217-hw-10

Part A (10 points)

Solve the following problems from the book:

6.1: 20, 54
6.2: 42, 50
6.3: 14
7.1: 12, 18, 42

6-1

In Exercises 11 through 22, use the determinant to find out for which values of the constant k the given matrix A is invertible.

A

20. $\begin{bmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{bmatrix}$

$$\det A = (k+1)(2k+4) + k(k+2) + k+2$$
$$- (k+1) - (k+2)^2 - k(2k+4)$$
$$= 2k^2+6k+4 + k^2+2k + k+2 - k - 1$$
$$- k^2 - 4k - 4 - 2k^2 - 4k$$
$$= 1 \neq 0$$

So A is invertible by WS4A. Thm 3

54. Is the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 7 & 1000 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1000 \\ 1 & 2 & 1000 & 3 & 4 \end{bmatrix}$$

positive or negative? How can you tell? Do not use technology.

We may see the inversions in the patterns of A

There are $5! = 120$ patterns

The one with all 1000 has 4 inversions so has positive sign.

So it contributes 10^{15} to $\det A$

And since we can not choose 4 1000 (4 1000 determines the 5th is also 1000)

The largest one of the other patterns has the absolute value $< 10^9 \cdot 10^2$ (9 is biggest of other entries.)

Even if all other patterns are negative.

$\Rightarrow$ $\det A > 10^{15} - 119 \cdot 10^9 \cdot 10^2 > 0$

6-2

42. Consider an $n \times m$ matrix

$$A = QR,$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an upper triangular $m \times m$ matrix with positive diagonal entries $r_{11}, \ldots, r_{mm}$. Express $\det(A^T A)$ in terms of the scalars r_{ii}. What can you say about the sign of $\det(A^T A)$?

$$\begin{aligned}
\det(A^T A) &= \det((QR)^T QR) \\
&= \det(R^T Q^T Q R) \\
&= \det(R^T I_m R) \\
&= \underline{\det(R^T R)} \\
&= \det(R^T)\det(R) \\
&= \det(R)\det(R) = \underline{\det^2(R) > 0}
\end{aligned}$$

50. Find the determinant of the matrix

$$M_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

for arbitrary n. (The ijth entry of M_n is the minimum of i and j.)

By recursively do:

1, do elementary operation: substract the last row by the $n-1^{th}$ row, we get M_n'

Note that the determinat does not change. $\det(M_n) = \det(M_n')$

2. Laplace expansion along the last row of M_n', we get $\det(M_n') = 1 \cdot \det(M_{n-1})$

$\Rightarrow \det(M_{n-1}) = \det(M_n)$

3. $n := n-1$

We can finally have $\det(M_n) = \det(M_1) = 1$

6-3

14. Find the 3-volume of the 3-parallelepiped defined by the vectors

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

$$V[P] = \sqrt{\left|\det\left(\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}^T\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}\right)\right|}$$

$$= \sqrt{\left|\det\begin{bmatrix}1&1&1\\1&4&10\\1&10&30\end{bmatrix}\right|} \quad = \sqrt{6}$$

7-1

12. Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$. Show that 2 and 4 are eigenvalues of A and find all corresponding eigenvectors. Find an eigenbasis for A and thus diagonalize A.

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Assume $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$

$$\Rightarrow \begin{bmatrix} 2v_1 \\ 3v_1 + 4v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$$\Rightarrow \underline{\lambda = 2} \text{ or } \overbrace{v_1 = 0}^{\lambda \neq 2,} \Rightarrow \lambda v_2 = 4v_2, \underline{\lambda = 4}$$

① if $\lambda = 2 \Rightarrow 3v_1 + 4v_2 = 2v_2$

$$\Rightarrow v_1 = -\frac{2}{3}v_2$$

$$\Rightarrow \underline{E_2 = \left\{ t\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}}$$

$\forall v \in E_2$, v is a 2-eigenvector

② if $\lambda = 4 \Rightarrow \underline{E_4 = \left\{ t\begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}}$

$\forall v \in E_4$, v is a 4-eigenvector

So An eigenbasis for A is $\left(\begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$,

So we can diagonalize A as $\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$

Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformations in Exercises 15 through 22. In each case, find an eigenbasis if you can, and thus determine whether the given transformation is diagonalizable.

18. Reflection about a plane V in $\mathbb{R}^3$

diagonalizable.

$\forall v \in V$, $T(v) = v \Rightarrow v$ is 1-eigenvector

Since $\dim(V) = 2$, we can choose a basis of V which is $\mathcal{B} = (v_1, v_2)$ where v_1, v_2 are 1-eigenvectors

And $\forall v \in V^\perp$, $T(v) = -v$
$\Rightarrow v$ is -1-eigenvector

$\Rightarrow$ We pick arbitrary nonzero vector $w \in V^\perp$, w is a -1-eigenvector

By definition, of $V^\perp$, (w, v_1, v_2) are linearly independent

So (w, v_1, v_2) is a basis of $\mathbb{R}^3$

42. Find a basis of the linear space V of all 3×3 matrices A for which both $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are eigenvectors, and thus determine the dimension of V.

let $A \in V$

$$\underbrace{A\begin{bmatrix}1\\0\\0\end{bmatrix} = \lambda_1\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\text{for some } \lambda_1 \in \mathbb{R}} \Rightarrow Ae_1 = \lambda_1\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\underbrace{A\begin{bmatrix}0\\0\\1\end{bmatrix} = \lambda_2\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{for some } \lambda_2 \in \mathbb{R}} \Rightarrow Ae_3 = \lambda_2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

So $A = \begin{bmatrix}a & b & 0\\0 & c & 0\\0 & d & e\end{bmatrix}$ for some $a, b, c, d, e \in \mathbb{R}$

So a basis for V is $\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$

Part B (25 points)

Problem 1. If V and W are vector spaces, a function $F : V \times W \to \mathbb{R}$ is said to be *bilinear* if all of the following hold:

- for all $\vec{x}, \vec{y} \in V$ and $\vec{z} \in W$, $F(\vec{x}+\vec{y}, \vec{z}) = F(\vec{x}, \vec{z}) + F(\vec{y}, \vec{z})$;
- for all $\vec{x} \in V$ and $\vec{y}, \vec{z} \in W$, $F(\vec{x}, \vec{y}+\vec{z}) = F(\vec{x}, \vec{y}) + F(\vec{x}, \vec{z})$;
- for all $\vec{x} \in V$ and $\vec{y} \in W$ and for all $a \in \mathbb{R}$, $F(a\vec{x}, \vec{y}) = aF(\vec{x}, \vec{y})$ and $F(\vec{x}, a\vec{y}) = aF(\vec{x}, \vec{y})$.

Furthermore, if $F : V \times V \to \mathbb{R}$ is a bilinear function, we say that F is *alternating* if $F(\vec{v}, \vec{v}) = 0$ for all $\vec{v} \in V$. Throughout this problem, let $F : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear function.

(a) Prove that F is alternating if and only if $F(\vec{u}, \vec{v}) = -F(\vec{v}, \vec{u})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

(b) Prove that if F is alternating and $F(\vec{e}_1, \vec{e}_2) = 1$, then

$$F(\vec{u}, \vec{v}) = \det[\vec{u}\ \vec{v}] \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^2.$$

(a) Proof

① Assume F is alternating

Let $u, v \in \mathbb{R}^2$ be arbitrary

Then $F(u+v, u+v) = 0$ since it is alternating

and $F(u+v, u+v) = F(u,v) + F(u,u) + F(v,v) + F(v,u)$ since it is bilinear.

$= F(u,v) + F(v,u)$

So $F(u,v) = -F(v,u)$

② Assume $\forall u, v \in \mathbb{R}^2$, $F(u,v) = -F(v,u)$

Then take arbitrary $v \in \mathbb{R}^2$

$\Rightarrow F(v,v) = -F(v,v)$

So $F(v,v) = 0$

By ①②, we have proved F is alternating if and only if $F(u,v) = -F(v,u)$ for all $u, v \in \mathbb{R}^2$

(b) <u>Proof</u>

Assume F is alternating and $F(\vec{e_1}, \vec{e_2}) = 1$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be arbitrary

So $\vec{u} = u_1\vec{e_1} + u_2\vec{e_2}$, $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2}$ for some $u_1, u_2, v_1, v_2 \in \mathbb{R}^2$

So $F(\vec{u}, \vec{v}) = F(u_1\vec{e_1} + u_2\vec{e_2}, v_1\vec{e_1} + v_2\vec{e_2})$

$= u_1 F(\vec{e_1}, v_1\vec{e_1} + v_2\vec{e_2}) + u_2 F(\vec{e_2}, v_1\vec{e_1} + v_2\vec{e_2})$

$= u_1(\underbrace{v_1 F(\vec{e_1}, \vec{e_1})}_{=0 \text{ by (a)}} + v_2 F(\vec{e_1}, \vec{e_2})) + u_2(v_1 F(\vec{e_1}, \vec{e_2}) + \underbrace{v_2 F(\vec{e_2}, \vec{e_2})}_{=0 \text{ by (a)}})$

$\Rightarrow = u_1 v_2 F(\vec{e_1}, \vec{e_2}) - u_2 v_1 F(\vec{e_2}, \vec{e_1})$

Since $F(\vec{e_1}, \vec{e_2}) = 1$ and F is alternating,

by (a) we have $F(\vec{e_2}, \vec{e_1}) = -F(\vec{e_1}, \vec{e_2}) = -1$

So $F(\vec{u}, \vec{v}) = \underline{u_1 v_2 - u_2 v_1}$

Since $\det[\vec{u}\ \vec{v}] = \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \underline{u_1 v_2 - u_2 v_1}$

Therefore $F(\vec{u}, \vec{v}) = \det[\vec{u}\ \vec{v}]$

Problem 2. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and consider the map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = AM$.

(a) Prove that T is a linear transformation.

(b) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E}$ is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$. Your answer should be in terms of the entries of M.

(c) Compute $\det[T]_{\mathcal{E}}$.

(d) Compute $\det[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the ordered basis of $\mathbb{R}^{2\times 2}$ given by

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pi & 1 \end{bmatrix}, \begin{bmatrix} 0 & 7 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right).$$

(e) Either prove that T is always diagonalizable no matter what M is, or provide an explicit example of a matrix M for which T is *not* diagonalizable and briefly explain why your example works.

(a) Pf. Let $A, B \in \mathbb{R}^{2\times 2}$ be arbitrary, $k \in \mathbb{R}$ be arbitrary.

Then $T(A+B) = (A+B)M = AM + BM = T(A) + T(B)$

$T(kA) = kAM = k(AM) = kT(A)$

So T is a linear transformation.

(b) $$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} & [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & | \\ [E_{11}M]_{\mathcal{E}} & [E_{12}M]_{\mathcal{E}} & [E_{21}M]_{\mathcal{E}} & [E_{22}M]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & | \\ \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]_{\mathcal{E}} & \left[\begin{smallmatrix} c & d \\ 0 & 0 \end{smallmatrix}\right]_{\mathcal{E}} & \left[\begin{smallmatrix} 0 & 0 \\ a & b \end{smallmatrix}\right]_{\mathcal{E}} & \left[\begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix}\right]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c) By Thm 6.1.5,

$$\det[T]_{\mathcal{E}} = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} - \det\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \det\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= (ad-bc)^2$$

(d) $\det[T]_{\mathcal{B}} = \det(S_{\mathcal{E}\to\mathcal{B}}) \det[T]_{\mathcal{E}} \det(S_{\mathcal{E}\to\mathcal{B}}^{-1})$

$$= \det[T]_{\mathcal{E}} = (ad-bc)^2$$

(e) Consider $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

So if for $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, $T(A) = AM = \lambda A$

$$\Rightarrow \begin{bmatrix} 2x_1 & x_1+2x_2 \\ 2x_3 & x_3+2x_4 \end{bmatrix} = \lambda \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

So either $\lambda = 2$ or $\lambda \neq 2$ but $x_1 = x_3 = 0$

$\Rightarrow x_1 = x_3 = 0$

$\Rightarrow 2x_2 = x_2,\ 2x_4 = x_4$

$\Rightarrow x_1 = x_2 = x_3 = x_4 = 0$

Since $\vec{0}$ cannot be eigenvector, impossible

So $\lambda = 2$ is the only possible eigenvalue

But $E_2 = \left\{ \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} \middle| \, m, n \in \mathbb{R} \right\}$

$\dim(E_2) = 2 < 4$

not enough eigenvectors to form an eigenbasis

So T is not diagonalizable.

Problem 3. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^4$. Define the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ by the rule $T(\vec{x}) = \det([\vec{x}\ \vec{u}\ \vec{v}\ \vec{w}])$ for all $\vec{x} \in \mathbb{R}^4$. (You do not have to prove that T is linear.)

(a) Prove that there exists a unique vector $\vec{z} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{z} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^4$, and find the components of $\vec{z}$ in terms of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. (Hint: $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$.)

(b) Find the vector $\vec{z}$ (as in part (a)) when $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{e}_2$, and $\vec{w} = \vec{e}_3$ are the first three standard basis vectors in $\mathbb{R}^4$.

(c) When is $\vec{z} = \vec{0}$? (Your answer should be in terms of $\vec{u}$, $\vec{v}$, and $\vec{w}$.)

(d) Prove that $\vec{z}$ is orthogonal to each of $\vec{u}, \vec{v}$ and $\vec{w}$, and find $\det([\vec{z}\ \vec{u}\ \vec{v}\ \vec{w}])$ in terms of $\|\vec{z}\|$.

(a) Since T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}$

By key theorem, $\exists$ matrix $A = [a_1\ a_2\ a_3\ a_4]$ for some $a_1, a_2, a_3, a_4 \in \mathbb{R}$

s.t. $T(\vec{x}) = A\vec{x} = A^T \cdot x$

Therefore take A^T as $\vec{z}$, $\forall \vec{x} \in \mathbb{R}^4$, $T(\vec{x}) = \vec{z} \cdot \vec{x}$

(b) $T(\vec{x}) = \det \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \\ x_4 & 0 & 0 & 0 \end{bmatrix}$

By the inversion method, $\det \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \\ x_4 & 0 & 0 & 0 \end{bmatrix} = (-1)^3 (x_4)$

$= -x_4$

So $T(\vec{x}) = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \vec{x}$

So $\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

(c) $\vec{z} = \vec{0}$ means $\forall \vec{x} \in \mathbb{R}^4$, $\det[\vec{x}\ \vec{u}\ \vec{v}\ \vec{w}] = 0$

which is equivalent to that $\forall \vec{x} \in \mathbb{R}^4$, $[\vec{x}\ \vec{u}\ \vec{v}\ \vec{w}]$ is invertible, by WS 21A-Thm3

Therefore $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent is a sufficient and necessary condition

i.e. $\exists a, b, c \in \mathbb{R}$ s.t. $a\vec{u} + b\vec{v} + c\vec{w} = 0$ where a, b, c are not all 0.

(d) $T(\vec{x}) = \det \begin{bmatrix} x_1 & u_1 & v_1 & w_1 \\ x_2 & u_2 & v_2 & w_2 \\ x_3 & u_3 & v_3 & w_3 \\ x_4 & u_4 & v_4 & w_4 \end{bmatrix} = x_1 \det \begin{bmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix}$

$- x_2 \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix} + x_3 \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{bmatrix} - x_4 \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$

$= \left(\det \begin{bmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix} - \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix} + \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{bmatrix} - \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right)$

$\cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = (\vec{u} \times \vec{v} \times \vec{w}) \cdot \vec{x} \implies$ so $\vec{z} = \vec{u} \times \vec{v} \times \vec{w}$

Therefore $\vec{J} \perp \vec{u}$, $\vec{J} \perp \vec{v}$, $\vec{J} \perp \vec{w}$

$\vec{J}$ is orthogonal to $\vec{u}, \vec{v}, \vec{w}$.

And $\det[\vec{J}\ \vec{u}\ \vec{v}\ \vec{w}] = \vec{J} \cdot \vec{J} = \|\vec{J}\|^2$

Problem 4. For a polynomial $p(x)$ and an $n \times n$ matrix A, let $p(A)$ denote the matrix obtained by plugging in A for x. For example, if $p(x) = x^3 + 2x^2 + 3$, then $p(A) = A^3 + 2A^2 + 3I_n$. (Note that I_n behaves like the constant "1" in $\mathbb{R}^{n \times n}$.)

(a) Prove that for every $n \times n$ matrix A and for every eigenvalue λ of A, the real number $p(\lambda)$ is an eigenvalue of the $n \times n$ matrix $p(A)$.

(b) Let p be a polynomial and let $n \in \mathbb{N}$. Prove that if S is an invertible $n \times n$ matrix, then for every $A \in \mathbb{R}^{n \times n}$ we have $p(S^{-1}AS) = S^{-1}p(A)S$.

(c) Let p be a polynomial and let A be an $n \times n$ matrix. Prove that if A is diagonalizable, then every eigenvalue of $p(A)$ is of the form $p(\lambda)$ for some eigenvalue λ of A.

(a) Select arbitrary $n \times n$ matrix A and let λ be an arbitrary eigenvalue of A

Then $p(A) = a_1 A + a_2 A^2 + \dots + a_n A^n$ for some $1, \dots, n \in \mathbb{Z}$ and scalars $a_1, \dots, a_n$

Since $\exists \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \lambda\vec{x}$. Fix $\vec{x}$

By WS22: then $\forall n \in \mathbb{Z}^+$, $A^n x = \overbrace{A \dots A}^{n-1} \cdot (A\vec{x}) = \lambda^n \vec{x}$

Then $p(A)\vec{x} = \left(\sum_{i=1}^{n} a_i A^i\right)\vec{x} = \sum_{i=1}^{n} (a_i A^i \vec{x})$

$= \sum_{i=1}^{n} a_i \lambda^i \vec{x} = \left(\sum_{i=1}^{n} a_i \lambda^i\right) \cdot \vec{x}$

So $p(\lambda) = \sum_{i=1}^{n} a_i \lambda^i$ is an eigenvalue of $p(A)$

(b) $p(x) = a_1 x + a_2 x^2 + \dots + a_n x^n$ for some $1, \dots, n \in \mathbb{Z}$ and scalars $a_1, \dots, a_n$

$$\Rightarrow p(S^{-1}AS) = a_1 S^{-1}AS + a_2 S^{-1}ASS^{-1}AS + \dots + a_n \underbrace{S^{-1}AS \dots S^{-1}AS}_{n \text{ times}}$$

$$= a_1 S^{-1}AS + a_2 S^{-1}A^2S + \dots + a_n S^{-1}A^n S$$

$$= S^{-1}\left(\sum_{i=1}^{n} a_i A^i S\right)$$

$$= S^{-1}\left(\sum_{i=1}^{n} a_i A^i\right) S$$

$$= S^{-1} p(A) S$$

(c) Assume A is diagonalizable

So $A = S^{-1}DS$ for some invertible matrix S and diagonal matrix $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A.

By (b) we know: $p(A) = p(S^{-1}DS) = S^{-1}p(D)S$

where $p(D) = a_1 D + a_2 D^2 + \dots + a_m D^m$ for some $1, \dots, m \in \mathbb{Z}$ and scalars $a_1, \dots, a_m$

And $p(D)$ is a diagonal matrix whose (i,i)th entry is $\sum_{j=1}^{m} a_j \lambda_i^j$

Since $p(D)$ is diagonal and S is invertible,

All elements on the diagonal are all eigenvalues of $p(A)$

So every eigenvalue of $p(A)$ is $\sum_{j=1}^{m} a_j \lambda_i^j$

for some $j \in \mathbb{Z}^+$, in the form of $p(\lambda_i)$

for some eigenvalue λ_i of A.

Problem 5.

(a) Let $A \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix such that $A^2 = I_2$. Prove that A is diagonalizable. (Hint: try factoring $A^2 - I_2$, and consider the possible ranks of the factors.)

(b) Does the same result hold for larger matrices? That is, if $A \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix for which $A^2 = I_n$, must A be diagonalizable? Either prove this or give a counterexample.

(a) Pf

$$A^2 = I_2 \Rightarrow \text{So } A^2 - I_2 = A^2 - I_2 A + I_2 A - I_n^2$$

$$= (A - I_2)(A + I_2) = 0.$$

$\Rightarrow$ Case ①: $A = I_2 \Rightarrow A$ is diagonalizable

Case ②: $A = -I_2 \Rightarrow A$ is diagonalizable

Case ③: $A \neq I_2$ and $A \neq -I_2$, so $(A - I_2) \neq 0$, $(A + I_2) \neq 0$

So $\text{rank}(A - I_n) \geq 1$, $\text{rank}(A + I_n) \geq 1$

Then $(A + I_2)$, $(A - I_2)$ must both not be invertible, since otherwise $\operatorname{rank}((A-I)(A+I)) \geq 1$, contradicts.

i.e. $\operatorname{rank}(A-I_2) \leq 1$, $\operatorname{rank}(A+I_2) \leq 1$

Therefore $\operatorname{rank}(A-I_2) = \operatorname{rank}(A+I_2) = 1$

So $\ker(A-I_2) \neq \{\vec{0}\}$, $\ker(A+I_2) \neq \{\vec{0}\}$

$\Rightarrow \exists$ nonzero $\vec{v}, \vec{w} \in \mathbb{R}^2$ s.t. $(A+I_2)\vec{v} = 0$, $(A-I_2)\vec{w} = 0$.

So 0 is an eigenvalue of both $A-I_2$ and $A+I_2$

$\Rightarrow \exists \vec{x} \in \mathbb{R}^2$ s.t. $(A-I_2)\vec{x} = 0$
nonzero

$\Rightarrow A\vec{x} = I_2\vec{x} = \vec{x}$

and $\exists \vec{y} \in \mathbb{R}^2$ s.t. $(A+I_2)\vec{y} = \vec{0}$
nonzero

$\Rightarrow A\vec{y} = -I_2\vec{y} = -\vec{y}$

So ± 1 are eigenvalues of A,

$\vec{x}, \vec{y}$ are two eigenvectors of A

And since $\dim(\mathbb{R}^2) = 2$, $(\vec{x}, \vec{y})$ is an eigenbasis of $\mathbb{R}^2$ for T_A

Therefore T_A is diagonalizable

So A is diagonalizable.

(b) Yes.

Proof. $A^2 = I_n \Rightarrow (A+I_n)(A-I_n) = 0$

Let $\vec{v}$ be arbitrary vector in $\mathbb{R}^n$.

$$A(A\vec{v}+\vec{v}) = A^2\vec{v} + A\vec{v} = \vec{v} + A\vec{v}$$
$$= A\vec{v} + \vec{v}$$

$\Rightarrow$ $A\vec{v}+\vec{v}$ is a 1-eigenvector of A

Also, $A(A\vec{v}-\vec{v}) = A^2\vec{v} - A\vec{v} = \vec{v} - A\vec{v}$
$$= -1(A\vec{v}-\vec{v})$$

$\Rightarrow$ $A\vec{v}-\vec{v}$ is a -1-eigenvector of A

So $\forall \vec{v} \in \mathbb{R}^n$, $\vec{v} = \left(\frac{1}{2}(A\vec{v}+\vec{v}) + \frac{1}{2}(A\vec{v}-\vec{v})\right)$

is the sum of a 1-eigenvector and a -1-eigenvector

Therefore every vector of $\mathbb{R}^n$ is a linear combination of a 1-eigenvector and a -1-eigenvector

So consider the basis $\mathcal{B}$ of E_1 and basis $\mathcal{U}$ of E_{-1}

$\mathcal{B} \cup \mathcal{U}$ is a basis of $\mathbb{R}^n$ which is for sure an eigenbasis

So A is diagonalizable.