2. How many element are in $M_2(\mathbb{Z}_2)$?
How many are units?

① $M_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \middle| a,b,c,d \in \mathbb{Z}_2 \right\}$

Since $\mathbb{Z}_2 = \{[0]_2, [1]_2\}$, $|\mathbb{Z}_2| = 2$

For each of a, b, c, d, there are 2 choices

So by combination, there are $4^2 = 16$ elements

② Units are elements with multiplicative inverse.

Since any combination of $[0]$, $[1]$ can only multiply to get $[0]$ $[1]$,

And modular arithmetic with results $\leq n$ is the same as integer arithmetiz,

the multiplication is just the same rule as in $M_2(\{0,1\})$

So the problem is to find invertible matrices in $M_2(\{0,1\})$ (i.e. $\det \neq 0$)

$\det = 0$ (not unit / not invertible):

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\det \neq 0$ (invertible, unit):

All other 6 matrices.

$\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$

So 6 units.

3. T/F: Let R be a ring, S, T be subrings of R
$\Rightarrow S \cap T$ is a subring of R.

True.

Pf. By worksheet 7, it suffices to show that

① $0_R, 1_R \in S \cap T$

② $S \cap T$ is closed under addition.

③ $S \cap T$ is closed under additive inverses.

④ S is closed under multiplication.

Since S, T are subrings of R,

$1_S = 1_R$, $0_S = 0_R$, $1_T = 1_R$, $0_T = 0_R$

So $1_R \in S$, $1_R \in T$, $0_R \in S$, $0_R \in T$

Therefore $1_R \in S \cap T$, $0_R \in S \cap T$,

we have proved ①

Let a, b be arbitrary elements in $S \cap T$

so $a, b \in S$, $a, b \in T$

Then $a+b \in S$, $a+b \in T$, $ab \in S$, $ab \in T$
(since S, T are rings (addition and multiplication are closed))

So $a+b \in S \cap T$, $ab \in S \cap T$

we have proved ②, ④

Since $a + b = 0_S = 0_R$ for some $b \in S$

$a + c = 0_T = 0_R$ for some $c \in T$

$b = c = 0_R - a = -a$ in R

So $-a \in S \cap T$, we have proved ③

Conclusion: $S \cap T$ is a subring of R