Def① 2.1.1 Linear Transformations

A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is called a linear transformation if

For $\vec{x} \in \mathbb{R}^m$, def $n \times m$ matrix A s.t.

$$\boxed{T(\vec{x}) = A\vec{x}}$$

Thm① 2.1.2

Consider a linear trans $T: \mathbb{R}^m \to \mathbb{R}^n$.

Let $\boxed{\vec{e_i}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to i^{th}$ component.

(def: $\vec{e_1}, \vec{e_2}, \dots, \vec{e_m}$ 被称为 $\mathbb{R}^m$ 的 standard vector.)

$$\boxed{A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_m) \\ | & | & & | \end{bmatrix}}$$

Def② (extra) Identity matrix

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \updownarrow n \quad \text{(其余元素为0)}$$

Thm② 2.1.3

A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear iff

(a) $\forall \vec{v}, \vec{w} \in \mathbb{R}^m$, $\boxed{T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})}$

(b) $\forall \vec{v} \in \mathbb{R}^m$ and scalar k, $\boxed{T(k\vec{v}) = kT(\vec{v})}$

Def 2.1.4 ③ Distribution vectors and transition matrices

A vector $\vec{x} \in \mathbb{R}^n$ is said to be a distribution vector if its components

1. 和为1
2. 全都 ≥ 0

A square matrix A 为一个 transition matrix if 它的每个 col vector 都是 distribution vector

Def④ 2.2.1 Orthogonal Projection

Consider a line L in coordinate plane, 穿过 $(0,0)$

任何 $\vec{x} \in \mathbb{R}^2$ 都可以写为 $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

其中 $\vec{x}^{\parallel} \parallel L$, $\vec{x}^{\perp} \perp L$.

而这个 $T(\vec{x}) = \vec{x}^{\parallel}$ 的 transformation 叫做 orthogonal projection of $\vec{x}$ onto L.

denoted: $\boxed{proj_L(\vec{x})}$

随意取 $\vec{w} \parallel L$, $\boxed{proj_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}}$

特别地，如果取 $\vec{w}$ 为 unit vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \parallel L$ 　($\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$)

可得 $\boxed{proj_L(\vec{x}) = (\vec{x}\cdot\vec{u})\cdot\vec{u}}$

这一 transformation 是 linear 的，with matrix

$$P = \boxed{\frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}} = \boxed{\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}}$$

Thm③ Scalings

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k.

Def⑤ 2.2.2 Reflection

Consider a line L in coordinate plane, 过 $(0,0)$.

Let $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel}$

$\Rightarrow \boxed{T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}}$ 为 reflection of $\vec{x}$ about L

denoted: $\boxed{ref_L(\vec{x})} = \vec{x}^{\parallel} - \vec{x}^{\perp}$

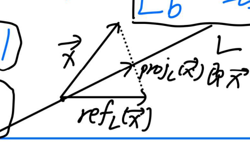
由 Def④ 2.2.1 可得 $ref_L(\vec{x}) = \vec{x}^{\parallel} - (\vec{x} - \vec{x}^{\parallel})$

$= 2proj_L(\vec{x}) - \vec{x} = \boxed{2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}}$

其 matrix of T 为 form: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$,

where $\boxed{a^2+b^2=1}$

$\left(\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}\right)$



Thm ④ 2.2.4 Rotations

$T(\vec{x}) = A\vec{x}$ 表示一个 $\boxed{\text{counterclockwise}}$ rotation in $\mathbb{R}^2$

$A = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$ (rotate by θ)

form: $\boxed{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}$, where $\boxed{a^2+b^2=1}$

Thm ⑤ 2.2.4 Rotations combined with a scaling

将 $\vec{v} \in \mathbb{R}^2$ counterwise rotate θ

并放大至 r 倍:

Let $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$\Rightarrow T(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Thm ⑦ 2.3.3 Matrix multiplication is noncommutative

In general $\boxed{AB \neq BA}$.

$\boxed{\text{有时 } AB = BA}$, 此时称 $\boxed{A, B \text{ commute}}$

Thm ⑧ 2.3.4 The entries of matrix product

$B \in \mathbb{R}^{n\times p}$, $A \in \mathbb{R}^{p\times m}$

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ip} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \end{bmatrix}$$

$(BA)^{n\times m}$ 的 $\boxed{\text{第 } ij \text{ th entry}}$:

$b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{ip}a_{pj} = \boxed{\sum_{k=1}^{p} b_{ik}a_{kj}}$

Def ⑥ 2.3.1 Matrix multiplication

a. Let $\boxed{B}$ be an $\boxed{n\times p}$ matrix

$\boxed{A}$ be an $\boxed{q\times m}$ matrix.

The product BA is $\boxed{\text{defined iff } p=q.}$

b. 接A, 此时 product BA is defined as the matrix of the linear trans

$\boxed{T(\vec{x}) = B(A\vec{x})}$ ← Def

This means:

$\boxed{\forall x \in \mathbb{R}^m,\ T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}}$

The product BA is an $\boxed{n\times m}$ matrix.

Thm ⑥ 2.3.2 matrix product.

Let $B: n\times p$, $A: p\times m$ where $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$

$\Rightarrow BA = B\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}$

Thm ⑨ 2.3.5 Multiplying with identity matrix

$A \in \boxed{\mathbb{R}^{n\times m}}$

$\Rightarrow \boxed{AI_m = I_nA = A}$

Thm ⑩ 2.3.6 Matrix multip is associative

$\forall A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times m}$

$\Rightarrow (AB)C = A(BC)$

我们可以简写 $(AB)C = A(BC) = ABC$

Thm ⑪ 2.3.7 Distributive property for matrices

$A, B \in \mathbb{R}^{n\times p}$, $C, D \in \mathbb{R}^{p\times m}$

$\Rightarrow \boxed{A(C+D) = AC+AD}$

$\boxed{(A+B)C = AC+BC}$

Thm ⑫ 2.3.8

$A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{p \times m}$, k is scalar

$\Rightarrow (kA)B = A(kB) = k(AB)$

Def ⑦ 2.4.1 Invertible Functions

function $T: X \to Y$ is called $\boxed{\text{invertible}}$

if $\forall y \in Y$, $T(x) = y$ has a unique sol in X

如果 T is invertible, define:

$\boxed{T^{-1}(y) = (\text{the unique } x \in X \text{ s.t. } T(x) = y)}$

即 $\boxed{x = T^{-1}(y) \Leftrightarrow y = T(x)}$

注意: $\forall x \in X$, $y \in Y$,

$\boxed{T^{-1}(T(x)) = x}$, $\boxed{T(T^{-1}(y)) = y}$

(conversely 如果 $L: Y \to X$ 使 $\forall x \in X$, $y \in Y$:

$L(T(x)) = x$, $T(L(x)) = y$

那么 T is invertible, 且 $T^{-1} = L$)

Def ⑧ 2.4.2 Invertible matrices

A $\boxed{\text{square}}$ matrix A is said to be invertible if the linear trans $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible.

若成立, denote T^{-1} 的 matrix 为 A^{-1}.

(若 $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible

$\Rightarrow \vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.)

Thm ⑬ 2.4.3 Invertibility

$A^{\boxed{(n\times n)}}$ is $\boxed{\text{invertible}}$ iff $\boxed{\text{rref}(A) = I_n}$

(equivalently, $\boxed{\text{rank}(A) = n}$)

证明: Ch3 中可证.

Thm ⑭ 2.4.4 Invertibility and Linear system

$A^{(n\times n)}$:

(a). Let $\vec{b} \in \mathbb{R}^n$

A is invertible $\Rightarrow A\vec{x} = \vec{b}$ 有unique sol $\vec{x} = A^{-1}\vec{b}$

A is noninvertible $\Rightarrow A\vec{x} = \vec{b}$ (由2.4.3自然得出) 有 inf sol / no sol.

(b). A is invertible $\Rightarrow A\vec{x} = \vec{0}$ 有唯一解 $\vec{x} = \vec{0}$ (无论怎么 elem. trans $\vec{b}$ 总为 $\vec{0}$)

A is noninvertible $\Rightarrow A\vec{x} = \vec{0}$ inf sol. (至少有一个变量随意)

Thm ⑮ 2.4.5 Find the inverse of matrix

Algo: 对于 $\boxed{A^{(n\times n)}}$, form $\boxed{n\overset{2n}{[A \,\vdots\, I_n]}}$

然后 compute $\boxed{\text{rref}[A \,\vdots\, I_n]}$

$\Rightarrow$ if $\text{rref}[A \vdots I_n] = [I_n \vdots B] \Rightarrow A$ 可逆, $A^{-1} = B$ (即 rref(A))

否则 (left, 即 $\text{rref}(A) \neq I_n$) A 不可逆

Thm ⑯ 2.4.6 if $A^{(n\times n)}$ invertible

$\Rightarrow A^{-1}A = AA^{-1} = I_n$

Thm ⑰ 2.4.7 if $A^{(n\times n)}$, $B^{(n\times n)}$ invertible

$\Rightarrow \boxed{BA \text{ inverible}}$

AND $\boxed{(BA)^{-1} = A^{-1}B^{-1}}$ (order matters)

Thm ⑱ 2.4.8 Criterion for invertibility

Let $A^{n\times n}$, $B^{n\times n}$ s.t. $BA = I_n$

$\Rightarrow$ (a) A, B invertible

(b) $A^{-1} = B$, $B^{-1} = A$

(c) $AB = I_n$.

Thm ⑲ 2.4.9 Inverse and determinat of 2x2 matrix

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff

$ad - bc \neq 0$

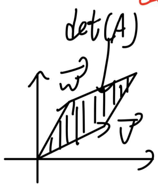
即 $\boxed{\det(A) \neq 0.}$

(b) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Thm ⑳ 2.4.10 2x2 matrix 的 determinant 的 geometrical interpretation

$\det A = \det[\vec{v}\ \ \vec{w}] = \|\vec{v}\| \sin\theta \|\vec{w}\|$ $(-\pi < \theta \leq \pi)$



即: $\boxed{\text{area spanned by } \vec{v}, \vec{w}}$

$\Rightarrow \det(A) = 0$ if $\vec{v} \parallel \vec{w}$

$\det(A) > 0$ if $0 < \theta < \pi$

$\det(A) < 0$ if $-\pi < \theta < 0$