Def. A sequence is a function with $\boxed{\text{domain}}$ $\boxed{\{n \in \mathbb{Z} \mid n \geq n_0\}}$ for some $n_0 \in \mathbb{Z}$ (∞)
The value of a seq. is call terms

if $s: \mathbb{N} \to \mathbb{R}$ 为一个 seq.，我们可以 write s_n 来表示它的 n^{th} term.

常也可以用 $\boxed{(s_n : n \in \mathbb{N})}$ 或 $\boxed{(s_n)_{n\in\mathbb{N}}}$ 或 $\boxed{(s_n)_{n=1}^{\infty}}$ 来表示这个 seq.

※ note: order matters in seq. !! $(s_n)_{n\in\mathbb{N}} \neq \{s_n \mid n \in \mathbb{N}\}$!!
(seq.: 有序的) (set: 无序)

ex

(1) constant seq. $s_n = c \in \mathbb{R}$ for all n
$(c, c, c, \dots)$

(2) the harmonic seq. $s_n = \frac{1}{n}$ for all $n \in \mathbb{N}$
$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (\frac{1}{n})_{n\in\mathbb{N}}$

(3) $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots) = (2^{-n})_{n\in\mathbb{N}\cup\{0\}}$

(4) the Fibonacci seq. $(1, 1, 2, 3, 5, 8, \dots)$
defined recursively by $s_1 = s_2 = 1$, $s_{n+2} = s_{n+1} + s_n$.

(5) $((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, \dots)$

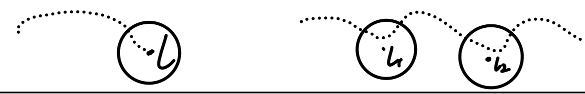
(6) $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$ the seq. of ration approximation of π

(7) $s_n = (1+\frac{1}{n})^n$

Def seq. (s_n) of real nums $\boxed{\text{converges to } L \in \mathbb{R}}$
if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|s_n - L| < \varepsilon$
whenever $n \geq N$
且此时称 L 为 s_n 的 limit
写作: $\boxed{\lim_{n\to\infty} s_n = L}$ or $\boxed{s_n \to L \text{ as } n \to \infty}$

不 converge 到 任何 $L \in \mathbb{R}$ 的 seq. 称为 diverge 的.
即: $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, ($\exists n \geq N$ s.t. $|s_n - L| \geq \varepsilon$)

converge to L diverge



Notation write $\boxed{\lim_{n\to\infty} s_n = +\infty}$ if $\forall M \in \mathbb{N}, \exists N \in \mathbb{N}$ s.t. whenever $n \geq N$, $s_n > M$

$\boxed{\lim_{n\to\infty} s_n = -\infty}$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. whenever $n \geq N$, $s_n < M$

ex.

(1) const seq. $s_n = c$ converge to c

(2) $\lim_{n\to\infty} \frac{1}{n} = 0$ by Archimedean property of $\mathbb{R}$

(3) $2^{-n} \to 0$ as $n \to \infty$

(4) the Fibonacci seq. "diverges to $+\infty$"
$\lim_{n\to\infty} s_i = +\infty$ ($+\infty$: not a real num)

(5) $\lim_{n\to\infty} (-1)^n$ DNE

(6) $(3, 3.14, 3.141, 3.1415, \dots) \to \pi$ as $n \to \infty$

(7) $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$ (def of e)

Fact every real num is the limit of a seq. in $\mathbb{Q}$

Pf Let $r \in \mathbb{R}$ be arbitrary.
By density of $\mathbb{Q}$ in $\mathbb{R}$,
$\forall n \in \mathbb{N}$, 都可以 choose $q_n \in \mathbb{Q}$ s.t. $r < q_n < r + \frac{1}{n}$
$\Rightarrow \lim_{n\to\infty} q_n = r$ by def.

Thm Uniqueness of limit
if seq (a_n) in $\mathbb{R}$ converge to L_1 and L_2
$\Rightarrow \boxed{L_1 = L_2}$

Pf we show that $\boxed{\forall \varepsilon > 0, |L_1 - L_2| < \varepsilon}$
Let $\varepsilon > 0$. Fix $N_1, N_2 \in \mathbb{N}$ s.t.
whenever $n \geq N_1$, $|a_n - L_1| < \frac{\varepsilon}{2}$
whenever $n \geq N_2$, $|a_n - L_2| < \frac{\varepsilon}{2}$
Then let $N = \max\{N_1, N_2\}$ (by def of limit, can choose any $\frac{\varepsilon}{2}$)
$\Rightarrow \forall n \geq N$, $|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |L_2 - a_n| < \varepsilon$

Some basic limits

(1) $\forall p>0, \lim_{n\to\infty} n^p = +\infty$

Pf. let $p>0$

let $M>0$

WT: find N s.t. $\forall n \geq N, n^p > M$ $(n > M^{\frac{1}{p}})$

consider: $N = M^{\frac{1}{p}}+1$

$\Rightarrow \forall n \geq N, \ n^p > (M^{\frac{1}{p}})^p = M$ $\square$

(2) $\forall p<0, \lim_{n\to\infty} n^p = 0$

Pf. let $p<0$

Let $\varepsilon>0$ and WT: find N s.t. $\forall n \geq N$, $n^p < \varepsilon$

Consider: $N = (\frac{1}{\varepsilon})^{-\frac{1}{p}}+1$

$\Rightarrow \forall n \geq N, \ \frac{1}{n} < (\frac{1}{\varepsilon})^{\frac{1}{p}} \Rightarrow (\frac{1}{n})^{-p} < (\frac{1}{\varepsilon})^{-1} = \varepsilon$

$\Rightarrow 0 \leq n^p < \varepsilon$ $\square$

(3) $\forall r>1, \lim_{n\to\infty} r^n = +\infty$

Pf let $r>1$, so $r = 1+a$ for some $a \in \mathbb{R}$

Then by Bernoulli's ineq.,

$\forall n, \ r^n = (1+a)^n \geq 1+na$

Given $M>0$, choose $N \in \mathbb{N}$ s.t. $Na > M-1$ (by Archimedean)

$\Rightarrow \forall n>N, \ r^n = (1+a)^n \geq 1+na > 1+Na > M$

(4) If $|r|<1$, then $\lim_{n\to\infty} r^n = 0$

Pf. suppose $0<r<1 \Rightarrow r = \frac{1}{1+a}$ for some $a>0, a\in\mathbb{R}$

Let $\varepsilon>0$

Take $N = \frac{1}{\varepsilon a}$

$\Rightarrow \forall n>N, \ 0<r^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na} < \varepsilon$

(by Bernoulli)

我们不用prove $-1<r<0$ 的部分,而是使用一个fact:

$\forall$ seq. (a_n) of real nums, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} |a_n| = 0$

($\Downarrow$ corollary)

$\forall$ seq. (a_n) of real nums, $\lim_{n\to\infty} a_n = L$ iff $\lim_{n\to\infty} |a_n - L| = 0$

Pf. 因为$\forall \varepsilon$, $|a_n - L| < \varepsilon \Leftrightarrow \big||a_n - L| - 0\big| < \varepsilon$

(5) $\forall c>0, \lim_{n\to\infty} c^{\frac{1}{n}} = 1$

pf (我参考: rudin 3.20)

取 $x_n = c^{\frac{1}{n}} - 1 \Rightarrow x_n > 0$

$(1+nx_n) \leq (1+x_n)^n = c$ by Bernoulli's ineq.

$\Rightarrow 0 < x_n \leq \frac{c-1}{n}$

因而 $\forall \varepsilon>0$, consider $N > (c-1)\varepsilon$

$\Rightarrow x_N \leq \varepsilon \Rightarrow \forall n>N, \ x_n < \varepsilon$

因而 $\lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} c^{\frac{1}{n}} = 1$

(6) $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$

Pf. (我参考: rudin 3.20)

Let $x_n = n^{\frac{1}{n}} - 1 \Rightarrow x_n \geq 0$

$\Rightarrow n = (1+x_n)^n$

且 by binomial thm $\Rightarrow n = (1+x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$

$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$

$\Rightarrow \forall \varepsilon (<1)$, take $N > \frac{2}{\varepsilon^2} - 1$,

$x_n < \varepsilon$ whenever $n \geq N$

选取这项是为了 $x_n \leq$ ⋯ 上分母的n次数更高,从而可以 bound 住 x_n

Fact. $\forall$ seq. of $\mathbb{R}$ $(a_n)_{n\in\mathbb{N}}$, 取任意 $N\in\mathbb{N}$

(a_n) converges $\Leftrightarrow$ $(a_n)_{n\geq N}$ converges.

$\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} (a_n)_{n\geq N} = L$ 可以无视前面任意N项.

$(a_n)_{n\geq N}$ 被称为 a tail of $(a_n)_{n\in\mathbb{N}}$