Math 412. Normal subgroups

DEFINITION: A subgroup N of a group G is **normal** if for all $g \in G$, the left and right N-cosets gN and Ng are the *same* subsets of G.

PROPOSITION: For any subgroup H of a group G, we have $|H| = |gH| = |Hg|$ for all $g \in G$.

THEOREM 8.11: A subgroup N of a group G is **normal** if and only if for all $g \in G$,

$$gNg^{-1} \subseteq N.$$

Here, the set $gNg^{-1} := \{gng^{-1} \mid n \in N\}$.

NOTATION: If $H \subseteq G$ is *any subgroup*, then G/H denotes the set of left cosets of H in G. It elements are *sets* denoted gH where $g \in G$. Recall that the cardinality of G/H is called the **index** of H in G. We sometimes write $H \trianglelefteq G$ to indicate that H is a normal subgroup of G.

Part 1: The essentials.

A. WARMUP

(1) Let $2\mathbb{Z}$ be the subgroup of even integers in $\mathbb{Z}$. Fix any $n \in \mathbb{Z}$. Describe the left coset $n + 2\mathbb{Z}$ (your answer will depend on the parity of n). Describe the right coset $2\mathbb{Z} + n$. Is $2\mathbb{Z}$ a **normal** subgroup of $\mathbb{Z}$? What is its index? Describe the partition of $\mathbb{Z}$ into left (respectively, right) $2\mathbb{Z}$-cosets.

(2) Let $K = \langle (2\,3) \rangle \subset \mathcal{S}_3$. Find the right coset $K(1\ 2)$. Find the left coset $(1\ 2)K$. Is K a normal subgroup of $\mathcal{S}_3$?

(3) Let $N = \langle (1\,2\,3) \rangle \subset S_3$. Find the right coset $N(1\ 2)$. Find the left coset $(1\ 2)N$. Describe the partition of $\mathcal{S}_3$ into left N-cosets. Compare to the partition into right N-cosets. Is $gN = Ng$ for *all* $g \in \mathcal{S}_3$? Is N a normal subgroup of $\mathcal{S}_3$?

Solution.

(1) The left coset $n + 2\mathbb{Z}$ is the set of odd numbers if n is odd and the set of even numbers if n is even. Ditto for the right coset $2\mathbb{Z} + n$. The subgroup $2\mathbb{Z}$ is normal because $n + 2\mathbb{Z} = 2\mathbb{Z} + n$ for all n. Index is two.

(2) Right coset $K(1\ 2)$ is $\{(12), (132)\}$. Left coset $(1\ 2)K$ is $\{(12), (123)\}$. Since $K(1\ 2) \neq (1\ 2)K$, K is not normal.

(3) The right coset is $N(1\ 2) = \{(12), (23), (13)\}$. The left coset is $(12)N = \{(12), (23), (13)\}$. We see that $(12)N = N(12)$. To find the partition into left cosets, we compute all left cosets. The only other coset is $eN = \{e, (123), (132)\}$. We know this because all left cosets have the same cardinality, so since the coset $(12)N$ has three element, so do all the others. But the cosets are disjoint! So there can be only one more coset, and it is eN. The partition into right cosets is the same! We know that $eN = Ne$, so one left/right coset is the set $\{e, (123), (132)\}$. The other left/right coset is the complement: $N(12) = (12)N = \{(12), (23), (13)\}$, which we can also write $N(13) = (13)N = N(23) = (33)N$. So yes, $gN = Ng$ for all g. So N is normal.

B. INTRODUCTORY PROOFS

(1) Prove that if G is abelian, then *every subgroup* K is normal.

(2) Prove that for any subgroup K, and any $g \in K$, we have $gK = Kg$.

(3) Find an example of subgroup H of G which is normal but *does not satisfy* $hg = gh$ for all $h \in H$ and all $g \in G$.

Solution.

(1) Take arbitrary $g \in G$. If G is abelian, we know $gK = \{gk \mid k \in K\} = \{kg|k \in K\} = Kg$. So K is normal.

(2) If $g \in K$, then $gk \in K$ for all $k \in K$, so $gK \subseteq K$. But also every $k \in K$ can be written $k = g(g^{-1}k) \in gK$, since $g^{-1}k \in K$ implies $g^{-1}k \in K$. So $K = gK$. A similar argument shows that $K = Kg$, so $Kg = gK$ for all $g \in K$.

(3) We saw an example already in A3.

C. Let G be the group $(S_5, \circ)$. Use Theorem 8.11 to determine which of the following are **normal** subgroups.

(1) The trivial subgroup e.
(2) The whole group S_5.
(3) The subgroup A_5 of *even* permutations.
(4) The subgroup H generated by $(1\,2\,3\,)$.
(5) The subgroup S_4 of permutations that fix 5.
(6) Use Lagrange's Theorem to compute the index of each subgroup in (1)–(5).

Solution.

(1) The trivial subgroup is normal.

(2) The whole group is a normal subgroup.

(3) The group A_n is normal, because given any $g \in S_n$, and any $h \in A_n$, we need to check that $ghg^{-1} \in A_n$. But h is a composition of an even number of transpositions, say $2k$ transpositions, and if g is a composition of d transpositions, then so is g^{-1}. So ghg^{-1} is a composition of $d + 2k + d = 2(d + k)$ transpositions, and hence is in A_n.

(4) The group $H = \{e, (123), (132)\}$ is not normal if $n \geqslant 4$: if we conjugate by (14), we get $(14)(123)(14) = (423)$ which is not in H.

(5) The subgroup S_{n-1} of permutations that fix n is not normal: the element (12) is in S_{n-1} but its conjugate by $(1n)$ is $(2n)$ which does not fix n.

(6) Lagrange's theorem tells us that $[S_n : A_n] = 2$ (we have even and odd permutations for the cosets) $[S_n : H] = n!/3$, and $[S_n : S_{n-1}] = n!/(n-1)! = n$.

D. Let $G \xrightarrow{\phi} H$ be a group homomorphism.

(1) Prove that the kernel of ϕ is a *normal subgroup* of G.
(2) Prove that the group $SL_n(\mathbb{Q})$ of determinant one matrices with entries in $\mathbb{Q}$ is a normal subgroup of $GL_n(\mathbb{Q})$.

Solution.

(1) Take $k \in \ker(\phi)$ and $g \in G$ arbitrary. We need to show that $gkg^{-1} \in \ker(\phi)$. Apply ϕ to get $\phi(g)\phi(k)\phi(g^{-1})$. Since $k \in \ker(\phi)$, this is $\phi(g)e\phi(g^{-1}) = \phi(gg^{-1}) = \phi(e_G) = e_H$. So $gkg^{-1} \in \ker(\phi)$.

(2) We only need to note that this is the kernel of the group homomorphism $\det$.

Part 2: Foreshadowing.

E. CONJUGATION. Let G be a group, and $g, h \in G$. We call the element ghg^{-1} is the **conjugate** of h by g. Let $c_g : G \to G$ be the function given by the rule $c_g(h) = ghg^{-1}$. We call this function **conjugation by** g.

(1) Show that, if $h_1, h_2 \in G$, then $c_g(h_1)c_g(h_2) = c_g(h_1h_2)$. Thus, c_g is a *group homomorphism* from G to itself.
(2) Show that $c_{g^{-1}} \circ c_g = c_g \circ c_{g^{-1}}$ is the identity on G. Conclude that c_g is an **automorphism** of G: a group isomorphism from G to itself.
(3) Let $G = \mathcal{S}_n$, and $h = (\,a\,b\,)$ be a 2-cycle. What is $c_g(h)$? If instead $h = (\,a_1\,a_2\,\cdots\,a_t\,)$ is a t-cycle, what do you think $c_g(h)$ is? If you know how to write h as a product of disjoint cycles, how can you write $c_g(h)$ as a product of disjoint cycles?
(4) Interpret the last problem as follows: $c_g(h)$ is "the same permutation as h up to relabeling the elements $\{1, \dots, n\}$ by g."
(5) Now let $G = \mathrm{GL}_n(\mathbb{R})$. If $g = S$ and $h = A$ are matrices in G, explain what is the geometric meaning of $c_g(h)$. Compare with the previous part.

Solution.

(1) $c_g(h_1)c_g(h_2) = (gh_1g^{-1})(gh_2g^{-1}) = gh_1(g^{-1}g)h_1g^{-1} = gh_1h_2g^{-1} = c_g(h_1h_2)$.
(2) $c_{g^{-1}} \circ c_g)(h) = c_{g^{-1}}(c_g(h)) = c_{g^{-1}}(ghg^{-1}) = g^{-1}(ghg^{-1})g = (g^{-1}g)h(g^{-1}g) = h$.

So $c_{g^{-1}} \circ c_g$ is the identity map. Applying the same argument with g^{-1} in place of g shows the other composition is the identity.

(3) We have that $g(\,a\,b\,)g^{-1} = (\,g(a)\,g(b)\,)$. Indeed, we can check that $g(a)$ goes to $g(b)$ by this permutation and vice versa. Moreover, let $x \in \{1, \dots, n\}$ be such that $x \neq g(a)$ and $x \neq g(b)$, and let $y \in \{1, \dots, n\}$ be such that $g(y) = x$. Note that, since g is a bijection, such y exists, is unique, and $y \neq a, b$. Hence $g(a, b)g^{-1}(x) = g(a, b)y = g(y) = x$, so every x different from $g(a)$ and $g(b)$ is fixed by $g(\,a\,b\,)g^{-1}$. This concludes the proof of the equality $g(\,a\,b\,)g^{-1} = (\,g(a)\,g(b)\,)$.

This generalizes to $g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1} = (\,g(a_1)\,g(a_2)\,\cdots\,g(a_t)\,)$. To check the equality, we deal with different cases. If $i = g(a_j)$ for some $j < t$, then

$$(g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1})(i) = (g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1})g(a_j) = (g(\,a_1\,a_2\,\cdots\,a_t\,))(a_j) = g(a_{j+1}).$$

If $i = g(a_t)$, then

$$(g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1})(i) = (g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1})g(a_t) = (g(\,a_1\,a_2\,\cdots\,a_t\,))(a_t) = g(a_1).$$

If $i \neq g(a_j)$ for any $j = 1, \dots, t$, then $g^{-1}(i) \neq a_j$ for any $j = 1, \dots, t$, so

$$(g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1})(i) = (g(\,a_1\,a_2\,\cdots\,a_t\,)g^{-1}(i) = g(g^{-1}(i)) = i.$$

Thus, this permutation agrees with the t-cycle $(\,g(a_1)\,g(a_2)\,\cdots\,g(a_t)\,)$.

Finally, since conjugation respects products, given a product of cycles, we can use the last rule to write the conjugate as a product of cycles (of the same lengths).

(4) OK!
(5) We called conjugtion *similarity* in linear algebra. It corresponds to change of basis. The matrix $c_g(h)$ gives the same linear transformation as h in the basis corresponding to the columns of g.

F. THE PROOF OF THEOREM 8.11. Let G be a group and H some subgroup. Prove that the following are equivalent by showing (1) implies (2) implies (3) implies (4) implies (5) implies (1).

(1) H is normal.
(2) $gHg^{-1} \subseteq H$ for *all* $g \in G$.

(3) $g^{-1}Hg \subseteq H$ for all $g \in G$.
(4) $g^{-1}Hg = H$ for all $g \in G$.
(5) $gHg^{-1} = H$ for all $g \in G$.

Solution. $(1) \Rightarrow (2)$: If H is normal, then for every $h \in H$ and every $g \in G$, $gh \in Hg$, so there exists $h' \in H$ such that $gh = h'g$, and $ghg^{-1} = h' \in H$.
$(2) \Rightarrow (3)$: Since the statement holds for any $g \in G$, it holds for g^{-1}.
$(2) \Leftrightarrow (3) \Leftrightarrow (4)$: Our proof that $(2) \Rightarrow (3)$ also gives $(3) \Rightarrow (2)$, which together make (4).
$(4) \Leftrightarrow (5)$: Replace g by g^{-1}, which we can do since the statements are written over all g.
$(5) \Leftrightarrow (1)$: given any $g \in G$ and any $h \in H$, $ghg^{-1} = h' \in H$, so $gh = h'g^{-1} \in Hg$; this shows that $gH \subseteq Hg$ for all $g \in G$. Similarly we can show that $Hg \subseteq gH$ for all $g \in G$, and thus H is normal.

G. Suppose that H is an index two subgroup of G.

(1) Prove that the partition of G up into left cosets is the disjoint union of H and $G \setminus H$.
(2) Prove that the partition of G up into right cosets is the disjoint union of H and $G \setminus H$.
(3) Prove that for every $g \in G$, $gH = Hg$.
(4) Prove the THEOREM: *Every subgroup of index two in G is normal.*

Solution.

(1) If the index is two, there are only two left cosets. Since one is $H = eH$, the other is its complement $G \setminus H$ (as they are disjoint).
(2) Ditto.
(3) Consider any $g \in G$; either $g \in H$ or $g \notin H$. If $g \in H$, then $gH = H = Hg$, since h is closed under multiplication and gH. If $g \notin H$, $gH \neq H$, so $gH = G \setminus H$. Similarly, $Hg = G \setminus H$, so $gH = Hg$.
(4) We just saw that $gH = Hg$ for all $g \in G$. By definition, H is normal.

Part 3: Bonus.

H. OPERATIONS ON COSETS: Let $(G, \circ)$ be a group and let $N \subseteq G$ be a *normal* subgroup.

(1) Explain why $Ng = gN$. Explain why both cosets contain g.
(2) Take arbitrary $ng \in Ng$. Prove that there exists $n' \in N$ such that $ng = gn'$.
(3) Take any $x \in g_1N$ and any $y \in g_2N$. Prove that $xy \in g_1g_2N$.
(4) Define a binary operation $\star$ on the set G/N of left N-cosets as follows:

$$G/N \times G/N \to G/N \qquad g_1N \star g_2N = (g_1 \circ g_2)N.$$

Think through the meaning: the elements of G/N are *sets* and the operation $\star$ combines two of these sets into a third set: how? Explain why the binary operation $\star$ is **well-defined.** Where are you using normality of N?

(5) Prove that the operation $\star$ in (4) is associative.
(6) Prove that N is an identity for the operation $\star$ in (4).
(7) Prove that every coset $gN \in G/N$ has an inverse under the operation $\star$ in (4).
(8) Conclude that $(G/N, \star)$ is a group.

Solution. This is 8.3 in the book. More next time. Please read 8.3 carefully, and then try this exercise on your own!