Math 412. The Division Algorithm

Let $\mathbb{Z}$ denote the set of all integers. There are at least a couple of related things that we mean by *divides / division* in $\mathbb{Z}$.

Division Algorithm Theorem: Let $n, d \in \mathbb{Z}$ with $d > 0$. There exists *unique* $q, r \in \mathbb{Z}$ such that

$$n = qd + r \qquad \text{and} \qquad 0 \leqslant r < d.$$

DEFINITION: Let $a, b \in \mathbb{Z}$. We say a *divides* b if there exists $q \in \mathbb{Z}$ such that $b = aq$. Equivalently, in this case, we say a is a *divisor* or *factor* of b, or that b *is divisible by* a, and write $a|b$.

Part 1: Getting acquainted.

A. DIVISION ALGORITHM WARM UP:

(1) Discuss with your team: *You have known the division algorithm since Grade 3. Explain. What did the uniqueness part mean in third grade? What words are used in third grade for q and r?*
(2) Drop the phrase "$0 \leqslant r < d$" from the statement of the theorem. Is it still true? Prove it or find a counterexample.
(3) Discuss with your team: Describe the main scaffolding (outline) of the proof of the Division Algorithm. What are the two main things to show?

ANSWER:

(1) q is the quotient, r is the remainder.
(2) No! Here's a counterexample to the uniqueness part. $85 = 21 \times 4 + 1$ and $85 = 20 \times 4 + 5$.
(3) Existence and Uniqueness are the two main pieces. We must show there exists $q, r \in \mathbb{Z}$ such that $n = qd + r$ with $0 \leqslant r < d$. Then separately, we must show that if there is another such expression $n = q'd + r'$, the $r = r'$ and $q = q'$.

B. DIVISOR WARM UP. True or False. Justify.

(1) The integer -4 is a factor of both 24 and 100.
(2) The integer 1 has exactly two divisors.
(3) The integer 0 has exactly one divisor.
(4) The integer 3 is a divisor of 4.
(5) If $n \in \mathbb{Z}$, then 5 divides $5n$.

(6) If $n \in \mathbb{R}$, then 5 divides $5n$.
(7) If a, b, c are integers, $a|b$, and $b|c$, then $a|c$.

ANSWER:

(1) True.
(2) True (± 1).
(3) False (every integer divides 0 since $d \times 0 = 0$ for every d).
(4) False.
(5) True.
(6) False.
(7) True.

C. THE CONNECTION BETWEEN "DIVIDES" AND "DIVISION ALGORITHM:" Let n, d be positive integers, and (using the division algorithm) write $n = qd + r$ where $0 \leqslant r < d$. Then d divides n if and only if $r = 0$.

ANSWER: We have to prove 'the "if" statement and the "only if" statement.

If $r = 0$, then $n = qd + 0 = qd$ for some integer q, so $d|n$ by definition of divides.

If $d|n$, then $n = md$ for some integer m. Taking $q = m$ and $r = 0$, this gives an expression for $n = qd+r$ with $0 \leqslant r < d$ as in the statement of the division algorithm. Thus, this must be the unique q, r satisfying the division algorithm, so $r = 0$.

Part 2: A deeper dive.

D. DIVISION ALGORITHM PROOF: EXISTENCE. Let n, d be integers with d positive. Define the set $\mathcal{S}$ as follows:

$$\mathcal{S} := \{n - dx \mid x \in \mathbb{Z} \text{ and } n - dx \geq 0\}.$$

(1) In the special case $n = 17, d = 5$, write out some explicit elements of $\mathcal{S}$. Ditto for $n = -33$ and $d = 8$.
(2) Prove that $\mathcal{S}$ is non-empty.[1]
(3) Explain why $\mathcal{S}$ has a **smallest element.**[2]
(4) Ler r be the smallest element of $\mathcal{S}$. Prove that $r < d$.
(5) Prove the **existence** part of the Division algorithm.

ANSWER: Read the textbook. proof of Theorem 1.1, pages 5-6, steps 1-3.

[1]Often, the easiest way to show a set is non-empty is to exhibit an element in it. Also, you can consider the case where $n \geq 0$ and $n < 0$ separately.]

[2]This follows from the obvious but fancy-sounding **Well-Ordering Principle**: every non-empty subset of integers which is bounded below has a minimal element. Like most axioms, this is formalized "common sense."

E. DIVISION ALGORITHM PROOF: UNIQUENESS. Let n, d be integers with d positive. Suppose that $n = qd+r$ and $n = q'd+r'$, where $q, r, q', r' \in \mathbb{Z}$ and $0 \leqslant r, r' < d$.

(1) Show that $d|(r - r')$.
(2) Show that $|r - r'| < d$.
(3) Show that $|d(q - q')| < d$. [Hint: Substitute.]
(4) Show that $q = q'$.
(5) Show that $r = r'$.
(6) Explain how Problem D above and your steps here complete the proof of the Division Algorithm.

ANSWER: Read the textbook. proof of Theorem 1.1, page 6, steps 4.

F. Complete the following takeaway sentences.

(1) To prove a set is nonempty ...
(2) To prove a solution is unique ...

ANSWER:

(1) To prove a set is nonempty, it is enough to find one element that satisfies the defining properties of the set.
(2) To prove a solution is unique, one can first assume there are two solutions and then prove that the two solutions are equal to each other.