Homework 6

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Thursday, October 12th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. Recall: an ideal $P \neq R$ in a commutative ring R is prime if $ab \in P$ implies $a \in P$ or $b \in P$.

   a) Prove that P is prime if and only if R/P is a domain.

   b) Use the first isomorphism theorem to show that the ideals (x) and $(2, x)$ in $\mathbb{Z}[x]$ are prime ideals.[1][2]

   c) Show that the ideal $(4, x)$ in $\mathbb{Z}[x]$ is not prime.

   d) Show that the ideal $(2, \sqrt{10})$ in $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$ is prime.

   e) Is the ideal (2) in $\mathbb{Z}[i]$ a prime ideal?

2. We say that a proper ideal I in a ring R is **maximal** if whenever $I \subsetneq J$ for some ideal J, we have $J = R$. For the next problems, assume R is a commutative ring and I is an ideal of R.

   (a) Prove that if I is a maximal ideal and $a \notin I$ then $a + I$ is a unit in R/I.

   (b) Prove that I is a maximal ideal if and only if R/I is a field.[3]

   (c) Use the First Isomorphism Theorem to show that the non-principal ideal $(2, x)$ in $\mathbb{Z}[x]$ is a maximal ideal.[4]

   (d) Show that the ideal $(4, x)$ in $\mathbb{Z}[x]$ is not maximal.

   (e) Show that the ideal $(2, \sqrt{10})$ in $\mathbb{Z}[\sqrt{10}] := \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$ is maximal.

   (f) Show that $I := \{a + bi \ : \ 3 \mid a \text{ and } 3 \mid b\}$ is a maximal ideal in $\mathbb{Z}[i]$.[5]

3. Polynomial rings in *many* variables. Let $R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ is a polynomial ring in variables $x_1, x_2, \ldots, x_n$; that is it contains all polynomials in finite terms that involve these variables.

   (a) Let $f_1, f_2, \ldots, f_k$ be polynomials in R_n. Prove that the set

   $$\langle f_1, f_2, \ldots f_k \rangle := \{g_1 f_1 + g_2 f_2 + \ldots + g_k f_k \mid g_1, g_2, \ldots, g_n \in R_n\}$$

   is an ideal of R_n.

   (b) Consider the ring homomorphism

   $$\varphi : R_4 \to \mathbb{Q}[t_1, t_2], \varphi(x_1) = t_1^3,\ \varphi(x_2) = t_1^2 t_2,\ \varphi(x_3) = t_1 t_2^2,\ \varphi(x_4) = t_2^3. \tag{1}$$

   (c) Explain why the above description fully determines $\varphi(f)$ for each polynomial $f \in R_4$.

   (d) It is given to you that $\ker(\varphi) = \langle f_1, f_2, f_3 \rangle$ for some polynomials $f_1, f_2, f_3 \in R_4$. Find f_1, f_2, f_3. *Hint: part (e) may help.*

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[1] Hint: For the first one, consider the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ "evaluate at zero".

[2] Reminder: $(2, x)$ refers to the ideal generated by 2 and x.

[3] Remark: Perhaps surprisingly, both directions of this "if and only if" are useful.

[4] Hint: Consider the homomorphism $f \colon \mathbb{Z}[x] \to \mathbb{Z}_2$ given by $f(x) \mapsto [f(0)]_2$

[5] Hint: If $r + si \notin I$, then $3 \not| r$ or $3 \not| s$. Show that 3 does not divide $r^2 + s^2 = (r + si)(r - si)$. Then show that and ideal containing $r + si$ and I also contains 1.

(e) Let h_1, h_2, h_3 be the 2 by 2 minors of the matrix $M = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$. Consider the ideal $I = \langle h_1, h_2, h_3 \rangle$. Show that I does not change if one applies elementary row operations to the matrix M.

(f) Take the ideal $J = \langle x_1x_4 - x_2x_3 \rangle$ in R_4. Express J as kernel of some ring homomorphism. You know such a homomorphism exists by WSH 10. You do not need to prove that the proposed homomorphism has J as its kernel. Hopefully you will be able to prove this by the end of the semester.

(g) Prove that the ideal J is not a maximal ideal.