Compact sets

<u>Def</u> By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets such that $E \subset \bigcup_\alpha G_\alpha$.

<u>Def</u> A subset K is <u>compact</u> if every open cover of K contains a finite subcover.

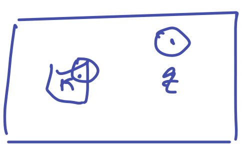
i.e. if $\{G_\alpha\}$ is an open cover of K, then can find finite indicies $\alpha_1, \alpha_2, \cdots, \alpha_k$ s.t.

$$K \subset \bigcup_{i=1}^{k} G_{\alpha_i}.$$

<u>Example</u> A finite set is compact

<u>Thm</u> Compact subsets of a metric space are closed.

<u>Proof</u>



$q \notin K$

$\forall p \in K$, $\exists$ nbd $V_p \ni p$, $W_p \ni q$ s.t. $V_p \cap W_p = \emptyset$.

Then $\{V_p\}$ is an open cover. By def, it contains a subcover $V_{p_1}, V_{p_2}, \cdots, V_{p_k}$.

Then $W_{p_1} \cap \cdots \cap W_{p_k}$ doesn't intersect $V_{p_1} \cup V_{p_2} \cup \cdots \cup V_{p_k}$. $q \in W_{p_1} \cap \cdots \cap W_{p_k}$ is an interior pt of K^c $\square$

Thm closed subsets of compact subsets are compact.

Proof E closed in a compact K. Any $\{G_\alpha\}$ open cover of E union E^c is an open cover of K. $\{G_\alpha\} \cup E^c$ has a finite cover of $K \Rightarrow \{G_\alpha\}$ has a finite cover of E.

Thm If E is an infinite subset of K, then E has a limit pt in K.

Proof If E has no limit pt in K, then $\forall q \in K$, $\exists$ nbd $V_q \ni q$ contains at most 1 element of E (i.e. q). No finite subcover of $\{V_q\}$ to cover E, so the same is true for K. A contradiction $\boxtimes$.

Goal: Thm (Heine-Borel) Let E be a subset of $\mathbb{R}^k$. Then the following are equivalent.

Thm Let $k \in \mathbb{Z}_{>0}$. If $\{I_n\}$ is a collection of k-cells such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

Proof Only do the case when $k=1$. $I_n = [a_n, b_n]$ $\quad \sup_n a_n \le \inf_m b_m \Rightarrow \exists x$ s.t.

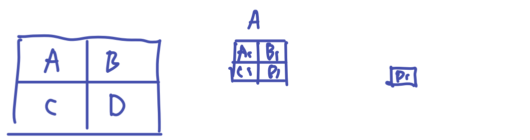
$$a_n \le \sup_n a_n \le x \le \inf_m b_m \le b_n$$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} I_n$. $\boxtimes$

Rmk For general k, $I_n = [a_{n1}, b_{n1}] \times [a_{n2}, b_{n2}] \times \cdots \times [a_{nk}, b_{nk}]$, the same argument works.

Thm Every k-cell is compact.

Proof Argue by contradiction. There is a cover $\{G_\alpha\}$ of a k-cell I without finite subcover. Only do the case $k=2$.



$I \supset I_1 \supset I_2 \supset \cdots$

Keep subdividing, get a sequence of k-cells $\{I_n\}$ such that $I_n \supset I_{n+1}$ and $\{G_\alpha\}$ has no finite subcover of each I_n. By the last Thm, $\exists\, x \in \bigcap_{n=1}^{\infty} I_n$. Say $x \in G_\alpha$. $\exists$ a nbd $V_r \ni x$ is contained in G_α. Can find some $I_m \subset V_r \subset G_\alpha$. A contradiction $\square$

Proof of Heine-Borel. a) $\Rightarrow$ b) : As E bounded, can find a k-cell containing E.

b) $\Rightarrow$ c) has already been proved.

c) $\Rightarrow$ a) $\square$

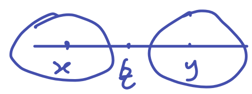
Thm (Weierstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Connected sets

Def A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated open sets.

Thm A subset E of $\mathbb{R}$ is connected if and only if $x \in E$, $y \in E$, $x < z < y$, then $z \in E$.

Proof If E connected, $(-\infty, z) \cup (z, +\infty)$ cannot cover E, $\Rightarrow z \in E$.

If E has this property and can be seperated by two open subsets, 

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