Midterm Exam 1

Graded

Student

Qiulin Fan

Total Points

92 / 100 pts

Question 1

Definitions **15** / 16 pts

1.1 **linear transformation** **4** / 4 pts

- ✔ **+ 4 pts** Correct

1.2 **injective function** **3** / 4 pts

- ✔ **+ 3 pts** Almost correct

2 for all x_1 and x_2 in X,

1.3 **dimension of V** **4** / 4 pts

- ✔ **+ 4 pts** Correct

1.4 **rank of T** **4** / 4 pts

- ✔ **+ 4 pts** Correct

Question 2

True / False **13** / 16 pts

2.1 matrix with dim ker A = 3 **4** / 4 pts

- ✔ **+ 1 pt** Correctly identifies as false
- ✔ **+ 3 pts** Valid justification

2.2 inverse image of subspace **4** / 4 pts

- ✔ **+ 1 pt** Correctly identifies as true
- ✔ **+ 1 pt** Proof that $0_V \in T^{-1}[U]$
- ✔ **+ 1 pt** Proof that $T^{-1}[U]$ is closed under addition
- ✔ **+ 1 pt** Proof that $T^{-1}[U]$ is closed under scalar multiplication

2.3 ker A = {0} **1** / 4 pts

- ✔ **+ 1 pt** True.

2.4 linearly dependent set **4** / 4 pts

- ✔ **+ 4 pts** Correct

Question 3

Matrix Algebra **12** / 13 pts

3.1 **3(a) Find k such that R is in rref** **2** / 2 pts

✓ **+ 2 pts** All three correct values ($k = \pm\sqrt{2}, k = 1$)

3.2 **3(b) Find k such that...** **2** / 3 pts

3(b)(i)... has no solutions

✓ **+ 1 pt** $k = -1$

3(b)(ii)... has exactly one solution

✓ **+ 1 pt** "Not possible"

3(b)(iii)... has a solutions set that is a line

✓ **+ 0 pts** No credit or no submission

3.3 **3(c) Solution set in parametric form** **4** / 4 pts

✓ **+ 1 pt** Expressed as a **set** of vectors using **correct notation**

✓ **+ 1 pt** Has one free parameter

✓ **+ 1 pt** Correct constant vector $[5\ \ -1\ 0\ 1/3]^\top$

✓ **+ 1 pt** Correct direction vector $[3\ \ -2\ 1\ 0]^\top$

3.4 **3(d) Interpret A as standard matrix of T** **4** / 4 pts

Part (i)

✓ **+ 2 pts** Correct

Part (ii)

✓ **+ 2 pts** Correct

Question 4

Composition of functions **10** / 10 pts

4.1 **4(a) Compute** **2** / 2 pts

✔ **+ 2 pts** All three linear transformations computed correctly, on the correct order, arriving at $[0\ 3]^\top$

4.2 **4(b) Find standard matrix** **4** / 4 pts

✔ **+ 1 pt** Uses answer from (a) as first column of matrix

✔ **+ 2 pts** Correctly finds $(h \circ g \circ f)(\vec{e}_2) = [1\ \ 1]^\top$

✔ **+ 1 pt** Uses $(f \circ g \circ f)(\vec{e}_2)$ as second column of matrix

4.3 **4(c) Invertible?** **4** / 4 pts

✔ **+ 4 pts** Correct answer with valid justification

Question 5

Commutator L(X) **7** / 7 pts

5.1 **Basis of ker L** **4** / 4 pts

✔ **+ 1 pt** Express $L(X)$ in terms of the entries of X.

✔ **+ 1 pt** Set $L(X) = 0$.

✔ **+ 1 pt** Solve $L(X) = 0$ correctly.

✔ **+ 1 pt** State a basis correctly, based on solutions of $L(X) = 0$.

5.2 **dim im L** **3** / 3 pts

✔ **+ 3 pts** Other complete solutions.

Question 6

Rotation and projection **12** / 12 pts

6.1 **Basis of im(T) and basis of ker(T)** **4** / 4 pts

✔ **+ 2 pts** Correct basis of im(T)

✔ **+ 2 pts** Correct basis of ker(T)

6.2 **Standard matrix of T** **4** / 4 pts

✔ **+ 4 pts** Correct

6.3 **Find theta such that...** **4** / 4 pts

✔ **+ 4 pts** Correct

Question 7

Intersection and union of subspaces **8** / 8 pts

7.1 **Intersection of subspaces** **4** / 4 pts

✓ **+ 4 pts** Correct

7.2 **Union of subspaces** **4** / 4 pts

✓ **+ 4 pts** Correct

Question 8

Composite maps R^n to R^m **5** / 8 pts

8.1 **Prove not surjective** **4** / 4 pts

✓ **+ 4 pts** Correct

1 When did we prove this?

8.2 **Prove existence of S and T** **1** / 4 pts

✓ **+ 1 pt** There is some basic idea upon which a correct proof could be based.

3 What are S and T? (Or A and B?)

Question 9

Three vectors x, y, z **10** / 10 pts

9.1 **Prove {x, y, z} linearly independent** **6** / 6 pts

✓ **+ 1 pt** Wrote an arbitrary relation ax+by+cz=0

✓ **+ 1 pt** showed a=0

✓ **+ 1 pt** showed b=0 (or y is not a LC of x and z)

✓ **+ 1 pt** showed c=0

✓ **+ 1 pt** concluded that the set is LI for a valid reason

✓ **+ 1 pt** used definition of span correctly throughout

9.2 **Scalar multiples?** **4** / 4 pts

✓ **+ 1 pt** (i) yes

✓ **+ 1 pt** (i) reasoning

✓ **+ 1 pt** (ii) no

✓ **+ 1 pt** (ii) reasoning

Math 217 – Midterm 1

Winter 2024

Time: 120 mins.

1. There are exam problems on both sides of the paper.
2. Answer each question in the space provided; we have left some pages blank if you need more space, but please indicate when you do so.
3. Ask us if you need more paper.
4. Your solutions will be graded for clarity, precision, and the correct use of mathematical notation.
5. You must solve all problems using methods that have been taught in this course.
6. You are free to quote results from the worksheets, the textbook, or homework as a step in proving something else, but indicate clearly when you are doing so. **Exception:** if an entire problem is asking you to reprove a result from class or homework, we expect you to reproduce a proof.
7. **No calculators,** notes, or other outside assistance allowed.
8. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
9. Even if a problem states that no justification is necessary, you may provide explanations if you wish; this could result in partial credit for an incorrect final answer.

Student ID Number: 58848733 Section: 005

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	16	16	13	10	7	12	8	8	10	100
Score:										

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1. Complete each partial sentence into a precise definition for, or precise mathematical characterization of, the *italicized* term in each part:

   For full credit, please remember to include all appropriate quantifiers, and write out fully what you mean instead of using shorthand phrases such as "preserves" or "closed under."

   (a) (4 points) Let V and W be vector spaces. A function $f : V \to W$ is called a *linear transformation* if ...

   for all $\vec{v_1}, \vec{v_2} \in V$ and $k \in \mathbb{R}$

   $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$

   $T(k\vec{v_1}) = kT(\vec{v_1})$

   (b) (4 points) Let X and Y be sets. A function $g : X \to Y$ is called *injective* if ...

   every $x \in X$ is mapped to a unique $y \in Y$ by g.

   (i.e. $x_1 = x_2$ whenever $g(x_1) = g(x_2)$)

(Problem 1 continues on the next page...)

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(Continuation of Problem 1: Complete each partial sentence into a precise definition for, or precise mathematical characterization of, the *italicized* term in each part.)

(c) (4 points) Let $V \subseteq \mathbb{R}^n$ be a vector space. The *dimension* of V is ...

the number of vectors in any of its basis.

(d) (4 points) Let $T : V \to W$ be a linear transformation. The *rank* of T is ...

the dimension of it image.
(dim(im T))

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2. State whether each statement is True or False, and justify your answer with either a **short proof** or an **explicit counterexample**.

(a) (4 points) There exists a matrix $A \in \mathbb{R}^{5\times 9}$ such that $\dim \ker A = 3$. $5\left[\quad\right]^{9}$

False

Assume for contradiction that there exists $A \in \mathbb{R}^{5\times 9}$ such that $\dim \ker A = 3$

dimension of source ($\mathbb{R}^9$) = 9.

rank =

by rank - nullity theorem, $\dim(\operatorname{im} A) = 9 - \dim \ker A$

$= 6$

since rank $\leq$ number of rows $= 5$

contradicts.

(b) (4 points) If $T : V \to W$ is a linear transformation, and $U \subseteq W$ is a subspace, then $T^{-1}[U]$ is a subspace of V. (Recall that if $f : A \to B$ is any function and $S \subseteq B$, then we define $f^{-1}[S] = \{x \in A : f(x) \in S\}$.)

True. $T^{-1}[U] = \{v \in V \mid T(v) \in U\}$

Let v_1, v_2 be two arbitrary elements of $T^{-1}[U]$.

since U is a subspace of W, $0_W \in U$

since by definition, $T(0_V) = 0_W$, $0_V \in T^{-1}[U]$

② $T(v_1+v_2) \in U$ by definition of subspace so $v_1+v_2 \in T^{-1}[U]$

③ Let $k \in \mathbb{R}$ be arbitrary,

$T(kv_1) \in U$ by def of subspace

so $kv_1 \in T^{-1}[U]$

so $T^{-1}[U]$ is subspace of V.

(Problem 2 continues on the next page...)

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(Problem 2, True-False, Continued).

(c) (4 points) If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times n}$ satisfy $\ker(AB) = \ker(B) = \{\vec{0}\}$, then $\ker(A) = \{\vec{0}\}$.

True. $\ker(AB) = \{\vec{0}\} \iff T_{AB} : \mathbb{R}^n \to \mathbb{R}^m$ is injective

$\ker(B) = \{\vec{0}\} \iff T_B : \mathbb{R}^n \to \mathbb{R}^n$ is injective by theorem.

so for arbitrary element $a \in \mathbb{R}^n$

$T_{AB}(a) = T_A(T_B(a))$

(d) (4 points) Suppose that

$$c_1v_1 + \cdots + c_nv_n = 0$$

is a relation on a set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V. If the relation above is trivial, then $\{v_1, \ldots v_n\}$ is a linearly independent set.

False. counterexample : consider $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} \in \mathbb{R}^2$

$0\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\0\end{bmatrix} = 0$

but $1\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}1\\0\end{bmatrix}$ also $= 0$

since exists non-trivial relation

$\Rightarrow$ it is linearly dependent.

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3. Suppose A is a 3×5 matrix that has been transformed by a sequence of elementary row operations into the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & k^2-1 & k-1 \end{bmatrix},$$

where k is a constant. Assume that A is the augmented matrix of the linear system $\mathcal{S}$.

(a) (2 points) Find all values of k such that R is in reduced row echelon form.

since $k^2 - 1 = 0$ or 1,

$k = \pm 1$, $\pm\sqrt{2}$ are possible

when $\underline{k^2-1=0}$, need to check $\underline{k-1=0 \text{ or } 1}$,

$k = \pm 1$ $\qquad$ $k = 1, 2$

since when $k=-1$, $k-1 \neq 0$ or 1

it is only possible that $\boxed{k = 1, \pm\sqrt{2}}$

(b) (3 points) Find all values of k such that $\mathcal{S}$:

(i) has no solutions

$\dfrac{k^2-1 = 0}{k = \pm 1}$ and $\dfrac{k-1 \neq 0}{k \neq 1}$

So $k = -1$

(ii) has exactly one solution

not possible since there are 4 variables and 3 equations.

(iii) has a solution set that is a line

$k^2-1 = k-1 = 0$

$\Rightarrow k = 1$,

(Problem 3 continues on next page...)

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(This is the continuation of Problem 3. For your reference, A is a 3×5 matrix that has been transformed by a sequence of elementary row operations into the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & k^2-1 & k-1 \end{bmatrix},$$

where k is a constant. Assume that A is the augmented matrix of the linear system $\mathcal{S}$.)

(c) (4 points) Assuming $k = 2$, find the solution set of $\mathcal{S}$, expressed in parametric vector form.

$$k=2, \quad R = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 1 \end{bmatrix} \div 3$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

so $x_1 = 5 + 3x_3$

$x_2 = -1 - 2x_3$

$x_4 = \frac{1}{3}$

So solution set is $\left\{ \begin{bmatrix} 5 \\ -1 \\ 0 \\ \frac{1}{3} \end{bmatrix} + r \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$

(d) (4 points) Now suppose A is the standard matrix of a linear transformation T.

(i) If $T : \mathbb{R}^n \to \mathbb{R}^m$, find the values of n and m.

$n = 5, \quad m = 3$

(ii) Find $T(\vec{e}_3)$, assuming $T(\vec{e}_1) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

$T(\vec{e}_1)$ first

$T(\vec{e}_2)$ second

$T(\vec{e}_3)$ is the third column vector of A, by theorem.

We can set $T(\vec{e}_3)$ by elementary operations on R

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & k^2-1 & k-1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & k^2-1 & k-1 \\ 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \end{bmatrix} \begin{matrix} +5\times III \\ \times 2 \\ -II \end{matrix}$$

$$\rightarrow \begin{bmatrix} 0 & 5 & 10 & k^2-1 & k-6 \\ 2 & 0 & -6 & 0 & 10 \\ -1 & 1 & 5 & 0 & -6 \end{bmatrix}$$

so $T(\vec{e}_3) = \begin{bmatrix} 10 \\ -6 \\ 5 \end{bmatrix}$

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4. Consider the following functions:

- Let $f\colon \mathbb{R}^2 \to \mathcal{P}_3$ be defined by $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^3 + bx$.
- Let $g\colon \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $g(p) = p'$.
- Let $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ be defined by $h(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

(a) (2 points) Find $h \circ g \circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

$$h\circ g\circ f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = h\circ g(x^3) = h((x^3)') = h(3x^2)$$
$$= \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(b) (4 points) The composition $h \circ g \circ f$ is a linear transformation. (You do not need to verify this.) What is its standard matrix?

Let $\begin{bmatrix} a \\ b \end{bmatrix}$ be an arbitrary element of $\mathbb{R}^2$ $(a \in \mathbb{R},\ b \in \mathbb{R})$

$$h\circ g\circ f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = h(3ax^2 + b) = \begin{bmatrix} b \\ 3a+b \end{bmatrix}.$$

So the standard matrix is $\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$.

(c) (4 points) Is $h \circ g \circ f$ invertible? Justify your answer.

invertible. By theorem, $h\circ g\circ f$ is invertible if and only if its standard matrix is invertible.

So it suffices to show $\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ is invertible

since $\mathrm{rref}\left(\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it is full rank matrix

so by theorem, it is invertible

Therefore $h\circ g\circ f$ is invertible.

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5. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices with real entries. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let L be a function from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ defined by

$$L(X) = AX - XA \quad \text{for all} \quad X \in \mathbb{R}^{2\times 2}.$$

Note that L is a linear transformation (you do not need to verify this fact).

(a) (4 points) Find a basis of the kernel of L.

$\ker(L) = \left\{ \underbrace{\quad}_{X} \;\middle|\; \quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary 2×2 matrix

Assume $L(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow b = c,\ d = a$

so $\ker(L) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

So a basis of $\ker(L)$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

(b) (3 points) Find the dimension of the image of L.

By a) we have reached that for arbitrary 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $(a, b, c, d \in \mathbb{R})$

$$L(X) = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix}$$

$$= \begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \quad (x, y \in \mathbb{R})$$

Since a, b, c, d is arbitrary in $\mathbb{R}$, x, y is also arbitrary in $\mathbb{R}$

So $\operatorname{im}(L) = \left\{ \begin{bmatrix} x & y \\ -y & -x \end{bmatrix} \middle| x, y \in \mathbb{R} \right\} = \left\{ x\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| x, y \in \mathbb{R} \right\}$

so $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ spans $\operatorname{im}(L)$

Note that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is also linearly independent because if we assume a linear relation on it, by the 0 entries in both matrices, the relation can only be trivial

so it is a basis of $\operatorname{im}(L)$

so $\dim(\operatorname{im}(L)) = 2$ by definition

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6. Let $\theta \in \mathbb{R}$ be a fixed angle (measured in radians). Suppose $R : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation about the origin by θ, and $P : \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal projection onto the y-axis. Let $T = P \circ R$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

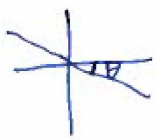
(a) (4 points) Find a basis of im(T) ~~and a basis of ker(T). Your answer may include the variable θ.~~

The standard matrix of R is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$,

the standard matrix of P is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

So the standard matrix of $T = P \circ R$ is $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix}$



$\text{im}(T) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in \mathbb{R} \right\}$

$= \left\{ y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| y \in \mathbb{R} \right\}$

So a basis of im(T) is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\ker(T) = \{$all vectors θ clockwisely from x-axis$\}$ $= \left\{ a \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \middle| a \in \mathbb{R} \right\}$, so a basis of $\ker(T)$ is $\left\{ \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \right\}$

(b) (4 points) Find the standard matrix of T. Your answer may include the variable θ.

$A = \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix}$ (in (a))

(c) (4 points) Find all angles θ in the interval $[0, 2\pi]$ such that T^2 is the zero map on $\mathbb{R}^2$.

for all $\vec{x} \in \mathbb{R}^2$, $T^2(\vec{x}) = A^2\vec{x} = \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$

$= \begin{bmatrix} 0 & 0 \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \vec{x}$

The question is to find all θ such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

that is, $\begin{cases} \sin\theta\cos\theta = 0 \Rightarrow \frac{1}{2}\sin 2\theta = 0 \Rightarrow \sin 2\theta = 0 \\ \cos^2\theta = 0 \Rightarrow \cos\theta = 0 \end{cases}$

so $\begin{cases} 2\theta \in \{k\pi \mid k \in \mathbb{Z}\} \\ \theta \in \{\frac{\pi}{2} + m\pi \mid m \in \mathbb{Z}\} \end{cases}$

So $\theta \in \{\frac{k\pi}{2} \mid k \in \mathbb{Z}\} \cap \{\frac{\pi}{2} + m\pi \mid m \in \mathbb{Z}\}$

Since $\{\frac{\pi}{2} + m\pi \mid m \in \mathbb{Z}\} \subseteq \{\frac{k\pi}{2} \mid k \in \mathbb{Z}\}$, $\theta \in \{\frac{\pi}{2} + m\pi \mid m \in \mathbb{Z}\}$. This is the set of all θ.

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7. Let U and V both be subspaces of the vector space W.

(a) (4 points) Prove or disprove: $U \cap V$ is a subspace of W.

Proof. Let w_1, w_2 be arbitrary elements in $U \cap V$
k be arbitrary scalar in $\mathbb{R}$.

So $w_1, w_2 \in U$ and $w_1, w_2 \in V$

so $w_1 + w_2 \in U$ since U is a subspace of W which means U is also a vector space

$w_1 + w_2 \in V$ since V is a subspace of W which means V is also a vector space.

Therefore $w_1 + w_2 \in U \cap V$ ①

Also since U, V are vector spaces,
$kw_1 \in U$, $kw_1 \in V$
so $kw_1 \in U \cap V$ ②

And also since U, V are vector spaces,
$0_W \in U$, $0_W \in V$
So $0_W \in U \cap V$ ③

By definition, $U \cap V$ is a vector space since ①, ②, ③.

(b) (4 points) Prove or disprove: $U \cup V$ is a subspace of W.

Disprove.

Counterexample: $W = \mathbb{R}^2$

$U = \{a\begin{bmatrix}1\\0\end{bmatrix} \mid a \in \mathbb{R}\}$,

$V = \{b\begin{bmatrix}0\\1\end{bmatrix} \mid b \in \mathbb{R}\}$ are two subspaces of W

$\begin{bmatrix}1\\0\end{bmatrix} \in U$, $\begin{bmatrix}0\\1\end{bmatrix} \in V$, but $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin U$ and also $\notin V$,

so $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in U \cup V$

So $\notin U \cap V$.

it violates the definition of a ~~vector~~ sub space.

So $U \cap V$ is not necessarily a subspace of W.

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8. (a) (4 points) Prove that for all linear transformations $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$, the composite map $T \circ S$ is not surjective.

Proof Let S, T be two arbitrary linear transformations such that $S : \mathbb{R}^5 \to \mathbb{R}^4$, $T : \mathbb{R}^4 \to \mathbb{R}^5$.

Assume for sake of contradiction that $T \circ S$ is surjective

by key theorem, S has some standard matrix $A \in \mathbb{R}^{4\times 5}$.

T has some standard matrix $B \in \mathbb{R}^{5\times 4}$.

so the standard matrix of $T \circ S$ is $B \cdot A \in \mathbb{R}^{5\times 5}$.

by rank-nullity theorem, $5 = \text{rank}(\text{im}(S)) + \text{nullity}(S)$

$= \text{rank}(A) + \text{nullity}(S)$

Similarly, $4 = \text{rank}(B) + \text{nullity}(T)$.

$5 = \text{nullity}(AB) + \text{rank}(AB)$

So $\text{rank}\, B \leq 4$

Since $\text{rank}(AB) \leq \text{rank}(B)$ by theorem

$\text{rank}\, AB \leq 4$, so $\text{im}(AB) \neq \mathbb{R}^5$

~~so nullity(AB) ≥ 1, dim(ker(AB))~~

so $T \circ S$ is not surjective.

(b) (4 points) Prove that for every linear transformation $U : \mathbb{R}^4 \to \mathbb{R}^4$, there exist linear transformations $S : \mathbb{R}^5 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^5$ such that $U = S \circ T$.

Proof

By key theorem, it suffices to prove that for any matrix $C \in \mathbb{R}^{4\times 4}$, there exist some matrix $A \in \mathbb{R}^{4\times 5}$, $B \in \mathbb{R}^{5\times 4}$ such that $A \cdot B = C$.

Since $C = \begin{bmatrix} U(e_1) & U(e_2) & U(e_3) & U(e_4) \end{bmatrix}$

$B = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4) \end{bmatrix}$

Consider

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9. Consider three vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ in a vector space V.

(a) (6 points) Prove that if $\vec{z} \neq \vec{0}$, $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$, and $\vec{x}+\vec{y} \notin \operatorname{Span}(\vec{x}, \vec{z})$, then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

~~We prove it by proving its contrapositive.~~

~~If $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent, then at least one of the three conditions are true~~

~~Assume $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent. Then there exist a non-trivial relation on $\{\vec{x}, \vec{y}, \vec{z}\}$. So $a\vec{x}+b\vec{y}+c\vec{z}=\vec{0}$ for some $a, b, c \in \mathbb{R}$ such that at least one of them are not 0.~~

~~Case 1: if $a=0$ Then $b\vec{y}+c\vec{z}=0$~~

~~We prove it by contradiction.~~

Assume the three conditions. Let $a\vec{x}+b\vec{y}+c\vec{z}=0$ be a relation on the set.

if $a \neq 0$, then $x = -\frac{b}{a}\vec{y} - \frac{c}{a}\vec{z}$ contradicts with $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$

so $a=0$ ①

if $b \neq 0$, then $\vec{y} = -\frac{a}{b}\vec{x} - \frac{c}{b}\vec{z}$, $x+\vec{y} = (-\frac{a}{b}+1)\vec{x} - \frac{c}{b}\vec{z} \in \operatorname{Span}(\vec{x}, \vec{z})$, contradicts so $b=0$ ②

so $c\vec{z}=\vec{0}$. if $c \neq 0$, then z can only be 0, contradicting with $z \neq 0$ so $c=0$ ③

Therefore by ①②③ $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent since every relation on it is trivial.

(b) (4 points) Suppose instead that $\vec{y} \in \operatorname{Span}(\vec{x}, \vec{z})$ but $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$. Fully justify your answers to the questions below.

(i) Must $\vec{y}$ be a scalar multiple of $\vec{z}$?

Yes. there exist $a, b \in \mathbb{R}$ such that $y = a\vec{x} + b\vec{z}$

Assume $\vec{y}$ is not a scalar multiple of $\vec{z}$ then $a \neq 0$,

so $\vec{x} = \frac{\vec{y}-b\vec{z}}{a} = \frac{1}{a}\vec{y} - \frac{b}{a}\vec{z} \in \operatorname{Span}(\vec{y}, \vec{z})$, contradicts. So $\vec{y}$ must be a scalar multiple of $\vec{z}$.

(ii) Must $\vec{z}$ be a scalar multiple of $\vec{y}$?

No.

~~Assume $\vec{z}$ is a scalar multiple of $\vec{y}$. then $\vec{z} = k\vec{y}$ for some $k \in \mathbb{R}$ since $y \in \operatorname{Span}(\vec{x}, \vec{z})$, $y = a\vec{x}+b\vec{z}$ for some $a, b \in \mathbb{R}$ so $a\vec{x} = y - kb\vec{y}$~~

Consider $\vec{y}=\vec{0}$ and $(\vec{x}, \vec{z})$ are linearly independent.

Explicit example: $y=\begin{bmatrix}0\\0\end{bmatrix}$, $x=\begin{bmatrix}1\\0\end{bmatrix}$, $z=\begin{bmatrix}0\\1\end{bmatrix}$.

it satisfies that $y \in \operatorname{Span}(\vec{x}, \vec{z})$ and $\vec{x} \notin \operatorname{Span}(\vec{y}, \vec{z})$ but $\vec{z}$ is not a scalar multiple of $\vec{y}$.

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$5 = \text{rank } A + \text{nullity } A$

$4 = \text{rank } B + \text{nullity } B$

$\text{rank } AB \leq \text{rank } B$.

$5 = \text{rank} AB$

$a\vec{x}^2 + b\vec{x} + c\vec{a} = 0$ $x + y + y$

$a \neq 0$

$x =$ ___

$b \neq 0$

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