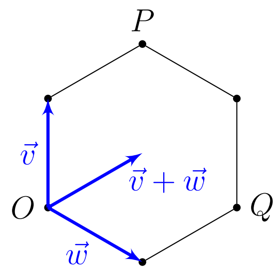
MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 6, SOLUTIONS

Part A

Solve the following problems from the book:
Section 3.4: 50, 70;
Section 4.1: 58;
Section 4.2: 46, 68.

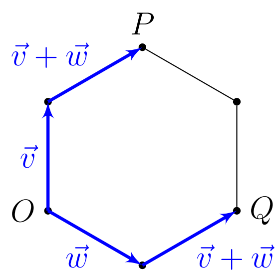
Solution.

3.4.50: It is helpful to note that the 'third direction' is $\vec{v}+\vec{w}$:



.

(a) We have $\overrightarrow{OP} = 2\vec{v}+\vec{w}$ and $\overrightarrow{OQ} = \vec{v}+2\vec{w}$:



.

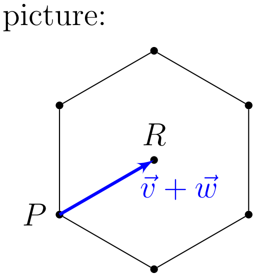
Therefore

$$\left[\overrightarrow{OP}\right]_{\mathfrak{B}} = \begin{bmatrix}2\\1\end{bmatrix}, \qquad \left[\overrightarrow{OQ}\right]_{\mathfrak{B}} = \begin{bmatrix}1\\2\end{bmatrix}.$$

(b) Note that

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (3\vec{v}+2\vec{w}) - (2\vec{v}+\vec{w}) = \vec{v}+\vec{w},$$

so we have the following picture:



.

Thus R is the center of a tile.

(c) Note that every vertex looks locally either like Y or ⅄. Starting at a ⅄ vertex and adding either $2\vec{v}+\vec{w}$ or $\vec{v}+2\vec{w}$ takes us to another ⅄ vertex. Since O is a ⅄ vertex and $17\vec{v}+13\vec{w} = 7(2\vec{v}+\vec{w})+3(\vec{v}+2\vec{w})$, we see that S is a ⅄ vertex.

3.4.70: No. Proceed by contradiction and suppose that there exists an ordered basis $\mathfrak{B} = (\vec{v}, \vec{w})$ of $\mathbb{R}^2$ such that $[T]_{\mathfrak{B}}$ is upper-triangular. Then we can write

$$[T]_{\mathfrak{B}} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{for some } a, b, c \in \mathbb{R}.$$

In particular, $T(\vec{v}) = a\vec{v}$. But note that T is rotation by $90°$ counterclockwise about $\vec{0}$, so the only vector sent to a scalar multiple of itself is $\vec{0}$. Thus $\vec{v} = \vec{0}$. But $\vec{0}$ cannot appear in a basis, which is a contradiction.

4.1.58: First observe that V is a subspace of vector space of functions from $\mathbb{R}$ to $\mathbb{R}$:
- the zero function $\vec{0}$ is in V;
- if $f, g \in V$, then $(f+g)'' = f''+g'' = -f+(-g) = -(f+g)$, so $f+g \in V$;
- if $f \in V$ and $c \in \mathbb{R}$, then $(cf)'' = cf'' = c(-f) = -(cf)$, so $cf \in V$.

(a) Let $g \in V$, so that $g'' = -g$. In order to show that $g^2+(g')^2$ is constant, it suffices to show that its derivative is the zero function $\vec{0}$. We have

$$(g^2+(g')^2)' = 2gg'+2g'g'' = 2gg'+2g'(-g) = \vec{0},$$

as desired.

(b) Suppose that $g \in V$ such that $g(0) = g'(0) = 0$. By part (a), the function $g^2+(g')^2$ is constant, and by assumption, it equals 0 when $x = 0$. Therefore $g^2+(g')^2 = \vec{0}$. Since the square of any function is everywhere nonnegative, we conclude that both g and g' are identically zero.

(c) Let $f \in V$. Observe that $\sin(x)$ and $\cos(x)$ are both in V, since

$$(\sin(x))'' = -\sin(x), \qquad (\cos(x))'' = -\cos(x).$$

Since V is closed under addition and scalar multiplication, get that $g \in V$, where by definition $g(x) := f(x) - f(0)\cos(x) - f'(0)\sin(x)$ for all $x \in \mathbb{R}$. We have

$$g(0) = f(0) - f(0)\cos(0) - f'(0)\sin(0) = f(0) - f(0) - 0 = 0,$$

and since $g'(x) = f'(x) + f(0)\sin(x) - f'(0)\cos(x)$, we have

$$g'(0) = f'(0) + f(0)\sin(0) - f'(0)\cos(0) = f'(0) + 0 - f'(0) = 0.$$

Thus by part (b), we get $g = \vec{0}$, i.e. $f(x) = f(0)\cos(x) + f'(0)\sin(x)$ for all $x \in \mathbb{R}$.

4.2.46: T is linear but not an isomorphism. To see that T is linear, note that for any $f(t), g(t) \in \mathcal{P}$ and $c \in \mathbb{R}$, we have

$$\begin{aligned} T((f+g)(t)) = (t-1)((f+g)(t)) &= (t-1)(f(t)+g(t)) \\ &= (t-1)f(t)+(t-1)g(t) = T(f(t))+T(g(t)), \end{aligned}$$

$$T((cf)(t)) = (t-1)((cf)(t)) = c(t-1)f(t) = cT(f(t)).$$

To show that T is not an isomorphism, we will prove that T is not surjective. Note that every polynomial $g(t) \in \mathcal{P}$ in the image of T satisfies $g(1) = 0$. However, not every element of $\mathcal{P}$ satisfies this condition, for example the constant polynomial 1.
[Note: T is injective. Therefore T is an example of a linear transformation from a vector space V to itself which is injective but not surjective. Such transformations exist only when V is infinite-dimensional.]

4.2.68: We claim that T is an isomorphism if and only if $k \neq 1, 5$. To prove this, we first observe that since T is a linear transformation from a finite-dimensional vector space to itself, it is injective if and only if it is surjective (this follows from the rank-nullity theorem). Therefore, T is an isomorphism if and only if it is injective, if and only if $\ker(T) = \{\vec{0}\}$. Thus we must show that $\ker(T) = \{\vec{0}\}$ if and only if $k \neq 1, 5$.

Note that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 3a & -b \\ (5-k)c & (1-k)d \end{bmatrix} \quad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

We see that if $k \neq 1, 5$, then $\ker(T) = \{\vec{0}\}$. Conversely, if $k = 1$, then $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is in $\ker(T)$, while if $k = 5$, then $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is in $\ker(T)$.

Part B

Problem 1. Let V be a vector space, and let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a list of vectors in V. Define the function $T : \mathbb{R}^n \to V$ by

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \text{for all } \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

(a) Prove that T is a linear transformation.
(b) Prove that T is injective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.
(c) Prove that T is surjective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V.
(d) Prove that T is an isomorphism if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V.

Solution.

(a) We must prove that T respects addition and scalar multiplication. To this end, let $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Then

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix}\right) = (c_1 + d_1)\vec{v}_1 + \cdots + (c_n + d_n)\vec{v}_n$$

$$= (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + \cdots + d_n\vec{v}_n) = T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) + T\left(\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}\right),$$

and

$$T\left(k\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\right) = T\left(\begin{bmatrix}kc_1\\ \vdots\\ kc_n\end{bmatrix}\right) = (kc_1)\vec{v}_1 + \cdots + (kc_n)\vec{v}_n$$

$$= k(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = kT\left(\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\right).$$

(b) We have the following chain of equivalent statements:

$$\begin{aligned} & T \text{ is injective} \\ \iff & \ker(T) = \{\vec{0}\} \\ \iff & \text{for all } c_1, \dots, c_n \in \mathbb{R}, \text{ we have } c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0} \text{ if and only if } c_1 = \cdots = c_n = 0 \\ \iff & (\vec{v}_1, \dots, \vec{v}_n) \text{ is linearly independent.} \end{aligned}$$

(c) We have that T is surjective if and only if $\mathrm{im}(T) = V$, and since $\mathrm{im}(T) = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_n)$, this is equivalent to $\mathrm{span}(\vec{v}_1, \dots, \vec{v}_n) = V$.

(d) We have the following chain of equivalent statements:

$$\begin{aligned} & T \text{ is an isomorphism} \\ \iff & T \text{ is injective and surjective} \\ \iff & (\vec{v}_1, \dots, \vec{v}_n) \text{ is linearly independent and spans } V \qquad \text{(by parts (b) and (c))} \\ \iff & (\vec{v}_1, \dots, \vec{v}_n) \text{ is an ordered basis of } V. \end{aligned}$$

Problem 2. For a 2×2 matrix $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, define the **transpose** of A to be the matrix

$$A^T = \begin{bmatrix}a & c\\ b & d\end{bmatrix}.$$

Consider the linear transformation

$$T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad T(A) = \frac{1}{2}(A + A^T).$$

(a) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where

$$\mathcal{E} = \left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)$$

is the standard ordered basis of $\mathbb{R}^{2\times 2}$.

(b) Find the $\mathfrak{C}$-matrix of T, where $\mathfrak{C}$ is the ordered basis

$$\mathfrak{C} = \left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)$$

(c) Compute the kernel of $[T]_{\mathcal{E}}$. This will be a subspace of the $\mathcal{E}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.

(d) Find a basis for the corresponding subspace of $\mathbb{R}^{2\times 2}$—that is, for the image of $\ker[T]_{\mathcal{E}}$ under the coordinate isomorphism $L_{\mathcal{E}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.

(e) Compute the kernel of the $\mathfrak{C}$-matrix. This will be a subspace of the $\mathfrak{C}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.
(f) Compute the image of the subspace $\ker[T]_{\mathfrak{C}}$ under the coordinate isomorphism $L_{\mathfrak{C}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.
(g) Compare your answers in (d) and (f). How are they related to $\ker T$?
(h) Find a basis for the image of T using **either** $\mathcal{E}$-coordinates or $\mathfrak{C}$-coordinates (which seems easier?) Don't forget to reinterpret vectors in the coordinate space as elements in $\mathbb{R}^{2\times 2}$!

Solution.

(a) Denote elements of $\mathcal{E}$ by $\mathcal{E} = (E_1, E_2, E_3, E_4)$. Then
- $[T(E_1)]_{\mathcal{E}} = [\frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}]_{\mathcal{E}} = [E_1 + 0E_2 + 0E_3 + 0E_4]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$,
- $[T(E_2)]_{\mathcal{E}} = [\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]_{\mathcal{E}} = [\frac{1}{2}E_2 + \frac{1}{2}E_3]_{\mathcal{E}} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T$,
- $[T(E_3)]_{\mathcal{E}} = [\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]_{\mathcal{E}} = [\frac{1}{2}E_2 + \frac{1}{2}E_3]_{\mathcal{E}} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T$,
- $[T(E_4)]_{\mathcal{E}} = [\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}]_{\mathcal{E}} = [E_4]_{\mathcal{E}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

Thus,
$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Let C_1, C_2, C_3, and C_4, respectively, be the elements of $\mathfrak{C}$. Note that $T(C_i) = C_i$ for $i = 1, 2, 3$ and $T(C_4) = \mathbf{0} \in \mathbb{R}^{\mathbf{2\times 2}}$. So the $\mathfrak{C}$-matrix is
$$[T]_{\mathfrak{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) The kernel of $[T]_{\mathcal{E}}$ is the Span of $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}^4$, by inspection, using our Chapter 1 skills.

(d) The corresponding space of $\mathbb{R}^{2\times 2}$ is Spanned by $L_{\mathcal{E}}^{-1}(\vec{v}) = E_2 - E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(e) The kernel of $[T]_{\mathfrak{C}}$ is the Span of $\vec{e}_4 \in \mathbb{R}^4$, by inspection.

(f) Since the kernel of $[T]_{\mathfrak{C}}$ is Spanned by $\vec{e}_4$, its image under $L_{\mathfrak{C}}^{-1}$ will be Spanned by $L_{\mathfrak{C}}^{-1}(\vec{e}_4)$, which is C_4. Thus the subspace of $\mathbb{R}^{2\times 2}$ corresponding to $\ker[T]_{\mathfrak{C}}$ is the set $\{\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \mid a \in \mathbb{R}\}$.

(g) Note that the answers in (d) and (f) are the same, and both are equal to $\ker T$. This makes sense, as the $\mathcal{E}$-coordinates and $\mathfrak{C}$-coordinates are just two different models of the same set-up.

(h) The $\mathfrak{C}$-coordinates are easier to work with. We see the image of $[T]_{\mathfrak{C}}$ in the $\mathfrak{C}$-coordinate space is Spanned by the columns of $[T]_{\mathfrak{C}}$, and these three columns are the $\mathfrak{C}$-coordinates of C_1, C_2, C_3. So $\operatorname{im} T$ is the Span of $\{C_1, C_2, C_3\}$. These must be a basis for $\operatorname{im} T$, since by rank-nullity theorem and the fact that the kernel is 1 dimensional, we know that $\dim \operatorname{im} T = \dim \mathbb{R}^{2\times 2} - \dim \ker T = 4 - 1 = 3$.

Problem 3. Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. In other words, every vector $f \in C^\infty(\mathbb{R})$ is a function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable k-times for all $k \in \mathbb{N}$. Let $f_1, \dots, f_6$ be the six functions in $C^\infty(\mathbb{R})$ defined by

$$f_1(x) = 1, \quad f_2(x) = \sin(2x), \quad f_3(x) = \cos(2x),$$

$$f_4(x) = \sin^2(x), \quad f_5(x) = \cos^2(x), \quad f_6(x) = \sin x \cos x.$$

Let $V = \operatorname{Span}(f_1, f_2, f_3, f_4, f_5, f_6)$, and let $\mathcal{B} = (f_1, f_2, f_4) = (1, \sin 2x, \sin^2 x)$.

(a) Prove that $\mathcal{B}$ is an ordered basis of V. [*Hint*: For linear independence, write a relation and evaluate it at one or more carefully-chosen values of x. For spanning, remember (or look up) some trig identities.]

(b) For each $i \in \{1, \dots, 6\}$, find $[f_i]_{\mathcal{B}}$.

(c) Show that for all $f \in V$, the derivative of f is also in V.

(d) As a result of (c), we can define the linear transformation $T : V \to V$ by $T(f) = f' + 2f$ for all $f \in V$. Compute the $\mathfrak{B}$-matrix $[T]_{\mathfrak{B}}$ of T.

(e) **Without using Calculus**, find $[T]_{\mathcal{B}}^{-1}$.

(f) Using **matrix methods only** (and **without directly using calculus**), find a function $f(x) \in V$ such that

$$f'(x) + 2f(x) = 4 + 8\sin^2(x)$$

Note: In (e) and (f) you will **not** receive credit for computing integrals using "Calc 2" methods (e.g., u-substitution) or methods from the theory of differential equations.

Solution.

(a) First we show that $\mathcal{B}$ is linearly independent. Let $a, b, c \in \mathbb{R}$, let $\vec{0}_V$ be the zero vector in V, and suppose $af_1 + bf_2 + cf_4 = \vec{0}_V$. Since $\vec{0}_V$ is the constant zero function, this means that

$$a + b\sin 2x + c\sin^2 x \;=\; 0 \quad \text{for all } x \in \mathbb{R}.$$

In particular, setting $x = 0$ gives us

$$0 \;=\; a + b\sin(2\cdot 0) + c\sin^2(0) \;=\; a + 0 + 0 \;=\; a.$$

Then, using $a = 0$ and setting $x = \frac{\pi}{2}$, we get

$$0 \;=\; b\sin(\pi) + c\sin^2\left(\tfrac{\pi}{2}\right) \;=\; 0 + c \;=\; c.$$

Finally, now that we know $a = c = 0$, setting $x = \frac{\pi}{4}$ gives us

$$0 \;=\; b\sin(2\cdot\frac{\pi}{4}) \;=\; b\sin(\frac{\pi}{2}) \;=\; b.$$

Thus $a = b = c = 0$, showing that $\mathcal{B}$ is indeed linearly independent.

Now we show that $\mathcal{B}$ Spans V. It will suffice to show $f_3, f_5, f_6 \in \mathrm{Span}(\mathcal{B})$. But this follows from trig identities:

$$\begin{aligned} f_3 &= \cos(2x) &= 1 - 2\sin^2 x &= 1f_1 - 2f_4 \\ f_5 &= \cos^2 x &= 1 - \sin^2 x &= 1f_1 - f_4 \\ f_6 &= \sin x \cos x &= \tfrac{1}{2}\sin(2x) &= \tfrac{1}{2} f_2. \end{aligned}$$

(b) Using our calculations in part (a), we see that

$$[f_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ [f_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ [f_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \ [f_4]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ [f_5]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ [f_6]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}.$$

(c) Since every vector in V can be expressed as a linear combination of f_1, f_2, f_4, and since differentiation respects sums and scalar multiplication, it is sufficient to find the derivatives of each of the functions f_1, f_2, f_4, and show that each derivative can be written as a linear combination of those same functions (which in turn shows that it lies in V). We compute:

$$\begin{aligned} f_1'(x) &= 0 = 0f_1(x) + 0f_2(x) + 0f_4(x) \\ f_2'(x) &= 2\cos 2x = 2(1 - 2\sin^2 x) = 2f_1(x) - 4f_3(x) \\ f_4'(x) &= 2\sin x \cos x = \sin 2x = f_2(x) \end{aligned}$$

(d) We find $T(f_i)$ for each or our basis vectors, and write the result as a linear combination of the basis vectors (using our answers from (c) to save time):

$$\begin{aligned} T(f_1)(x) &= f_1'(x) + 2f_1(x) \\ &= 0 + 2(1) \\ &= 2f_1(x) \\ T(f_2)(x) &= f_2'(x) + 2f_2(x) \\ &= (2f_1(x) - 4f_3(x)) + 2f_2(x) \\ &= 2f_1(x) + 2f_2(x) - 4f_3(x) \\ T(f_3(x)) &= f_3'(x) + 2f_3(x) \\ &= f_2(x) + 2f_3(x) \end{aligned}$$

Consequently the $\mathcal{B}$-matrix of T is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & 2 \end{bmatrix}$$

(e) We perform row operations on an augmented matrix to find $[T]_{\mathcal{B}}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 8 & 0 & -4 & 4 & -4 & 0 \\ 0 & 8 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & -2 & 1 \\ 0 & 8 & 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 1/4 & -1/8 \\ 0 & 0 & 1 & 0 & 1/2 & 1/4 \end{array}\right]$$

We conclude that

$$[T]_{\mathcal{B}}^{-1} = \begin{bmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/4 & -1/8 \\ 0 & 1/2 & 1/4 \end{bmatrix}$$

(f) We want to find a function f such that $T(f) = 4 + 8\sin^2(x)$. By the Generalized Key Theorem, for such a function f we would have

$$[T]_{\mathcal{B}}[f]_{\mathcal{B}} = [4 + 8\sin^2(x)]_{\mathcal{B}}$$

That is, we wish to solve the matrix equation

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & -4 & 2 \end{bmatrix} [f]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$$

We could solve this equation by performing row operations on the augmented matrix $\begin{bmatrix} 2 & 2 & 0 & | & 4 \\ 0 & 2 & 1 & | & 0 \\ 0 & -4 & 2 & | & 8 \end{bmatrix}$, but since we already have found $[T]_{\mathcal{B}}$, we will just use that:

$$[f]_{\mathcal{B}} = \begin{bmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/4 & -1/8 \\ 0 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Since $[f]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, we conclude that

$$f(x) = 3(1) - 1(\sin 2x) + 2(\sin^2 x)$$

is the desired function.

* * *

Let A and B be sets. Recall from the handout *More Joy of Sets* that we define the *Cartesian product* of A and B to be the set

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

If X and Y are vector spaces, then $X \times Y$ is also a vector space, with addition and scalar multiplication given by

$$(\vec{x}, \vec{y}) + (\vec{x}', \vec{y}') = (\vec{x} + \vec{x}', \vec{y} + \vec{y}'), \qquad c(\vec{x}, \vec{y}) = (c\vec{x}, c\vec{y})$$

for all $(\vec{x}, \vec{y}), (\vec{x}', \vec{y}') \in X \times Y$ and $c \in \mathbb{R}$.

Problem 4. Let X and Y be finite-dimensional vector spaces.

(a) Describe the zero vector of $X \times Y$. (*No justification necessary.*)

(b) Let $\{\vec{x}_1, \ldots, \vec{x}_m\}$ be a basis of X, and let $\{\vec{y}_1, \ldots, \vec{y}_n\}$ be a basis of Y. Prove that

$$\{(\vec{x}_1, \vec{0}_Y), \ldots, (\vec{x}_m, \vec{0}_Y), (\vec{0}_X, \vec{y}_1), \ldots, (\vec{0}_X, \vec{y}_n)\}$$

is a basis of $X \times Y$.

(c) Determine $\dim(X \times Y)$ in terms of $\dim(X)$ and $\dim(Y)$.

Solution.

(a) The zero vector of $X \times Y$ is $(\vec{0}_X, \vec{0}_Y)$.

(b) Let $\mathcal{B} := \{(\vec{x}_1, \vec{0}_Y), \dots, (\vec{x}_m, \vec{0}_Y), (\vec{0}_X, \vec{y}_1), \dots, (\vec{0}_X, \vec{y}_n)\}$. First we show that $\mathcal{B}$ is linearly independent. To this end, suppose that $c_1, \dots, c_m, d_1, \dots, d_n$ are scalars such that

$$c_1(\vec{x}_1, \vec{0}_Y) + \dots + c_m(\vec{x}_m, \vec{0}_Y) + d_1(\vec{0}_X, \vec{y}_1) + \dots + d_n(\vec{0}_X, \vec{y}_n) = (\vec{0}_X, \vec{0}_Y).$$

Then

$$(c_1\vec{x}_1 + \dots + c_m\vec{x}_m, d_1\vec{y}_1 + \dots + d_n\vec{y}_n) = (\vec{0}_X, \vec{0}_Y),$$

so

$$c_1\vec{x}_1 + \dots + c_m\vec{x}_m = \vec{0}_X \quad \text{and} \quad d_1\vec{y}_1 + \dots + d_n\vec{y}_n = \vec{0}_Y.$$

Since $\{\vec{x}_1, \dots, \vec{x}_m\}$ and $\{\vec{y}_1, \dots, \vec{y}_n\}$ are both linearly independent, we get $c_1 = \dots = c_m = 0$ and $d_1 = \dots = d_n = 0$. Thus $\mathcal{B}$ is linearly independent.

Now we show that $\mathcal{B}$ spans $X \times Y$. Let $(\vec{x}, \vec{y})$ be any vector in $X \times Y$. Since $\{\vec{x}_1, \dots, \vec{x}_m\}$ spans X, we can write

$$\vec{x} = c_1\vec{x}_1 + \dots + c_m\vec{x}_m \quad \text{for some } c_1, \dots, c_m \in \mathbb{R}.$$

Similarly, since $\{\vec{y}_1, \dots, \vec{y}_n\}$ spans Y, we can write

$$\vec{y} = d_1\vec{y}_1 + \dots + d_n\vec{y}_n \quad \text{for some } d_1, \dots, d_n \in \mathbb{R}.$$

Then

$$\begin{aligned}(\vec{x}, \vec{y}) &= (c_1\vec{x}_1 + \dots + c_m\vec{x}_m, d_1\vec{y}_1 + \dots + d_n\vec{y}_n) \\ &= c_1(\vec{x}_1, \vec{0}_Y) + \dots + c_m(\vec{x}_m, \vec{0}_Y) + d_1(\vec{0}_X, \vec{y}_1) + \dots + d_n(\vec{0}_X, \vec{y}_n) \in \operatorname{span}(\mathcal{B}).\end{aligned}$$

(c) Take a basis $\{\vec{x}_1, \dots, \vec{x}_m\}$ of X and a basis $\{\vec{y}_1, \dots, \vec{y}_n\}$ of Y. By part (b), $\{(\vec{x}_1, \vec{0}_Y), \dots, (\vec{x}_m, \vec{0}_Y), (\vec{0}_X, \vec{y}_1), \dots, (\vec{0}_X, \vec{y}_n)\}$ is a basis of $X \times Y$. Thus

$$\dim(X \times Y) = m + n = \dim(X) + \dim(Y).$$

Problem 5. Let V be a vector space, and let X and Y be subspaces of V. Define the function $T : X \times Y \to X + Y$ by

$$T(\vec{x}, \vec{y}) := \vec{x} + \vec{y} \quad \text{for all } (\vec{x}, \vec{y}) \in X \times Y.$$

(a) Prove that T is a linear transformation and that T is surjective.
(b) Prove that $\ker(T)$ is isomorphic to $X \cap Y$.
(c) Assuming that X and Y are finite-dimensional, prove that

$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y).$$

(d) Let X and Y be 3-dimensional subspaces of $\mathbb{R}^5$. Is it possible that $X \cap Y = \{\vec{0}\}$? Now instead assume that X and Y are 3-dimensional subspaces of $\mathbb{R}^6$. Is it possible that $X \cap Y = \{\vec{0}\}$? Prove your answers.

Solution.

(a) First we show that T is linear. Let $(\vec{x},\vec{y}),(\vec{x}',\vec{y}') \in X \times Y$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} T((\vec{x},\vec{y})+(\vec{x}',\vec{y}')) = T(\vec{x}+\vec{x}',\vec{y}+\vec{y}') &= (\vec{x}+\vec{x}')+(\vec{y}+\vec{y}') \\ &= (\vec{x}+\vec{y})+(\vec{x}'+\vec{y}') = T(\vec{x},\vec{y})+T(\vec{x}',\vec{y}'), \end{aligned}$$

$$T(c(\vec{x},\vec{y})) = T(c\vec{x},c\vec{y}) = c\vec{x}+c\vec{y} = c(\vec{x}+\vec{y}) = cT(\vec{x},\vec{y}).$$

Thus T is linear. To see that T is surjective, note that any element of $X+Y$ can be written as $\vec{x}+\vec{y}$ for some $\vec{x} \in X, \vec{y} \in Y$, and $\vec{x}+\vec{y} = T(\vec{x},\vec{y})$.

(b) Note that if $\vec{x} \in X \cap Y$, then $-\vec{x} \in Y$ (since Y is closed under scalar multiplication), and $(\vec{x},-\vec{x}) \in \ker(T)$ (since $T(\vec{x},-\vec{x}) = \vec{x}+(-\vec{x}) = \vec{0}$). Therefore we may define the function $f : X \cap Y \to \ker(T)$ by

$$f(\vec{x}) := (\vec{x},-\vec{x}) \quad \text{for all } \vec{x} \in X \cap Y.$$

Let us show that f is an isomorphism, from which it follows that $\ker(T)$ is isomorphic to $X \cap Y$. First note that f is linear, since for all $\vec{x},\vec{x}' \in X \cap Y$ and $c \in \mathbb{R}$, we have

$$f(\vec{x}+\vec{x}') = (\vec{x}+\vec{x}',-(\vec{x}+\vec{x}')) = (\vec{x},-\vec{x})+(\vec{x}',-\vec{x}') = f(\vec{x})+f(\vec{x}'),$$

$$f(c\vec{x}) = (c\vec{x},-(c\vec{x})) = c(\vec{x},-\vec{x}) = cf(\vec{x}).$$

Also, f is injective, since if $\vec{x} \in \ker(f)$, then $(\vec{x},-\vec{x}) = (\vec{0},\vec{0})$, whence $\vec{x} = \vec{0}$. Finally, to see that f is surjective, let $(\vec{x},\vec{y}) \in \ker(T)$. Then $T(\vec{x},\vec{y}) = \vec{0}$, so $\vec{y} = -\vec{x}$. Thus $(\vec{x},\vec{y}) = (\vec{x},-\vec{x}) = f(\vec{x}) \in \operatorname{im}(f)$.

(c) Since T is surjective, we have $\operatorname{rank}(T) = \dim(X+Y)$. Since $\ker(T)$ is isomorphic to $X \cap Y$, we have $\operatorname{nullity}(T) = \dim(X \cap Y)$. By Problem 4, part (c), we have $\dim(X \times Y) = \dim(X)+\dim(Y)$. Thus the desired equality is simply $\operatorname{rank}(T)+\operatorname{nullity}(T) = \dim(X \times Y)$, which is the rank-nullity theorem applied to T.

(d) If X and Y are 3-dimensional subspaces of $\mathbb{R}^5$, it is not possible that $X \cap Y = \{\vec{0}\}$. This follows from part (c): since $X+Y$ is a subspace of $\mathbb{R}^5$, we have $\dim(X+Y) \leq 5$, so

$$\dim(X \cap Y) = \dim(X)+\dim(Y)-\dim(X+Y) \geq 3+3-5 = 1.$$

However, there do exist 3-dimensional subspaces X and Y of $\mathbb{R}^6$ such that $X \cap Y = \{\vec{0}\}$. For example, let X have the basis $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$, and let Y have the basis $\{\vec{e}_4,\vec{e}_5,\vec{e}_6\}$. Then by construction $\dim(X) = \dim(Y) = 3$, and $X \cap Y = \{\vec{0}\}$ since $\{\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_6\}$ is linearly independent.