HW 7

DUE FRIDAY OCTOBER 11 AT 7PM (BONUS 24 HOURS LATER)

For hints see office door. But try without the hints first. There are no IBL problems this week.

Problem A: Suppose $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is differentiable at $x \in A$. Show that if u is a unit vector in $\mathbb{R}^n$, then

$$D_u f(x) \leq |Df(x)|,$$

with equality if and only if $u = Df(x)/|Df(x)|$. Show that $D_u f(x) = 0$ if and only if u is orthogonal to $Df(x)$.

Remark: Keeping in mind that for functions to $\mathbb{R}$ the Jacobian matrix is also called the gradient, this shows that the gradient is the direction of fastest change of the function. Similarly the set of directions where the function does not change (to first order) is the perp space of the gradient.

Problem B: Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ and $c : U \to \mathbb{R}$ are C^1. Set

$$M = c^{-1}(0).$$

Assume that f restricted to M has a local minimum at p, and that Dc is surjective at p. Then prove that there exist a real number λ such that

$$Df(p) = \lambda Dc(p).$$

(This means the gradients of f and p are parallel at p. The number λ is called a Lagrange multiplier.)

Problem C: In at most a few sentences, give a non-rigorous, intuitive explantion for Problem B.

Problem D: Using Problem B, find the mimum of the function $f(x, y) = 3x + y$ on the unit circle centered at the origin in $\mathbb{R}^2$.

Problem E: Formulate and prove a generalization of Problem B when c maps to $\mathbb{R}^k$ rather than k. (Whereas Problem B allows you to do certain optimization problems subject to one constraint, this lets you do some optimization problems subject to k constraints. Your generalization will feature numbers $\lambda_1, \ldots, \lambda_k$.)

Remark: You must do B first and then E. You may not reference E in your solution to B.

Problem F: Prove that the set of positive definite matrices is open in the set of n by n symmetric matrices. (You may not use the bonus from HW5.)

Problem G: Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open. Suppose that x_0 is critical point of f and the Hessian of f is positive definite at x_0. Prove that x_0 is a strict local minimum for f.

Problem H: Let A be an invertible n by n matrices. Let C be its cofactor matrix, so $C_{ij} = (-1)^{i+j} \det(A_{ij})$, where A_{ij} is the $n-1$ by $n-1$ matrix obtained by deleting row i and column j from A. Prove the following version of Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C^T denotes the transpose of C. (You may use the cofactor expansion of the determinant.)

Problem I: Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable. Show that

$$(f \cdot g)'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t),$$

where $\cdot$ denotes dot product and $f'(t)$ denotes $Df(t)$ (which in this case is a vector).

Bonus: Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open and convex. Show that the region above f, i.e.

$$\{(x, y) \in A \times \mathbb{R} : y \geq f(x)\},$$

is convex if and only if the Hessian of f is positive semi-definite at each point of A.