1-3

26. Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

Since $A\vec{x} = \vec{b}$ has a unique solution,
and A is a 4×3 matrix
By $\boxed{\text{example 3}}$ on 1-3,
we know then $\underline{\text{rank}(A) = 3}$
and $\text{rank}([A \vdots \vec{b}]) = 3$
Therefore the question falls into 2 cases.

<u>Case 1</u>: By the same elementary transformations we do to turn $[A \vdots \vec{b}]$ into its $\text{rref}([A \vdots \vec{b}])$, we get $\text{rref}([A \vdots \vec{c}])$ to be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 0 \end{array}\right]$
for some $\alpha, \beta, \gamma \in \mathbb{R}$.
In this case, we have $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$,
the system has exactly 1 solution.

<u>Case 1</u>: By the same elementary transformations we do to turn $[A \vdots \vec{b}]$ into its $\text{rref}([A \vdots \vec{b}])$

we get $\mathrm{rref}([A \vdots \vec{c}])$ to be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & \varepsilon \end{array}\right]$

for some $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$

where $\varepsilon \neq 0$

Note that we have $[0\,0 \ldots \vdots\, \varepsilon]$, $\varepsilon \neq 0$ in the system

Hence it is inconsistent with no solution by $\boxed{\text{Theorem 1.3.1}}$

34. We define the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$.

a. For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

compute $A\vec{e}_1$, $A\vec{e}_2$, and $A\vec{e}_3$.

b. If B is an $n \times 3$ matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, what are $B\vec{e}_1$, $B\vec{e}_2$, and $B\vec{e}_3$?

a. By <u>Theorem 1.3.8</u>,

$$A\vec{e}_1 = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$$

$$= \vec{v}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$$

$$A\vec{e_2} = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0\vec{v_1} + 1\vec{v_2} + 0\vec{v_3}$$

$$= \vec{v_2} = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$$

$$A\vec{e_3} = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0\vec{v_1} + 0\vec{v_2} + 0\vec{v_3}$$

$$= \vec{v_3} = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$$

b. By Theorem 1.3.8, same as a,

$$B\vec{e_1} = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v_1}$$

$$B\vec{e_2} = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v_2}$$

$$B\vec{e_3} = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v_3}$$

48. Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

a. If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

b. If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

a. Since $\vec{x_h}$ is a solution to $A\vec{x}=\vec{0}$, we have $A\vec{x_h}=\vec{0}$

Since $\vec{x_1}$ is a solution to $A\vec{x}=\vec{b}$, we have $A\vec{x_1}=\vec{b}$

Then $A\vec{x_1}-A\vec{x_h}=\vec{b}-\vec{0}=\vec{b}$ ①

By $\boxed{\text{Theorem 1.3.10}}$,

$A\vec{x_1}+A\vec{x_h}=A(\vec{x_1}+\vec{x_h})$ ②

Combining ①, ②, $A(\vec{x_1}+\vec{x_h})=\vec{b}$

Therefore $\vec{x_1}+\vec{x_h}$ is a solution to $A\vec{x}=\vec{b}$.

b. Similar to a, we have $A\vec{x_2}=\vec{b}$

Then we have $A\vec{x_2}-A\vec{x_1}=\vec{b}-\vec{b}=\vec{0}$ ①

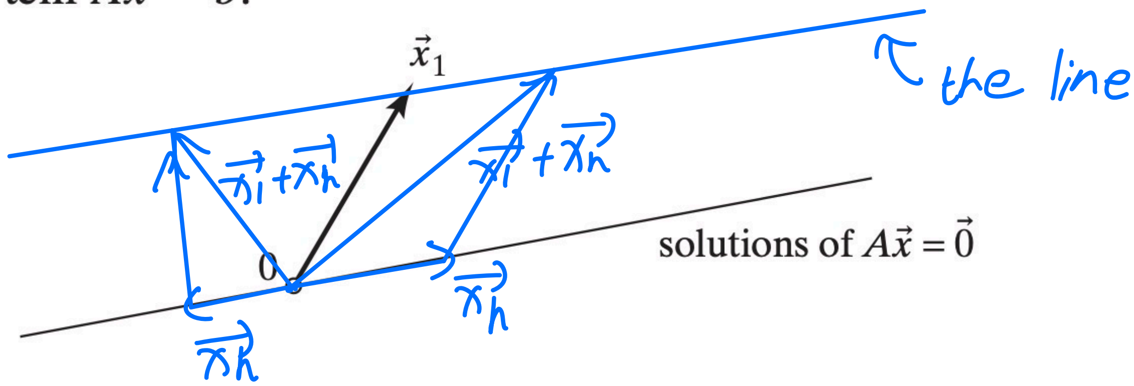
By Theorem 1.3.10,

$A\vec{x_2}-A\vec{x_1}=A(\vec{x_2}-\vec{x_1})$ ②

Combining ①, ②, we have $A(\vec{x_2}-\vec{x_1})=\vec{0}$

Therefore $\vec{x_2}-\vec{x_1}$ is a solution of $A\vec{x}=\vec{0}$

c. Now suppose A is a 2×2 matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$.



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

From a we have proved that
$\vec{x}_1$ is a solution of $A\vec{x} = \vec{b}$
AND $\vec{x}_h$ is a solution of $A\vec{x} = \vec{0}$
implies $\vec{x}_1 + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$
AND since all vectors starting from o, on the line in the sketch is a solution to $A\vec{x} = \vec{0}$.

Let a be an arbitrary point on the line, then the vector tailed at o, pointing to a is an instance of $\vec{x}_h$. Using the vector addition's rule, the $\vec{x}_1 + \vec{x}_h$ must point towards the line parallel to the sketch line, running through the head of $\vec{x}_1$ tailed at o.

Therefore all solution form the line parallel to the sketch line of solutions of $A\vec{x}=0$, running through the head of $\vec{x_1}$ tailed at 0.

2-1

6. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ by } \boxed{\text{Theorem 1.3.8}}$$

So for any $\vec{x}, \vec{y} \in \mathbb{R}^2$, $T(\vec{x}+\vec{y}) = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}(\vec{x}+\vec{y}) = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\vec{y}$ $\boxed{\text{by Theorem 1.2.10}}$

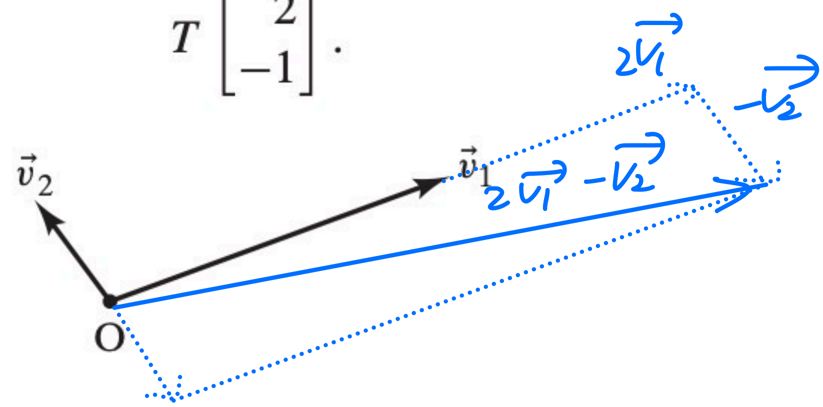
and, for any scalar k, $kT(\vec{x}) = k\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\vec{x}\right) = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}(k\vec{x})$ by $\boxed{\text{Teorem 1.3.10}}$

Therefore it is a linear transformation by definition on Worksheet 4.

Its matrix A is $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ by key Teorem on Worksheet 4.

38. The two column vectors $\vec{v}_1$ and $\vec{v}_2$ of a 2×2 matrix A are shown in the accompanying sketch. Consider the linear transformation $T(\vec{x}) = A\vec{x}$, from $\mathbb{R}^2$ to $\mathbb{R}^2$. Sketch the vector

$$T\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$



$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$

Therefore $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= 2\vec{v}_1 - \vec{v}_2$ $\boxed{\text{by Theorem 1.3.8}}$

44. The cross product of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

See Definition A.9 and Theorem A.11 in the Appendix. Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

Let $\vec{v}$ be $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{x}$ be $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$(v_1, v_2, x_3, x_1, x_2, x_3 \in \mathbb{R})$

By Theorem A.11.

$$\vec{v} \times \vec{x} = \begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix} = \begin{bmatrix} 0 - v_3 x_2 + v_2 x_3 \\ v_3 x_1 + 0 - v_1 x_3 \\ -v_2 x_1 + v_1 x_2 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \vec{x}$$

Denote $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$ by A and by Theorem 1.3.10,

① for any $\vec{x_1}, \vec{x_2} \in \mathbb{R}^3$, $T(\vec{x_1} + \vec{x_2}) = \vec{v} \times (\vec{x_1} + \vec{x_2}) = A[\vec{x_1} + \vec{x_2}]$
$= A\vec{x_1} + A\vec{x_2}$
$= T(\vec{x_1}) + T(\vec{x_2})$

② AND for any scalar k, $kT(\vec{x}) = k\vec{v} \times \vec{x} = kA\vec{x} = A(k\vec{x})$

Therefore the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ is linear, by definition on Worksheet 4.

its matrix A is $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$ by key Theorem on Worksheet 4.

46. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$. See Exercise 45. *Hint*: Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

Since $T(\vec{x}) = B(A\vec{x})$

$$T(\vec{e_1}) = B(A\vec{e_1}) = B\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = B \cdot \begin{bmatrix} a \\ c \end{bmatrix}$$

$$= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ap + cq \\ ar + cs \end{bmatrix}$$

$$T(\vec{e_2}) = B(A\vec{e_2}) = B\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = B \begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} bp + dq \\ br + ds \end{bmatrix}$$

By Theorem 2.1.2, $A = \begin{bmatrix} | & | \\ T(\vec{e_1}) & T(\vec{e_2}) \\ | & | \end{bmatrix}$

$$= \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix}$$

Part B (25 points)

Let X and Y be sets. Recall that a **function** f **from** X **to** Y is a rule which assigns a unique element $f(x) \in Y$ to each element $x \in X$. We call X the **domain** or **source** of f, we call Y the **codomain** or **target space** of f, and we write $f : X \to Y$ to indicate that f is a function from X to Y. The **image** of f is defined to be the set $\text{im}(f) = \{f(x) \mid x \in X\}$. (Note that the codomain and the image of f are not necessarily equal.) We say that the function $f : X \to Y$ is:

- **surjective** or **onto** if for all $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$;
- **injective** or **one-to-one** if for all $y \in \text{im}(f)$ there exists at most one $x \in X$ such that $f(x) = y$;
- **bijective** if f is both injective and surjective.

Problem 1. In parts (a) – (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

(a) the function $f : [0, 4] \to [0, 18]$ defined by $f(x) = x^2 + 2$;
(b) the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x - 5$;
(c) the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = 2x^2 + 5y^2$;
(d) the function $q : \mathbb{N} \to \mathbb{N}$ defined by $q(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$

(a) the function is injective but not surjective

Proof. (1) f is not surjective.

Counterexample: consider $0 \in Y$,

There does not exist $x \in [0,4]$

such that $f(x) = x^2+2 = 0$

since $x^2 > 0$, $x^2+2 > 2$.

(2) Proof of f is injective.

We want to show that for any $x_1, x_2 \in [0,4]$,

$x_1 = x_2$ whenever $f(x_1) = f(x_2)$ ①

Assume that for $a, b \in [0,4]$,

$f(a) = f(b)$ ($\in Im(f)$)

Then $a^2+2 = b^2+2$, so $a = \pm b$

If $a \neq b$, then $a \neq b \neq 0$, one of

a, b must be positive and the

other one must be negative.

But since $X = [0,4]$, $a, b \geq 0$, so impossible.

Therefore we have proved statement ①.

Hence f is injective.

(b) The function is bijective.

Proof. (1) Proof of g is surjective.

Let y be an arbitrary element of $\mathbb{R}$

Consider $x = \frac{y+5}{2}$,

Note that $x \in \mathbb{R}$ and $g(x) = 2x - 5 = y$

Since the selection of y is arbitrary, we have proved that g is surjective by definition.

(2) Proof of g is injective.

We want to show that for all $a, b \in \mathbb{R}$, if $g(a) = g(b)$, then $a = b$

Assume that there exists $x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2) \in Im(g)$

Then $2x_1 - 5 = 2x_2 - 5$, $x_1 = x_2$.

Therefore we have proved the statement

(3) Since g is injective and surjective, it is bijective.

(c) h is neither injective nor surjective.

Counterexample Consider: $h(1, 2) = h(-1, -2) = 22$

So h is not injective

Counterexample Consider: $-1 \in \mathbb{R}$

For any $x, y \in \mathbb{R}$, $x^2, y^2 \geq 0$,

so $h(x, y) \geq 0$ thus can not equal to -1

Therefore h is not surjective.

(d) q is surjective but not injective.

(1) q is not injective

Counterexample Consider $n=1$, $m=2 \in \mathbb{N}$

$q(n)=1$, $q(m)=\frac{2}{2}=1$, $q(n)=q(m)$

So q is not injective

(2) Proof of q is surjective.

Let y be an arbitrary natural number

Then it falls into two cases.

Case 1 y is even

Consider $x=2y$, then x is even and $x \in \mathbb{N}$.

we have $q(x)=\frac{x}{2}$

Case 2 y is odd

Consider $x=y$. then x is odd and $x \in \mathbb{N}$

we have $q(x)=y$

In both cases, exists some $x \in \mathbb{N}$ such that $q(x)=y$, covering all situations.

So q is surjective by definition.

Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define the **direct image** or **forward image of** A **under** f to be the set

$$f[A] = \{f(x) \mid x \in A\}.$$

Similarly, if $B \subseteq Y$ then we define the **preimage** or **inverse image of** B **under** f to be the set

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$

Note that $f[X] = \text{im}(f)$.

Problem 2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

(a) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.
(b) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$.
(c) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f^{-1}[f[A]] = A$.
(d) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f[X \setminus A] = Y \setminus f[A]$.
(e) For every bijective function $f : X \to Y$ and all $A, B \subseteq X$, we have $f[A \cap B] = f[A] \cap f[B]$.

(a) False.

Counterexample. $f : [-2, 2] \to [0, 4]$

$f(x) = x^2$

Let $A = [-2, -1]$, $B = [1, 2]$

Note that $A \cap B = \emptyset$

but $f(-2) = f(2) = 4$

This means that $f[A] \cap f[B]$ at least have 4 as an element, so here $f[A] \cap f[B] \neq \emptyset$.

(b) True.

We prove it by contradiction.

Let $f : X \to Y$ be a function and $A, B \subseteq X$ with $f[A] \cap f[B] = \emptyset$.

Assume for sake of contradiction that $A \cap B \neq \emptyset$. This means that there exists some element $a \in A \cap B$, then we know $a \in A$ and $a \in B$

Since $a \in A$, $f(a) \in f[A]$ by definition
Since $a \in B$, $f(a) \in f[B]$ by definition.
Therefore $f[A]$ and $f[B]$ has at least $f(a)$ as a common element.
So $f[A] \cap f[B] \neq \phi$, which contradicts with the assumption.
Then we have proved the statement.

(c) False

Counterexample Consider

$$f: [-2,2] \to [0,4]$$
$$f(x) = x^2$$

Let $A = [0,2]$

By definition $f^{-1}[f[A]] = \{x \in [-2,2] \mid x^2 \in f[A]\}$

Since $A = [0,2]$, $f[A] = [0,4]$

So $f^{-1}[f[A]] = \{x \in [-2,2] \mid x^2 \in [0,4]\}$
$= [-2,2] \neq A$.

(d) False

Counterexample Still consider

$$f: [-2,2] \to [0,4]$$
$$f(x) = x^2$$

And let $A = (0,2]$

So $f[X \setminus A] = f[[-2,0]] = [0,4]$

$Y \setminus f[A] = [0,4] \setminus (0,4] = \{0\}$

$f[X \setminus A] \neq Y \setminus f[A]$

(e) True.

Proof (1) Want to show:

$\boxed{f[A \cap B] \subseteq f[A] \cap f[B]}$

Let $m \in f[A \cap B] = \{f(x) \mid x \in A \cap B\}$

This implies that for some $x \in A \cap B$, $f(x) = m$

By definition of intersection,

$x \in A$ and $x \in B$.

So $m \in \{f(x) \mid x \in A\}$ and $\{f(x) \mid x \in B\}$

So by definition of intersection

$m \in \{f(x) \mid x \in A\} \cap \{f(x) \mid x \in B\}$

So $m \in f[A] \cap f[B]$

Hence $\boxed{f[A \cap B] \subseteq f[A] \cap f[B]}$ ①

(2) Want to show:

$\boxed{f[A] \cap f[B] \subseteq f[A \cap B]}$

Let $m \in f[A] \cap f[B]$

So by definition of intersection, $m \in f[A]$ and $m \in f[B]$

that is, $m \in \{f(x) \mid x \in A\}$ and $\{f(y) \mid y \in B\}$

Therefore for some $x \in A$, $y \in B$

$f(x) = m$, $f(y) = m$

Since the function is bijective, it is injective

injective, and this implies $x = y$ by definition.

So $x = y \in A$ and $x = y \in B$

By definition of intersection $x = y \in A \cap B$

Therefore $m \in \{f(x) \mid x \in A \cap B\}$

that is $m \in f[A \cap B]$

Hence $\boxed{f[A] \cap f[B] \subseteq f[A \cap B]}$ ②

Combining ①, ②, we have $f[A] \cap f[B] = f[A \cap B]$

We call a function $T : \mathbb{R}^m \to \mathbb{R}^n$ a *linear transformation* if it satisfies:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$; and

(2) $T(k\vec{x}) = kT(\vec{x})$ for all vectors $\vec{x} \in \mathbb{R}^m$ and all scalars $k \in \mathbb{R}$.

(Note that this definition differs from the one given in Section 2.1 of the textbook.)

Problem 3.

(a) Prove that for every function $f : \mathbb{R} \to \mathbb{R}$, if $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, prove that every function $f : \mathbb{R} \to \mathbb{R}$ that preserves scalar multiplication is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.)

(b) Give an example to show that the argument you gave in part (a) cannot work in 2 dimensions. That is, explicitly describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is *not* a linear transformation but has the property that $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Remember to prove that your example works!

(a) Proof. We will denote the property $f(cx)=cf(x)$ for all $x\in\mathbb{R}$ and $c\in\mathbb{R}$ as ①

By ① we know $f(0x) = 0f(0) = 0$

$0x=0$, so $f(0)=0$ ②

And $f(-x)=-f(x)$ ③

(1) First, we prove $f(x+y)= f(x)+f(y)$ when $x+y=0$. By ②, $f(x+y) = f(0) = 0$

Since $x+y=0$, $y=-x$, by ③ we know $f(y)=-f(x)$, so $f(x)+f(y)=f(x)-f(x)=0$.

So we have proved $f(x+y)=f(x)+f(y)$ whenever $x+y=0$.

(2) Now we prove it for any $x,y\in\mathbb{R}$ such that $x+y\neq 0$.

Let x,y be arbitrary real numbers. with $y\neq -x$

Take $c= x+y$,

then $f(cx) = f(x(x+y)) = (x+y)f(x)$

$f(cy) = f(y(x+y)) = (x+y)f(y)$

Since $f(x(x+y)) = xf(x+y)$ by ①

$f(y(x+y)) = yf(x+y)$

We have $xf(x+y)+yf(x+y)=(x+y)f(x)+(x+y)f(y)$

Therefore $(x+y)f(x+y)=(x+y)(f(x)+f(y))$

then $f(x+y)=f(x)+f(y)$ by dividing $(x+y)$ on both sides.

Now we have proved the complete statement.

(b) Example: $f(\vec{x})=||\vec{x}||$ (norm)

Then all $\vec{x}\in\mathbb{R}^2$, $c\in\mathbb{R}$

$$f(c\vec{x})=||c\vec{x}||=c||\vec{x}||$$
$$=cf(\vec{x})$$

But for any $\vec{y}\in\mathbb{R}^2$

$$f(\vec{x}+\vec{y})=||\vec{x}+\vec{y}||$$

$$f(\vec{x})+f(\vec{y})=||\vec{x}||+||\vec{y}||$$

By triangle inequality, $||\vec{x}+\vec{y}||=||\vec{x}||+||\vec{y}||$ only when $\vec{x}\,//\,\vec{y}$. For example, take $\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\2\end{bmatrix}$, then $||\vec{x}+\vec{y}||=5$

$||\vec{x}||=\sqrt{3}$, $||\vec{y}||=2$,

So in this case, $||\vec{x}+\vec{y}||\neq||\vec{x}||+||\vec{y}||$.

$f(c\vec{x})=cf(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$, $c\in\mathbb{R}^2$ but

$f(x+y)=f(x)+f(y)$ does not hold true for all $x,y\in\mathbb{R}^2$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, suppose that f preserves addition).

(a) Prove that $f(0) = 0$.
(b) Prove that for all $x \in \mathbb{R}$, $f(-x) = -f(x)$.
(c) Use induction to prove that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(nx) = nf(x)$.
(d) Prove that for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, $f(mx) = mf(x)$.
(e) (**RECREATIONAL**) Prove that for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$, $f(qx) = qf(x)$.

Remark: It will perhaps come as a surprise that the property in (c)-(e) cannot be extended to include arbitrary real scalars. That is, there exist functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ but also $f(cx) \neq cf(x)$ for some $c, x \in \mathbb{R}$. Put yet another way, there exist functions $f : \mathbb{R} \to \mathbb{R}$ that preserve addition but not scalar multiplication. *This* fact is actually rather difficult to prove!)

We will denote the property $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ as ①

(a) Taking $x, y = 0$,
By ①, $f(0) = f(0) + f(0) = 2f(0)$
Deducting $f(0)$ from both sides we have $f(0) = 0$.

(b) Let x be an arbitrary real number
Take $y = -x$ and by ①, $f(x+y) = f(x) + f(y)$
$= f(x) + f(-x)$
Since $f(x+y) = f(0) = 0$,
we have $f(x) + f(-x) = 0$, so $f(x) = -f(-x)$
Since x is arbitrary, we have proved the statement.

(c) Base case: $n=1$, for any $x \in \mathbb{R}$, $f(nx) = f(x)$
$= n f(x)$.

Inductive step: We want to show that
if for $n \in \mathbb{N}$, $f(nx) = nf(x)$ for any $x \in \mathbb{R}$,
then for $n+1$, $f((n+1)x) = (n+1) f(x)$ for any $x \in \mathbb{R}$

Let $x \in \mathbb{R}$, $n \in \mathbb{N}$ be arbitrary.

Assume that $f(nx) = nf(x)$

then $f((n+1)x) = f(nx+x)$
$= f(nx) + f(x)$ by ①
$= nf(x) + f(x)$
$= (n+1) f(x)$

Then we have prove the inductive step

Therefore by induction, we have proved that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(nx) = nf(x)$

(d) Since we have prove that $f(0) = 0$ and $f(nx) = nf(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$

We only need to prove $f(mx) = mf(x)$ for all $m \in \mathbb{Z}_{<0}$, since $\mathbb{Z} = \mathbb{Z}_{<0} \cup \{0\} \cup \mathbb{N}$.

Since in (b) we have proved that for all $x \in \mathbb{R}$, $f(-x) = -f(x)$

Then let $m \in \mathbb{Z}$ be an arbitrary negative integer

We have $-m \in \mathbb{N}$, and $\underbrace{f(-(mx)) = -f(mx)}_{②}$ by (b)

Since $-m \in \mathbb{N}$, $\underbrace{f[(-m)x] = -m f(x)}_{③}$ by (c)

Combining ②, ③, we have $-f(mx) = -mf(x)$

so $f(mx) = mf(x)$

Since m, x is arbitrary, we have proved the statement for $m \in \mathbb{Z}_{<0}$

And combing the conclusion with (a), (c),

we have prove $f(mx) = mf(x)$ for all $x \in \mathbb{R}$

for all $m \in \mathbb{Z}$.

(e) Since we have proved that $\underbrace{f(mx) = mf(x) \text{ for all } m \in \mathbb{Z},\ x \in \mathbb{R}}_{④}$

Then let n be an arbitrary integer.

m be an arbitrary integer with $m \neq 0$.

Let x be an arbitrary integer

Take $y = \frac{n}{m}x$. By ④, $f(my) = mf(y)$

so $f(nx) = mf(\frac{n}{m}x)$

Since $f(nx) = nf(x)$ by ④

$nf(x) = mf(\frac{n}{m}x)$

So $f(\frac{n}{m}x) = \frac{n}{m}f(x)$.

Since x, m, n are arbitrary integers, any $q \in \mathbb{Q}$ except 0 is covered (and we have prove the statement for $m=0$), we have extended the statement to any $m \in \mathbb{Q}$.