Math 412 Homework 3

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Thursday, February 8th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. Let $R = \mathrm{Fun}(\mathbb{R}, \mathbb{R})$ be the ring in exercise D2 of the "Ring Basics" adventure sheet. 0_R and 1_R are the constant functions zero and one.

   Show which of the following subsets of R are subrings of R. If they are not subrings, show whether they are rings (with a different multiplicative identity than 1_R, but endowed with the same operations as in R) or not.

   (a) The set C of constant functions.

   (b) The set S of those functions f such that $f(q) = 0$ for any $q \in \mathbb{Q}$.

   (c) The set T consisting of 0_R, together with those functions with no zeros, or only a finite number of zeros. (A zero of a function $f \in R$ is an element $x \in \mathbb{R}$ such that $f(x) = 0$)

2. An element x in a ring R is said to be *nilpotent* if $x^m = 0_R$ for some positive integer m. Generalizing the definition on page 40 of our text, a *unit* u in a ring R is an element with a multiplicative inverse, meaning there exists $s \in R$ such that $su = us = 1_R$.

   (a) Prove that if $x \in R$ is nilpotent (and R is not the zero ring), then x cannot be a unit.

   (b) Prove that if $x \in R$ is nilpotent, then $(1_R - x)$ is a unit. (Hint: One approach to showing something is a unit is to write down its inverse. In this case, it could help to recall geometric series from Calculus.)

   (c) Describe all the nilpotent elements in $\mathbb{Z}_n$ in terms of their prime factorization.

3. An element $r \neq 0$ in a commutative ring R is said to be a *zerodivisor* if there exists a nonzero element $s \in R$ such that $rs = 0$.

   (a) Given a nonzero element $r \in R$, prove that r is not a zerodivisor if and only if the map $R \longrightarrow R$ given by multiplication by r, meaning the map $s \mapsto rs$, is injective.

   (b) Describe all the zerodivisors in $\mathbb{Z}_n$ in terms of the prime factorization of n or their greatest common divisor with n.

4. For two rings, R and S a function $\varphi\colon R \to S$ is a *ring homomorphism* if $\varphi(1_R) = 1_S$, and for all $x, y \in R$ the following conditions hold

$$\varphi(x +_R y) = \varphi(x) +_S \varphi(y), \text{ and } \varphi(x \times_R y) = \varphi(x) \times_S \varphi(y)$$

   (a) Let R be any ring (recalling how our class convention differs from that of the book!). Prove that there exists a unique ring homomorphism $\mathbb{Z} \to R$.

   (b) Let $n > 1$ be an integer. Prove that there does not exist a ring homorphism $\mathbb{Z}_n \to \mathbb{Z}$.

   (c) Suppose R and S are two rings, and $f : R \to S$ is a ring isomorphism; in particular, f is a bijection and so has an inverse function $g : S \to R$. Prove that g is also a ring homomorphism.

(d) Prove the following theorem that appears in our worksheets:

> If $f : R \to S$ is a ring homomorphism, then f is injective if and only if $\ker f = \{0_R\}$.