Math 217: "Prove or Disprove" Practice for Final Exam

On the Exam and on this review, the words "eigenvalue, eigenvector, eigenbasis, and eigenspace" all refer to the these concepts over the **real numbers** unless otherwise stated. The word "diagonalizable" means **diagonalizable over the real numbers**, unless we explicitly say "diagonalizable over the complex numbers."

1. A square matrix is invertible if and only if zero is not an eigenvalue.

2. If $T : V \to V$ and $S : V \to V$ are linear transformations, both with eigenvalue 5, then $T \circ S$ also has eigenvalue 5.

3. If A and B are 2×2 matrices, both with eigenvalue 5, then $A + B$ also has eigenvalue 5.

4. A square matrix has determinant zero if and only if zero is an eigenvalue.

5. If B is the $\mathfrak{B}$-matrix of some linear transformation $V \xrightarrow{T} V$. Then for all $\vec{v} \in V$, we have $B[\vec{v}]_{\mathfrak{B}} = [T(\vec{v})]_{\mathfrak{B}}$.

6. If $V \xrightarrow{T} W \xrightarrow{S} V'$ are linear transformations, then $\mathrm{im}(ST) \subset \mathrm{im}S$.

7. If $V \xrightarrow{T} W \xrightarrow{S} V'$ are linear transformations, then $\ker(T) \subset \ker(ST)$.

8. Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathfrak{B} = (f_1, f_2, f_3)$. Then f_1 is an eigenvector.

9. Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathfrak{B} = (f_1, f_2, f_3)$. Then $T(f_1 + f_2 + f_3)$ is $6f_1 + 2f_2 + f_3$.

10. If A and B are similar, then they have the same trace and determinant.

11. Let $T : V \to V$ be a linear transformation, and suppose that T has a $\mathcal{B}$-matrix which is lower triangular for some $\mathcal{B} = (f_1, \dots, f_n)$. Then T has at least one eigenvector.

12. There exists a linear transformation with exactly 6 eigenvectors.

13. The polynomials $x + 1, x^3, x^2$ span the vector space $\mathcal{P}_3$.

14. The polynomials $x + 1, x^3, x^2, x^3 + x^2 + x + 1$ span the vector space $\mathcal{P}_3$.

15. The set of polynomials $\{x + 1, x^3, x^2, x^3 + x^2 + x + 1\}$ is a linearly independent set in $\mathcal{P}_5$.

16. Suppose that T is a linear transformation of rank 5 from the space $U^{3\times 3}$ of upper triangular matrices to itself. If the characteristic polynomial of T is $(x-1)(x-2)(x-3)(x-4)(x-5)(x-b)$, then it is possible to find the exact value of b.

17. The rank of $\frac{d^2}{dx^2}$ on $\mathcal{P}_{17}$ is 16.

18. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ form a basis for the space of symmetric 2×2 matrices.

19. If $f, g, h \in P_6$ are eigenvectors for a linear transformation $T : \mathcal{P}_6 \to \mathcal{P}_6$ with eigenvalues $3, 4, 0$ respectively, then $T(2f - 3g + h) = 6f - 12g$.

20. The only rotation $\mathbb{R}^2 \to \mathbb{R}^2$ which has a real eigenvalue are rotations that induce the identity transformation (so through $\pm 2\pi, \pm 4\pi$, etc).

21. If the change of basis matrix $S_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} \vec{e}_4 & \vec{e}_3 & \vec{e}_2 & \vec{e}_1 \end{bmatrix}$, then the elements of $\mathcal{A}$ are the same as the element of $\mathcal{B}$, but in a different order.

22. The map assigning $\langle A, B \rangle$ to trace(AB^T) is an inner product on the space of all $\mathbb{R}^{2\times 2}$ matrices.

23. If $T : \mathbb{R}^{7\times 8} \to \mathbb{R}^{3\times 8}$ is a linear transformation whose 0-eigenspace has dimension 33, then T is surjective.

24. An orthogonal matrix must have at least one real eigenvalue.

25. The determinant of the differentiation map of $\mathcal{P}_3$ is zero.

26. If A is a 3×4 matrix, then the matrix $A^T A$ is similar to a diagonal matrix with three or less non-zero entries.

27. If A is similar to both D_1 and D_2, where D_1 and D_2 are diagonal, then $D_1 = D_2$.

28. If A and B are similar to Q, then A is similar to B.

29. Let u and v be any two orthonormal vectors in an inner product space. Then $||u - v|| = \sqrt{2}$.

30. If $\langle x, y \rangle = -\langle y, x \rangle$ in some inner product space, then x is orthogonal to y.

31. A linear transformation of a 7-dimensional space to itself has at least one real eigenvalue.

32. Let $V \xrightarrow{T} V$ be a linear transformation, and suppose that x and y are linearly independent eigenvectors with *different* eigenvalues. Then $x + y$ is NOT an eigenvector.

33. If $\langle x, y \rangle = \langle x, z \rangle$ for vectors x, y, z in an inner product space, then $y - z$ is orthogonal to x.

34. For any matrix A and any column vector $\vec{b}$, the system $A^T A\vec{x} = A^T\vec{b}$ is consistent.

35. If A is the $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ matrix of a transformation T and $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are the $\mathfrak{B}$-coordinates of $\vec{x}$, then $T(\vec{x}) = 2\vec{v}_1 + \vec{v}_3$.

36. If A is the $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ matrix of a transformation T and $T(\vec{v}_3) = \vec{v}_1 + \vec{v}_3$, then $A\vec{e}_3 = \vec{e}_1 + \vec{e}_3$.

37. For any $n \times n$ matrix A, and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have $A\vec{x} \cdot \vec{y} = \vec{x} \cdot A^T\vec{y}$.

38. For any symmetric $n \times n$ matrix A, and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have $A\vec{x} \cdot \vec{y} = A\vec{y} \cdot \vec{x}$.

39. If an 5×5 matrix P has eigenvalues $1, 2, 4, 8$ and 16, then P is similar to a diagonal matrix.

40. If A is an orthogonal matrix, then its only real eigenvalues are ± 1.

41. The functions $\sin x$ and $\cos x$ are orthogonal in the inner product defined by $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} fg dx$.

42. Suppose we have an inner product space V and w and v are orthonormal vectors in V. Then for any $f \in V$, the element $\langle w, f \rangle w + \langle v, f \rangle v$ is the closest vector to f in the span of v and w.

43. In any inner product space, $||f|| = \langle f, f \rangle$ for all f.

44. Consider $\mathbb{R}^{2\times 2}$ as an inner product space with the inner product $\langle A, B \rangle = \text{trace } A^T B$. Then $|| \begin{bmatrix} a & b \\ c & d \end{bmatrix} || = \sqrt{(a^2 + b^2 + c^2 + d^2)}$.

45. The matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are orthonormal in the inner product $\langle A, B \rangle = \text{trace } A^T B$ on $\mathbb{R}^{2\times 2}$.

46. If f and g are elements in an inner product space satisfying $||f|| = 2, ||g|| = 4$ and $||f+g|| = 5$, then it is possible to find the exact value of $\langle f, g\rangle$

47. If $(\vec{v}_1, \ldots, \vec{v}_d)$ is a basis for the subspace V of $\mathbb{R}^n$ and $\vec{b} \in V$, then the least squares solutions of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$ are exact solutions to $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$.

48. Let V and W be distinct planes in $\mathbb{R}^3$ and let $\phi_V : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi_W : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projections onto V and W, respectively. Then the matrices of ϕ_V and ϕ_W in the standard basis are similar.

49. If $(\vec{v}_1, \ldots, \vec{v}_d)$ is a basis for the subspace V of $\mathbb{R}^n$ and $\vec{b} \in V^{\perp}$, then the only least squares solutions of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_d \end{bmatrix} \vec{x} = \vec{b}$ is the zero vector.

50. Suppose a is an eigenvalue of an invertible matrix A. Then a^{-1} is an eigenvalue of A^{-1}.

51. If A is upper triangular, then A is diagonalizable.

52. The matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ has an orthonormal eigenbasis.

53. The matrix $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ has an orthonormal eigenbasis.

54. Every lower triangular matrix with pairwise distinct diagonal entries has an eigenbasis.

55. There are no surjective maps $\mathcal{P}_4 \to \mathbb{R}^{10}$.

56. There are no injective maps $\mathcal{P}_{14} \to \mathbb{R}^{10}$.

57. Consider the inner product on $\mathbb{R}^{2\times 2}$ defined by $\langle A, B\rangle = \text{trace}(A^T B)$. Then the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are orthonormal.

58. The rank of the map $\mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ sending $A \mapsto A - A^T$ is three.

59. Using the inner product from the previous problem, the closest diagonal matrix to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

60. There is a matrix which has determinant 6 and trace 5.

61. If a 2×2 matrix A has characteristic polynomial $x^2 + bx + c$, then it has an eigenvalue of algebraic multiplicity two if and only if $b^2 = 4c$.

62. A 2×2 matrix A has no real eigenvalues if and only if $(\text{trace } A)^2 < 4 \det A$.

63. If some eigenspace of an $n \times n$ matrix A has dimension n, then A is a scalar multiple of the identity matrix.

64. Let S be an orthogonal 3×3 matrix. The linear transformation $\mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}$ sending $X \mapsto S^T X S$ is invertible.

65. Let S be an invertible 3×3 matrix. The only eigenvalues of the linear transformation $\mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}$ sending $X \mapsto S^{-1} X S$ are 0 and 1.

66. For any $n \times n$ matrix A, the determinant of kA is $k^n \det A$.

67. There exists an orthogonal matrix with eigenvalues $3, 2$ and 1.

68. There exists a symmetric matrix with no real eigenvalues.

69. Let S be an orthogonal matrix and D be diagonal of the same size as S. Then $S^{-1} D S$ is symmetric.

70. If a square matrix B has an orthonormal eigenbasis, then B is symmetric.

71. If an 7×7 matrix Q has eigenvalues 1 of geometric multiplicity 3 and 2 of geometric multiplicity 4, then Q is invertible.

72. There is a 10 by 10 matrix with eigenvalues $1, 2, \ldots, 10$.

73. There is noninvertible 10 by 10 matrix with eigenvalues $1, 2, \ldots, 10$.

74. There is non-diagonalizable 10 by 10 matrix with eigenvalues $1, 2, \ldots, 5$, each of algebraic multiplicity 2.

75. There is 10 by 10 matrix with eigenvalues $1, 2, \ldots, 5$, each of geometric multiplicity 2, which does not have an eigenbasis.

76. There is non-zero 10 by 10 matrix with an eigenvalue 0 of algebraic multiplicity 10.

77. There is non-zero 10 by 10 matrix with an eigenvalue 0 of geometric multiplicity 10.

78. There is a 10 by 10 matrix with an eigenvalue λ of geometric multiplicity 5 and algebraic multiplicity 2.

79. The only matrix similar to the zero matrix is the zero matrix itself.

80. The only matrix similar to the identity matrix is the identity matrix itself.

81. There is a non-diagonal matrix similar to kI_n for some $k \in \mathbb{R}$.

82. Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space V of finite dimension, and let linear transformation $T : V \to V$ be an arbitrary linear transformation. Then $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{B}}$ have the same eigenvalues.

83. A matrix has an eigenbasis if and only if all eigenvalues have geometric multiplicity one.

84. Every non-zero matrix in $\mathbb{R}^{17\times 231}$ is an eigenvector for the transformation $A \mapsto 5A$.

85. If two matrices have the same characteristic polynomial, then they are similar.

86. There exists a non-zero 5×5 matrix with eigenvalue 0 of geometric multiplicity 5.

87. If v is a eigenvector of T, then v is also an eigenvector of T^n for all $n \geq 1$.

88. There exists a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^5$ whose kernel consists of exactly two points.

89. Let $A, B \in \mathbb{R}^{2\times 2}$ and let $C, D \in \mathbb{R}^{4\times 4}$ be the block matrices $\begin{bmatrix} A & 0_{\mathbb{R}^{2\times 2}} \\ 0_{\mathbb{R}^{2\times 2}} & B \end{bmatrix}$ and $\begin{bmatrix} 0_{\mathbb{R}^{2\times 2}} & B \\ A & 0_{\mathbb{R}^{2\times 2}} \end{bmatrix}$, respectively. Then $\det C = \det D$.

90. If T has no real eigenvalues, then also T^2 has no real eigenvalues.

91. The collection of functions $f_k(x) = \sin(kx) + \cos(kx)$, as k ranges through all positive real numbers, form an infinite set of linearly independent eigenvectors for $\frac{d^2}{dx^2}$.

92. A linear transformation $V \xrightarrow{T} V$ (of a finite dimensional vector space) has eigenvalue zero if and only if $\det T = 0$.

93. For any $n \times m$ matrix B, the matrix $B^T B$ has an orthonormal eigenbasis.

94. A symmetric matrix $n \times n$ matrix has exactly n distinct eigenvalues.

95. Let λ be an eigenvalue of a symmetric $n \times n$ matrix. Then the geometric and algebraic multiplicities of λ must be equal.

96. A $n \times n$ matrix is diagonalizable if and only if it has n distinct eigenvalues.

97. There exists a matrix with one real eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.

98. Let W be the subspace of diagonal matrices in $\mathbb{R}^{n\times n}$, with the inner product $\langle A, B\rangle = \text{trace}(A^T B)$. Then $W^\perp$ has dimension $n(n-1)$.

99. There exists linear transformation $\mathbb{R}^{2\times 3} \to \mathbb{R}^{2\times 3}$ whose distinct eigenspaces have dimensions 2, 2, and 3, respectively.

100. Math 217 is awesome.