This is the Diagonalization Gateway test. Passing on this test is ALL FIVE of the five problems on the test.

Problem 1. **(1 point)**

Identify the diagonalizability of each of the following matrices.

1. $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$
   A is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

2. $B = \begin{bmatrix} 6 & -1 \\ 2 & 4 \end{bmatrix}$
   B is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

3. $C = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$
   C is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

Solution: For each matrix, we determine diagonalizability from the existence of a complete eigenspace. For A, we see that eigenvalues are given by

$$(-\lambda)(4-\lambda) - 2\cdot 0 = \lambda^2 - 6\lambda + 8 = 0.$$

Completing the square, we have

$$(\lambda-3)^2 - 1 = 0,$$

so that there are two distinct real eigenvalues, so that A is Diagonalizable over real and complex numbers.

Similarly, for B, we have

$$(\lambda-5)^2 + 1 = 0,$$

so that there are two complex conjugate eigenvalues, so that B is Diagonalizable over complex numbers but not reals.

Similarly, for C, we have

$$(\lambda-3)^2 + 3 = 0,$$

so that there are two complex conjugate eigenvalues, so that C is Diagonalizable over complex numbers but not reals.

Answer(s) submitted:

- Diagonalizable over real and complex numbers
- Diagonalizable over real and complex numbers
- Diagonalizable over real and complex numbers

submitted: (incorrect)
recorded: (incorrect)

Correct Answers:

- Diagonalizable over real and complex numbers
- Diagonalizable over complex numbers but not reals
- Diagonalizable over complex numbers but not reals

Problem 2. **(1 point)**

The matrix $A = \begin{bmatrix} 7 & 48 \\ -1 & -9 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues of A satisfy

$$\begin{aligned} \det(\begin{bmatrix} 7-\lambda & 48 \\ -1 & -9-\lambda \end{bmatrix}) &= (7-\lambda)(-9-\lambda) - (48)(-1) \\ &= \lambda^2 + 2\lambda - 15 \\ &= (\lambda-3)(\lambda+5) = 0. \end{aligned}$$

Thus $\lambda = 3$ or $\lambda = -5$.

If $\lambda = 3$, the eigenvector satisfies

$$\begin{bmatrix} 4 & 48 \\ -1 & -12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 48 \\ -4 \end{bmatrix}$.

Similarly, if $\lambda = -5$, the eigenvector satisfies

$$\begin{bmatrix} 12 & 48 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 48 \\ -12 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 48 & 48 \\ -4 & -12 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$$

Answer(s) submitted:

- --; --

submitted: (incorrect)
recorded: (incorrect)

Correct Answers:

- $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}; \begin{bmatrix} 48 & 48 \\ -4 & -12 \end{bmatrix}$

Problem 3. **(1 point)**

The matrix $A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues satisfy

$$\begin{aligned} \det(\begin{bmatrix} 4-\lambda & -1 \\ 3 & 0-\lambda \end{bmatrix}) &= (4-\lambda)(0-\lambda) - (-1)(3) \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda-3)(\lambda-1) = 0. \end{aligned}$$

Thus $\lambda = 3$ or $\lambda = 1$.

If $\lambda = 3$, the eigenvector satisfies

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly, if $\lambda = 1$, the eigenvector satisfies

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer(s) submitted:

- --; --

submitted: (incorrect)
recorded: (incorrect)

Correct Answers:

- $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

Problem 4. **(1 point)**

Identify the diagonalizability of each of the following matrices.

1. $A = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}$
   A is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

2. $B = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$
   B is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

3. $C = \begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix}$
   C is

• ?
• Diagonalizable over real and complex numbers
• Diagonalizable over complex numbers but not reals
• Not diagonalizable over either real or complex numbers
• Diagonalizable over real numbers but not complex numbers

Solution: For each matrix, we determine diagonalizability from the existence of a complete eigenspace. For A, we see that eigenvalues are given by

$$(-\lambda)(1-\lambda) - 0 \cdot 7 = \lambda^2 - 2\lambda + 1 = 0.$$

Completing the square, we have

$$(\lambda - 1)^2 + 0 = 0,$$

so that there are two repeated real eigenvalues, and because the matrix (after subtracting λI) will have nonzero entries and so cannot have two eigenvectors, so that A is Not diagonalizable over either real or complex numbers.

Similarly, for B, we have

$$(\lambda - 3)^2 + 0 = 0,$$

so that there are two repeated real eigenvalues, and because the matrix (after subtracting λI) will have nonzero entries and so cannot have two eigenvectors, so that B is Not diagonalizable over either real or complex numbers.

Similarly, for C, we have

$$(\lambda + 2)^2 + 0 = 0,$$

so that there are two repeated real eigenvalues, and because the matrix (after subtracting λI) will have nonzero entries and so cannot have two eigenvectors, so that C is Not diagonalizable over either real or complex numbers.

Answer(s) submitted:

- Not diagonalizable over either real or complex numbers
- Not diagonalizable over either real or complex numbers
- Not diagonalizable over either real or complex numbers

submitted: (correct)
recorded: (correct)
Correct Answers:

- Not diagonalizable over either real or complex numbers
- Not diagonalizable over either real or complex numbers
- Not diagonalizable over either real or complex numbers

Problem 5. **(1 point)**

The matrix $A = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. Find a diagonal matrix D and an invertible matrix S with real or complex entries such that $D = S^{-1}AS$.

$D = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

$S = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$

Solution: We know that $S^{-1}AS = D$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (and $\lambda_{1,2}$ are the eigenvalues of the matrix A) and $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ (with $\mathbf{v}_{1,2}$ being the corresponding basis for the eigenspace).

Eigenvalues satisfy

$$\begin{aligned} \det\left(\begin{bmatrix} 0-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix}\right) &= (0-\lambda)(-1-\lambda)-(2)(3) \\ &= \lambda^2+1\lambda-6 \\ &= (\lambda+3)(\lambda-2) = 0. \end{aligned}$$

Thus $\lambda = -3$ or $\lambda = 2$.

If $\lambda = -3$, the eigenvector satisfies

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Similarly, if $\lambda = 2$, the eigenvector satisfies

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Thus our diagonalization is $S^{-1}AS = D$, with

$$S = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

Answer(s) submitted:

- $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$; $\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$

submitted: (correct)
recorded: (correct)

Correct Answers:

- $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$; $\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$