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Assignment WebWork4-Definition_of_Rings due 02/12/2024 at 11:59pm EST

Problem 1. **(6 points)**
Which of the following rings have no non-zero zero divisors?

a. $\mathbb{Z}_{24}[x]$ [?/Yes/No]
b. $\mathbb{Z}_4[x]$ [?/Yes/No]
c. $M_2(\mathbb{Q})$ [?/Yes/No]
d. $\mathbb{Z}_{11}[x]$ [?/Yes/No]
e. $\mathbb{Z}_3[x]$ [?/Yes/No]

Problem 2. **(6 points)**
For each of the following maps, say whether or not it is a Ring Homomorphism (using our class's definition of a ring Homomorphism, not the book's!).

a. $\mathbb{Z}_8 \times \mathbb{Z}_8 \to \mathbb{Z}_8$
$([n]_8, [m]_8) \mapsto [m]_8$ [?/Yes/No]

b. $\mathbb{R}[x] \to \mathbb{R}$
$f(x) \mapsto f(\pi)$ [?/Yes/No]

c. $\mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_3$
n
$_2 \mapsto ([n]_2, [0]_3)$ [?/Yes/No]

d. $\mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3$
n
$_3 \mapsto ([1]_2, [n]_3)$ [?/Yes/No]

e. The inclusion of $\{[0]_6, [3]_6\}$ in $\mathbb{Z}_6$ [?/Yes/No]

Problem 3. **(1 point)**
Which of the following are ring homomorphisms? Answer Yes if it is and No otherwise.

1. $\phi : \mathbb{Z}_7 \to \mathbb{Z}_{28}$ such that $\phi([c]_7) = [c]_{28}$ [?/Yes/No]

2. $\phi : \mathbb{Z}_{54} \to \mathbb{Z}_6$ such that $\phi([c]_{54}) = [c]_6$ [?/Yes/No]

3. $\phi : \mathbb{Z}_3 \to \mathbb{Z}_3[x]$ such that $\phi([c]_3) = [c]_3$ [?/Yes/No]

4. $\phi : \mathbb{Z}_{24}[x] \to \mathbb{Z}_{24}$ such that $\phi(f) = [f(2)]_{24}$ [?/Yes/No]

5. $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ such that $\phi(f) = [f(2)]_8$ [?/Yes/No]

6. $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}[x]$ such that $\phi([c]_{10}) = [c]_{10}x$ [?/Yes/No]

Problem 4. **(9 points)**
Recall that a reduced fraction is a fraction a/b where $\gcd(a,b) = 1$. It is a fact that

$$\mathbb{Z}_{(7)} := \{ \text{ all reduced fractions whose denominators are powers of } 7\}$$
$$\text{(including the zero-th power)}$$

is a subring of $\mathbb{Q}$, the ring of rational numbers.

For this problem we require that ring homomorphisms take the multiplicative identity to the multiplicative identity.

(a) Determine the number of ring homomorphisms $\mathbb{Z}_{(7)} \to \mathbb{Z}_7$.

_____

(b) Determine the number of ring homomorphisms $\mathbb{Z}_7 \to \mathbb{Z}_{(7)}$.

_____

Solution: Solution:
Note that $1/7$ is a non-zero element in the ring $\mathbb{Z}_{(7)}$, and $1_{\mathbb{Z}_{(7)}} = (1/7) \times 7$. So if $f : \mathbb{Z}_{(7)} \to \mathbb{Z}_7$ is a ring homomorphism that takes the multiplicative identity to the multiplicative identity, then
$1_{\mathbb{Z}_7} = f(1/7)f(7)$.
But since f is a ring homomorphism,
$f(7) = 7f(1_{\mathbb{Z}_{(7)}}) = 7 \times 1_{\mathbb{Z}_7} = 0_{\mathbb{Z}_7}$.
Combine these two and we arrive at a contradition, so f does not exist.

Next, suppose $g : \mathbb{Z}_7 \to \mathbb{Z}_{(7)}$ is a ring homomorphism that takes the multiplicative identity to the multiplicative identity. Then $0_{\mathbb{Z}_7} = 7 \times 1_{\mathbb{Z}_7}$, so $0_{\mathbb{Z}_{(7)}} = g(7 \times 1_{\mathbb{Z}_7}) = 7 \times g(1_{\mathbb{Z}_7}) = 7 \times 1_{\mathbb{Z}_{(7)}}$, a contradiction. So g does not exist.

Problem 5. **(9 points)**

Determine the number of possible ring homomorphisms for each pair of rings:

(a) $\mathbb{Z}_{13} \to \mathbb{Z}_{13}$: ____

(b) $\mathbb{Z}_{80} \to \mathbb{Z}_{16}$: ____

(c) $\mathbb{Z}_{127} \to \mathbb{Z}_{21}$: ____

Note: Recall that ring homomorphisms take the multiplicative identity to multiplicative identity.

Solution: Solution:

If $f : \mathbb{Z}_m \to \mathbb{Z}_n$ is a ring homomorphism then

$$\begin{aligned}(**)\ f(a1_{\mathbb{Z}_m}) &= af(1_{\mathbb{Z}_m}) \\ &= a1_{\mathbb{Z}_n}\end{aligned}$$

Since every element of $\mathbb{Z}_m$ is of the form $a1_{\mathbb{Z}_m}$ for some integer a, the requirement

$$(*) \text{ a ring homomorphism } f : A \to B \text{ takes } 1_A \text{ to } 1_B$$

means that for any integers m,n there is at most one ring homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n$. On the other hand, by properties of ring homomorphisms we get

$$\begin{aligned}0_{\mathbb{Z}_n} &= f(0_{\mathbb{Z}_m}) \\ &= f(m1_{\mathbb{Z}_m}) \\ &= mf(1_{\mathbb{Z}_m} \\ &= m1_{\mathbb{Z}_n}\end{aligned}$$

so in order for f to be a ring homomorphism we need n to divide m. And if n does divide m, we check that $(**)$ does satisfy the definition of ring homomorphisms. Actually – and this is important – there are two things to check:

(A) f is well-defined, and

(B) f satisfies the conditions related to addition and multiplication.

(A) is important and often neglected: Recall that elements of $\mathbb{Z}_m$ are congruence classes, so every element has infinitely many representatives. Specificially,

$$\begin{aligned}& a \text{ and } a' \text{ represent the same congruence class in } \mathbb{Z}_m \\ \Leftrightarrow \quad & a' \equiv a \pmod{m} \\ \Leftrightarrow \quad & m \text{ divides } (a'-a) \quad (@@)\end{aligned}$$

Now, by $(**)$ we have

$$f(a'1_{\mathbb{Z}_m}) = a'1_{\mathbb{Z}_n}.$$

In order for the right side to represent the same congruence class in $\mathbb{Z}_n$, we need

$$a' \equiv a \pmod{n}.$$

Equivalently,

$$n \text{ divides } (a'-a).$$

If n divides m then this follows from $(@@)$, and so f is well-defined.

Compared to the above, (B) is easy to check. For example:

$$\begin{aligned}f(a1_{\mathbb{Z}_m} + b1_{\mathbb{Z}_m}) &= f((a+b)1_{\mathbb{Z}_m}) \\ &= (a+b)f(1_{\mathbb{Z}_m}) \\ &= (a+b)1_{\mathbb{Z}_n} \\ &= a1_{\mathbb{Z}_n} + b1_{\mathbb{Z}_n} \\ &= af(1_{\mathbb{Z}_m}) + bf(1_{\mathbb{Z}_m}) \\ &= f(a1_{\mathbb{Z}_m}) + f(b1_{\mathbb{Z}_m})\end{aligned}$$

The condition for multiplication is checked similarly.