Math 412. The Orbit–Stabilizer Theorem

Fix a group action of the group G on the set X.
DEFINITION: The **orbit** of an element $x \in X$ is the subset of X

$$O(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

DEFINITION: The **stabilizer** of an element $x \in X$ is the subgroup of G

$$\text{Stab}(x) = \{g \in G \mid g(x) = x\} \subset G.$$

ORBIT-STABILIZER THEOREM: *If a finite group G acts on a set X, then for every $x \in X$, we have*

$$|G| = |O(x)| \cdot |\text{Stab}(x)|.$$

Part 1: The essentials.

A. Let D_4 be the symmetry group of the square. Consider the natural action of D_4 on the square with vertices $(\pm 1, \pm 1)$ by rotations and reflections.

(1) Complete the following chart which records, for different points of the square, the orbit, stabilizer, and cardinalities of each.

$(x,y) \in \mathbb{R}^2$	$O(x,y)$	stab(x,y)	# $O(x,y)$	# stab(x,y)
$(0,0)$				
$(1,0)$				
$(1,1)$				
$(1,\frac{1}{10})$				
$(\frac{1}{2},\frac{1}{3})$				

(2) Verify the orbit stabilizer theorem for each of the five points in your chart.

Solution.

(1) We will use the notation $\{e, r, r^2, r^3, l_v, l_h, l_{(1,1)}, l_{(1,-1)}\}$, where l_v is the reflection on the line $x = 0$, l_h is the reflection on the line $y = 0$, $l_{(1,1)}$ is the reflection on the line $x = y$ and $l_{(1,-1)}$ is the reflection on the line $y = -x$.

$(x,y) \in \mathbb{R}^2$	$O(x,y)$	stab(x,y)	# $O(x,y)$	# stab(x,y)
$(0,0)$	$\{(0,0)\}$	D_4	1	8
$(1,0)$	$\{(\pm 1,0),(0,\pm 1)\}$	$\{e, l_h\}$	4	2
$(1,1)$	$\{(\pm 1,\pm 1)\}$	$\{e, l_{(1,1)}\}$	4	2
$(1,\frac{1}{10})$	$\{(\pm 1,\pm\frac{1}{10}),(\pm\frac{1}{10},\pm 1)\}$	$\{e\}$	8	1
$(\frac{1}{2},\frac{1}{3})$	$\{(\pm\frac{1}{2},\pm\frac{1}{3}),(\pm\frac{1}{3},\pm\frac{1}{2})\}$	$\{e\}$	8	1

(2) Easy: the number of elements in the orbit times the number of elements in the stabilizer is the same, always 8, for each point.

B. THE STABILIZER OF EVERY POINT IS A SUBGROUP. Assume a group G acts on a set X. Let $x \in X$.

(1) Prove that the stabilizer of x is a **subgroup** of G.

(2) Use the Orbit-Stabilizer theorem to prove that the cardinality of every orbit divides $|G|$.

(3) Let G be a group of order 17 and let X be a set with 16 elements. Explain why there is no nontrivial action of G on X. [The trivial action is the one in which $g \cdot x = x$ for all $g \in G$ and all $x \in X$.]

Solution.

(1) We need to show that $Stab(x) = \{g \in G \mid g \cdot x = x\}$ is a subgroup of G. It suffices to check:

(a) $Stab(x)$ is non-empty; this is easy since $e_G \cdot x = x$ (by definition of action), so $e_G \in Stab(x)$.

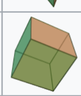
(b) If $a, b \in Stab(x)$, then $ab \in Stab(x)$. Also pretty easy: take arbitrary $a, b \in Stab(x)$. We compute $(a \circ b) \cdot x = a(\cdot(b \cdot x)$ by definition of action. Since $b \cdot x = x$, this becomes $(a \circ b) \cdot x = a(\cdot(b \cdot x)) = a \cdot x$; and since $a \cdot x = x$, we conclude that $(a \circ b) \cdot x = x$. Thus $a \circ b \in Stab(x)$.

(c) if $a \in Stab(x)$, then $a^{-1} \in Stab(x)$. For this, take arbitrary $a \in Stab(x)$. This means $a \cdot x = x$. Apply a^{-1} to both sides to get $a^{-1} \cdot (a \cdot x) = a^{-1} \cdot x$. By definition of action, $a^{-1} \cdot (a \cdot x) = (a^{-1} \circ a) \cdot x = e \cdot x = x$. So we have $a^{-1} \cdot x = x$, and $a^{-1} \in Stab(x)$.
QED.

(2) This is clear: $|O(x)|$ divides $|G|$ since $|G| = |O(x)| \times |Stab(x)|$.

(3) Since 17 is prime, the only divisors of $|G|$ are 1 and 17. So any orbit of any G action can only have size 1 or 17. Since in this case, X has only 16 points total, no orbit can have 17 points! So all orbits have one point. This means $g \cdot x = x$ for all g and all x. The only such action is the trivial action.

C. SYMMETRY GROUPS OF PLATONIC SOLIDS. There are exactly five convex regular solid figures in $\mathbb{R}^3$.

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Each is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. The chart describes each of these platonic solids. Each platonic solid has a symmetry group which acts naturally on the solid. In particular, each symmetry group also acts on the set of vertices, the set of edges and the set of faces, of the corresponding solid. By analyzing these three actions, we can better understand the symmetry group of each solid.

For each of the 5 platonic solids, complete the following chart:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces			
on edges			
on vertices			

For each of the three actions, does it matter which point $x \in X$ (i.e., which face, edge, or vertex) you use to compute the orbit? Why is the order of the stabilizer the same for each $x \in X$ in each of the three actions? Is this true in general for a group acting on a set? What is special in this case?

Solution.

(1) For the symmetry group of the cube, we have:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces	6	4	24
on edges	12	2	24
on vertices	8	3	24

(2) For the symmetry group of the tetrahedron we have:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces	4	3	12
on edges	6	2	12
on vertices	4	3	12

Note that here, it is a bit tricky to find the stabilizer of an edge, but since we know there are 2 elements in the stabilizer from the Orbit-Stabilizer theorem, we can look.

(3) For the Octahedron, we have

Action	# orbit	# stab	$\lvert G\rvert$
on Faces	8	3	24
on edges	12	2	24
on vertices	6	4	24

(4) For the symmetry group of the dodecahedron, we have:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces	12	5	60
on edges	30	2	60
on vertices	20	3	60

(5) For the symmetry group of the icosahedron, we have:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces	20	3	60
on edges	30	2	60
on vertices	12	5	60

D. Consider the group Cube of symmetries of the cube.

(1) Observe that Cube acts on the set of 4 diagonals (from one vertex to its opposite) of the cube.
(2) Show that this action is faithful.[1]
(3) Show that Cube is isomorphic to $\mathcal{S}_4$.[2]
(4) Conclude that the orders of the elements in Cube are exactly $1, 2, 3, 4$, and that Cube is generated by two elements.

Solution.

(1) A symmetry must take a diagonal to a diagonal; it is clear that this is compatible with composition.
(2) Following the hint, once we label the diagonals, every face is determined by the order in which the diagonals meet its vertices. Thus, if an element of the group fixes all four diagonals, then it fixes all of the faces, so it can only be the identity.

[1]Hint: Label the diagonals as $1, 2, 3, 4$. Note that every face has one vertex on each diagonal. For each face, list the diagonal of each vertex, conterclockwise, starting with 1. Note that each face has a different list.

[2]Hint: Use the homomorphism $\mathrm{ad} : G \to \mathrm{Bij}(X)$ from the last worksheet.

(3) By the last part, we obtain an injective homomorphism from Cube to $\mathcal{S}_4$. Since these groups have the same order, this map must be bijective.
(4) This follows from the fact that these statements hold in $\mathcal{S}_4$.

E. THE PROOF OF THE ORBIT-STABILIZER THEOREM: Let G act on X. Fix a point $x \in X$.

(1) Show that there is a surjective map of sets $G \to O(x)$ sending each $g \in G$ to $g \cdot x$.
(2) Show that g and h have the same image under this map if and only if $g^{-1}h \in \text{Stab}(x)$.
(3) For each $g \cdot x \in O(x)$, show that the set of elements in G mapping to $g \cdot x$ is the left coset gK where $K = \text{Stab}(x)$.
(4) Show that this map induces a bijection between the set $G/\text{Stab}(x)$ of left cosets of $\text{Stab}(x)$ in G and the orbit $O(x)$.
(5) Prove the Orbit Stabilizer Theorem.

Solution.
(1) Given any $y \in O(x)$, $y = g \cdot x$ for some g, so y is in the image of our map.
(2) Given any $g, h \in G$, $g \cdot x = h \cdot x$ if and only if $x = (g^{-1}h) \cdot x$, or equivalently $g^{-1}h \in \text{Stab}(x)$.
(3) Given $g \in O(x)$, we have shown that for each $h \in G$, $h \cdot x = g \cdot x$ if and only if $h^{-1}g \in \text{Stab}(x)$, which is equivalent to $g \in hK$. Since the left cosets form a partition, this is the same as $h \in gK$.
(4) We have already showed that two elements g and h have the same image if and only if $gK = hK$. This means that the map $G/\text{Stab}(x) \longrightarrow O(x)$ given by $g\text{Stab}(x) \mapsto g \cdot x$ is well-defined and injective. As in par (1), this map is surjective.
(5) Our bijection shows that $|O(x)| = [G : \text{Stab}(x)]$. The Orbit-Stabilizer theorem now follows by Lagrange's Theorem.

Part 2: Bonus.
F. LINEAR ACTIONS. Consider the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$ by matrix multiplication (where elements of $\mathbb{R}^2$ are written as columns).

(1) Describe the action in mathematical symbols and prove it is really an action.
(2) What is the stabilizer of the point $\begin{bmatrix}1\\0\end{bmatrix}$?
(3) What is the orbit of the point $\begin{bmatrix}1\\0\end{bmatrix}$?

Solution.
(1) For $A \in GL_2(\mathbb{R})$ and $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$, we have $A \cdot \begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$. It is an action because $I_2\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ and $A \cdot (B \cdot \begin{bmatrix}x\\y\end{bmatrix}) = A(B\begin{bmatrix}x\\y\end{bmatrix}) = (AB)\begin{bmatrix}x\\y\end{bmatrix}$, by the associativity of matrix multiplication. So the two axioms of a group action are satisfied.
(2) What $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ stabilize $\begin{bmatrix}1\\0\end{bmatrix}$? Since $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix}$, a necessary and sufficient condition is that $\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$. So the stabilizer of $\begin{bmatrix}1\\0\end{bmatrix}$ is the subgroup of matrices $\{\begin{bmatrix}1 & b\\0 & d\end{bmatrix} \mid b, d, \in \mathbb{R}\}$.

(3) The orbit of $\begin{bmatrix}1\\0\end{bmatrix}$ under matrix multiplication is $\mathbb{R}^2 \setminus \vec{0}$. Indeed, we can get an arbitrary non-zero $\begin{bmatrix}x\\y\end{bmatrix}$ by multiplying $\begin{bmatrix}x & b\\y & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$. Note that as long as $\begin{bmatrix}x\\y\end{bmatrix}$ is not the zero vector, we can always find $b, d \in \mathbb{R}$ such that the matrix $\begin{bmatrix}x & b\\y & d\end{bmatrix}$ is invertible.

G. SYMMETRY GROUPS OF PLATONIC SOLIDS AGAIN.

(1) Show that the symmetry group of the tetraheadron is isomorphic to $\mathcal{A}_4$.
(2) Show that the symmetry group of the octahedron is isomorphic to $\mathcal{S}_4$.
(3) Can you compute the symmetry groups of the dodecahedron and the icosahedron?

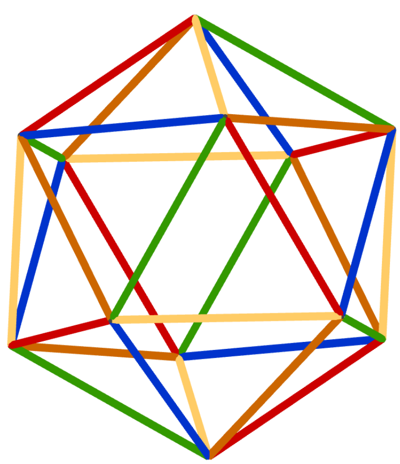
Solution.

(1) Inspired by our calculation of the symmetry group of the cube, we look for a set of four elements for this group to act on. An obvious choice is the set of faces. It is clear that this action is faithful: if a symmetry of the tetrahedron fixes all fo the faces, it fixes the whole tetrahedron. We then get an injective homomorphism from $\mathrm{Tet} \hookrightarrow \mathcal{S}_4$. We know $|\mathrm{Tet}| = 12$ and $|\mathcal{S}_4| = 24$, so this is not surjective.

We can show directly that the image consists of even permutations. For each vertex, there are two $120°$ rotations. These fix one face and cycle around the others, so these give the 8 three cycles. Also, there are three $180°$ rotations that switch two pairs of faces. These are the three pairs of disjoint 2-cycles. This plus the identity makes for all twelve elements.

(2) The argument is similar to that for the cube, except let Oct act on the set of *pairs of opposite faces* of the octahedron. We challenge you to fill in the details!

(3) Let's start with the icosahedron. We want to show that the symmetry group is isomorphic to $\mathcal{A}_5$ by cleverly having our symmetry group act on a set of five elements. The key geometric insight we will use is the following: any collection of six edges such that no pair shares a vertex must be the same up to symmetry—if it contains an edge e, it contains its opposite edge e', as well as the two other edges that are parallel to the plane P containing e and e', and the two edges that are perpendicular to P. We note also that such a set of edges has the property that every vertex meets exactly one of these edges. Let's call this set of edges the *minimal spanning set* determined by e, denoted $[e]$. Since any edge is in exactly one of these, the minimal spanning sets form a partition of the set of edges. That is, "minimal spanning sets" partition the edge set into five subsets $[a], [b], [c], [d], [e]$ that each contain six edges.



We can think of the minimal edge sets as colors, like in the picture above.

Now, if we have any symmetry of the icosahedron, it must take a set of six edges such that no pair shares a vertex to another set of six edges such that no pair shares a vertex. That is, the symmetry group of the icosahedron acts on the set of minimal spanning sets.

We claim that this action is faithful. One way to see this is as follows. For any vertex, look at the edges that meet it. Read off the minimal spanning set of each edge, starting at the edge in $[a]$ and going clockwise. That is, for each vertex, we get a sequence like $[a], [d], [b], [e], [c]$. Dropping brackets, I got

$$\{abcde, abdec, abecd, acbed, acdbe, acedb, adbce, adceb, adebc, aebdc, aecbd, aedcb\},$$

which in particular are distinct. Thus, if a symmetry fixes all of the sets $\{[a], [b], [c], [d], [e]\}$, it must fix each vertex (since you can solve for a vertex from its sequence in that list), and thus fix the whole solid.

Now, since we have a faithful action, the adjoint homomorphism from the icosahedron group to $\mathcal{S}_5$ (the symmetries of a set of five elements) is injective, so the icosahedron group is isomorphic to the image of this homomorphism. We need to compute the image.

Let's try to compute the image. For each face, look at the minimal spanning sets of the edges that meet the face. If we do this, we see that every combination of three minimal spanning sets occurs in exactly two faces. Now consider what happens if you rotate around face: say the minimal spanning sets of its edges are $[a]$, $[b]$, $[c]$. If you rotate around, the a-edge goes to the b-edge, the b-edge goes to the c-edge, and the c-edge goes to the a-edge. Looking beyond the face, you can track that a $[d]$-edge goes to another $[d]$-edge and a $[e]$-edge goes to another $[e]$-edge. (It suffices to check just one other edge actually! Why?) Thus, the adjoint of this symmetry is the 3-cycle $(a\,b\,c)$. Since we can find all triples on a face, and we can square each 3-cycle, we obtain all 20 3-cycles this way.

Now we remember that the 3-cycles in $\mathcal{S}_5$ generate $\mathcal{A}_5$ (from the quiz, or if you didn't do that, from a little later in the textbook, in section 8.5). Thus, the image of our homomorphism, which is a subgroup of $\mathcal{S}_5$ containing all the 3-cycles, must contain $\mathcal{A}_5$. But the image has order 60, and so does $\mathcal{A}_5$, so the image must be equal to $\mathcal{A}_5$. QED.

For the dodecahedron, we claim that its symmetry group is isomorphic to that of the icosahedron. This is because they are "dual solids." That is, if you take the center of each face of an icosahedron, and connect those, you get a dodecahedron. Any symmetry of an icosahedron then yields a symmetry of a dodecahedron, and conversely.

H. Representations of a group. A representation of a group G on a vector space V is a group homomorphism $G \longrightarrow GL(V)$ to the set of bijective linear transformations $V \longrightarrow V$.

(1) Explain why a group representation is equivalent to an action of G on V where for each $g \in G$, $\mathrm{ad}(g)$ is a linear transformation.
(2) Show that a group representation is a faithful action if and only if $G \longrightarrow GL(V)$ is injective.
(3) Give an example of a representation of D_n on $\mathbb{R}^2$.

Solution.

(1)
(2) We have shown this in the previous worksheet in a more general setting.
(3) We have essentially already seen this example in Problem Set 8:

$$r \mapsto \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} \text{ and send the reflection about the } y\text{-axis to } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$