Name:

Math 412 Fall 2021 Final Exam–Solutions

Time: 120 mins.

(a) Answer each question in the space provided. If you require more space, you may use the blank page at the end of this exam, but you must clearly indicate in the provided answer space that you have done so.

(b) You may use any results proved in class, on the homework, or in the textbook, except for the specific question being asked. You should clearly state any facts you are using.

(c) Remember to show all your work.

(d) No calculators, notes, or other outside assistance allowed.

Best of luck!

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Problem 1 (3 points each). Give examples as requested. No explanation needed:

(a) An abelian group which is not cyclic.

(b) A subgroup of a group which is not normal.

(c) A non-zero proper subgroup of $Z_5^\times$.

Problem 2 (4 points each)**.** For a given group G and an element $g \in G$, compute the order of g (and give a justification).

(a) $G = \mathbb{Z}_{12}$ and $g = [3]_{12}$,

(b) $G = S_7$ and $g = (123)(345)(67)$,

(c) $G = S_4/N$ where $N = \{e, (12)(34), (13)(24), (14)(23)\}$ and $g = (1234)N$.

Problem 3 (4 points each). Let $R = \mathbb{Z}_3[x]/(x^2 + 2x + 1)$.

(a) Find the number of elements of R.

(b) Find the number of elements of $R^\times$.

(c) Is $R^\times$ a cyclic group? Justify your answer.

Problem 4 (3 points each)**.** For each of the questions below, indicate clearly whether the statement is *true* or *false*, and give a short justification.

(a) There exists a group of order 9 acting on a set X such that such that some $x \in X$ has orbit of cardinality 5.

(b) The map $f\colon \mathbb{Z}_5 \longrightarrow \mathbb{Z}_5$ given by $f(x) = x^3$ is a ring isomorphism.

(c) If an action of a group G on a set X is faithful, then $\mathrm{Stab}(x) = \{e\}$ for every $x \in X$.

(d) The only group homomorphism $\varphi\colon D_3 \to \mathbb{Z}_3$ is the trivial homomorphism $\varphi(x) = [0]_3$ for all $x \in D_3$.

Problem 5 (2 points each). Write the following groups as products of cyclic groups $\mathbb{Z}_n$ (e.g. $\mathbb{Z}_5^\times$ is isomorphic to the cyclic group $\mathbb{Z}_4$). No justification required.

(a) $\mathbb{Z}_{12}^\times$

(b) $\mathbb{Z}_{16}^\times$

(c) $\mathbb{Z}_{36}/4\mathbb{Z}_{36}$

(d) S_4/A_4

(e) $<(12),(34),(56)>$ as a subgroup of S_6

(f) $<\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}>$ as a subgroup of $\mathbb{GL}_2(\mathbb{Z}_8)$

(g) $Z(D_3)$

(h) $D_4/Z(D_4)$.

Problem 6 (6 points each)**.** Let R be a ring. Consider an operation $\star$ on the set $R \times R$ defined as follows:

$$(x_0, x_1) \star (y_0, y_1) := (x_0 + y_0, x_1 + y_1 + x_0 y_0).$$

where $x_0, x_1, y_0, y_1 \in R$.

(a) Show that $a \star (b \star c) = (a \star b) \star c$ for $a, b, c \in R \times R$.

(b) Prove that $(R \times R, \star)$ is a group.

Problem 7 (12 points)**.** Let p be an odd prime number and let $a \in \mathbb{Z}_p$ be nonzero. Show that $a = x^2$ for some $x \in \mathbb{Z}_p$ if and only if $a^{\frac{p-1}{2}} = 1$.

Problem 8 (6 points each). Let H be a subgroup of a group G.

(a) The *normalizer* $N_G(H)$ is given by

$$N_G(H) = \{g \in G \mid gH = Hg\}.$$

Show that $N_G(H)$ is a subgroup of G and H is a normal subgroup of $N_G(H)$.

(b) Two subgroups H_1 and H_2 of G are *conjugate* if there is $g \in G$ such that $H_2 = gH_1g^{-1}$. Show that if G is finite, then the number of subgroups of G conjugate to H is equal to $[G : N_G(H)]$.