Homework 7: Due Monday, June 24, at 11:59pm, on Gradescope

(1) Prove that if the function f is continuous on $[a, b]$, then there is $c \in [a, b]$ such that

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f(x)dx.$$

(2) (a) Let $f : [a, b] \to \mathbb{R}$ be nonnegative and continuous. Prove that if $f(x) > 0$ for some $x \in [a, b]$, then $\int_a^b f > 0$.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(x) \le g(x)$ for all $x \in [a, b]$. Prove that if $\int_a^b f = \int_a^b g$, then $f = g$.

(3) (a) Prove that if the function f is integrable on $[a, b]$, then there is $c \in [a, b]$ such that $\int_a^c f = \int_c^b f$.

(b) Give an example to show that in part (a) we may not have $c \in (a, b)$.

(4) Compute the following limits:

$$\text{(a)}\ \lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}\,dt \qquad\qquad \text{(b)}\ \lim_{h\to 0}\int_3^{3+h} e^{t^2}\,dt$$

(5) For all $x \ge 0$ and $n \in \mathbb{N}$, let $f_n(x) = \dfrac{x^n}{1+x^n}$.

(a) Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

(b) Show that for all $0 < b < 1$, f_n converges uniformly on $[0, b]$.

(c) Does f_n converge uniformly on $[0, 1]$? Prove your claim.

(6) Prove that if (f_n) is a sequence of uniformly continuous functions on the interval (a, b) such that $f_n \to f$ uniformly on (a, b), then f is also uniformly continuous on (a, b).

(7) Give an example of a sequence (f_n) of continuous functions from $[0, 1]$ to $\mathbb{R}$ that converges pointwise but *not* uniformly to a continuous limit function $f : [0, 1] \to \mathbb{R}$.

(8) Let (f_n) be a sequence in of C^1 functions on $[0, 1]$ such that (f_n') converges uniformly. Prove that if $(f_n(a))$ converges for *some* $a \in [0, 1]$, then $(f_n(x))$ converges for *all* $x \in [0, 1]$.

(9) A function $f : [a, b] \to \mathbb{R}$ is called a *step function* if there is a partition $\mathcal{P} = (x_k)_{k=0}^n$ of $[a, b]$ such that f is constant on (x_{k-1}, x_k) for each k. Prove that for every continuous function $f : [a, b] \to \mathbb{R}$, there is a sequence $f_n : [a, b] \to \mathbb{R}$ of step functions such that $f_n(x) \le f(x)$ for all $x \in [a, b]$ and $f_n \to f$ uniformly on $[a, b]$.

(10) Suppose $\sum c_n x^n$ is a power series such that $\lim\limits_{n\to\infty}\left|\dfrac{c_{n+1}}{c_n}\right| = L > 0$. Prove that $\sum c_n x^n$ converges for all $x \in (-R, R)$ and diverges for all $x \in \mathbb{R} \setminus [-R, R]$, where $R = \frac{1}{L}$.

(11) For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

$$\text{(a)}\ \sum n^2 x^n \qquad\qquad \text{(b)}\ \sum\left(\frac{2^n}{n^2}\right)x^n \qquad\qquad \text{(c)}\ \sum\left(\frac{2^n}{n!}\right)x^n$$

(12) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$.

(a) Prove by induction that for all $n \in \mathbb{N}$ and $x \neq 0$, $f^{(n)}(x)$ exists and has the form $f^{(n)}(x) = p(\frac{1}{x})f(x)$ where p is a polynomial.

(b) Show that for every polynomial p, $\lim_{x \to 0} p(\frac{1}{x})f(x) = 0$. *Remark: you may freely use without proof the fact from calculus that* $\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$ *for every polynomial* p.

(c) Show by induction that $f^{(n)}(0)$ exists and is equal to 0 for all integers $n \geq 0$.

(d) Give an example of a C^∞ function g whose Taylor series expansion about 0 converges to g for all $x \leq 0$ and converges but *not* to g for all $x > 0$. *(No justification needed.)*

Optional Challenge Problems:

(13) (a) For each $n \in \mathbb{N}$, define the function $f_n : (-1, 1) \to \mathbb{R}$ by

$$f_n(x) \;=\; \begin{cases} -x - 2^{-n-1} & \text{if } -1 < x < 2^{-n} \\ 2^{n-1}x^2 & \text{if } -2^{-n} \leq x \leq 2^{-n} \\ x - 2^{-n-1} & \text{if } 2^{-n} < x < 1 \end{cases}$$

Show that each f_n is differentiable on $(-1, 1)$, and that (f_n) converges uniformly to the absolute value function on $(-1, 1)$.

(b) For each $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = \dfrac{\sin(nx)}{n}$. Show that (g_n) converges uniformly on $\mathbb{R}$ to a differentiable function whose derivative is *not* $\lim_{n\to\infty} g_n'$.

(14) Let $\mathbb{Q} = \{q_n \;:\; n \in \mathbb{N}\}$, and for each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \setminus \{q_n\} \to \mathbb{R}$ be the function defined by

$$f_n(x) \;=\; 4^{-n} \sin\left(\frac{1}{x - q_n}\right).$$

For each $x \in \mathbb{R} \setminus \mathbb{Q}$, let $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

(a) Prove that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges. Thus $\text{dom}(f) = \mathbb{R} \setminus \mathbb{Q}$.

(b) Prove that f is continuous.

(c) Prove that for every $q \in \mathbb{Q}$, $\lim_{x \to q} f(x)$ does not exist. (cf: HW 6, #17)