Math 217 Worksheet: Symmetric and Skew Symmetric Matrices

Definition: A matrix $A \in \mathbb{R}^{n\times n}$ is said to be **symmetric** if $A = A^\top$, and **skew-symmetric** if $A = -A^\top$.

Let $\mathrm{Sym}^{n\times n}$ and $\mathrm{Skew}^{n\times n}$ denote the sets of symmetric and skew symmetric $n \times n$ matrices, respectively.

Problem 1.

(a) Prove that $\mathrm{Sym}^{n\times n}$ and $\mathrm{Skew}^{n\times n}$ are subspaces of $\mathbb{R}^{n\times n}$.

(b) Prove that the diagonal entries of a skew symmetric matrix are zero.

(c) Find a basis for $\mathrm{Skew}^{2\times 2}$ and a basis for $\mathrm{Sym}^{2\times 2}$. What are their dimensions?

(d) Prove that $\mathrm{Skew}^{3\times 3}$ is three dimensional.

(e) Prove that the dimension of $\mathrm{Skew}^{n\times n}$ is $\sum_{i=1}^{n-1} i$, or equivalently (as you should show using induction), $\frac{(n-1)n}{2}$.

Problem 2. Consider the map $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $T(A) = A + A^\top$.

(a) Is T linear?

(b) Prove that $\mathrm{im}T = \mathrm{Sym}^{n\times n}$.

(c) What is the kernel of T?

(d) Find a formula for the dimension of $\mathrm{Sym}^{n\times n}$ in terms of n.

Problem 3. Does there exist a linear transformation $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ with kernel the space of symmetric matrices and image the space of skew symmetric matrices?

Problem 4. Is every matrix in $\mathbb{R}^{n\times n}$ the sum of a symmetric and skew symmetric matrix? If so, is it uniquely such a sum?