Homework 10

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Sunday, April 14th at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. Let G be a group and let N and K be normal subgroups of G.
   (a) Show that $N \cap K \lhd K$.
   (b) Prove that $NK = \{nk | n \in N, k \in K\}$ is a normal subgroup of G.
   (c) Prove that $N \lhd NK$.
   (d) Prove that the function $f : K \to NK/N$ given by $f(k) = Nk$ is a surjective homomophism with kernel $K \cap N$.
   (e) Prove that $K/(N \cap K) \cong NK/N$.

2. In the following problem, it may help to use the first isomorphism theorem.
   (a) Prove that $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$ (HINT: consider the function $e^{2\pi i z}$).
   (b) Prove that $\mathbb{R}/\mathbb{Z} \cong S^1$.
   (c) Prove that the subset $N = \{e, (12)(34), (13)(24), (14)(23)\} \subset A_4$ is a normal subgroup. What familiar group is the quotient A_4/N isomorphic to?

3. Let p be a prime number. The goal of this problem is to prove that any group G of order p^2 is abelian.
   (a) Let the group G act on itself by the conjugacy action defined in the previous problem set. Prove that $h \in Z(G)$ if and only if the orbit (AKA the conjugacy class) of h has exactly 1 element.
   (b) Use the Class Equation to deduce that p divides $|Z(G)|$. (Thus there are two possibilities $|Z(G)| = p$ or $|G|$, in the latter case G is abelian.)
   (c) Suppose that $|Z(G)| = p$ and let $g \in G$ with $g \notin Z(G)$. Define $\langle Z(G), g\rangle$ to be the group generated by g and every element of $Z(G)$. Show that $\langle Z(G), g\rangle$ is abelian.
   (d) Suppose that $|Z(G)| = p$ and let $g \in G$ with $g \notin Z(G)$. Define $\langle Z(G), g\rangle$ to be the group generated by g and every element of $Z(G)$. Show that $\langle Z(G), g\rangle = G$.
   (e) Deduce (in one line) that G is abelian.
   (f) Give an example of a group with p^3 elements that is not abelian.
   (g) Use the Class Equation to conclude that any p-group[1] H satisfies p divides $|Z(H)|$.

> Theorem 9.7: Fundamental Structure Theorem for Finite Abelian Groups: Let G be a finite abelian group. Then G is isomorphic to a group of the form
>
> $$\mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \mathbb{Z}_{p_3^{a_3}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}$$
>
> where $p_1, p_2, \ldots p_n$ are (not necessarily distinct!) prime numbers. Moreover, the product is unique, up to re-ordering the factors.

[1] A p-group H is a group such that $|H| = p^k$ for some $k > 0$ and p a prime number.

4. (a) Suppose that G is abelian and has order 8. Use the Structure Theorem for Finite Abelian Groups to show that up to isomorphism, G must be isomorphic to one of three possible groups, each a product of cyclic groups of prime power order.

   (b) Determine the number of abelian groups of order 18, up to isomorphism.

   (c) For p prime, how many isomorphism types of abelian groups of order p^4?

   (d) If an abelian group of order 100 has no element of order 4, prove that G contains a Klein 4-group.