Spring 2024 Math 451: Practice Problems

(1) Prove that for all sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.

Sample Solution: Let A and B be sets, and for the forward direction suppose $A \subseteq B$. Note that we always have $A \cup B \supseteq B$, so in order to show $A \cup B = B$ we need only show $A \cup B \subseteq B$. Thus let $x \in A \cup B$. Since $A \subseteq B$, regardless of whether $x \in A$ or $x \in B$ we will have $x \in B$, so we conclude that $A \cup B \subseteq B$ as desired. For the backward direction, suppose $A \cup B = B$, and let $x \in A$ be arbitrary. Then $x \in A \cup B = B$, so $x \in B$, which shows $A \subseteq B$ as desired. We conclude that $A \subseteq B$ if and only if $A \cup B = B$.

(2) DeMorgan's Laws state that for all sets A, B, and C, we have $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. Choose one of these equations and prove it.

Sample Solution: Let A, B, and C be sets. We show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ by proving that each set is a subset of the other. For the forward inclusion, let $x \in A \setminus (B \cup C)$ be arbitrary. Then $x \in A$ but $x \notin B \cup C$, so $x \notin B$ and $x \notin C$, which implies $x \in A \setminus B$ and $x \in A \setminus C$, and therefore $x \in (A \setminus B) \cap (A \setminus C)$. For the reverse inclusion, let $x \in (A \setminus B) \cap (A \setminus C)$ be arbitrary. Then $x \in A \setminus B$ and $x \in A \setminus C$, so we have $x \in A$ and $x \notin B$ and $x \notin C$. This implies $x \in A$ and $x \notin A \cup B$, so $x \in A \setminus (B \cup C)$ as desired. This completes the proof that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(3) DeMorgan's Laws also hold for indexed families of sets, even if the indexing family is infinite. For instance, let A be a set and suppose that B_n is a set for every $n \in \mathbb{N}$. Then we have

$$A \setminus \left(\bigcup_{n \in \mathbb{N}} B_n \right) = \bigcap_{n \in \mathbb{N}} (A \setminus B_n) \qquad \text{and} \qquad A \setminus \left(\bigcap_{n \in \mathbb{N}} B_n \right) = \bigcup_{n \in \mathbb{N}} (A \setminus B_n).$$

Prove whichever version you did not choose in (2).

Sample Solution: Let A be a set, and suppose B_n is a set for each $n \in \mathbb{N}$. We show that

$$A \setminus \left(\bigcup_{n \in \mathbb{N}} B_n \right) = \bigcap_{n \in \mathbb{N}} (A \setminus B_n)$$

by showing that each side is contained in the other. For the forward inclusion, let $x \in A \setminus \left(\bigcup_{n \in \mathbb{N}} B_n \right)$, so that $x \in A$ but $x \notin \bigcup_{n \in \mathbb{N}} B_n$ and thus $x \notin B_n$ for any $n \in \mathbb{N}$. This implies that $x \in A \setminus B_n$ for all $n \in \mathbb{N}$, which means $x \in \bigcap_{n \in \mathbb{N}} (A \setminus B_n)$. Conversely, let $x \in \bigcap_{n \in \mathbb{N}} (A \setminus B_n)$, so that $x \in A \setminus B_n$ for all $n \in \mathbb{N}$. Then $x \in A$, and also for every $n \in \mathbb{N}$ we have that $x \notin B_n$, which implies $x \notin \bigcup_{n \in \mathbb{N}} B_n$. Thus $x \in A \setminus \left(\bigcup_{n \in \mathbb{N}} B_n \right)$, as desired.

(4) Let X and Y be sets, and let $f : X \to Y$ be a function. Prove that for all $A, B \subseteq X$ and $C, D \subseteq Y$, the following are true:

(a) $f[f^{-1}[C]] \subseteq C$

(b) $f^{-1}[f[A]] \supseteq A$

(c) $f[A \cup B] = f[A] \cup f[B]$

(d) $f[A \cap B] \subseteq f[A] \cap f[B]$

(e) $f[A \setminus B] \supseteq f[A] \setminus f[B]$
(f) $f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D]$
(g) $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$
(h) $f^{-1}[C \setminus D] = f^{-1}[C] \setminus f^{-1}[D]$

Sample Solution:

(a) Let $y \in f[f^{-1}[C]]$, say $y = f(x)$ where $x \in f^{-1}[C]$. Since $x \in f^{-1}[C]$, we have $y = f(x) \in C$, as desired.

(b) For every $x \in A$, we have $f(x) \in f[A]$, so $x \in f^{-1}[f[A]]$ by definition of preimage.

(c) Let $y \in Y$, and first suppose $y \in f[A \cup B]$, say $y = f(x)$ where $x \in A \cup B$. If $x \in A$ then $f(x) \in f[A]$, and if $x \in B$ then $f(x) \in f[B]$, so either way we see $y = f(x) \in f[A] \cup f[B]$. This shows $f[A \cup B] \subseteq f[A] \cup f[B]$, For the reverse inclusion, suppose $y \in f[A] \cup f[B]$. If $y = f(x)$ where $x \in A \subseteq A \cup B$, then $y = f(x) \in f[A \cup B]$, and if $y = f(x)$ where $x \in B \subseteq A \cup B$, then $y = f(x) \in f[A \cup B]$. Either way we see that $y = f(x) \in f[A \cup B]$. This completes the proof.

(d) Let $y \in f[A \cap B]$, say $y = f(x)$ where $x \in A \cap B$. Then $x \in A$, so $f(x) \in f[A]$, and $x \in B$, so $f(x) \in f[B]$. Therefore $y = f(x) \in f[A] \cap f[B]$, as desired.

(e) Let $y \in f[A] \setminus f[B]$, say $y = f(x)$ where $x \in A$, and note that $x \notin B$. Then $x \in A \setminus B$, so $y = f(x) \in f[A \setminus B]$.

(f) For all $x \in X$, we have

$$\begin{aligned} x \in f^{-1}[C \cup D] &\iff f(x) \in C \cup D \iff f(x) \in C \text{ or } f(x) \in D \\ &\iff x \in f^{-1}[C] \text{ or } x \in f^{-1}[D] \iff x \in f^{-1}[C] \cup f^{-1}[D]. \end{aligned}$$

(g) For all $x \in X$, we have

$$\begin{aligned} x \in f^{-1}[C \cap D] &\iff f(x) \in C \cap D \iff f(x) \in C \text{ and } f(x) \in D \\ &\iff x \in f^{-1}[C] \text{ and } x \in f^{-1}[D] \iff x \in f^{-1}[C] \cap f^{-1}[D]. \end{aligned}$$

(h) For all $x \in X$, we have

$$\begin{aligned} x \in f^{-1}[C \setminus D] &\iff f(x) \in C \setminus D \iff f(x) \in C \text{ and } f(x) \notin D \\ &\iff x \in f^{-1}[C] \text{ and } x \notin f^{-1}[D] \iff x \in f^{-1}[C] \setminus f^{-1}[D]. \end{aligned}$$

(5) Give conditions (on f) under which the containments in (a), (b), (d), and (e) from (4) above are in fact equalities.

Sample Solution:

(a) For any $C \subseteq Y$, we will have $f[f^{-1}[C]] = C$ iff $C \subseteq \operatorname{ran}(f)$. So $f[f^{-1}[C]] = C$ for *all* $C \subseteq Y$ iff f is surjective.

(b) For any $A \subseteq X$, we will have $f^{-1}[f[A]] = A$ iff for all $y \in f[A]$ we have $x \in A$ whenever $f(x) = y$. So $f^{-1}[f[A]] = A$ for *all* $A \subseteq X$ iff f is injective.

(d) $f[A \cap B] = f[A] \cap f[B]$ for all $A, B \subseteq X$ iff f is injective.

(e) $f[A \setminus B] = f[A] \setminus f[B]$ for all $A, B \subseteq X$ iff f is injective.

(6) Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove the following:

(a) f is injective if and only if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

(b) f is surjective if and only if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

(c) f is bijective if and only if f is invertible.

Sample Solution:

(a) For the forward direction, suppose f is injective. Fix $x_0 \in X$, and define the function $g : Y \to X$ by letting $g(y)$ be the unique $x \in X$ such that $f(x) = y$ if $y \in \mathrm{ran}(f)$, and $g(y) = x_0$ otherwise. Then $g \circ f = \mathrm{id}_X$. Conversely, suppose $g : Y \to X$ is a function such that $g \circ f = \mathrm{id}_X$. Let $x_1, x_2 \in X$ and suppose $x_1 \neq x_2$. Then $f(x_1) \neq f(x_0)$, since otherwise we would have $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$. This shows that f is injective.

(b) For the forward direction, suppose f is surjective. For each $y \in Y$ there is $x \in X$ such that $f(x) = y$, so for each $y \in Y$ we choose a particular element $g(y) \in X$ such that $f(g(y)) = y$. This defines a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. Conversely, suppose $g : Y \to X$ is a function such that $f \circ g = \mathrm{id}_Y$. Then given arbitrary $y \in Y$, we have $f(g(y)) = y$, so there is indeed $x \in X$ (namely $x = g(y)$) such that $f(x) = y$. This shows f is surjective, and completes the proof.

(c) If f is invertible, then it follows immediately from (a) and (b) that f is injective and surjective, hence bijective. Conversely, if f is bijective then for every $y \in Y$ we can let $g(y)$ be the unique $x \in X$ for which $f(x) = y$. Then g is the inverse of f, so f is invertible.

(7) Let X, Y, and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following:

(a) If f and g are injective, then so is $g \circ f$.

(b) If f and g are surjective, then so is $g \circ f$.

(c) If f and g are bijective, then so is $g \circ f$.

(d) If $g \circ f$ is injective, then so is f.

(e) If $g \circ f$ is surjective, then so is g.

Sample Solution:

(a) Suppose f and g are injective, let $x_1, x_2 \in X$, and suppose $x_1 \neq x_2$. Then since f is injective, we have $f(x_1) \neq f(x_2)$, and therefore since g is injective we have $g(f(x_1)) \neq g(f(x_2))$. This shows that $g \circ f$ is injective.

(b) Suppose f and g are surjective, and let $z \in Z$. Using the fact that g is surjective, we can fix $y \in Y$ such that $g(y) = z$, and then using the fact that f is surjective, we can fix $x \in X$ such that $f(x) = y$ and therefore $g(f(x)) = z$. This shows that $g \circ f$ is surjective.

(c) This follows immediately from (a) and (b) and the definition of bijective.

(d) We prove the contrapositive. Suppose f is not injective, so we can fix $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. But then $g(f(x_1)) = g(f(x_2))$, which shows that $g \circ f$ is not injective.

(e) Again we prove the contrapositive. Suppose g is not surjective, so we can fix $z \in Z$ such that $z \notin \mathrm{ran}(g)$. But $\mathrm{ran}(g \circ f) \subseteq \mathrm{ran}(g)$, so z does not belong to $\mathrm{ran}(g \circ f)$ either and thus $g \circ f$ is also not surjective.

(8) Recall[1] Kuratowski's set-theoretic definition of ordered pair: $(a, b) := \{\{a\}, \{a, b\}\}$. Using this definition, prove that for all a, b, c, d we have $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Sample Solution: If $a = c$ and $b = d$ then clearly $(a, b) = (c, d)$, so we only need to show the converse. Suppose $(a, b) = (c, d)$. We consider two cases: first, suppose $a = b$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$, so the set $\{a, b\}$ has one element in it. This means the set $(c, d) = \{\{c\}, \{c, d\}\}$ must have one element in it too, so $\{c\} = \{c, d\}$. Hence $c = d$, so $(c, d) = \{\{c\}\}$, and we see that in fact $a = b = c = d$, as desired. For the second case, we suppose $a \neq b$, so $(a, b) = \{\{a\}, \{a, b\}\}$ has two elements. Thus $(c, d) = \{\{c\}, \{c, d\}\}$ also has two elements, so $c \neq d$. Furthermore, since $\{a\}$ and $\{c\}$ both have one element but $\{a, b\}$ and $\{c, d\}$ have two elements, we must have $a = c$, and therefore $b = d$, completing the proof.

(9) Use induction to prove the following formulas:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Sample Solution:

(a) For the base case $n = 1$, both sides are equal to 1. For the inductive step, let $n \in \mathbb{N}$ and suppose for inductive hypothesis that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then

$$\sum_{k=1}^{n+1} k = n + 1 + \frac{n(n+1)}{2} = \frac{2n + 2 + n^2 + n}{2} = \frac{(n+1)(n+2)}{2},$$

completing the induction.

(b) For the base case $n = 1$, both sides are equal to 1. For the inductive step, let $n \in \mathbb{N}$ and suppose for inductive hypothesis that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Then

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} = n^2 + 2n + 1 + \frac{2n^3 + 3n^2 + n}{6} \\ &= \frac{(6n^2 + 12n + 6) + (2n^3 + 3n^2 + n)}{6} = \frac{2n^3 + 9n^2 + 13n + 6}{6} = \frac{(n+1)(n+2)(2(n+1)+1)}{6},\end{aligned}$$

completing the induction.

(10) Prove that $\sqrt{3}$ is irrational without using the Rational Roots Theorem. Then show how the irrationality of $\sqrt{3}$ follows from the Rational Roots Theorem.

Sample Solution: Suppose for contradiction that there is $r \in \mathbb{Q}$ such that $r^2 = 3$, and fix $m, n \in \mathbb{N}$ with no common factors (greater than 1) such that $(m/n)^2 = 3$, so that $m^2 = 3n^2$. Then m^2 is divisible

[1] From the *More Joy of Sets* handout.

by 3, which means m itself must be dvisible by 3, say $m = 3k$ where $k \in \mathbb{N}$. Then

$$3n^2 = m^2 = (3k)^2 = 9k^2,$$

so $n^2 = 3k^2$ and thus n^2 is divisible by 3. But this implies n is divisible by 3 as well, contradicting our assumption that m and n have no common factors. We conclude that no such $r \in \mathbb{Q}$ exists.

Towards applying the Rational Roots Theorem, consider the polynomial $p(x) = x^2 - 3$. By the Theorem, if $r \in \mathbb{Q}$ is a root of p then $r \in \{\pm 1, \pm 3\}$. Since none of these four numbers is a root of p (by inspection), we conclude that p has no rational root, so there is no rational number x such that $x^2 - 3 = 0$. In other words, $\sqrt{3}$ is irrational.

(11) Suppose $<$ is a linear order on the set X. Using nothing but the linear order axioms, prove that for all $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.

Sample Solution: Let $a, b \in X$, suppose $a \leq b$ and $b \leq a$, and assume for contradiction that $a \neq b$. Then $a < b$ and $b < a$, which by transitivity implies $a < a$, contradicting irreflexivity. Thus $a = b$ as desired.

(12) Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$, and suppose that $b = \max A$ is the greatest element of A. Prove that $b = \sup A$.

Sample Solution: Let $A \subseteq \mathbb{R}$, let $b \in \mathbb{R}$, and suppose $b = \max A$. By definition of greatest element, we have $a \leq b$ for all $a \in A$, so b is an upper bound of A. Now let u be any other upper bound of A. Since $b \in A$ (again by definition of greatest element), we have $b \leq u$. Thus b is not just an upper bound of A, but is in fact the *least* upper bound of A; that is, $b = \sup A$ as desired.

(13) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below, and let L be the set of all lower bounds of A in $\mathbb{R}$. Prove that $\sup L = \inf A$.

Sample Solution: Every $a \in A$ is an upper bound of L, so $\sup L \leq a$ for all $a \in A$, and thus $\sup L$ is a lower bound of A. But also $\sup L \geq \ell$ for every lower bound ℓ of A, so $\sup L$ is the *greatest* lower bound of A, as desired.

(14) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below, and let $-A = \{-a : a \in A\}$. Prove that $\inf A = -\sup(-A)$.

Sample Solution: Let $a \in A$ be arbitrary. Then $-a \in -A$, so $-a \leq \sup(-A)$, which implies $a \geq -\sup(-A)$. This shows that $-\sup(-A)$ is a lower bound of A. To show that it is the *greatest* lower bound of A, let ℓ be an arbitrary lower bound of A. Then for all $a \in A$ we have $\ell \leq a$ and therefore $-\ell \geq -a$. Thus $-\ell \geq \sup(-A)$, so $\ell \leq -\sup(-A)$ as desired.

(15) *Show[2] that if we were to drop the Distributive Law (Axiom 9) from the field axioms, we would no longer be able to prove that $0 \cdot x = 0$ for all x.

[2]All these practice problems are optional, but ones with *'s are even more optional! (ie, don't worry if you don't know how to do them.)

Sample Solution: For instance, consider the two-element set $\mathbb{F} = \{0, 1\}$ with addition and multiplication operations given by

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

Then $(\mathbb{F}, +, \times)$ satisfies Axioms 1–8 but not Axiom 9, so $\mathbb{F}$ is not a field. (This illustrates the importance of the Distributive Axiom. No one wants $0 \times 0 = 1$ to be true!)

(16) Prove that for all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$.

Sample Solution: Let $x, y \in \mathbb{R}$. Write $a = x - y$ and $b = y$. Then by the triangle inequality we have

$$|x| \;=\; |a + b| \;\leq\; |a| + |b| \;=\; |x - y| + |y|,$$

so $|x| - |y| \leq |x - y|$. Swapping x and y gives $|y| - |x| \leq |y - x| = |x - y|$, and then combining these results gives us $||x| - |y|| \leq |x - y|$, as desired.

(17) Let $a \in \mathbb{R}$ and let $\epsilon > 0$. Prove that for all $x, y \in V_\epsilon(a)$, we have $|x - y| < 2\epsilon$.

Sample Solution: Let $a \in \mathbb{R}$ and $\epsilon > 0$, and let $x, y \in V_\epsilon(a)$. Then, using the triangle inequality, we have

$$|x - y| \;=\; |x - a + a - y| \;\leq\; |x - a| + |y - a| \;<\; \epsilon + \epsilon \;=\; 2\epsilon.$$

(18) A function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing [decreasing]* if for all $x, y \in \mathbb{R}$, $x < y$ implies $f(x) < f(y)$ $[f(x) > f(y)]$, and *strictly monotone* if f is either strictly increasing or strictly decreasing.

(a) Prove that every strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is injective.

(b) Show by example that a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ need not be bijective.

(c) Show by example that a bijective function $f : \mathbb{R} \to \mathbb{R}$ need not be strictly monotone.

Sample Solution:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Let $x, y \in \mathbb{R}$ and suppose $x \neq y$, say without loss of generality that $x < y$. Then $f(x) < f(y)$, so in particular $f(x) \neq f(y)$. This shows f is injective.

(b) For instance, the function $f(x) = e^x$ is strictly increasing but not surjective.

(c) For instance, the function $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 1 \end{cases}$ is bijective but not strictly monotone.

(19) Determine whether the given function is injective, surjective, both, or neither:

(a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + |x|$.

(b) The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x|x|$.

(c) The function $h : \mathbb{R} \to (0, \infty)$ defined by $h(x) = e^x$.

(d) The function $p : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = x + y$.

(e) The function $m : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{3\}$ defined by $m(x) = \frac{3x+5}{x+2}$.

(f) The function $s : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ defined by $s(n) = \{k \in \mathbb{N} : k \le n\}$, where the set $\mathcal{P}(\mathbb{N})$ is called the *powerset* of $\mathbb{N}$ and is defined by $\mathcal{P}(\mathbb{N}) = \{A : A \subseteq \mathbb{N}\}$.

Sample Solution:

(a) The function f is neither injective nor surjective. To see that f is not injective, note that f is constant on the set of negative numbers, since for all $x < 0$ we have $f(x) = x + |x| = x - x = 0$. To see that f is not surjective, note that there are no negative numbers in $\mathrm{ran}(f)$, since $f(x) = 0$ if $x < 0$ and $f(x) = 2x \ge 0$ if $x \ge 0$.

(b) The function g is both injective and surjective (i.e., bijective). To see this, note that $g(x) = -x^2$ for all $x < 0$ and $g(x) = x^2$ for all $x \ge 0$, so g is strictly increasing on $\mathbb{R}$, and therefore is bijective.

(c) The function h is both injective and surjective (i.e., it is bijective). It is injective because h is strictly increasing (i.e., $x < y$ implies $h(x) < h(y)$), and it is surjective because for every $y \in (0, \infty)$ there is $x \in \mathbb{R}$, namely $x = \ln y$, such that $h(x) = y$.

(d) The function p is surjective but not injective. To see that p is surjective, note that for all $y \in \mathbb{R}$, we have $p(0, y) = y$. To see that f is not injective, note that $p(0, 2) = p(1, 1) = 2$.

(e) The function m is bijective since it is invertible. Indeed, we can find a formula for the inverse of m by starting with the equation $y = \frac{3x+5}{x+2}$ and solving for x in terms of y. This produces the formula $m^{-1}(y) = \frac{5-2y}{y-3}$.

(f) The function s is injective but not surjective. To see that s is injective, note that if $k < n$ then $n \in s(n)$ but $n \notin s(k)$, so $s(n) \ne s(k)$. However, s is not surjective since for instance $\mathbb{N} \in \mathcal{P}(\mathbb{N})$ but $\mathbb{N} \notin \mathrm{ran}(s)$.

(20) For any sets X and Y and subset $R \subseteq X \times Y$, define $R^{-1} := \{(y, x) \in Y \times X : (x, y) \in R\}$. Prove that for any function $f : X \to Y$, the set f^{-1} is a function if and only if f is injective. Assuming f is injective, what is $\mathrm{dom}(f^{-1})$?

Sample Solution: Let $f : X \to Y$ be a function, and view $f^{-1} = \{(y, x) : x \in X \text{ and } f(x) = y\}$ as a subset of $Y \times X$. If f is injective, then for every $y \in \mathrm{ran}(f)$ there is unique $x \in X$ such that $(y, x) \in f^{-1}$, which shows that f^{-1} is a function from $\mathrm{ran}(f)$ to X. Conversely, if f is *not* injective then there exist $x_1 \ne x_2$ in X such that $y = f(x_1) = f(x_2)$, so both (y, x_1) and (y, x_2) belong to f^{-1}, which means f^{-1} is not a function. Assuming f is injective, $\mathrm{dom}(f^{-1}) = \mathrm{ran}(f)$.

(21) For any sets X and Y, we define $X^Y = \{f : f \text{ is a function from } Y \text{ to } X\}$. Recalling that, as a set, $2 = \{0, 1\}$, show that for every set X we have $\mathcal{P}(X) \approx 2^X$.

Sample Solution: We define a function $f : \mathcal{P}(X) \to 2^X$ as follows: for every $A \subseteq X$ and $x \in X$, let $f(A)(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ Conversely, define $g : 2^X \to \mathcal{P}(X)$ as follows: given $\alpha \in 2^X$ and $x \in X$, let $x \in g(\alpha)$ iff $\alpha(x) = 1$. Then $g \circ f = \mathrm{id}_{\mathcal{P}(X)}$ and $f \circ g = \mathrm{id}_{2^X}$, so f and g are inverses of each other, which shows f and g are bijective. We conclude that $\mathcal{P}(X) \approx 2^X$.

(22) Define the function $f : 2^{\mathbb{N}} \to \mathbb{R}$ by $f(\alpha) = \displaystyle\sum_{k=1}^{\infty} \frac{2\alpha(k)}{3^k}$.

(a) What does $\operatorname{ran}(f)$ look like? Try[3] to draw a picture.

(b) Show that f is injective.

(c) Show that $\mathcal{P}(\mathbb{N}) \preceq \mathbb{R}$.

Sample Solution:

(a) The range of f is the subset of $[0,1]$ consisting of all real numbers in $[0,1]$ whose base 3 expansion only contains 0s and 2s, but never contains a 1. This is a famous set called the *Cantor set*; it has lots of interesting properties, and there are plenty of good attempts at drawing it to be found on the internet.

(b) Let $\alpha, \beta \in 2^X$, and suppose $\alpha \neq \beta$. Let n be least such that $\alpha(n) \neq \beta(n)$; wlog say $\alpha(n) = 0 < 1 = \beta(n)$. Then

$$f(\beta) - f(\alpha) \;=\; \frac{2}{3^n} + \sum_{k=n+1}^{\infty} \frac{\beta(k) - \alpha(k)}{3^k} \;\geq\; \frac{2}{3^n} - \sum_{k=n+1}^{\infty} \frac{2}{3^k} \;=\; \frac{2}{3^n} - \frac{1}{3^n} \;=\; \frac{1}{3^n},$$

so $f(\alpha) \neq f(\beta)$.

(c) By (21) we know there is a bijection $h : \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}}$, and by part (b) we know $f : 2^{\mathbb{N}} \to \mathbb{R}$ is injective, so the composite function $f \circ h : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is injective, which shows $\mathcal{P}(\mathbb{N}) \preceq \mathbb{R}$.

(23) (a) Show that $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q})$.

(b) Show that $\mathbb{R} \preceq \mathcal{P}(\mathbb{Q})$.

(c) Show that $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$.

Sample Solution:

(a) We already know $\mathbb{N} \approx \mathbb{Q}$, so let $g : \mathbb{N} \to \mathbb{Q}$ be a bijection. Now define $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Q})$ by $f(A) = \{g(n) : n \in A\}$. Then f is bijective since g is bijective.

(b) Define the function $f : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ by $f(x) = \{q \in \mathbb{Q} : q \leq x\}$. Then f is injective by density of $\mathbb{Q}$, so $\mathbb{R} \preceq \mathcal{P}(\mathbb{Q})$.

(c) From parts (a) and (b) we get $\mathbb{R} \preceq \mathcal{P}(\mathbb{N})$, and from 22(c) we know $\mathcal{P}(\mathbb{N}) \preceq \mathbb{R}$, so it follows from the Cantor-Schroder-Bernstein Theorem that $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$.

(24) (a) Show that $2^{\mathbb{N}} \preceq \mathbb{N}^{\mathbb{N}}$.

(b) Show that $\mathbb{N}^{\mathbb{N}} \preceq 2^{\mathbb{N}}$.

(c) Show that $2^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$.

Sample Solution:

(a) The function $f : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $f(\alpha)(n) = \alpha(n)$ is injective, so $2^{\mathbb{N}} \preceq \mathbb{N}^{\mathbb{N}}$.

(b) Define the function $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ as follows: given $\alpha = (\alpha(1), \alpha(2), \dots) \in \mathbb{N}^{\mathbb{N}}$, let

$$f(\alpha) \;=\; 1^{\alpha(1)} 0 1^{\alpha(2)} 0 1^{\alpha(3)} 0 \cdots \;\in\; 2^{\mathbb{N}}, \qquad \text{where } 1^k = \overbrace{1 \cdots 1}^{k\text{-times}} \text{ for each } k.$$

[3] Hint: spend a few minutes reading about the *Cantor set* on Wikipedia!

Then f is injective, so $\mathbb{N}^{\mathbb{N}} \preceq 2^{\mathbb{N}}$.

(c) This follows immediately from parts (a) and (b) and the Cantor-Schroder-Bernstein Theorem.

(25) A sequence $f : \mathbb{N} \to X$ is *eventually constant* if there is $x \in X$ and $N \in \mathbb{N}$ such that $f(n) = x$ for all $n \geq N$.

(a) Prove that there are only countably many eventually constant sequences in $2^{\mathbb{N}}$.

(b) How many eventually constant sequences are there in $\mathbb{N}^{\mathbb{N}}$?

Sample Solution:

(a) For each $n \in \mathbb{N}$, let $A_n = \{\alpha \in 2^{\mathbb{N}} : \alpha(i) = \alpha(j) \text{ for all } i, j \geq n\}$. Then each A_n is finite (in fact, $|A_n| = 2^n$), and the set Q of all eventually constant sequences in $2^{\mathbb{N}}$ is just $\bigcup_{n \in \mathbb{N}} A_n$. Thus Q is a countable union of countable (in fact finite) sets, so Q is countable.

(b) Also countable! Following the proof of (a), if we let $B_n = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \alpha(i) = \alpha(j) \text{ for all } i, j \geq n\}$, then each B_n is countable (why?), so the set of all eventually constant sequences in $\mathbb{N}^{\mathbb{N}}$ is a countable union of countable sets, and is therefore countable.

(26) Evaluate the limits $\lim\limits_{n\to\infty} \frac{n^2}{2^n}$ and $\lim\limits_{n\to\infty} \frac{2^n}{n^2}$. (Don't bother proving your claims, but take a moment to consider how you would proceed.)

Sample Solution: $\lim\limits_{n\to\infty} \frac{n^2}{2^n} = 0$ and $\lim\limits_{n\to\infty} \frac{2^n}{n^2} = \infty$. To show this, we could prove by induction that $2^n > n^3$ for all $n \geq 10$, which implies $\frac{n^2}{2^n} \leq \frac{1}{n}$ and $\frac{2^n}{n^2} \geq n$ for all $n \geq 10$.

(27) Let (a_n) be a sequence in $\mathbb{R}$. Prove that if $\lim a_n = L \in \mathbb{R}$, then $\lim |a_n| = |L|$.

Sample Solution: Suppose $\lim a_n = L \in \mathbb{R}$. Let $\epsilon > 0$, and fix $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq N$. Then for all $n \geq N$, we have $||a_n| - |L|| \leq |a_n - L| < \epsilon$. This shows $\lim |a_n| = |L|$.

(28) Can a sequence of positive real numbers converge to a negative number? Can a sequence of positive real numbers converge to a number that is not positive? Justify your claims.

Sample Solution: No, a sequence of positive real numbers cannot converge to a negative number. To see this, let (a_n) be an arbitrary sequence of real numbers, and suppose (a_n) converges to the negative real number L. Fix $\epsilon = |L|$, along with $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. Then $a_n < 0$ for all $n \geq N$, establishing our claim. On the other hand, a sequence of positive real numbers *can* converge to a number that is not positive: for instance, the sequence $(\frac{1}{n})$ converges to zero.

(29) Prove that if $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim a_n b_n = -\infty$.

Sample Solution: Suppose $\lim a_n = \infty$ and $\lim b_n = -\infty$. Let $M > 0$ be arbitrary. Using $\lim a_n = \infty$, fix $N_1 \in \mathbb{N}$ such that $a_n > 1$ whenever $n \geq N_1$, and using $\lim b_n = -\infty$, fix $N_2 \in \mathbb{N}$ such that $b_n < -M$ whenever $n \geq N_2$. Let $N = \max(N_1, N_2)$, and suppose $n \geq N$. Then $a_n > 1$ and $b_n < -M$, so $a_n b_n < -M$. Since $M > 0$ was arbitrary, this shows $\lim a_n b_n = -\infty$.

(30) Prove that if $\lim a_n = L \in \mathbb{R}$ and $\lim b_n = \infty$, then $\lim(a_n - b_n) = -\infty$.

Sample Solution: Suppose $\lim a_n = L \in \mathbb{R}$ and $\lim b_n = \infty$. Let $M \in \mathbb{R}$ be arbitrary. Using $\lim b_n = \infty$, fix $N_1 \in \mathbb{N}$ such that $b_n > L + 1 - M$ whenever $n \geq N_1$. Using $\lim a_n = L$, fix $N_2 \in \mathbb{N}$ such that $|a_n - L| < 1$ whenever $n \geq N_2$. Let $N = \max(N_1, N_2)$, and suppose $n \geq N$. Then $a_n < L + 1$ and $b_n > L + 1 - M$, so $a_n - b_n < L + 1 - (L + 1 - M) = M$. Since $M \in \mathbb{R}$ was arbitrary, this shows $\lim(a_n - b_n) = -\infty$.

(31) Let (a_n) be a sequence in $\mathbb{R}$. Prove in detail that (a_n) converges *iff* some tail of (a_n) converges *iff* every tail of (a_n) converges. [Hint: there is a "logically efficient" way of proving these implications; can you find it?]

Sample Solution: Let (a_n) be a sequence in $\mathbb{R}$. Since (a_n) is a tail of itself, the implications

"*every* tail of (a_n) converges $\implies$ (a_n) converges $\implies$ *some* tail of (a_n) converges"

are trivial, so it suffices to prove the remaining implication: if *some* tail of (a_n) converges, then *every* tail of (a_n) converges. So assume some tail of (a_n) converges, fix N such that the sequence (b_n) defined by $b_n = a_{N+n}$ converges (say, to $L \in \mathbb{R}$), and let $(c_n) = (a_{M+n})$ be an arbitrary tail of (a_n). Let $\epsilon > 0$, and fix $K \in \mathbb{N}$ such that for all $n \geq K$, $|b_n - L| < \epsilon$. Then for all $n \geq K + N$, we have $|c_n - L| = |a_{M+n} - L| = |b_{M+n-N} - L| < \epsilon$, since $M + n - N \geq n - N \geq K$. This shows that $\lim c_n = L$, as desired.

(32) *Let (a_n) and (b_n) be two sequences such that for all $n \in \mathbb{N}$, we have $a_n < b_n$ if n is even and $a_n > b_n$ if n is odd. Prove that if (a_n) and (b_n) both converge, then $\lim a_n = \lim b_n$.

Sample Solution: Suppose $\lim a_n = L$ and $\lim b_n = M$. Then $\lim a_{2n} = \lim(a_{2n-1}) = L$ and $\lim b_{2n} = \lim(b_{2n-1}) = M$. Since $a_{2n} < b_{2n}$ for all n we have $L \leq M$, and since $a_{2n-1} > b_{2n-1}$ for all n, we have $L \geq M$. Thus $L = M$. [Note: this proof uses subsequences, which we will meet soon; a direct proof without using subsequences is possible, but harder!]

(33) Prove that if $a_n \leq b_n$ for all n and $\lim a_n = \infty$, then also $\lim b_n = \infty$.

Sample Solution: Suppose $a_n \leq b_n$ for all n and that $\lim a_n = \infty$. Let $M > 0$ be arbitrary, and using $\lim a_n = \infty$ fix N such that for all $n \geq N$ we have $a_n > M$. then $b_n \geq a_n > M$ for all $n \geq N$, which shows $\lim b_n = \infty$.

(34) In lecture we showed that if (a_n) and (b_n) are convergent sequences of real numbers for which $a_n \leq b_n$ for all n, then $\lim a_n \leq \lim b_n$. Can these nonstrict inequalities be replaced by strict ones? That is, if (a_n) and (b_n) are convergent sequences of real numbers for which $a_n < b_n$ for all n, does it necessarily follow that $\lim a_n < \lim b_n$?

Sample Solution: No, if (a_n) and (b_n) converge and $a_n < b_n$ for all n, it does not necessarily follow that $\lim a_n < \lim b_n$. For instance, consider the sequences $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$ for $n \geq 2$.

(35) Prove the Squeeze Theorem directly using the definition of limit, but without using lim inf and lim sup. (The Squeeze Theorem says: if $a_n \le s_n \le b_n$ for all n and $\lim a_n = \lim b_n = L \in \mathbb{R}$, then $\lim s_n = L$.)

Sample Solution: Suppose $a_n \le s_n \le b_n$ for all n, and suppose $\lim a_n = \lim b_n = L \in \mathbb{R}$. We will show $\lim s_n = L$. Let $\epsilon > 0$, and fix $N_1, N_2 \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$ whenever $n \ge N_1$ and $|b_n - L| < \frac{\epsilon}{2}$ whenever $n \ge N_2$. Let $N = \max(N_1, N_2)$ and suppose $n \ge N$. Then

$$L - \frac{\epsilon}{2} \;<\; a_n \;\le\; s_n \;\le\; b_n \;<\; L + \frac{\epsilon}{2},$$

so $|s_n - L| < \epsilon$. This shows $\lim s_n = L$.

(36) Find the lim inf and lim sup of the sequences whose nth terms are given as follows:

(a) $2^{n(-1)^n}$

(b) $1 + (-1)^n(1 - \frac{1}{n})$

(c) $\sin\left(\frac{\pi n}{3}\right)\cos\left(\frac{\pi n}{4}\right)$

Sample Solution:

(a) $\liminf = 0$ and $\limsup = +\infty$.

(b) $\liminf = 0$ and $\limsup = 2$.

(c) $\liminf = -\frac{\sqrt{3}}{2}$, $\limsup = \frac{\sqrt{3}}{2}$ (the other subsequential limits are 0 and $\pm\frac{\sqrt{6}}{4}$)

(37) Prove that for every sequence (a_n) in $\mathbb{R}$, if $\lim a_n = \infty$ then $\liminf a_n = \infty$ and $\limsup a_n = \infty$.

Sample Solution: Let (a_n) be a sequence in $\mathbb{R}$, and suppose $\lim a_n = \infty$. Then by definition we have $\liminf a_n = \infty$. Furthermore, if $\lim a_n = \infty$ then (a_n) is unbounded above, so by definition $\limsup(a_n) = \infty$.

(38) Prove that for all $L \in \mathbb{R}$ and for every bounded sequence (a_n) in $\mathbb{R}$, $\limsup(a_n) = L$ if and only if for every $\epsilon > 0$ the set $\{n \in \mathbb{N} : a_n > L - \epsilon\}$ is infinite and $\{n \in \mathbb{N} : a_n > L + \epsilon\}$ is finite.

Sample Solution: Let (a_n) be a bounded sequence in $\mathbb{R}$ and let $L \in \mathbb{R}$. Suppose first that $\limsup(a_n) = L$, and let $\epsilon > 0$. Then by definition of lim sup, we can fix $N \in \mathbb{N}$ such that for all $n \ge N$,

$$L - \epsilon \;<\; \sup\{a_k : k \ge n\} \;<\; L + \epsilon.$$

Then in particular we have $a_n < L + \epsilon$ for all $n \ge N$, so the set $\{n \in \mathbb{N} : a_n > L + \epsilon\}$ is indeed finite. On the other hand, if the set $\{n \in \mathbb{N} : a_n > L - \epsilon\}$ were finite, then we could fix $N' \ge N$ large enough so that $\sup\{a_k : k \ge N'\} \le L - \epsilon$, a contradiction.

Now for the converse, suppose that for every $\epsilon > 0$ the set $\{n \in \mathbb{N} : a_n > L - \epsilon\}$ is infinite and $\{n \in \mathbb{N} : a_n > L + \epsilon\}$ is finite. Let $\epsilon > 0$, and, using the fact that $\{n \in \mathbb{N} : a_n > L + \frac{\epsilon}{2}\}$ is finite, fix $N \in \mathbb{N}$ large enough so that $a_n \le L + \frac{\epsilon}{2} < L + \epsilon$ for all $n \ge N$. Let $n \ge N$, so we have $\sup\{a_k : k \ge N\} < L + \epsilon$. But since $\{m \in \mathbb{N} : a_m > L - \epsilon\}$ is infinite, there must be $n' \ge n$ such that $a_{n'} > L - \epsilon$, so $\sup\{a_k : k \ge n\} > L - \epsilon$. We have now show that for all $n \ge N$,

$$L - \epsilon \;<\; \sup\{a_k : k \ge n\} \;<\; L + \epsilon.$$

We conclude that $\limsup(a_n) = L$, as desired.

(39) Evaluate the following limits:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

(b) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$

(c) (s_n), where $s_1 = 2$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{3}{s_n}\right)$ for each $n \in \mathbb{N}$.

Sample Solution:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^2 = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^2 = e^2.$

(b) Note that $e = \lim\left(1+\frac{1}{n}\right)^n = \lim\left(\frac{n+1}{n}\right)^n = \lim\left(\frac{n}{n-1}\right)^{n-1} = \lim\left(\frac{n}{n-1}\right)^n$. Since $1-\frac{1}{n} = \frac{n-1}{n}$, this implies $\lim\left(1-\frac{1}{n}\right)^n = \lim\left[\frac{1}{\left(\frac{n}{n-1}\right)}\right]^n = \lim\frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\lim\left(\frac{n}{n-1}\right)^n} = \frac{1}{e}$.

(c) Let us start by assuming the limit exists, say $\lim(s_n) = L \in \mathbb{R}$. Then

$$L = \lim_{n\to\infty}(s_n) = \lim_{n\to\infty}(s_{n+1}) = \lim_{n\to\infty}\frac{1}{2}\left(s_n + \frac{3}{s_n}\right) = \frac{1}{2}\cdot\left(\lim(s_n) + \frac{3}{\lim s_n}\right) = \frac{1}{2}\left(L + \frac{3}{L}\right).$$

Thus $2L = L + \frac{3}{L}$, so $2L^2 = L^2 + 3$, and therefore $L = \pm\sqrt{3}$. Since (s_n) is bounded below by 0, this implies $L = \sqrt{3}$ if (s_n) converges. But it can be shown by induction* that (s_n) is decreasing, and therefore it converges since it is bounded.

*This is actually more difficult than I had realized. Here is one argument: first show by induction that $s_n > \sqrt{3}$ for all n, where for the inductive step $s_n > \sqrt{3}$ implies $0 < (s_n - \sqrt{3})^2 = s_n^2 - 2\sqrt{3}s_n + 3$, so $2\sqrt{3}s_n < s_n^2 + 3$, which implies $\sqrt{3} < \frac{s_n^2+3}{2s_n} = s_{n+1}$. Then for all n we have

$$\frac{s_{n+1}}{s_n} = \frac{\frac{1}{2}(s_n + \frac{3}{s_n})}{s_n} = \frac{1}{2}\left(1 + \frac{3}{s_n^2}\right) < 1,$$

so $s_{n+1} < s_n$.

(40) Go back and prove (32), now that you know about subsequences.

Sample Solution: Suppose $\lim a_n = L$ and $\lim b_n = M$. Then $\lim a_{2n} = \lim(a_{2n-1}) = L$ and $\lim b_{2n} = \lim(b_{2n-1}) = M$. Since $a_{2n} < b_{2n}$ for all n we have $L \leq M$, and since $a_{2n-1} > b_{2n-1}$ for all n, we have $L \geq M$. Thus $L = M$.

(41) Suppose (a_n) is a bounded sequence in $\mathbb{R}$. Prove that (a_n) diverges if and only if (a_n) has two subsequences that converge to different limits.

Sample Solution: We know from lecture that if (a_n) converges, say to L, then every subsequence of (a_n) also converges to L; this is the contrapositive of the backward implication. For the forward implication, suppose (a_n) diverges. Since (a_n) is bounded, both $\liminf(a_n)$ and $\limsup(a_n)$ are real numbers. If $\liminf(a_n) = \limsup(a_n)$, then we know from lecture that (a_n) converges to their common value, so we must have $\liminf(a_n) < \limsup(a_n)$. But we also know that (a_n) has a subsequence converging to $\liminf(a_n)$ and a subsequence converging to $\limsup(a_n)$, completing the proof.

(42) Prove that if $\lim_{n\to\infty} a_n = L$, then for every bijection $\pi : \mathbb{N} \to \mathbb{N}$, $\lim_{n\to\infty} a_{\pi(n)} = L$. (Is this still true if we replace the word *bijection* with *injection*? How about *surjection*?

Sample Solution: Suppose $\lim a_n = L$, and let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection. Let $\epsilon > 0$, and fix $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq N$. The set $\{\pi^{-1}(n) : n < N\}$ is finite, so fix $N' \in \mathbb{N}$ such that $N' > \pi^{-1}(n)$ for all $n < N$. Let $n \geq N'$. Then $\pi(n) \geq N$, so $|a_{\pi(n)} - L| < \epsilon$. This shows $\lim_{n\to\infty} a_{\pi(n)} = L$.

(43) Let (a_n) be a sequence of real numbers. Prove that if every subsequence of (a_n) diverges, then for all $M > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \notin [-M, M]$.

Sample Solution: We prove the contrapositive. Suppose there is $M > 0$ such that $a_n \in [-M, M]$ for infinitely many $n \in \mathbb{N}$. If we let (a_{n_k}) be the subsequence of (a_n) consisting of all the terms in $[-M, M]$, then (a_{n_k}) is a bounded sequence and thus it has a convergent subsequence $(a_{n_{k_\ell}})$ by Bolzano-Weierstrass. But then $(a_{n_{k_\ell}})$ is a convergent subsequence of (a_n), completing the proof.

(44) (a) Prove that if U and V are open subsets of $\mathbb{R}$, then $U \cap V$ is also open.
(b) Prove that if U are V are open subsets of $\mathbb{R}$, then $U \cup V$ is also open.

Sample Solution: Let U and V be open subsets of $\mathbb{R}$.
(a) Let $x \in U \cap V$. Fix $\epsilon_U > 0$ and $\epsilon_V > 0$ such that $(x - \epsilon_U, x + \epsilon_U) \subseteq U$ and $(x - \epsilon_V, x + \epsilon_V) \subseteq V$. Let $\epsilon = \min(\epsilon_U, \epsilon_V)$. Then $(x - \epsilon, x + \epsilon) \subseteq U \cap V$. This shows that $U \cap V$ is open.
(b) Let $x \in U \cup V$, and wlog say $x \in U$. Fix $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. Then $(x-\epsilon, x+\epsilon) \subseteq U \cup V$. This shows that $U \cup V$ is open.

(45) Is the previous problem still true if you replace "open" with "closed"?

Sample Solution: Yes! This follows from DeMorgan's Laws: for instance, if A and B are closed subsets of $\mathbb{R}$, so that $\mathbb{R} \setminus A$ and $\mathbb{R} \setminus B$ are open, then $\mathbb{R} \setminus (A \cap B) = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$ is open by (b), so $A \cap B$ is closed.

(46) Prove that if the subset $C \subseteq \mathbb{R}$ is closed and bounded, then every sequence (a_n) in C has a subsequence that converges to a limit in C.

Sample Solution: Suppose $C \subseteq \mathbb{R}$ is closed and bounded, and let (a_n) be a sequence in C. By Bolzano-Weierstrass, (a_n) has a convergent subsequence, say (a_{n_k}) with limit L. But C is closed, so $L \in C$ by the theorem from lecture.

(47) Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Prove that $a_k > 0$ for infinitely many k and $a_k < 0$ for infinitely many k.

Sample Solution: Suppose $\sum a_k$ is conditionally convergent. If $a_k < 0$ for at most finitely many k, then $\sum a_k$ and $\sum |a_k|$ have a tail in common, so they both either converge or diverge, contradicting our assumption that $\sum a_k$ converges conditionally; thus $a_k < 0$ for infinitely many k. By symmetry, it follows that $a_k > 0$ for infinitely many k as well.

(48) Suppose $a_k \geq 0$ for all k, and let $f : \mathbb{N} \to \mathbb{N}$ be any bijection. For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} a_{f(k)}$. Prove that $\sup\{s_n : n \in \mathbb{N}\} = \sup\{t_n : n \in \mathbb{N}\}$.

Sample Solution: Since f^{-1} is also a bijection, by symmetry it will suffice to show that $\sup\{s_n : n \in \mathbb{N}\} \geq \sup\{t_n : n \in \mathbb{N}\}$. Let $n \in \mathbb{N}$ and consider $t_n = \sum_{k=1}^{n} a_{f(k)}$. Fix $m \geq n$ such that $m \geq f(k)$ for all $1 \leq k \leq n$. Then $\{f(k) : k \leq n\} \subseteq \{k \in \mathbb{N} : 1 \leq k \leq m\}$, so

$$t_n = \sum_{k=1}^{n} a_{f(k)} \leq \sum_{k=1}^{m} a_k = s_m \leq \sup\{s_n : n \in \mathbb{N}\}$$

since each $a_k \geq 0$. Thus $t_n \leq \sup\{s_n : n \in \mathbb{N}\}$ for every $n \in \mathbb{N}$, which shows $\sup\{t_n : n \in \mathbb{N}\} \leq \sup\{s_n : n \in \mathbb{N}\}$ as desired.

(49) Let $\sum_{k=1}^{\infty} a_k$ be an infinite series of real numbers, and let (t_k) be a strictly increasing sequence of natural numbers such that $t_1 = 1$. For each $n \in \mathbb{N}$ let $b_n = \displaystyle\sum_{k=t_n}^{t_{n+1}-1} a_k$. (Write out a simple example to understand what is going on here, and how $\sum a_k$ and $\sum b_n$ are related to each other.)

(a) Supposing that $\sum a_k$ converges, show that $\sum b_n$ also converges and that $\sum a_k = \sum b_n$.

(b) Show by example that $\sum b_n$ could converge even if $\sum a_k$ does not converge.

Sample Solution:

(a) Suppose $\sum a_k$ converges, say $\sum a_k = L \in \mathbb{R}$. Let $s_n = \sum_{k=1}^{n} a_k$, and let $u_n = \sum_{k=1}^{n} b_k$. We must show $\lim u_n = L$. Let $\epsilon > 0$, and fix $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ whenever $n \geq N$. Suppose $n \geq N$. Note that $n \leq t_n \leq t_{n+1} - 1$ since (t_k) is strictly increasing. Thus

$$|u_n - L| = \left| \sum_{k=1}^{n} b_k - L \right| = \left| \sum_{k=1}^{t_{n+1}-1} a_k - L \right| = |s_{t_{n+1}-1} - L| < \epsilon,$$

which shows $\lim u_n = L$ as desired.

(b) For instance, let $a_k = (-1)^k$ and $t_k = 2k - 1$ for each $k \in \mathbb{N}$. Then $\sum a_k$ diverges but $b_n = 0$ for each n, so $\sum b_n$ converges.

(50) Determine whether the following infinite series converge or diverge, with justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \sin(n)}{n^3 + 3}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{e^n}$

(c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n \ln(n)^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \sin(n)$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 7}$

(g) $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi) \ln\left(1 + \frac{1}{n}\right)$

(h) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$

(i) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^5 - 2n^2 + 1}{4n^5 + 9n^4 + \sqrt{n}} \right)^n$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$ (k) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ (l) $\displaystyle\sum_{n=1}^{\infty} \sin(e^{-n})$

Sample Solution:

(a) Diverges, by the Limit Comparison Test.
(b) Diverges, by the Ratio Test (or nth Term Test).
(c) Converges, by the Integral Test.
(d) Diverges, by the nth Term Test.
(e) Convergent geometric series.
(f) Converges, by Comparison with a convergent p-series.
(g) Converges, by the Alternating Series Test.
(h) Diverges, by the Limit Comparison Test.
(i) Converges, by the Root Test.
(j) Diverges, by the Ratio Test.
(k) Converges, by the Integral Test (twice), or by Comparison Test (eg, with $\frac{1}{n\ln(n)^2}$)
(l) Converges, by Comparison with a convergent geometric series.

(51) Of the series from the previous problem that converge, which ones (if any) converge conditionally?

Sample Solution: Only (g) converges conditionally. To see that $\sum \ln(1+\frac{1}{n})$ diverges, note that the partial sums are

$$s_M = \sum_{n=1}^{M} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{M} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{M} (\ln(n+1) - \ln(n)) = \ln(M+1) - \ln(1) = \ln(M+1).$$

(52) Find the limit points of the following subsets of $\mathbb{R}$:

(a) $\{0, 1\}$
(b) $(0, 1)$
(c) $[0, 1]$
(d) $\{m \pm \frac{1}{n} : m, n \in \mathbb{N}\}$
(e) $\bigcup_{n\in\mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$
(f) $\{\frac{m}{n} : m \in \mathbb{Z}$ and $n = 2^k$ for some $k \in \mathbb{N}\}$
(g) the set of transcendental real numbers
(h) the set of partial sums of the harmonic series

Sample Solution:

(a) $\emptyset$
(b) $[0, 1]$
(c) $[0, 1]$
(d) $\mathbb{N}$
(e) $[0, 1]$
(f) $\mathbb{R}$
(g) $\mathbb{R}$
(h) $\emptyset$

(53) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. We call c a *closure point* of A if $c \in \mathrm{cl}(A) = A \cup A'$, where A' is the set of all limit points of A.

(a) Show that c is a limit point of A iff there is a sequence (a_n) in $A \setminus \{c\}$ converging to c.
(b) Show that c is a closure point of A iff there is a sequence (a_n) in A converging to c.

Sample Solution:

(a) For the forward implication, suppose c is a limit point of A. For each $n \in \mathbb{N}$, choose $a_n \in \big(A \cap V_{\frac{1}{n}}(c)\big) \setminus \{c\}$. Then (a_n) is a sequence in $A \setminus \{c\}$ that converges to c. For the reverse implication, let (a_n) be a sequence in $A \setminus \{c\}$ that converges to c, and let $\epsilon > 0$. Since $a_n \to c$, we can fix n such that $a_n \in V_\epsilon(c)$. Since (a_n) is a sequence in $A \setminus \{c\}$, we know $a_n \neq c$. This completes the proof.

(b) For the forward implication, suppose $c \in A \cup A'$ is a closure point of A. If $c \in A$, then of course there is a sequence in A converging to c, namely the constant sequence with value c, and if $c \in A'$ then there is a sequence in A converging to c by part (a). Conversely, suppose (a_n) is a sequence in A converging to c. If $c \in A$ then $c \in \mathrm{cl}(A)$ by definition, and if $c \notin A$ then (a_n) is a sequence in $A \setminus \{c\}$ converging to c, so $c \in A' \subseteq \mathrm{cl}(A)$ by part (a).

(54) Let (a_n) be a sequence in $\mathbb{R}$ and let $A = \{a_n \,:\, n \in \mathbb{N}\}$.

(a) Show that every limit point of A is a subsequential liimit of (a_n).

(b) Show by example that not every subsequential limit of (a_n) need be a limit point of A.

Sample Solution:

(a) Let L be a limit point of A. Then for every $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $0 < |a_n - L| < \epsilon$. So we can define inductively a subsequence (a_{n_k}) of (a_n) by setting $n_1 = 1$ and then given n_k choosing $n_{k+1} > n_k$ such that $0 < |a_{n_{k+1}} - L| < \frac{1}{k+1}$. Then (a_{n_k}) is a subsequence of (a_n) that converges to L, so L is a subsequential limit of (a_n).

(b) If we let (a_n) be the constant sequence $a_n = L$ for all n, then L is a subsequential limit of (a_n) but the set $A = \{L\}$ has no limit points.

(55) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, let $c \in \mathbb{R}$, and suppose $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} f(x) = M$. Prove directly, without using sequences, that $\lim\limits_{x \to c} \big(f(x) + g(x)\big) = L + M$.

Sample Solution: Let $\epsilon > 0$. Using $\lim\limits_{x \to c} f(x) = L$, fix $\delta_1 > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta_1$, and using $\lim\limits_{x \to c} f(x) = M$, fix $\delta_2 > 0$ such that $|f(x) - M| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, and suppose $0 < |x - c| < \delta$. Then

$$|f(x) + g(x) - (L + M)| \;=\; |(f(x) - L) + (g(x) - M)| \;\leq\; |f(x) - L| + |g(x) - M| \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

(56) State the precise ϵ/δ-style definitions of the following:

(a) $\lim\limits_{x \to c^-} f(x) = +\infty$

(b) $\lim\limits_{x \to \infty} f(x) = -\infty$

Sample Solution:

(a) For every $M \in \mathbb{R}$ there exists $\delta > 0$ such that $f(x) > M$ whenever $x \in \mathrm{dom}(f)$ and $0 < c - x < \delta$.

(b) For every $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that $f(x) < M$ whenever $x \in \mathrm{dom}(f)$ and $x > N$.

(57) Find the following limits[4]:

(a) $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$

(b) $\lim_{x\to 3} \frac{x^2+x-6}{x-3}$

(c) $\lim_{x\to 3^-} \frac{x^2+x-6}{x-3}$

(d) $\lim_{x\to 3} \frac{x^2+x-6}{(x-3)^2}$

(e) $\lim_{x\to -\infty} e^x$

(f) $\lim_{x\to \infty} e^x$

(g) $\lim_{x\to 0^+} \ln x$

(h) $\lim_{x\to 1} \ln x$

(i) $\lim_{x\to \infty} \ln x$

(j) $\lim_{x\to 0} \frac{\sin x}{x}$

(k) $\lim_{x\to 0} \frac{\sin x}{|x|}$

(l) $\lim_{x\to 0^-} \frac{\sin x}{|x|}$

(m) $\lim_{x\to 0^+} \frac{\sin x}{|x|}$

Sample Solution:

(a) $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = 5$

(b) $\lim_{x\to 3} \frac{x^2+x-6}{x-3}$ DNE

(c) $\lim_{x\to 3^-} \frac{x^2+x-6}{x-3} = -\infty$

(d) $\lim_{x\to 3} \frac{x^2+x-6}{(x-3)^2} = +\infty$

(e) $\lim_{x\to -\infty} e^x = 0$

(f) $\lim_{x\to \infty} e^x = \infty$

(g) $\lim_{x\to 0^+} \ln x = -\infty$

(h) $\lim_{x\to 1} \ln x = 0$

(i) $\lim_{x\to \infty} \ln x = \infty$

(j) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(k) $\lim_{x\to 0} \frac{\sin x}{|x|}$ DNE

(l) $\lim_{x\to 0^-} \frac{\sin x}{|x|} = -1$

(m) $\lim_{x\to 0^+} \frac{\sin x}{|x|} = 1$

(58) Define the function $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = \begin{cases} x^3 - 2x^2 & \text{if } x \in \mathbb{Q}; \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Find all points $c \in \mathbb{R}$ for which $\lim_{x\to c} f(x)$ exists, and for each such point c, find $\lim_{x\to c} f(x)$.

Sample Solution: $\lim_{x\to c} f(x)$ exists if and only if $c^3 - 2c^2 = c$, i.e., if and only if $c \in \{0, 1 \pm \sqrt{2}\}$. In each of these three cases, the limit of f at c is c.

(59) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let $c \in \mathbb{R}$ be a limit point of A, and let $L \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(c+h) = L$.

Sample Solution: Assume the hypotheses, and for the forward direction assume $\lim_{x\to c} f(x) = L$. Let $\epsilon > 0$. Fix $\delta > 0$ such that for all $x \in A$, if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$. Let h be such that $c + h \in A$ and $0 < |h| < \delta$. Then $0 < |(c + h) - c| < \delta$, so $|f(c + h) - L| < \epsilon$.

For the converse, assume $\lim_{h\to 0} f(c+h) = L$ and let $\epsilon > 0$. Fix $\delta > 0$ such that for all h satisfying $c + h \in A$ and $0 < |h| < \delta$, we have $|f(c + h) - L| < \epsilon$. Let $x \in A$ and suppose $0 < |x - c| < \delta$. Then

[4] This means find the limit, if it exists, or else state that the limit is equal to $+\infty$ or $-\infty$, if that is the case, or else state that the limit does not exist. Note that since $\pm\infty$ are not real numbers, cases where the limit is equal to $\pm\infty$ are cases where the limit "does not exist;" however, in these cases, the limit fails to exist in a particular way and it is better to say that the limit is $+\infty$ or $-\infty$ rather than just saying that it does not exist, since doing so provides more information. Note that, as always in this class, "∞" means the same thing as "$+\infty$."

$x-c$ satisfies $c+(x-c) \in A$ and $0<|x-c|<\delta$, so we must have $|f(x)-L| = |f(c+(x-c))-L| < \epsilon$, completing the proof.

(60) *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be functions, and let $a,b,L\in\mathbb{R}$. Assuming that $\lim_{x\to a} f(x)=b$ and $\lim_{x\to b} g(x)=L$, does it necessarily follow that $\lim_{x\to a} g(f(x))=L$?

Sample Solution: No, it does not! Finding a counterexample is a problem on HW 4.

(61) Let (X,d) be a metric space and let $F\subseteq X$. Prove directly from the definitions that F is closed if and only if every limit point of F in X belongs to F.

Sample Solution: For the forward direction, suppose $F'\not\subseteq F$, and fix $c\in F'\setminus F$. Since $c\in F'$, we have $V_\epsilon(c)\cap F\neq\emptyset$ for every $\epsilon>0$, which shows that $X\setminus F$ is not open, so F is not closed. Conversely, suppose $F'\subseteq F$. Let $x\in X\setminus F$. Then $x\notin F'$, so we can fix $\epsilon>0$ such that $V_\epsilon(x)\subseteq X\setminus F$. This shows $X\setminus F$ is open, so F is closed.

(62) Let $A\subseteq\mathbb{R}$, let $a\in A$, and let $f:A\to\mathbb{R}$ be a function. Show that f is continuous at a if and only if for every open neighborhood V of $f(a)$ there is an open neighborhood U of a such that $f[U\cap A]\subseteq V$.

Sample Solution: For the forward direction, suppose f is continuous at a, and let V be an open neighborhood of $f(a)$. Fix $\epsilon>0$ such that $V_\epsilon(f(a))\subseteq V$. Using continuity, fix $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $x\in A$ and $|x-a|<\delta$. Then $f[V_\delta(a)\cap A]\subseteq V_\epsilon(f(a))\subseteq V$, so we can let $U=V_\delta(a)$.

For the backward direction, assume that for every open neighborhood V of $f(a)$ there is an open neighborhood U of a such that $f[U\cap A]\subseteq V$, and let $\epsilon>0$. Using our assumption, obtain an open neighborhood U of a such that $f[U\cap A]\subseteq V_\epsilon(f(a))$, and fix $\delta>0$ such that $V_\delta(a)\subseteq U$. Let $x\in A$ and suppose $|x-a|<\delta$. Then $x\in V_\delta(a)\subseteq U$, so $f(x)\in V_\epsilon(f(a))$, which means $|f(x)-f(a)|<\epsilon$, as desired.

(63) Prove directly (using the ϵ/δ definition and without using sequences) that if $f,g:\mathbb{R}\to\mathbb{R}$ are continuous at $a\in\mathbb{R}$, then also fg is continuous at a.

Sample Solution: Let $f,g:\mathbb{R}\to\mathbb{R}$ be functions that are continuous at $a\in\mathbb{R}$. Let $\epsilon>0$. Fix $\delta_1>0$ such that $|g(x)-g(a)|<1$ whenever $|x-a|<\delta_1$. Then fix $\delta_2>0$ and $\delta_3>0$ such that $|g(x)-g(a)|<\frac{\epsilon}{2(|f(a)|+1)}$ whenever $|x-a|<\delta_2$ and $|f(x)-f(a)|<\frac{\epsilon}{2(|g(a)|+1)}$ whenever $|x-a|<\delta_3$. Let $\delta=\min(\delta_1,\delta_2,\delta_3)$, and suppose $|x-a|<\delta$. Then

$$\begin{aligned}
|f(x)g(x)-f(a)g(a)| &= \big|\big(f(x)-f(a)\big)g(x)+f(a)\big(g(x)-g(a)\big)\big| \\
&\leq |g(x)||f(x)-f(a)|+|f(a)||g(x)-g(a)| \\
&< (|g(a)|+1)\cdot\frac{\epsilon}{2(|g(a)|+1)}+|f(a)|\cdot\frac{\epsilon}{2(|f(a)|+1)} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

This shows that fg is continuous at a, as desired.

(64) Note that the square root function $f(x) = \sqrt{x}$ is continuous at $x = 9$. Given $0 < \epsilon < 1$, find the largest $\delta > 0$ such that $|\sqrt{x} - 3| < \epsilon$ whenever $|x - 9| < \delta$.

Sample Solution: $\delta = 9 - (3 - \epsilon)^2$.

(65) A function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* if for all $x, y \in \mathbb{R}$, $x \leq y$ implies $f(x) \leq f(y)$, *decreasing* if for all $x, y \in \mathbb{R}$, $x \leq y$ implies $f(x) \geq f(y)$, and *monotone* if it is either increasing or decreasing.

(a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone and f is discontinuous at $a \in \mathbb{R}$, then f has a jump discontinuity at a.

(b) Prove that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has at most countably many discontinuities.

Sample Solution:

(a) Wlog suppose f is increasing, and assume f has a discontinuity at a. Let $L^- = \sup\{f(x) : x < a\}$ and let $L^+ = \inf\{f(x) : x > a\}$. We claim $L^- = \lim_{x \to a^-} f(x)$ $\big($and, dually, $L^+ = \lim_{x \to a^+} f(x)\big)$. To see this, let $\epsilon > 0$ be arbitrary, and fix $c < a$ such that $f(c) > L^- - \epsilon$. Let $\delta = a - c$. Then since f is increasing, for all $x \in (a - \delta, a)$ we have $L^- - \epsilon > f(c) \leq f(x) \leq L^-$. This shows $L^- = \lim_{x \to a^-} f(x)$, as claimed, and dually we have $L^+ = \lim_{x \to a^+} f(x)$. Thus both one-sided limits of f at a exist, so f has a jump discontinuity at a.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function (wlog increasing), and let $D = \{a \in \mathbb{R} : f \text{ is discontinuous at } a\}$. By part (a), for each $a \in D$ we have that

$$L^-(a) := \sup\{f(x) : x < a\} = \lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x) = \inf\{f(x) : x > a\} =: L^+(a).$$

Using density of $\mathbb{Q}$, for each $a \in D$ chose $q_a \in \mathbb{Q} \cap \big(L^-(a), L^+(a)\big)$. Since f is increasing, we have that $L^+(a) \leq L^-(b)$ for all $a < b$ in D. Thus the map $a \mapsto q_a$ from D to $\mathbb{Q}$ is injective, which shows that D is countable as desired.

(66) *Define *Thomae's function* $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ where } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ are coprime;} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

(a) Prove that for every $x \in \mathbb{Q}$, T is discontinuous at x.

(b) Prove that for every $x \in \mathbb{R} \setminus \mathbb{Q}$, T is continuous at x.

Sample Solution:

(a) Let $x \in \mathbb{Q}$, so $T(x) > 0$. Since the irrationals are dense in $\mathbb{R}$, we can find a sequence (x_n) of irrationals converging to x. But then $T(x_n) = 0$ for each n, so $T(x_n) \not\to T(x)$, and thus T is not continuous at x.

(b) Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and let $\epsilon > 0$. Fix $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then the set

$$S = \left\{\frac{m}{n} : n \in \mathbb{N} \text{ and } n < N \text{ and } m \in \mathbb{Z}\right\} \cap [x - 1, x + 1]$$

is finite, so we can let $\delta = \min\{|x - q| : q \in S\}$. Then $|T(y)| < \epsilon$ whenever $|x - y| < \delta$, which shows that T is continuous at x.

(67) Give an example of a bounded continuous function on $(0,1)$ that is not uniformly continuous.

Sample Solution: For instance, $y = \sin \frac{1}{x}$.

(68) In each part, determine whether the given function is uniformly continuous on the given set. (No rigorous proofs required.)

(a) $y = x^{3/5}$ on $\mathbb{R}$

(b) $y = x^{5/3}$ on $\mathbb{R}$

(c) $y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $y = \ln x$ on $[1, \infty)$

(e) $y = \ln x$ on $(0, 1]$

(f) $y = \sin x^2$ on $\mathbb{R}$

(g) $y = (\sin x)^2$ on $\mathbb{R}$

(h) $y = e^{-x} \sin x^2$ on $[0, \infty)$

Sample Solution:

(a) yes

(b) no

(c) no

(d) yes

(e) no

(f) no

(g) yes

(h) yes (note that the derivative is bounded)

(69) Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Then f is called *Lipschitz continuous* if there is $K \geq 0$ such that for all $x, y \in A$ we have $|f(x) - f(y)| \leq K|x - y|$.

(a) Prove that every Lipschitz continuous function is uniformly continous.

(b) Show by example that not every uniformly continuous function is Lipschitz continuous.

Sample Solution:

(a) Let $f : A \to \mathbb{R}$ and suppose $K \geq 0$ is such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in A$. If $K = 0$ then f is constant, hence uniformly continuous, so we may assume $K > 0$. Let $\epsilon > 0$, and set $\delta = \frac{\epsilon}{K}$. Let $x, y \in A$, and suppose $|x - y| < \delta$. Then $|f(x) - f(y)| \leq K|x - y| < K\delta = K \cdot \frac{\epsilon}{K} = \epsilon$.

(b) For instance, consider the square root function $f(x) = \sqrt{x}$ on $[0, 1]$. We claim that f is not Lipschitz continuous. To see this, let $K > 0$ be arbitrary, and let $x = \frac{1}{K^2}$ and $y = \frac{1}{(K+1)^2}$. Then

$$K|x - y| = \frac{K}{K^2(K+1)^2} = \frac{1}{K(K+1)^2} < \frac{1}{K(K+1)} = |f(x) - f(y)|,$$

so $|f(x) - f(y)| \not\leq K|x - y|$.

(70) Suppose the function $f : (0, 1) \to \mathbb{R}$ is continuous but not uniformly continuous. Show that at least one of the limits $\lim_{x \to 0^+} f(x)$ or $\lim_{x \to 1^-} f(x)$ does not exist.

Sample Solution: We prove the contrapositive: assuming that both $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist, we will show that f is uniformly continuous. Define the function $g : [0, 1] \to \mathbb{R}$ by setting $g(0) = \lim_{x \to 0^+} f(x)$, $g(1) = \lim_{x \to 1^-} f(x)$, and $g(x) = f(x)$ for all $x \in (0, 1)$. Then g is continuous on $[0, 1]$, so g is uniformly continuous on $[0, 1]$, which means that $f = g \restriction (0, 1)$ is uniformly continuous too, as desired.

(71) Let (a_n) be the sequence defined recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for all $n \in \mathbb{N}$. Show that (a_n) converges and prove that $\lim(a_n) < 2$.

Sample Solution: First we show by induction that (a_n) is strictly increasing. For the base, note that $a_1 = \sqrt{2} < \sqrt{2 + \sqrt[4]{2}} = a_2$. For the inductive step, let $n \in \mathbb{N}$ and assume $a_{n+1} > a_n$. Then $a_{n+2} = \sqrt{2 + \sqrt{a_{n+1}}} > \sqrt{2 + \sqrt{a_n}} = a_{n+1}$. Next we prove, again by induction, that $a_n < \sqrt{2 + \sqrt{2}}$ for all n. The base case $n = 1$ is clear, and assuming $a_n < \sqrt{2 + \sqrt{2}} < 2$, we have $a_{n+1} = \sqrt{2 + \sqrt{a_n}} < \sqrt{2 + \sqrt{2}}$ as desired. Thus (a_n) is bounded and increasing, so it converges, and since $a_n < \sqrt{2 + \sqrt{2}} < 2$ for all n, we have $\lim a_n < 2$.

(72) Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions, suppose f is bounded, and suppose $\lim\limits_{x \to -\infty} h(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = 0$. Prove directly that $\lim\limits_{x \to -\infty} [f(x) \cdot g(h(x))] = 0$.

Sample Solution: Let $\epsilon > 0$. Fix $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Fix $N_1 \in \mathbb{N}$ such that $|g(x)| < \frac{\epsilon}{M}$ whenever $x > N_1$. Fix $N_2 \in \mathbb{N}$ such that $h(x) > N_1$ whenever $x < -N_2$. Suppose $x < -N_2$. Then $h(x) > N_1$, so $g(h(x)) < \frac{\epsilon}{M}$, so

$$|f(x) \cdot g(h(x))| \;=\; |f(x)| \cdot |g(h(x))| \;<\; M \cdot \frac{\epsilon}{M} \;=\; \epsilon.$$

(73) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that if $f(x) \neq 0$ for all $x \in [0,1]$, then there is $\epsilon > 0$ such that either $f(x) < -\epsilon$ for all $x \in [0,1]$ or $\epsilon < f(x)$ for all $x \in [0,1]$.

Sample Solution: Let $f : [0,1] \to \mathbb{R}$ be continuous, and suppose $f(x) \neq 0$ for all $x \in [0,1]$. By the IVT, f either takes on only positive values or only negative values; wlog, say $f(x) > 0$ for all $x \in [0,1]$. By the EVT, we may fix $x_0 \in [0,1]$ such that $0 < f(x_0) \leq f(x)$ for all $x \in [0,1]$. So we set $\epsilon = \frac{f(x_0)}{2}$.

(74) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. We say that f *dominates* g if $g(x) < f(x)$ for all $x \in \mathbb{R}$. Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that neither one dominates the other, then $f(x) = g(x)$ for some $x \in \mathbb{R}$.

Sample Solution: Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and suppose that neither dominates the other, so there exist $a, b \in \mathbb{R}$ such that $f(a) \leq g(a)$ and $g(b) \leq g(a)$. Wlog we have $a < b$, $f(a) < g(a)$, and $g(b) < g(a)$. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. Then h is continuous, $h(a) < 0$, and $h(b) > 0$, so by the IVT there is $c \in (a,b)$ such that $h(c) = 0$, which implies $f(c) = g(c)$ as desired.

(75) Let $A \subseteq \mathbb{R}$, let $a \in A \cap A'$, and let $f : A \to \mathbb{R}$ be a function. Prove that f is differentiable at a if and only if there is a function $\varphi : A \to \mathbb{R}$ that is continuous at a and has the property that for all $x \in A$,

$$\varphi(x)(x - a) = f(x) - f(a).$$

Sample Solution: Assume the hypotheses. For the forward direction, define $\varphi : A \to \mathbb{R}$ by

$$\varphi(x) \;=\; \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \in A \setminus \{a\}; \\ f'(a) & \text{if } x = a. \end{cases}$$

Then φ is continuous at a since by definition of derivative we have

$$\varphi(a) \;=\; f'(a) \;=\; \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; \lim_{x\to a} \varphi(x).$$

Conversely, suppose φ is given as stated. Then by continuity of φ at a, we have

$$f'(a) \;=\; \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; \lim_{x\to a} \varphi(x) \;=\; \varphi(a),$$

so that f is indeed differentiable at a with $f'(a) = \varphi(a)$.

(76) Find the derivative of the function $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(x) \;=\; \frac{e^{\sin x^2}\left(x^{2/5} - \sqrt{x^2+1}\right)}{\cos(\ln(x))\, e^{e^x}}.$$

Sample Solution:

$$\left[\left(2x\cos(x^2)e^{\sin x^2}\left(x^{2/5} - \sqrt{x^2+1}\right) + e^{\sin x^2}\left(\frac{2}{5}x^{-3/5} - \frac{x}{\sqrt{x^2+1}}\right)\right)\cos(\ln x)e^{e^x} \right.$$
$$\left. - e^{\sin x^2}\left(x^{2/5} - \sqrt{x^2+1}\right)\left(\frac{-\sin(\ln x)}{x}e^{e^x} + \cos(\ln x)e^{e^x}e^x\right)\right] \Big/ \left(\cos(\ln(x))e^{e^x}\right)^2$$

(77) (a) Give an example of a function $f : (-1,1) \to \mathbb{R}$ that is C^1 but not twice-differentiable.
(b) Give an example of a function $f : (-1,1) \to \mathbb{R}$ that is twice-differentiable but not C^2.

Sample Solution:
(a) eg, $f(x) = x^3 \sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$.
(b) eg, $f(x) = x^4 \sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$.

(78) Suppose $f : (a,b) \to \mathbb{R}$ is differentiable. In the Increasing/Decreasing Test, we stated that:
(i) if $f'(x) \geq 0$ for all $x \in (a,b)$, then f is increasing on (a,b);
(ii) if $f'(x) \leq 0$ for all $x \in (a,b)$, then f is decreasing on (a,b);
(iii) if $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b);
(iv) if $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on (a,b);
For which of these statements is the converse true? Prove those that are true, and give counterexamples for those that can fail.

Sample Solution: The converses of (i) and (ii) hold, while the converses of (iii) and (iv) can fail. To see that the converses of (iii) and (iv) can fail, consider the functions $y = \pm x^3$. To prove the converse of (i), suppose f is increasing on (a,b) and let $x \in (a,b)$. Then, using the fact that $f(x) \leq f(y)$ for all $y \in (x,b)$, we have

$$f'(x) \;=\; \lim_{y\to x^+} \frac{f(y)-f(x)}{y-x} \;\geq\; 0.$$

The proof of the converse of (ii) is similar.

(79) Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Show that if f' is bounded on I, then f is uniformly continuous. Then show by example that the converse can fail.

Sample Solution: Fix $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Let $\epsilon > 0$, set $\delta = \frac{\epsilon}{M}$, and let $x, y \in I$ and suppose $|x - y| < \delta$. Using the MVT, fix c between x and y such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. Then $|f(x) - f(y)| = |f'(c)||x - y| < M\delta = \epsilon$. This shows that f is uniformly continuous on I. To see that the converse can fail, note that $y = \sqrt{x}$ is differentiable and uniformly continuous on $(0, 1)$ but has unbounded derivative on $(0, 1)$.

(80) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let $a \in A \cap A'$, and suppose f is differentiable at a. Show the following:

(a) If $f'(a) > 0$, then there is $\delta > 0$ such that for all $x, y \in A$, if $a - \delta < x < a < y < a + \delta$ then $f(x) < f(a) < f(y)$.

(b) If $f'(a) < 0$, then there is $\delta > 0$ such that for all $x, y \in A$, if $a - \delta < x < a < y < a + \delta$ then $f(x) > f(a) > f(y)$.

Sample Solution:

(a) Assume the hypotheses, and in particular suppose $f'(a) > 0$. Let $\epsilon = \frac{f'(a)}{2}$, and fix δ such that for all $x \in A$, if $0 < |x - a| < \delta$ then $\left|\dfrac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon$, so in particular we have $\frac{f(x)-f(a)}{x-a} > 0$. Let $x, y \in A$ and suppose $a - \delta < x < a < y < a + \delta$. Then since $x < a$ and $\frac{f(x)-f(a)}{x-a} > 0$, we must have $f(x) < f(a)$, and likewise since $a < y$ and $\frac{f(y)-f(a)}{y-a} > 0$ we must have $f(a) < f(y)$.

(b) Dual to the proof of (a).

(81) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, let $c \in \mathbb{R}$, and suppose $f'(c) > 0$.

(a) Prove that if f is C^1, then there is an open neighborhood of c on which f is injective.

(b) Show by example that the result in (a) can fail if we do not assume f is C^1.

Sample Solution:

(a) Using continuity of f' at c, fix $\delta > 0$ such that $|f'(x) - f'(c)| < \frac{f'(c)}{2}$ whenever $|x - c| < \delta$. Then $f'(x) > f'(c) - \frac{f'(c)}{2} = \frac{f'(c)}{2} > 0$ for all $x \in V_\delta(c)$, which implies that f is strictly increasing, and therefore injective, on $V_\delta(c)$.

(b) For instance, the function $f(x) = \frac{1}{2}x + \frac{1}{x^2}\sin\frac{1}{x}$ is differentiable at zero with $f'(0) = \frac{1}{2} > 0$, but there is no open neighborhood of 0 on which f is injective. To see this, let $\delta > 0$ be arbitrary, fix $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, and consider the points $x_1 = \frac{1}{2(n+1)\pi+\frac{\pi}{2}}$, $x_2 = \frac{1}{(2n+1)\pi}$, and $x_3 = \frac{1}{2\pi n+\frac{\pi}{2}}$ in $V_\delta(0)$. Then $x_1 < x_2 < x_3$, but $0 = f(x_2) < f(x_1) < f(x_3)$, so the IVT implies that f is not injective on $V_\delta(0)$.

(82) Let $f(x) = x - 12x^{1/3}$.

(a) Find the largest interval I containing 5 on which f is injective.

(b) Find $\big((f \restriction I)^{-1}\big)'(11)$.

(c) Find all points in the range of $f \restriction I$ at which $(f \restriction I)^{-1}$ is not differentiable.

Sample Solution:

(a) Note that $f'(x) = 1 - 4x^{-2/3}$ has zeros at $x = \pm 8$, that $f'(x) < 0$ for all $x \in (-8, 8)$, and that $f'(x) > 0$ for $x \in \mathbb{R} \setminus [-8, 8]$. It follows that f is injective on $I = [-8, 8]$ but is not injective on any interval strictly containing $[-8, 8]$.

(b) Since $f(-1) = 11$, we have $\big((f \restriction I)^{-1}\big)'(11) = \frac{1}{f'(-1)} = \frac{1}{1-4(-1)^{-2/3}} = -\frac{1}{3}$.

(c) $\operatorname{ran}(f \restriction I) = [-16, 16]$, and the only points in this interval at which f^{-1} fails to be differentiable are the endpoints ± 16.

(83) Prove directly from the definitions that for all $a < b$, the identity function $f(x) = x$ is Darboux integrable on $[a, b]$.

Sample Solution: For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the regular partition of $[a, b]$ with n subintervals. Then

$$\begin{aligned} L(f, \mathcal{P}_n) &= \sum_{k=0}^{n-1} \left(a + \frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \\ &= \frac{b-a}{n}\left(na + \frac{b-a}{n}\sum_{k=0}^{n-1} k\right) \\ &= a(b-a) + \left(\frac{b-a}{n}\right)^2 \left(\frac{(n-1)n}{2}\right) \\ &= a(b-a) + (b-a)^2 \left(\frac{n-1}{2n}\right) \end{aligned}$$

and

$$\begin{aligned} U(f, \mathcal{P}_n) &= \sum_{k=1}^{n} \left(a + \frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \\ &= \frac{b-a}{n}\left(na + \frac{b-a}{n}\sum_{k=1}^{n} k\right) \\ &= a(b-a) + \left(\frac{b-a}{n}\right)^2 \left(\frac{n(n+1)}{2}\right) \\ &= a(b-a) + (b-a)^2 \left(\frac{n+1}{2n}\right), \end{aligned}$$

so

$$\lim_{n\to\infty} L(f, \mathcal{P}_n) = \lim_{n\to\infty} U(f, \mathcal{P}_n) = a(b-a) + \frac{(b-a)^2}{2} = \frac{1}{2}(b^2 - a^2).$$

This shows that $f(x) = x^2$ is integrable on $[a, b]$ with $\int_a^b x^2 dx = \frac{1}{2}(b^2 - a^2)$.

(84) Does there exist a function $f : [0, 1] \to \mathbb{R}$ such that $|f|$ is integrable on $[0, 1]$ but f is not?

Sample Solution: Yes — for instance, we could let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

(85) Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a nonnegative integrable function such that $f(x) > 0$ for some $x \in [a, b]$.

(a) Show by example that we could have $\int_a^b f = 0$.

(b) Prove that if f is continuous, then $\int_a^b f > 0$.

Sample Solution:

(a) For instance, let $a = 0$ and $b = 1$, and define f by $f(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } x \neq 0. \end{cases}$

(b) Assume the hypotheses, and in particular assume f is continuous. Fix $x_0 \in [a, b]$ such that $f(x_0) > 0$, and using continuity of f at x_0, fix $\delta > 0$ such that $[x_0, x_0 + \delta) \subseteq [a, b]$ or $(x_0 - \delta, x_0] \subseteq [a, b]$ and also $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$ whenever $|x - x_0| \leq \delta$. Let $\mathcal{P} = (x_k)_{k=0}^n$ be any partition of $[a, b]$ for which the subinterval containing x_0 has width δ. Fix k such that $x_0 \in [x_{k-1}, x_k]$. Then

$$\int_a^b f \;\geq\; L(f) \;\geq\; L(f, \mathcal{P}) \;=\; \big(\inf f[I_k]\big)\delta + \sum_{j \neq k} (\inf f[I_j])\Delta x_j \;\geq\; \frac{f(x_0)}{2}\delta + \sum_{j \neq k} \inf f[I_j]\delta x_j \;\geq\; \frac{f(x_0)}{2}\delta \;>\; 0.$$

(86) Suppose the function $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Show by example that F' need not be integrable on $[a, b]$. *(This shows that the assumption of integrability in the statement of the FTOC cannot be removed.)*

Sample Solution: For instance, let $F(x) = x^2 \sin \frac{1}{x^2}$ and $F(0) = 0$ for $x \in [a, b] = [-1, 1]$. Then F is differentiable on $[a, b]$ but F' is unbounded on every open interval containing 0, so F' is not integrable on $[-1, 1]$.

(87) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Is f integrable on $[-1, 1]$? Prove your claim.

Sample Solution: Yes. To show this, we will use the fact that f is integrable on $[-1, 1]$ iff for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[-1, 1]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$. Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{6}$. Using the fact that f is continuous (hence integrable) on $[\delta, 1]$ and on $[-1, -\delta]$, fix a partition $\mathcal{P}_1$ of $[-1, -\delta]$ and a partition $\mathcal{P}_2$ of $[\delta, 1]$ such that $U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) < \delta$ and $U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2) < \delta$. Now let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, so that $\mathcal{P}$ is a partition of $[-1, 1]$. Since $-1 \leq \sin x \leq 1$ for all $x \in I := [-\delta, \delta]$, we have

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \big[U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)\big] + \big[\sup f[I] - \inf f[I]\big](2\delta) + \big[U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)\big] \\ &\leq \delta + 2(2\delta) + \delta \;=\; 6\delta \;=\; \epsilon. \end{aligned}$$

This shows that f is indeed integrable on $[-1, 1]$.

(88) Let $a < b$ and $c < d$ be real numbers, and suppose the functions $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ are integrable. Does it follow that $g \circ f$ is integrable? Either prove this or give a counterexample.

Sample Solution: No, it does not. For instance, let $a = 0$ and $b = 1$, let f be Thomae's function, and let $g = \chi_{(0,\infty)}$ be the characteristic function of $(0, \infty)$. Then $g \circ f$ is Dirichlet's function, which is not (Riemann) integrable.

(89) *Does $\displaystyle\lim_{x \to 0} \left(\frac{1}{x} \int_0^x \sin(\tfrac{1}{t})\, dt \right)$ exist? If so, evaluate it.

Sample Solution: We claim $\lim_{x\to 0} \frac{1}{x}\int_0^x \sin(1/t)dt = 0$. Toward showing this, let $0 < x < \frac{1}{\pi}$, and fix $n \in \mathbb{N}$ such that $\frac{1}{n\pi} \le x < \frac{1}{(n-1)\pi}$, so $n \ge 2$. Then

$$\left|\int_{-x}^0 f\right| = \left|\int_0^x f\right| \le \left|\int_0^{\frac{1}{n\pi}} f\right| + \left|\int_{\frac{1}{n\pi}}^x f\right| \le \left|\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} f\right| + \left|\int_{\frac{1}{n\pi}}^{\frac{1}{(n-1)\pi}} f\right|$$
$$\le \left(\frac{1}{n\pi} - \frac{1}{(n+1)\pi}\right) + \left(\frac{1}{(n-1)\pi} - \frac{1}{n\pi}\right) = \frac{2\pi^{-1}}{n^2-1} < \frac{2}{n^2} < 2\pi^2 x^2.$$

The desired limit follows.

(90) For all $x \ge 0$ and $n \in \mathbb{N}$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Determine whether (f_n) converges uniformly to f on $[0,1]$.

(c) Determine whether (f_n) converges uniformly to f on $[0,\infty)$.

Sample Solution:

(a) For each $x \in [0,\infty)$, $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0$.

(b) $f_n \to f$ uniformly on $[0,1]$. To see this, let $\epsilon > 0$ and fix $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for all $x \in [0,1]$ and $n \ge N$,

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n} \le \frac{1}{n} < \epsilon.$$

(c) (f_n) does *not* converge uniformly to f on $[0,\infty)$. To see this, set $\epsilon = 1$, let $n \in \mathbb{N}$ be arbitrary, and let $x \ge n$. Then

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n} \ge 1 = \epsilon.$$

(91) Show that $\lim_{n\to\infty} \int_1^2 e^{-nx^2}\, dx = 0$.

Sample Solution: We claim that e^{-nx^2} converges uniformly on $[1,2]$ to the constant 0 function. To see this, let $\epsilon > 0$, and fix $N \in \mathbb{N}$ large enough so that $-N < \log(\epsilon)$, and thus $e^{-N} < \epsilon$. Then for all $n \ge N$ and $x \in [1,2]$, we have

$$e^{-nx^2} \le e^{-n} \le e^{-N} < \epsilon.$$

Thus $e^{-nx^2} \to 0$ uniformly on $[1,2]$, so by 8.2.4,

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = \int_1^2 \lim_{n\to\infty} e^{-nx^2} dx = \int_1^2 0\, dx = 0.$$

(92) Find a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that:

(i) for each $n \in \mathbb{N}$, f_n is discontinuous at every point $x \in \mathbb{R}$; and

(ii) the sequence (f_n) converges uniformly to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

Sample Solution: For each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

(93) For each $n \in \mathbb{N}$, define the function $f_n : (-1, 1) \to \mathbb{R}$ by

$$f_n(x) \;=\; \begin{cases} -x & \text{if } -1 < x < -2^{-n} \\ 2^{n-1}x^2 + 2^{-(n+1)} & \text{if } -2^{-n} \leq x \leq 2^{-n} \\ x & \text{if } 2^{-n} < x < 1 \end{cases}$$

Show that each f_n is differentiable on $(-1, 1)$, and that (f_n) converges uniformly to the absolute value function on $(-1, 1)$.

Sample Solution: Let $n \in \mathbb{N}$. Clearly f_n is differentiable on $(-1, -2^{-n})$, $(-2^{-n}, 2^{-n})$, and $(2^{-n}, 1)$, so we need only check differentiability at the points $x = \pm 2^{-n}$. Note first that

$$\lim_{x \to (-2^{-n})^-} f_n(x) \;=\; 2^{-n} \;=\; f_n(-2^{-n}) \;=\; \lim_{x \to (-2^{-n})^+} f_n(x)$$

and

$$\lim_{x \to (2^{-n})^+} f_n(x) \;=\; 2^{-n} \;=\; f_n(2^{-n}) \;=\; \lim_{x \to (2^{-n})^-} f_n(x),$$

so f_n is continuous at $\pm 2^{-n}$. Therefore, since

$$\lim_{x \to (-2^{-n})^-} f'(x) \;=\; -1 \qquad \text{and} \qquad \lim_{x \to (2^{-n})^+} f'(x) \;=\; 1,$$

it will suffice (by L'Hôpital's Rule) to show

$$\lim_{x \to (-2^{-n})^+} f'(x) \;=\; -1 \qquad \text{and} \qquad \lim_{x \to (2^{-n})^-} f'(x) \;=\; 1,$$

But this follows from the fact that the derivative of $2^{n-1}x^2 + 2^{-(n+1)}$ at $\pm 2^{-n}$ is ± 1. Thus f_n is differentiable on all of $(-1, 1)$.

Now, to see that (f_n) converges uniformly to $y = |x|$ on $(-1, 1)$, let $\epsilon > 0$, fix $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$, suppose $n \geq N$, and let $x \in (-1, 1)$. If $x \in (-1, -2^{-n}] \cup [2^{-n}, 1)$, then $|f_n(x) - |x|| = 0 < \epsilon$, so assume $x \in (-2^{-n}, 2^n)$. Then $0 \leq |x| \leq f_n(x) \leq 2^{-n}$, so $|f_n(x) - |x|| \leq 2^{-n} < \epsilon$ as needed.

(94) For each $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = \dfrac{\sin(nx)}{n}$. Show that (g_n) converges uniformly on $\mathbb{R}$ to a differentiable function whose derivative is *not* $\lim\limits_{n \to \infty} g_n'$.

Sample Solution: We claim that (g_n) converges uniformly on $\mathbb{R}$ to the constant zero function. Indeed, given $\epsilon > 0$, for every $x \in \mathbb{R}$ we have $|g_n(x)| < \epsilon$ for all $n > \frac{1}{\epsilon}$, since $\left|\frac{\sin(nx)}{n}\right| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. However, we have $g_n'(x) = \cos(nx)$ for each n, so (g_n') does not converge.

(95) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let (f_n) be a sequence of continuous functions from A to $\mathbb{R}$ that converges uniformly on A to f. Prove that for every $a \in A$ and sequence (x_n) in A that converges to a, we have $\lim\limits_{n \to \infty} f_n(x_n) = f(a)$.

Sample Solution: Let $a \in A$ and let (x_n) be a sequence in A that converges to a. Let $\epsilon > 0$. Using the fact that uniform limits of sequences of continuous functions are continuous, fix $\delta > 0$ such that $|f(x) - f(a)| < \frac{\epsilon}{2}$ whenever $x \in A$ and $|x - a| < \delta$. Using the convergence of (x_n) to a, fix $N_1 \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq N_1$, and using uniform convergence of (f_n) to f, fix $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Let $N = \max(N_1, N_2)$ and suppose $n \geq N$. Then

$$|f_n(x_n) - f(a)| \;\leq\; |f_n(x_n) - f(x_n)| + |f(x_n) - f(a)| \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

(96) Find the intervals of convergence of the power series:

$$\text{(a)} \sum_{n=0}^{\infty} \left(\frac{(n!)^3}{(3n)!}\right) x^n \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{n^n}{n!} x^n \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{5^{n+1}}{\sqrt{n} \cdot 3^{2n}}\right) x^n$$

Can you find the interval of convergence in (c)?

Sample Solution:

(a) We have $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{\left((n+1)!\right)^3}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3}\right) = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} = \frac{1}{27}$. Thus the radius of convergence is 27 by Hadamard's Theorem.

(b) We have $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right) = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$. Thus the radius of convergence is $\frac{1}{e}$ by Hadamard's Theorem.

(c) We have $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{5^{n+2}}{\sqrt{n+1} \cdot 3^{2(n+1)}} \cdot \frac{\sqrt{n} \cdot 3^{2n}}{5^{n+1}}\right) = \lim_{n\to\infty} \frac{5\sqrt{n}}{9\sqrt{n+1}} = \frac{5}{9}$. Thus the radius of convergence is $\frac{9}{5}$ by Hadamard's Theorem. Testing the endpoints, we see that

$$\sum_{n=1}^{\infty} \left(\frac{5^{n+1}}{\sqrt{n} \cdot 3^{2n}}\right) \left(\frac{-9}{5}\right)^n \;=\; \sum_{n=1}^{\infty} \frac{5(-1)^n}{\sqrt{n}}$$

which converges by the Alternating Series Test, but $\sum \frac{5}{\sqrt{n}}$ diverges by comparison with the harmonic series. Thus the interval of convergence is $[-\frac{9}{5}, \frac{9}{5})$.

(97) (a) Using the fact that $\frac{d}{dx} \ln x = \frac{1}{x}$ for all $x > 0$, calculate the Taylor Series of the natural log function centered at $x = 1$.

(b) Using the fact (which you may assume without proof) that the Taylor Series you found in part (a) converges to the natural log function on $(0, 2]$, calculate the limit of the alternating harmonic series.

Sample Solution:

(a) Writing $f(x) = \ln x$, we have $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ for all $n \in \mathbb{N}$ and $x > 0$. Thus the Taylor series for f centered at 1 is

$$\ln(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

(b) Write $T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$, so $T(x) = \ln x$ for all $x \in (0, 2]$. Then

$$\ln(2) = T(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(2-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

(98) Let $A \subseteq \mathbb{R}$. Prove that A is open if and only if A can be expressed as a disjoint union of countably many open intervals.

Sample Solution: Let $A \subseteq \mathbb{R}$. We have already shown that open intervals are open and that unions of open sets are open, so the backward direction holds. For the forward direction, suppose A is open, and let $x \in A$. Define $a_x := \sup\{y < x : y \notin A\}$ and $b_x := \inf\{y > x : y \notin A\}$. Note that a_x might be $-\infty$ and/or b_x might be $+\infty$, but since A is open we have $a_x < x < b_x$ and also $a_x, b_x \notin A$. Furthermore, we know $(a_x, b_x) \subseteq A$ by definition of a_x and b_x. Thus

$$A = \bigcup_{x \in A} (a_x, b_x).$$

Next, note that for all $x, y \in A$, we have that $(a_x, b_x) \cap (a_y, b_y) \neq \emptyset$ implies $(a_x, b_x) = (a_y, b_y)$, so any two *distinct* open intervals in the union above are actually disjoint. Finally, there can be only countably many distinct open intervals in the union, since each such interval contains a rational by density of $\mathbb{Q}$, and $\mathbb{Q}$ is countable.

(99) Let $V \subseteq \mathbb{R}$ be an open set, and write $V = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$, where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for all $m \neq n$.[5] Define the *measure* of V to be $\mu(V) = \sum_{n=1}^{\infty} (b_n - a_n)$. In the problems below, you may use without proof the (geometrically obvious) fact that for any open set $V \subseteq \mathbb{R}$ and sequence of open intervals (a_n, b_n) in $\mathbb{R}$, if $V \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ then $\mu(V) \leq \sum_{n=1}^{\infty} (b_n - a_n)$.

(a) Prove that for every $\epsilon > 0$, there is an open subset of $\mathbb{R}$ that contains $\mathbb{Q}$ and has measure less than ϵ.

(b) Does there exist an open set $V \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq V$ and $\mathbb{R} \setminus V$ is uncountable?

Sample Solution:

(a) Let $\epsilon > 0$. Using the fact that $\mathbb{Q}$ is countable, write $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\delta_n = \epsilon \cdot 2^{-(n+2)}$, and let $a_n = q_n - \delta_n$ and $b_n = q_n + \delta_n$. Then $q_n \in (a_n, b_n)$ for each n, so $V := \bigcup_{n \in \mathbb{N}} (a_n, b_n) \supseteq \mathbb{Q}$. But $\sum_{n=1}^{\infty} (b_n - a_n) = \sum_{n=1}^{\infty} \epsilon 2^{-(n+1)} = \frac{\epsilon}{2}$, so $\mu(V) \leq \frac{\epsilon}{2} < \epsilon$.

(b) Yes! We can take $\epsilon = 1$ and apply part (a) to obtain an open set $V \supseteq \mathbb{Q}$ such that $\mu(V) < 1$. If $\mathbb{R} \setminus V$ were countable, then $\mathbb{R} \setminus V$ would have measure zero, which would make $\mu(\mathbb{R}) = \mu(V) + \mu(\mathbb{R} \setminus V) < \epsilon + 0 = \epsilon$. But $\mu(\mathbb{R}) = \infty$, so this is impossible!

Remark: it is an excellent exercise to meditate on what such a set V could possibly look like. Remember that the vast majority of real numbers are not in V!

[5]Note: we might need to take $a_n = b_n$ for infinitely many n here.

(100) *For each pair of real numbers $\alpha, \beta \in \mathbb{R}$, define the function $f_{\alpha,\beta} : [0, \infty) \to \mathbb{R}$ as follows: if $\alpha, \beta \geq 0$, then $f_{\alpha,\beta}(x) = x^\alpha \sin x^\beta$, and if $\alpha < 0$ or $\beta < 0$ then

$$f_{\alpha,\beta} = \begin{cases} x^\alpha \sin x^\beta & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

(Note that for some $\alpha, \beta \in \mathbb{R}$, including all $\alpha, \beta \geq 0$, we can also define $f_{\alpha,\beta}(x)$ for $x < 0$, but to avoid certain complications we will just work on $[0, \infty)$ in this problem.)

(a) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is continuous.

(b) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is differentiable.

(c) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is C^1.

Sample Solution:

(a) $f_{\alpha,\beta}$ is continuous on $[0, \infty)$ iff $[\alpha > 0 \text{ or } (\alpha = 0 \text{ and } \beta \geq 0)]$.

(b) $f_{\alpha,\beta}$ is differentiable on $[0, \infty)$ iff $[\alpha > 1 \text{ or } (\alpha = 1 \text{ and } \beta \geq 0)]$.

(c) $f_{\alpha,\beta}$ is C^1 on $[0, \infty)$ iff $[\alpha > 1 \text{ and } \alpha + \beta > 1]$ or $[\alpha = 1 \text{ and } \beta \geq 0]$.