Math 451 Midterm Exam

Spring 2024

You have 110 minutes to complete this exam. You may not use notes, textbooks, or electronic devices of any kind. Write your answers clearly on the exam itself in the space provided for you. Circle your answers where appropriate.

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Problem 1 17/18

Problem 2 17/17

Problem 3 20/21

Problem 4 19/23

Problem 5 18/21

Total Score 91

Problem 1: Definitions (18 points)

Write clear, precise definitions or statements of the following italicized terms or phrases.

(a) the sequence (a_n) in $\mathbb{R}$ *converges* to the real number L

if For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ whenever $n \geq N$

3

(b) $f : X \to Y$ is an *injective* (or *one-to-one*) function

if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$

3

(c) the real number m is the *infimum* of the set $A \subseteq \mathbb{R}$

if m is lower bound of A and for all $x > m$, x is not a lower bound of A.

3

(d) the subset V of the metric space X with metric d is *open*

if for all $x \in V$, there exists some open neighborhood $U_\varepsilon(x)$ (i.e. $\{y \mid d(y,x) < \varepsilon\}$) s.t. $U_\varepsilon(x) \subseteq V$.

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(e) State the *Completeness Axiom* for $\mathbb{R}$

for all $A \subseteq \mathbb{R}$, if A is bounded above then $\sup(A) \in \mathbb{R}$

nonempty

2

(f) State the *Bolzano-Weierstrass Theorem* for sequences of real numbers

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

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Problem 2: Short Answer (17 points)

(a) (5 pts) Can an ordered field have a positive element ϵ such that $\epsilon < \frac{1}{n}$ for every natural number n? Does $\mathbb{R}$ have any such elements? Briefly explain.

An ordered field can have such element, ~~for example~~ but $\mathbb{R}$ does not has such element since $\mathbb{R}$ is an Archimedean ordered field.

which means that for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$

5 ✓

(b) (4 pts) Briefly explain how you know that some real numbers are transcendental.

By hw 1 we have proved: $\bar{\mathbb{Q}}$ is countable and we know that $\mathbb{R}$ is uncountable. ~~$\forall r \in \mathbb{R}$, r is either algebraic or transcendental~~ if all real numbers are algebraic, then ~~so there must exists $r \in \mathbb{R}$ s.t. $r \in$~~ $\mathbb{R} \subseteq \bar{\mathbb{Q}}$ which contradicts. (if $\mathbb{R} \subseteq \bar{\mathbb{Q}} \Rightarrow \mathbb{R} \precsim \bar{\mathbb{Q}}$)

4 ✓

(c) (4 pts) Can a sequence of negative numbers converge to a positive number? Justify your answer either with an example or with a short argument.

No. ✓

Proof. Let (a_n) be a sequence of negative numbers.

Assume for contradiction that (a_n) converges to a positive number. L

Then consider $\varepsilon = L - 0 = L$

By our assumption, $\exists N \in \mathbb{N}$ s.t. $\forall a_n \geq N$, $|a_n - L| < \varepsilon \Rightarrow -a_n + L < \varepsilon \Rightarrow a_n > L - \varepsilon = 0$ contradicts with a_n be negative ✓

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(d) (4 pts) In each part, give a concrete example of a set with the given properties, or else write IMPOSSIBLE if no such set exists. *No justification necessary.*

(i) An infinite subset of $\mathbb{R}$ that has no limit points in $\mathbb{R}$.

$\mathbb{N}$. ✓

4

(ii) A countable subset of $\mathbb{R}$ that has uncountably many limit points in $\mathbb{R}$.

$\mathbb{Q}$. ✓

Problem 3: Computational Problems (21 points)

(a) (4 pts) Determine the following limit using the limit laws and any particular limits from lecture or the text. You may simply carry out the steps in the computation *without* citing what laws you are using, but you should show some work and circle your final answer.

$$\lim_{n\to\infty} \frac{(2n)^{1/n}(n^2+1)}{\pi n^2}$$

$$= \lim_{n\to\infty} \frac{1}{\pi}\cdot\frac{(n^2+1)}{n^2}\cdot (2n)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\pi}\cdot \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)\cdot \lim_{n\to\infty} 2^{\frac{1}{n}} \cdot \lim_{n\to\infty} \sqrt[n]{n}$$

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$$= \frac{1}{\pi}\cdot 1\cdot 1\cdot 1 = \frac{1}{\pi}$$

(b) (5 pts) Let (a_n) be a sequence of real numbers such that $a_1 = \frac{1}{2}$ and for all $n \in \mathbb{N}$, $a_{n+1} = \frac{(n+1)^2}{n(n+2)} \cdot a_n$. Carefully prove by induction that $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

we prove by induction on $n \in \mathbb{N}$

Proof Base case: $n=1$

$a_1 = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1}$, the statement holds true.

Inductive step:

Assume for $n=k\in\mathbb{N}$, $a_n = \frac{n}{n+1} = \frac{k}{k+1}$

Then

(no need to use k here)

$$a_{n+1} = \frac{(n+1)^2}{n(n+2)}a_n = \frac{(k+1)^2}{k(k+2)}\cdot\frac{k}{k+1} = \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1} = \frac{(n+1)}{(n+1)+1},$$

the statement holds true

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This finishes the proof that $a_n = \frac{n}{n+1}$ for all $n\in\mathbb{N}$.

Problem 3 (Continued)

(c) (4 pts) Find the supremum and the infimum in $\mathbb{R} \cup \{\pm\infty\}$ of each of the following sets of real numbers. If the sup or inf is $+\infty$ or $-\infty$, say so instead of writing "DNE." *No justification needed.*

(i) $\bigcap_{n\in\mathbb{N}} \left(1 - \frac{1}{n}, \frac{n^2+1}{n^2}\right)$

Supremum $= 1$

infimum $= 1$

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(ii) $\bigcup_{n\in\mathbb{N}} \left(-\frac{1}{n}, n + \frac{1}{n}\right)$

infimum $= -1$,

supremum $= +\infty$

(d) (4 pts) Find the min / max of the given set as indicated, if it exists; if it does not exist, write DNE. *No justification needed.*

(i) $\min \{|x - y| : x, y \in \mathbb{R} \text{ and } x \neq y\}$ DNE

(ii) $\max\, [0, \sqrt{2}] \cap \mathbb{Q}$ DNE

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(iii) $\max\, [0, \sqrt{2}] \setminus \mathbb{Q}$ $= \sqrt{2}$

(iv) $\max \bigcap_{n\in\mathbb{N}} (-n, \frac{1}{n}]$ DNE

(e) (4 pts) Find the lim sup and lim inf in $\mathbb{R} \cup \{\pm\infty\}$ of the given sequences. If the lim sup or lim inf is $+\infty$ or $-\infty$, say so instead of writing "DNE." *No justification needed.*

(i) the sequence $(a_n)_{n\in\mathbb{N}}$ whose nth term is $a_n = (-1)^n + \frac{1}{n}$ for each $n \in \mathbb{N}$.

$\limsup (a_n) = 1$. $\liminf (a_n) = -1$

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(ii) the sequence $(b_n)_{n\in\mathbb{N}}$ whose nth term is $b_n = (1 + \frac{1}{n})^{n(-1)^n}$ for each $n \in \mathbb{N}$.

$\limsup (b_n) = e$

$\liminf (b_n) = \frac{1}{e}$

Problem 4: Limits (23 points)

(a) (3 pts) Give a precise statement in terms of ϵ and δ of the fact that $\lim_{x\to 2} \frac{1}{x-1} = 1$.

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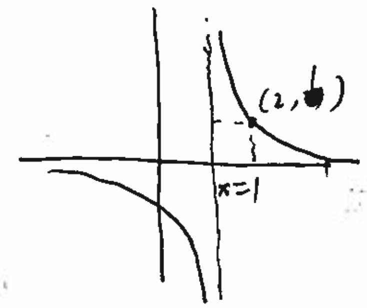
For all $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|\frac{1}{x-1} - 1\right| < \varepsilon$ whenever $0 < |x-2| < \delta$

(b) (6 pts) Give a direct proof using ϵ and δ of the fact that $\lim_{x\to 2} \frac{1}{x-1} = 1$.



Proof Let $\varepsilon > 0$

~~Consider~~ $\cancel{\delta = \min\left\{1, \frac{1}{2}\left(\frac{1}{1-\varepsilon} - \frac{1}{1+\varepsilon}\right)\right\}}$

~~note that~~ $\cancel{\frac{1}{1-\varepsilon} - 1 < \delta < 1 - \frac{1}{1+\varepsilon}}$

what if $\varepsilon = 1$?

2

We want $\delta > 0$ such that whenever $0 < |x-2| < \delta$, $\frac{1}{x-1} \in (1-\varepsilon, 1+\varepsilon)$

but this could be negative (eg, if $\varepsilon = \frac{1}{2}$)

we can take $\delta = 1 - \frac{1}{2}\left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right)$, then $\delta < 1 - \frac{1}{1+\varepsilon}$ and $\delta > 1 - \frac{1}{1-\varepsilon}$

$\Rightarrow \frac{1}{1+\delta} > 1-\varepsilon$, $\frac{1}{1-\delta} < 1+\varepsilon$ then if $0 < |x-2| < \delta \Rightarrow x \in (2-\delta, 2+\delta) \Rightarrow$

$\frac{1}{x-1} \in \left(\frac{1}{1+\delta}, \frac{1}{1-\delta}\right)$

$\Rightarrow \frac{1}{x-1} \in (1-\varepsilon, 1+\varepsilon)$

$\Rightarrow 0 < \left|\frac{1}{x-1} - 1\right| < \varepsilon$

(c) (7 pts) Let $a, L \in \mathbb{R}$. Prove that for any functions $f, g : \mathbb{R} \to \mathbb{R}$, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = L$, then $\lim_{x\to a} g(f(x)) = L$.

Proof. Let $\varepsilon > 0$

Since $\lim_{x\to\infty} g(x) = L$, there exists $N \in \mathbb{R}$ s.t.

for all $x \geq N$ $|g(x) - L| < \varepsilon$ ①

(fix such N)

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Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta \in \mathbb{R}$ st. $f(x) \geq N$ whenever $0 < |x-a| < \delta$

~~for all $0 < |x-a| < \delta$, $f(x) > N$~~

②

Take the same δ, combing ① and ② we have

$|g(f(x)) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$

This finishes the proof that $\lim_{x\to a} g(f(x)) = L$

good

Problem 4 (Continued)

(d) (7 pts) Suppose (a_n) and (b_n) are sequences of real numbers. Prove that if (a_n) converges and $\lim b_n = \infty$, then $\lim(a_n + b_n) = \infty$.

Proof. Let $M > 0$.

Since (a_n) converges, (a_n) is bounded ✓

So $k_1 < a_n < k_2$ for all $n \in \mathbb{N}$, for some $k_1, k_2 \in \mathbb{R}$

Since $\lim b_n = \infty$, there exists $N \in \mathbb{N}$ s.t. $b_n > M - k_1$ for all $n \geq N$

Then $a_n + b_n > (M - k_1) + k_1 = M$ for all $n \geq N$

This finishes the proof that $\lim(a_n + b_n) = \infty$ ✓

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Problem 5: General Proofs (21 points)

(a) (7 pts) Prove that every convergent sequence of real numbers is Cauchy.

Proof. Let (a_n) be a convergent sequence in $\mathbb{R}$. Write $\lim_{n\to\infty} (a_n) = L$ ✓

Let $\varepsilon > 0$ ✓

Since (a_n) converges, there exists $N \in \mathbb{N}$ such that

for all $n \geq N$, $|a_n - L| < \frac{\varepsilon}{2}$ ✓ Fix such N.

Take arbitrary $m, n \in \mathbb{N}$ such that $m, n \geq N$

Then $|a_n - L| < \frac{\varepsilon}{2}$, $|a_m - L| < \frac{\varepsilon}{2}$

$\Rightarrow a_n \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$, $a_m \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$

$\Downarrow$

$-a_m \in (-\frac{\varepsilon}{2} - L, \frac{\varepsilon}{2} - L)$

So $a_n - a_m < \varepsilon$, $a_n - a_m > -\varepsilon$

$\Rightarrow |a_n - a_m| < \varepsilon$

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Since ε, m, n are arbitrary, (a_n)

This finishes the proof that every convergent sequence of real numbers is Cauchy. ✓

Problem 5 (Continued)

(b) (7 pts) Let (a_n) be an increasing sequence of real numbers and (b_n) a decreasing sequence of real numbers such that $a_m \le b_n$ for all $m, n \in \mathbb{N}$. Prove that

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset.$$

Since $a_m \le b_n$ for all $m,n \in \mathbb{N}$, $a_m \le b_1$ for all $m \in \mathbb{N}$ and $b_n \ge a_1$ for all $n \in \mathbb{N}$.

This means that (a_n) is bounded above and (b_n) is bounded belo ①

Since (a_n) is increasing and (b_n) is decreasing,

$a_n \ge a_1$ and $b_n \le b_1$ for all $n \in \mathbb{N}$

Therefore (a_n) is bounded below and (b_n) is bounded above ②

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By ①②, (a_n) and (b_n) are bounded, together with the fact that they are monotone, this shows (a_n) and (b_n) are convergent. write $\lim(a_n) = a$ and $\lim(b_n) = b$

Since $a_n \le b_n$ for all $n \in \mathbb{N}$ $\Longrightarrow$ $\lim(a_n) \le \lim(b_n)$ $\Longrightarrow$ $\boxed{a \le b}$

And since (a_n) is increasing and (b_n) is decreasing, $a_n \le a$ and $b_n \ge b$ for all $n \in \mathbb{N}$

$\Longrightarrow$ for all $n \in \mathbb{N}$, $[a,b] \subseteq [a_n, b_n]$

so $[a,b] \subseteq \bigcap_{n\in\mathbb{N}} [a_n, b_n]$

Since $a \le b$, $[a,b] \neq \emptyset$

This finishes the pr

good

(c) (7 pts) Prove that if the function $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for every subset $A \subseteq \mathbb{R}$,

$$f[\mathrm{cl}(A)] \subseteq \mathrm{cl}(f[A]).$$

(Recall that $\mathrm{cl}(A)$ is the *closure* of A, defined by $\mathrm{cl}(A) = A \cup A'$ where A' is the set of all limit points of A.)

Proof Since f is continuous, f is continuous on $A \subseteq \mathrm{dom}(f)$ and thus continuous on every point in A

So $\forall a \in A$, either $\lim_{x\to a} f(x) = f(a)$, or a is an isolated point (i.e. $a \in A \setminus A'$)

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Let $a \in \mathrm{cl}(A)$ be arbitrary

Case ① $a \in A' \setminus A$ $\Longrightarrow$ $f(a) = \lim_{x\to a} f(x) \in \mathrm{cl}(f[A])$ why?

Case ② $a \in A$ $\Longrightarrow$ then either $\lim_{x\to a} f(x) = f(a)$ or $a \in A \setminus A'$

If $\lim_{x\to a} f(x) = f(a)$ $\Longrightarrow$ $\lim_{x\to a} f(x) \in f[A] \subseteq \mathrm{cl}(f[A])$ a

if a is an isolated point, then

(you're making this case more complicated than it needs to be)