Def ⑤ (正交变换)

$T:\mathbb{R}^n\to\mathbb{R}^n$ is orthogonal: 保留 dot products

即 $\forall \vec{x},\vec{y}\in\mathbb{R}^n$, $\boxed{\vec{x}\cdot\vec{y} = T(\vec{x})\cdot T(\vec{y})}$

Theorem A

$T:\mathbb{R}^n\to\mathbb{R}^n$ is orthogonal iff it preserves the length of vector 即: $\forall \vec{x}\in\mathbb{R}^n$, $\boxed{\|T(\vec{x})\| = \|\vec{x}\|}$

因而 linear trans 保留 dot product iff 保留 length.

P2

(a) orthogonal trans T is injective

Pf $T(\vec{x})=\vec{0} \Rightarrow \|T(\vec{x})\|=0 \Rightarrow \|\vec{x}\|=\vec{0} \Rightarrow \vec{x}=\vec{0}$

$\Rightarrow \ker T=\vec{0} \Rightarrow$ inj

(b) T is an isomorphism.

because same dim linear trans, inj $\Leftrightarrow$ surj (isomorphic)

因而叫 orthogonal trans

(c) the standard matrix ($\mathcal{E}$-matrix) of T 的 columns 是 orthonormal 的

P4

(a) show: 写A作 $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

求证: $A^TA = AA^T = \begin{bmatrix} \vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 & \cdots & \vec{v}_1\cdot\vec{v}_n \\ \vec{v}_2\cdot\vec{v}_1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & & \vdots \\ \vec{v}_n\cdot\vec{v}_1 & \cdots & \ddots & \vec{v}_n\cdot\vec{v}_n \end{bmatrix}$

即: $\boxed{ij^{th} \text{ entry of } A \text{ 为 } \vec{v}_i\cdot\vec{v}_j}$ ⊕

Pf 我们知道 by definition,

the ij^{th} entry of matrix multiplication

AB is (i^{th} row of A)$\cdot$(j^{th} col of B)

$\Rightarrow$ (ij) of A^TA is (i^{th} row of A^T)$\cdot$(j^{th} col of A)

$A^T = \begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$ $= \vec{v}_i$; $\vec{v}_j$

$\Longrightarrow \underline{A^TA = \vec{v}_i\cdot\vec{v}_j}$

(Attention: 通常 $A^TA \neq AA^T$! 但它们都 symmetric
但是 if A orthogonal $\Rightarrow A^T=A^{-1}$, $\boxed{AA^T = A^TA = I_n}$)

$[T]_{\mathcal{E}} = \begin{bmatrix} | & & | \\ [T(\vec{e}_1)]_{\mathcal{E}} & \cdots & [T(\vec{e}_n)]_{\mathcal{E}} \\ | & & | \end{bmatrix}$

$= \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$

Since $\forall 1\le i,j\le n$, $\begin{cases} e_i\cdot e_j = 1 & \text{if } i=j \\ e_i\cdot e_j = 0 & \text{if } i\neq j \end{cases}$

$\Longrightarrow \forall 1\le i,j\le n$, $\begin{cases} T(\vec{e}_i)\cdot T(\vec{e}_j) = 1 & \text{if } i=j \\ T(\vec{e}_i)\cdot T(\vec{e}_j) = 0 & \text{if } i\neq j \end{cases}$

(d) the composition of orthogonal trans is orthogonal. (显然)

$\|T_k\circ T_{k-1}\circ\cdots\circ T_1(\vec{x})\| = \|T_{k-1}\circ T_{k-2}\circ\cdots\circ T_1(\vec{x})\|$

$= \cdots = \|T(\vec{x})\| = \|\vec{x}\|$

$\Longrightarrow$ orthogonal.

Def ⑥

一个 square matrix A 是 orthogonal 的 if $A^TA = I_n$ (即: $A^T = A^{-1}$)

(c) (引理) 一个 square matrix 为 orthogonal 的 iff 它的 cols 为 orthonormal 的

Pf. ① A orthogonal ($A^TA = I_n$)

$\Longleftrightarrow \vec{v}_i\cdot\vec{v}_j = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$

$\Longleftrightarrow \{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ orthonormal

(我们也由此知道: 由 orthonormal basis 组成的 matrix A $\Longrightarrow A^{-1}$ 就是 A^T.)

P9: 用P4的方式证明 $\forall A\in\mathbb{R}^{n\times d}$, $B\in\mathbb{R}^{d\times p}$, $(AB)^T = B^TA^T$.

$AB = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}$ ($n \times p$) $\quad (AB)^T = \begin{bmatrix} - & (A\vec{b}_1)^T & - \\ - & (A\vec{b}_2)^T & - \\ & \vdots & \\ - & (A\vec{b}_p)^T & - \end{bmatrix}$ ($p \times n$)

the ij^{th} entry of $(AB)^T$ is $\vec{a}_{row_j}\cdot\vec{b}_i$ ①

$$B^T = \begin{bmatrix} — \vec{b_1} — \\ — \vec{b_2} — \\ \vdots \\ — \vec{b_p} — \end{bmatrix} \qquad B^TA^T = \underbrace{\begin{bmatrix} | & & | \\ B^T(\vec{a_{row_1}}) & \cdots & B^T(\vec{a_{row_n}}) \\ | & & | \end{bmatrix}}_{n} \Big\updownarrow p$$

the ij^{th} entry of B^TA^T is $\vec{b_i}\cdot\vec{a_{row_j}}$ ②

① = ② $\Rightarrow (AB)^T = B^TA^T$

$\Downarrow$

P5 If A, B 为 orthogonal $n\times n$ matrices, 则 AB 也 orthogonal

Ø : orthogonal matrices 的 matrix product 也为 orthogonal 的.

A, B orthogonal $\Rightarrow AA^T = A^TA = I_n$

$BB^T = B^TB = I_n$

$\Rightarrow (AB)(B^TA^T) = A(BB^T)A^T = I_n$

$\Rightarrow B^TA^T = (AB)^{-1}$

$\Rightarrow (AB)^T = (AB)^{-1}$

$\Rightarrow$ AB orthogonal

If A orthogonal $\Rightarrow A^{-1}$ (也是 A^T) 也 orthogonal

$A^TA = I_n \Rightarrow (A^T)^T(A^T) = AA^T = I_n$

$\Rightarrow A^T$ 也是 orthogonal 的

结论 关键信息: 这意味着若 A 的 cols 是 orthonormal 的, 那么 A 的 rows 也是 orthonormal 的. Ø

P6 Proof of Thm B:

$T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $[T]_\mathcal{E}$ is orthogonal.

结论关键信息: 正交 trans ① $\Longleftrightarrow$ 正交 standard matrix ②. Ø

Pf. ① → ②:

orthogonal trans T $\xRightarrow{\text{by P2}}$ $[T]_\mathcal{E}$ columns orthonormal $\xRightarrow{\text{by P4}}$ orthogonal $[T]_\mathcal{E}$

② → ①:

orthogonal $[T]_\mathcal{E} \Rightarrow [T]_\mathcal{E}^T = [T]_\mathcal{E}^{-1}$

let $\vec a, \vec b \in \mathbb{R}^n$. $T(\vec a)\cdot T(\vec b) = ([T]_\mathcal{E}\vec a)\cdot([T]_\mathcal{E}\vec b)$

这里把 dot product 转化为 matrix product Ø

$= ([T]_\mathcal{E}\vec a)^T([T]_\mathcal{E}\vec b)$

$= \vec a^T \underbrace{[T]_\mathcal{E}^T}_{= [T]_\mathcal{E}^{-1}} [T]_\mathcal{E}\vec b$

GPT 的 巧妙办法 Ø

$= \vec a^T \vec b$

$= \vec a \cdot \vec b$

$\Rightarrow T(\vec a)\cdot T(\vec b) = \vec a\cdot\vec b$

Q.E.D. □

P6 (b) 我们已经证明了 Thm B.

① $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $[T]_\mathcal{E}$ is orthogonal.

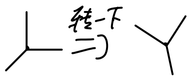
现在 generalize 它:

② Generalization: $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $[T]_\mathcal{B}$ is orthogonal, $\mathcal{B}$ 为任意 orthonormal basis. Ø

Claim ①: 从任意一组 orthonormal basis $\mathcal{A}$ 到另一组 orthonormal basis $\mathcal{B}$ 的变换 ($T_{\mathcal{A}\to\mathcal{B}}$) (其 standard matrix 为 $S_{\mathcal{A}\to\mathcal{B}}$) 为 orthogonal transformation.

也说明了如果 $\mathcal{A}$, $\mathcal{B}$ 是 orthonormal basis, $S_{\mathcal{A}\to\mathcal{B}}$, $S_{\mathcal{B}\to\mathcal{A}}$ 是 orthogonal matrix

首先这在几何直观上非常显然



且易证: by fact ① on WS16:

$$S_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} | & & | \\ [\vec a_1]_\mathcal{B} & \cdots & [\vec a_n]_\mathcal{B} \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec a_1\cdot\vec b_1 & \vec a_2\cdot\vec b_1 & \cdots & \vec a_n\cdot\vec b_1 \\ \vec a_1\cdot\vec b_2 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \vec a_1\cdot\vec b_n & \cdots & \cdots & \vec a_n\cdot\vec b_n \end{bmatrix}$$

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} | & & | \\ [\vec b_1]_\mathcal{A} & \cdots & [\vec b_n]_\mathcal{A} \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec b_1\cdot\vec a_1 & \vec b_2\cdot\vec a_1 & \cdots & \vec b_n\cdot\vec a_1 \\ \vec b_1\cdot\vec a_2 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \vec b_1\cdot\vec a_n & \cdots & \cdots & \vec b_n\cdot\vec a_n \end{bmatrix}$$

$\Rightarrow S_{\mathcal{A}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{A}}^T$

又 $S_{\mathcal{A}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{A}}^{-1}$

得 $S_{\mathcal{B}\to\mathcal{A}}^T = S_{\mathcal{B}\to\mathcal{A}}^{-1}$

Claim ②: orthogonal matrices 的 product 也是 orthogonal matrix.

显然，这是因为 orthogonal transformations 的 composition 也是 orthogonal transformation (fact ②)
且它们各自被一个 orthogonal matrix 代表.
那么 by Thm B，这些 matrices 的 product 也是 orthogonal 的

Claim ③: 如果 T orthogonal，则 $[T]_\mathcal{B}$ orthogonal
$\mathcal{B}$ 为任意 orthonormal basis.

$$[T]_\mathcal{B} = \underbrace{S_{\mathcal{E}\to\mathcal{B}}}_{\text{orthogonal}} \overbrace{[T]_\mathcal{E}}^{\text{orthogonal}} \underbrace{S_{\mathcal{B}\to\mathcal{E}}}_{\text{orthogonal.}}$$

By claim ②, $[T]_\mathcal{B}$ is orthogonal.

Claim ④: $\mathcal{B}$ 为任意 orthonormal basis.
如果 $[T]_\mathcal{B}$ orthogonal，则 T orthogonal.

因为 $[T]_\mathcal{B} = S_{\mathcal{E}\to\mathcal{B}}[T]_\mathcal{E}\, S_{\mathcal{B}\to\mathcal{E}}$

$\Rightarrow$ $[T]_\mathcal{E} = S_{\mathcal{B}\to\mathcal{E}}[T]_\mathcal{B}\, S_{\mathcal{E}\to\mathcal{B}}$

orthogonal

$\Rightarrow$ $[T]_\mathcal{E}$ orthogonal $\Rightarrow$ T orthogonal.

QED: 对于 orthonormal basis $\mathcal{B}$,
T orthogonal $\Leftrightarrow$ $[T]_\mathcal{B}$ orthogonal.

7. Proof of Thm A: T is orthogonal
iff $\forall \vec{x}\in\mathbb{R}^n$, $\|T(\vec{x})\| = \|\vec{x}\|$.

Assume $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}\in\mathbb{R}^n$

则 $\|T(\vec{x}+\vec{y})\| = \|\vec{x}+\vec{y}\|$ for all $\vec{x},\vec{y}\in\mathbb{R}^n$

$T(\vec{x}+\vec{y})\cdot T(\vec{x}+\vec{y}) = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$(T(\vec{x})+T(\vec{y}))\cdot(T(\vec{x})+T(\vec{y})) = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$$\|T(\vec{x})\|^2 + 2T(\vec{x})\cdot T(\vec{y}) + \|T(\vec{y})\|^2 = \|\vec{x}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2$$

$\Rightarrow$ $T(\vec{x})\cdot T(\vec{y}) = \vec{x}\cdot\vec{y}$
for all $\vec{x},\vec{y}\in\mathbb{R}^n$

$\Rightarrow$ T is orthogonal

由下往上可双向推导出 T is orthogonal
$\Rightarrow \forall \vec{x}\in\mathbb{R}^n$, $\|T(\vec{x})\| = \|\vec{x}\|$.

总结: T is orthogonal (保留 dot product)

= T 保留 length = T 保留 distance

= T 把 $\mathbb{R}^n$ 的某个 orthonormal basis map 到另一个 orthonormal basis

= $[T]_\mathcal{B}$ 为 orthogonal 的，$\mathcal{B}$ 为任意 orthonormal basis
$([T]_\mathcal{B}^T = [T]_\mathcal{B}^{-1})$

= $[T]_\mathcal{B}$ 的 rows 为一个 orthogonal basis of $\mathbb{R}^n$

= $[T]_\mathcal{B}$ 的 cols 为一个 orthonormal basis of $\mathbb{R}^n$