Functions on Intervals (5.3, 5.6, 6.1)

Standing assumption: let $I \subseteq \mathbb{R}$ be a nondegenerate interval, and $f : I \to \mathbb{R}$ a function.

Theorem: If f is strictly increasing, then:

(i) f is injective;

(ii) f^{-1} is also strictly increasing;

(iii) if $c \in I$ is not the right endpt of I, then $\lim\limits_{x \to c^+} f(x)$ exists;

(iv) if $c \in I$ is not the left endpt of I, then $\lim\limits_{x \to c^-} f(x)$ exists;

(v) f has at most countably many discontinuities, and they are all jumps;

(vi) if $f[I]$ is an interval, then f is continuous.

pf sketch: For (iii), the set $S = f[I \cap (c, \infty)]$ is nonempty and bounded below by $f(c)$, so write $L = \inf(S)$. Let $\epsilon > 0$, and fix $0 < \delta$ st $c + \delta \in I$ and $f(c+\delta) < L + \epsilon$. Then $L \leq f(x) \leq f(c+\delta) < L + \epsilon$ for all $x \in (c, c+\delta)$ since f is increasing, which shows $\lim\limits_{x \to c^+} f(x) = L$. The proof of (iv) is similar, and (iii) & (iv) imply that every discontinuity of f is a jump.

For (vi), we prove the contrapositive. Suppose f has a discontinuity, say at $c \in I$, which by (v) is a jump. If c is not an endpt of I, the limits $\ell = \lim\limits_{x \to c^-} f(x)$ and $L = \lim\limits_{x \to c^+} f(x)$ exist, with $\ell < L$. Then

$$(-\infty, l] \cap f[I] \neq \varnothing \qquad \text{and} \qquad [L, \infty) \cap f[I] \neq \varnothing,$$

but $(l, L) \not\subseteq f[I]$ since $(l, L) \cap f[I] \subseteq \{f(c)\}$. So $f[I]$ is not an interval. A similar argument can be made if c is an endpt of I.

The remaining proofs are left as exercises. □

Remark: the dual also holds if f is stricly decreasing.

Theorem: If f is continuous, then:

(i) $f[I]$ is an interval;

(ii) if I is closed and bounded, so if $f[I]$;

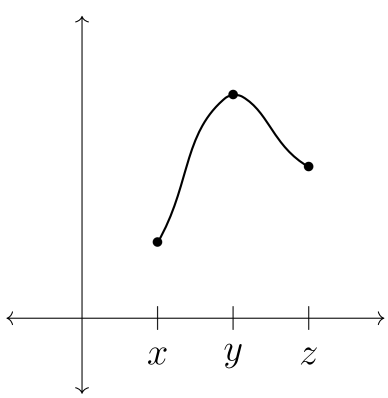
(iii) f is strictly monotone if and only if f is injective;

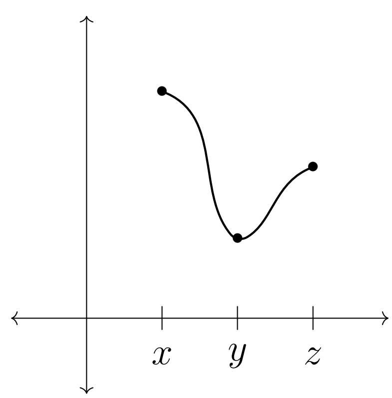
(iv) if f is injective, then f^{-1} is also continuous.

pf sketch: We already proved (i) and (ii). For the backward direction of (iii), assuming f is *not* strictly monotone we can find, wlog, $x < y < z$ in I st

$$f(x) < f(y) \quad \text{and} \quad f(y) > f(z)$$

$$\underline{\text{or}} \quad f(x) > f(y) \quad \text{and} \quad f(y) < f(z).$$





In each case, the IVT implies that f is not one-to-one.

Finally, for (iv), assume f is injective, hence strictly monotone. Then f^{-1} is also strictly monotone. Therefore, since $I = f^{-1}[f[I]]$ is an interval, f^{-1} is continuous by our previous theorem. □

Corollary: If f is injective, then:

(i) f is strictly increasing iff f^{-1} is strictly increasing;

(ii) f is strictly decreasing iff f^{-1} is strictly decreasing;

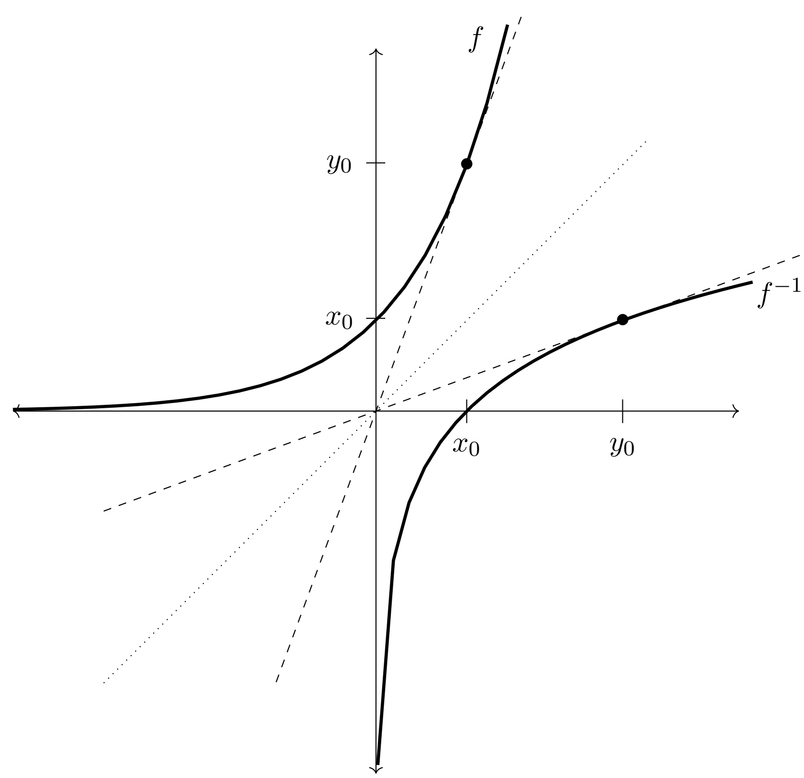
(iii) f is continuous iff f^{-1} is continuous.

Question: Could we add: "f is differentiable iff f^{-1} is differentiable"?

Answer: not quite, since, e.g., $f(x) = x^3$ is injective and differentiable on $(-1, 1)$ but f^{-1} is not differentiable at $f(0) = 0$.

Theorem (Inverse Function Theorem)**:** Suppose f is continuous and injective on an open interval I, and let $x_0 \in I$. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}.$$



pf: Write $g = f^{-1}$, and let $\epsilon > 0$. Since $f'(x_0) \neq 0$ and $f(x) \neq f(x_0)$ whenever $x \neq x_0$, $\lim_{x \to x_0} \dfrac{x - x_0}{f(x) - f(x_0)} = \dfrac{1}{f'(x_0)}$.

Fix $\delta_0 > 0$ such that $\left| \dfrac{x - x_0}{f(x) - f(x_0)} - \dfrac{1}{f'(x_0)} \right| < \epsilon$ whenever $0 < |x - x_0| < \delta_0$.

Using continuity of g at y_0, fix $\delta_1 > 0$ such that $|g(y) - g(y_0)| < \delta_0$ whenever $|y - y_0| < \delta_1$. Suppose $0 < |y - y_0| < \delta_1$. Then

$$|g(y) - g(y_0)| \ < \ \delta_0,$$

so

$$\left| \frac{g(y) - g(y_0)}{f(g(y)) - f(g(y_0))} - \frac{1}{f'(x_0)} \right| \ < \ \epsilon,$$

which implies

$$\left| \frac{g(y) - g(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| \ < \ \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows

$$\frac{1}{f'(x_0)} \ = \ \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \ = \ g'(y_0) \ = \ \left(f^{-1}\right)'(y_0).$$

Corollary: If f is differentiable and $f' \neq 0$ on the open interval I, then f is injective on I, f^{-1} is differentiable on $f[I]$, and $(f^{-1})' = \frac{1}{f' \circ f^{-1}}$. [Prove? Fix? Skip? 6.1.9]

Example: Define the invertible, differentiable function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \;=\; \frac{e^x}{x^2+1} + x^3 + 2x$$

for all $x \in \mathbb{R}$. Find $(f^{-1})'(1)$.

Solution: Note that $f(0) = 1$ and that

$$f'(x) \;=\; \frac{e^x(x^2+1) - 2xe^x}{(x^2+1)^2} + 3x^2 + 2 \;=\; \frac{e^x(x-1)^2}{(x^2+1)^2} + 3x^2 + 2.$$

So $(f^{-1})'(1) \;=\; \frac{1}{f'\left(f^{-1}(1)\right)} \;=\; \frac{1}{f'(0)} \;=\; \frac{1}{3}$.

L'Hôpital's Rule

Lemma (Cauchy's Mean Value Theorem)**:** Let $a < b$, and suppose the functions $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Then there is $c \in (a,b)$ such that

$$\big(f(b) - f(a)\big)g'(c) \;=\; \big(g(b) - g(a)\big)f'(c).$$

pf sketch: Apply the MVT to the function

$$h(x) \;=\; \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x)$$

on $[a,b]$. $\square$

Theorem (L'Hôpital's Rule)**:** Let $a < b$, and let $f, g : (a,b) \to \mathbb{R}$ be differentiable functions such that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. If

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

exists and is equal to $L \in \mathbb{R}$, then also

$$\lim_{x\to a^+} \frac{f(x)}{g(x)}$$

exists and is equal to L.

pf: Extend f and g to functions $F, G : [a, b) \to \mathbb{R}$ by setting $F(a) = G(a) = 0$, so F and G are continuous at a. Applying Rolle's Theorem to G on $[a, b)$, we see that not just g' but also g itself is never 0 on (a, b). Let (x_n) be a sequence in (a, b) with limit a. Using Cauchy's MVT, for each n fix $y_n \in (a, x_n)$ s.t.

$$F'(y_n)\big[G(x_n) - G(a)\big] \;=\; G'(y_n)\big[F(x_n) - F(a)\big].$$

Then $y_n \to a$, and $\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)}$ for all n, so from $\lim\limits_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ we get

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} \;=\; \lim_{n\to\infty} \frac{f'(y_n)}{g'(y_n)} \;=\; L.$$

Since (x_n) was an arbitrary sequence in (a, b) converging to a, it follows that $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = L$, as desired. $\square$

Remark: L'Hôpital's Rule also holds for two-sided limits and for limits at $\pm\infty$. It also holds for indeterminate limits of the form $\frac{\pm\infty}{\pm\infty}$, and can be adapted to handle forms such as $\infty - \infty$, $0 \cdot \infty$, 1^∞, 0^0, and ∞^0 (see 6.3.) Skip the rest?

Examples: • $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$

• For all $a > 0$, $\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{1}{ax^a} = 0.$

• For all $a > 0$, $\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{x\to\infty} \frac{ax^{a-1}}{e^x} = \cdots = 0.$

Corollary: Let $a \in \mathbb{R}$, and let I be an open interval containing a. Let $f : I \to \mathbb{R}$ be a continuous function and suppose f is differentiable on $I \setminus \{a\}$. If $\lim_{x\to a} f'(x)$ exists, then f is differentiable at a and $\lim_{x\to a} f'(x) = f'(a)$.

pf: Assume the hypotheses. Letting $F(x) = f(x) - f(a)$ for all $x \in I$, we have that F is differentiable on $I \setminus \{a\}$ and $\lim_{x\to a} F(x) = 0$. Now let $G(x) = x - a$, so that $G'(x) = 1$ for all $x \in I$ and $\lim_{x\to a} G(x) = 0$. Then

$$\lim_{x\to a} \frac{F'(x)}{G'(x)} = \lim_{x\to a} \frac{f'(x)}{1} = \lim_{x\to a} f'(x)$$

exists, so using the definition of the derivative and L'Hôpital's Rule, we have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{F(x)}{G(x)} \overset{\text{LH}}{=} \lim_{x\to a} \frac{F'(x)}{G'(x)} = \lim_{x\to a} f'(x)$$

as claimed. $\square$

This says that the derivative of f cannot have a removable discontinuity at a point where f is continuous.

Example: Let $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$

Then just from the fact that f is continuous at 0 and differentiable everywhere *except* at 0, we know $\lim_{x \to 0} f'(x)$ cannot exist.