Math 412 Homework 3

Submission Instructions: You are responsible to read these instructions. Failure to submit correctly as described below will result in point deductions or loss of credit for entire problems. Submit these problems on Gradescope by Thursday, February 15th, at 11:59pm. Each problem should be on a separate page (or pages). **You will need to scan a PDF of the assignment AND select the pages belonging to each problem when you submit on gradescope.**

1. (a) If $f : R \to S$ is a homomorphism of rings, show for any $r \in R$ and $n \in \mathbb{Z}$, $f(nr) = nf(r)$.
   (b) Prove that isomorphic rings have the same characteristic.
   (c) If $f : R \to S$ is a homomorphism of rings, must R and S have the same characteristic?

2. Let V be a vector space. Recall that a function $T : V \to V$ is a *linear transformation* if for all $v, w \in V$ and all $\lambda \in \mathbb{R}$, we have $T(v + w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$.
   (a) Show that the set of linear transformations from V to V, with usual addition, and *composition of functions* as multiplication, forms a ring.
   (b) Consider the vector space $\mathbb{R}[x]$ and let $\mathcal{L}(\mathbb{R}[x])$ be the ring of linear transformations of $\mathbb{R}[x]$ as defined in the previous part. Consider the element $\frac{d}{dx} \in \mathcal{L}(\mathbb{R}[x])$. Show that there is an element $F \in \mathcal{L}(\mathbb{R}[x])$ such that $\frac{d}{dx}F = 1_{\mathcal{L}(\mathbb{R}[x])}$, but there is no element $G \in \mathcal{L}(\mathbb{R}[x])$ such that $G\frac{d}{dx} = 1_{\mathcal{L}(\mathbb{R}[x])}$.

3. Let d be an integer.
   (a) Prove that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$ is an integral domain.
   (b) Show that $\mathbb{Z}_7[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}_7\}$ is a field.
   (c) Now assume d is also positive and p is a prime. Determine a necessary and sufficient condition for $\mathbb{Z}_p[\sqrt{d}]$ to be a field.

4. Let R be a commutative ring in which $a^2 = 0$ only if $a = 0$. Show that if $q(x) \in R[x]$ is a zero divisor in $R[x]$, then if:
$$q(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$
there is an element $b \neq 0$ in R such that $ba_0 = ba_1 = \cdots = ba_n = 0$.