Math 412. The symmetric group

DEFINITION: The **symmetric group** $\mathcal{S}_n$ is the group of bijections from any set of n objects, which we usually call simply $\{1, 2, \ldots, n\}$, to itself. An element of this group is called a **permutation** of $\{1, 2, \ldots, n\}$. The group operation in $\mathcal{S}_n$ is *composition* of mappings.

PERMUTATION STACK NOTATION: The notation $\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ denotes the permutation that sends i to k_i for each i.

CYCLE NOTATION: The notation $(a_1\ a_2\ \cdots\ a_t)$ refers to the (special kind of!) permutation that sends a_i to a_{i+1} for $i < t$, a_t to a_1, and fixes any element other than the a_i's. A permutation of this form is called a **t-cycle.** A 2-cycle is also called a **transposition.**

Remember that a cycle is a function, so if we have cycles side-by-side, this refers to composition of functions, where the composition as usual goes from right to left.

THEOREM 7.24: Every permutation can be written as a product of *disjoint cycles* — cycles that all have no elements in common. Disjoint cycles commute.

THEOREM 7.26: Every permutation can be written as a product of *transpositions,* not necessarily disjoint.

Part 1: The essentials.

A. WARM-UP WITH ELEMENTS OF $\mathcal{S}_n$

(1) Write the permutation $(1\ 3\ 5)(2\ 7) \in \mathcal{S}_7$ in permutation stack notation.

(2) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 2 & 4 & 7 & 5 \end{pmatrix} \in \mathcal{S}_7$ in cycle notation.

(3) If $\sigma = (1\ 2\ 3)(4\ 6)$ and $\tau = (2\ 3\ 4\ 5\ 6)$ in $\mathcal{S}_7$, compute $\sigma\tau$; write your answer in stack notation. Now also write it as a product of disjoint cycles.

(4) With σ and τ as in (4), compute $\tau\sigma$. Is $\mathcal{S}_7$ abelian?

(5) List all elements of $\mathcal{S}_3$ in cycle notation. What is the order of each?

(6) What is the inverse of $(1\ 2\ 3)$? What is the inverse of $(1\ 2\ 3\ 4)$? How about $(1\ 2\ 3\ 4\ 5)^{-1}$?

Solution.

(1) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 5 & 4 & 1 & 6 & 2 \end{pmatrix}$

(2) $(1\ 3)(2\ 6\ 7\ 5\ 4)$

(3) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 6 & 5 & 4 & 3 & 7 \end{pmatrix}$. Same as $(1\ 2)(3\ 6)(\ 4\ 5)$.

(4) $(1\ 3)(2\ 4)(\ 5\ 6)$. No, not abelian as $\sigma\tau \neq \tau\sigma$.

(5) e, $(1\ 2), (2\ 3), (1\ 3)$, $(1\ 2\ 3), (3\ 2\ 1)$. These have orders $1, 2, 2, 2, 3, 3$. Each divides the order of $\mathcal{S}_3$, which is 3! or 6.
(6) $(1\ 2\ 3)^{-1} = (3\ 2\ 1)$? $(1\ 2\ 3\ 4)^{-1} = (4\ 3\ 2\ 1)$? $(1\ 2\ 3\ 4\ 5)^{-1} = (5\ 4\ 3\ 2\ 1)$.

B. THE SYMMETRIC GROUP $\mathcal{S}_4$

(1) What is the order of $\mathcal{S}_4$?
(2) List all 2-cycles in $\mathcal{S}_4$. How many are there?
(3) List all 3-cycles in $\mathcal{S}_4$. How many are there?
(4) List all 4-cycles in $\mathcal{S}_4$. How many are there?
(5) List all 5-cycles in $\mathcal{S}_4$.
(6) How many permutations in $\mathcal{S}_4$ are not cycles? Find them all.
(7) Find the order of each element in $\mathcal{S}_4$. Why are the orders the same for permutations with the same "cycle type"?
(8) Find cyclic subgroups of $\mathcal{S}_4$ of orders 2, 3, and 4.
(9) Find a subgroup of $\mathcal{S}_4$ isomorphic to the Klein 4-group. List out its elements.
(10) List out all elements in the subgroup $H = \langle (1\ 2\ 3), (2\ 3) \rangle$ of $\mathcal{S}_4$ generated by $(1\ 2\ 3)$ and $(2\ 3)$. What familiar group is this isomorphic to? Can you find four different subgroups of $\mathcal{S}_4$ isomorphic to $\mathcal{S}_3$?

Solution.

(1) $4!$ or 24.
(2) The transpositions are $(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)$. There are six.
(3) The 3-cycles are $(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)$. There are eight.
(4) The 4-cycles are $(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)$. There are six.
(5) There are no 5-cycles!
(6) We have found 20 permutations of 24 total permutations in $\mathcal{S}_4$. So there must be 4 we have not listed. The identity e is one of these, but let's say it is a 0-cycle. The permutations that are not cycles are $(1\ 2)(3\ 4)$ and $(1\ 3)(2\ 4)$ and $(1\ 4)(2\ 3)$.
(7) The order of the 2-cycles is 2, the order of the 3 cycles is 3, the order of the 4-cycles is 4. The order of the four permutations that are products of disjoint transpositions is 2.
(8) An example of a cyclic subgroup of order 2 is $\langle (1\ 2) \rangle = \{e, (1\ 2)\}$. An example of a cyclic subgroup of order 3 is $\langle (1\ 2\ 3) \rangle = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$. An example of a cyclic subgroup of order 4 is $\langle (1\ 2\ 3\ 4) \rangle = \{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$.
(9) A subgroup isomorphic to the Klein 4 group is $\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.
(10) The subgroup $\langle (1\ 2\ 3), (2\ 3) \rangle = \{e, (1\ 2\ 3), (1\ 3\ 2), (2\ 3), (1\ 2), (1\ 3)\}$, which is $\mathcal{S}_3$. We can get four different subgroups inside $\mathcal{S}_4$ that are isomorphic to $\mathcal{S}_3$, just by looking at the sets of permutations that FIX one of the four elements. The one we just looked at fixes 4. But we could have just as easily looked only

at permutations that fix 1: these would be the permutations of the set $\{2, 3, 4\}$, which is also $\mathcal{S}_3$. Likewise, the permutation group of $\{1, 3, 4\}$ and the permutation group of $\{1, 2, 4\}$ are also subgroups of $\mathcal{S}_3$ isomorphic to $\mathcal{S}_3$.

C. EVEN AND ODD PERMUTATIONS. A permutation is **odd** if it is a composition of an odd number of transposition, and **even** if it is a product of an even number of transpositions.

(1) Explain why a definition like this might be problematic. Problem G below justifies this definition.
(2) Write the permutation (123) as a product of transpositions. Is (123) even or odd ?
(3) Write the permutation (1234) as a product of transpositions. Is (1234) even or odd ?
(4) Write the $\sigma = (12)(345)$ a product of transpositions in two different ways. Is σ even or odd ?
(5) Prove that every 3-cycle is an even permutation.

Solution.

(1) Note that the definition of even/odd permutation is problematic: how do we know it is well-defined? That is, if Waleed writes out a certain permutation σ as a product of 17 transposition, but Linh writes out the same permutation σ as a product of 22 transposition, is σ even or odd? By problem G, this cannot happen.
(2) $(1\ 2\ 3) = (1\ 2)(2\ 3)$, even.
(3) $(1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4)$, odd.
(4) $(1\ 2)(3\ 4\ 5) = (1\ 2)(3\ 4)(4\ 5) = (4\ 5)(1\ 2)(4\ 5)(3\ 4)(4\ 5)$. odd.
(5) The 3-cycle $(i\ j\ k) = (i\ j)(j\ k)$ so it is even.

D. THE ALTERNATING GROUPS

(1) Prove that the subset of even permutations in S_n is a subgroup. This is the called the **alternating group** A_n.
(2) List out the elements of A_2. What group is this?
(3) List out the elements of A_3. To what group is this isomorphic?
(4) How many elements in A_4? Is A_4 abelian? What about A_n?

Solution.

(1) To check that A_n is a subgroup, we need to prove that for arbitrary $\tau, \sigma \in \mathcal{A}_n$.
    (a) $\tau \circ \sigma \in \mathcal{A}_n$.
    (b) $\sigma^{-1} \in A_n$.

    For (1): Assume σ and τ are both even. we need to show $\sigma \circ \tau$ is even. Write τ and σ as a composition of (an even number of) transpositions. So the composition $\sigma\tau$ is the composition of all these...still an even number of them.

    For (2): Note that if σ is a product $\tau_1 \circ \tau_2 \cdots \tau_n$, then the inverse of σ is $\tau_n \circ \tau_{n-1} \cdots \tau_2 \circ \tau_1$. This has the same number of transpositions, so σ is even if and only if its inverse is even. That is, if $\sigma \in A_n$, then so is σ^{-1}. QED.

(2) We have $A_2 = \{e\}$, the trivial group.
(3) We have $A_3 = \{e, (1\ 2)(2\ 3), (1\ 3)(2\ 3)\} = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$. This is a cyclic group of order 3.
(4) This is order 12, not abelian. In general, A_n has order $n!/2$ and is not abelian if $n \geqslant 4$.s

$$A_4 = \{e, (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}.$$

Part 2: More practice.

E. THE SYMMETRIC GROUP $\mathcal{S}_5$

(1) Find one example of each type of element in S_5 or explain why there is none:
  (a) A 2-cycle
  (b) A 3-cycle
  (c) A 4-cycle
  (d) A 5-cycle
  (e) A 6-cycle
  (f) A product of disjoint transpositions
  (g) A product of 3-cycle and a disjoint 2-cycle.
  (h) A product of 2 disjoint 3 cycles.
(2) For each example in (1), find the order of the element.
(3) What are all possible orders of elements in $\mathcal{S}_5$?
(4) What are all possible orders of cycle subgroups of $\mathcal{S}_5$.
(5) For each example in (1), write the element as a product of transpositions. Which are even and which are odd?

Solution.
(1) $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2\ 3\ 4)$, $(1\ 2\ 3\ 4\ 5)$, No six cycles!, $(1\ 2)(3\ 5)$, $(1\ 2)(3\ 4\ 5)$, no triple products of disjoint 2 cycles exist in $\mathcal{S}_5$....only 5 objects to permute.
(2) The orders are 2, 3, 4, 5, none, 2, 6.
(3) The orders above, and 1, are all possible orders because these exhaust all possible cycle-types of permutations.
(4) There are cyclic subgroups of all the orders listed in (2), and the trivial subgroup $\{e\}$ which is cyclic of order 1.
(5) $(1\ 2)$, $(1\ 2\ 3) = (1\ 2)(2\ 3)$, $(1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4)$, $(1\ 2\ 3\ 4\ 5) = (1\ 2)(2\ 3)(3\ 4)(4\ 5)$, No six cycles!, $(1\ 2)(3\ 5)$, $(1\ 2)(3\ 4\ 5) = (1\ 2)(3\ 4)(4\ 5)$. To determine even/odd just count the number of transpositions in each.

F. Discuss with your workmates how one might prove Theorem 7.26.[1]

[1]Hint: Imagine lining everyone in the class up in a straight line. How can we put the class in alphabetical order by a sequence of swaps?

Solution. In $\mathcal{S}_2$, every element is a transposition (12) or a product of transpositions $(12)(12) = e$.

In $\mathcal{S}_3$, every element is a transposition, or a product of transpositions such as $(12)(12) = e$, or $(123) = (12)(23)$.

In $\mathcal{S}_4$, the previous cases handle every thing which is a 1-cycle, 2-cycle or 3-cycle. The remaining elements are either products of two disjoint transpositions, such as $(12)(34)$, in which case we're done, or four cycles such as (1234). The latter can be written $(12)(23)(34)$.

In $\mathcal{S}_n$, we write an arbitrary element as a product of cycles. Then, it comes down to writing each cycle as a product of transpositions. But for example $(i_1 i_2 i_3 \cdots i_t) = (i_1 i_2) \circ (i_2 i_3) \circ \cdots (i_{t-1} i_t)$. So it is a clear that this can be done.

Alternatively, we can proceed by induction on n. The base case $n = 1$ is trivial. Given a general permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$, note that $(k_n n)\sigma$ fixes n, so can be considered as a permutation of $n-1$ elements. By the induction hypothesis, it is a product of transpositions. Then $\sigma = (k_n n)(k_n n)\sigma$ is a product of transpositions as well.

G. Permutation Matrices. We say that an $n \times n$ matrix is a **permutation matrix** if it has exactly one 1 in each row and each column, and the other entries 0. If $\sigma \in \mathcal{S}_n$ is a permutation, let P_σ be the $n \times n$ permutation matrix with $(\sigma(i), i)$ entry 1 for all i, and all other entries 0.

(1) Show that $P_\sigma e_i = e_{\sigma(i)}$ for any permutation σ, where e_j is the jth standard basis vector.
(2) Show that $P_\sigma P_\tau = P_{\sigma\circ\tau}$.
(3) Show that the set of permutation matrices is a subgroup of $GL_n(\mathbb{R})$ that is isomorphic to $\mathcal{S}_n$.
(4) Show that the determinant of $P_{(ij)}$ is -1.
(5) Show that if σ is a product of an even number of transpositions, then the determinant of P_σ is 1, and if σ is a product of an odd number of transpositions, then the determimant of P_σ is -1. Conclude that the sign of a permutation is well-defined.

Solution.

(1) The product $P_\sigma e_i$ is the ith column of P_σ, which has a one in the $\sigma(i)$ row ans zeroes elsewhere: this is $e_{\sigma(i)}$.
(2) Since matrix multiplication corresponds to composition of linear transformations, we have $(P_\tau P_\sigma)e_i = P_\tau e_{\sigma(i)} = P_{\tau(\sigma(i))}$ for all i. Thus, $(P_\tau P_\sigma)e_i = P_{\tau\circ\sigma} e_i$ for all i, so the matrices $P_\tau P_\sigma$ and $P_{\tau\circ\sigma}$ must be equal.
(3) We see from the previous part that this subset is closed under composition. Since every $\sigma \in \mathcal{S}_n$ has finite order, one of its positive powers is its inverse. It follows that the set of permutation matrices is closed under inverses. The previous part also shows that the map sending σ to P_σ is an isomorphism.

(4) Follows from Math 217.
(5) This follows from the fact that $\det$ is a homomorphism. Since $\det$ is well-defined, a permutation matrix can only be an even product of transpositions OR (exclusive) an odd product of transpositions.