Homework 9

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Thursday, April 4th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. (a) Prove Fermat's Little Theorem: if p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \mod p$.
   (b) If G is a group of prime order p, then G is cyclic.
   (c) A nontrivial group G has no nontrivial proper subgroups if and only if G is finite and of order p where p is prime.

2. For each of the following parts, K is a subgroup of the group G. Write down every element of every (distinct) right coset AND every distinct left coset. You do not need to prove that you have found every coset.
   (a) $K = \{r_{0^\circ}, r_{90^\circ}, r_{180^\circ}, r_{270^\circ}\}$, G is D_4, the set of symmetries of the square.
   For the sake of notation, denote the reflections of the square as s_v, s_h, s_{NW} and s_{NE}.
   (b) $K = \{e, (12)\}$, $G = S_3$
   (c) $K = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$, $G =_2 (\mathbb{Z}_2)$. For your convenience, the elements of $_2(\mathbb{Z}_2)$ are given below.
   You may use the given letters to refer to them:
   $$_2(\mathbb{Z}_2) = \left\{ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, c = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} f = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$
   (d) $K = \langle 5 \rangle$, $G = \mathbb{Z}_{12}^\times$

3. Any group G acts on itself by conjugation: $g \cdot h = ghg^{-1}$. The orbits of this action are called **conjugacy classes**.
   1. Show $h \in Z(G)$ if and only h is a fixed point of the conjugation action.
   2. Show a subgroup H of G is normal if and only if it is a disjoint union of conjugacy classes.
   3. Describe the partition of $\mathcal{S}_5$ into its conjugacy classes.
   4. Show that the only nontrivial normal subgroup of $\mathcal{S}_5$ is $\mathcal{A}_5$.[1]

4. Let p be a prime, and G be a finite group with $p \,\big|\, |G|$. Consider the set
   $$X = \{(g_1, \ldots, g_p) \in \underbrace{G \times \cdots \times G}_{p-\text{times}} \mid g_1 g_2 \cdots g_p = e\}.$$
   The group $\mathbb{Z}_p$ acts on X by rotating elements: $[i]_p \cdot (g_1, \ldots, g_p) = (g_{1+i}, \ldots, g_p, g_1, \ldots, g_i)$.
   1. Show that X has $|G|^{p-1}$ elements, so $p \,\big|\, |X|$.
   2. Show that the orbits of the action of $\mathbb{Z}_p$ on X either have 1 or p elements, and the orbits of order 1 are either $(e, e, \ldots, e)$ or of the form $(g, g, \ldots, g)$ with $|g| = p$.
   3. Show that G contains an element of order p.

[1] Hint: By (b), a normal subgroup is a union of conjugacy classes, one of which is the identity. Use the sizes of these conjugacy classes from (c), plus Lagrange's Theorem, to narrow down the list, and finally show that on your shortlist, the only collection closed under products is $\mathcal{A}_5$.