Homework: Sample Solutions

Math 451: Spring 2024

Homework 1

(1) (a) TRUE. Let $x \in (A \cup B) \setminus C$, so $x \in A$ or $x \in B$, and $x \notin C$. If $x \in A$ then $x \in A \cup (B \setminus C)$, and if $x \in B$ then $x \in B \setminus C$, so either way $x \in A \cup (B \setminus C)$. This proves that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

(b) FALSE; e.g., $A = B = C = \mathbb{R}$.

(c) FALSE; e.g., $A = \mathbb{R}$, $B = (0, 1)$, $C = (1, 2)$.

(d) TRUE. Suppose $A \subseteq B$. If $x \in A$, then also $x \in B$ since $A \subseteq B$, so $x \in A \cap B$. This shows $A \subseteq A \cap B$. Since $A \cap B \subseteq A$, we have that in fact $A \cap B = A$. Conversely, suppose that $A \cap B = A$. Then $x \in A$ implies $x \in A \cap B$ which implies $x \in B$, showing that $A \subseteq B$. We conclude that $A \subseteq B$ if and only if $A \cap B = A$.

(2) (a) $A_2 \cap A_3 = A_6$.

(b) $\bigcup_{n=2}^{\infty} A_n = \mathbb{N} \setminus \{1\}$ and $\bigcap_{n=2}^{\infty} A_n = \emptyset$.

(3) (a) $1 = 1$, $1+3 = 4$, $1+3+5 = 9$, and $1+3+5+7 = 16$. Since 1, 4, 9, and 16 are the first four perfect squares, it is reasonable to conjecture that for all $n \in \mathbb{N}$, $1+3+\cdots+(2n-1) = n^2$.

(b) We prove by induction that $1 + 3 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$. For the induction basis, both sides of the equation equal 1. For the inductive step, fix $n \geq 1$ and suppose that

$$1 + 3 + \cdots + (2n - 1) \;=\; n^2.$$

We must show that

$$1 + 3 + \cdots + (2n - 1) + (2(n + 1) - 1) \;=\; (n + 1)^2.$$

Using the inductive hypothesis, we have

$$\begin{aligned} 1 + 3 + \cdots + (2n - 1) + (2(n + 1) - 1) &= n^2 + (2(n + 1) - 1) \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2, \end{aligned}$$

as desired. It now follows by the Principle of Induction that the equation is true for all $n \in \mathbb{N}$.

(4) The inequality $2^n > n^2$ is true for 0, 1, and every integer greater than or equal to 5. For 0 and 1 we have

$$0^2 = 0 < 1 = 2^0 \quad \text{and} \quad 1^2 = 1 < 2 = 2^1.$$

We prove that $n^2 < 2^n$ for all integers $n \geq 5$ by induction on n with base $n = 5$. For the induction base, we have

$$5^2 = 25 \;<\; 32 = 2^5.$$

For the inductive step, fix $n \geq 5$ and suppose that $n^2 < 2^n$, so that $n^2 + 1 \leq 2^n$. Since $n \geq 5 > 2$, we have $2n < n \cdot n = n^2$. Therefore

$$\begin{aligned}(n+1)^2 \;=\; n^2 + 2n + 1 \;&\leq\; 2^n + 2n \\ &<\; 2^n + n^2 \\ &<\; 2^n + 2^n \;=\; 2^{n+1}.\end{aligned}$$

It follows by induction that $n^2 < 2^n$ for all integers $n \geq 5$. Finally, that this inequality fails for $n \in \{2, 3, 4\}$ can be verified by direct computation.

(5) (a) bounded below but not above; $\inf = 1$
(b) bounded above and below; $\inf = 0$, $\sup = 1$
(c) bounded above and below; $\inf = 2$, $\sup = 7$
(d) bounded above and below; $\inf = e$, $\sup = \pi$
(e) bounded above and below; $\inf = 0$, $\sup = 1$
(f) bounded above and below; $\inf = \sup = 0$
(g) bounded above and below; $\inf = 0$, $\sup = 3$
(h) bounded below but not above; $\inf = 2$
(i) bounded above and below; $\inf = 0$, $\sup = 1$
(j) bounded above and below; $\inf = 2/3$, $\sup = 1$
(k) bounded below but not above; $\inf = 0$,
(l) bounded above but not below; $\sup = 2$
(m) bounded above and below; $\inf = -2$, $\sup = 2$
(n) bounded above and below; $\inf = -\sqrt{2}$, $\sup = \sqrt{2}$
(o) bounded above but not below; $\sup = 0$
(p) bounded above and below; $\inf = 1$, $\sup = 10$
(q) bounded above and below; $\inf = 0$, $\sup = 16$
(r) bounded above and below; $\inf = \sup = 1$
(s) bounded above and below; $\inf = 0$, $\sup = 1/2$
(t) bounded above but not below; $\sup = 2$
(u) bounded below but not above; $\inf = 0$
(v) bounded above and below; $\inf = -1$, $\sup = 1$
(w) bounded below but not above; $\inf = 0$
(x) bounded below but not above; $\inf = 1$

(6) Suppose for contradiction that $<$ is a linear order relation on $\mathbb{C}$ that makes $\mathbb{C}$ an ordered field. By Fact 22 from the handout, we know $0 < 1$, which by Axiom 13 implies $-1 < 0$. On the other hand, $i^2 = -1$, so by Fact 21 we have $0 < -1$. Thus by transitivity of $<$ (Axiom 10) we get $0 < 0$, which contradicts Axiom 11 (irreflexivity of $<$). It follows that there can be no linear order relation $<$ on $\mathbb{C}$ that satisfies the ordered field axioms.

(7) (a) Let $a, b \in \mathbb{R}$. We want to prove that if $a \leq c$ for every $c > b$, then $a \leq b$. We will instead prove the contrapositive of this statement, which is equivalent to it. The contrapositive is: "if $a > b$ then $a > c$ for some $c > b$." So assume $a > b$. Then if we let $r = \frac{a+b}{2}$, we will have $b < r < a$, so there is some $c > b$ (namely $c = r$) such that $a > c$, and we are done.

(b) Let $A \subseteq \mathbb{R}$, let $L \in \mathbb{R}$, and suppose L is an upper bound of A. Assume first that $L = \sup A$, and let $\epsilon > 0$. Then $L - \epsilon < L$, so $L - \epsilon$ is not an upper bound of A, which means there is $a \in A$ such that $L - \epsilon < a$ (and, of course, $a \leq L$ since $a \in A$ and $L = \sup A$). Conversely, assume that for every $\epsilon > 0$ there is $a \in A$ such that $L - \epsilon < a \leq L$. Let u be an arbitrary upper bound of A; we want to show $L \leq u$. By (a), it will suffice to show $L \leq u + \epsilon$ for all $\epsilon > 0$. But for each $\epsilon > 0$, we can fix $a \in A$ such that $L - \epsilon < a$ and therefore $L < a + \epsilon$, which implies $L < u + \epsilon$ since $a \leq u$. This completes the proof.

(8) Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Using $S \neq \emptyset$, fix $a \in S$. Then by definition of $\inf S$ and $\sup S$ we have

$$\inf S \ \leq \ a \ \leq \ \sup S,$$

and therefore $\inf S \leq \sup S$ by transitivity of $\leq$.

(b) Suppose that $S \subseteq T$. We claim that

$$\inf T \ \leq \ \inf S \ \leq \ \sup S \ \leq \ \sup T.$$

To see this, first note that $\inf T \leq s$ for all $s \in S$ by the definition of infimum and the fact that $S \subseteq T$. So $\inf T$ is a lower bound for S, and hence $\inf T \leq \inf S$ again by definition of infimum. Similarly, $\sup T \geq s$ for all $s \in S$, so $\sup T$ is an upper bound for S, which implies $\sup S \leq \sup T$. The fact that $\inf S \leq \sup S$ was shown in (a).

(c) We have $\sup S \leq \sup(S \cup T)$ and $\sup T \leq \sup(S \cup T)$ by part (b) above, so

$$\max\{\sup S, \sup T\} \ \leq \ \sup(S \cup T).$$

Now for contradiction suppose $\max\{\sup S, \sup T\} < \sup(S \cup T)$. Then there is $x \in S \cup T$ such that $\sup S < x$ and $\sup T < x$, which implies $x \notin S$ and $x \notin T$ even though $x \in S \cup T$, a contradiction.

(9) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let

$$S = A + B = \{a + b \ : \ a \in A \text{ and } b \in B\}.$$

Let $a_0 = \sup A$ and $b_0 = \sup B$. Then for all $a \in A$ and $b \in B$, we have $a \leq a_0$ and $b \leq b_0$, which gives us $a + b \leq a_0 + b_0$. Hence $a_0 + b_0$ is an upper bound for S. In particular S is bounded above, so it has a supremum, say $s_0 = \sup S$. Since $a_0 + b_0$ is an upper bound for S and s_0 is the *least* upper bound for S, we know that $s_0 \leq a_0 + b_0$.

Now, suppose for contradiction that $s_0 < a_0 + b_0$. Then there is a real number r such that $s_0 < r < a_0 + b_0$. Hence $r - a_0 < b_0$, so we can fix $b_1 \in B$ such that $r - a_0 < b_1$. Then $r - b_1 < a_0$, so we can fix $a_1 \in A$ such that $r - b_1 < a_1$. But then $s_0 < r < a_1 + b_1 \in S$, contradicting the fact that s_0 is an upper bound for S. We conclude that $s_0 = a_0 + b_0$.

(10) Let a and b be real numbers such that $a < b$, so that also $a - \sqrt{2} < b - \sqrt{2}$. Using density of $\mathbb{Q}$ in $\mathbb{R}$, choose $q \in \mathbb{Q}$ such that $a - \sqrt{2} < q < b - \sqrt{2}$, and let $r = q + \sqrt{2}$. Then $a < r < b$, and r is irrational since otherwise $\sqrt{2} = r - q$ would be rational. This show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

(11) (a) Let A be a finite subset of $\mathbb{R}$, and let n be the natural number such that A has n elements. (If $A = \emptyset$ then A is vacuously discrete, so we may assume $n \geq 1$). Then we can write $A = \{a_1, \ldots, a_n\}$. Let $\epsilon = \min\{|a_i - a_j| \,:\, 1 \leq i, j \leq n\}$. Then $V_\epsilon(a) \cap A = \{a\}$ for every $a \in A$.

(b) This statement is false. For a counterexample, let $A = \{\frac{1}{n} \,:\, n \in \mathbb{N}\}$. Then A is discrete, since for each n we have $V_\epsilon(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$ where $\epsilon = \frac{1}{n} - \frac{1}{n+1}$. But there is no $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for all $a \neq b \in A$, since given any $\epsilon > 0$ we can choose $n > \frac{1}{\epsilon}$, and then $|\frac{1}{n} - \frac{1}{n+1}| < \frac{1}{n} < \epsilon$.

Homework 2

(1) For the induction base $n = 1$, we have that for all $a_n \in \mathbb{R}$,

$$\left|\sum_{k=1}^{1} a_k\right| = |a_1| \le |a_1| = \sum_{k=1}^{1} |a_k|.$$

For the inductive step, let $n \in \mathbb{N}$ and assume for inductive hypothesis that for all $a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\left|\sum_{k=1}^{n} a_k\right| \le \sum_{k=1}^{n} |a_k|.$$

Let $a_1, \ldots, a_n, a_{n+1} \in \mathbb{R}$. Then, using the Triangle inequality and inductive hypothesis, we have

$$\left|\sum_{k=1}^{n+1} a_k\right| = \left|\left(\sum_{k=1}^{n} a_k\right) + a_{n+1}\right| \le \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| \le \left(\sum_{k=1}^{n} |a_k|\right) + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k|.$$

This completes the inductive step, and we are done.

(2) Let $A \subseteq \mathbb{R}$ be bounded, and let $c \in \mathbb{R}$. We claim that

$$\sup(cA) = \begin{cases} c \cdot \sup A & \text{if } c \ge 0 \\ c \cdot \inf A & \text{if } c < 0. \end{cases}$$

To see this, first suppose $c > 0$. Let $x \in cA$ be arbitrary, and fix $a \in A$ such that $x = ca$. Then $a \le \sup A$, so $x = ca \le c \cdot \sup A$. This shows that $c \cdot \sup A$ is an upper bound of cA. Now let u be an arbitrary upper bound of cA, and let $a \in A$. Then $ca \in cA$, so $ca \le u$, so $a \le \frac{u}{c}$. As $a \in A$ was arbitrary, this implies $\sup A \le \frac{u}{c}$, so $c \cdot \sup A \le u$. We conclude that $c \cdot \sup A$ is indeed the supremum of cA, as desired.

Next, note that if $c = 0$ then we have $cA = \{0\}$ so $\sup(cA) = 0$, and also $c \cdot \sup A = 0 \cdot \sup A = 0$, as desired. Finally, suppose $c < 0$. We can mimic our argument from above. Let $x \in cA$ be arbitrary, and fix $a \in A$ such that $x = ca$. Then $a \ge \inf A$, so $x = ca \le c \cdot \inf A$. This shows that $c \cdot \inf A$ is an upper bound of cA. Now let u be an arbitrary upper bound of cA, and let $a \in A$. Then $ca \in cA$, so $ca \le u$, so $a \ge \frac{u}{c}$. As $a \in A$ was arbitrary, this implies $\inf A \ge \frac{u}{c}$, so $c \cdot \inf A \le u$. As before, we conclude that $c \cdot \inf A$ is the supremem of cA, as desired. This completes the proof that our expression is correct.

Finally, the dual statement is:

$$\inf(cA) = \begin{cases} c \cdot \inf A & \text{if } c \ge 0 \\ c \cdot \sup A & \text{if } c < 0. \end{cases}$$

(3) (a) Suppose $f : X \to Y$ and $g : Y \to Z$ are injective. Let $x_1, x_2 \in X$, and suppose $x_1 \ne x_2$. Then since f is injective, $f(x_1) \ne f(x_2)$. Therefore, since g is injective, $g(f(x_1)) \ne g(f(x_2))$. It follows that $g \circ f$ is injective.

(b) To see that $\preceq$ is reflexive, note that for every set X, the identity map $\mathrm{id}_X : X \to X$ is an injective function from X to itself, so $X \preceq X$. To see that $\preceq$ is transitive, suppose $X \preceq Y$ and $Y \preceq Z$, and fix injective functions $f : X \to Y$ and $g : Y \to Z$. By part (a), $g \circ f$ is an injective function from X to Z, which shows $X \preceq Z$.

(4) (a) Let A and B be nonempty sets, and suppose $A \subseteq B$. Let $f : A \to B$ be the function defined by $f(a) = a$ for all $a \in A$. Then f is injective, so $A \preceq B$.

(b) Let A and B be nonempty sets. Suppose $f : A \to B$ is an injective function. Using $A \neq \emptyset$, fix $a_0 \in A$. Define $g : B \to A$ so that $g(b)$ is the unique element $a \in A$ such that $f(a) = b$ if $b \in \mathrm{ran}(f)$, and $g(b) = a_0$ otherwise. Then $g : B \to A$ is surjective, because for every $a \in A$ we have $g(f(a)) = a$. Conversely, suppose $g : B \to A$ is a surjective function. Using the fact that g is surjective, for each $a \in A$ choose an element $b \in B$ such that $g(b) = a$, and let $b = f(a)$. This defines a function $f : A \to B$, and f is injective since g is a function.

(5) (a) Suppose A is infinite and that $A_0 = \{x_1, \ldots, x_n\}$ is a finite subset of A. Since a union of two finite sets is finite, $A \setminus A_0$ is infinite, so we may fix an injective function $g : \mathbb{N} \to A \setminus A_0$. Let $B = \mathrm{ran}(g)$. Define the function $f : A \to A \setminus A_0$ by

$$f(x) = \begin{cases} x & \text{if } x \in A \setminus (B \cup A_0); \\ g(n) & \text{if } x = x_n \in A_0; \\ g\big(g^{-1}(x) + n\big) & \text{if } x \in B. \end{cases}$$

Then f is injective, so $A \preceq A \setminus A_0$. Since $A \setminus A_0 \preceq A$ by (2), we conclude that $A \approx A \setminus A_0$ by the Cantor-Schroder-Bernstein Theorem.

(b) Suppose A is uncountable and that A_0 is a countable subset of A. Fix a surjection $h : \mathbb{N} \to A_0$. Since a union of two countable sets is countable, $A \setminus A_0$ is uncountable, in particular infinite, so we may fix an injective function $g : \mathbb{N} \to A \setminus A_0$. Let $B = \mathrm{ran}(g)$. Define the function $f : A \setminus A_0 \to A$ by

$$f(x) = \begin{cases} x & \text{if } x \in A \setminus (B \cup A_0); \\ g\big(g^{-1}(b)/2\big) & \text{if } b \in B \text{ and } g^{-1}(b) \text{ is even;} \\ h\big((g^{-1}(b) + 1)/2\big) & \text{if } b \in B \text{ and } g^{-1}(b) \text{ is odd.} \end{cases}$$

Then f is surjective, so $A \preceq A \setminus A_0$. Since $A \setminus A_0 \preceq A$ by (2), we conclude that $A \approx A \setminus A_0$ by the Cantor-Schroder-Bernstein Theorem.

(6) (a) Recall that a number belongs to $\overline{\mathbb{Q}}$ iff it is a root of a polynomial with integer coefficients. For each $n \in \mathbb{N}$, let P_n be the set of polynomials with integer coefficients of degree n. Then for each n, the function

$$f_n : \mathbb{Z}^{n+1} \to P_n$$

defined by

$$f(a_0, \ldots, a_n) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a bijection from a finite Cartesian product of copies of $\mathbb{Z}$ to P_n. Since $\mathbb{Z}$ is countable and finite Cartesian products of countable sets are countable, each P_n is countable. Since a union of countably many countable sets is countable, the set

$$P = \bigcup_{n\in\mathbb{N}} P_n$$

of all polynomials with integers coefficients is countable. Finally, by the Fundamental Theorem of Algebra, every polynomial has finitely many roots. For each $p \in P$, let R_p be the finite set of (complex) roots of p. Then $\overline{\mathbb{Q}} = \bigcup_{p\in P} R_p$ is a countable union of finite sets, and is therefore countable. Finally, since $\overline{\mathbb{Q}}$ is countable, so is $\overline{\mathbb{Q}} \cap \mathbb{R}$. But then $\mathbb{R} \setminus \overline{\mathbb{Q}}$ must be uncountable, since $\mathbb{R}$ is uncountable. Hence there are uncountably many transcendental real numbers!

(b) Suppose $a < b$. From lecture we know that (a, b) is uncountable. Since $\overline{\mathbb{Q}}$ is countable by part (a), we know $\overline{\mathbb{Q}} \cap (a, b)$ is countable as well. Then $(a, b) \setminus \overline{\mathbb{Q}}$ must be uncountable by the same reasoning used in part (a).

(7) (a) Define the function $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^{\mathbb{R}}$ as follows. For each $A \subseteq \mathbb{R}$, let $F(A)$ be the function $F(A) : \mathbb{R} \to \mathbb{R}$ given by

$$F(A)(x) \;=\; \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

We claim that F is injective. Let $A, B \subseteq \mathbb{R}$ with $A \neq B$. Then there is some $x \in \mathbb{R}$ such that $x \in (A \setminus B) \cup (B \setminus A)$, so $F(A)(x) \neq F(B)(x)$, and therefore $F(A) \neq F(B)$. Thus F is injective, so $\mathcal{P}(\mathbb{R}) \preceq \mathbb{R}^{\mathbb{R}}$.

(b) By part (a) and 4(b), there is a surjection $g : \mathbb{R}^{\mathbb{R}} \to \mathcal{P}(\mathbb{R})$. Suppose for contradiction that there is a surjective function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{R}}$. Then $g \circ f$ would be a surjective function from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$, which is impossible by Cantor's Theorem.

(8) (a) Let $\epsilon > 0$. Using the Archimedian property of $\mathbb{R}$, fix $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$,

$$\left| \frac{(-1)^n}{n} - 0 \right| \;=\; \frac{1}{n} \;\leq\; \frac{1}{N} \;<\; \epsilon.$$

This shows that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

(b) Let $\epsilon > 0$, and fix $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$,

$$\left| \frac{n}{n+1} - 1 \right| \;-\; \left| \frac{n}{n+1} - \frac{n+1}{n+1} \right| \;=\; \left| \frac{-1}{n+1} \right| \;=\; \frac{1}{n+1} \;\leq\; \frac{1}{N} \;<\; \epsilon.$$

This show that $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$.

(9) Let (a_n) be a sequence in $\mathbb{R}$ that converges to $L \in \mathbb{R}$, and let $\epsilon > 0$. Fix $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Then for all $n \geq N$ we have $||a_n| - |L|| \leq |a_n - L| < \epsilon$ by the Triangle Inequality. This shows $\lim\limits_{n\to\infty} |a_n| = |L|$.

(10) Suppose $\lim a_k = L$. We prove by induction on n that $\lim a_k^n = L^n$ for all $n \in \mathbb{N}$. The base case $n = 1$ is given. Fix $n \geq 1$ and suppose $\lim a_k^n = L^n$. Then

$$\lim a_k^{n+1} \;=\; \lim a_k^n \cdot a_k \;=\; (\lim a_k^n) \cdot (\lim a_k) \;=\; L^n \cdot L \;=\; L^{n+1}.$$

(11) Suppose $\lim a_n = L$. Let $\epsilon > 0$. Fix N such that $|a_n - L| < \epsilon/2$ whenever $n \geq N$. Then for all $n \geq N$,

$$|s_n - 0| \;=\; |a_{n+1} - a_n| \;\leq\; |a_{n+1} - L| + |L - a_n| \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\lim s_n = 0$.

(12) Let S be a bounded nonempty subset of $\mathbb{R}$, and let $L = \sup S$. For every $n \in \mathbb{N}$ there is $a_n \in S \cap \left(L - \frac{1}{n}, L\right]$, since otherwise $L - \frac{1}{n}$ would be an upper bound for S, contradicting the fact that L is the *least* upper bound for S. So we may fix, for each n, an element $a_n \in S \cap \left(L - \frac{1}{n}, L\right]$. Then (a_n) is a sequence in S and $\lim a_n = L$, as desired.

Homework 3

(1) The forward implication is true, while the backward implication is false.

Proof of forward implication: Suppose $\lim a_n = \infty$. Let $\epsilon > 0$. Fix $N \in \mathbb{N}$ such that $a_n \geq \frac{1}{\epsilon}$ whenever $n \geq N$. Then for all $n \geq N$ we have $0 < \frac{1}{a_n} < \epsilon$. Since $\epsilon > 0$ was arbitrary, this shows $\lim \frac{1}{a_n} = 0$.

Counterexample for backward implication: Let $a_n = n(-1)^n$ for each $n \in \mathbb{N}$, so $(a_n) = (-1, 2, -3, 4, -5, 6, \ldots)$. Then $\lim a_n \neq \infty$ but $\lim \frac{1}{a_n} = 0$.

(2) Suppose $\lim a_n = 0$ and (b_n) is bounded, say $|b_n| < M$ for all n. Let $\epsilon > 0$, and fix N such that $|a_n| < \epsilon/M$ whenever $n \geq N$. The for all $n \geq N$,

$$|a_n b_n - 0| \;=\; |a_n| \cdot |b_n| \;<\; \frac{\epsilon}{M} \cdot M \;=\; \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\lim |a_n b_n| = 0$, and therefore $\lim a_n b_n = 0$.

(3) (a) We prove by induction on n that $n \cdot 2^n < n!$ for all $n \geq 6$. For the induction base, $6 \cdot 2^6 = 384 < 720 = 6!$. Fixing $n \geq 6$ and supposing $n \cdot 2^n < n!$, we have

$$(n+1) \cdot 2^{n+1} \;=\; 2(n \cdot 2^n + 2^n) \;<\; 2(n! + 2^n) \;<\; 2(n! + n!) \;=\; 4n! \;<\; (n+1)!$$

where the first two inequalities follow from the inductive hypothesis. Now, let $\epsilon > 0$ and fix N such that $\frac{1}{N} < \epsilon$. The for all $n \geq N$ we have

$$\left| \frac{2^n}{n!} - 0 \right| \;=\; \frac{2^n}{n!} \;<\; \frac{1}{n} \;\leq\; \frac{1}{N} \;<\; \epsilon.$$

This shows $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

(b) We prove by induction that for all $n \geq 1$, $n^n \geq n \cdot n!$. The inductive base is clear. Fixing $n \geq 1$ and supposing $n^n \geq n \cdot n!$, we have

$$(n+1)^{n+1} \;=\; (n+1)(n+1)^n \;>\; (n+1)n^n \;\geq\; (n+1)n \cdot n! \;=\; n \cdot (n+1)!$$

as needed. Thus $\dfrac{n^n}{n!} \geq n$ for all $n \in \mathbb{N}$, which shows $\displaystyle\lim_{n\to\infty} \frac{n^n}{n!} = \infty$.

(c) First we show by induction that $b_n \geq \sqrt{2}$ for all n. For the base case, we have $b_1 = 2 > \sqrt{2}$. For the induction, fix $n \geq 1$ and suppose $b_n \geq \sqrt{2}$. Then

$$b_n^2 - 2\sqrt{2} b_n + 2 \;=\; (b_n - \sqrt{2})^2 \;\geq\; 0,$$

which implies $b_n^2 + 2 \geq 2\sqrt{2} b_n$, and hence

$$b_{n+1} \;=\; \frac{b_n^2 + 2}{2b_n} \;\geq\; \sqrt{2}$$

as claimed. Now, this result implies that $b_n^2 \geq 2$, and hence $2b_n^2 \geq b_n^2 + 2$, for all $n \in \mathbb{N}$. Dividing both sides by $2b_n$, we get

$$b_{n+1} \;=\; \frac{b_n^2 + 2}{2b_n} \;\leq\; b_n$$

for all $n \in \mathbb{N}$. We have now shown that (b_n) is decreasing and bounded below, thus it converges, say to L. To find L, note that

$$L \;=\; \lim b_n \;=\; \lim b_{n+1} \;=\; \lim \frac{b_n^2+2}{2b_n} \;=\; \frac{L^2+2}{2L}.$$

Hence $L^2 = 2$, so $L = \sqrt{2}$.

(4) Let A be a discrete subset of $\mathbb{R}$ and let (a_n) be a convergent sequence of numbers in A. In order to prove that (a_n) is eventually constant or $\lim a_n \notin A$, we will assume that $\lim a_n \in A$ and show that this forces (a_n) to be eventually constant. Thus suppose $\lim a_n = L \in A$. Using the fact that A is discrete, fix $\epsilon > 0$ such that $V_\epsilon(L) \cap A = \{L\}$. Using the fact that $\lim a_n = L$, fix N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. Then for all $n \geq N$, we have both $a_n \in A$ and $a_n \in V_\epsilon(L)$, which implies $a_n = L$. So $a_n = L$ for all $n \geq N$, which means (a_n) is eventually constant, as desired.

(5) Let M be a positive integer, and let $\mathbb{Q}_M$ be the set of all rational numbers m/n where $|m| \leq M$. We claim that for every $\epsilon > 0$, the set $\mathbb{Q}_M \setminus V_\epsilon(0)$ is finite. To see this, let $\epsilon > 0$ be arbitrary and fix $N \in \mathbb{N}$ large enough so that $M/N < \epsilon$. Then $|m| \leq M$ and $|n| < N$ for every rational number m/n belonging to $\mathbb{Q}_M \setminus V_\epsilon(0)$, showing that $\mathbb{Q}_M \setminus V_\epsilon(0)$ has at most $2N(2M+1)$ elements and hence is indeed finite.

Now, let (a_n) be a sequence of distinct numbers in $\mathbb{Q}_M$, and let $\epsilon > 0$. Since $\mathbb{Q}_M \setminus V_\epsilon(0)$ is finite and the terms of (a_n) are distinct, there cannot be infinitely many different indices n such that $a_n \in \mathbb{Q}_M \setminus V_\epsilon(0)$; hence there is N such that $a_n \in V_\epsilon(0)$ whenever $n \geq N$. Since ϵ was arbitrary, this shows that $\lim a_n = 0$; in particular, (a_n) converges. As (a_n) was arbitrary, we have shown that every sequence of distinct numbers in $\mathbb{Q}_M$ converges.

(6) (a) Suppose $a_n < b_n$ for all n, and assume $\lim a_n = \infty$. Let $M > 0$, and fix N such that $a_n > M$ whenever $n \geq N$. Then $b_n > a_n > M$ for all $n \geq N$. Since M was arbitrary, this shows $\lim b_n = \infty$ as well.
(b) For instance, let $a_n = -2/n$ and $b_n = -1/n$ for each n.

(7) Suppose $\lim \frac{a_{n+1}}{a_n} = L > 1$, and fix r such that $1 < r < L$. Write $r = 1 + a$, so $a > 0$. Let $M > 0$, and fix N large enough so that $\frac{a_{n+1}}{a_n} > r$ and $1 + na > \frac{M}{a_0}$ whenever $n \geq N$. Using Bernoulli's inequality, we have that for all $n \geq N$,

$$a_n \;>\; a_0 r^n \;=\; a_0(1+a)^n \;\geq\; a_0(1+na) \;>\; a_0 \cdot \frac{M}{a_0} \;=\; M.$$

Since M was arbitrary, this shows that $\lim a_n = \infty$.

(8) (a) $\liminf(a_n) = -1$, $\limsup(a_n) = 1$
(b) $\liminf(b_n) = 0$, $\limsup(b_n) = 0$
(c) $\liminf(c_n) = -\infty$, $\limsup(c_n) = \infty$
(d) $\liminf(d_n) = \infty$, $\limsup(d_n) = \infty$

(9) In class we showed that (s_n) is Cauchy, so it converges; let $L = \lim s_n$. Note that $s_2 + 2s_3 = a + 2b$, and that for all $n \geq 1$, if $s_{n+1} + 2s_{n+2} = a + 2b$ then

$$s_{n+2} + 2s_{n+3} \;=\; s_{n+2} + 2\left(\frac{s_{n+1} + s_{n+2}}{2}\right) \;=\; s_{n+1} + 2s_{n+2} \;=\; a + 2b.$$

Therefore $s_{n+1} + 2s_{n+2} = a + 2b$ for all $n \in \mathbb{N}$ by induction. But then

$$3L \;=\; \lim s_{n+1} + 2\lim s_{n+2} \;=\; \lim\big(s_{n+1} + 2s_{n+2}\big) \;=\; a + 2b,$$

so $L = \frac{a+2b}{3}$.

(10) For instance, let $a_n = \begin{cases} n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

(11) (a) By HW 1 #9, we have $\sup\{a_i + b_j \,:\, i, j \geq n\} = \sup\{a_i \,:\, i \geq n\} + \sup\{b_j \,:\, j \geq n\}$ for each $n \in \mathbb{N}$. By HW 1 #8(b), we have $\sup\{a_k + b_k \,:\, k \geq n\} \leq \sup\{a_i + b_j \,:\, i, j \geq n\}$ for each $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \limsup(a_n + b_n) &= \lim_{n\to\infty} \sup\{a_k + b_k \,:\, k \geq n\} \\ &\leq \lim_{n\to\infty} \sup\{a_i + b_j \,:\, i, j \geq n\} \\ &= \lim_{n\to\infty} \Big(\sup\{a_i \,:\, i \geq n\} + \sup\{b_j \,:\, j \geq n\}\Big) \;=\; \limsup(a_n) + \limsup(b_n). \end{aligned}$$

(b) For instance, let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, so $\limsup(a_n + b_n) = 0$ but $\limsup(a_n) + \limsup(b_n) = 2$.

(c) Suppose $\lim a_n = L$, so also $\limsup(a_n) = L$, and let (b_{n_k}) be a subsequence of (b_n) converging to $\limsup(b_n)$. Then $\lim a_{n_k} = L$, so

$$\limsup(a_n) + \limsup(b_n) \;=\; \lim(a_{n_k} + b_{n_k}) \;\leq\; \limsup(a_n + b_n).$$

We showed the reverse inequality in part (a), so we conclude that $\limsup(a_n) + \limsup(b_n) = \limsup(a_n + b_n)$.

(12) Let $s : \mathbb{N} \to \mathbb{Q}$ be any surjection, and let $r \in \mathbb{R}$. Let $n_1 = 1$, fix $k \geq 1$, and suppose n_i has been defined for all $1 \leq i \leq k$ so that $n_i < n_j$ whenever $1 \leq i < j \leq k$ and such that $|s_{n_i} - r| < \frac{1}{i}$ for each $1 \leq i \leq k$. Then $\mathbb{Q} \cap (r - \frac{1}{k+1}, r + \frac{1}{k+1})$ is infinite, so we may choose $n_{k+1} > n_k$ such that $|s_{n_{k+1}} - r| < \frac{1}{k+1}$. Then $(s_{n_k}) \to r$.

(13) (a) neither

(b) closed

(c) open

(d) closed

(e) neither

(f) closed

(14) The statement is FALSE. For instance, for each $n \in \mathbb{N}$ let $a_n = \sum_{k=1}^{n} \frac{1}{k}$, and let $A = \{a_n : n \in \mathbb{N}\}$. Then A is closed and discrete, but there is no $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$.

(15) Suppose $A \subseteq \mathbb{R}$ is infinite, bounded, and discrete. Since A is infinite, we may fix an injective function $s : \mathbb{N} \to A$, which we view as a sequence in A. Then (s_n) is bounded, and thus has a convergent subsequence (s_{n_k}) by the Bolzano-Weierstrass Theorem. Write $L = \lim s_{n_k}$. By the definition of convergence, for every $\epsilon > 0$ there is $k \in \mathbb{N}$ such that the distinct points s_{n_k} and $s_{n_{k+1}}$ both belong to $V_\epsilon(L)$. This shows that for every $\epsilon > 0$, $V_\epsilon(L)$ contains some point of $A \setminus \{L\}$, and hence $L \notin A$ since A is discrete.

Homework 4

(1) No, (a_n) might diverge even if $\lim(a_{n+1} - a_n) = 0$. For instance, consider the sequence of partial sums of the harmonic series, i.e., let $a_n = \sum_{k=1}^n \frac{1}{k}$ for each $n \in \mathbb{N}$. Then $a_{n+1} - a_n \to 0$ but $a_n \to \infty$.

(2) Let (a_n) be a sequence in $\mathbb{R}$, and let $S \subseteq \mathbb{R}$ be its set of real subsequential limits. We show that $U = \mathbb{R} \setminus S$ is open, and hence S is closed. Let $x \in U$. Then there is no subsequence of (a_n) converging to x, so we may fix $\epsilon > 0$ and $N \in \mathbb{N}$ such that $a_n \notin V_{2\epsilon}(x)$ for all $n \geq N$. Now let $y \in V_\epsilon(x)$ be arbitrary. Then $V_\epsilon(y) \subseteq V_{2\epsilon}(x)$, so $a_n \notin V_\epsilon(y)$ for all $n \geq N$, which implies $y \notin S$. This shows $V_\epsilon(x) \subseteq U$, completing the proof.

Alternative proof sketch: Let (s_n) be a convergent sequence in S with limit L. It will suffice to show $L \in S$. Fix n_1 such that $|a_{n_1} - s_1| < 1$, and assuming $n_1 < n_2 < \cdots < n_m$ have been defined so that $|a_{n_k} - s_k| < \frac{1}{k}$ for all $1 \leq k \leq m$, choose $n_{m+1} > n_m$ such that $|a_{n_{m+1}} - s_{m+1}| < \frac{1}{m+1}$. Then $L = \lim\limits_{k\to\infty} a_{n_k}$ (why?) so $L \in S$.

(3) (a) Let $A \subseteq \mathbb{R}$, let $U = \mathbb{R} \setminus A'$, and let $x \in U$, so x is not a limit point of A. Thus we may fix $\epsilon > 0$ such that $|x - a| \geq \epsilon$ for all $a \in A \setminus \{x\}$. We claim $V_\epsilon(x) \subseteq U$. Indeed, given $y \in V_\epsilon(x) \setminus \{x\}$ let $\delta(y) = \min\{|y-(x+\epsilon)|, |y-(x-\epsilon)|, |y-x|\}$. Then $V_{\delta(y)}(y) \subseteq V_\epsilon(x) \setminus \{x\}$, and hence $V_{\delta(y)}(y) \cap A = \emptyset$, so $y \in U$.

(b) Let $A \subseteq \mathbb{R}$, let $U = \mathbb{R} \setminus \mathrm{cl}(A)$, and let $x \in U$. Then $x \notin A$ and x is not a limit point of A. Hence there is $\epsilon > 0$ such that $V_{2\epsilon}(x)$ does not contain any points of A. But now for any $y \in V_\epsilon(x)$, we have $V_\epsilon(y) \subseteq V_{2\epsilon}(x)$, so that $V_\epsilon(y)$ does not contain any points in A and hence y is neither in A nor a limit point of A. Thus $V_\epsilon(x) \subseteq U$, and since $x \in U$ was arbitrary this shows that U is open and hence $\mathrm{cl}(A)$ is closed.

Alternative proof sketch: Let $A \subseteq \mathbb{R}$ and let (a_n) be a convergent sequence of points in $\mathrm{cl}(A)$ with limit L. Then by definition of convergence, every open neighborhood of L contains points of A. So the only way L can fail to be a limit point of A is if some open neighborhood of L contains no points of A other than L itself, which means $L \in A$. Either way we have $L \in A \cup A' = \mathrm{cl}(A)$. This shows that $\mathrm{cl}(A)$ is closed.

(c) Let $A \subseteq \mathbb{R}$ and let F be any closed set containing A. Let $U = \mathbb{R} \setminus F$, so U is open, and let $x \in U$. Then there is $\epsilon > 0$ such that $V_\epsilon(x) \subseteq U$, and hence $V_\epsilon(x) \cap A = \emptyset$. Thus x is neither in A nor a limit point of A, so $x \notin \mathrm{cl}(A)$. Since $x \in U$ was arbitrary, this shows $U \subseteq \mathbb{R} \setminus \mathrm{cl}(A)$, and hence $\mathrm{cl}(A) \subseteq F$.

(4) (a) Let $\epsilon > 0$, and set $\delta = \min\{1, \epsilon/19\}$. Suppose $|x - 2| < \delta$, so in particular $1 < x < 3$ and hence $7 < |x^2 + 2x + 4| < 19$. Then $|x^3 - 8| = |x - 2| \cdot |x^2 + 2x + 4| < 19\delta \leq \epsilon$. This shows $\lim\limits_{x\to 2} x^3 = 8$.

(b) $\delta = \sqrt[3]{8 + \epsilon} - 2$.

(c) Let $\epsilon > 0$, and set $\delta = \min\{3, 3\epsilon\}$. Suppose $|x-4| < \delta$, so in particular $1 < x < 7$ and hence $|\sqrt{x}+2| > 3$. Then $|\sqrt{x}-2| = \dfrac{|x-4|}{|\sqrt{x}+2|} < \dfrac{\delta}{3} \leq \epsilon$. This shows $\lim_{x\to 4} \sqrt{x} = 2$.

(d) for $\epsilon \leq 2$, $\delta = 4-(2-\epsilon)^2$; for $\epsilon > 2$, $\delta = (2+\epsilon)^2 - 4$.

(5) Let $\epsilon > 0$ be arbitrary, and using the fact that $\lim_{x\to\infty} g(x) = L$ fix $N > c$ such that $|g(x)-L| < \epsilon$ whenever $x > N$. Using the fact that $\lim_{x\to a^+} f(x) = \infty$, fix $\delta > 0$ such that $f(x) > N$ whenever $0 < x-a < \delta$. Then $|(g\circ f)(x) - L| < \epsilon$ whenever $0 < x-a < \delta$, and we conclude that $\lim_{x\to a^+} (g\circ f)(x) = L$.

(6) For instance, let $a = b = L = 0$, let $f(x) = 0$ for all $x \in \mathbb{R}$, and let $g(x) = \begin{cases} 0 & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = L$, but $\lim_{x\to a} (g\circ f) = 1 \neq L$.

(7) Let (a_n) be a sequence of nonzero real numbers. Let $L > \limsup \left|\frac{a_{n+1}}{a_n}\right|$ be arbitrary, and fix N such that $\left|\frac{a_{n+1}}{a_n}\right| < L$ for all $n \geq N$. Then for any $n > N$,

$$|a_n| \;=\; \left|\frac{a_n}{a_{n-1}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|,$$

so $|a_n| < L^{n-N} \cdot |a_N| = L^n\big(|a_N|/L^N\big)$. Therefore

$$|a_n|^{1/n} \;<\; L\left(\frac{a_N}{L^N}\right)^{1/n}$$

for all $n > N$. Since $a_N/L^N > 0$ and $c^{1/n} \to 1$ as $n \to \infty$ for all $c > 0$, we obtain

$$\limsup |a_n|^{1/n} \;<\; \limsup L\left(\frac{a_N}{L^N}\right)^{1/n} \;=\; \lim_{n\to\infty} L\left(\frac{a_N}{L^N}\right)^{1/n} \;=\; L.$$

As $L > \limsup \left|\frac{a_{n+1}}{a_n}\right|$ was arbitrary, the claim follows.

(8) Assume the hypotheses, and let $\eta = f(a)/2$. Using continuity of f at a, fix $\epsilon > 0$ such that $|f(x)-f(a)| < \eta$ whenever $|x-a| < \epsilon$. The for all $x \in A \cap V_\epsilon(a)$, we have

$$0 \;<\; \frac{f(a)}{2} \;=\; f(a) - \eta \;<\; f(x) \;<\; f(a) + \eta.$$

Thus f is positive and bounded on $V_\epsilon(a)$.

(9) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Let $x \in \mathbb{R}$ be arbitrary, and using density of $\mathbb{Q}$ let (x_n) be a sequence of rationals converging to x. Using continuity of f and g at x and the fact that $f(x_n) = g(x_n)$ for each n, we have $f(x) = \lim f(x_n) = \lim g(x_n) = g(x)$. Since x was arbitrary, this shows $f = g$.

(10) Suppose $A \subseteq \mathbb{R}$ is not closed, and fix $x_0 \in A' \setminus A$. Define $f : A \to \mathbb{R}$ by $f(x) = \frac{1}{|x-x_0|}$ for all $x \in A$. Since $x_0 \notin A$, f is well-defined. Since polynomials and the absolute value function are continuous and since quotients and compositions of continuous functions are continuous, f is continuous. Finally, we have $\lim_{x\to x_0} f(x) = +\infty$ since x_0 is a limit point of A, so f is unbounded on A.

(11) First suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and let $V \subseteq \mathbb{R}$ be open. Let $a \in f^{-1}(V)$, so $f(a) \in V$. Using the fact that V is open, fix $\epsilon > 0$ such that $V_\epsilon(f(a)) \subseteq V$. Using continuity of f, fix $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Then $V_\delta(a) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is open.

Conversely, suppose $f^{-1}(V)$ is open for every open set $V \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$, and let $\epsilon > 0$. Then $f^{-1}\big(V_\epsilon(f(a))\big)$ is open and contains a, so we may fix $\delta > 0$ such that $V_\delta(a) \subseteq f^{-1}\big(V_\epsilon(f(a))\big)$. Then $|f(x) - f(a)| < \epsilon$ whenenver $|x - a| < \delta$. This shows that f is continuous at a, and since a was arbitrary, this shows that f is continuous.

<u>Homework 5</u>

(1) Let $\{I_i : i \in I\}$ be a family of nonempty open sets in $\mathbb{R}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Using density of $\mathbb{Q}$, for each $i \in I$ fix $q_i \in \mathbb{Q} \cap I_i$. Then the map $i \mapsto q_i$ is an injective function from I to $\mathbb{Q}$, so I must be countable since $\mathbb{Q}$ is countable.

(2) (a) $y = x^3$ is uniformly continuous on $[0, 1]$ since $[0, 1]$ is closed and bounded.

(b) $y = x^3$ is uniformly continuous on $(0, 1)$ by (a) since $(0, 1) \subseteq [0, 1]$

(c) $y = x^3$ is not uniformly continuous on $\mathbb{R}$. Set $\epsilon = 1$, let $\delta > 0$, and set $a = \frac{1}{\delta}$. Then

$$|(a + \delta)^3 - a^3| = |(a + \delta) - a| \cdot |(a + \delta)^2 + a + a^2| = \delta \cdot |(a + \delta)^2 + a + a^2| \geq \delta \cdot \frac{1}{\delta} = \epsilon.$$

So for arbitrary $\delta > 0$ we have found two points within δ of each other whose values under the function are at least ϵ away from each other.

(d) $y = 1/x^3$ is not uniformly continuous on $(0, 1]$, since the harmonic sequence $(a_n) = (\frac{1}{n})$ is a Cauchy sequence in $(0, 1]$ such that $(\frac{1}{a_n^3}) = (n^3)$ is not Cauchy.

(3) Let $a > 0$, and suppose f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$. Then f is uniformly continuous on $[0, a]$ since $[0, a]$ is closed and bounded.

Now let $\epsilon > 0$ be arbitrary, and fix $\delta_1, \delta_2 > 0$ such that for all $x, y \in [0, a]$, $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta_1$, and for all $x, y \in [a, \infty)$, $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$, and let $x, y \in [0, \infty)$ be arbitrary with $|x - y| < \delta$. If $x, y \leq a$ then $|f(x) - f(y)| < \epsilon/2 < \epsilon$ since $\delta_1 \leq \delta$, and if $x, y \geq a$ then again $|f(x) - f(y)| < \epsilon/2 < \epsilon$ since $\delta_2 \leq \delta$. Finally, if one of x, y belongs to $[0, a]$ and one to $[a, \infty)$, then $|x - a| < \delta$ and $|y - a| < \delta$ so

$$|f(x) - f(y)| \ \leq \ |f(x) - f(a)| + |f(y) - f(a)| \ < \ \frac{\epsilon}{2} + \frac{\epsilon}{2} \ = \ \epsilon.$$

In all cases we see that $|f(x) - f(y)| < \epsilon$, and we conclude that f is uniformly continuous on $[0, \infty)$.

(4) (a) Let (a_n) and (b_n) be sequences in A that converge to $a \in A' \setminus A$. Fix $\epsilon_0 > 0$ such that f is uniformly continuous on $V_{\epsilon_0}(a) \cap A$. Let $\epsilon > 0$ be arbitrary. Using uniform continuity of f on $V_{\epsilon_0}(a) \cap A$, fix $\delta > 0$ such that for all $x, y \in V_{\epsilon_0}(a) \cap A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Fix $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\delta}{2}$ for all $n \geq N_1$ and $|b_n - a| < \frac{\delta}{2}$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, and suppose $m, n \geq N$. Then $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \delta$, so $|f(a_n) - f(a_m)| < \epsilon$, and likewise $|f(b_n) - f(b_m)| < \epsilon$. This shows $f(a_n)$ and $f(b_n)$ are Cauchy, and therefore converge. But also $|a_n - b_n| \leq |a_n - a| + |b_n - a| < \delta$, so $|f(a_n) - f(b_n)| < \epsilon$, which shows that $f(a_n) - f(b_n) \to 0$, so we conclude $\lim f(a_n) = \lim f(b_n)$.

(b) Again let $\epsilon_0 > 0$ be such that f is uniformly continuous on $V_{\epsilon_0}(a) \cap A$. Since $a \in A' \setminus A$, we can choose a sequence (a_n) in $V_{\epsilon_0}(a) \cap A$ that converges to a. Then (a_n) is Cauchy, so the sequence $(f(a_n))$ is also Cauchy by uniform continuity of f on $V_{\epsilon_0}(a) \cap A$. Thus

$(f(a_n))$ converges, and we define $g(a) = \lim f(a_n)$. By part (a), this definition of $g(a)$ is independent of the sequence (a_n) that was chosen. Then for all $x \in A$, let $g(x) = f(x)$, so $g \upharpoonright A = f$. Then g is continuous on A since f is, and g is continuous at a by construction.

(5) Let $\epsilon > 0$. Using uniform continuity of g, fix δ_0 such that $|g(x)-g(y)| < \epsilon$ whenever $|x-y| < \delta_0$ in B. Using uniform continuity of f, fix $\delta > 0$ such that $|f(x)-f(y)| < \delta_0$ whenever $|x-y| < \delta$ in A. Then for all $x, y \in A$ such that $|x - y| < \delta$, we have $|f(x) - f(y)| < \delta_0$, which implies $|g(f(x)) - g(f(y))| < \epsilon$. Thus $g \circ f$ is uniformly continuous.

(6) (a) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h\to 0} \dfrac{x-(x+h)}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-1}{x(x+h)} = -x^{-2}$.

(b) $\dfrac{d}{dx}(x^3) = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h} = \lim\limits_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$.

(7) f is continuous at 0 and 1 but nowhere else, and differentiable at 0 but nowhere else.

(8) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \;\leq\; (x-y)^2 \;=\; |x-y| \cdot |x-y|$$

for all $x, y \in \mathbb{R}$. Dividing both sides by $|x - y|$, this implies that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \;\leq\; |x - y|$$

for all $x, y \in \mathbb{R}$. In particular, for all $a \in \mathbb{R}$ we have

$$0 \;\leq\; \lim_{x\to a} \left|\frac{f(x) - f(a)}{x - a}\right| \;\leq\; \lim_{x\to a} |x - a| \;=\; 0,$$

which means that for all $a \in \mathbb{R}$ we have $f'(a) = 0$. Thus f is differentiable on $\mathbb{R}$ with constant zero derivative, so f must be constant on $\mathbb{R}$.

(9) Let f and g be differentiable on $\mathbb{R}$, and suppose that $f(0) = g(0)$ and that $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Suppose for contradiction that there is $b > 0$ such that $f(b) > g(b)$. Then $f - g$ is continuous on $[0, b]$ and differentiable on $(0, b)$, so by the Mean Value Theorem there is $c \in (0, b)$ such that

$$f'(c) - g'(c) \;=\; (f-g)'(c) \;=\; \frac{(f-g)(b) - (f-g)(0)}{b - 0} \;=\; \frac{f(b) - g(b)}{b} \;>\; 0,$$

contradicting the assumption that $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$.

(10) (a) That the statement is true is an immediate consequence of the EVT along with the fact that differentiable functions are continuous.

(b) The statement could fail. For instance, let $f(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$, and set $f(0) = 0$. Then

$$f'(0) \;=\; \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x^2} - 0}{x - 0} \;=\; \lim_{x\to 0} x \sin \frac{1}{x} \;=\; 0,$$

and for $x \neq 0$,

$$f'(x) \;=\; 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}.$$

Thus f is differentiable on $\mathbb{R}$, but f' is unbounded on $[0, 1]$.

(c) Let f be the same function given in the answer to (b). Then f is differentiable and bounded on $(0,1)$, but f' is unbounded on $(0,1)$.

(d) This statement is true. Suppose f' is bounded on (a,b), say $|f'(x)| < M$ for all $x \in (a,b)$. Fix $c \in (a,b)$, and let $x \in (a,b)$ be arbitrary. Using the MVT, choose d between c and x such that

$$\frac{f(x)-f(c)}{x-c} \;=\; f'(d).$$

Then $|f(x)-f(c)| < |f'(d)||x-c| < M(x-c) < M(b-a)$. This shows that on (a,b), f is bounded above by $f(c)+M(b-a)$ and below by $f(c)-M(b-a)$.

(11) (a) The converse is true. Suppose f is increasing on (a,b) and let $x \in (a,b)$. Then

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \;\geq\; 0,$$

since both the numerator and denominator of the quotient are nonnegative. But since f is differentiable at a, the limit above must equal $f'(x)$, so $f'(x) \geq 0$ as desired.

(b) The converse is false; eg, $f(x) = x^3$ is strictly increasing on $(-1,1)$ but $f'(0) = 0$.

(12) Suppose $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} f'(x) = M$. Let $\epsilon > 0$, and choose $a, b \in \mathbb{R}$ such that $|f(x)-L| < \frac{\epsilon}{4}$ whenever $x \geq a$ and $|f'(x)-M| < \frac{\epsilon}{4}$ whenever $x \geq b$. Let $c = \max(a,b)$, and let $d = c+1$. Then

$$\left|\frac{f(d)-f(c)}{d-c}\right| \;=\; |f(d)-f(c)| \;\leq\; |f(d)-L| + |f(c)-L| \;<\; \frac{\epsilon}{2}.$$

Therefore, using the MVT we can fix $x_0 \in (c,d)$ such that $|f'(x_0)| < \frac{\epsilon}{2}$, and thus for all $x > c$ we have

$$|f'(x)| \;<\; \frac{\epsilon}{2} + |f'(x)-f'(x_0)| \;\leq\; \frac{\epsilon}{2} + |f'(x)-M| + |f'(x_0)-M| \;\leq\; \epsilon.$$

Since ϵ was arbitrary, this shows that $\lim\limits_{x\to\infty} f'(x) = 0$.

(13) (a) This is true. Suppose $f'(a) > 0$, and let $\epsilon = \frac{f'(a)}{2}$. Choose $\delta > 0$ such that

$$\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| \;<\; \epsilon$$

whenever $a < x < a+\delta$. Then for all $x \in (a, a+\delta)$, we have

$$\frac{f(x)-f(a)}{x-a} \;>\; 0$$

and therefore $f(x) > f(a)$.

(b) This statement could fail. For a counterexample, consider the function g defined by $g(x) = x + x^2 \sin\frac{1}{x^2}$ for $x \neq 0$, and $g(0) = 0$. Using the computations from Problem 1(b), we see that $g'(0) = 1 > 0$, while for $x \neq 0$ we have

$$g'(x) \;=\; 1 + 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}.$$

Note that for any $\delta > 0$, g' takes on negative values on $(0,\delta)$, so there can be no $\delta > 0$ such that g is strictly increasing on $(0,\delta)$.

Homework 6

(1) We use the fact, proved in lecture, that $\limsup(s_n)$ is the largest subsequential limit of (s_n). Write $B = \limsup(b_n)$. Fix a subsequence (b_{n_k}) of (b_n) that converges to B. Then $\lim a_{n_k} = \lim a_n = A$, so $\lim(a_{n_k} b_{n_k}) = AB$. This shows that $AB \leq \limsup(a_n b_n)$. Conversely, let $(a_{n_k} b_{n_k})$ be an arbitrary convergent subsequence of $(a_n b_n)$, say with limit L. We must show $L \leq AB$. Then since $(a_{n_k}) \to A$, we have

$$\lim_{k\to\infty} b_{n_k} \;=\; \lim_{k\to\infty} \frac{a_{n_k} b_{n_k}}{a_{n_k}} \;=\; \frac{\lim(a_{n_k} b_{n_k})}{\lim a_{n_k}} \;=\; \frac{L}{A} \;\leq\; B,$$

so $L \leq AB$. We conclude that $\limsup(a_n b_n) = AB$.

(2) (a) We claim that for each $n \in \mathbb{N}$, the nth derivative of $y = x^n$ is the constant function $y = n!$. For the base case $n = 1$, we know that the derivative of $y = x$ is the constant function $y = 1 = 1!$. For the inductive step, let $n \in \mathbb{N}$ as assume for inductive hypothesis that the nth derivative of $y = x^n$ is the constant function $y = n!$. Then the derivative of $y = x^{n+1}$ is

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) \;&=\; \frac{d^{n+1}}{dx^{n+1}}(xx^n) \;=\; \frac{d^n}{dx^n}(1 \cdot x^n + x \cdot nx^{n-1}) \\ &=\; \frac{d^n}{dx^n}((n+1)x^n) \;=\; (n+1) \cdot \frac{d^n}{dx^n}(x^n) \;=\; (n+1) \cdot n! \;=\; (n+1)!.\end{aligned}$$

This completes the induction step, and shows that for all $n \in \mathbb{N}$, the nth derivative of $y = x^n$ is the constant function $y = n!$.

(b) Let $n \in \mathbb{N}$. By an argument similar to that given in part (a), we have

$$f_{n+1}^{(n)}(x) \;=\; \begin{cases} (n+1)!x & \text{if } x \geq 0; \\ -(n+1)!x & \text{if } x < 0, \end{cases}$$

so f_{n+1} is n-times differentiable on $\mathbb{R}$. However,

$$\lim_{x\to 0^-} f_{n+1}^{(n)}(x) \;=\; -(n+1)! \qquad \text{and} \qquad \lim_{x\to 0^+} f_{n+1}^{(n)}(x) \;=\; (n+1)!,$$

so the derivative of $f_{n+1}^{(n)}$ at zero does not exist. Thus f_{n+1} is not $(n+1)$-times differentiable.

(3) Since $f'(x) = 0$ for all $x \in \mathbb{R}$ if and only if f is a constant function, the induction base $n = 0$ is true. Suppose now that the claim holds for n, and consider the inductive step $n+1$. We must show that if $f^{(n+2)}$ is constantly zero on $\mathbb{R}$, then f is a polynomial of degree at most $n+1$. So assume $f^{(n+2)}$ exists and is constantly zero. Then $f^{(n+1)}$ must be constant, say $f^{(n+1)}(x) = a$ for all $x \in \mathbb{R}$. The function

$$g(x) = \frac{ax^{n+1}}{(n+1)!}$$

also has the property that $g^{(n+1)}(x) = a$ for all $x \in \mathbb{R}$, so $f^{(n+1)} - g^{(n+1)}$ is constantly zero. Thus by inductive hypothesis $f - g$ is a polynomial of degree at most n. Since g is a monomial of degree $n+1$, this implies that f is a polynomial of degree at most $n+1$, as desired.

(4) We use the integral test in order to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. By making the substitution $u = \ln x$, we have

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_{\ln 2}^b \frac{1}{u}\, du \\ &= \lim_{b\to\infty} \left(\ln u \Big]_{\ln 2}^b \right) = \lim_{b\to\infty} \Big[\ln b - \ln(\ln 2) \Big] = \infty, \end{aligned}$$

which shows that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. On the other hand, for all $\epsilon > 0$ we have

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^{1+\epsilon}}\, dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^{1+\epsilon}}\, dx = \lim_{b\to\infty} \int_{\ln 2}^b \frac{1}{u^{1+\epsilon}}\, du \\ &= \lim_{b\to\infty} \left(-\epsilon^{-1} u^{-\epsilon} \Big]_{\ln 2}^b \right) = \lim_{b\to\infty} \frac{b^{-\epsilon} - (\ln 2)^{-\epsilon}}{-\epsilon} = \frac{1}{\epsilon (\ln 2)^{\epsilon}} \in \mathbb{R}, \end{aligned}$$

which shows that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\epsilon}}$ converges by the Integral Test.

(5) Since $(\frac{1}{n^{1+1/n}})$ is a strictly decreasing sequence of positive numbers converging to 0, we know that $\sum \frac{(-1)^n}{n^{1+1/n}}$ converges by the Alternating Series Test. (The best way to see that $(\frac{1}{n^{1+1/n}})$ is strictly decreasing is to take the derivative of $x^{1+\frac{1}{x}}$ and note that it is always positive.)

Now, since $n^{1/n} \to 1$ as $n \to \infty$, we know $n^{1/n} \leq 2$, and thus $n^{1+1/n} \leq 2n$, for sufficiently large n. But $\sum \frac{1}{2n}$ diverges, so $\sum \frac{1}{n^{1+1/n}}$ diverges as well by the Comparison Test.

(6) For instance, we could let $a_n = \frac{1}{n^2}$ if n is odd, and $a_n = \frac{1}{n}$ if n is even. Then (a_n) is a sequence of positive real numbers converging to 0, and since $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ converges but $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ diverges, we have $\sum (-1)^n a_n = +\infty$.

(7) (a) Applying the Ratio Test, we have

$$\lim_{n\to\infty} \left| \frac{(n+1)!/e^{n+1}}{n!/e^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!e^n}{n!e^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{e} \right| = \infty,$$

which implies that $\sum \frac{n!}{e^n}$ DIVERGES.

(b) Note that $e^{1/n} \to 1$ as $n \to \infty$, so $\lim_{n\to\infty} (-1)^n e^{1/n}$ DNE. Therefore $\sum (-1)^n e^{1/n}$ DIVERGES by the nth term test.

(c) Since $\sum \frac{1}{n}$ diverges and

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

we have that $\sum \sin \frac{1}{n}$ DIVERGES by the limit comparison test.

(d) Note that if we let $a_n = \ln(1+\frac{1}{n})$ for each n, then (a_n) is a strictly decreasing sequence of positive numbers that converges to zero, and $(\cos \pi n)\ln(1+\frac{1}{n}) = (-1)^n a_n$. Thus $\sum(\cos \pi n)\ln(1+\frac{1}{n})$ CONVERGES by the Alternating Series Test.

(e) Applying the Ratio Test, we have

$$\lim_{n\to\infty}\left|\frac{e^{(n+1)^2}/(n+1)!}{e^{n^2}/n!}\right| = \lim_{n\to\infty}\left|\frac{e^{(n+1)^2}n!}{e^{n^2}(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{e^{n^2+2n+1}}{ne^{n^2}}\right| = \lim_{n\to\infty}\left|\frac{e^{2n+1}}{n}\right| = \infty,$$

which shows that $\sum \frac{e^{n^2}}{n!}$ DIVERGES.

(8) Suppose $\sum a_k^2$ and $\sum b_k^2$ both converge. For each k, let $m_k = \max\{|a_k|, |b_k|\}$ and

$$c_k = \begin{cases} |a_k| & \text{if } |a_k| \geq |b_k| \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad d_k = \begin{cases} |b_k| & \text{if } |a_k| < |b_k| \\ 0 & \text{otherwise} \end{cases}$$

so $c_k + d_k = m_k$ for all k. Note that $c_k^2 \leq a_k^2$ and $d_k^2 \leq b_k^2$ for all k, so $\sum c_k^2$ and $\sum d_k^2$ both converge by the Comparison Test, which means $\sum m_k^2 = \sum(c_k^2 + d_k^2)$ converges. But $|a_kb_k| \leq m_k^2$ for each k, so $\sum |a_kb_k|$ converges by the Comparison Test. We conclude that $\sum a_kb_k$ converges absolutely, as desired.

Alternative proof: Suppose $\sum a_k^2$ and $\sum b_k^2$ both converge, say $\sum_{k=1}^\infty a_k^2 = L$ and $\sum_{k=1}^\infty b_k^2 = M$. Then, using the Cauchy-Schwarz inequality, for each $n \in \mathbb{N}$ we have

$$0 \leq \sum_{k=1}^n |a_kb_k| = \sum_{k=1}^n |a_k||b_k| \leq \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\left(\sum_{k=1}^n |b_k|^2\right)^{1/2} = \left(\sum_{k=1}^n a_k^2\right)^{1/2}\left(\sum_{k=1}^n b_k^2\right)^{1/2}.$$

Taking a limit as $n \to \infty$, we obtain

$$0 \leq \sum_{k=1}^\infty |a_kb_k| \leq \lim_{n\to\infty}\left(\sum_{k=1}^n a_k^2\right)^{1/2}\left(\sum_{k=1}^n b_k^2\right)^{1/2} = \left(\lim_{n\to\infty}\sum_{k=1}^n a_k^2\right)^{1/2}\left(\lim_{n\to\infty}\sum_{k=1}^n b_k^2\right)^{1/2} = \sqrt{L}\sqrt{M},$$

showing that $\sum a_kb_k$ converges absolutely.

(9) Suppose that f is integrable on $[a,b]$ and let $[c,d] \subseteq [a,b]$ be any subinterval. Let $\epsilon > 0$, and fix a partition P of $[a,b]$ such that $U(f;P) - L(f;P) < \epsilon$. Let $Q = P \cup \{c,d\}$, and write $Q = (a = x_0, \ldots, c = x_l, \ldots, d = x_m, \ldots, b = x_n)$. Let $Q_0 = (x_l, \ldots, x_m)$ be the restriction of Q to the interval $[c,d]$. Since Q is a refinement of P, on $[a,b]$ we have

$$L(f;P) \leq L(f;Q) \leq U(f;Q) \leq U(f;P),$$

so $U(f;Q) - L(f;Q) < \epsilon$. But then

$$\begin{aligned} U(f;Q_0) - L(f;Q_0) &= \sum_{k=l+1}^m \Big(\sup f(I_k)\Delta x_k - \inf f(I_k)\Delta x_k\Big) \\ &\leq \sum_{k=1}^n \Big(\sup f(I_k)\Delta x_k - \inf f(I_k)\Delta x_k\Big) \\ &= U(f;Q) - L(f;Q) < \epsilon. \end{aligned}$$

Hence f is integrable on $[c,d]$.

(10) Suppose $S \subseteq [a,b]$ is infinite, and let $(a_n)_{n\in\mathbb{N}}$ be an infinite sequence of distinct points in S. Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) \;=\; \begin{cases} f(x) & \text{if } x \in [a,b] \setminus \{a_n \,:\, n \in \mathbb{N}\} \\ n & \text{if } x = a_n \end{cases}$$

Then $g(x) = f(x)$ for all $x \in [a,b] \setminus S$, but g is not integrable on $[a,b]$ since g is not bounded on $[a,b]$.

(11) Suppose that f is bounded on $[a,b]$ and continuous everywhere in $[a,b]$ except at the point $x_0 \in (a,b)$. Fix M such that $|f(x)| < M$ for all $x \in [a,b]$. Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{12M}$. Then since f is continuous on both $[a, x_0 - \delta]$ and $[x_0 + \delta, b]$, f is integrable on each of these intervals and so we can fix a partition $\mathcal{P}$ of $[a, x_0 - \delta]$ and a partition $\mathcal{Q}$ of $[x_0 + \delta, b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \;<\; \frac{\epsilon}{3} \qquad \text{and} \qquad U(f,\mathcal{Q}) - L(f,\mathcal{Q}) \;<\; \frac{\epsilon}{3}$$

But now $\sup\{f(x) \,:\, |x - x_0| \le \delta\} - \inf\{f(x) \,:\, |x - x_0| \le \delta\} \le 4\delta M = \dfrac{\epsilon}{3}$, so $\mathcal{P} \cup \mathcal{Q}$ is a partition of $[a,b]$ such that

$$U(f,\mathcal{P}\cup\mathcal{Q}) - L(f,\mathcal{P}\cup\mathcal{Q}) \;<\; U(f,\mathcal{P}) - L(f,P) + \frac{\epsilon}{3} + U(f,\mathcal{Q}) - L(f,\mathcal{Q}) \;<\; \epsilon.$$

This shows that f is integrable on $[a,b]$.

(12) Suppose f and g are continuous on $[a,b]$, and suppose $\int_a^b f(x)dx = \int_a^b g(x)dx$. Let $h = f - g$. Then also h is continuous and integrable on $[a,b]$ and $\int_a^b h(x)dx = 0$. Using the Extreme Value Theorem, fix $x_0, y_0 \in [a,b]$ such that $h(x_0) = m = \min\{h(x) \,:\, a \le x \le b\}$ and $h(y_0) = M = \max\{h(x) \,:\, a \le x \le b\}$, so that $m \le h(x) \le M$ for all $a \le x \le b$. Then

$$m \;=\; \frac{1}{b-a}\int_a^b m\,dx \;\le\; \frac{1}{b-a}\int_a^b h(x)dx \;\le\; \frac{1}{b-a}\int_a^b M\,dx \;=\; M,$$

and hence $m \le 0 \le M$. The Intermediate Value Theorem now implies that there is x between x_0 and y_0 (inclusive) such that $f(x) - g(x) = h(x) = 0$.

<u>**Homework 7**</u>

(1) Let $f : [a,b] \to \mathbb{R}$ be continuous, and define $g(x) = \int_a^x f$ for all $x \in [a,b]$, so that g is continuous on $[a,b]$ and differentiable on (a,b) by the FTOC. Applying the MVT to g on $[a,b]$, we obtain a point $c \in (a,b)$ such that

$$f(c) \;=\; g'(c) \;=\; \frac{g(b)-g(a)}{b-a} \;=\; \frac{1}{b-a}\int_a^b f,$$

so $\int_a^b f = f(c)(b-a)$ as desired.

Alternative Solution: Suppose f is continuous, hence integrable, on $[a,b]$. Let $\mu = \dfrac{1}{b-a}\displaystyle\int_a^b f$. Using the EVT, fix $x_0, y_0 \in [a,b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a,b]$. Then

$$f(x_0)(b-a) \;=\; \int_a^b f(x_0) \;\le \int_a^b f \;\le\; \int_a^b f(y_0) \;=\; f(y_0)(b-a),$$

so after dividing by $b-a$ we see that $f(x_0) \le \mu \le f(y_0)$. Thus by the IVT there is $c \in [a,b]$ (between x_0 and y_0) such that $f(c) = \mu$, as desired.

(2) (a) Assume the hypotheses, and in particular assume f is continuous. Fix $x_0 \in [a,b]$ such that $f(x_0) > 0$, and using continuity of f at x_0, fix $\delta > 0$ such that $[x_0, x_0+\delta) \subseteq [a,b]$ or $(x_0-\delta, x_0] \subseteq [a,b]$ and also $|f(x)-f(x_0)| < \frac{f(x_0)}{2}$ whenever $|x-x_0| \le \delta$. Let $\mathcal{P} = (x_k)_{k=0}^n$ be any partition of $[a,b]$ for which the subinterval containing x_0 has width δ. Fix k such that $x_0 \in [x_{k-1}, x_k]$. Then

$$\begin{aligned}\int_a^b f \;\ge\; L(f) \;\ge\; L(f,\mathcal{P}) \;&=\; \big(\inf f[I_k]\big)\delta + \sum_{j\neq k}(\inf f[I_j])\Delta x_j \\ &\ge\; \frac{f(x_0)}{2}\delta + \sum_{j\neq k}\inf f[I_j]\delta x_j \;\ge\; \frac{f(x_0)}{2}\delta \;>\; 0.\end{aligned}$$

(b) Assume the hypotheses, and let $h = g - f$, so h is continuous and nonnegative on $[a,b]$. Suppose $\int_a^b f = \int_a^b g$. Then $\int_a^b h = \int_a^b (g-f) = \int_a^b g - \int_a^b f = 0$. By part (a), this implies h is never strictly positive on $[a,b]$, so h must be constantly zero on $[a,b]$, and thus $f = g$ on $[a,b]$.

(3) (a) Suppose f is integrable on $[a,b]$ and define the function $g : [a,b] \to \mathbb{R}$ by $g(x) = \int_a^x f(t)dt = \int_x^b f(t)dt$. Then g is continuous by FTOC. Note that $g(a) = 0$ if and only if $g(b) = 0$, and if $g(a) = 0$ then we are done, so assume $g(a) \neq 0$. Then $g(a)$ and $g(b)$ have opposite signs, so by the IVT there is $c \in (a,b)$ such that $g(c) = 0$ and hence $\int_a^c f = \int_c^b f$.

(b) For instance, let $[a,b] = [0,2\pi]$ and let $f(t) = \sin t$. Then for every $x \in (0,2\pi)$ we have

$$\int_0^x \sin t\,dt \;>\; \int_x^{2\pi} \sin t\,dt.$$

(4) Let $g(x) = \int_0^x e^{t^2}\,dt$, so $g'(x) = e^{x^2}$. Then

$$(a)\ \lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt \ = \ g'(0) \ = \ e^{0^2} \ = \ 1,$$

and

$$(b)\ \lim_{h\to 0} \int_3^{3+h} e^{t^2}\,dt \ = \ \lim_{h\to 0} hg'(3) \ = \ 0.$$

(5) (a) First note that $f_n(1) = \frac{1}{2}$ for all $n \in \mathbb{N}$, so $\lim f_n(1) = \frac{1}{2}$. Next, note that for all $0 \le x < 1$, we have $x^n \to 0$, so $f_n(x) \to \frac{0}{1+0} = 0$. Finally, for all $x > 1$ we have $x^n \to \infty$ as $n \to \infty$, so $\frac{1}{x^n} \to 0$, and thus

$$\lim_{n\to\infty} f_n(x) \ = \ \lim_{n\to\infty} \frac{x^n}{1+x^n} \ = \ \lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+1} \ = \ \frac{\lim 1}{\lim\left(\frac{1}{x^n}+1\right)} \ = \ \frac{1}{1} \ = \ 1.$$

Thus

$$f(x) \ = \ \lim_{n\to\infty} f_n(x) \ = \ \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(b) Let $0 < b < 1$, and let $\epsilon > 0$. Using the fact that $b^{-n} \to \infty$, fix $N \in \mathbb{N}$ large enough so that $\frac{1}{\epsilon} < 1 + \frac{1}{b_N}$, and thus $\frac{b^N}{1+b^N} < \epsilon$. Then for all $0 \le x \le b$ and $n \ge N$ we have

$$\frac{x^n}{1+x^n} \ \le \ \frac{x^N}{1+x^N} \ \le \ \frac{b^N}{1+b^N} \ < \ \epsilon.$$

Thus $f_n \to f$ uniformly on $[0, b]$.

(c) No, (f_n) does not converge uniformly on $[0, 1]$, since each f_n is continuous on $[0, 1]$ but the limit function f is not continuous on $[0, 1]$.

(6) Let (f_n) be a sequence of uniformly continuous functions on (a, b), and suppose that $f_n \to f$ uniformly on (a, b). Let $\epsilon > 0$ and let $x, y \in (a, b)$. Fix N such that for all $x \in (a, b)$ and $n \ge N$, $|f_n(x) - f(x)| < \epsilon/3$. Fix also $\delta > 0$ such that for all $x, y \in (a, b)$, $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \epsilon/3$. Then for all $x, y \in (a, b)$, $|x - y| < \delta$ implies

$$|f(x) - f(y)| \ \le \ |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \ < \ \epsilon.$$

(7) We use an example from lecture: for each $x \in [0, 2]$, let $f_n(x) = \begin{cases} n^2x & \text{if } 0 \le x \le \frac{1}{n} \\ 2n - n^2x & \text{if } \frac{1}{n} < x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \end{cases}$

Then each f_n is continuous (since each f_n is piecewise linear with the endpoints of the line segments matching up), and f_n converges pointwise to the constant zero function on $[0, 2]$, since $f_n(0) = 0$ for each n and for every $c \in (0, 2]$ we have $f_n(c) = 0$ for all $n > \frac{1}{c}$. However, the convergence is not uniform, since for every $n \in \mathbb{N}$ we have $|f_n(\frac{1}{n}) - 0| = n \ge 1$.

(8) Assume the hypotheses, and in particular fix $a \in [0, 1]$ and $L \in \mathbb{R}$ such that $\lim f_n(a) = L$. For each $n \in \mathbb{N}$ and $x \in [0, 1]$, let $F_n(x) = \int_a^x f_n'(t)dt = f_n(x) - f_n(a)$. Let h be the uniform limit of

(f_n'). Then h is continuous, hence integrable, and $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x h(t)dt$. Thus for each $x \in [a,b]$ we have

$$\lim_{n\to\infty} f_n(x) \;=\; \lim_{n\to\infty} \big(F_n(x) + f_n(a)\big) \;=\; \int_a^x h(t)\,dt + L,$$

which shows that $(f_n(x))$ converges for each $x \in [a,b]$ as desired.

Alternate Solution: Assume the hypotheses, and in particular fix $a \in [0,1]$ such that $(f_n(a))$ converges. Let $x \in [0,1] \setminus \{a\}$ be arbitrary, wlog say $a < x$. Let $\epsilon > 0$. Fix N_1 such that $|f_n(a) - f_m(a)| < \frac{\epsilon}{2}$ whenever $m, n \geq N_1$, and fix N_2 such that for all $x \in [0,1]$, $|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2|x-a|}$. Let $N = \max(N_1, N_2)$, and suppose $m, n \geq N$. Using the MVT, fix $c \in (a,x)$ such that

$$[f_n(x) - f_m(x)] - [f_n(a) - f_m(a)] \;=\; \big(f_n'(c) - f_m'(c)\big)(x-a).$$

Then

$$|f_n(x) - f_m(x)| \;\leq\; |[f_n(x) - f_m(x)] - [f_n(a) - f_m(a)]| + |f_n(a) - f_m(a)| \;<\; \frac{\epsilon}{2|x-a|}|x-a| + \frac{\epsilon}{2} \;=\; \epsilon.$$

(9) Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $n \in \mathbb{N}$. Using the fact that f is uniformly continuous on $[a,b]$, fix $\delta > 0$ such that for all $x, y \in [a,b]$, $|f(x) - f(y)| < \frac{1}{n}$ whenever $|x - y| \leq \delta$. Fix $M_n \in \mathbb{N}$ such that $\frac{b-a}{M_n} < \delta$, and let $\mathcal{P}_n = \big(x_k^{(n)}\big)_{k=0}^{M_n}$ be the regular partition of $[a,b]$ into M_n subintervals. Now define $f_n : [a,b] \to \mathbb{R}$ by $f_n(x) = \inf f[I_k]$, where k is least such that $x \in I_k = \big[x_{k-1}^{(n)}, x_k^{(n)}\big]$. Then f_n is a step function such that $f_n(x) \leq f(x)$ for all $x \in [a,b]$, and $|f_n(x) - f(x)| < \frac{1}{n}$ for all $x \in [a,b]$. This is true for all $n \in \mathbb{N}$, so $(f_n) \to f$ uniformly on $[a,b]$, as desired.

(10) Assume $\lim |\frac{c_{n+1}}{c_n}| = L > 0$, and let $R = \frac{1}{L}$. Let $x \in (-R, R)$, and fix r such that $|x| < r < R$, so $L < \frac{1}{r}$. Fix N such that $|\frac{c_{n+1}}{c_n}| < \frac{1}{r}$ for all $n \geq N$. Then $|c_{N+k}| \leq \frac{|c_N|}{r^k}$ for all $k \geq 0$. So for all k we have $|c_{N+k}x^{N+k}| \leq |\frac{c_N x^N}{r^k} x^k| = |c_N x^N| \cdot |\frac{x}{r}|^k$. Since $|\frac{x}{r}| < 1$, the series $\sum |c_N x^N| \cdot |\frac{x}{r}|^k$ is a convergent geometric series, which shows $\sum |c_{N+k}x^{N+k}|$ converges by the Comparison Test, and therefore $\sum c_n x^n$ converges as well.

Now suppose $x \in \mathbb{R} \setminus [-R, R]$, and fix r such that $|x| > r > R = \frac{1}{L}$, so $\frac{1}{r} < L$. Fix N such that $|\frac{c_{n+1}}{c_n}| > \frac{1}{r}$ for all $n \geq N$. Then $|c_{N+k}| \geq \frac{|c_N|}{r^k}$ for all $k \geq 0$. So for all k we have $|c_{N+k}x^{N+k}| \geq |\frac{c_N x^N}{r^k} x^k| = |c_N x^N| \cdot |\frac{x}{r}|^k$. Since $|\frac{x}{r}| > 1$, it follows that $\sum c_n x^n$ diverges by the nth term test.

[Note: it is fine if you directly use the Ratio Test, which we are essentially reproving here.]

(11) (a) $\sum n^2 x^n$. $R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$. The series diverges for $x = \pm 1$, so the interval of convergence is $(-1, 1)$.

(b) $\sum \frac{2^n}{n^2} x^n$. $R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2}$. The series converges for $x = \pm\frac{1}{2}$, so the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

(c) $\sum \frac{2^n}{n!}x^n$. $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2}{n+1} = 0$, so $R = \infty$, and the interval of convergence is $(-\infty, \infty)$, or all of $\mathbb{R}$.

(12) (a) Using the Chain Rule, for $x \neq 0$ we have $f'(x) = 2x^{-3}e^{-1/x^2}$, which has the desired form with $p(x) = 2x^3$. So the base case $n = 1$ holds, and now we let $n \in \mathbb{N}$ be arbitrary and suppose for inductive hypothesis that for all $x \neq 0$, $f^{(n)}$ exists and has the form $f^{(n)}(x) = q(\frac{1}{x})f(x)$ for some polynomial q. Then, letting $x \neq 0$ and using the Product and Chain Rules, we have

$$f^{(n+1)}(x) \;=\; \frac{d}{dx}\left[q(\tfrac{1}{x})e^{-1/x^2}\right] \;=\; -x^{-2}q'(\tfrac{1}{x})e^{-1/x^2} + 2x^{-3}q(\tfrac{1}{x})e^{-1/x^2},$$

which again has the desired form with $p(x) = 2x^3q(x) - x^2q'(x)$.

(b) Let p be an arbitrary polynomial. Writing $t = \frac{1}{x}$, we have

$$0 \;\leq\; \lim_{x\to 0^+}\left|p(\tfrac{1}{x})e^{-1/x^2}\right| \;=\; \lim_{t\to\infty}\left|p(t)e^{-t^2}\right| \;\leq\; \lim_{t\to\infty}\left|\frac{p(t)}{e^t}\right| \;=\; 0$$

and likewise

$$0 \;\leq\; \lim_{x\to 0^-}\left|p(\tfrac{1}{x})e^{-1/x^2}\right| \;=\; \lim_{t\to\infty}\left|p(-t)e^{-t^2}\right| \;\leq\; \lim_{t\to\infty}\left|\frac{p(-t)}{e^t}\right| \;=\; 0.$$

Thus $\lim\limits_{x\to 0} p(\frac{1}{x})f(x) = 0$, as desired.

(c) The base case $n = 0$ is trivial. For the inductive step, let $n \in \mathbb{N}$ and assume $f^{(n)}(0) = 0$. Using (a), fix a polynomial p such that $f^{(n)}(x) = p(\frac{1}{x})f(x)$ for all $x \neq 0$. Then

$$f^{(n+1)}(0) \;=\; \lim_{x\to 0}\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} \;=\; \lim_{x\to 0}\frac{f^{(n)}(x)}{x} \;=\; \lim_{x\to 0}\tfrac{1}{x}p(\tfrac{1}{x})f(x) \;=\; 0,$$

where the final equality follows from part (b).

(d) For instance, let $g(x) = 0$ for all $x < 0$, and $g(x) = f(x)$ for all $x \geq 0$.