1.

Thm Closure properties of Continuous functions

If f and g are ctn at $x=a$,
then so are $f+g$, $f-g$, fg, f/g
and cf for $\forall c \in \mathbb{R}$

note: $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$
$\Rightarrow$ domain of $f+g$, $f-g$ and fg is $A \cap B$
domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$

Proof Using the sequence def of continuity.
Suppose f, g are ctn at $a \in A \cap B$
if (a_n) is a seq. in $A \cap B$ conv. to a

$$\Rightarrow \lim_{n\to\infty} (f+g)(a_n) = \lim_{n\to\infty} f(a_n) + \lim_{n\to\infty} g(a_n)$$
$$= f(a) + g(a) = (f+g)(a)$$

So $(f+g)$ is ctn at a. (另三个相似.)

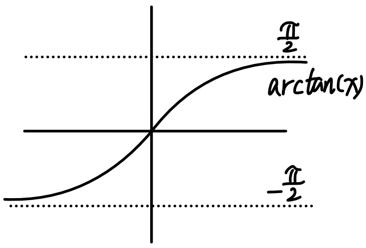
Thm (前提 $f: A \to \mathbb{R}$; $g: B \to \mathbb{R}$; $a \in (\text{dom}(g \circ f))'$)
if f ctn. at a 且 g is ctn at $\lim_{x\to a} f(x)$,
$$\Rightarrow \boxed{\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)}$$

note: $\text{dom}(g \circ f) = \{x \in A \mid f(x) \in B\} \subseteq A$

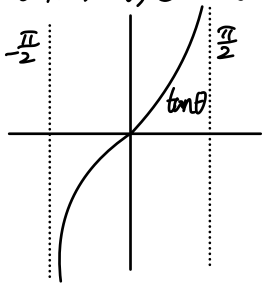
Proof Let (a_n) be a seq. in $\text{dom}(g\circ f)$, conv. to a
Since f is ctn at a, $f(a) = \lim_{x\to a} f(x)$ and $\boxed{f(a_n) \to f(a)}$
By continuity of g at b, we have
$$\lim_{n\to\infty} g\circ f(a_n) = \lim_{n\to\infty} g(f(a_n)) = g(f(a)) = g\circ f(a)$$
Since (a_n) is arbitrary, we conclude that
$\lim_{x\to a} g(f(x)) = g(b)$, as claimed.

Remark The previous thm $\boxed{\text{admits}}$ variants
where the lim. at a is replaced
throughout by the limit at $\boxed{a^+, a^-, \pm\infty}$
(pf: hw4, #5)

ex1
$$\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$
as $x \to 0^+$, $\frac{1}{x} \to +\infty$
as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}$



ex2
$$\lim_{\theta\to\frac{\pi}{2}^-} e^{-\tan\theta} = 0$$
as $\theta \to \frac{\pi}{2}^-$, $\tan\theta \to +\infty$
as $x \to -\infty$, $e^x \to 0$



Corollary① If $f: A \to \mathbb{R}$ 在 $a \in A$ 处 ctn,
且 $g: B \to \mathbb{R}$ 在 $b = f(a)$ 处 ctn
$\Rightarrow g\circ f$ 在 a 处 ctn.
(ctn 点 的 composition 也 ctn)

pf directly follows from thm.

Corollary② If $f: A \to \mathbb{R}$ 和 $g: B \to \mathbb{R}$ 是 ctn. function.
则 $g\circ f$, $f\circ g$ 也是 ctn. function
(ctn functions 的 composition 也是 ctn function)

pf 使用 topology 快速证出.
Review:

Def of ctn function in metric space
$f: X \to Y$ is ctn if $f^{-1}[V]$ is open in X
for every open set $V \subseteq Y$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$\forall$ open set $V \subseteq Z$
Since g ctn $\Rightarrow g^{-1}[V]$ is open in Y
$\Rightarrow$ Since f ctn $\Rightarrow f^{-1}[g^{-1}[V]]$ is open in X
$\square$

Exs usings properties of ctn functions

ex

(1) 使用 df of ctnity,
pf: $y = x^2$ is ctn at $a=2$

we must show: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.
$|x^2 - 4| < \varepsilon$ whenever $|x-2| < \delta$

Pf $|x^2-4| = |x-2||x-2|$

let $\varepsilon > 0$ be arbi.

Set $\delta = \min\{1, \frac{\varepsilon}{5}\}$

Suppose $|x-2| < \delta \implies 1 < x < 3$

Then $|x^2-4| = \underbrace{|x+2|}_{<5}\underbrace{|x-2|}_{<\delta} < 5\delta \le \varepsilon$ □

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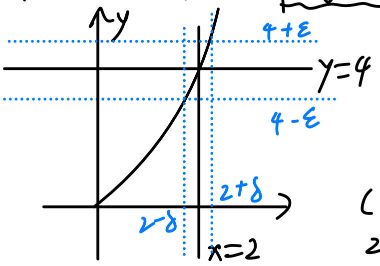
* Give $\varepsilon > 0$, find $\boxed{\text{largest } \delta > 0}$ s.t. $|x^2-4| < \varepsilon$ whenever $|x-2| < \delta$?



$(2+\delta)^2 = 4+\varepsilon$

$\implies \delta = \sqrt{4+\varepsilon} - 2$

(* note: 选取 $2+\delta$ 这边而不是 $2-\delta$ 这边，因为 $2+\delta$ 这边更陡峭，如果只 bound $2-\delta$ 这边，则会不满足 $2+\delta$ 这边的 $|f(x)-f(a)| < \varepsilon$)

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ex3 Show: $\forall a \in \mathbb{R}$, $y = x^2$ is ctn. at $x = a$

pf Given $a \in \mathbb{R}$ and $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{2|a|+1}\}$

Suppose $|x-a| < \delta$. Then $|x^2-a^2| = |x-a||x+a| < \delta(2|a|+1) = \varepsilon$

* note that $\boxed{\delta \text{ depend on both } a \text{ and } \varepsilon}$

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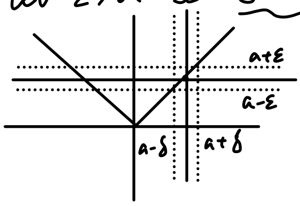
ex4 Show: $y = |x|$ is ctn. everywhere

Pf let $\varepsilon > 0$. set $\delta = \varepsilon$



Then $\forall a \in \mathbb{R}$, $|x-a| < \delta$ implies $\big||x|-|a|\big| \le |x-a| < \delta = \varepsilon$

$\boxed{\text{* Note: here } \delta \text{ depend on } \varepsilon \text{ but not } a}$

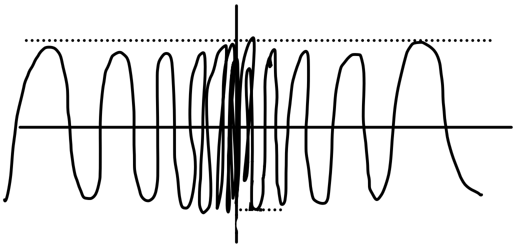
因为 $f'(x) = \pm 1$ 处处一样 a 的变化不改变 δ 的选择.

ex5 $\boxed{f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0 \end{cases}}$ is ctn everywhere except at 0

ex6 $\boxed{g(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}}$ is $\boxed{\text{ctn everywhere except at 0}}$

(但 $f(x) = \sin\frac{1}{x}$ 是 ctn everywhere 的)

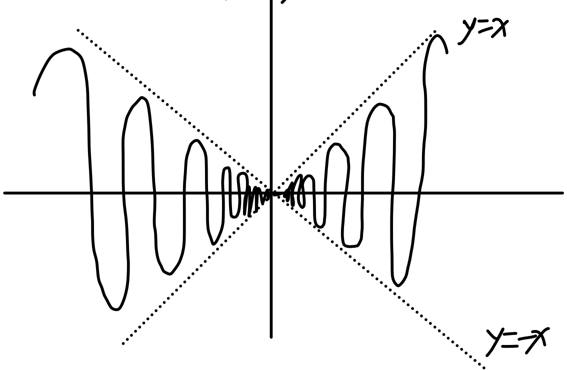
这是因为在 $x \ne 0$ 处，$g(x)$ 为 a composition of ctn. functions $\implies$ ctn. 而 $\lim_{x\to 0} g(x)$ DNE



ex7 $h(x) = \begin{cases} x\sin\frac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$ is ctn. everywhere.

因为在 $x \ne 0$ 时 h 为 a product of composition of ctn. functions 而 $x = 0$ 处 $\lim_{x\to 0} h(x) = 0 = h(0)$ by squeeze thm.

$(-|x| \le x\sin\frac{1}{x} \le |x|)$



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(8) Dirichlet's function $D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

is disctn everywhere

因为 $\forall q \in \mathbb{Q}$ 都有 a seq. of irrationals conv. to q

$\forall r \in \mathbb{R}\setminus\mathbb{Q}$ there is a seq. of rationals conv. to r.

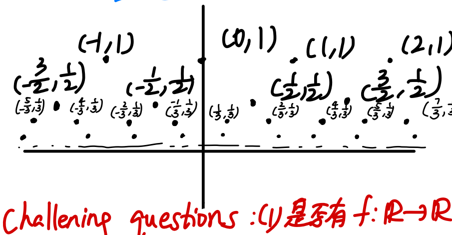
(9) Let $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

Then $|f(x)| \le |x|$ for all x, 因而 f 在 0 处 ctn ($\delta = \varepsilon$)

However, f is disctn. everywhere else

(10) Thoma's function

$T(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ in "lowest terms"} \\ 0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$



$\boxed{T \text{ is ctn. at } x \text{ iff } x \in \mathbb{R}\setminus\mathbb{Q}}$

Pf 5.1.6(h)

Challening questions: (1) 是否有 $f: \mathbb{R} \to \mathbb{R}$ 使 f ctn at x iff $x \in \mathbb{Q}$?

(2) is T diffable anywhere?