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Math 217: §Vector Spaces and Linear Transformations

Matrices of the same size can be added together and scaled by real numbers; similarly, functions can be added together and scaled by real numbers (as in $2\cos x + \sin x$). These operations have certain natural properties like commutativity and associativity. Structures like this are called *vector spaces*.

Definition. Let V be a set. Suppose an *addition operation* $+$ is defined on V, so that for every pair of elements v, w of V there is associated another element $v+w$ of V. Suppose also that a *scalar multiplication* by real numbers is defined on V, so that for every $c \in \mathbb{R}$ and $v \in V$ there is associated an element cv in V. Then V is called a **vector space** if all the following *axioms* hold:

VS-1: For all $u, v, w \in V$, $(u+v)+w = u+(v+w)$;

VS-2: For all $u, v \in V$, $u+v = v+u$;

VS-3: There is an element $0_V \in V$ such that $v + 0_V = v$ for all $v \in V$;

VS-4: For all $v \in V$ there is a unique element $-v \in V$ such that $v + (-v) = 0_V$;

VS-5: For all $a \in \mathbb{R}$ and $v, w \in V$, $a(v+w) = av + aw$;

VS-6: For all $a, b \in \mathbb{R}$ and for all $v \in V$, $(a+b)v = av + bv$;

VS-7: For all $a, b \in \mathbb{R}$ and for all $v \in V$, $a(bv) = (ab)v$;

VS-8: For all $v \in V$, $1v = v$.

Definition. An element of a vector space is called a *vector*.

Problem 1: Coordinate Vector Spaces. Verify that $\mathbb{R}^n$ satisfies the axioms of a vector space. The vectors in $\mathbb{R}^n$ are called **column vectors**. We denote them by $\vec{v}$ (with an arrow on top).

Problem 2: Matrices as Vector Spaces.

(a) Does the set $\mathbb{R}^{m\times n}$ of $m \times n$ matrices satisfy the axioms of a vector space? How does scalar multiplication work? What is the zero element in $\mathbb{R}^{2\times 3}$?

(b) Write out an arbitrary vector in $\mathbb{R}^{m\times 1}$. Explain why $\mathbb{R}^{m\times 1}$ is the *same* as $\mathbb{R}^m$.

(c) Write an arbitrary vector in $\mathbb{R}^{1\times n}$. These are *row vectors.* Note: $\mathbb{R}^{1\times n}$ is *not* the same* as $\mathbb{R}^n$.

(d) The set of *all* matrices (of any size) is *not* a vector space. Why not?

Solution: (a) Yes, it satisfies all the axioms.

(b) An arbitrary vector in $\mathbb{R}^{m\times 1}$ is a $m \times 1$ matrix $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$. These are just column vectors, or elements in $\mathbb{R}^m$.

(c) An arbitrary vector in $\mathbb{R}^{1\times n}$ is a $1 \times n$ matrix $\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$. This is a row, which is obviously not literally the same as a column. However, the transpose map $\mathbb{R}^{1\times n} \to \mathbb{R}^{n\times 1}$

*Although, there is a certain "sameness" which can be formalized using the transpose map.

sending $\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ to $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is a bijective linear transformation that "identifies" rows with the corresponding columns. We call this an **isomorphism.** So even though $\mathbb{R}^{1\times n}$ is NOT equal to $\mathbb{R}^{n\times 1}$, it is true that $\mathbb{R}^{1\times n}$ is **isomorphic** to $\mathbb{R}^{n\times 1}$,

(d) You can't add matrices of different dimensions. So there is no way to add arbitrary elements in the set of *all* matrices.

Problem 3: Polynomials as Vector Spaces. For each $n \in \mathbb{N}$, let P_n be the set of all *polynomial functions* of degree exactly n and let $\mathcal{P}_n$ be the set of all polynomial functions of degree *at most* n. For example, $3x^4 - \pi x^2 + x - 1$ is in P_4 and is in $\mathcal{P}_n$ for all $n \geq 4$.

(a) For which n is $x^2 + 1$ in P_n? In $\mathcal{P}_n$?

(b) How could you write an arbitrary element of $\mathcal{P}_n$? Of P_n?

(c) Is the set $\mathcal{P}_n$, together with the usual way to add and scalar multiply polynomials, a vector space?

(d) What about P_n? [HINT: Is there a zero? What happens adding $x^2 - x$ and $1 - x^2$?]

(e) Is the set $\mathcal{P}$ of all polynomials (of all degrees) a vector space? Justify your answer.

Solution: (a) The polynomial $x^2 + 1$ is in P_n for $n = 2$. It is in $\mathcal{P}_n$ for $n \geq 2$.

(b) An arbitrary element of $\mathcal{P}_n$: is $a_n x^n + a_{n-1}x^{n-1} + ...a_1 x_1 + a_0$ where $a_n, \dots, a_0 \in \mathbb{R}$. An arbitrary element of P_n is $a_n x^n + a_{n-1}x^{n-1} + ...a_1 x_1 + a_0$ where $a_n, \dots, a_0 \in \mathbb{R}$ and $a_n \neq 0$.

(c) Yes, adding polynomials and scalar multiplying like we always have since high school (so that if $f = x^2 + 1$ and $g = 3x - 2$, then $f + g = x^2 + 3x - 1$, for example), you can check that all the axioms of a vector space are satisfied. The zero element is the constant polynomial 0.

(d) No, $x^n + x^{n-1} \in P_n$ and $-x^n + x^{n-1} \in P_n$, but $x^n + x^{n-1} + (-x^n + x^{n-1}) = 2x^{n-1} \notin P_n$

(e) Yes, this is similar to $\mathcal{P}_n$.

While there are many different examples of vector spaces, our original examples $\mathbb{R}^n$ end up playing a special role. For reasons that will become clearer later, the special vector spaces $\mathbb{R}^n$ are called *coordinate vector spaces*, or simply *coordinate spaces*. The coordinate n-space $\mathbb{R}^n$ **is** a vector space, but remember that there are many other examples of vector spaces. Just because something is called a "vector" does not necessarily mean that it belongs to some $\mathbb{R}^n$—depending on context, it could be a row vector, matrix, polynomial, or something else.

Definition. Suppose that V and W are vector spaces. A map $T : V \to W$ is called a *linear transformation* (or is said to be *linear*) if for all $u, v \in V$ and for all $c \in \mathbb{R}$,

$$T(u + v) = T(u) + T(v) \qquad \text{and} \qquad T(cv) = cT(v).$$

Problem 4: Some Examples and Non-Examples. Consider the map $F : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $F(p) = x + p(x) \mapsto x + p(x)$, and the map $G : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $p(x) \mapsto xp(x)$.

(a) Write out three explicit elements in the *source* of F, and then write out their images (outputs) under F. Then, write out their images under G.

(b) Prove that F is *not* a linear transformation.

(c) Prove that G is a linear transformation.

Solution: (a). For example, if the three elements in the source are 1, $2x$ and x^2+3, then we have $F(1) = x+1, F(2x) = 3x$ and $F(x^2+3) = x^2+x+3$. Also, $G(1) = x, G(2x) = 2x^2$ and $G(x^2+3) = x^3+3x$.

(b). F does not respect addition. For example, $F(1+1) = F(2) = x+2$, but $F(1)+F(1) = (x+1)+(x+1) = 2x+2 \neq x+2$.

(c). Proof: for all polynomials $q, p \in \mathcal{P}_2$, we have $G(q+p) = x(q+p) = xq+xp = G(q)+xG(p)$ so G respects addition. Also $G(\lambda q) = x(\lambda q) = \lambda(xq) = \lambda G(q)$ for all scalars λ, so G respects scalar multiplication, too.

Problem 5: Smooth Functions. Let C^∞ be the set of all *smooth functions* from $\mathbb{R}$ to $\mathbb{R}$, where "smooth" means n-times differentiable for all $n \in \mathbb{N}$. You have been adding such functions together and scalar multiplying them since high school; discuss how, and verify that C^∞ is a vector space. Is the map $\frac{d}{dx} : C^\infty \to C^\infty$ a linear transformation?

Solution: If we have functions $f(x)$ and $g(x)$ for which all derivatives are defined, then the sum function $(f+g)(x) = f(x)+g(x)$ also have all derivatives defined, because in general $(f+g)' = f'+g'$. The same is true for scalar multiplying: $(kf)' = kf'$. All the axioms of a vector space follow from the corresponding properties of real numbers, because we define the addition just by adding the outputs for each x (which are real numbers). You've been using since high school, for example, $f(x)+g(x)$ is the same function as $g(x)+f(x)$, and $h(x)+(f(x)+g(x))$ os the same function as $(f(x)+g(x))+h(x)$, and so on. The constant zero function is in C^∞ since all its derivatives are just zero (hence exist). The differentiation map is a linear transformation, because $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$ for all $f, g \in \mathcal{C}^\infty$ and $\frac{d}{dx}(kf) = k\frac{df}{dx}$ for all $f \in \mathcal{C}^\infty$ and scalars k.

Problem 6: More examples and Non-Examples of Linear Transformations. For each of the mappings between vector spaces given below, determine whether or not the function is linear.

(a) $F : \mathcal{P} \to \mathcal{P}$ defined by $F(p) = p^2$. No.

(b) $D : C^\infty \to C^\infty$ defined by $D(g) = \frac{d^2}{dx^2}(g)$. Yes.

(c) $\iota : C^\infty \to C^\infty$ defined by $\iota(g)(x) = \int_0^x g(t)\,dt$. Yes.

(d) $\det : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. No.

(e) $\mathrm{tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\mathrm{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$. Yes.

(f) $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ where A is a fixed $m \times n$ matrix. Yes.

(g) $\phi : C^\infty \to C^\infty$ defined by $\phi(g)(x) = g(1-x)$ for all $x \in \mathbb{R}$. Yes.

(h) $E : C^\infty \to C^\infty$ defined by $E(g)(x) = e^{g(x)}$ for all $x \in \mathbb{R}$. No.

Problem 7: Coordinate free space. You may have seen in physics or multivariable calculus that "vectors" are directed magnitudes, represented by arrows which can be added by placing them "head to tail". What is scalar multiplication for such "vectors"? What is the zero-vector? What is the additive inverse of this type of vector? Draw a sketch illustrating why the commutativity axiom (VS-2) holds. Let $\mathbb{E}^2$ denote the set of all such vectors, say, in the plane. Is $\mathbb{E}^2$ a vector space? Can you find a *bijection* $\mathbb{E}^2 \to \mathbb{R}^2$? Is your bijection a *linear transformation*?

Solution: Scaling stretches/shrinks the arrow representing the vector by the scalar—for example, $2\vec{v}$ is represented by an arrow in the same direction but of twice the length. The zero-vector can be represented by a dot, or an arrow of length zero which points in every/no direction. The additive inverse of a vector $\vec{v}$ is represented by an arrow of the same length but in the opposite direction. To illustrate the commutative law, we draw a parallelogram. Yes, $\mathbb{E}^2$ is a vector space. We can define a bijection $\mathbb{E}^2 \to \mathbb{R}^2$ as follows: for a given $\vec{v} \in \mathbb{E}^2$, represent it by an arrow whose tail is at the origin in $\mathbb{R}^2$. The coordinates of the head of this arrow are a point in $\mathbb{R}^2$, which we represent by a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. The map $\vec{v} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ is a well defined map from $\mathbb{E}^2$ to $\mathbb{R}^2$.

Each vector $\vec{v}$ in $\mathbb{E}^2$ has a different head (if all tails are placed at the origin), so gives a different point in $\mathbb{R}^2$. That is, the map $\mathbb{E}^2$ to $\mathbb{R}^2$ is injective. It is bijective because the inverse map is defined: to a column vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ we assign the vector in $\mathbb{E}^2$ represented by an arrow with tail at the origin and head at $\begin{bmatrix} x \\ y \end{bmatrix}$. Yes, the map $\mathbb{E}^2 \to \mathbb{R}^2$ is a linear transformation: you can check that addition has the same meaning whether in $\mathbb{E}^2$ or $\mathbb{R}^2$ and likewise for scalar multiplication.