Homework 8

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Sunday, March 24th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. An isomorphism from a group G to itself is called an *automorphism*. Let $\mathrm{Aut}(G)$ denote the set of automorphisms of a group G.
   (a) Let $f\colon G_1 \to G_2$ and $g\colon G_2 \to G_3$ be group homomorphisms. Prove that the composition $g \circ f\colon G_1 \to G_3$ is a group homomorphism.
   (b) Let $f\colon G \to H$ be a group isomorphism. Prove that the inverse function $f^{-1}\colon H \to G$ is also a group isomorphism.
   (c) Prove that $\mathrm{Aut}(G)$ is a group with operation given by composition.
   (d) Prove that $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$.
   (e) Prove that $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.

2. Let G be a group. The **center** of G is the set $Z(G) = \{g \in G \mid gh = hg \ \ \forall h \in G\}$.
   1. Prove that $Z(G)$ is an abelian subgroup of G.
   2. Compute the center of D_4.
   3. Compute the center of S_3.
   4. Compute the center of $GL_2(\mathbb{R})$.

3. Consider the symmetric group S_n, with $n \geq 3$. The goal of this problem is to prove that S_n can be generated by only two elements.
   (a) Let $\tau \in S_n$ be a permutation, and $(\,a\,b\,)$ be a transposition. Show that $\tau(\,a\,b\,)\tau^{-1} = (\,\tau(a)\,\tau(b)\,)$, the transposition changing $\tau(a)$ and $\tau(b)$.
   (b) Show that $(\,i\,j\,) = (\,1\,i\,)(\,1\,j\,)(\,1\,i\,)$. Conclude that every element of S_n is the product of transpositions of the form $(\,1\,i\,)$.
   (c) Let σ be the $(n-1)$-cycle $(2\,3\,\cdots\,n\,)$. Show that $(\,1\,i\,) = \sigma^{i-2}(\,1\,2\,)(\sigma^{-1})^{i-2}$ for all $i = 2, \ldots, n$. Conclude that $S_n = \langle (\,1\,2\,), (2\,3\,\cdots\,n\,) \rangle$.

4. Consider the alternating group A_n, that is, the subgroup of S_n consisting of all the even permutations of S_n, for $n \geq 3$. Let $i, j, k, l \in \{1, 2, \ldots, n\}$, with $i \neq j$ and $k \neq l$.
   (a) Suppose that $(i\,j)$ and $(k\,l)$ are not disjoint cycles. Show that $(i\,j)(k\,l)$ is either the identity or a 3-cycle.
   (b) Suppose that $(i\,j)$ and $(k\,l)$ are disjoint cycles. Show that $(i\,j)(k\,l)$ is the product of two 3-cycles.
   (c) Prove that A_n is generated by the set of all 3-cycles of S_n.