A

(1) inclusion mapping

$\varphi: \mathbb{Z} \to \mathbb{Q}$

$z \mapsto \frac{z}{1}$ 是 hom 但不 iso

(比如对于 $\frac{3}{2}$，没有 $\varphi(z) = \frac{3}{2}$)

(2) doubling mapping

$\varphi: \mathbb{Z} \to \mathbb{Z}$

$z \mapsto 2z$ 不是 hom，$1 \mapsto 2$ 不是 $1_{\mathbb{Z}}$.

(3) residue mapping

$\varphi: \mathbb{Z} \to \mathbb{Z}_n$

$z \mapsto [z]_n$ 是 hom，因为 modular arithmetic 且 $[0]$, $[1]$ 符合.

(但显然不是 iso，完全是两个不同的 ring)

(4) "evaluation at 0" mapping

$\varphi: \mathbb{R}[x] \mapsto \mathbb{R}$

$f(x) \mapsto f(0)$

$\mathbb{R}[x]$: $\begin{cases} 0_R: f(x) = 0 + 0x + 0x^2 + \dots = 0 \\ 1_R: f(x) = 1 + 0x + 0x^2 + \dots = 1 \end{cases}$

这里把 φ 写作 eval

eval $(f(x) + g(x))$

即取 $h(x) = (f(x) + g(x))$ 的 $x=0$ 值，

即 constant，

那么一定等于 $f(x)$ 的 const + $g(x)$ 的 const。

$= \text{eval}(f(x)) + \text{eval}(g(x))$

eval $(f(x) \cdot g(x))$ 同理

$= a_0 \cdot b_0 = \text{eval}(f(x)) \cdot \text{eval}(g(x))$,

(5) differentiation map

$\varphi: \mathbb{R}[x] \mapsto \mathbb{R}[x]$ 不是 hom，因为 $1 \mapsto 0$].

(6) $\varphi: \mathbb{R} \to M_2(\mathbb{R})$

$\lambda \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ 是 hom.

$\varphi(a+b) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix} = \varphi(a) + \varphi(b)$

$\varphi(a)\varphi(b) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} = \varphi(ab)$

(7) $\varphi: M_2(\mathbb{Z}) \to \mathbb{R}$

$A \mapsto \det(A)$ 不是 hom. $\det(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \neq \det(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) + \det(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$

B.

(1) ① hom preserves 0_R

即: $\varphi(0_S) = 0_T$.

Pf (对于 $\varphi: S \to T$)

$0_S + 0_S = 0_S$

$\Rightarrow \varphi(0_S + 0_S) = \varphi(0_S)$

又 $\varphi(0_S + 0_S) = \varphi(0_S) + \varphi(0_S)$

$\Rightarrow \varphi(0_S) = \varphi(0_S) + \varphi(0_S)$

令 $b = -\varphi(0_S)$, then $\varphi(0_S) + b = 0_T$.

$\xRightarrow{\text{两边}+b} 0_T = \varphi(0_S) + 0_T \Rightarrow \varphi(0_S) = 0_T$

(2) hom preserves $+^{-1}$

即: $-\varphi(x) = \varphi(-x)$

Pf. $x \in S \Rightarrow x + (-x) = 0_S$.

$\therefore \varphi(x + (-x)) = \varphi(0_S) = 0_T$

又 $\varphi(x + (-x)) = \varphi(x) + \varphi(-x) \Rightarrow \square$.

(3) hom preserves unit.

即: $\varphi: S \to T$

if $u \in S$ is unit

$\Rightarrow \varphi(u) \in T$ is unit

Pf. Let u be unit

$\Rightarrow \exists u^{-1} \in S$ s.t. $uu^{-1} = u^{-1}u = 1$

$\Rightarrow \varphi(uu^{-1}) = \varphi(1_S) = 1_T$

又 $\varphi(uu^{-1}) = \varphi(u)\varphi(u^{-1})$

$\Rightarrow \varphi(u^{-1})\varphi(u) = 1_T$，即

$\varphi(u)$, $\varphi(u^{-1})$ 为 unit.

C (1) hom 的 kernel 一定非空

(因为 $\varphi(0_S) = 0_R$，至少 0_S 这个元素在 $\ker\varphi$ 中.

(4) 任意 hom $\varphi: R\times S \to R$, $(r,s)\mapsto r$ 的 kernel

$\ker\varphi = \{(0_R, s) \mid s\in S\}$

D) $\boxed{\text{(Pf of Thm⑧)}}$

Pf. $\varphi: R\to S$

① 设 φ injective

取 $\forall x\in\ker\varphi$

$\Rightarrow \varphi(x) = \varphi(0) = 0$

By injectivity, $\ker\varphi = \{0\}$

② 设 $\ker\varphi = \{0\}$

Suppose $\exists x,y\in R$ 使 $\varphi(x)=\varphi(y)$

$\Rightarrow \varphi(x) + (-\varphi(y)) = 0_S$

$\Rightarrow \varphi(x) + \varphi(-y) = 0_S$ (by preserve $+^{-1}$)

$\Rightarrow \varphi(x+(-y)) = 0_S \Rightarrow x+(-y)\in\ker\varphi$

$\Rightarrow x+(-y)=0 \Rightarrow x=y \Rightarrow$ injective

$\boxed{\text{Extra Thm: isomorphism 的逆也是 isomorphism.}}$

显然，不证。

(E) 简单而言，isomorphism preserves 基本 everything

(1) $\boxed{\text{units}}$

(2) $\boxed{\text{zero-divisors}}$

(3) 是否为 field, domain.

...

(而 hom φ 只要 surjective, 也 preserve 所有 x^{-1} 和 $\boxed{\text{units}}$)

(F) 这里的重点是：两个 ring 之间可以有不止一个 isomorphism

尽管 isomorphism implies 代数结构基本相同 ring 也可以有不同的自 isomorphism

H. $\boxed{\text{任何都有一个 unique 的从 } \mathbb{Z} \text{ 到它的 hom. } \varphi:\mathbb{Z}\to R.}$

这个 hom 叫做 canonical ring homomorphism.

如果有 hom φ

$\Rightarrow$

因为 $\varphi(1) = 1_R$, $\varphi(0) = 0_R$

$$\Rightarrow \begin{cases} \varphi(n) = \varphi(\underbrace{1+1+\dots+1}_{n \text{ times}}) = n\varphi(1) = n\cdot 1_R & \text{for } n\ge 1 \\ \varphi(0) = 0_R \\ \varphi(n) = \varphi(\underbrace{-1-1-\dots-1}_{n \text{ times}}) = -n\varphi(1) = -n\cdot 1_R & \text{for } n\le -1 \end{cases}$$

因而 φ 只可能存在有这一个情况。现在我们证明这个情况一定存在，即 φ 一定为一个 hom.

$\varphi(n+m) = (n+m)\cdot 1_R = n\cdot 1_R + m\cdot 1_R = \varphi(n)+\varphi(m)$

$\varphi(nm) = (n\cdot m)\cdot 1_R = \varphi(n)\varphi(m)$

$\square$