Part A
4-3

In Exercises 5 through 40, find the matrix of the given linear transformation T with respect to the given basis. If no basis is specified, use the standard basis: $\mathfrak{A} = (1, t, t^2)$ for P_2,

$$\mathfrak{A} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

for $\mathbb{R}^{2\times2}$, and $\mathfrak{A} = (1, i)$ for $\mathbb{C}$. For the space $U^{2\times2}$ of upper triangular 2×2 matrices, use the basis

$$\mathfrak{A} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

unless another basis is given. In each case, determine whether T is an isomorphism. If T isn't an isomorphism, find bases of the kernel and image of T, and thus determine the rank of T.

14. $T(M) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} M$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$, with respect to the basis

$$\mathfrak{B} = \left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right)$$

Sol $[T(M)]_{\mathfrak{B}} = \begin{bmatrix} | & | & | & | \\ [T(b_1)]_{\mathfrak{B}} & [T(b_2)]_{\mathfrak{B}} & [T(b_3)]_{\mathfrak{B}} & [T(b_4)]_{\mathfrak{B}} \\ | & | & | & | \end{bmatrix}$

$T(b_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow [T(b_1)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(b_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow [T(b_2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(b_3) = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} \Rightarrow [T(b_3)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

$T(b_4) = \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} \Rightarrow [T(b_4)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

So $[T(M)]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow \text{rref}([T(M)]_{\mathfrak{B}})$

$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So $[\ker T]_M = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$[\text{im} T]_M = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$

So a basis for $\ker T$ is $\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right)$

a basis for $\text{im} T$ is $\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right)$

$\text{rank} T = \dim(\text{im} T) = 2.$

28. $T(f(t)) = f(2t-1)$ from P_2 to P_2, with respect to the basis $\mathfrak{B} = (1, t-1, (t-1)^2)$

Sol. $[T]_{\mathfrak{B}} = \begin{bmatrix} | & | & | \\ [T(b_1)]_{\mathfrak{B}} & [T(b_2)]_{\mathfrak{B}} & [T(b_3)]_{\mathfrak{B}} \\ | & | & | \end{bmatrix}$

$= \begin{bmatrix} | & | & | \\ [1]_{\mathfrak{B}} & [2t-2]_{\mathfrak{B}} & [4(t-1)^2]_{\mathfrak{B}} \\ | & | & | \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Since $[T]_{\mathfrak{B}}$ has full rank, $[T]_{\mathfrak{B}}$ is invertible

so T is invertible linear transformation, so T is isomophism by definition. $\text{rank}(T) = 3.$

60. In the plane V defined by the equation $2x_1 + x_2 - 2x_3 = 0$, consider the bases

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right)$$

and

$$\mathfrak{B} = (\vec{b}_1, \vec{b}_2) = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right).$$

a. Find the change of basis matrix S from $\mathfrak{B}$ to $\mathfrak{A}$.

b. Find the change of basis matrix from $\mathfrak{A}$ to $\mathfrak{B}$.

c. Write an equation relating the matrices $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$, $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$, and $S = S_{\mathfrak{B}\to\mathfrak{A}}$.

a. $S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix} | & | \\ [\vec{b}_1]_{\mathfrak{A}} & [\vec{b}_2]_{\mathfrak{A}} \\ | & | \end{bmatrix}$

$[\vec{b}_1]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[\vec{b}_2]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b. $S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix} | & | \\ [\vec{a}_1]_{\mathfrak{B}} & [\vec{a}_2]_{\mathfrak{B}} \\ | & | \end{bmatrix}$

$[\vec{a}_1]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[\vec{a}_2]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(c) $[\vec{a}_1 \ \vec{a}_2] = S[\vec{b}_1 \ \vec{b}_2]$

5-1 *Find the angle θ between each of the pairs of vectors $\vec{u}$ and $\vec{v}$ in Exercises 4 through 6.*

6. $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-3}{\sqrt{540}} = \frac{-3}{6\sqrt{15}} = \frac{-1}{2\sqrt{15}}$$

$$\text{So } \theta = \arccos\frac{-\sqrt{15}}{30} \simeq 1.7 \text{ rad} = 97.4^\circ$$

17. Find a basis for $W^\perp$, where

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}\right).$$

Sol Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{cases}$$

The matrix form of the system:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right]$$

$$\xRightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$\Rightarrow x_1 = x_3 + 2x_4,\ x_2 = -2x_3 - 3x_4$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

So the solution space which is $W^\perp$ is

$$\left\{ r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, r, s \in \mathbb{R} \right\}$$

So a basis for $W^\perp$ is $\left(\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$

26. Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ spanned by

$$\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Sol $\vec{u_1} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\vec{u_2} = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$

So by Thm 5.1.5, $\text{proj}_V \vec{x} = (\vec{u_1} \cdot \vec{x})\vec{u_1} + (\vec{u_2} \cdot \vec{x})\vec{u_2}$

$$= (2+3+6)\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + (3-6+2)\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}$$

Problem 1. Let W be an n-dimensional vector space with ordered bases $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.

(a) Prove that $S_{\mathcal{C}\to\mathcal{A}} = S_{\mathcal{B}\to\mathcal{A}}\, S_{\mathcal{C}\to\mathcal{B}}$.

(b) Show that $S_{\mathcal{C}\to\mathcal{A}}\, S_{\mathcal{B}\to\mathcal{C}}\, S_{\mathcal{A}\to\mathcal{B}} = I_n$.

(a) Proof.

Denote C by $(c_1, c_2, \dots, c_n)$

Take $f \in W$.

So $[f]_A = S_{C\to A}[f]_C = S_{\mathcal{B}\to A}[f]_{\mathcal{B}}$

$[f]_{\mathcal{B}} = S_{C\to\mathcal{B}}[f]_C$ by def 4.3.3

So $S_{C\to A}[f]_C = S_{\mathcal{B}\to A}(S_{C\to\mathcal{B}}[f]_C)$

$= (S_{\mathcal{B}\to A} S_{C\to\mathcal{B}})[f]_C$ by associativity of matrix multiplication

By taking f such that $[f]_C = e_i$, $1 \le i \le n$

we have $S_{C\to A}\, e_i = (S_{\mathcal{B}\to A} \cdot S_{C\to\mathcal{B}})\, e_i$ for all $1 \le i \le n$

which indicates every column of $S_{C\to A}$ and $S_{\mathcal{B}\to A} S_{C\to\mathcal{B}}$ equals.

Therefore $S_{C\to A} = S_{\mathcal{B}\to A} S_{C\to\mathcal{B}}$.

(b) By (a), $S_{C\to A} S_{\mathcal{B}\to C} S_{A\to\mathcal{B}} = S_{\mathcal{B}\to A} S_{C\to\mathcal{B}} \cdot S_{\mathcal{B}\to C} \cdot S_{A\to\mathcal{B}}$

$= S_{\mathcal{B}\to A}(S_{C\to\mathcal{B}} \cdot S_{\mathcal{B}\to C}) S_{A\to\mathcal{B}}$ by associativity of matrix multiplication

By def 4.3.3, $S_{C\to B} S_{B\to C} = I_n$,

$S_{B\to A} S_{A\to B} = I_n$

So $S_{C\to A} S_{B\to C} S_{A\to B} = S_{B\to A}(S_{C\to B} \cdot S_{B\to C}) S_{A\to B}$

$= (S_{B\to A} I_n) S_{A\to B}$

$= S_{B\to A} S_{A\to B}$

$= I_n$

Problem 2. Let f_1, f_2, f_3 be the smooth functions defined by

$$f_1(x) = \sin 2x,\ f_2(x) = \cos 2x,\ f_3(x) = e^{3x}$$

and consider the subspace $V \subseteq C^\infty(\mathbb{R})$ spanned by the basis $\mathcal{B} = (f_1, f_2, f_3)$. (You may assume without proof that these three functions are linearly independent.) Now consider the linear transformation $D : V \to V$ defined by differentiation, i.e. for any function $g \in V$, $D(g)(x) = \frac{dg}{dx}$.

(a) Find $[D]_{\mathcal{B}}$.

(b) Give a geometric interpretation of the matrix $[D]_{\mathcal{B}}$. That is, how does it act on $\mathbb{R}^3$?

(a) $[D]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [D(f_1)]_{\mathcal{B}} & [D(f_2)]_{\mathcal{B}} & [D(f_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$

$D(f_1) = 2\cos 2x$, so $[D(f_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$D(f_2) = -2\sin 2x$, so $[D(f_2)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

$D(f_3) = 3e^{3x}$, so $[D(f_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

So $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) For any element $f = af_1 + bf_2 + cf_3 \in V$

D maps every f_1 to $2f_2$, every f_2 to $-2f_1$

and every f_3 to $3f_3$.

So for $[D]_{\mathcal{B}}$, it rotates every vector in $\mathbb{R}^3$

for $\frac{\pi}{2}$ counterclockwisely, and stretch the

x, y coordinates to be twice long and the

z coordinate to be three times long.

Problem 3. Let V be a vector space with ordered bases $\mathcal{B} = (b_1, \dots, b_n)$ and $\mathcal{C} = (c_1, \dots, c_n)$. Let $T : V \to V$ be a linear transformation, with $B = [T]_{\mathcal{B}}$ and $C = [T]_{\mathcal{C}}$. Give a proof or counterexample for each of the following statements:

(a) For all integers $k \geq 1$, B^k and C^k are similar.
(b) $\ker(B) = \ker(C)$.
(c) $\dim(\ker(B)) = \dim(\ker(C))$.

(a) By change of Basis theorem for transformations,

Proof. $[T]_C = S_{C\to B}^{-1} [T]_B S_{C\to B}$

Take $S_{C\to B}$ as S, we have $C = S^{-1}BS$

We will prove the statement by induction on k

Claim: for all integer $k \geq 1$, B^k and C^k are similar

(and furthermore, $C^k = S^{-1}B^kS$ where S is the same one as $S_{C\to B}$.)

Base case: $k=1$. $C^1 = S^{-1}B^1S$, so B^k and C^k are similar by definition

Inductive step: Assume $B^k = S^{-1}C^kS$

So $B^{k+1} = B^k \cdot B = (S^{-1}C^kS)(S^{-1}CS)$

$= S^{-1}C^k(SS^{-1})CS$

$= S^{-1}C^kI_nCS$

$= S^{-1}C^{k+1}S$

Therefore B^{k+1} is similar to C^{k+1} by definition.

So we have proved that for all integer $k \geq 1$, B^k is similar to C^k by induction

(b) This is not correct.

Counterexample: consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ are two ordered basis.

So $[A]_{\mathcal{E}} = A$, $[A]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$[A]_{\mathcal{E}}$ is similar to $[A]_{\mathcal{B}}$ since $\underbrace{[A]_{\mathcal{E}} = S^{-1}_{\mathcal{E}\to\mathcal{B}}[A]_{\mathcal{B}}S_{\mathcal{E}\to\mathcal{B}}}_{\text{by definition}}$

But $\ker[A]_{\mathcal{E}} = \left\{ \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, $\ker[A]_{\mathcal{B}} = \left\{ \begin{bmatrix} 0 \\ r \\ -r \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$

$\ker[A]_{\mathcal{E}} \neq \ker[A]_{\mathcal{B}}$. By taking $T: \mathbb{R}^n \to \mathbb{R}^n$ represented B then $B = [A]_{\mathcal{E}}$ and $C = [A]_{\mathcal{B}}$, so we have disproved the statement.

(c) Proof:

By rank-nullity theorem, $\dim(\ker B) = \dim(V) - \dim(\operatorname{im} B)$
$= n - \operatorname{rank} B$

Similarly we get $\dim(\ker C) = n - \operatorname{rank} C$

Claim ①: For any $m \times n$ matrices M, N, $\operatorname{rank}(MN) \leq \operatorname{rank}(N)$ and $\operatorname{rank}(MN) \leq \operatorname{rank}(M)$

Proof of claim ①:

By rank-nullity theorem, $n = \operatorname{rank}(MN) + \dim(\ker MN)$
$= \operatorname{rank}(N) + \dim(\ker N)$

Let $\vec{v} \in \ker N$, so $N\vec{v} = \vec{0}$, so $(MN)\vec{v} = (M N)\vec{0} = \vec{0}$

Therefore $\ker N \subseteq \ker MN$, so $\dim(\ker MN) \geq \dim(\ker N)$

So $\operatorname{rank}(MN) \leq \operatorname{rank}(N)$ (1)

Also, $\operatorname{rank}(MN) = \operatorname{rank}((MN)^T) = \operatorname{rank}(N^T M^T)$

by (1), $\operatorname{rank}(N^T M^T) \leq \operatorname{rank}(M^T) = \operatorname{rank}(M)$

so $\operatorname{rank}(MN) \leq \operatorname{rank}(M)$ (2)

Claim ②: For any $m \times n$ matrices M, N: If M is invertible, then $\operatorname{rank}(MN) = \operatorname{rank}(N)$
$\operatorname{rank}(NM) = \operatorname{rank}(N)$ also

Proof of claim ②:

Since M is invertible, $N = NMM^{-1}$

So $\text{rank}(N) = \text{rank}(I_n N) = \text{rank}(M^{-1} M N)$

Since $\text{rank}(M^{-1}MN) \leq \text{rank}(MN)$ by claim ①,

$\underline{\text{rank}(N) \leq \text{rank}(MN)}$ (3)

So by (1)(3) we have proved $\underline{\text{rank}(MN) = \text{rank}(N)}$

Also, $\text{rank}(N) = \text{rank}(NMM^{-1}) \leq \text{rank}(NM)$ by claim ①
$\leq \text{rank}(N)$

So $\underline{\text{rank}(N) = \text{rank}(NM)}$

Claim ③: $\underline{\text{rank}(B) = \text{rank}(C)}$

Now since $B = S^{-1}CS$, $\text{rank}(B) = \text{rank}(S^{-1}CS)$

Since S^{-1}, S are invertible, by claim ②,

$\text{rank}(S^{-1}CS) = \text{rank}(CS) = \underline{\text{rank}(C)}$

So by the rank-nullity theorem, $\dim(\ker B) = \dim(\ker C)$

Problem 4. Let $T : U \to W$ be a linear transformation between vector spaces U and W. Suppose that $\mathcal{B} = (u_1, u_2, \ldots, u_k)$ is a basis for the source U and $\mathcal{C} = (w_1, w_2, \ldots, w_d)$ is a basis for the target W. As usual, let $L_{\mathcal{B}}$ denote the coordinate isomorphism $U \to \mathbb{R}^k$ and let $L_{\mathcal{C}}$ denote the coordinate isomorphism $W \to \mathbb{R}^d$.

(a) Show that there exists a linear transformation $T' : \mathbb{R}^k \to \mathbb{R}^d$ such that $T' \circ L_{\mathcal{B}} = L_{\mathcal{C}} \circ T$. [Hint: A diagram showing four vector spaces and four maps between them, similar to those immediately before and after Definition 4.3.1 in the textbook, might be useful.]

(b) Let $[T]_{(\mathcal{B},\mathcal{C})}$ denote the standard matrix of the transformation T' you described in (a). Prove that for all $u \in U$,

$$[T(u)]_{\mathcal{C}} = [T]_{(\mathcal{B},\mathcal{C})}[u]_{\mathcal{B}}.$$

(c) Describe, with explanation, the columns of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ in terms of the bases $\mathcal{B}$ and $\mathcal{C}$.

(a)

$$\begin{tikzcd} u \arrow[r, mapsto, "T"] \arrow[d, "L_{\mathcal{B}}"'] & w \arrow[d, "L_C"] \\ {[u]_{\mathcal{B}}} \arrow[r, mapsto, "T'"] & {[w]_C} \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}}"'] & W \arrow[d, "L_C"] \\ \mathbb{R}^k \arrow[r, "T'"] & \mathbb{R}^d \end{tikzcd}$$

Consider T' sending $\underline{[u]_{\mathcal{B}} \longmapsto [w]_C}$

whenever $\underline{T(u) = w}$

Let $u \in U$ with $T(u) = w$

then $T' \circ L_{\mathcal{B}}(u) = T'([u]_{\mathcal{B}}) = [w]_C$

$L_C \circ T(u) = L_C(w) = [w]_C$

So $T' \circ L_{\mathcal{B}} = L_C \circ T$

(b) Let u be a vector in U.

Then $[T(u)]_C = L_C \circ T(u)$ by definition

Also, by (a), $L_C \circ T(u) = T' \circ L_{\mathcal{B}}(u) = T'([u]_{\mathcal{B}})$

Since T' is represented by $[T]_{(\mathcal{B}, C)}$,

$T'([u]_{\mathcal{B}}) = [T]_{(\mathcal{B},C)} [u]_{\mathcal{B}}$

Therefore $\underline{[T(u)]_C = [T]_{(\mathcal{B},C)} [u]_{\mathcal{B}}}$

(c) Take $u = a_1\vec{u_1} + a_2\vec{u_2} + \dots + a_k\vec{u_k}$

By (b), $[T(u)]_C = [T]_{(\mathcal{B},C)}[u]_{\mathcal{B}}$

Since $[T(u)]_C = L_C \circ T(a_1\vec{u_1} + \dots + a_k\vec{u_k})$

$$= a_1[T(\vec{u_1})]_C + a_2[T(\vec{u_2})]_C + \dots + a_k[T(\vec{u_k})]_C$$

and $[u]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$,

So consider $\begin{bmatrix} | & | & & | \\ [T(u_1)]_C & [T(u_2)]_C & \dots & [T(u_k)]_C \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = [T(u)]_C$

it satisfies the equivalence.

Since the standard matrix of a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^d$ is unique.

Therefore $[T]_{(\mathcal{B},C)} = \begin{bmatrix} | & | & & | \\ [T(u_1)]_C & [T(u_2)]_C & \dots & [T(u_k)]_C \\ | & | & & | \end{bmatrix}$

Problem 5. Let f_1, f_2, f_3 be the functions defined by

$$f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = e^x,$$

which you may assume without proof are linearly independent. Consider the subspace V of $\mathcal{C}^\infty$ spanned by the set $\{f_1, f_2, f_3\}$. Recall from Calculus that every function in V may be expressed as a Taylor series that converges for all real numbers. For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots,$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

Let $T : V \to \mathcal{P}_3$ be the linear transformation that assigns to each function $f \in V$ the third-degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ for f, a polynomial approximation to f.

(a) Find a basis $\mathcal{C}$ for $\mathcal{P}_3$ such that

$$[T(f_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, [T(f_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [T(f_3)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Let $\mathcal{C}$ be as in (a), and let $\mathcal{B} = (f_1 + f_2, f_1 - f_2, f_3 + f_1)$. Find $[T]_{(\mathcal{B},\mathcal{C})}$ (see Problem 4).

(a) $T(f_1) = x - \frac{x^3}{6}$

$T(f_2) = 1 - \frac{x^2}{2}$

$T(f_3) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

Since $[T(f_3)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$C = \{1, x, \frac{x^2}{2}, \frac{x^3}{6}\}$

(b) $[T]_{(\mathcal{B},C)} = \begin{bmatrix} | & | & | \\ [T(f_1+f_2)]_C & [T(f_1-f_2)]_C & [T(f_3+f_1)]_C \\ | & | & | \end{bmatrix}$

By theorem 4.1.4,

$$[T(f_1+f_2)]_C = [T(f_1)]_C + [T(f_2)]_C = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$[T(f_1-f_2)]_C = [T(f_1)]_C - [T(f_2)]_C = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$[T(f_3+f_1)]_C = [T(f_3)]_C + [T(f_1)]_C = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $[T]_{(\mathcal{B},C)} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Problem 6. Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

(a) Show that for all $\vec{v} \in V$, $A\vec{v} \in V$.
(b) Find a basis for $V^\perp$, and show that for all $\vec{w} \in V^\perp$, $A\vec{w} \in V^\perp$.
(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_\mathcal{B}$ is diagonal, and write the matrix $[T]_\mathcal{B}$ explicitly.
(d) Calculate $[T^{10}]_\mathcal{B}$. [HINT: Leave numbers like 7^{13} in that form; do not attempt to multiply them out.]
(e) Calculate $[T^{10}]_\mathcal{E}$. [HINT: Leave the entries as numerical expressions; do not attempt to simplify.]

(a) Take arbitrary $\vec{v} \in V$, so $\vec{v} = a\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for some scalar a

so $A\vec{v} = a\begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = a\begin{bmatrix} -78 \\ -52 \end{bmatrix} = -26a\begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \operatorname{span}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

So $A\vec{v} \in V$.

Therefore for all $\vec{v} \in V$, $A\vec{v} \in V$

(b) Take arbitrary $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in V^\perp$, so $\underbrace{3x+2y=0}_{\text{by definition}} \Rightarrow y = -\frac{3x}{2}$

So $V^\perp = \left\{ \begin{bmatrix} x \\ -\frac{3}{2}x \end{bmatrix} \middle| x \in \mathbb{R} \right\} = \left\{ r\begin{bmatrix} -2 \\ 3 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$

So a basis for $V^\perp$ is $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Then take arbitrary $\vec{w} \in V^\perp$, $\vec{w} = \begin{bmatrix} -2r \\ 3r \end{bmatrix}$ for some $r \in \mathbb{R}$

So $A\vec{w} = r\begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = r\begin{bmatrix} -78 \\ 117 \end{bmatrix}$

So for all $\vec{w} \in V^\perp$, $A\vec{w} \in V^\perp$ $= -39r\begin{bmatrix} -2 \\ 3 \end{bmatrix} \in V^\perp$

(c) Consider $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$

Since $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, by Thm 5.1.3,

the two vectors are linearly independent.

And since $\dim(\mathbb{R}^2) = 2$, by Thm 3.3.4,

$\mathcal{B}$ is a basis of $\mathbb{R}^2$

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | \\ \left[T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)\right]_{\mathcal{B}} & \left[T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)\right]_{\mathcal{B}} \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} \left[26\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right]_{\mathcal{B}} & \left[-39\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right]_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 0 \\ 0 & 39 \end{bmatrix}$$ by Thm 3.4.3.

(d) $[T^{10}]_{\mathcal{B}} = ([T]_{\mathcal{B}})^{10}$ by Thm 3.4.2

$$= \begin{bmatrix} 26^{10} & 0 \\ 0 & 39^{10} \end{bmatrix}$$

(e) $[T^{10}]_{\mathcal{E}} = [T$ ^{10} by Thm 3.4.2

By the change of basis thm for matrix,

$$[T]_{\mathcal{E}} = S_{\mathcal{E}\to\mathcal{B}}^{-1} [T]_{\mathcal{B}} S_{\mathcal{E}\to\mathcal{B}}$$

$$= S_{\mathcal{B}\to\mathcal{E}} [T]_{\mathcal{B}} S_{\mathcal{B}\to\mathcal{E}}^{-1}$$

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} | & | \\ [b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}} \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$$

So $[T^{10}]_{\mathcal{E}} = [T]_{\mathcal{E}}^{10} = (S_{\mathcal{B}\to\mathcal{E}} [T]_{\mathcal{B}} S_{\mathcal{B}\to\mathcal{E}}^{-1})^{10}$

$$= \underbrace{S_{\mathcal{B}\to\mathcal{E}} [T]_{\mathcal{B}}^{10} S_{\mathcal{B}\to\mathcal{E}}^{-1}}$$

$$S_{\mathcal{B}\to\mathcal{E}}^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix}$$

So $[T^{10}]_{\mathcal{E}} = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 26^{10} & 0 \\ 0 & 39^{10} \end{bmatrix} \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 26^{10} & -2\cdot 39^{10} \\ 2\cdot 26^{10} & 3\cdot 39^{10} \end{bmatrix} \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix} = \begin{bmatrix} \frac{9\cdot 26^{10}+4\cdot 39^{10}}{13} & \frac{6\cdot 26^{10}-6\cdot 39^{10}}{13} \\ \frac{6\cdot 26^{10}-6\cdot 39^{10}}{13} & \frac{4\cdot 26^{10}+9\cdot 39^{10}}{13} \end{bmatrix}$$