Worksheet 12: The Rank-Nullity Theorem and Isomorphism

The Rank-Nullity Theorem. Let $T : V \to W$ be a linear transformation between finitely dimensional vector spaces. Then

$$\dim \ker T + \dim \operatorname{im} T = \dim \operatorname{Source} T.$$

Theorem A. Let V be finite dimensional vector space, and let $W \subseteq V$ be a *subspace*. Then

$$\dim W \leq \dim V.$$

Furthermore, $\dim W = \dim V$ if and only if $W = V$.

Problem 1. Use Theorem A to determine all possible dimensions of subspaces of the vector space $\mathcal{P}_2$ of polynomials of degree at most two. How many different subspaces exist of each dimension?

Solution: Because $\mathcal{P}_2$ is three dimensional, Theorem A says that a subspace of $\mathcal{P}_2$ must have dimension $0, 1, 2$, or 3. Each of these actually occur: the set $\{0_{\mathcal{P}_2}\}$ consisting only of the zero polynomial is a subspace of dimension zero. It is the *only* subspace of dimension zero, since subspaces always contain the zero element. The set of constant polynomials is a one dimensional space, and $\mathcal{P}_1$ is a two dimensional space, while $\mathcal{P} - 2$ itself is a subspace of dimension three.

There is only one subspace of dimension zero, namely $\{0_{\mathcal{P}_2}\}$, and one subspace of dimension 3 (by the second statement in Theorem A). But there are infinitely subspaces of dimensions 1 and 2. To see there are infinitely spaces of dimension 1, take any non-zero polynomial, such as $x + c$ for some $c \in \mathbb{R}$, and note that its span is a one-dimensional space; these are all distinct for different values of c. Similarly, to see that there are infinitely spaces of dimension 2, consider $\operatorname{Span}(x^2, x + c)$. These are all two dimensional and distinct for different values of c.

Problem 2. Compare the Rank-Nullity Theorem to the Rank-Nullity Theorem in the book.* Deduce the statement in the book as a special case of the Rank Nullity Theorem above.

Solution: If we think of the matrix A as a linear transformation of coordinate spaces, the source is $\mathbb{R}^m$, the kernel is $\ker A$ and the image is spanned by the columns and has dimension equal to $\operatorname{rank} A$. So by the Rank-Nullity Theorem above, we recover $\dim \ker A + \operatorname{rank} A = m$, the book version.

Problem 3. Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ be the **trace** map sending a matrix $A = [a_{ij}]$ to the sum $\sum_{i=1}^{n} a_{ii}$ of its diagonal elements. Use the Rank-Nullity Theorem to compute the dimension of the kernel. Find a basis for the kernel when $n = 3$. Does your answer generalize to arbitrary n?
[HINT: A good PROBLEM SOLVING STRATEGY is to start with a smaller case, like $n = 2$.]

*The book's Rank-Nullity Theorem 3.3.7: *For any $n \times m$ matrix A,* $\dim \ker(A) + \textit{rank}(A) = m$.

Solution: The trace map is obviously surjective. So by Rank-Nullity, the kernel is dimension $n^2 - 1$. In the 3×3 case, the kernel is dimension 8, and a basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

For arbitrary n, a basis could be the matrices with zeros everywhere except 1 in the spot ij wherever $i \neq j$ (there are $n^2 - n$ of these) TOGETHER WITH the $(n-1)$ matrices in which we have 1 in the 11 spot and -1 in the ii spot for $i = 2, \ldots, n$. These are obviously linear independent, so they must be a basis since there are $n^2 - 1$ of them.

Definition. An **isomorphism** of vector spaces is a bijective linear transformation. If there exist an isomorphism $T : V \to W$, we say also that V is **isomorphic to** W, and write $V \cong W$.

Problem 4. Let $T : V \to W$ be a linear transformation of vector spaces *with the same (finite) dimension*. Use the two theorems above to prove that the following are equivalent:

(i) T is surjective;

(ii) T is injective;

(iii) T is an isomorphism.

Solution: We will show (i) implies (ii) and (ii) implies (iii). Since (iii) obviously implies (i), this will complete the proof.

Assume (i). If T is surjective, then $\mathrm{im} T = W$. So $\dim \mathrm{im} T = \dim W = \dim V$. By Rank-Nullity, $\dim \ker T = 0$. Thus $\ker T = \{0_V\}$, which means T is injective. This proves (ii).

Assume (ii). So $\ker T = \{0_V\}$, so by Rank-Nullity, $\dim \mathrm{im} T = \dim V = \dim W$. Since $\mathrm{im} T \subset W$ and has the same dimension as W, Theorem A tells us $\mathrm{im} T = W$, and T is surjective. Thus T is injective and surjective and hence an isomorphism, proving (iii).

Problem 5. Let $T : V \to W$ be an isomorphism of finite dimensional vector spaces. Use the rank nullity theorem to prove that V and W have the same dimension. Under what conditions can $\mathbb{R}^n$ be isomorphic to $\mathbb{R}^m$?

Solution: Since T is an isomorphism, we have $\ker T = \{0\}$ (injectivity) and $\mathrm{im} T = W$ (surjectivity). So $\dim\{0\} + \dim W = \dim V$ and so $\dim W = \dim V$. The converse is also true! Can you prove it? $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$ if and only if $n = m$.

Problem 6. Say $c \neq 0$. Let $V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid ax + by + cz = 0\}$ be a plane through the origin in $\mathbb{R}^3$. Find (with proof) an explicit isomorphism from V to $\mathbb{R}^2$. Does it matter if $c = 0$?
[Hint: Visualize V in $\mathbb{R}^3$ and think geometrically about projection onto the xy-plane.]

Solution: The projection $\mathbb{R}^3 \to \mathbb{R}^2$ sending $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$ restricts to a bijection on V, since for each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, the vector $\begin{bmatrix} x \\ y \\ \frac{-1}{c}(x+y) \end{bmatrix}$ is the unique element on V mapping to it. So it is an isomorphism. If $c = 0$, we need to project in one of the other directions.

Problem 7. Let V_1, V_2 and V_3 be vector spaces. Prove that

(a) If V_1 is isomorphic to V_2, then V_2 is isomorphic to V_1; and

(b) If V_1 is isomorphic to V_2 and V_2 is isomorphic to V_3, then V_1 is isomorphic to V_3.

Solution: For (a), observe that if $T : V_1 \to V_2$ is an isomorphism, then so is $T^{-1} : V_2 \to V_1$. For details see Proposition 2.11 in the handout Theory of Linear Algebra.

For (b), observe that if $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are isomorphisms, then so is $S \circ T : V_1 \to V_3$, because a composition of linear transformations is linear, and a a composition of bijective maps is bijective.

Problem 8. Let V be a vector space of dimension n. Fix any ordered basis $(v_1, \ldots, v_n)$ for V, and consider the map

$$\phi : \mathbb{R}^n \to V \quad \text{defined by} \quad \phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 v_1 + \cdots + a_n v_n.$$

(a) Show that ϕ is a linear transformation.

(b) Show that ϕ is an *isomorphism*. [Hint: Consider using Problem 4 to shorten your task.]

(c) Prove that two vector spaces of the same dimension are isomorphic.

Solution:

(a) Check

$$\begin{aligned}
\phi\left(\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) + \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)\right) &= \phi\left(\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}\right) \\
&= (a_1 + b_1)v_1 + \ldots (a_n + b_n)v_n \\
&= a_1 v_1 + \cdots + a_n v_n + b_1 v_1 + \cdots + b_n v_n \\
&= \phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) + \phi\left(\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right),
\end{aligned}$$

so ϕ respects vector addition. Similarly,

$$\phi\left(\begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}\right) = \lambda(a_1 v_1 + \cdots + a_n v_n) = \lambda\phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right),$$

so ϕ respects scalar multiplication.

(b) To see that the map ϕ is injective, we can show that $\ker\phi = 0$. Say $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \ker\phi$. This means $a_1v_1 + \cdots + a_nv_n = 0$. But since $\{v_1, \ldots, v_n\}$ is linearly independent, we conclude that $a_1 = \cdots = a_n = 0$. This means ϕ is injective. To see that that ϕ is surjective, observe that we can write an arbitrary $v \in V$ as a linear combination $v = a_1v_1 + \cdots + a_nv_n$. Now it's easy to check that $\phi(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}) = v$. We could have also used Problem 4 to check *either* injective or surjective (we don't need to check both).

(c) Let V and W be two vector spaces of dimension n. By (c), we know $\mathbb{R}^n \cong V$ and $\mathbb{R}^n \cong W$. So by Problem 7, $V \cong W$.

Problem 9. The Proof of Theorem A. Discuss with your group how Theorem A follows from the characterization of basis proved on Worksheet 10: a basis for a vector space V is **equivalently** a *minimal spanning set* **or** a *maximal linearly independent set.*

Solution: Suppose the subspace W of V has basis $\{w_1, \ldots, w_d\}$. Then the set $\{w_1, \ldots, w_d\}$ is a linearly independent set of vectors in W (and hence V). It is maximal in W, but it might be possible to extend it by vectors in V to a larger linearly independent set in V. In any case, $\dim V \geq d$ since there are d linearly independent vectors in V. If $\dim W = \dim V$, then $\{w_1, \ldots, w_d\}$ is a *maximal* linear independent set in V, and hence a basis for V. So $\mathrm{Span}(w_1, \ldots, w_d) = W = V$, and $W = V$.

Problem 10. The Proof of Rank-Nullity.[†] Let $T : V \to W$ be a linear transformation.
Let $\{v_1, \ldots, v_m\}$ be a set of vectors in V such that $\{T(v_1), \ldots, T(v_m)\}$ is a basis of $\mathrm{im} T$.
Let $\{u_1, \ldots, u_n\}$ be a basis of $\ker T$.

(a) Prove that $v_i \neq u_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.
[Hint: Try proof by contradiction, and recall that no set containing the zero vector is linearly independent.]

(b) Prove that $\{v_1, \ldots, v_m, u_1, \ldots, u_n\}$ is a linearly independent set of vectors in V.
[Hint: Apply T to a relation on them.]

(c) Prove that $\{v_1, \ldots, v_m, u_1, \ldots, u_n\}$ spans V.

(d) Prove the *Rank-Nullity Theorem* as stated on the front page using (a), (b) and (c).

(e) Can you imagine another way to prove the Rank-Nullity Theorem using the fact that all finite dimensional vector spaces are isomorphic to coordinate spaces proved in Problem 8, together with work done on Worksheet 11?

[†] For coordinate spaces, you have already proved the Rank-Nullity Theorem on Worksheet 9.

Solution:

(a) For the sake of contradiction, assume $v_i = u_j$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$. But then $T(v_i) = T(u_j) = 0_W$. But this contradicts the hypotheses that $\{T(v_1), \cdots, T(v_m)\}$ is linearly independent since any set of vectors containing the zero vector is not linearly independent.

(b) Consider the relation:

$$\sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} c_j u_j = 0_V. \tag{1}$$

We will see that the only possible solution is the trivial one. We apply T to both sides of the equation (1), to obtain the equation:

$$0_W = T(0_V) = T\left(\sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} c_j u_j\right) = \sum_{i=1}^{m} a_i T(v_i),$$

where the last equality holds because T is linear and $u_1, \cdots u_n$ are all in $\ker(T)$. But then $0_W = \sum_{i=1}^{m} a_i T(v_i)$, and since the $T(v_1), \cdots, T(v_m)$ are linearly independent, all the $a_i = 0$. Plugging back in to 1, we see that each $c_j = 0$, by linear independence of the u_j. Thus, the relation is trivial.

(c) Let $\vec{v} \in V$. Because $T(v_1), \cdots, T(v_m)$ span $\operatorname{im}(T)$, the image $T(\vec{v})$ can be written as some linear combination $T(\vec{v}) = b_1 T(v_1) + \cdots + b_m T(v_m)$.

Now let $\vec{w} = b_1 v_1 + \cdots + b_m v_m$ (where the b_i's are the same as above). We see that $T(\vec{v} - \vec{w}) = T(\vec{v}) - T(\vec{w}) = b_1 T(v_1) + \cdots + b_m T(v_m) - (T(b_1 v_1 + \cdots + b_m v_m)) = b_1 T(v_1) + \cdots + b_m T(v_m) - (b_1 T(v_1) + \cdots + b_m T(v_m)) = 0$. Therefore $\vec{v} - \vec{w} \in \ker(T)$. Because $u_1, \cdots, u_n$ span $\ker(T)$, we have that $\vec{v} - \vec{w}$ can be written as a linear combination of the u_i's:

$$\vec{v} - \vec{w} = d_1 u_1 + \cdots + d_n u_n.$$

But then $\vec{v} = \vec{w} + d_1 u_1 + \cdots + d_n u_n = b_1 v_1 + \cdots + b_m v_m + d_1 u_1 + \cdots + d_n u_n$. Therefore $\vec{v} \in \operatorname{Span}\{v_1, \cdots, v_m, u_1, \cdots, u_n\}$. Since this was true for any arbitrary $\vec{v}$, we have that $(v_1, \cdots, v_m, u_1, \cdots, u_n)$ span V.

(d) First we observe that $\dim(\operatorname{im}(T)) = m$ and $\dim(\ker(T)) = n$.

By combining the above parts, we see that $(v_1, \cdots, v_m, u_1, \cdots, u_n)$ is a basis for V, therefore $\dim(V) = m + n = \dim(\operatorname{im}(T)) + \dim(\ker(T))$.

Problem *11. Any two bases of a vector space have the same cardinality. We claimed this earlier without proof, and have been using it unquestioned (though you proved it on Worksheet 11 for *coordinate spaces*; see Problem 3 there). Observe that without this fact, the *dimension* would not be *well-defined*: if V has two bases, one with 3 elements and one with 17, should the dimension be 3 or 17? (Or maybe some other number corresponding to the number of vectors in some third basis?) Prove now, for a vector space V with a finite basis $\{v_1, \ldots, v_n\}$, that any other basis also has n elements.[‡] Note that you should avoid using rank-nullity or any other statement involving

[‡]Even if V does not have a finite basis, all bases have the same cardinality. There are different kinds of "infinities", some bigger than others—the different kinds of infinities are said to be "cardinalities." (The finite cardinalities are just the counting numbers, $0, 1, 2, 3, 4 \ldots$). All bases of a given vector space will have the same cardinality, at least

dimension, since what are trying to show is that the definition of dimension is sensible in the first place.

Solution: For this proof, we will use only the definition of isomorphism, basis, and facts we proved in Chapter 2. Suppose that V has one basis with n elements and another with m elements. By Problem 8 part (a&b), there are isomorphisms $\phi : \mathbb{R}^n \to V$ and $\psi : \mathbb{R}^m \to V$ (observe that 8a and b can be proved with out much machinery; we proved 8b using only the definitions of isomorphism and basis). The map $\psi^{-1} \circ \phi : \mathbb{R}^n \to \mathbb{R}^m$ is also an isomorphism by Problem 7. Its standard matrix is therefore invertible, as you proved on Worksheet 7. But then the standard matrix must be square! This says $n = m$.

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if you believe in the "Axiom of Choice".