Math 217 Worksheet 5: Linear transformations and geometry (§2.2)

Definition: A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a map (a.k.a. mapping or function) such that for all vectors $\vec{x}$ and $\vec{y}$ in the source $\mathbb{R}^m$ and all scalars $c \in \mathbb{R}$, both

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) \qquad \text{and} \qquad T(a\vec{x}) = aT(\vec{x}).$$

Key Theorem. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, let A be the matrix whose j-th column is $T(\vec{e}_j)$. Then for all $\vec{x} \in \mathbb{R}^n$, we have $T(\vec{x}) = A\vec{x}$.

The matrix A is called the **standard matrix of** T.

Problem 1. Warmup: Finding the standard matrix. Assuming each of the given maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, *use the Key Theorem* to find a matrix A such that $T\vec{x} = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

(a) T is *dilation* by a factor of three, sending each vector $\vec{v}$ to $3\vec{v}$.

(b) T is *rotation* in the clockwise direction by 90° (fixing the origin).

(c) T is *reflection* over the line $y = x$.

(d) T is *projection* to the x-axis taking each $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ 0 \end{bmatrix}$.

Solution: The four matrices, in order are $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 2. Rotations. For each $\theta \in \mathbb{R}$, let $\text{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counter-clockwise rotation about the origin through an angle of θ (measured in radians).

(a) Give an intuitive *geometric* argument that Rot_θ is linear. [Draw some pictures, reasoning physics/Math 215 style with arrows representing vectors, but do not write any equations or try to write out a formal proof.]

(b) Draw and label a sketch showing where Rot_θ sends the vectors $\vec{e}_1$ and $\vec{e}_2$.

(c) Use the Key Theorem and some trigonometry to find the standard matrix A_θ of Rot_θ.
[Hint: Your answer will involve sine and cosine of θ. Do not just repeat a memorized formula from the book.]

(d) Given a pair of rotations Rot_θ and Rot_ϕ, what sort of transformation (geometrically speaking) is the composite transformation $\text{Rot}_\phi \circ \text{Rot}_\theta$? In general, are $\text{Rot}_\phi \circ \text{Rot}_\theta$ and $\text{Rot}_\theta \circ \text{Rot}_\phi$ equal, or different?

(e) Can you think of two different ways to compute $(\text{Rot}_\phi \circ \text{Rot}_\theta)(\vec{x})$ using matrix-vector products?

Solution: For (c), the first column of R_θ is $R_\theta(\vec{e}_1)$, which is $\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$ (by the Key Theorem) and the second is $R_\theta(\vec{e}_2) = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$. So $A_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

For (d), note that Rot_θ and Rot_ϕ are rotations, then $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta$ is also a rotation, namely

$$\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta = \mathrm{Rot}_{\phi+\theta}.$$

Since addition is commutative, it follows that $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta = \mathrm{Rot}_\theta \circ \mathrm{Rot}_\phi$.

For (e), $\mathrm{Rot}_\phi \circ \mathrm{Rot}_\theta(\vec{x}) = A_\phi(A_\theta\vec{x})$ but it is also $A_{\phi+\theta}\vec{x}$.

Problem 3: Orthogonal projections. Let L be a line through the origin in $\mathbb{R}^2$. Consider the mapping

$$\mathbb{R}^2 \xrightarrow{\pi_L} \mathbb{R}^2 \qquad \vec{x} \mapsto \text{“the projection of } \vec{x} \text{ onto } L.\text{”}$$

(a) Draw a sketch to illustrate π_L. Write a formula for $\pi_L(\vec{x})$ using dot product and a unit vector $\vec{u}$ in the direction of L. [You may assume basic facts about dot product.]

(b) The textbook writes $\vec{x}^{\parallel}$ for $\pi_L(\vec{x})$. Why? It also writes $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$. Draw a picture to explain what this means.

(c) *Using the definition of linear transformation,* prove that π_L is a linear transformation. [You may assume basic facts about dot product.]

(d) Remember that every linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ can be described by matrix multiplication. Find the matrix of π_L in terms of $\vec{u} = \begin{bmatrix}u_1\\ u_2\end{bmatrix}$, a unit vector parallel to L. Confirm, in the special case where L is the x-axis, that your answer matches the formula in Problem 1(d).

(e*) Find the matrix for the projection onto line of slope m though $\begin{bmatrix}0\\ 0\end{bmatrix}$.

Solution: For (a), we have $\pi_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$. For (b), the notation $\vec{x}^{||}$ means ”component of $\vec{x}$ parallel to L” and $\vec{x}^{\perp}$ is the component of $\vec{x}$ perpendicular to $\vec{x}$.

For (c), we need to check both linearity conditions. Let $\vec{x}$ and $\vec{y}$ be arbitrary vectors in $\mathbb{R}^2$ and $c \in \mathbb{R}$ an arbitrary scalar. We have

1. $\pi_L(\vec{x}+\vec{y}) = [(\vec{x}+\vec{y})\cdot\vec{u}]\vec{u} = (\vec{x}\cdot\vec{u})\vec{u} + (\vec{y}\cdot\vec{u})\vec{u} = \pi_L(\vec{x}) + \pi_L(\vec{y})$; and
2. $\pi_L(c\vec{x}) = (c\vec{x}\cdot\vec{u})\vec{u} = c(\vec{x}\cdot\vec{u})\vec{u} = c\pi_L(\vec{x})$.

So π_L is linear.

For (d), we use the Key Theorem. We need to compute $\pi_L(\vec{e}_1)$ and $\pi_L(\vec{e}_2)$. Using the formula from (a), these are $u_1\vec{u}$ and $u_2\vec{u}$, respectively, so the matrix is $\begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}$. When L is the x-axis, we can take $\vec{u} = \vec{e}_1$, and we get $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

For (e), we compute that $\vec{u} = \frac{1}{\sqrt{m^2+1}}\begin{bmatrix}1\\ m\end{bmatrix}$. So the matrix is $\begin{bmatrix}\frac{1}{m^2+1} & \frac{m}{m^2+1}\\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1}\end{bmatrix}$.

Problem 4. Reflection. Let L be a line through the origin in $\mathbb{R}^2$. Consider the mapping

$$\mathbb{R}^2 \xrightarrow{\rho_L} \mathbb{R}^2 \quad \vec{x} \mapsto \text{"the reflection of } \vec{x} \text{ over } L\text{."}$$

(a) Draw a picture illustrating $\rho_L(\vec{x})$. Include vectors $\vec{x}$, $\vec{x}^{||}$, $\vec{x}^{\perp}$, as well as the line L.

(b) Write down a formula for ρ_L using the dot product and a unit vector $\vec{u}$ in the direction of L.

(c) Prove that ρ_L is linear.

(d) Find the matrix of ρ_L.

(e*) Write the matrix for the linear transformation "reflection over the line through the origin of slope m." Does your formula give the correct answer when $m = 0$? Why?

Solution: For (b), if you draw the picture, you see that $\rho_L(\vec{x}) = \vec{x}^{||} - \vec{x}^{\perp}$. From above, this is

$$\begin{aligned}\rho_L(\vec{x}) &= (\vec{x} \cdot \vec{u})\vec{u} - (\vec{x} - (\vec{x} \cdot \vec{u})\vec{u}) \\ &= 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}\end{aligned}$$

For (c), we check both linearity conditions.

1. $\rho_L(\vec{x} + \vec{y}) = 2[(\vec{x} + \vec{y}) \cdot \vec{u}]\vec{u} - (\vec{x} + \vec{y}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x} + 2(\vec{y} \cdot \vec{u})\vec{u} - \vec{y} = \rho_L(\vec{x}) + \rho_L(\vec{y})$;
2. $\rho_L(c\vec{x}) = (c\vec{x} \cdot \vec{u})\vec{u} - c\vec{x} = c[(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}] = c\rho_L(\vec{x})$.

For (d), we use the Key Theorem, plugging in $\rho_L(\vec{e}_1) = 2(\vec{e}_1 \cdot \vec{u})\vec{u} - \vec{e}_1 = \begin{bmatrix} 2u_1^2 - 1 \\ 2u_1u_2 \end{bmatrix}$ and $\rho_L(\vec{e}_2) = 2(\vec{e}_2 \cdot \vec{u})\vec{u} - \vec{e}_2 = \begin{bmatrix} 2u_1u_2 \\ 2u_2^2 - 1 \end{bmatrix}$, we see the matrix of ρ_L is $\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$.

For (e), we substitute $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{m^2+1}} \\ \frac{m}{\sqrt{1+m^2}} \end{bmatrix}$. We get $\begin{bmatrix} \frac{2}{m^2+1} - 1 & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{2m^2}{m^2+1} - 1 \end{bmatrix}$. This can be simplified. When $m = 0$, it is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which indeed is the matrix of reflection over the x axis.

Rotations, projections, and reflections are important examples of linear transformations: they respect vector addition and scalar multiplication. This is true also in higher dimension. Like any linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, you can describe these geometric transformations by *matrix multiplication*. Be sure you know how to find the matrix of given linear transformation.

Problem 5. Geometric meaning of Determinant. Let Q be the unit square in $\mathbb{R}^2$, that is $Q = \{c_1\vec{e}_1 + c_2\vec{e}_2 \,|\, 0 \leq c_i \leq 1\}$. In this problem, we consider what happens to Q under a linear transformation $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$.

(a) Sketch $\vec{e}_1$, $\vec{e}_2$ and Q, which we will view in the source $\mathbb{R}^2$.

(b) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which stretches each vector by 2 in the horizontal direction and by 3 in the vertical direction. Find the matrix A such that $T_1 = T_A$.

(c) Describe $T_1(Q)$ in set-builder notation, and sketch the image of Q under T_1 in the target $\mathbb{R}^2$.

(d) Compute the area of $T_1(Q)$, comparing to the determinant of the matrix A.

(e) Now repeat (c) and (d) for the linear transformation $T_2 = T_B$ where $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

(f) Repeat (b), (c) and (d) for projection T_3 onto the x-axis (See Problem 1 (d)).

(g) Let A be any 2×2 matrix. What kind of shape can $T_A(Q)$ be? Conjecture a formula for the area of $T(Q)$? We will prove your conjecture in Chapter 6 (if it's correct).

Solution: For (b), $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ so $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

For (c), The image $T(Q)$ is a rectangle of height 3 and width 2, squared up against the x and y axis in the first quadrant. In set builder notation, $T(Q) = \{\{c_1 T(\vec{e}_1) + c_2 T(\vec{e}_2) \mid 0 \le c_i \le 1\} = \{\{c_1(\begin{bmatrix} 2 \\ 0 \end{bmatrix}) + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mid 0 \le c_i \le 1\}$ or alternatively, $T(Q) = \{\begin{bmatrix} 2c_1 \\ 3c_2 \end{bmatrix} \mid 0 \le c_i \le 1\}$.

For (d), the area of $T(Q)$ is 6, same as the determinant of A.

For (e), the unit square is stretched and pulled into a parallelogram with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Its area is 4, same as the determinant of the matrix.

In (f), the unit square is squashed onto a line segment whose endpoints are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The "area" of the line segment is zero, same as the determinant of the matrix!

CONJECTURE: A linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ takes the unit square to a parallelogram (possibly degenerated to a segment) of area $|det A|$, where A is the matrix of T. This is in fact a theorem, and it will work in higher dimension too (suitably generalized). You will eventually be able to prove this.