Math 217: Geometry in Three Space

Theorem: *A composition of two rotations in $\mathbb{R}^3$ (around lines through the origin) is another rotation (around some third line through the origin).*

The purpose of this worksheet is to prove this non-trivial theorem.

Problem 1. Warm-up: Rotations in the plane.

(a) Is the composition of two rotations (around the origin) a rotation in $\mathbb{R}^2$? If ρ_1 is rotation through θ_1 and ρ_2 is rotation through θ_2, describe the composite explicitly.

(b) Is the map given by multiplication by $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ a rotation? Through which angle?

(c) Show that a 2×2 matrix A represents a rotation in the plane (in standard coordinates) if and only if A is orthogonal with determinant 1. [Hint: given one column of A, find the other explicitly. Now find the desired θ.]

Solution: Yes, the composition is the rotation through $\theta_1 + \theta_2$. The matrix in (b) is rotation through 45^0.

For (c), let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Because A is orthogonal, the columns have dot product zero. This means that that A must have the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Also, $a^2 + b^2 = 1$, so that (a, b) is a point on the unit circle. Let θ be the corresponding angle, so that $(\cos\theta, \sin\theta) = (a, b)$. Now it is easy to check that A is the matrix of the rotation through θ.

Problem 2: Rotations in 3-space. In $\mathbb{R}^3$, a rotation has some *axis*— a line L through the origin— around which the rotation takes place. Think about two rotations in $\mathbb{R}^3$, called ρ_1 and ρ_2, around L_1 and L_2 respectively, and through angles θ_1 and θ_2 respectively. Discuss why the argument that a composition of rotations is a rotation in dimension two does not immediately generalize to 3D. Can you find a special case where it does?

Solution: The theorem in $\mathbb{R}^2$ generalizes only if the two axes of rotation are the same.

Problem 3: An example. Let T_1 be rotation be $\pi/2$ counterclockwise around the z-axis and let T_2 be rotation around the x-axis by $\pi/2$ (counterclockwise).

(a) Find the (standard) matrix of the composition $T_1 \circ T_2$.

(b) Find the eigenvalues and the algebraic and geometric multiplicity of each eigenvalue.

(c) Find a basis for the eigenspace of each eigenvalue.

(d) According to the Theorem, we know that $T_1 \circ T_2$ is some rotation. What is the axis of this rotation? [Hint: Think geometrically—what does this axis have to do with eigenvectors?]

Solution:

(a) Thinking geometrically, we see where the standard basis vectors go under T_2, then T_1. It helps to draw pictures both in $\mathbb{R}^3$ and the (copy of) $\mathbb{R}^2$ perpendicular to the axis of rotation. $\vec{e}_1 \mapsto \vec{e}_1 \mapsto \vec{e}_2, \vec{e}_2 \mapsto \vec{e}_3 \mapsto \vec{e}_3$ and $\vec{e}_3 \mapsto -\vec{e}_2 \mapsto \vec{e}_1$. So the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(b) The char poly is $(-x)^3 + (-x)$, which factors as $-(x-1)(x^2+1)$. The zeros of this polynomial are $1, i, -i$. Of these, only 1 is real. So the only (real) eigenvalue is 1. The geometric multiplicity is 1 because we have three different roots of the char poly, so each has algebraic mutliplicity 1, hence geometric multiplicity 1. Alternatively, if you are not used to thinking about complex eigenvalues, we directly compute the kernel of $A - I_3 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

(c) The eigenvalue 1 has eigenspace spanned by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

(d) Since the line spanned by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is the only line fixed by this transformation, it must be the axis of rotation!

Problem 4: Explain why rotation ρ (around L, through θ) in $\mathbb{R}^3$ is an orthogonal transformation. Discuss how to find a convenient basis for understanding it. Prove that the determinant of ρ is 1. [Hint: find the matrix in a well-chosen orthonormal basis.]

Solution: Rotations preserve the length of every vector, so they are orthogonal. A convenient basis would have three vectors: one along the axis L of rotation and the other 2 perpendicular to the axis of rotation (in $L^{\perp|}$). We can even make these two perpendicular to each other if we want, and make all length 1. If we do this, the matrix of ρ will have block form: $\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$ where B is the matrix of the rotation in the plane $L^{\perp|}$. The determinant of this matrix is $\det B$ which is 1, since B is just a rotation in the plane.

Problem 5: Let T be the composition of two rotations in $\mathbb{R}^3$ around (possibly) different axes of rotation through the origin.

(a) Prove that the composition T of two rotations (around possibly different axes!) in $\mathbb{R}^3$ is orthogonal of determinant one.

(b) Prove that 1 is an eigenvalue of T (or more generally, of any orthogonal transformation with determinant 1). [Hint: Compute $\det(A - I_3) = \det(A - A^\top A)$.]

Solution:

(a) If A_1 and A_2 are the corresponding matrices of the rotations, we know each has determinant 1. So the composition, which has matrix A_1A_2 has determinant $\det(A_1A_2) =$

$\det A_1 \det A_2 = 1$. Also, it is orthogonal, since a composition of orthogonal transformations is orthogonal.

(b) Let A be the matrix of the composition T. We know A is orthogonal with determinant 1. Compute $\det(A - I_3)$. We have $\det(A - I_3) = \det(A - A^T A) = \det((I_3 - A^T)A) = \det(I_3 - A^T)\det A = \det(I_3 - A)^T \det A = \det(I_3 - A)\det A = \det(I_3 - A)$ since $\det A = 1$. But now we have $\det(A - I_3) = -\det(A - I_3)$. This says that $\det(A - I_3) = 0$, which means the 1-eigenspace is non-zero and so 1 is an eigenvalue.

Problem 6: Let T be the composition of two rotations in $\mathbb{R}^3$. From Problem 5b, we know that T has an eigenvector with eigenvalue 1. Let L be the line spanned by such a 1-eigenvector.

(a) Let $W = L^\perp$. Prove that $T(\vec{v}) \in W$ for all $\vec{v} \in W$. Discuss why this means that the restriction of T to W is an *orthogonal* linear transformation $W \xrightarrow{T|_W} W$.

(b) Show that the determinant of $W \xrightarrow{T|_W} W$ is also 1. [HINT: chose a convenient basis for W which extends to a convenient basis for $\mathbb{R}^3$ for analyzing T.]

(c) Conclude (using the Problem 1) that $W \xrightarrow{T|_W} W$ is a rotation.

(d) Prove the Theorem at the top of the first page.

Solution:

(a) Let $\vec{v} \in L^\perp$. We need to show $T(\vec{v}) \in L^\perp$. Since L is spanned by $\vec{w}$, we have $\vec{v} \in L^\perp$ if and only if $\vec{v} \cdot \vec{w} = 0$. Becuase T is orthogonal, it preserves dot products. So also $T(\vec{v}) \cdot T(\vec{w}) = 0$. Since $\vec{w}$ is an eigenvector, also $T(\vec{v}) \cdot \vec{)} = 0$. This means $\vec{w} \in L^\perp$.

(b) Chose a basis for $\mathbb{R}^3$ consisting of $\vec{w}$ (a basis for L) and then $\vec{v}_1, \vec{v}_2$ a basis for $W = L^\perp$. The matrix of T in this basis will be a block matrix, with $\vec{e}_1$ as the first column and some 2×2 in the lower right corner. This 2×2 matrix is the matrix of $T_{|W}$, so it is orthogonal. Its determinant is 1 since the determinant of T is one. So it is rotation, by Problem 1.

(c)

(d) Let A be the matrix of the composition. We have seen that this matrix is orthogonal and has determinant 1. By Problem 6, it has eigenvalue 1. Let $\vec{w}$ be an eigenvector. Then T does not move the line L spanned by $\vec{w}$. Also, if W is the plane perpendicular to L, then $T|_W$ is a rotation, so the matrix in the basis $(\vec{w}, \vec{v}_1, \vec{v}_2)$ is a block matrix with a 1 in the upper left and a 2×2 rotation matrix in the bottom right. So this is rotation around L.