HW 8

DUE FRIDAY OCTOBER 18 AT 7PM (BONUS 24 HOURS LATER)

Problem A: Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be of class C^1 and such that $f(1,2,3) = 0$ and

$$Df(1,2,3) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(1) Does the equation $f(x,y,z) = 0$ define and implicitly a function of some of the variables in terms of the rest? If so, what variables can be expressed in terms what others? Discuss all the possibilities.

(2) Suppose there is a function $g : B \to \mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}$ such that $f(x, g(x)) = 0$ for $x \in B$ and $g(1) = (2,3)$. Compute $Dg(1)$.

Problem B: Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1 and suppose that $f(a) = 0$ and $Df(a)$ has rank n. Show that if $c \in \mathbb{R}^n$ is sufficiently close to 0, then the equation $f(x) = c$ has a solution.

Problem C: Let B be a closed box, and $f : B \to \mathbb{R}$ is a continuous function. Show that f is integrable.

Problem D: Write up Problem D from the IBL worksheet (on countable sets).

Problem E: Write up Problem G from the IBL worksheet (on countable sets).

Bonus: A map T from a metric space (X, d) to itself is called a contraction mapping if there is a $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c \cdot d(x, y)$$

for all $x, y \in X$.

(1) Show that every contraction mapping of a complete metric space has a unique fixed point.
(2) Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is C^1, and $Df(0)$ is invertible. Show that for $\epsilon > 0$ sufficiently small, if B is the closed ball of radius ϵ around 0, then there is a $\delta > 0$ such that if $|y| < \delta$

$$T(x) = Df(0)(x) + y - f(x)$$

defines a contraction mapping from B to itself. (As part of this, you'll have to show $T(B) \subset B$.)
(3) Explain why this immediately implies that the image of f contains a neighbourhood of $f(0)$.

Remark: The point of this problem is to give the idea for a different proof of the Inverse Function Theorem. (This proof can be found in many textbooks, but don't look!) Studying the proof from class will help you solve this question, and you can use the lemmas from class if you want to.