Math 217 – Midterm 2

Winter 2019

Solutions

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	15	
3	12	
4	12	
5	16	
6	12	
7	10	
8	11	
Total:	100	

1. (12 points) Write complete, precise definitions for, or precise mathematical characterizations of, each of the following (italicized) terms.

(a) The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B

Solution: The $n \times n$ matrix A is *similar* to the $n \times n$ matrix B if there exists an invertible $n \times n$ matrix S such that $A = S^{-1}BS$.

(b) The *orthogonal complement* of the subspace W of the inner product space V, with inner product $\langle \cdot, \cdot \rangle$

Solution: The *orthogonal complement* of the subspace W of the inner product space V is the set

$$W^{\perp} = \{v \in V \ : \ \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

(c) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal* transformation

Solution: The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal* transformation if $T(v) \cdot T(w) = v \cdot w$ for all $v, w \in \mathbb{R}^n$.

(d) The *norm* (or *magnitude*, or *length*) of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$

Solution: The *norm* of the vector $\vec{v}$ in the inner product space $(V, \langle \cdot, \cdot \rangle)$ is the scalar $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

2. State whether each statement is True or False and provide a short proof of your claim.

(a) (3 points) For every orthogonal matrix A, we have $|\det(A)| = 1$.

Solution: TRUE. Let $A \in \mathbb{R}^{n\times n}$ be orthogonal, so $A^\top A = I_n$. Then

$$1 \;=\; \det I_n \;=\; \det(A^\top A) \;=\; \det(A^\top)\det(A) \;=\; (\det A)^2.$$

Thus $\det A = \pm 1$, so $|\det A| = 1$.

(b) (3 points) For every square matrix A, if $\ker(A) = \operatorname{im}(A)^\perp$, then A is symmetric.

Solution: FALSE. For instance let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\ker(A) = \{\vec{0}\}$ and $\operatorname{im}(A) = \mathbb{R}^2$, so $\ker(A) = \operatorname{im}(A)^\perp$, but A is not symmetric. (In general, any non-symmetric invertible matrix will be a counterexample.)

(c) (3 points) For all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$, if both S and $S \circ T$ are orthogonal transformations, then T is an orthogonal transformation.

Solution: TRUE. Suppose S and $S \circ T$ are orthogonal transformations. We have $T = S^{-1} \circ (S \circ T)$. But inverses and compositions of orthogonal transformations are orthogonal, so T is orthogonal.

(Problem 2, Continued).

(d) (3 points) The map $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\langle \vec{x}, \vec{y} \rangle = \det \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$$

is an inner product on $\mathbb{R}^2$.

Solution: FALSE. We know that $\det[\vec{x}\ \vec{y}] = -\det[\vec{y}\ \vec{x}]$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$, so the map $\langle \cdot, \cdot \rangle$ defined as above is not symmetric, hence not an inner product. For an explicit counterexample, take $\vec{x} = \vec{e}_1$ and $\vec{y} = \vec{e}_2$. Then

$$\langle \vec{e}_1, \vec{e}_2 \rangle \ = \ \det I_n \ = \ 1 \qquad \text{but} \qquad \langle \vec{e}_2, \vec{e}_1 \rangle \ = \ \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ = \ -1.$$

(e) (3 points) For all finite-dimensional vector spaces V, ordered bases $\mathcal{B}$ and $\mathcal{C}$ of V, and linear transformations $T : V \to V$,

$$\det[T]_{\mathcal{B}} = \det[T]_{\mathcal{C}}.$$

Solution: TRUE. Let V be a finite-dimensional vector space, let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases of V, and let $T : V \to V$ be a linear transformation. Then, using the fact that $S_{\mathcal{C}\to\mathcal{B}} = S^{-1}_{\mathcal{B}\to\mathcal{C}}$, we have

$$\begin{aligned} \det[T]_{\mathcal{B}} \ &= \ \det \left(S_{\mathcal{C}\to\mathcal{B}}[T]_{\mathcal{C}} S_{\mathcal{B}\to\mathcal{C}} \right) \\ &= \ \det \left(S^{-1}_{\mathcal{B}\to\mathcal{C}} \right) \det[T]_{\mathcal{C}} \det \left(S_{\mathcal{B}\to\mathcal{C}} \right) \\ &= \ \det \left(S_{\mathcal{B}\to\mathcal{C}} \right)^{-1} \det[T]_{\mathcal{C}} \det \left(S_{\mathcal{B}\to\mathcal{C}} \right) \ = \ \det[T]_{\mathcal{C}}. \end{aligned}$$

3. Let $\mathcal{P}_3$ be the vector space of polynomial functions of degree at most 3 in the variable x, and consider the ordered bases $\mathcal{E} = (1, x, x^2, x^3)$ and $\mathcal{B} = (1, x+1, 2x^2, x^3+x)$ of $\mathcal{P}_3$.

   (All solutions in this problem should be matrices or vectors having numerical entries. No justification is required on this problem, but including it might give you partial credit for incorrect final answers.)

   (a) (4 points) Let $f \in \mathcal{P}_3$ be the polynomial given by $f(x) = 2 + 3x + 4x^2 + 5x^3$. Find the coordinate vectors $[f]_{\mathcal{E}}$ and $[f]_{\mathcal{B}}$.

   Solution: $[f]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and $[f]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 5 \end{bmatrix}$.

   (b) (4 points) Find the change-of-coordinates matrices $S_{\mathcal{B}\to\mathcal{E}}$ and $S_{\mathcal{E}\to\mathcal{B}}$.

   Solution: $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $S_{\mathcal{E}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{E}}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (c) (4 points) Find the $\mathcal{B}$-matrix $[D]_{\mathcal{B}}$ of D, where $D : \mathcal{P}_3 \to \mathcal{P}_3$ is the differentiation map defined by $D(f) = f'$ for all $f \in \mathcal{P}_3$.

   Solution:

$$
\begin{aligned}
[D]_{\mathcal{B}} &= \begin{bmatrix} | & | & | & | \\ [D(1)]_{\mathcal{B}} & [D(x+1)]_{\mathcal{B}} & [D(2x^2)]_{\mathcal{B}} & [D(x^3+x)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | \\ [0]_{\mathcal{B}} & [1]_{\mathcal{B}} & [4x]_{\mathcal{B}} & [3x^2+1]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

4. Let $\mathcal{P}_1$ be the vector space of polynomials of degree at most 1 in the variable x. Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_1$ by

$$\langle p, q \rangle \;=\; \int_0^1 p(x)q(x)\,dx.$$

Let $u_1(x) = 1$ and $u_2(x) = \sqrt{3}(2x-1)$ be polynomials in $\mathcal{P}_1$, and let $\mathcal{B} = (u_1, u_2)$.

(a) (5 points) Prove that $\mathcal{B}$ is an orthonormal basis of $\mathcal{P}_1$.

Solution: We have

$$\langle u_1, u_1 \rangle \;=\; \int_0^1 (1)(1)dx \;=\; 1$$

and

$$\langle u_2, u_2 \rangle \;=\; \int_0^1 u_2^2\,dx \;=\; \int_0^1 (12x^2 - 12x + 3)\,dx \;=\; 4x^3 - 6x^2 + 3x\Big]_0^1 \;=\; 1,$$

so u_1 and u_2 are unit vectors. To check that u_1 and u_2 are orthogonal, we have

$$\langle u_1, u_2 \rangle \;=\; \int_0^1 \sqrt{3}(2x-1)\,dx \;=\; \sqrt{3}x^2 - \sqrt{3}x\Big]_0^1 \;=\; \sqrt{3} - \sqrt{3} \;=\; 0.$$

Thus $\mathcal{B}$ is orthonormal. But we know that orthonormal lists of vectors are linearly independent, so $\mathcal{B}$ is a linearly independent list of 2 vectors in $\mathcal{P}_1$. Since $\dim \mathcal{P}_1 = 2$, it follows that $\mathcal{B}$ is a basis of $\mathcal{P}_1$.

(b) (3 points) Find, in terms of a and b, the polynomial $p \in \mathcal{P}_1$ whose $\mathcal{B}$-coordinate vector is $\begin{bmatrix} a \\ b \end{bmatrix}$.

Solution: We have

$$p(x) \;=\; a(1) + b(\sqrt{3}(2x-1)) \;=\; a + 2\sqrt{3}bx - \sqrt{3}b \;=\; 2\sqrt{3}bx + (a - \sqrt{3}b).$$

(Problem 4, Continued).

As above, let $\mathcal{P}_1$ be the inner product space of polynomials of degree at most 1 with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q \rangle \;=\; \int_0^1 p(x)q(x)\,dx.$$

Also let $\mathcal{B} = (u_1, u_2)$, where $u_1(x) = 1$ and $u_2(x) = \sqrt{3}(2x-1)$.

(c) (4 points) Define the linear transformation $T : \mathcal{P}_1 \to \mathbb{R}$ by $T(ax+b) = a+b$. Find a polynomial $w(x) = cx+d$ in $\mathcal{P}_1$ such that $T(p) = \langle w, p \rangle$ for all $p \in \mathcal{P}_1$.

Solution: Note $T(u_1) = 1$ and $T(u_2) = \sqrt{3}$. For any $p \in \mathcal{P}_1$, we can write $p = tu_1 + su_2$. Hence

$$T(p) = tT(u_1) + sT(u_2) = t + \sqrt{3}s.$$

Define $w = u_1 + \sqrt{3}u_2$. Then

$$\begin{aligned}\langle w, p \rangle &= \langle u_1 + \sqrt{3}u_2, tu_1 + su_2 \rangle \\ &= t + \sqrt{3}s \\ &= T(p).\end{aligned}$$

Thus,

$$w(x) = u_1(x) + \sqrt{3}u_2(x) = 1 + \sqrt{3}(\sqrt{3}(2x-1)) = 6x - 2.$$

Solution: Alternatively, one can solve

$$\begin{aligned}\langle w, p \rangle = \langle cx+d,\, ax+b \rangle \;&=\; \int_0^1 (cx+d)(ax+b)\,dx \\ &= \int_0^1 acx^2 + (ad+bc)x + bd\,dx \\ &= \frac{ac}{3} + \frac{ad+bc}{2} + bd.\end{aligned}$$

Setting $T(ax+b) = a+b = \langle w, p \rangle$ and simplifying, we obtain

$$\left(\frac{c}{3} + \frac{d}{2} - 1\right) a + \left(d + \frac{c}{2} - 1\right) b = 0.$$

Since this must be true for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\frac{c}{3} + \frac{d}{2} &= 1 \\ d + \frac{c}{2} &= 1.\end{aligned}$$

Solving this system of linear equations, we obtain $c = 6$ and $d = -2$.

5. Let $A \in \mathbb{R}^{5\times 3}$ be a 5×3 matrix, and suppose that $A^\top A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix}$.

   (a) (3 points) Show that $\operatorname{rank}(A) = 3$.

Solution: Note that $\det(A^\top A) = 4$, so $A^\top A$ is invertible, which means $\ker(A) = \ker(A \top A) = \{\vec{0}\}$. Thus $\dim(\ker(A)) = 0$, so by Rank-Nullity we conclude that $\operatorname{rank}(A) = 3$.

Solution: We show that $\operatorname{rank}(A^\top A) = 3$ by proving that $\operatorname{rref}(A^\top A) = I_3$.

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to I_3.$$

By Problem 5b, Part B, Homework 5, we know that $\operatorname{rank}(A^\top A) \leq \operatorname{rank}(A)$. On the other hand, as $3 \leq \operatorname{rank}(A) = \operatorname{rank}(A^\top) \leq 3$ (cf. Problem 2d, Part B, Homework 7), we have that $\operatorname{rank}(A) = 3$.

(Problem 5, Continued).

As above, let A be a 5×3 matrix such that $A^\top A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix}$.

Also, for the rest of this problem, suppose that $A = QR$ is the QR-factorization of A.

(b) (2 points) Circle the correct statement below. *(No justification necessary.)*

- Q is orthogonal.
- $\boxed{Q \text{ is not orthogonal.}}$
- There is not enough information given to determine whether Q is orthogonal.

(c) (3 points) Circle all matrices below that are equal to the standard matrix P of orthogonal projection onto $\operatorname{im} A$ in $\mathbb{R}^5$. If none of these are equal to P, circle *none*. *(No justification necessary.)*

$A^\top A$ $\quad Q^\top Q$ $\quad R^\top R$ $\quad AA^\top$ $\quad \boxed{QQ^\top}$ $\quad RR^\top$ $\quad$ *none*

(d) (4 points) Find the following determinants. *(No justification necessary.)*

- $\det Q^\top Q = \boxed{1}$
- $\det R^\top R = \boxed{4}$
- $\det A^\top A = \boxed{4}$
- $\det AA^\top = \boxed{0}$

(e) (4 points) Find R.

Solution: Note that $A^\top A = (QR)^\top(QR) = (R^\top Q^\top)(QR) = R^\top(Q^\top Q)R = R^\top R$. So we write

$$R^\top R = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$

and solve for a, b, c, d, e, f, recalling that $a, d, f > 0$ by definition of QR-factorization. Thus $a = 2$, $b = -1$, $c = 0$, $d = 1$, $e = 3$, and $f = 1$. So

$$R = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Let A be the 3×3 matrix $A = \begin{bmatrix} -1 & 0 & r \\ p & -1 & 1 \\ 1 & q & -2 \end{bmatrix}$, where $p, q, r \in \mathbb{R}$. In parts (a) – (c) below, there exist values of p, q, and r that satisfy the given conditions. You may assume this fact without proof. In each of (a) – (c), find values of p, q, and r that satisfy the given conditions. *No justification is needed in parts (a) – (c).*

(a) (3 points) Both $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$ are least-squares solutions of $A\vec{x} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$.

Solution: It follows that $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in \ker(A)$, which gives $p = 0$, $q = 3$, and $r = -1$.

(b) (3 points) The linear systems $A\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ have exactly the same least-squares solutions.

Solution: It follows that $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ have the same orthogonal projection onto $\operatorname{im}(A)$, so their difference belongs to $\operatorname{im}(A)^\perp = \ker(A^\top)$, which gives $p = 2$, $q = -1$, and $r = -3$.

(c) (3 points) For all $\vec{x}, \vec{y} \in \mathbb{R}^3$, we have $\vec{x} \cdot A\vec{y} = A\vec{x} \cdot \vec{y}$.

Solution: It follows that $\vec{x}^\top A\vec{y} = \vec{x} \cdot A\vec{y} = A\vec{x} \cdot \vec{y} = (A\vec{x})^\top \vec{y} = \vec{x}^\top A^\top \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, which implies that $A^\top = A$. Thus $p = 0$, $q = 1$, and $r = 1$.

(d) (3 points) Do there exist values of p, q, and r for which the function $\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A\vec{w}$ defines an inner product on $\mathbb{R}^3$? Briefly justify your answer.

Solution: This is impossible, since $\langle \vec{e}_1, \vec{e}_1 \rangle = -1$ no matter what a, b, and c are, but inner products must be positive-definite.

7. Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_k \\ | & & | \end{bmatrix}$ be an $n \times k$ matrix with orthonormal columns.

(a) (4 points) Prove directly from the definitions of *orthonormal* and *linearly independent* that the list $(\vec{a}_1, \ldots, \vec{a}_k)$ is linearly independent.

Solution: Let $c_1, \ldots, c_k \in \mathbb{R}$, and suppose $\sum_{i=1}^{k} c_i \vec{a}_i = \vec{0}$. Let $1 \le j \le k$. Then, using the fact that $(\vec{a}_1, \ldots, \vec{a}_k)$ is an orthonormal list, we have

$$0 \;=\; \vec{a}_j \cdot \vec{0} \;=\; \vec{a}_j \cdot \sum_{i=1}^{k} c_i \vec{a}_i \;=\; \sum_{i=1}^{k} c_i (\vec{a}_j \cdot \vec{a}_i) \;=\; c_j.$$

Thus $c_j = 0$ for each $1 \le j \le k$, and we conclude that $(\vec{a}_1, \ldots, \vec{a}_k)$ is linearly independent.

(b) (3 points) Prove that if $n = k$, then A has orthonormal rows. (You may use properties of matrices that were proved in class or stated in the textbook.)

Solution: If $n = k$, then A is an orthogonal matrix, so $AA^\top = I_n$, which means that A has orthonormal rows.

(c) (3 points) Prove that if A has orthonormal rows, then $n = k$. (You may use properties of matrices that were proved in class or stated in the textbook.)

Solution: Since A has orthonormal columns, $A^\top A = I_k$. Since A has orthonormal rows, $AA^\top = I_n$. It follows that A and $A^\top$ are inverses of each other, so both are square and $n = k$.

Alternatively, since A has orthonormal columns and the k columns of A are linearly independent elements of $\mathbb{R}^n$, then $k \le n$. Similarly, since A has orthonormal rows, then $A^\top$ has orthonormal columns. So the columns of $A^\top$ are n linearly independent elements of $\mathbb{R}^k$. Thus $n \le k$. Therefore, $n = k$.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A, so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) (4 points) Prove that $\ker(A^\top) = (\operatorname{im} A)^\perp$.

Solution: For all $\vec{x} \in \mathbb{R}^n$,

$$\begin{aligned} \vec{x} \in \ker(A^\top) &\iff A^\top \vec{x} = \vec{0} \\ &\iff \vec{x} \cdot A\vec{e}_j = 0 \quad \text{for all } 1 \le j \le n \\ &\iff \vec{x} \cdot \vec{y} = 0 \quad \text{for all } \vec{y} \in \operatorname{im}(A) \\ &\iff \vec{x} \in \operatorname{im}(A)^\perp. \end{aligned}$$

Solution: First, let $\vec{x} \in \ker A^\top$. Let $\vec{a}_i$ be the ith column of A, so that $\vec{a}_i^\top$ is the ith row of $A^\top$. Then $\vec{0} = A^\top \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_n \cdot \vec{x} \end{bmatrix}$ implies $\vec{a}_i \cdot \vec{x} = 0$ for all $i = 1, \dots, n$. Since $\vec{a}_1, \dots, \vec{a}_n$ span $\operatorname{im} A$, $\vec{x}$ is orthogonal to every vector in $\operatorname{im} A$, so $\vec{x} \in (\operatorname{im} A)^\perp$.
Now assume $\vec{x} \in (\operatorname{im} A)^\perp$. Then $\vec{a}_i = A\vec{e}_i \in \operatorname{im} A$, so $\vec{x}$ is orthgonal to $\vec{a}_i$ for all $i = 1, \dots, n$ so $A^\top \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_n \cdot \vec{x} \end{bmatrix} = \vec{0}$ and $\vec{x} \in \ker A^\top$.
Alternatively, see that $\dim(\operatorname{im} A^\perp) = n - \dim \operatorname{im} A = n - \operatorname{rank} A = n - \operatorname{rank} A^\top = \dim \ker A^\top$, and $(\operatorname{im} A)^\perp \subset \ker A^\top$ so we conclude they are equal.

Solution: Let $\vec{x} \in \ker A^\top$. Then for all $\vec{y} \in \operatorname{im} A$, writing $\vec{y} = A\vec{z}$ for some $\vec{z} \in \mathbb{R}^n$,

$$\vec{y} \cdot \vec{x} = (A\vec{z}) \cdot \vec{x} = (A\vec{z})^\top \vec{x} = \vec{z}^\top A^\top \vec{x} = \vec{z}^\top \vec{0} = 0,$$

so indeed $\vec{x} \in (\operatorname{im} A)^\top$.
Then, let $\vec{x} \in (\operatorname{im} A)^\perp$. Then for all $\vec{z} \in \mathbb{R}^n$, $A\vec{z} \in \operatorname{im} A$ so

$$0 = (A\vec{z}) \cdot \vec{x} = (A\vec{z})^\top \vec{x} = \vec{z}^\top A^\top \vec{x}$$

which implies $A^\top \vec{x} = \vec{0}$, since $\vec{z}$ can be anything in $\mathbb{R}^n$. Therefore, $\vec{x} \in \ker A^\top$.

   (b) (4 points) Assume that $A^2 = A$ and $A^\top = A$.

      (i) Prove that $A\vec{y} = \vec{y}$ for all $\vec{y} \in \operatorname{im}(A)$.

      (ii) Prove that $A\vec{x} = \vec{0}$ for all $\vec{x} \in (\operatorname{im} A)^\perp$.

Solution:

(i) Let $\vec{y} \in \operatorname{im}(A)$, say $\vec{y} = A\vec{x}$ where $\vec{x} \in \mathbb{R}^n$. Then $A\vec{y} = A(A\vec{x}) = A^2\vec{x} = A\vec{x} = \vec{y}$. Since $\vec{y} \in \operatorname{im}(A)$ was arbitrary, we have shown that $A\vec{y} = \vec{y}$ for all $\vec{y} \in \operatorname{im}(A)$.

(ii) Let $\vec{x} \in (\operatorname{im} A)^\perp$. We know that $(\operatorname{im} A)^\perp = \ker A^\top$, and since $A = A^\top$ it follows that $(\operatorname{im} A)^\perp = \ker A$. Thus $\vec{x} \in \ker A$, so $A\vec{x} = \vec{0}$.

(c) (3 points) Give an example of a rank 1 matrix $A \in \mathbb{R}^{2\times 2}$ such that $A^2 = A$ and $A^\top = A$. *(No justification required.)*

Solution: For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is such a matrix.