Recall:

- Every convergent seq. is bounded
- Limit laws
- extension of limit to $\pm\infty$
- Every bounded monotone seq. converges
- def: $\limsup(a_n) = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$
- $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ (Corollary: $\lim_{n\to\infty} (\frac{1}{n})^{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} (n)^{\frac{1}{n}}} = \frac{1}{1} = 1$)

Thm Let (a_n) be a seq. in $\mathbb{R}$

$\Rightarrow$ (i) if (a_n) converges $\Rightarrow \liminf(a_n) = \limsup(a_n) = \lim_{n\to\infty} a_n$

(ii) if $\limsup(a_n) = \liminf(a_n) = L \in \mathbb{R} \Rightarrow \lim_{n\to\infty} a_n = L$

(因而 (a_n) converges $\Leftrightarrow \limsup = \liminf$)

Pf (i) suppose $\lim_{n\to\infty} a_n = L$

let $\varepsilon > 0$, fix N s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$

$\Rightarrow \forall n \geq N$, $L - \varepsilon \leq \inf\{a_k \mid k \geq n\} \leq \sup\{a_k \mid k \geq n\} \leq L + \varepsilon$

$\Rightarrow \limsup(a_n) = \liminf(a_n) = L$ $\square$

(ii) Suppose $\liminf(a_n) = \limsup(a_n) = L \in \mathbb{R}$

Let $\varepsilon > 0$. Fix N_1, N_2 s.t.

$|\inf\{a_k \mid k \geq n\} - L| < \frac{\varepsilon}{2}$ whenever $n \geq N_1$

and $|\sup\{a_k \mid k \geq n\} - L| < \frac{\varepsilon}{2}$ whenever $n \geq N_2$

$\Rightarrow$ for $N = \max(N_1, N_2)$, $\forall n \geq N$,

$L - \varepsilon \leq \inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\} \leq L + \varepsilon$

So $\lim_{n\to\infty} a_n = L$

Remark: this extends to limits of $\pm\infty$

($\lim a_n = \infty$ (−∞) $\Leftrightarrow \liminf a_n = \limsup a_n = \infty$ (−∞))

Thm Let (a_n) and (b_n) be convergent seq.s in $\mathbb{R}$, st. $a_n < b_n$ for all $n \in \mathbb{N}$,

$\Rightarrow \lim a_n \leq \lim b_n$

Pf Write $L = \lim a_n$ and $M = \lim b_n$

Let $\varepsilon > 0$

Fix N_1, N_2 s.t. $\forall n \geq N_1$, $|a_n - L| < \frac{\varepsilon}{2}$

$\forall n \geq N_2$, $|b_n - L| < \frac{\varepsilon}{2}$

Let $N = \max\{N_1, N_2\}$

$\Rightarrow L < a_N + \frac{\varepsilon}{2} \leq b_N + \frac{\varepsilon}{2} < M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = M + \varepsilon$

So $\forall \varepsilon > 0$, $L < M + \varepsilon \Rightarrow L \leq M$.

Corollary Let (a_n) and (b_n) be bounded seq.s

s.t. $\boxed{\forall n \in \mathbb{N}, a_n \leq b_n}$

$\Rightarrow \boxed{\limsup(a_n) \leq \limsup(b_n)}$ 且 $\boxed{\liminf(a_n) \leq \liminf(b_n)}$

Pf (显然) $\forall n, a_n \leq b_n \Rightarrow \sup\{a_k \mid k \geq n\} \leq \sup\{b_k \mid k \geq n\}$

$\inf\{a_k \mid k \geq n\} \leq \inf\{b_k \mid k \geq n\}$

So $\limsup(a_n) \leq \limsup(b_n)$, ~

Corollary Squeeze Thm

if $\forall n$, $a_n \leq s_n \leq b_n$ 且 $\lim a_n = \lim b_n = L$

$\Rightarrow \boxed{\lim_{n\to\infty} s_n = L}$

Pf Assuming the hypothesis,

$\Rightarrow L = \liminf(a_n) \leq \liminf(s_n) \leq \limsup(s_n)$

$\leq \limsup(b_n) = L$

$\Rightarrow \liminf(s_n) = \limsup(s_n) = L$

$\Rightarrow \lim_{n\to\infty} s_n = L$ $\square$

Corollary Let (a_n) be a seq. of positive reals

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1 \Rightarrow \lim_{n\to\infty} a_n = 0$

但其实这也说明: if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1 \Rightarrow \lim_{n\to\infty} |a_n| = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$. 因而不用positive条件, 只需 $\left|\lim_{n\to\infty} \frac{a_{n+1}}{a_n}\right| < 1$ 即可.

hw: if (a_n) is a seq. of positive reals s.t.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, 则 $\lim_{n\to\infty} a_n = +\infty$

(a_n) converges $\Rightarrow \lim |a_n - a_{n+1}| = 0$ ✓
$\not\Leftarrow$ ✗ but

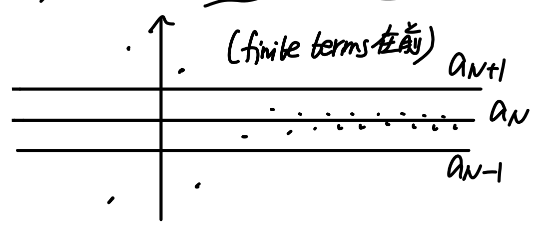
☆☆

Def A seq. (a_n) in $\mathbb{R}$ is called a $\boxed{\text{Cauchy seq.}}$
if $\forall \varepsilon > 0$, $\exists N$ s.t. $\boxed{|a_m - a_n| < \varepsilon \text{ whenever } m,n \geq N}$

Lemma Every Cauchy seq. in $\mathbb{R}$ is bounded

Pf 令 (a_n) 为一个 Cauchy seq.
随便选的
Fix $N \in \mathbb{N}$ s.t. $|a_m - a_n| < 1$ whenever $n,m \geq N$

$\Rightarrow \forall n \geq N$, $a_N - 1 \leq a_m \leq a_N + 1$



Let M be large enough s.t.

$-M \leq \min\{a_k \mid k \leq N\} - 1$
$\leq \max\{a_k \mid k \leq N\} + 1 \leq M$

$\Rightarrow -M \leq a_n \leq M$ for all $n \in \mathbb{N}$

So (a_n) is bounded.

Thm ☆☆☆
A seq. (a_n) in $\mathbb{R}$ converges iff it is Cauchy.
(这是 completeness axiom 的结果)

Pf ① Suppose $\lim a_n = L$
Let $\varepsilon > 0$
Fix $N \in \mathbb{N}$ s.t. $|a_n - L| < \frac{\varepsilon}{2}$ whenever $n \geq N$
$\Rightarrow \forall m,n \geq N$, $|a_m - a_n| = |a_m - L + L - a_n|$
$\leq |a_m - L| + |L - a_n| < \varepsilon$

② Suppose (a_n) is Cauchy
$\Rightarrow$ By Lemma, (a_n) is bounded
$\Rightarrow -\infty < \liminf(a_n) \leq \limsup(a_n) < \infty$

So it suffices to show: $\liminf(a_n) = \limsup(a_n)$

Let $\varepsilon > 0$, fix $N \in \mathbb{N}$ s.t. $|a_m - a_n| < \frac{\varepsilon}{2}$
whenever $m,n \geq N$

$\Rightarrow \forall m \geq N$, $a_N - \frac{\varepsilon}{2} < a_m < a_N + \frac{\varepsilon}{2}$

$\Rightarrow a_N - \frac{\varepsilon}{2} \leq \inf\{a_n \mid n \geq N\} \leq \liminf(a_n)$
$\leq \limsup(a_n) \leq \sup\{a_n \mid n \geq N\} \leq a_N + \frac{\varepsilon}{2}$

即 $\forall \varepsilon > 0$, $|\limsup(a_n) = \liminf(a_n)|$

$\Leftrightarrow \limsup(a_n) = \liminf(a_n)$ $(= \lim a_n)$ □

Def ☆☆ Complete metric space
Cauchy seq. 的 def 可推广至任意 metric space: $\forall \varepsilon > 0$, 有 $N \in \mathbb{N}$ 使 $d(a_m, a_n) < \varepsilon$ whenever $m,n \geq N$.
A metric space (X,d) is said to be complete if every Cauchy seq. in X converges.

ex. $\mathbb{C}$ is complete $\left(d(a+bi, c+di) = \sqrt{(a-c)^2 + (b-d)^2}\right)$

ex2. Let $a < b$, $S_0 = a$, $S_1 = b$, $S_{n+2} = \frac{S_n + S_{n+1}}{2}$ $(\forall n)$
$\Rightarrow |S_{n+2} - S_{n+1}| = \frac{1}{2}|S_{n+1} - S_n| \Rightarrow = \frac{b-a}{2^n}$

$\Rightarrow \forall\, 0 < m < n$

$|S_m - S_n| \leq \left|\sum_{k=m}^{n-1}(S_{k+1} - S_k)\right| \leq \sum_{k=m}^{n-1}|S_{k+1} - S_k|$
$= \sum_{k=m}^{n-1} \frac{b-a}{2^k} \leq \frac{b-a}{2^{m-1}}$

So (S_n) is Cauchy $\Rightarrow$ convergent.
(Hw: find the limit)

Def contractive seq.
A seq. (a_n) in $\mathbb{R}$ is contractive
if $\exists c \in (0,1)$ s.t. $\forall n \in \mathbb{N}$,
$|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$
即:前后两项差越来越小 ($|\text{ratio}| < 1$)

Fact Every contractive seq. in $\mathbb{R}$ is Cauchy
(so convergent)

Pf. 3.5.8

ex $a_1 = 1,\ a_{n+1} = \sqrt{2+a_n}$

首先算出如果 converge，会收敛于什么值

if (a_n) converges ($\lim_{n\to\infty} a_n = l$)

$\Rightarrow l^2 = (\lim a_n)^2 = (\lim a_{n+1})^2 = \lim a_{n+1}^2 = \lim(2+a_n) = 2+l$

$\Rightarrow l = 2$ or -1

Since $\forall n,\ a_n > 0$

$\Rightarrow l = 2$

然后证明的确 converge 于 $l=2$

(by induction : (a_n) bounded 且 increasing $\Rightarrow$ converge $\lim_{n\to\infty} a_n = 2$)

ex2 Given $0 < a < b$, let $S_0 = a,\ S_1 = b,\ S_{n+2} = \sqrt{S_n S_{n+1}}$

then (S_n) converge 且 show $\lim S_n$

① 取 $(a_n)_{n\in\mathbb{N}} = (\ln S_n)$, then $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$

② 取 $(d_n)_{n\in\mathbb{N}} = (a_n - a_{n-1})$, then $d_n = \frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1} = \frac{1}{2}(a_{n-2} - a_{n-1}) = -\frac{1}{2} d_{n-1}$

$\Rightarrow \lim d_n = 0$ by ratio test

$\Rightarrow a_n = a_1 + \sum_{k=1}^{n-1} d_{k+1} \Rightarrow \lim a_n = a + \frac{b-a}{1-(\frac{-1}{2})} = \frac{2}{3}(b-a)+a = \frac{2}{3}b + \frac{1}{3}a$

因而 $\lim S_n = \lim e^{a_n} = e^{\lim a_n} = e^{\frac{2}{3}b + \frac{1}{3}a}$

ex3 (Important: definition of e)

Let $a_n = (1+\frac{1}{n})^{n+1} = (\frac{n+1}{n})^{n+1}$ for $n \in \mathbb{N}$

$\Rightarrow \forall n \in \mathbb{N},\ a_n > 1$ 因而 a_n is bounded below

现在 show (a_n) decreases:

$\forall n > 1,\ \frac{a_{n-1}}{a_n} = \frac{(\frac{n}{n-1})^n}{(\frac{n+1}{n})^{n+1}} = \left(\frac{n^2}{n^2-1}\right)^{n+1}\left(\frac{n-1}{n}\right)$

$= \left(1+\frac{1}{n^2-1}\right)^{n+1}\left(\frac{n-1}{n}\right)$, by Bernoulli, $\geq \left(1+\frac{n+1}{n^2-1}\right)\left(\frac{n-1}{n}\right)$

$= \left(1+\frac{1}{n-1}\right)\left(\frac{n-1}{n}\right) = \frac{n}{n-1}\cdot\frac{n-1}{n} = 1$

因而 (a_n) is weakly decreasing + bounded

$\Rightarrow (a_n)$ converges

We define $\boxed{e = \lim_{n\to\infty}(1+\frac{1}{n})^n}$ $(= \lim(a_n))$

note: $e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty}\frac{(1+\frac{1}{n})^{n+1}}{1+\frac{1}{n}} = \frac{\lim(a_n)}{\lim(1+\frac{1}{n})} = \lim(a_n)$

总结：

① $\mathbb{R}$ 中一系列有用 Thms:

(conv.) 如果 $\forall n,\ a_n \leq b_n \Rightarrow \lim(a_n) \leq \lim(b_n)$

$\xRightarrow{\text{coro}}$ (bounded) 如果 $\forall n,\ a_n \leq b_n \Rightarrow \limsup/\liminf(a_n) \leq \limsup/\liminf(b_n)$

$\xRightarrow{\text{coro}}$ squeeze thm: 如果 $\forall n,\ a_n \leq b_n \leq c_n$ 且 $\lim(a_n) = \lim(c_n) = l \Rightarrow \lim(b_n) = l$

$\xRightarrow{\text{coro}}$ ratio test thm: 如果 $\forall n,\ \lim\left|\frac{a_{n+1}}{a_n}\right| < 1$，则 (a_n) conv.; >1，则 $\lim a_n = \infty / -\infty$

① 一个 metric space 中的一个 Cauchy seq 意思是:
$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t. $\forall m,n \geq N,\ |a_m - a_n| < \varepsilon$

(取任意小的 dist，只要项数足够大时，之后任意两项的 dist 都能小于这一 dist)

② 如果一个 metric space 中任意 Cauchy seq. 都 conv. (即 lim 存在于这个 metric space 中)，则这是个 complete metric space.)

③ $\mathbb{R}$ 和 $\mathbb{C}$ 是 complete metric space (任意 seq. in $\mathbb{C}$ conv. iff. Cauchy.)

④ contractive seq: $\forall n \in \mathbb{N},\ d(a_{n+2}, a_{n+1}) \leq c\, d(a_{n+1}, a_n)$
其中 c 为一个 $(0,1)$ 的数.

$\mathbb{R}$ 中, contractive $\Rightarrow$ Cauchy ($\Leftrightarrow$ conv.)