. To facilitate computations, note that for all $p(t) = a_0 + a_1t + a_2t^2$ and $q(t) = b_0 + b_1t + b_2t^2$ in $\mathcal{P}_2$, we have

$$\langle p, q\rangle \;=\; \langle a_0 + a_1t + a_2t^2,\; b_0 + b_1t + b_2t^2\rangle \;=\; a_0b_0 + a_1b_1 + 2a_2b_2.$$

The vector

$$g^{\perp} \;=\; g - \frac{\langle g, f\rangle}{\langle f, f\rangle}f \;=\; g - \frac{2}{2}f \;=\; (2 - t^2) - (1 + t) \;=\; 1 - t - t^2$$

is orthogonal to f. Normalizing f and $g^{\perp}$, we obtain the orthonormal basis

$$\left(\frac{f}{\|f\|}, \frac{g^{\perp}}{\|g^{\perp}\|}\right) \;=\; \left(\frac{1+t}{\|1+t\|}, \frac{1-t-t^2}{\|1-t-t^2\|}\right) \;=\; \left(\frac{1+t}{\sqrt{2}}, \frac{1-t-t^2}{2}\right)$$

of W.

(b) (6 points) Let $\operatorname{proj}_W : \mathcal{P}_2 \to \mathcal{P}_2$ be the orthogonal projection onto W in $\mathcal{P}_2$. Find a polynomial $h \in \mathcal{P}_2$ such that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the $\mathcal{B}$-matrix of proj_W, where $\mathcal{B}$ is the ordered basis of $\mathcal{P}_2$ given by $\mathcal{B} = (f, g, h)$.

Solution: The matrix of proj_W with respect to the basis $\mathcal{B} = (f, g, h)$ will have the given form if and only if h is a nonzero vector in $W^{\perp}$, so we need $\langle f, h\rangle = \langle g, h\rangle = 0$. Writing $h(t) = a + bt + ct^2$, this leads to the equations

$$\begin{aligned} 0 \;&=\; \langle f, h\rangle \;=\; \langle 1 + t, a + bt + ct^2\rangle \;=\; a + b \\ 0 \;&=\; \langle g, h\rangle \;=\; \langle 2 - t^2, a + bt + ct^2\rangle \;=\; 2a - 2c. \end{aligned}$$

Solving this linear system for a, b, and c, we find that $c = a = -b$, so we can take, for instance,

$$h(t) \;=\; 1 - t + t^2.$$

7. (a) (5 points) Prove that for every $n \times n$ matrix A, if $A^\top A = AA^\top$ then $\|Ax\| = \|A^\top x\|$ for all $x \in \mathbb{R}^n$. (Here length is defined with respect to the dot product on $\mathbb{R}^n$).

Solution: Suppose that A is an $n \times n$ matrix such that $A^\top A = AA^\top$. Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\|Ax\|^2 &= (Ax) \cdot (Ax) \\
&= (Ax)^\top Ax \\
&= x^\top A^\top Ax \\
&= x^\top AA^\top x \\
&= (A^\top x)^\top A^\top x \\
&= (A^\top x) \cdot (A^\top x) \\
&= \|A^\top x\|^2
\end{aligned}$$

and thus $\|Ax\| = \|A^\top x\|$.

(b) (7 points) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, and let $T : V \to V$ be a linear transformation. Prove that if $\langle T(v), w \rangle = \langle v, T(w) \rangle$ for all $v, w \in V$, then $\ker(T) = (\mathrm{im}(T))^\perp$.

Solution: Suppose that $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $T : V \to V$ is a linear transformation such that $\langle T(v), w \rangle = \langle v, T(w) \rangle$ for any $v, w \in V$. We will show that $\ker(T) \subseteq (\mathrm{im}(T))^\perp$ and that $(\mathrm{im}(T))^\perp \subseteq \ker(T)$, which together implies $\ker(T) = (\mathrm{im}(T))^\perp$.

Let $x \in \ker(T)$, which means that $T(x) = 0$. Then for any $y \in \mathrm{im}(T)$ there exists $w \in V$ with $T(w) = y$ and hence

$$\langle x, y \rangle = \langle x, T(w) \rangle = \langle T(x), w \rangle = \langle 0, w \rangle = 0.$$

This implies that $x \in (\mathrm{im}(T))^\perp$. Thus, $\ker(T) \subseteq (\mathrm{im}(T))^\perp$.

Now let $x \in (\mathrm{im}(T))^\perp$. Then for any $y \in V$,

$$\langle T(x), y \rangle = \langle x, T(y) \rangle = 0.$$

In particular, when $y = T(x)$, we have $\langle T(x), T(x) \rangle = 0$. Since the inner product is positive definite, this implies that $T(x) = 0$; i.e. $x \in \ker(T)$. Thus, $(\mathrm{im}(T))^\perp \subseteq \ker(T)$.

8. (10 points) Let $n \in \mathbb{N}$, let V be a subspace of $\mathbb{R}^n$, and let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto V. Prove that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, $P \circ T = T \circ P$ if and only if $T[V] \subseteq V$ and $T[V^\perp] \subseteq V^\perp$.

   $\big($Recall that, by definition, $T[X] = \{T(x) : x \in X\}$ for any subset $X \subseteq \mathbb{R}^n\big)$.

Solution:

(⇒) Suppose that $P \circ T = T \circ P$.

To prove that $T[V] \subseteq V$, let $\vec{y} \in T[V]$. This means that $\vec{y} = T(\vec{x})$ for some $\vec{x} \in V$. Then we will have that $P(\vec{x}) = \vec{x}$, and hence

$$P(\vec{y}) = P(T(\vec{x})) = (P \circ T)(\vec{x}) = (T \circ P)(\vec{x}) = T(P(\vec{x})) = T(\vec{x}) = \vec{y}.$$

Thus $P(\vec{y}) = \vec{y}$, and this means that $\vec{y} \in V$. Therefore $T[V] \subseteq V$.

Now, to prove that $T[V^\perp] \subseteq V^\perp$, let $\vec{y} \in T[V^\perp]$. This means that $\vec{y} = T(\vec{x})$ for some $\vec{x} \in V^\perp$. Then we will have that $P(\vec{x}) = \vec{0}$, and hence

$$P(\vec{y}) = P(T(\vec{x})) = (P \circ T)(\vec{x}) = (T \circ P)(\vec{x}) = T(P(\vec{x})) = T(\vec{0}) = \vec{0}.$$

Thus $P(\vec{y}) = \vec{0}$, meaning that $\vec{y} \in \ker(P) = V^\perp$. Therefore $T[V^\perp] \subseteq V^\perp$.

(⇐) Assume that $T[V] \subseteq V$ and $T[V^\perp] \subseteq V^\perp$. Let $\vec{x} \in \mathbb{R}^n$ be arbitrary. Then $\vec{x} = \vec{v} + \vec{w}$ for some $\vec{v} \in V$ and $\vec{w} \in V^\perp$; note that $P(\vec{x}) = \vec{v}$. Our assumptions imply that $T(\vec{v}) \in V$ and $T(\vec{w}) \in V^\perp$, and therefore $P(T(\vec{v})) = T(\vec{v})$ and $P(T(\vec{w})) = \vec{0}$. We therefore have that

$$(P \circ T)(\vec{x}) = P(T(\vec{x})) = P(T(\vec{v}+\vec{w})) = P(T(\vec{v})) + P(T(\vec{w})) = T(\vec{v}) + \vec{0} = T(\vec{v}),$$

and, on the other hand,

$$(T \circ P)(\vec{x}) = T(P(\vec{x})) = T(P(\vec{v} + \vec{w})) = T(\vec{v}).$$

Putting together these two equations we obtain that

$$(T \circ P)(\vec{x}) = T(\vec{x}) = (P \circ T)(\vec{x}).$$

Since this holds for every $\vec{x} \in \mathbb{R}^n$, we can conclude that $P \circ T = T \circ P$, and the proof is complete.