**Definition:** A sequence $(a_n)$ of real numbers is *eventually constant* if there is $c \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $a_n = c$ for all $n \geq N$.

(1) Let $(a_n)$ be a sequence in $\mathbb{R}$, and consider the bi-implication: "$\lim a_n = \infty \iff \lim \frac{1}{a_n} = 0$." For each direction of this implication, either prove that direction if it is true, or else give a counterexample if it is false.

Forward direction:

Proof Suppose $\lim a_n = \infty$

Let $\varepsilon > 0$

Consider $M = \frac{1}{\varepsilon}$, then for some $N > M$,

$a_n > M$ whenever $n \geq N$

So $a_n > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{a_n}\right| < \varepsilon$ whenever $n \geq N$

So $\lim \frac{1}{a_n} = 0$

Backward direction:

Counterexample Consider $a_n = -n$

So $\lim \frac{1}{a_n} = \lim \frac{-1}{n} = 0$

But $\lim a_n = -\infty$

$\square$

(2) Let $(a_n)$ and $(b_n)$ be sequences of real numbers. Prove that if $\lim a_n = 0$ and $(b_n)$ is bounded, then $\lim a_n b_n = 0$.

Proof. Since $(b_n)$ is bounded, $(|b_n|)$ is also bounded

Consider the constant sequence $S_n = \sup(|b_n|)$

Then $\lim(S_n) = \sup|b_n|$

Since $\lim(a_n) = 0 \Rightarrow \lim(|a_n|) = 0$

So $\lim|a_n S_n| = \lim|a_n| \cdot \lim|S_n| = 0$

Since $S_n = \sup|b_n|$, $0 \le |b_n| \le |S_n|$ for all $n \in \mathbb{N}$

$\Rightarrow 0 \le |a_n b_n| \le |a_n S_n|$ for all $n \in \mathbb{N}$

Since $\lim 0 = \lim|a_n S_n| = 0$,

by the squeeze thm, $\lim|a_n b_n| = 0$

Therefore $\lim a_n b_n = \lim|a_n b_n| = 0$

$\square$

(3) Determine the limits (in $\mathbb{R} \cup \{\pm\infty\}$) of the following sequences, and prove your results:

(a) $\lim_{n\to\infty} \frac{2^n}{n!}$ (b) $\lim_{n\to\infty} \frac{n^n}{n!}$ (c) $\lim_{n\to\infty} b_n$, where $b_1 = 2$ and $b_{n+1} = \frac{b_n^2+2}{2b_n}$

(a) $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

Proof Consider $a_n = \frac{2^n}{n!}$

Take $n \in \mathbb{N}$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{2^{n+1} n!}{(n+1)!\, 2^n}$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

So $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ $\square$

So for any $n \in \mathbb{N}$

$a_{n+1} \leq a_n$.

(b) $\lim_{n\to\infty} \frac{n^n}{n!} = +\infty$

Proof $\frac{n^n}{n!} = \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \ldots \cdot \frac{n}{1}$
$> n$

Let $M > 0$

Consider $N = \lceil M \rceil$

let $n \geq N$, then $\frac{n^n}{n!} > n \geq N \geq M$

So $\lim_{n\to\infty} \frac{n^n}{n!} = +\infty$ $\square$

(c) Assume $\lim b_n = l$

then $\lim b_{n+1} = \lim \frac{b_n^2+2}{2b_n}$

$= \lim \frac{b_n}{2} + \lim \frac{1}{b_n}$

$\Rightarrow l = \frac{1}{2} l + \frac{1}{l}$

$\frac{1}{2} l = \frac{1}{l} \Rightarrow \boxed{l = \sqrt{2}}$

Since $b_n > 0$ for all $n \in \mathbb{N}$

Therefore $\lim_{n\to\infty} b_n$ can only be $\sqrt{2}$ if it exists.

Now we prove that $(b_n)$ does converge

Let $n \in \mathbb{N}$, $b_{n+1} = \frac{b_n}{2} + \frac{1}{b_n}$

$\geq 2\sqrt{\frac{b_n}{2} \cdot \frac{1}{b_n}} = \sqrt{2}$

Since $b_1 = 2 \Rightarrow \forall n \in \mathbb{N}, b_n \geq \sqrt{2}$

So $\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{b_n^2}\right) \leq 1$

Therefore $(b_n)$ is decreasing and bounded below $\Rightarrow$ $(b_n)$ converges

Therefore $\lim(b_n) = \sqrt{2}$ $\square$

(4) Suppose $A$ is a discrete[2] subset of $\mathbb{R}$, and let $(a_n)$ be a convergent sequence of numbers in $A$. Prove that either $(a_n)$ is eventually constant or $\lim a_n \notin A$.

Proof. We prove it by contradiction

Write $\lim a_n = L$

Assume $(a_n)$ is not eventually constant and $\lim a_n \in A$

Since $A$ is discrete, there exists some $\varepsilon > 0$ such that $(L-\varepsilon, L+\varepsilon) \cap A \setminus \{L\} = \emptyset$

Since $\lim a_n = L$, there exists some $N \in \mathbb{N}$ s.t for all $n \geq N$, $|a_n - L| < \varepsilon$, and since $(a_n)$ is not eventually constant, there exists $n \geq N$ s.t. $a_n \neq L$ and $|a_n - L| < \varepsilon$ i.e. $a_n \in (L-\varepsilon, L+\varepsilon)$

So $a_n \in (L-\varepsilon, L+\varepsilon) \cap A \setminus \{L\}$, contradicting with $(L-\varepsilon, L+\varepsilon) \cap A \setminus \{L\} = \emptyset$

This finishes the proof that $(a_n)$ is either eventually constant or $\lim(a_n) \in A$

$\square$

(5) For each positive integer $M$, let $\mathbb{Q}_M$ be the set of all rational numbers $m/n$ where $m, n \in \mathbb{Z}$ and $|m| \leq M$. Prove that for all $M \in \mathbb{N}$, every sequence of distinct numbers in $\mathbb{Q}_M$ converges.

Proof Let $(a_n)$ be an arbitrary sequence in $\mathbb{Q}_M$

Let $\varepsilon > 0$

Since for each $q \in \mathbb{Z}$, there are only finitely many terms of $(a_n)$ that has $q$ as denominator,

Consider $N = \max\left\{ k : a_k = \frac{p}{q} \text{ for some } p \leq M \text{ and } q \leq \lceil \frac{M}{\varepsilon} \rceil \right\}$

Take arbitrary $n \geq N+1$

then $a_n = \frac{M}{q}$ where $q > \frac{M}{\varepsilon}$

So $a_n \leq \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$

So $\lim\limits_{n\to\infty}(a_n) = 0$

This finishes the proof that every sequence of distinct numbers in $\mathbb{Q}_M$ converges

$\square$

(6) Let $(a_n)$ and $(b_n)$ be sequences of real numbers such that $a_n < b_n$ for all $n$.

(a) Show that if $\lim a_n = \infty$, then $\lim b_n = \infty$.

(b) Given an example to show that $(a_n)$ and $(b_n)$ could converge to the same real number.

(a) Suppose $\lim a_n = \infty$

Let $M > 0$ and fix it

Then for some $N \in \mathbb{N}$, $a_N > M$ whenever $n \geq N$

Since $a_n < b_n$ for all $n \Rightarrow$

$\underline{b_N > a_N > M \text{ for all } n \geq N}$

Therefore $\underline{\lim_{n\to\infty} (b_n) = \infty}$ $\square$

(b) Consider $a_n = \frac{1}{n}$, $b_n = \frac{2}{n}$ for all $n \in \mathbb{N}$

So $a_n < b_n$ for all $n \in \mathbb{N}$

But $\lim a_n = \lim b_n = 0$

(7) Let $(a_n)$ be a sequence of positive real numbers. Show that if $\lim \frac{a_{n+1}}{a_n} = L > 1$, then $\lim a_n = \infty$.

Proof Let $\varepsilon = \frac{L-1}{2}$

Since $\lim \frac{a_{n+1}}{a_n} = L$, there is some $N_1 \in \mathbb{N}$

s.t. $\left|\frac{a_{n+1}}{a_n} - L\right| < \varepsilon$ for all $n \geq N_1$

i.e. $L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon$

$\Rightarrow a_{n+1} > \left(\frac{L}{2} + \frac{1}{2}\right) a_n$ for all $n \geq N_1$

Let $M > 0$.

Then there exists some $N_2 \geq N_1$

s.t. $\left(\frac{L}{2} + \frac{1}{2}\right)^{N_2} a_{N_2} > M$, since $\frac{L}{2} + \frac{1}{2} > 1$

then $\forall\, n \geq N_2$, $a_n \geq \left(\frac{L}{2} + \frac{L}{2}\right)^{N_2} a_{N_2} > M$

Therefore $\lim a_n = \infty$

$\square$

(8) Find the lim sup and lim inf of the following sequences. (No justification is needed).

(a) $(a_n)_{n \geq 1}$, where $a_n = (-1)^{n+1} + \frac{(-1)^n}{n}$ $\limsup(a_n) = 1$, $\liminf(a_n) = -1$

(b) $(b_n)_{n \geq 1}$, where $b_n = \sin \frac{1}{n}$ $\limsup(b_n) = \liminf(b_n) = 0$

(c) $(c_n)$, where $c : \mathbb{N} \to \mathbb{Q}$ is any bijection $\limsup(c_n) = +\infty$, $\liminf(c_n) = -\infty$

(d) $(d_n)$, where $d_n = \ln n + \cos n$

$\limsup(d_n) = \liminf(d_n) = +\infty$

(9) Let $a, b \in \mathbb{R}$ with $a < b$. Find the limit of the sequence $(s_n)$ defined recursively by $s_1 = a$, $s_2 = b$, and for all $n \in \mathbb{N}$,

$$s_{n+2} = \frac{s_n + s_{n+1}}{2}.$$

Prove your claim.

$\lim_{n\to\infty} S_n = \frac{2}{3}b + \frac{1}{3}a$

Proof

Let $d_n = S_{n+1} - S_n$ for all $n \in \mathbb{N}$

Then $\begin{cases} d_1 = S_2 - S_1 = b - a \\ d_n = \frac{S_{n-1} + S_n}{2} - S_n, \text{ if } n \geq 2 \end{cases}$

$= -\left(\frac{1}{2}S_n - \frac{1}{2}S_{n-1}\right) = -\frac{1}{2}d_{n-1}$

Note that for all $n \in \mathbb{N}$, $S_{n+1} = \left(\sum_{i=1}^{n} S_{n+1} - S_n\right) + S_1$

$= S_1 + \sum_{i=1}^{n} d_n = a + \frac{1 - (-\frac{1}{2})^n}{1 - (-\frac{1}{2})} d_1 = a + \frac{2}{3}(1 - (-\frac{1}{2})^n)(b-a)$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} = \lim\left(a + \frac{2}{3}(b-a) - \frac{2}{3}(-\frac{1}{2})^n(b-a)\right)$

$= \frac{2}{3}b + \frac{1}{3}a - \frac{2}{3}(b-a)\lim_{n\to\infty}(-\frac{1}{2})^n$ by limit law

Since $|-\frac{1}{2}| < 1$, $\lim_{n\to\infty}(-\frac{1}{2})^n = 1$

So $\lim_{n\to\infty} S_n = \frac{2}{3}b + \frac{1}{3}a$ $\square$

(10) Give an example of a divergent sequence $(a_n)$ in $\mathbb{R}$ with a convergent subsequence such that all convergent subsequences of $(a_n)$ converge to the same limit.

Consider: $a_n = n^{(-1)^n}$ i.e. $(a_n) = (1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots)$

Then $(a_n)$ diverges but every convergent subsequence of $(a_n)$ convergent to $L=0$

Proof

① Consider $(a_{n_k} : k \text{ is odd}) = (1, \frac{1}{3}, \frac{1}{5}, \ldots) \to 0$

② Every convergent subsequence of $(a_n)$ converges to 0

Let $(a_{n_k})$ a convergent subsequence of $(a_n)$

Then $k \xmapsto{g} n_k$ is a strictly increasing function

(i) $\boxed{\text{Suppose there are infinitely } k \in \mathbb{N} \text{ s.t. } n_k \text{ is even}}$

Now we show that $(a_{n_k})$ diverges, so this is impossible

Let $L \in \mathbb{R}$. Take $M = 1$. Let $N \in \mathbb{N}$ and fix it.

If there is no $n_k > N$ s.t. $a_{n_k} > L+1$

then there are only finitely many even $n_k$,

contradicts, which indicates there must exists

$n_k > N$ s.t. $|a_{n_k} - L| > M$

$\Rightarrow (a_{n_k})$ diverges, contradicts

Therefore there can only be finitely many $k \in \mathbb{N}$ s.t. $n_k$ is even

So we can cut the tail and then all remaining $n_k$ ($k \in \mathbb{N}$) ~~are~~ odd $\Rightarrow (a_{n_k})$ converges to 0.

(11) Let $(a_n)$ and $(b_n)$ be bounded sequences of positive real numbers.

(a) Show that $\limsup(a_n + b_n) \le \limsup(a_n) + \limsup(b_n)$.

(b) Give an example to show that $\limsup(a_n+b_n)$ might not equal $\limsup(a_n)+\limsup(b_n)$.

(c) Show that if $(a_n)$ converges, then $\limsup(a_n + b_n) = \limsup(a_n) + \limsup(b_n)$.

(a) Proof. $\limsup(a_n+b_n) = \lim\limits_{n\to\infty} \sup\{a_k+b_k \mid k \ge n\}$

Let $n \in \mathbb{N}$. Let $l_n = \sup\{a_k+b_k \mid k \ge n\}$

$l_{n_1} = \sup\{a_k \mid k \ge n\}$

Let $\varepsilon > 0$ $\quad$ $l_{n_2} = \sup\{b_k \mid k \ge n\}$

then $(\forall k \ge n)$ $a_k < l_{n_1} + \frac{\varepsilon}{2}$ and $b_k < l_{n_2} + \frac{\varepsilon}{2}$

$\Rightarrow \underline{a_k + b_k < l_{n_1} + l_{n_2} + \varepsilon}$ (whenever $\varepsilon > 0$)

So $l_n \le l_{n_1} + l_{n_2}$

Since $n$ is arbitrary, $l_n \le l_{n_1} + l_{n_2}$ for all $n \in \mathbb{N}$

Therefore $\lim l_n \le \lim l_{n_1} + \lim l_{n_2}$

i.e. $\limsup(a_n+b_n) \le \limsup(a_n) + \limsup(b_n)$

(b) counterexample

$a_n = 1 + (-1)^n \Rightarrow \limsup(a_n) = 2$

$b_n = 1 + (-1)^{n+1} \Rightarrow \limsup(b_n) = 2$

but $\limsup(a_n+b_n) = 1+1 = 2 < \limsup(a_n) + \limsup(b_n)$

(c) write $\lim a_n = l$. $\limsup(a_n) = \lim a_n = l$ since $a_n$ converges.

Then $\limsup(a_n) + \limsup(b_n) = l + \limsup(b_n) = l + \lim l_{n_2}$

$= \lim(l + l_{n_2}) = \lim(l_{n_1} + l_{n_2}) = \limsup(a_1 + a_2)$

(12) Prove that there exists a sequence $(a_n)$ in $\mathbb{R}$ such that for every $r \in \mathbb{R}$ there is a subsequence of $(a_n)$ that converges to $r$.

Proof Since $\mathbb{N} \simeq \mathbb{Q}$, there exists a surjective function $S: \mathbb{N} \to \mathbb{Q}$.

Note that $(S_n)$ is a sequence.

Let $r \in \mathbb{R}$ be arbitrary real number

Then there exists a sequence in $\mathbb{Q}$ $(q_n)$ s.t. $(q_n) \to r$

Since $S: \mathbb{N} \to \mathbb{Q}$ is surjective,

consider the subsequence $(S_{n_k})$ of $(S_n)$

defined by $S_{n_k} = q_m$ for some $m \in \mathbb{N}$, for all $k \in \mathbb{N}$

Then Take a monotonic subsequence of $(S_{n_k})$ as $(S_m)$

$(S_m)$ is a subsequence of $(S_{n_k})$, so it is also a subsequence of $(S_n)$

Let $\varepsilon > 0$.

Then there is some $N \in \mathbb{N}$ s.t. $|q_n - r| < \varepsilon$ whenever $n \geq N$

Since there is some term $S_M$ s.t. $S_M = q_N$

and since $(S_m)$ is monotonic, $|S_m - r| < \varepsilon$ whenever $m \geq M$

Therefore $(S_m) \to r$

(13) Determine whether the following sets are open, closed, both, or neither (no justification needed):

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$ neither

(b) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ closed & not open

(0,3) (c) $\bigcup_{n\geq 1} \left[\frac{1}{n}, 3-\frac{1}{n}\right]$ open & not closed

(d) $\mathbb{Z}$ closed & not open

(e) $\mathbb{Q}$ neither

0 (f) $\bigcap_{n\geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$ closed & not open

(14) Either prove the following if it is true, or else give a counterexample if it is false: if $A \subseteq \mathbb{R}$ is closed and discrete, then there is $\epsilon > 0$ such that $|a-b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$. [cf: HW 1, #11(b)]

Counterexample

Consider $S_n = \sum_{k=1}^{n} \frac{1}{k}$, which is a partial sum of harmonic series.

$A = \{S_n : n \in \mathbb{N}\}$

There is no subsequential limit in $S_n$, so A has no limit point $\Rightarrow A' \subseteq A \Rightarrow$ A is closed

And for each $S_n$ ($n \in \mathbb{N}$), consider

$\frac{1}{\varepsilon} \; n\varepsilon = \frac{1}{n}$

$\varepsilon = \frac{1}{n+1}$, then $V_\varepsilon(S_n) \cap A \setminus \{S_n\} = \emptyset$

so A is discrete

But there is no $\varepsilon > 0$ s.t. $|a-b| > \varepsilon$ for each pair of $a, b \in A$, since if we take $\varepsilon > 0$

$S_{\lceil \frac{1}{\varepsilon} \rceil + 1} - S_{\lceil \frac{1}{\varepsilon} \rceil} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$

(15) Suppose the set $A \subseteq \mathbb{R}$ is infinite, bounded, and discrete. Prove that there is a convergent sequence in $A$ whose limit is not in $A$.

Proof Take an arbitrary seq. $(a_n)$ in $A$ s.t.

$\forall m,n \in \mathbb{N},\ a_m \neq a_n$

By the BW theorem, there exists a subsequence of $(a_n)$ that converges. Denote that subsequence by $(a_{n_k})$ and write $\lim a_{n_k} = l$

Claim: $l \notin A$

Suppose $l \in A$.

Since $l$ is the limit of a sequence in $A$,

$l$ is a limit point of $A$

$\Rightarrow \forall \varepsilon > 0,\ \exists x \in A\ (x \neq l)$ s.t. $0 < |x - l| < \varepsilon$

$\Rightarrow x \in V_\varepsilon(l) \Rightarrow x \in V_\varepsilon(l) \cap A$ ①

Since $A$ is discrete and $l \in A$, there exists some $\varepsilon > 0$ s.t. $V_\varepsilon(l) \cap A = \{l\}$ ②

① ② contradicts

So $l \notin A$

$\square$