Math 217: "Prove or Disprove" Practice for Final Exam

On the Exam and on this review, the words "eigenvalue, eigenvector, eigenbasis, and eigenspace" all refer to the these concepts over the **real numbers** unless otherwise stated. The word "diagonalizable" means **diagonalizable over the real numbers**, unless we explicitly say "diagonalizable over the complex numbers."

1. A square matrix is invertible if and only if zero is not an eigenvalue.

   Solution note: True. Zero is an eigenvalue means that there is a non-zero element in the kernel. For a square matrix, being invertible is the same as having kernel zero.

2. If $T : V \to V$ and $S : V \to V$ are linear transformations, both with eigenvalue 5, then $T \circ S$ also has eigenvalue 5.

   Solution note: False. This is silly. Let $A = B = 5I_2$, so T and S are both the "dilation by 5" map of $\mathbb{R}^2$. Then the eigenvalues of AB are 25. The composition of T and S dilates by 25.

3. If A and B are 2×2 matrices, both with eigenvalue 5, then $A + B$ also has eigenvalue 5.

   Solution note: False. This is silly. Let $A = B = 5I_2$. Then the eigenvalues of $A+B$ are 10.

4. A square matrix has determinant zero if and only if zero is an eigenvalue.

   Solution note: True. Both conditions are the same as the kernel being non-zero.

5. If B is the $\mathfrak{B}$-matrix of some linear transformation $V \xrightarrow{T} V$. Then for all $\vec{v} \in V$, we have $B[\vec{v}]_{\mathfrak{B}} = [T(\vec{v})]_{\mathfrak{B}}$.

   Solution note: True. This is the definition of $\mathfrak{B}$-matrix.

6. If $V \xrightarrow{T} W \xrightarrow{S} V'$ are linear transformations, then $\text{im}(ST) \subset \text{im}S$.

   Solution note: True: An arbitrary element in $\text{im}(ST)$ has the form $S(T(v))$ for some $v \in V$. This is in $\text{im}S$ since it is S applied to the element $T(v)$ in W.

7. If $V \xrightarrow{T} W \xrightarrow{S} V'$ are linear transformations, then $\ker(T) \subset \ker(ST)$.

   Solution note: True: Take $x \in \ker(T)$. Then $T(x) = 0$, so clearly also $S(T(x)) = S(0) = 0$, whence $x \in \ker(ST)$.

8. Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathfrak{B} = (f_1, f_2, f_3)$. Then f_1 is an eigenvector.

Solution note: True. It has eigenvalue 1. The first column of the $\mathfrak{B}$-matrix is telling us that $T(f_1) = f_1$.

9. Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathfrak{B} = (f_1, f_2, f_3)$. Then $T(f_1 + f_2 + f_3)$ is $6f_1 + 2f_2 + f_3$.

Solution note: TRUE! The $\mathfrak{B}$-coordinates of $f_1 + f_2 + f_3$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. To get the $\mathfrak{B}$-coordinates of $T(f_1 + f_2 + f_3)$ we just multiply by the matrix $[T]_{\mathfrak{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. This represents the vector $6f_1 + 2f_2 + f_3$.

10. If A and B are similar, then they have the same trace and determinant.

Solution note: True. Similar matrices have the same eigenvalues. The trace is the sum of the eigenvalues, the determinant is the product.

11. Let $T : V \to V$ be a linear transformation, and suppose that T has a $\mathcal{B}$-matrix which is lower triangular for some $\mathcal{B} = (f_1, \dots, f_n)$. Then T has at least one eigenvector.

Solution note: True. If the matrix is lower triangular, it means that the last column is a scalar multiple of e_n. The last column tells us the $\mathcal{B}$-coordinates of $T(f_n)$. So $T(f_n)$ is a scalar multiple of f_n.

12. There exists a linear transformation with exactly 6 eigenvectors.

Solution note: False! If there is one eigenvector, there are infinitely many, since every non-zero scalar multiple of an eigenvector is also an eigenvector.

13. The polynomials $x + 1, x^3, x^2$ span the vector space $\mathcal{P}_3$.

Solution note: False! $\mathcal{P}_4$ is four dimensional, so no set of three vectors can span it.

14. The polynomials $x + 1, x^3, x^2, x^3 + x^2 + x + 1$ span the vector space $\mathcal{P}_3$.

Solution note: False! Since $x^3 + x^2 + x + 1$ is the sum of the other three, the span of these four polynomials is the same as the span of the first three, which we have already seen is not $\mathcal{P}_3$.

15. The set of polynomials $\{x+1, x^3, x^2, x^3+x^2+x+1\}$ is a linearly independent set in $\mathcal{P}_5$.

Solution note: False! Here is a non-trivial relation: $(x+1)+(x^3)+(x^2)+-1\ (x^3+x^2+x+1)$.

16. Suppose that T is a linear transformation of rank 5 from the space $U^{3\times 3}$ of upper triangular matrices to itself. If the characteristic polynomial of T is $(x-1)(x-2)(x-3)(x-4)(x-5)(x-b)$, then it is possible to find the exact value of b.

Solution note: True! Since T has a 6 dimensional source and a five dimensional image, it's kernel has dimension 1. So its kernel is not zero, which means that 0 is an eigenvalue. So b must be zero!

17. The rank of $\frac{d^2}{dx^2}$ on $\mathcal{P}_{17}$ is 16.

Solution note: True! The kernel is the space spanned by $1, x$. By rank-nullity, since $\mathcal{P}_{17}$ has dimension 18, we know that the dimension of the image is 2. This is the rank.

18. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ form a basis for the space of symmetric 2×2 matrices.

Solution note: TRUE. They are clearly all in the space of symmetric matrices and are linearly independent. But the space of symmetric matrices has dimension less than 4, since not every matrix is symmetric. So it must have dimension 3, in which case these are a basis.

19. If $f, g, h \in P_6$ are eigenvectors for a linear transformation $T : \mathcal{P}_6 \to \mathcal{P}_6$ with eigenvalues $3, 4, 0$ respectively, then $T(2f-3g+h) = 6f-12g$.

Solution note: True. $T(2f-3g+h) = 2T(f)-3T(g)+T(h) = 2(3f)-3(4)g+0h$.

20. The only rotation $\mathbb{R}^2 \to \mathbb{R}^2$ which has a real eigenvalue are rotations that induce the identity transformation (so through $\pm 2\pi, \pm 4\pi$, etc).

Solution note: FALSE! Rotation through π has eigenvalue -1.

21. If the change of basis matrix $S_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} \vec{e}_4 & \vec{e}_3 & \vec{e}_2 & \vec{e}_1 \end{bmatrix}$, then the elements of $\mathcal{A}$ are the same as the element of $\mathcal{B}$, but in a different order.

Solution note: True. The matrix tells us that the first element of $\mathcal{A}$ is the fourth element of $\mathcal{B}$, the second element of basis $\mathcal{A}$ is the third element of $\mathcal{B}$, the third element of basis $\mathcal{A}$ is the second element of $\mathcal{B}$, and the the fourth element of basis $\mathcal{A}$ is the first element of $\mathcal{B}$.

22. The map assigning $\langle A, B\rangle$ to trace(AB^T) is an inner product on the space of all $\mathbb{R}^{2\times 2}$ matrices.

Solution note: TRUE. It satisifies the four axioms in 5.5.

23. If $T : \mathbb{R}^{7\times 8} \to \mathbb{R}^{3\times 8}$ is a linear transformation whose 0-eigenspace has dimension 33, then T is surjective.

> *Solution note:* False. By rank nullity, T is surjective if and only if the dimension of the kernel is dimension 32.

24. An orthogonal matrix must have at least one real eigenvalue.

> *Solution note:* False! Rotation through 90 degrees is orthogonal but has no real eigenvalues!

25. The determinant of the differentiation map of $\mathcal{P}_3$ is zero.

> *Solution note:* True! A linear transformation $T : V \to V$ has zero determinant if and only if it is *not* an isomorphism. Since the kernel here includes all constant polynomials, the map is not injective and hence not an isomorphism.

26. If A is a 3×4 matrix, then the matrix $A^T A$ is similar to a diagonal matrix with three or less non-zero entries.

> *Solution note:* True! The matrix $A^T A$ is symmetric, so by the spectral theorem, it is similar to a diagonal matrix. But also, its rank is at most 3 since we had a homework exercise in which we checked that the rank of a matrix can not go up when we multiply by any other matrix, so rank $A^T A$ can't be more than rank A which is at most 3 since A is 3×4. So the 4×4 matrix $A^T A$ has rank at most 3 which means it is not invertible. This means zero is an eigenvalue. Since the eigenvalues are the elements on the diagonal of this diagonal matrix, this means there is a zero on the diagonal, so at most 3 non-zero entries.

27. If A is similar to both D_1 and D_2, where D_1 and D_2 are diagonal, then $D_1 = D_2$.

> *Solution note:* False! The elements on the diagonal are the eigenvalues, but they could be arranged in different orders. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ are similar, taking $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

28. If A and B are similar to Q, then A is similar to B.

> *Solution note:* True! Suppose $SA = QS$ and $BT = TQ$ for some invertible S and T. Then substituting $Q = T^{-1}BT$, we have $SA = QS = (T^{-1}BT)S$, so $A = A^{-1}T^{-1}B(TS) = (TS)^{-1}B(TS)$.

29. Let u and v be any two orthonormal vectors in an inner product space. Then $||u - v|| = \sqrt{2}$.

> *Solution note:* True. $||u-v||$ is the square root of $(u-v)\cdot(u-v) = u\cdot u - 2u\cdot v + v\cdot v = 2$.

30. If $\langle x, y\rangle = -\langle y, x\rangle$ in some inner product space, then x is orthogonal to y.

Solution note: True! We know $\langle x, y\rangle = \langle y, x\rangle$ by the symmetric property of inner products, so the hypothesis forces $\langle x, y\rangle = 0$.

31. A linear transformation of a 7-dimensional space to itself has at least one real eigenvalue.

Solution note: True! The characteristic polynomial is degree 7. An odd degree polynomial always has at least one root.

32. Let $V \xrightarrow{T} V$ be a linear transformation, and suppose that x and y are linearly independent eigenvectors with *different* eigenvalues. Then $x + y$ is NOT an eigenvector.

Solution note: TRUE! Say $T(x) = k_1 x$ and $T(y) = k_2 y$. Suppose $T(x+y) = k_3(x+y)$. Then $T(x+y) = T(x) + T(y)$ so $k_3(x+y) = k_1 x + k_2 y$. Rewriting, we have $(k_3 - k_1)x + (k_3 - k_2)y = 0$. Since x and y are linearly independent, this relation must be trivial so $(k_3 - k_1) = (k_3 - k_2) = 0$. This implies $k_1 = k_2 = k_3$.

33. If $\langle x, y\rangle = \langle x, z\rangle$ for vectors x, y, z in an inner product space, then $y - z$ is orthogonal to x.

Solution note: True: $0 = \langle x, y\rangle - \langle x, z\rangle = \langle x, y - z\rangle$, so the x and $y - z$ are orthogonal.

34. For any matrix $A \in \mathbb{R}^{n\times d}$ and any column vector $\vec{b} \in \mathbb{R}^n$, the system $A^T A\vec{x} = A^T\vec{b}$ is consistent.

Solution note: True! The solutions are the least squares solutions.

35. If A is the $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ matrix of a transformation T and $\begin{bmatrix}2\\0\\1\\0\end{bmatrix}$ are the $\mathfrak{B}$-coordinates of $\vec{x}$, then $T(\vec{x}) = 2\vec{v}_1 + \vec{v}_3$.

Solution note: False!! By definition of $\mathfrak{B}$-matrix, we know $[T]_{\mathfrak{B}}[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix}T(\vec{v}_1) & T(\vec{v}_2) & T(\vec{v}_3) & T(\vec{v}_4)\end{bmatrix}\begin{bmatrix}2\\0\\1\\0\end{bmatrix} = 2T(\vec{v}_1) + T(\vec{v}_3)$. This does not have to equal $2\vec{v}_1 + \vec{v}_3$. The zero transformation for T is an explicit counterexample.

36. If A is the $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ matrix of a transformation T and $T(\vec{v}_3) = \vec{v}_1 + \vec{v}_3$, then $A\vec{e}_3 = \vec{e}_1 + \vec{e}_3$.

Solution note: True. The third column of A tells us the $\mathfrak{B}$-coordinates of $T(\vec{v}_3)$. This should be $[1\ \ 0\ \ 1\ \ 0]^T$. Also the third column of A is $A\vec{e}_3$.

37. For any $n \times n$ matrix A, and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have $A\vec{x}\cdot\vec{y} = \vec{x}\cdot A^T\vec{y}$.

Solution note: True! $A\vec{x}\cdot\vec{y} = (A\vec{x})^T y = \vec{x}^T A^T y = \vec{x}\cdot A^T\vec{y}$.

38. For any symmetric $n \times n$ matrix A, and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have $A\vec{x} \cdot \vec{y} = A\vec{y} \cdot \vec{x}$.

> *Solution note:* True! As in previous, using $A = A^T$.

39. If an 5×5 matrix P has eigenvalues $1, 2, 4, 8$ and 16, then P is similar to a diagonal matrix.

> *Solution note:* Yes! There are 5 *different* eigenvalues and the matrix is size 5×5. So the geometric multiplicity of each is (at least) 1, and the sum is (at least) 5. So the matrix is diagonalizable by the theorem.

40. If A is an orthogonal matrix, then its only real eigenvalues are ± 1.

> *Solution note:* True. Say v is an eigenvector, so that $A\vec{v} = av$ for some a. Since A preserves lengths, we know $||v|| = ||Av|| = ||av|| = |a|||v||$ so $|a| = 1$.

41. The functions $\sin x$ and $\cos x$ are orthogonal in the inner product defined by $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} fg dx$.

> *Solution note:* True. Check $\langle \sin x, \cos x \rangle = 0$. This is easy since $\sin x \cos x$ is an odd function.

42. Suppose we have an inner product space V and w and v are orthonormal vectors in V. Then for any $f \in V$, the element $\langle w, f \rangle w + \langle v, f \rangle v$ is the closest vector to f in the span of v and w.

> *Solution note:* True! This is the formula for the projection of f onto the span of the $\{v, w\}$ (because they are an orthonormal basis!). The projection is the closest vector.

43. In any inner product space, $||f|| = \langle f, f \rangle$ for all f.

> *Solution note:* False! Must square root!

44. Consider $\mathbb{R}^{2\times 2}$ as an inner product space with the inner product $\langle A, B \rangle = \text{trace } A^T B$. Then $|| \begin{bmatrix} a & b \\ c & d \end{bmatrix} || = \sqrt{(a^2 + b^2 + c^2 + d^2)}$.

> *Solution note:* True. Compute $||A|| = \sqrt{\langle A, A \rangle} = \sqrt{trace(A^T A)} =$ $\sqrt{trace \begin{bmatrix} a^2 + b^2 & - \\ - & c^2 + d^2 \end{bmatrix}}$.

45. The matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are orthonormal in the inner product $\langle A, B \rangle = \text{trace } A^T B$ on $\mathbb{R}^{2\times 2}$.

> *Solution note:* False. They are perpendicular (orthogonal) but not of length one. Each has $||A|| = \sqrt{2}$.

46. If f and g are elements in an inner product space satisfying $||f|| = 2$, $||g|| = 4$ and $||f+g|| = 5$, then it is possible to find the exact value of $\langle f, g \rangle$

> *Solution note:* True: $5^2 = ||f + g||^2 = \langle f + g, f + g \rangle = ||f||^2 + 2\langle f, g \rangle + ||g||^2 = 2^2 + 2(\langle f, g \rangle) + 4^2$. This can be solved for $\langle f, g \rangle$, which is $5/2$.

47. If $(\vec{v}_1, \ldots, \vec{v}_d)$ is a basis for the subspace V of $\mathbb{R}^n$ and $\vec{b} \in V$, then the least squares solutions of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$ are exact solutions to $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$.

Solution note: True: since $\vec{b} \in V$, it is in the span of the columns of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix}$. So it is consistent, so the least squares solutions are actual solutions.

48. Let V and W be distinct planes in $\mathbb{R}^3$ and let $\phi_V : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi_W : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projections onto V and W, respectively. Then the matrices of ϕ_V and ϕ_W in the standard basis are similar.

Solution note: True! Both transformations are diagonalizable with eigenvalues $1, 1, 0$. So both matrices are similar to a diagonal matrix with diagonal $1, 1, 0$.

49. If $(\vec{v}_1, \ldots, \vec{v}_d)$ is a basis for the subspace V of $\mathbb{R}^n$ and $\vec{b} \in V^\perp$, then the only least squares solutions of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$ is the zero vector.

Solution note: True. Because $\vec{b} \in V^\perp$, the projection of $\vec{b}$ to V is zero. This means that the least squares solutions are the actual solutions to $A\vec{x} = \vec{0}$. Since the columns of A are a basis, they are linearly independent, which means that A is invertible. So the only solution of $A\vec{x} = \vec{0}$ is 0.

50. Suppose a is an eigenvalue of an invertible matrix A. Then a^{-1} is an eigenvalue of A^{-1}.

Solution note: True! We have $A\vec{v} = a\vec{v}$. Apply A^{-1} to both sides and divide by a (note it is not zero since an invertible matrix has non-zero eigenvalues). This gives $A^{-1}\vec{v} = \frac{1}{a}\vec{v}$.

51. If A is upper triangular, then A is diagonalizable.

Solution note: False! The eigenvalues are the elements of the diagonal. But we don't know that the geometric multiplicities sum up to the desired value. A counterexample is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

52. The matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ has an orthonormal eigenbasis.

Solution note: False! Spectral theorem.

53. The matrix $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ has an orthonormal eigenbasis.

Solution note: True! Spectral theorem.

54. Every lower triangular matrix with pairwise distinct diagonal entries has an eigenbasis.

Solution note: True! Since the matrix is lower triangular, all eigenvalues are real and the set of eigenvalues is exactly the set of diagonal entries. Since they are distinct, the geometric multiplicity of each is one, and these sum to the size of the matrix.

55. There are no surjective maps $\mathcal{P}_4 \to \mathbb{R}^{10}$.

> *Solution note:* True! Rank nullity! Dimension of $\mathcal{P}_4$ is 5, so the largest the dimension of the image can be is 5, never 10.

56. There are no injective maps $\mathcal{P}_{14} \to \mathbb{R}^{10}$.

> *Solution note:* True! Rank nullity. Kernel has dimenion at least 5.

57. Consider the inner product on $\mathbb{R}^{2\times 2}$ defined by $\langle A, B\rangle = \text{trace}(A^T B)$. Then the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are orthonormal.

> *Solution note:* True! Compute $\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle = trace(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = 1$,
> $\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rangle = trace(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = 1$,
> and $\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rangle = trace(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 0$. So these are orthonormal.

58. The rank of the map $\mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ sending $A \mapsto A - A^T$ is three.

> *Solution note:* True. The kernel consists of the symmetric matrices, which form a six dimension space. By rank nullity, the image is 3-dimensional. That is, the map is rank 3.

59. Using the inner product from the previous problem, the closest diagonal matrix to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

> *Solution note:* True! We need to compute the projection onto the space of diagonal matrices. Since we already found an orthonormal basis in 38, this is easy. Call this basis (E_{11}, E_{22}). The projection is $\langle A, E_{11}\rangle E_{11} + \langle A, E_{22}\rangle E_{22}$. Computing this coeeficients we have $\langle A, E_{11}\rangle = trac(A^T E_{11}) = a$ and $\langle A, E_{22}\rangle = trac(A^T E_{22}) = d$. So the closest matrix is $aE_{11} + dE_{22}$.

60. There is a matrix which has determinant 6 and trace 5.

> *Solution note:* True: A diagonal 2×2 with 2 and 3 on the diagonal.

61. If a 2×2 matrix A has characteristic polynomial $x^2 + bx + c$, then it has an eigenvalue of algebraic multiplicity two if and only if $b^2 = 4c$.

> *Solution note:* True! Having an eigenvalue of multiplicity two means that the characteristic polynomial has one double root. According to the quadratic formula, this happens if and only if $b^2 - 4c = 0$. The eigenvalue in this case is the root $-b/2$.

62. A 2×2 matrix A has no real eigenvalues if and only if $(\text{trace } A)^2 < 4 \det A$.

> *Solution note:* True! The char poly is $x^2 + bX + c$ where $-b$ is the trace and c is the determinant. Again, using the quadratic formula, we see that this has no real roots if and only if $b^2 - 4c < 0$.

63. If some eigenspace of an $n \times n$ matrix A has dimension n, then A is a scalar multiple of the identity matrix.

> *Solution note:* True! If A has an eigenspace of dimension n, then the eigenspace is all of $\mathbb{R}^n$. This means that $A\vec{v} = k\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^n$. So the map is just scalar multiplication by k and the matrix A must be kI_n.

64. Let S be an orthogonal 3×3 matrix. The linear transformation $\mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}$ sending $X \mapsto S^TXS$ is invertible.

> *Solution note:* True! The inverse map is $Y \mapsto SYS^T$. Note that $S^T(SYS^T)S = Y$ and $S(S^TXS)S^T = X$ for all X and all Y.

65. Let S be an invertible 3×3 matrix. The only eigenvalues of the linear transformation $\mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}$ sending $X \mapsto S^{-1}XS$ are 0 and 1.

> *Solution note:* False! This map is invertible, so zero is not an eigenvalue.

66. For any $n \times n$ matrix A, the determinant of kA is $k^n \det A$.

> *Solution note:* True. Multiplinearity of determinant.

67. There exists an orthogonal matrix with eigenvalues $3, 2$ and 1.

> *Solution note:* False! Orthogonal matrices can have eigenvalues ± 1 only! Or no eigenvalues, like a rotation.

68. There exists a symmetric matrix with no real eigenvalues.

> *Solution note:* False. Spectral Theorem.

69. Let S be an orthogonal matrix and D be diagonal of the same size as S. Then $S^{-1}DS$ is symmetric.

> *Solution note:* TRUE! We check that $S^{-1}DS = (S^{-1}DS)^T$. Note that $S^{-1} = S^T$. So $(S^{-1}DS)^T = S^TD^T(S^T)^T = S^{-1}DS$.

70. If a square matrix B has an orthonormal eigenbasis, then B is symmetric.

> *Solution note:* True! This is (one direction of) the spectral theorem. Or, you can interpret as the same as the previous problem.

71. If an 7×7 matrix Q has eigenvalues 1 of geometric multiplicity 3 and 2 of geometric multiplicity 4, then Q is invertible.

> *Solution note:* True! The geometric multiplicities sum to $3+4=7$, so the matrix has an eigenbasis, which means that it is diagonal.

72. There is a 10 by 10 matrix with eigenvalues $1, 2, \ldots, 10$.

> *Solution note:* True! Just take the diagonal matrix with $1-10$ on the diagonal.

73. There is noninvertible 10 by 10 matrix with eigenvalues $1, 2, \ldots, 10$.

> *Solution note:* False! Non-invertible would mean that 0 is an eigenvalue. But there can be at most 10 eigenvalues for a 10 by 10 matrix, and we know that they are 1–10 in this case. None is zero.

74. There is non-diagonalizable 10 by 10 matrix with eigenvalues $1, 2, \ldots, 5$, each of algebraic multiplicity 2.

> *Solution note:* True! Take $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$. We can compute using rank-nullity that all the geometric multiplicities are 1.

75. There is 10 by 10 matrix with eigenvalues $1, 2, \ldots, 5$, each of geometric multiplicity 2, which does not have an eigenbasis.

> *Solution note:* False! The sum of the geometric multiplicities in this case is 10, so we must have an eigenbasis.

76. There is non-zero 10 by 10 matrix with an eigenvalue 0 of algebraic multiplicity 10.

> *Solution note:* Here is it: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

77. There is non-zero 10 by 10 matrix with an eigenvalue 0 of geometric multiplicity 10.

> *Solution note:* False. If the geometric multiplicity is 10, then the matrix is diagonalizable. But since the only eigenvalue is 0, it would be diaogonalizable to the zero matrix. This is impossible-the only matrix similar to the zero matrix is the zero matrix.

78. There is a 10 by 10 matrix with an eigenvalue λ of geometric multiplicity 5 and algebraic multiplicity 2.

> *Solution note:* False! Gemu ¡ almu for all eigenvalues!

79. The only matrix similar to the zero matrix is the zero matrix itself.

> *Solution note:* True! $S^{-1}0S = 0$.

80. The only matrix similar to the identity matrix is the identity matrix itself.

> *Solution note:* True! $S^{-1}I_nS = I_n$.

81. There is a non-diagonal matrix similar to kI_n for some $k \in \mathbb{R}$.

> *Solution note:* False! $S^{-1}kI_nS = kI_n$.

82. Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space V of finite dimension, and let linear transformation $T : V \to V$ be an arbitrary linear transformation. Then $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{B}}$ have the same eigenvalues.

> *Solution note:* True!

83. A matrix has an eigenbasis if and only if all eigenvalues have geometric multiplicity one.

> *Solution note:* False! This is silly. The identity matrix A_2 has an eigenbasis (any basis for $\mathbb{R}^2$ is an eigenbasis) but the geometric mutltiplicity 2 of the eigenvalue 1 is two,

84. Every non-zero matrix in $\mathbb{R}^{17\times 231}$ is an eigenvector for the transformation $A \mapsto 5A$.

> *Solution note:* True! Definition!

85. If two matrices have the same characteristic polynomial, then they are similar.

> *Solution note:* False! The zero matrix and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ both have char poly x^2.

86. There exists a non-zero 5×5 matrix with eigenvalue 0 of geometric multiplicity 5.

> *Solution note:* False! If the geometric multiplicity of 0 is 5, it means the kernel is five dimensional, the kernel is all of $\mathbb{R}^5$, which means the map is the zero map and the matrix is the 0 matrix.

87. If v is a eigenvector of T, then v is also an eigenvector of T^n for all $n \geq 1$.

> *Solution note:* True! $T(\vec{v}) = a\vec{v}$ means $T^n(\vec{v}) = T(T^{n-1}(\vec{v})) = T(a^{n-1}\vec{v}) = a^n\vec{v}$.

88. There exists a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^5$ whose kernel consists of exactly two points.

> *Solution note:* False! We proved that the kernel of a transformation is a subspace. The only subspaces of $\mathbb{R}^2$ are $\{0\}$, a line through the origin, or all of $\mathbb{R}^2$.

89. Let $A, B \in \mathbb{R}^{2\times 2}$ and let $C, D \in \mathbb{R}^{4\times 4}$ be the block matrices $\begin{bmatrix} A & 0_{\mathbb{R}^{2\times 2}} \\ 0_{\mathbb{R}^{2\times 2}} & B \end{bmatrix}$ and $\begin{bmatrix} 0_{\mathbb{R}^{2\times 2}} & B \\ A & 0_{\mathbb{R}^{2\times 2}} \end{bmatrix}$, respectively. Then $\det C = \det D$.

> *Solution note:* True! C is obtained from D by swapping rows 1 and 3, then swapping rows 2 and 4. Since each row swap multiples the determinant by -1, an even number of swaps will not change the determinant.

90. If T has no real eigenvalues, then also T^2 has no real eigenvalues.

> *Solution note:* False! Rotation by $\pi/2$ has no real eigenvalues but if we do it twice, we have rotation by π which has eigenvalue -1. Alternatively, the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues but $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalue -1.

91. The collection of functions $f_k(x) = \sin(kx) + \cos(kx)$, as k ranges through all positive real numbers, form an infinite set of linearly independent eigenvectors for $\frac{d^2}{dx^2}$.

> *Solution note:* True! $\frac{d^2}{dx^2} f_k = \frac{d}{dx} k\cos(kx) - k\sin(kx) = -k^2\sin(kx) - k^2\cos(kx) = -k^2 f_k$. So f_k is an eigenvector with eigenvalue $-k^2$, and since the $-k^2$ are all distinct, the f_k are all linearly independent.

92. A linear transformation $V \xrightarrow{T} V$ (of a finite dimensional vector space) has eigenvalue zero if and only if $\det T = 0$.

> *Solution note:* True. Eigenvalue 0 and det $= 0$ are both equivalent to the transformation being non-invertible.

93. For any $n \times m$ matrix B, the matrix $B^T B$ has an orthonormal eigenbasis.

> *Solution note:* True. By spectral theorem, since $B^T B$ is symmetric.

94. A symmetric matrix $n \times n$ matrix has exactly n distinct eigenvalues.

> *Solution note:* False! The identity matrix of size $n \geq 2$ is symmetric but does not have n distinct eigenvalues.

95. Let λ be an eigenvalue of a symmetric $n \times n$ matrix. Then the geometric and algebraic multiplicities of λ must be equal.

Solution note: True! A symmetric matrix is diagonalizable, so the almu and gemu is the same of every eigenvalue.

96. A $n \times n$ matrix is diagonalizable if and only if it has n distinct eigenvalues.

Solution note: False! The 2×2 identity matrix is a counterexample.

97. There exists a matrix with one real eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.

Solution note: True. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

98. Let W be the subspace of diagonal matrices in $\mathbb{R}^{n \times n}$, with the inner product $\langle A, B \rangle = \text{trace}(A^T B)$. Then $W^\perp$ has dimension $n(n-1)$.

Solution note: W has dimension n, so $W^\perp$ has complementary dimension inside the ambient space $\mathbb{R}^{n \times n}$. That is, $W^\perp$ has dimension $n^2 - n$.

99. There exists linear transformation $\mathbb{R}^{2\times 3} \to \mathbb{R}^{2\times 3}$ whose distinct eigenspaces have dimensions 2, 2, and 3, respectively.

Solution note: False! Concatenating the eigenbases produces a linearly independent set, but $\mathbb{R}^{2\times 3}$ is six dimensional.

100. Math 217 is awesome.

Solution note: True!