Math 412 Homework 5

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Thursday, February 22nd, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. Consider the ring $M_2(\mathbb{R})$.

   (a) Take any nonzero 2×2 matrix A. Show that by multiplying A on the left by matrices of the form
   $$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
   we can do any elementary row operation to A.

   (b) State a way of interpreting column operations using matrix multiplication.

   (c) Prove that the only ideals in $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.

2. Let $S_{\text{odd}} \subset \mathbb{Q}$ be the subset of rational numbers with odd denominators (when expressed in lowest terms).

   (a) Show that S_{odd} is a subring of $\mathbb{Q}$.

   (b) Let $I \subseteq S_{\text{odd}}$ be the subset of rational numbers with even numerator (when expressed in lowest terms). Prove that I is an ideal of S_{odd}.

   (c) Define a ring homomorphism $\phi\colon S_{\text{odd}} \to \mathbb{Z}_2$. What is the kernel?

3. Let F be a field, and let $f \in \mathbb{F}[x]$. Two polynomials $g, h \in \mathbb{F}[x]$ are **congruent modulo** f if $f|(g-h)$. We write $g \equiv h \mod f$. The set of all polynomials congruent to g modulo f is written $[g]_f$. For this problem, we fix a polynomial $f \in \mathbb{F}[x]$ of degree $d > 0$.

   (a) Prove that every congruence class $[g]_f$ contains a *unique* polynomial in the set $S = \{h(x) \in F[x] \ : \ h(x) = 0 \text{ or } \deg h(x) < d\}$.

   (b) How many distinct congruence classes are there for $\mathbb{Z}_2[x]$ modulo $x^3 + x$?

   (c) How many distinct congruence classes are there for $\mathbb{Z}_3[x]$ modulo $x^2 + x$?

4. What are the subrings of $\mathbb{Q}$? We have $\mathbb{Z}$, $\mathbb{Q}$, and according to the previous problem, the subring S of rational numbers with odd denominators.

   (a) Prove that the subset
   $$\mathbb{Z}[1/2] := \left\{ \frac{a}{2^m} \middle| a, m \in \mathbb{Z} \text{ and } m \geq 0 \right\} \subset \mathbb{Q}$$
   is a subring.

   (b) Let $R \subset \mathbb{Q}$ be a subring. Define:
   $$\Pi(R) := \left\{ p \text{ a positive prime} \middle| \frac{1}{p} \in R \right\} \subset \mathcal{P} \text{ (the set of positive primes)}.$$
   Compute $\Pi(\mathbb{Z})$, $\Pi(\mathbb{Q})$, $\Pi(\mathbb{Z}[1/2])$, $\Pi(S_{\text{odd}})$ (no proof needed).

(c) (Tricky!) Given set of the positive prime numbers $\Gamma \subset \mathcal{P}$, define a subring denoted $\mathbb{Z}[1/\Gamma]$ such that $\Pi(\mathbb{Z}[1/\Gamma]) = \Gamma$.

(d) (This is also hard!) Prove that two subrings $R_1, R_2 \subset \mathbb{Q}$ are equal $\iff \Pi(R_1) = \Pi(R_2)$. Conclude, that the subrings of $\mathbb{Q}$ are in bijection with the subsets of the positive prime numbers!