Fields

<u>Def</u>. A <u>field</u> is a set F with two operations "$+$", "$\cdot$" called addition and multiplication

satisfying "field axioms" (A) (M) (D).

(A) Axioms for addition (A1) $x, y \in F$ $x+y \in F$ (A2) $x+y = y+x$

(A3) $(x+y)+z = x+(y+z)$ (A4) $\exists\, 0 \in F$, $x+0=x$

(A5) $x \in F$, $\exists\, y \in F$ s.t. $x+y=0$ (write $y=-x$)

(M) Axioms for multiplication (M1) $x, y \in F$, $x \cdot y \in F$ (M2) $xy = yx$, $x, y \in F$ (M3) $(xy)z = x(yz)$

(M4) $1 \in F$, $x \cdot 1 = x$ (M5) $x \in F, x \neq 0$, $\exists\, y \in F$ s.t. $xy = 1$
write $y = 1/x$

(D) The distributive law: $x(y+z) = xy + xz$ holds for $x, y, z \in F$.

<u>Rmk</u>: $\mathbb{Q}$ is a field. All familiar properties for $\mathbb{Q}$ should hold for a field

For example: Cancellation law: $x+y = z+y \Rightarrow x = y$.

<u>Def</u> An ordered field is a field which is also an ordered set s.t.

(i) $x+y < x+z$ if $x \in F$, $y < z$.

(ii) $xy > 0$ if $x > 0$, $y > 0$

We will call x positive (negative) if $x > 0$ ($x < 0$), Rmk: $\mathbb{Q}$ is an ordered field $x<y \Leftrightarrow y-x>0$.

Prop For x,y in an ordered field

(a) if $x>0$, then $-x<0$ (b) $x>0$, $y<z \Rightarrow xy<xz$ (c) $x<0$, $y<z \Rightarrow x\cdot y > x\cdot z$

(d) If $x\neq 0$ then $x^2>0$; in particular, $1>0$ (e) $0<x<y \Rightarrow 0<1/y<\frac{1}{x}$

Proof: (a) $0<x \Rightarrow 0+(-x)<x+(-x) \Rightarrow -x<0$. (b) $0<(z-y)\cdot x$

(c) $(z-y)(-x)>0$
$-zx+yx>0 \Rightarrow xy>zx$.

d) $x>0 \Rightarrow x\cdot x>0$
$-x>0 \Rightarrow (-x)\cdot(-x)>0$, $(-x)\cdot(-x) = x^2$

e) $0<\frac{x}{y}<1$
$\Rightarrow 0<\frac{1}{y}<\frac{1}{x}$.

Thm There exists an ordered field $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield, satisfying the least-upperbound property

We call the member in $\mathbb{R}$ a real number.

Construction: Dedekind cuts α:
① $\alpha\neq\mathbb{Q}$ non-empty ② $\forall s\in\alpha$, $t\in\alpha$ if $t<s$ ③ no maximal rational in α.

Can define addition: $\alpha+\beta=\{s+t : s\in\alpha, t\in\beta\}$

multiplication is more tedious. First define positive cuts

Order: $\alpha<\beta$ if $\alpha\subset\beta$.

Least-upper-bound propert: If E is bounded above by β.

Let $s=\bigcup_{\alpha\in E}\alpha\subset\beta$. Can show this is $s=\sup E$. Say $t<s$
$\exists \alpha\notin t$ $t\in s$.

Archimedean property and Denseness of $\mathbb{Q}$ in $\mathbb{R}$.

(a) If $x \in \mathbb{R}_{>0}$, $y \in \mathbb{R}$, there is a positive int. n s.t. $n \cdot x > y$

(b) If $x, y \in \mathbb{R}$, $x < y$, then there is a rat'l number r s.t. $x < r < y$.

Proof (a) If for any $n \in \mathbb{Z}_{>0}$, $nx \leq y$, the set $\{nx\}$ has an upper bound, Then by LUP, let α be the least upper bound of $\{nx\}$. Then $\alpha - x$ is not an upper bound of $\{nx\}$, i.e. $\exists\, n_0 x > \alpha - x \Rightarrow (n_0 + 1) x > \alpha$ #.

(b) WTF: $x < \frac{m}{n} < y$. $\exists n \in \mathbb{Z}$ s.t. $n(y-x) > 1 \Leftrightarrow ny > nx + 1 > nx$. Just find m in between nx and ny. $\exists$ integers m_1, m_2 $m_1 < nx < m_2$ $\exists$ unique m s.t. $nx < m < nx+1$ ☒

<u>Decimals</u> of $x \in \mathbb{R}$. having chosen $n_0, \dots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Then $x = \sup E$, $E = \{n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}\}$. The decimal expression of x is $n_0.n_1n_2\cdots$

<u>Thm</u>. for $\forall x \in \mathbb{R}_{>0}$, there is a unique $y \geq 0$ s.t. $y^n = x$.

Idea: $E = \{t \in \mathbb{R} : t^n \leq x\}$ $y = \sup E \Rightarrow y^n = x$

(Such number y is denoted by $\sqrt[n]{x}$ or $x^{1/n}$)

The symbol $+\infty, -\infty$. We define $-\infty < x < \infty$ for all $x \in \mathbb{R}$

Euclidean space. $\mathbb{R}^k$.

$\mathbb{R}^k \ni \vec{x} = (x_1, x_2, \cdots, x_k)$ s.t.

$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_k + y_k)$

$\alpha \in \mathbb{R}$, $\alpha \cdot \vec{x} = (\alpha x_1, \alpha x_2, \cdots, \alpha x_k)$

Inner product: $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_k y_k$

Define: $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$.

Cauchy-Schwarz Inequality: $\vec{x} \cdot \vec{y} \leq |\vec{x}| \cdot |\vec{y}|$

Proof Notation: $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$.

Want to show:

$$\sum_{i=1}^{n} x_i \cdot y_i \leq \sqrt{\left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right)}$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

$(\lambda \vec{x} - \vec{y}) \cdot (\lambda \vec{x} - \vec{y}) \geq 0$ for any λ.

$\lambda^2 |\vec{x}|^2 - 2\lambda \vec{y} \cdot \vec{x} + |\vec{y}|^2 \geq 0$

$\lambda^2 \cdot A - 2\lambda \cdot B + C \geq 0$

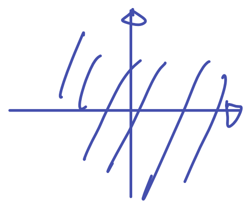
$\Rightarrow A \cdot C \geq B^2$

$A\left(\lambda - \frac{B}{A}\right)^2 + \frac{AC - B^2}{4} \geq 0$ $\boxed{\text{Q.E.D.}}$

Def the Distance between $\vec{x}$ and $\vec{y}$ is $|\vec{x}-\vec{y}| = \sqrt{\Sigma(x_i - y_i)^2}$

Example: $\mathbb{R}^1$:

$\mathbb{R}^2$:



$\mathbb{R}^3$: