Worksheet 18: Orthogonal Transformations (§5.3)

Definition: A linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ is **orthogonal** if it preserves dot products—that is, if $\vec{x} \cdot \vec{y} = T(\vec{x}) \cdot T(\vec{y})$ for all vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

Theorem A: Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. Then T is orthogonal if and only if it preserves the length of every vector—that is, $||T(\vec{x})|| = ||\vec{x}||$ for all $\vec{x} \in \mathbb{R}^n$.

Problem 1. Which of the following maps are orthogonal transformations? Short, geometric justifications are preferred, where possible.

(a) The identity map $\mathbb{R}^3 \to \mathbb{R}^3$.

(b) Rotation counterclockwise through θ in $\mathbb{R}^2$.

(c) The reflection $\mathbb{R}^3 \to \mathbb{R}^3$ over a plane (though the origin).

(d) The projection $\mathbb{R}^3 \to \mathbb{R}^3$ onto a subspace V of dimension 2.

(e) Dilation $\mathbb{R}^3 \to \mathbb{R}^3$ by a factor of 3.

(f) Multiplication by $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$.

Problem 2. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be an orthogonal transformation.

(a) Prove that T is **injective.** [Hint: consider the kernel.]

(b) Prove that T is an isomorphism. [Hint: Note that the source and target here have the same dimension.]

(c) Prove that the matrix of T (in standard coordinates) has columns that are *orthonormal.*

(d) Prove the composition of orthogonal transformations is orthogonal. [Hint: Use the Theorem!]

Definition: An $n \times n$ matrix A is **orthogonal** if $A^\top A = I_n$—i.e., if its transpose is its inverse.

Problem 3. Which of the following matrices are orthogonal?

(i) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Problem 4. Suppose that $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ is a 3×3 matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$.

(a) Recalling that $A^\top = \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vec{v}_3^\top \end{bmatrix}$, show that

$$A^\top A = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot \vec{v}_3 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \vec{v}_2 \cdot \vec{v}_3 \\ \vec{v}_3 \cdot \vec{v}_1 & \vec{v}_3 \cdot \vec{v}_2 & \vec{v}_3 \cdot \vec{v}_3 \end{bmatrix}.$$

(b) Does the argument work for any size square matrix?

(c) Use (a)/(b) to prove a square matrix is orthogonal if and only if its columns are orthonormal.

(d) Is $B = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ orthogonal? Find B^{-1}. [Be clever! There are easy ways and hard ways!]

Problem 5. Show that if A and B are orthogonal $n \times n$ matrices, then the matrices $A^\top$, A^{-1}, and AB are also orthogonal.

> **Theorem B:** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is orthogonal if and only if A is orthogonal.*

Problem 6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

(a) Prove Theorem B just above. [Hint: Use Problems 2 and 4.]

(b) If $\mathcal{B}$ is an arbitrary basis, is it still true that T is orthogonal if and only if $[T]_\mathcal{B}$ is orthogonal? What about if $\mathcal{B}$ is an *orthonormal basis*?

Problem 7. Prove Theorem A from page 1. [Hint: For the harder direction, consider $T(\vec{x} + \vec{y})$.]

Problem 8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let A be the standard matrix of T. Which of following are equivalent?

(a) T *preserves length*, i.e., $\|T(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$.

(b) T *preserves distance*, i.e., $\|T(v) - T(w)\| = \|v - w\|$ for all $v, w \in \mathbb{R}^n$.

(c) T is an orthogonal transformation, i.e., T preserves the dot product.

(d) T maps any orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.

(e) T maps the standard basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.

(f) The columns of A form an orthonormal basis of $\mathbb{R}^n$.

(g) $A^\top A = I_n$.

(h) $AA^\top = I_n$.

(i) A is an orthogonal matrix.

(j) The rows of A form an orthonormal basis of $\mathbb{R}^n$.

Problem 9. Let $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{d \times p}$. Prove that $(AB)^\top = B^\top A^\top$ using the ideas from Problem 4. [Note: You already proved this in the homework, most likely, a clumsier way!]