MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 6, DUE Thursday, March 7 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

Part A

Solve the following problems from the book:
Section 3.4: 50, 70;
Section 4.1: 58;
Section 4.2: 46, 68.

Part B

Problem 1. Let V be a vector space, and let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a list of vectors in V. Define the function $T : \mathbb{R}^n \to V$ by

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \text{for all } \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

(a) Prove that T is a linear transformation.
(b) Prove that T is injective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.
(c) Prove that T is surjective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V.
(d) Prove that T is an isomorphism if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V.

Problem 2. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define the **transpose** of A to be the matrix

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Consider the linear transformation

$$T \colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad T(A) = \frac{1}{2}(A + A^T).$$

(a) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where

$$\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

is the standard ordered basis of $\mathbb{R}^{2\times 2}$.

(b) Find the $\mathfrak{C}$-matrix of T, where $\mathfrak{C}$ is the ordered basis

$$\mathfrak{C} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

(c) Compute the kernel of $[T]_{\mathcal{E}}$. This will be a subspace of the $\mathcal{E}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.

(d) Find a basis for the corresponding subspace of $\mathbb{R}^{2\times 2}$—that is, for the image of $\ker[T]_{\mathcal{E}}$ under the coordinate isomorphism $L_{\mathcal{E}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.

(e) Compute the kernel of the $\mathfrak{C}$-matrix. This will be a subspace of the $\mathfrak{C}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.

(f) Compute the image of the subspace $\ker[T]_{\mathfrak{C}}$ under the coordinate isomorphism $L_{\mathfrak{C}}^{-1} : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$.

(g) Compare your answers in (d) and (f). How are they related to $\ker T$?

(h) Find a basis for the image of T using **either** $\mathcal{E}$-coordinates or $\mathfrak{C}$-coordinates (which seems easier?) Don't forget to reinterpret vectors in the coordinate space as elements in $\mathbb{R}^{2\times 2}$!

Problem 3. Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. In other words, every vector $f \in C^\infty(\mathbb{R})$ is a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable k-times for all $k \in \mathbb{N}$. Let $f_1, \ldots, f_6$ be the six functions in $C^\infty(\mathbb{R})$ defined by

$$f_1(x) = 1, \quad f_2(x) = \sin(2x), \quad f_3(x) = \cos(2x),$$

$$f_4(x) = \sin^2(x), \quad f_5(x) = \cos^2(x), \quad f_6(x) = \sin x \cos x.$$

Let $V = \operatorname{Span}(f_1, f_2, f_3, f_4, f_5, f_6)$, and let $\mathcal{B} = (f_1, f_2, f_4) = (1, \sin 2x, \sin^2 x)$.

(a) Prove that $\mathcal{B}$ is an ordered basis of V. [*Hint*: For linear independence, write a relation and evaluate it at one or more carefully-chosen values of x. For spanning, remember (or look up) some trig identities.]

(b) For each $i \in \{1, \ldots, 6\}$, find $[f_i]_{\mathcal{B}}$.

(c) Show that for all $f \in V$, the derivative of f is also in V.

(d) As a result of (c), we can define the linear transformation $D : V \to V$ by $D(f) = f'$ for all $f \in V$. Compute the $\mathfrak{B}$-matrix $[D]_{\mathfrak{B}}$ of D.

(e) **Without using calculus**, compute $[D]_{\mathfrak{B}}^{-1}$.

(f) Using $[D]_{\mathfrak{B}}^{-1}$, but **without directly using calculus**, find an antiderivative of

$$3\cos^2(x) + 4\sin(x)\cos(x) + \sin(2x).$$

Note: In (e) and (f) you will **not** receive credit for computing integrals using "Calc 2" methods (e.g., u-substitution or related techniques.

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Let A and B be sets. Recall from the handout *More Joy of Sets* that we define the *Cartesian product* of A and B to be the set

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

If X and Y are vector spaces, then $X \times Y$ is also a vector space, with addition and scalar multiplication given by

$$(\vec{x}, \vec{y}) + (\vec{x}', \vec{y}') = (\vec{x} + \vec{x}', \vec{y} + \vec{y}'), \qquad c(\vec{x}, \vec{y}) = (c\vec{x}, c\vec{y})$$

for all $(\vec{x}, \vec{y}), (\vec{x}', \vec{y}') \in X \times Y$ and $c \in \mathbb{R}$.

Problem 4. Let X and Y be finite-dimensional vector spaces.

(a) Describe the zero vector of $X \times Y$. (*No justification necessary.*)

(b) Let $\{\vec{x}_1, \dots, \vec{x}_m\}$ be a basis of X, and let $\{\vec{y}_1, \dots, \vec{y}_n\}$ be a basis of Y. Prove that
$$\{(\vec{x}_1, \vec{0}_Y), \dots, (\vec{x}_m, \vec{0}_Y), (\vec{0}_X, \vec{y}_1), \dots, (\vec{0}_X, \vec{y}_n)\}$$
is a basis of $X \times Y$.

(c) Determine $\dim(X \times Y)$ in terms of $\dim(X)$ and $\dim(Y)$.

Problem 5. Let V be a vector space, and let X and Y be subspaces of V. Define the function $T : X \times Y \to X + Y$ by
$$T(\vec{x}, \vec{y}) := \vec{x} + \vec{y} \quad \text{for all } (\vec{x}, \vec{y}) \in X \times Y.$$

(a) Prove that T is a linear transformation and that T is surjective.

(b) Prove that $\ker(T)$ is isomorphic to $X \cap Y$.

(c) Assuming that X and Y are finite-dimensional, prove that
$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y).$$

(d) Let X and Y be 3-dimensional subspaces of $\mathbb{R}^5$. Is it possible that $X \cap Y = \{\vec{0}\}$? Now instead assume that X and Y are 3-dimensional subspaces of $\mathbb{R}^6$. Is it possible that $X \cap Y = \{\vec{0}\}$? Prove your answers.