Midterm 2

Graded

Student

Qiulin Fan

Total Points

82 / 100 pts

Question 1

Definitions **16** / 16 pts

1.1 **B-coordinate vector** **4** / 4 pts

- ✓ **+ 4 pts** Correct

1.2 **Orthonormal basis** **4** / 4 pts

- ✓ **+ 4 pts** Correct

1.3 **Least-squares solution** **4** / 4 pts

- ✓ **+ 4 pts** Correct

1.4 **Isomorphic vector spaces** **4** / 4 pts

- ✓ **+ 1 pt** T:V->W
- ✓ **+ 1 pt** T linear
- ✓ **+ 1 pt** T injective
- ✓ **+ 1 pt** T surjective

Question 2

True / False **10** / 16 pts

2.1 **ker(A) = ker(AA^\top A)** **2** / 4 pts

- ✔ **+ 1 pt** $\ker(A) \subseteq \ker\left(AA^\top A\right)$
- ✔ **+ 1 pt** Use $\vec{v} \cdot \vec{w} = \vec{v}^\top \vec{w}$, $\ker\left(B^\top\right) = (\mathrm{im}(B))^\perp$, or $\ker\left(B^\top B\right) = \ker(B)$ for $\supseteq$

2.2 **im(A) orthogonal to ker(A)** **1** / 4 pts

- ✔ **+ 1 pt** Stated it is false, but didn't show enough justification/had significant errors, or presented a wrong counterexample

2.3 **If x = 0 is a least-squares solution...** **4** / 4 pts

- ✔ **+ 1 pt** Correctly identifies as true
- ✔ **+ 3 pts** Valid justification

2.4 **Orthogonality-preserving LT** **3** / 4 pts

- ✔ **+ 3 pts** Correct truth value and counterexample, but correctness of the counterexample requires a bit more justification

1 Why? Can you show this?

2 What if the rotation is by a multiple of 2π and the scaling is by ± 1? (Try to use concrete, explicit counterexamples.)

Question 3

Transformation on R^{2x2} **8** / 8 pts

3.1 **Find [T]_B** **4** / 4 pts

- ✔ **+ 4 pts** Correct

3.2 **Find ordered basis such that....** **4** / 4 pts

- ✔ **+ 4 pts** Correct

Question 4

Hyperplane in R^4 — **5** / 12 pts

4.1 **Find B-matrix of reflection** — **3** / 4 pts

- ✔ **+ 3 pts** Correct answer, but without justifying it by mentioning that v_1 is orthogonal to W.

4.2 **Find basis such that projection is...** — **2** / 4 pts

- ✔ **+ 2 pts** First two vectors correct, but third is either not in V or not orthogonal to Z (or both)

4.3 **Find matrix such that...** — **0** / 4 pts

- ✔ **+ 0 pts** Incorrect or blank

Question 5

Space of functions — **14** / 18 pts

5.1 **Fnd [T]_B** — **3** / 3 pts

- ✔ **+ 3 pts** All 3 columns correct: $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

5.2 **Find a function such that...** — **1** / 3 pts

- ✔ **+ 1 pt** Correct problem setup, incorrect or unfinished solution

5.3 **Verify inner product** — **4** / 4 pts

- ✔ **+ 1 pt** Correctly verified symmetry
- ✔ **+ 1 pt** Correctly verified linearity (both addition and scalar multiplication)
- ✔ **+ 2 pts** Correctly verified positive definiteness

5.4 **Find orthonormal basis** — **4** / 4 pts

- ✔ **+ 1 pt** Correctly calculates $\|xe^x\| = \sqrt{\frac{2}{3}}$
- ✔ **+ 1 pt** Correctly calculates $\langle xe^x, x^2e^x \rangle = 0$
- ✔ **+ 1 pt** Correctly calculates $\|x^2e^x\| = \sqrt{\frac{2}{5}}$
- ✔ **+ 1 pt** Divides given basis vectors by their calculated magnitudes (even if incorrect)

5.5 **Find orthogonal projection onto W** — **2** / 4 pts

- ✔ **+ 1 pt** Some presence of a correct formula or method for projecting onto an orthonormal basis
- ✔ **+ 1 pt** Calculates $\langle x^3e^x, x^2e^x \rangle = 0$ (or equivalent calculation)

Question 6

Fill-in Problem **9** / 10 pts

6.1 (a) **1** / 1 pt

✓ **+ 1 pt** Correct

6.2 (b) **1** / 1 pt

✓ **+ 1 pt** Correct

6.3 (c) **1** / 1 pt

✓ **+ 1 pt** Correct

6.4 (d) **1** / 1 pt

✓ **+ 1 pt** Correct

6.5 (e) **1** / 1 pt

✓ **+ 1 pt** Correct

6.6 (f) **1** / 1 pt

✓ **+ 1 pt** Correct

6.7 (g) **1** / 1 pt

✓ **+ 1 pt** Correct

6.8 (h) **1** / 1 pt

✓ **+ 1 pt** Correct

6.9 (i) **0** / 1 pt

✓ **+ 0 pts** Incorrect

6.10 (j) **1** / 1 pt

✓ **+ 1 pt** Correct

Question 7

Transformation with two ordered bases **10** / 10 pts

7.1 **Prove T is an isomorphism** **5** / 5 pts

- ✔ **+ 1 pt** Tried to show that T is bijective, injective or surjective using a correct definition.
- ✔ **+ 1 pt** Made correct assumption to show injectivity or surjectivity
- ✔ **+ 1 pt** Applied linearity of T to an appropriate vector in V to show injectivity or surjectivity
- ✔ **+ 1 pt** Mentioned that showing injective or surjective is enough here by rank nullity or by worksheet problem, OR gave a correct or partially correct proof of bijectivity.
- ✔ **+ 1 pt** Concluded bijectivity, injectivity or surjectivity based on a correct definition

7.2 **Prove [T]_B = [T]_C** **5** / 5 pts

- ✔ **+ 1 pt** Wrote generalized key thm or equivalent correctly
- ✔ **+ 1 pt** Did some work toward finding [T]_B or [T]_C
- ✔ **+ 1 pt** wrote c_i as a LC of b_i or vice versa, or wrote correct expression for c.o.b. matrix
- ✔ **+ 1 pt** Correctly related a B vector to a C vector
- ✔ **+ 1 pt** Completed the problem

Question 8

Symmetric matrix in different bases **10** / 10 pts

8.1 **Prove: symmetric in standard basis if and only if symmetric in all orthonormal bases** **5** / 5 pts

- ✓ **+ 1 pt** $[T]_{\mathcal{B}} = S_{\mathcal{E}\to\mathcal{B}} A S_{\mathcal{B}\to\mathcal{E}}$
- ✓ **+ 1 pt** $[T]_{\mathcal{B}}^{\top} = S_{\mathcal{B}\to\mathcal{E}}^{\top} A^{\top} S_{\mathcal{E}\to\mathcal{B}}^{\top}$
- ✓ **+ 1 pt** $S_{\mathcal{B}\to\mathcal{E}}$ is orthogonal
- ✓ **+ 1 pt** $S_{\mathcal{B}\to\mathcal{E}}^{\top} = S_{\mathcal{E}\to\mathcal{B}}$
- ✓ **+ 1 pt** $\Leftarrow$

8.2 **Show false if "orthonormal" not included in hypotheses** **5** / 5 pts

- ✓ **+ 1 pt** $\mathcal{B}$ is a basis of $\mathbb{R}^n$
- ✓ **+ 1 pt** $\mathcal{B}$ is not orthonormal
- ✓ **+ 1 pt** A and $\mathcal{B}$
- ✓ **+ 1 pt** $[T]_{\mathcal{B}}$ and $\mathcal{B}$
- ✓ **+ 1 pt** Correct

Math 217 – Midterm 2

Winter 2024

Time: 120 mins.

1. There are exam problems on both sides of the paper.
2. Answer each question in the space provided; we have left some pages blank if you need more space, but please indicate when you do so.
3. Ask us if you need more paper.
4. Your solutions will be graded for clarity, precision, and the correct use of mathematical notation.
5. You must solve all problems using methods that have been taught in this course.
6. You are free to quote results from the worksheets, the textbook, or homework as a step in proving something else, but indicate clearly when you are doing so. **Exception:** if an entire problem is asking you to reprove a result from class or homework, we expect you to reproduce a proof.
7. **No calculators,** notes, or other outside assistance allowed.
8. Remember to show all your work and justify all your answers, unless the problem explicitly states that no justification is necessary.
9. Even if a problem states that no justification is necessary, you may provide explanations if you wish; this could result in partial credit for an incorrect final answer.

Student ID Number: 58848133 Section: 005

Qinlin Fan

Question:	1	2	3	4	5	6	7	8	Total
Points:	16	16	8	12	18	10	10	10	100
Score:									

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1. Complete each partial sentence into a precise definition for, or precise mathematical characterization of, the *italicized* term in each part:

For full credit, please write out fully what you mean instead of using shorthand phrases such as "preserves" or "closed under."

(a) (4 points) If $\mathcal{B} = (b_1, b_2, \ldots, b_k)$ is an ordered basis for a vector space V, and $v \in V$ is any vector, then the *$\mathcal{B}$-coordinate vector* for v (denoted $[v]_\mathcal{B}$) is ...

the unique coefficients $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ that $v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$

(b) (4 points) If V is a vector space with an inner product $\langle -, - \rangle$, then an *orthonormal basis* for V is ...

a subset $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ of V such that for all $v_i, v_j \in \mathcal{B}$,

$\langle v_i, v_j \rangle = \delta_{ij}$

$\left(= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \right)$

(c) (4 points) Let A be an $n \times m$ matrix, and let $\vec{v}$ be a vector in $\mathbb{R}^n$. The vector $\vec{x}^*$ is a *least-squares solution* to a linear system $A\vec{x} = \vec{b}$ if ...

$\vec{x}$ is the solution to the linear system $A\vec{x} = \text{proj}_{\text{im}A} \vec{b}$

(d) (4 points) Vector spaces V and W are called *isomorphic* (denoted $V \cong W$) if ...

there exists an isomorphism (bijective linear transformation) from V to W.

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2. State whether each statement is True or False, and justify your answer with either a **short proof** or an **explicit counterexample**.

(a) (4 points) If $A \in \mathbb{R}^{m\times n}$, then $\ker(A) = \ker(AA^\top A)$.

False

we know : $\ker(A) = \ker(A^TA)$

if $\vec{x} \in \ker A$, then $A^TA\vec{x} = \vec{0}$

So $AA^TA\vec{x} = A\vec{0} = \vec{0}$

if $\vec{x} \in \ker(AA^TA)$, then $AA^TA\vec{x} = 0$

$\Rightarrow A^TA\vec{x} \in \ker A$

$\Rightarrow A^TA\vec{x} \in \ker A^TA$

(b) (4 points) For any square matrix A, the vectors in $\mathrm{im}(A)$ are orthogonal to the vectors in $\ker(A)$.

False

let $\vec{v} \in \mathrm{im}(A)$

So $\vec{v} = A\vec{x}$ for some $\vec{x}$

$\vec{w} \in \ker(A)$, so $A\vec{w} = 0$

Problem 2 continues on the next page.

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(Problem 2, True-False, Continued).

(c) (4 points) If $\vec{x} = \vec{0}$ is a least-squares solution of the linear system $A\vec{x} = \vec{b}$, then $\vec{b} \in (\operatorname{im} A)^{\perp}$.

True

$$\underbrace{A^TA\vec{0}}_{=\vec{0}} = A^T\vec{b}$$

$$\Rightarrow \vec{b} \in \ker(A^T)$$

$$\Rightarrow \vec{b} \in (\operatorname{im} A)^{\perp}$$

(d) (4 points) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that preserves orthogonality (that is, such that $\vec{v} \perp \vec{w} \implies T(\vec{v}) \perp T(\vec{w})$) then T is an orthogonal transformation.

False

Consider rotation with scaling. It preserves orthogonality but it does not preserve length, which by Thm implies is not orthogonal transformation

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3. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & 0 \\ 2b & 3c \end{bmatrix}$$

$\begin{matrix} 0 & 0 \\ 0 & 3 \end{matrix}$

and let

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right)$$

be an ordered basis for $\mathbb{R}^{2\times 2}$. You do not need to show this is a basis.

(a) (4 points) Find $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{B}} & \cdots & [T(b_4)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

$$= \left[\left[\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}\right]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 3 \\ -3 & -3 & 0 & 0 \\ -3 & -3 & 0 & 0 \end{bmatrix}$$

(b) (4 points) Find an ordered basis $\mathcal{A}$ for $\mathbb{R}^{2\times 2}$ such that $[T]_{\mathcal{A}} = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

let $A = \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \cdots \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix}\right)$

So $[T]_A = \begin{bmatrix} | & & | \\ \left[\begin{bmatrix} d_1 & 0 \\ 2b_1 & 3c_1 \end{bmatrix}\right]_A & \cdots & \left[\begin{bmatrix} d_4 & 0 \\ 2b_4 & 3c_4 \end{bmatrix}\right]_A \\ | & & | \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} d_1 & 0 \\ 2b_1 & 3c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow b_1 = c_1 = d_1 = 0$

$\begin{bmatrix} d_2 & 0 \\ 2b_2 & 3c_2 \end{bmatrix} = \begin{bmatrix} 6a_1 & 6b_1 \\ 6c_1 & 6d_1 \end{bmatrix} \Rightarrow b_2 = c_2 = 0, \ d_2 = 6a_1$

$\begin{bmatrix} d_3 & 0 \\ 2b_3 & 3c_3 \end{bmatrix} = \begin{bmatrix} 6a_2 & 6b_2 \\ 6c_2 & 6d_2 \end{bmatrix} \Rightarrow b_3 = 0, \ c_3 = 2d_2 = 12a_1, \ d_3 = 6a_2$

$\begin{bmatrix} d_4 & 0 \\ 2b_4 & 3c_4 \end{bmatrix} = \begin{bmatrix} 6a_3 & 6b_3 \\ 6c_3 & 6d_3 \end{bmatrix} \Rightarrow d_4 = 6a_3, \ b_4 = 3c_3 = 36a_1$

So consider

$A = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 12 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 36 \\ 0 & 0 \end{bmatrix}\right)$

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4. Let $\vec{v}_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, and let $V \subseteq \mathbb{R}^4$ be the subspace defined by $V = \{\vec{x} \in \mathbb{R}^4 : \vec{x} \cdot \vec{v}_0 = 0\}$.

Further, let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$, so that $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an ordered basis of V. (You do not need to verify that this is a basis.)

(a) (4 points) Let $W = \operatorname{Span}(\vec{v}_2, \vec{v}_3)$. Find the $\mathcal{B}$-matrix of the linear map $\operatorname{Ref}_W : V \to V$ that reflects each vector in V through the plane W in V.

~~$\operatorname{Ref}_W\ V \to V$ is represented by~~

By Ref_W, each vector $\vec{v}$ in V with $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

is mapped to $\vec{v}'$ with $[\vec{v}']_{\mathcal{B}} = \begin{bmatrix} -a \\ b \\ c \end{bmatrix}$

since the $\vec{v_2}, \vec{v_3}$ coordinate is preserved while $\vec{v_1}$ coordinate is flipped

So the $\mathcal{B}$-matrix of Ref_W is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) (4 points) Let $Z = \operatorname{Span}(\vec{v}_1, \vec{v}_2)$, and let $P_Z : V \to V$ be the linear map that projects each vector in V orthogonally onto the subspace Z of V. Find an ordered basis $\mathcal{C}$ of V such that

$$[P_Z]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

let $C = (\vec{C_1}, \vec{C_2}, \vec{C_3})$

$$[P_Z]_C = \begin{bmatrix} | & | & | \\ [P_Z(\vec{C_1})]_C & [P_Z(\vec{C_2})]_C & [P_Z(\vec{C_3})]_C \\ | & | & | \end{bmatrix}$$

So $P_Z(\vec{C_1}) = C_1$, $P_Z(\vec{C_2}) = C_2$, $P_Z(\vec{C_3}) = \vec{0}$

So $\vec{C_1}, \vec{C_2} \in Z$, $\vec{C_3} \in Z^\perp$

Consider $\vec{C_1} = \vec{U_1}$, $\vec{C_2} = \vec{U_2}$, and we apply Gram-Schmidt to $\vec{v_1}, \vec{v_2}, \vec{v_3}$ to find $\vec{C_3}$

Since $\vec{v_1} \perp \vec{v_2}$, we only need to scale:

$\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{\vec{v_1}}{\sqrt{3}}$

$\vec{u_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{\vec{v_2}}{\sqrt{3}}$

$\vec{C_3} = \vec{U_3} = \frac{v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2}{\|v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2\|}$

$= \frac{v_3 - \sqrt{3}u_2}{\|v_3 - \sqrt{3}u_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

So $C = \left(\vec{v_1}, \vec{v_2}, \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}\right)$

(Problem 4 continues on the next page ...)

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Problem 4 continues below. For your convenience, part of the problem is reproduced here: $\vec{v}_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $V \subseteq \mathbb{R}^4$ is the subspace defined by $V = \{\vec{x} \in \mathbb{R}^4 : \vec{x} \cdot \vec{v}_0 = 0\}$, and $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an ordered basis for V, with $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$.

(c) (4 points) Find a 4×4 matrix A such that $\ker(A) = V$ and $\operatorname{im}(A) = V^{\perp}$. Your answer should be an explicit matrix with numerical entries.

let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, $\vec{x} \cdot \vec{v}_0 = \vec{0} \Rightarrow a - b + d = 0 \Rightarrow V = \left\{ \begin{bmatrix} b-d \\ b \\ c \\ d \end{bmatrix} \middle| b, c, d \in \mathbb{R} \right\}$

5. Let $\mathcal{B} = (e^x, xe^x, x^2e^x)$ be an ordered basis for a subspace $V \subseteq C^{\infty}[-1, 1]$, and let $T : V \to V$ be the linear transformation defined by $T(f) = f' - 2f$.

(a) (3 points) Find $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [T(b_1)]_{\mathcal{B}} & [T(b_2)]_{\mathcal{B}} & [T(b_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ [e^x - 2e^x]_{\mathcal{B}} & [e^x - xe^x]_{\mathcal{B}} & [2xe^x - x^2e^x \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) (3 points) Using matrix methods, find a function $f \in V$ such that $f'(x) - 2f(x) = (2x^2 - 5x - 2)e^x$.

this is : $[T(f)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}$

$\Rightarrow [T]_{\mathcal{B}} [f]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} [f]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix} \Rightarrow [f]_{\mathcal{B}} = \begin{bmatrix} - \\ - \\ -2 \end{bmatrix}$

(Problem 5 continues on the next page ...)

let $[f]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{bmatrix} -a + b \\ -b + 2c \\ -c \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$ $\Rightarrow f = -7e^x - 9xe^x - 2 e$

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(This is a continuation of Problem 5 from the previous page. For reference, $V = \text{span}(e^x, xe^x, x^2e^x) \subseteq C^\infty[-1,1]$.)

(c) (4 points) Let $\langle f,g\rangle = \displaystyle\int_{-1}^{1} e^{-2x} f(x)g(x)\,dx$. Verify this is an inner product on $C^\infty[-1,1]$.

① Symmetry: $\langle f,g\rangle = \int_{-1}^{1} e^{-2x} f(x)g(x)\,dx = \int_{-1}^{1} e^{-2x} g(x)f(x)\,dx = \langle g,f\rangle$

② Linearity in the first argument, $\forall a,b\in\mathbb{R}$, $f(x), g(x), h(x) \in C^\infty[-1,1]$

$\langle af+bh, g\rangle = \int_{-1}^{1} e^{-2x}(af+bh)g\,dx = a\int_{-1}^{1} e^{-2x} fg\,dx + b\int_{-1}^{1} e^{-2x} hg\,dx = a\langle f,g\rangle + b\langle f,g\rangle$

③ Linearity in the second argument is guaranteed by ① + ②

④ Positive define: $\forall f\in C^\infty[-1,1]$, $\langle f,f\rangle = \int_{-1}^{1} e^{-2x}(f(x))^2\,dx$. Since $e^{-2x}>0$, $f(x)^2\geq 0$, $\langle f,f\rangle \geq 0$, where (=) holds only when $f(x)$ is the 0 vector ($f(x)=0$)

So $\langle f,g\rangle$ is an inner product on $C^\infty[-1,1]$

(d) (4 points) Find an orthonormal basis for $W = \text{span}(xe^x, x^2e^x)$ with respect to the inner product defined above.

$\vec{u}_1 = \dfrac{xe^x}{\|xe^x\|} = \dfrac{xe^x}{\sqrt{\int_{-1}^{1} e^{-2x}\cdot x^2 e^{2x}\,dx}} = \dfrac{xe^x}{\sqrt{\frac{x^3}{3}\Big|_{-1}^{1}}} = \sqrt{\frac{3}{2}}\,xe^x$

$\vec{u}_2 = \dfrac{x^2e^x - \langle x^2e^x, \vec{u}_1\rangle \vec{u}_1}{\|x^2e^x - \langle x^2e^x, \vec{u}_1\rangle\vec{u}_1\|}$, where $\langle x^2e^x, \vec{u}_1\rangle = \int_{-1}^{1} e^{-2x}x^2e^x\cdot\sqrt{\frac{3}{2}}\,xe^x\,dx = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^3 = \sqrt{\frac{3}{2}}\cdot\left[\frac{x^4}{4}\right]_{-1}^{1} = 0$

So $\vec{u}_2 = \dfrac{x^2e^x}{\|x^2e^x\|} = \dfrac{x^2e^x}{\sqrt{\int_{-1}^{1} x^4\,dx}} = \sqrt{\frac{5}{2}}\,x^2e^x$

$\Rightarrow \left(\sqrt{\frac{3}{2}}xe^x, \sqrt{\frac{5}{2}}x^2e^x\right)$ is an orthonormal basis of W

(e) (4 points) Find the orthogonal projection of $h(x) = x^3e^x$ into W.

$\text{proj}_W h(x) = \langle h(x), \vec{u}_1\rangle \vec{u}_1 + \langle h(x), \vec{u}_2\rangle \vec{u}_2$

$= \left(\int_{-1}^{1} (x^3e^x)\left(\sqrt{\frac{3}{2}}xe^x\right)dx\right)\vec{u}_1 + \left(\int_{-1}^{1} (x^3e^x)\left(\sqrt{\frac{5}{2}}x^2e^x\right)\right)\vec{u}_2$

$= \left(\sqrt{\frac{3}{2}}\int_{-1}^{1} x^4e^{2x}\,dx\right)\sqrt{\frac{3}{2}}xe^x$

$+\underline{\left(\sqrt{\frac{5}{2}}\int_{-1}^{1} x^5e^{2x}\,dx\right)\sqrt{\frac{5}{2}}x^2e^x}$ $\Rightarrow 0$ since it's odd function in $[-1,1]$

$= \underline{\frac{3}{2}xe^x\int_{-1}^{1} x^4e^{2x}\,dx}$

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6. (10 points) Throughout this problem, let $m, n \in \mathbb{N}$ with $m > n$, let $(\vec{v}_1, \dots, \vec{v}_n)$ be a linearly independent list of vectors in $\mathbb{R}^m$, and let A be the (non-square) $m \times n$ matrix whose columns are $\vec{v}_1, \dots, \vec{v}_n$. Also let $A = QR$ be the QR-factorization of A, where Q has columns $\vec{u}_1, \dots, \vec{u}_n$, so

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}}_{Q} R. \quad m \times n$$

In each part below, determine whether the given statement *must be True* or *must be False*, or if there is *Not Enough Information to tell*, based on the given information. Indicate your answers by writing "T", "F", or "NEI" in the blank.

(a) NEI $A^\top A = I_n$.

(b) F $AA^\top = I_m$.

(c) T $Q^\top Q = I_n$.

(d) F $QQ^\top = I_m$.

(e) NEI $AA^\top = [\text{proj}_{\text{im}\,A}]_{\mathcal{E}}$.

(f) T $QQ^\top = [\text{proj}_{\text{im}\,A}]_{\mathcal{E}}$.

(g) T $\text{im}\,A = \text{im}\,Q$.

(h) T $\ker A = \ker Q$.

(i) T Q is orthogonal.

(j) NEI R is orthogonal.

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7. Let $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_n)$ be ordered bases of the vector space V, and suppose $T : V \to V$ is a linear transformation such that $T(\vec{b}_i) = \vec{c}_i$ for each $1 \leq i \leq n$.

   (a) (5 points) Prove that T is an isomorphism.

Pf: Claim ①: T is a linear transformation.

Proof: Fix $\vec{v_1}, \vec{v_2} \in V$.

So $v_1 = k_1\vec{b_1} + \ldots + k_n\vec{b_n}$
$v_2 = q_1\vec{b_1} + \ldots + q_n\vec{b_n}$ for some scalars $k_1 \ldots k_n$ and $q_1 \ldots q_n$ by definition of basis.

So $T(\vec{v_1}+\vec{v_2}) = T((k_1+q_1)b_1 + \ldots + (k_n+q_n)b_n)$
$= T((k_1+q_1)b_1) + \ldots + T((k_n+q_n)b_n)$
$= (k_1+q_1)T(b_1) + \ldots + (k_n+q_n)T(b_n)$
$= (k_1+q_1)c_1 + \ldots + (k_n+q_n)c_n$
$= \sum_{i=1}^{n} k_i c_i + \sum_{i=1}^{n} q_i c_i$
$= T(\vec{v_1}) + T(\vec{v_2})$

Claim ②: T is surjective

Proof: $\forall \vec{v} \in V$,
$\vec{v} = \sum_{i=1}^{n} k_i c_i$ for some $k_1 \ldots k_n \in \mathbb{R}$

So consider $\vec{w} = \sum_{i=1}^{n} k_i b_i \Rightarrow T(\vec{w}) = \vec{v}$

By claim ①②, T is a surjective linear transformation.

And since $T: V \to V$, $\dim(V) = \dim(V) = n$

So T is an isomorphism.

   (b) (5 points) Prove that $[T]_{\mathcal{B}} = ~~[T]_{\mathcal{C}}~~$.

Pf. $[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{B}} & \cdots & [T(b_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [c_1]_{\mathcal{B}} & \cdots & [c_n]_{\mathcal{B}} \\ | & & | \end{bmatrix}$

$[T]_{\mathcal{C}} = \begin{bmatrix} [T(c_1)]_{\mathcal{C}} \end{bmatrix}$

$= \begin{bmatrix} \cdots & | & \cdots \\ [k_{i1}\vec{c_1} + k_{i2}\vec{c_2} + \ldots + k_{in}\vec{c_n}]_{\mathcal{C}} \\ & | & \end{bmatrix} = \begin{bmatrix} k_{11} & \cdots & k_{n1} \\ k_{12} & & k_{n2} \\ \vdots & & \vdots \\ k_{1n} & \cdots & k_{nn} \end{bmatrix} = [T]_{\mathcal{B}}$

the i^{th} column

Therefore $[T]_{\mathcal{B}} = [T]_{\mathcal{C}}$

Let $\vec{c_1} = k_{11}\vec{b_1} + k_{12}\vec{b_2} + \ldots + k_{1n}\vec{b_n}$
$\vdots$
$\vec{c_n} = k_{n1}\vec{b_1} + k_{n2}\vec{b_2} + \ldots + k_{nn}\vec{b_n}$

So $[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{B}} & \cdots & [T(b_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$

$= \begin{bmatrix} | & & | \\ [c_1]_{\mathcal{B}} & \cdots & [c_n]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} k_{11} & \cdots & k_{n1} \\ k_{12} & & k_{n2} \\ \vdots & & \vdots \\ k_{1n} & \cdots & k_{nn} \end{bmatrix}$

$[T]_{\mathcal{C}} = \begin{bmatrix} | & & | \\ [T(c_1)]_{\mathcal{C}} & \cdots & [T(c_n)]_{\mathcal{C}} \\ | & & | \end{bmatrix} = \begin{bmatrix} [T(k_{11}b_1 + \ldots + k_{1n}b_n)]_{\mathcal{C}} & \cdots \\ | & \end{bmatrix}$

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8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, with standard matrix A.

   (a) (5 points) Prove that A is symmetric if and only if $[T]_\mathcal{B}$ is symmetric for every orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$.

Pf. ① Claim: if A is symmetric, then $\forall$ orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$, $[T]_\mathcal{B}$ is symmetric

Proof. Assume A is symmetric, then $A^T = A$

Select arbitrary orthonormal basis $\mathcal{B}$

Note that $[T]_\mathcal{B} = S_{\mathcal{B}\to\mathcal{E}}^{-1} A S_{\mathcal{B}\to\mathcal{E}}$ by Thm.

So $([T]_\mathcal{B})^T = (S_{\mathcal{B}\to\mathcal{E}}^{-1} A S_{\mathcal{B}\to\mathcal{E}})^T = S_{\mathcal{B}\to\mathcal{E}}^T A^T (S_{\mathcal{B}\to\mathcal{E}}^{-1})^T$

Since $S_{\mathcal{B}\to\mathcal{E}}$ is the change of basis matrix from an orthonormal ~~basis~~ basis to another, $= S_{\mathcal{B}\to\mathcal{E}}^T A^T (S_{\mathcal{B}\to\mathcal{E}}^T)^{-1}$

& $S_{\mathcal{B}\to\mathcal{E}}$ is orthogonal, so $S_{\mathcal{B}\to\mathcal{E}}^T = S_{\mathcal{B}\to\mathcal{E}}^{-1} \Rightarrow ([T]_\mathcal{B})^T = S_{\mathcal{B}\to\mathcal{E}}^{-1} A^T S_{\mathcal{B}\to\mathcal{E}} = [T]_\mathcal{B}$

□

   (b) (5 points) Give an example that shows that if the word "orthonormal" is removed from the statement in (a), then the statement is not necessarily true.

Consider $\mathcal{B} = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right)$

$A = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ is symmetric

But $[T]_\mathcal{B} = \left[\begin{bmatrix}1\\1\end{bmatrix}_\mathcal{B} \begin{bmatrix}3\\1\end{bmatrix}_\mathcal{B}\right] = \begin{bmatrix}\frac{1}{2} & \frac{5}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ is not symmetric

((a) continued here ↓)

② Claim: if $\forall$ orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$, $[T]_\mathcal{B}$ is symmetry, then A is symmetry

Proof. ~~Consider the standard matrix $\mathcal{E}$~~. take arbitrary orthonormal basis $\mathcal{B}$

So $A = S_{\mathcal{E}\to\mathcal{B}}^{-1} [T]_\mathcal{B} S_{\mathcal{E}\to\mathcal{B}}$ So $[T]_\mathcal{B}^T = [T]_\mathcal{B}$

$A^T = (S_{\mathcal{E}\to\mathcal{B}}^{-1} [T]_\mathcal{B} S_{\mathcal{E}\to\mathcal{B}})^T = S_{\mathcal{E}\to\mathcal{B}}^T [T]_\mathcal{B}^T (S_{\mathcal{E}\to\mathcal{B}}^{-1})^T$

Same logic as claim ①, we have $S_{\mathcal{E}\to\mathcal{B}}^T = S_{\mathcal{E}\to\mathcal{B}}^{-1}$

$\Rightarrow A^T = S_{\mathcal{E}\to\mathcal{B}}^{-1} [T]_\mathcal{B} S_{\mathcal{E}\to\mathcal{B}} = A$ □

By Claim ①②, we have proved ~~that~~ the if-and-only-if statement.

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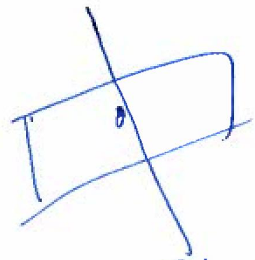
$[T]_{\mathcal{B}} =$



$S_{\mathcal{B}\to\mathcal{C}} = [\,[b_1]_{\mathcal{C}}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\int_{-1}^{1} e^{-x} x^4 e^{2x}$

$A b_1$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\int_{-1}^{1} e^{2x} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{x^5}{5}\Big|_{-1}^{1}$ $\frac{2}{5}$ $(xe^x)'$

$\frac{1}{2}$ $= e^x + xe^x$

$\frac{1}{2}(\mathbb{R}^n \to \mathbb{R}^m)$ $x^2 e^x$ $\left(\frac{x^3}{3}\right)\Big|_{-1}^{1}$ $-2xe^x$

$\frac{5}{2}$ $\frac{1}{2}$ $+2xe^x$ $\frac{1}{3} - \left(-\frac{1}{3}\right)$ $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

$-b-4=5$ $-2x^2e^x$ 0 $\frac{2}{3}$ $\frac{\sqrt{3}}{3} + 2\cdot\frac{\sqrt{3}}{3}$

-2 1

$-b=9$ -1 1 $=\sqrt{}$

$-a+b=-2$ $c=$ $\frac{\sqrt{3}}{3}$

$-b+2c=5$ $=\frac{x^4}{4}$ 1 1 1 1 -2 -1 0 1

$-c = 2$ $[T]_{\mathcal{B}} = S_{\mathcal{C}\to\mathcal{B}}[T]_{\mathcal{C}}S_{\mathcal{B}\to\mathcal{C}}$ $\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} -$ 0 0

$c=-2$ $a=b+2$ -1

$-b=5+4$ $b=-9$ $[\,[b_1]_{\mathcal{C}}$ -1

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$\vec{x}$ A ~~x~~ $\ker A$

$\vec{x}$ A in $(A^T)^\perp$