Homework 7

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Sunday, Mar. 17th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. Let G and H be groups.
   (a) Give an example where G and H are both cyclic, but $G \times H$ is not.
   (b) If $G \times H$ is a cyclic group, prove that G and H are both cyclic.
   (c) Recall that $\mathbb{R}^\times$ is the multiplicative group of units of $\mathbb{R}$. Define an explicit isomorphism $f : \mathbb{R}^\times \to \mathbb{R} \times \mathbb{Z}_2$.

2. Let $S^1 \subset \mathbb{C}$ be the unit circle; that is
$$S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}.$$
   (a) Prove that S^1 is a subgroup of $\mathbb{C}^\times$.
   (b) For every positive integer n, find an element of order n in S^1.
   (c) Find an element of infinite order in S^1.

3. Let R be a commutative ring, and consider the group $\mathrm{GL}_2(R)$ of units in the ring of 2×2 matrices $\mathrm{M}_2(R)$ with coefficients in R.
   (a) Suppose that
$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathrm{M}_2(R)$$
   and all the entries are in an ideal $I \subsetneq R$. Prove that A is not a unit.
   (b) Prove that for any matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{M}_2(R)$$
   there is a matrix B such that
$$AB = BA = \det(A) I_2.$$
   (c) (This is the hard problem) Prove that a matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{M}_2(R)$$
   is a unit in $\mathrm{M}_2(R)$ if and only if $\det(A)$ is a unit.

4. Let p be a prime number and consider the field $\mathbb{Z}_p$.
   (a) Show that a 2×2 matrix $A \in \mathrm{M}_2(\mathbb{Z}_p)$ is not a unit if and only if "the columns are linearly dependent."

(b) Show that the set of upper triangular invertible matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ forms a subgroup of order $p(p-1)^2$, which is non-abelian when $p \neq 2$.

(c) Compute the order of $\mathrm{GL}_2(\mathbb{Z}_p)$.

(d) Show that the diagonal invertible matrices form an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of order $(p-1)^2$.

(e) Find an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of order p. Make sure to show this is a subgroup.