L 6 (1): limit laws.

ex. $\lim_{n\to\infty} \frac{3n+1}{4n-1}$

Formal Proof: $\left|\frac{3n+1}{4n-1} - \frac{3}{4}\right| = \left|\frac{4(3n+1)-3(4n-1)}{4(4n-1)}\right| = \frac{7}{4(4n-1)}$

So we need $\frac{7}{4(4n-1)} < \varepsilon$

$(4n-1) > \frac{7}{4\varepsilon}$

$n > \frac{7}{16\varepsilon} + \frac{1}{4}$

Now we start the proof:

Let $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $N > \frac{7}{16\varepsilon} + \frac{1}{4}$

So $\frac{7}{4\varepsilon} < 4N-1$

$\Rightarrow$ take $n \geq N$, $\frac{7}{4\varepsilon} < 4n-1$

So $\left|\frac{3n+1}{4n-1} - \frac{3}{4}\right| = \frac{7}{4(4n-1)} < \varepsilon$

So $\lim_{n\to\infty} \frac{3n+1}{4n-1} = \frac{3}{4}$ $\square$

现在我们引入 limit law 来简化这些运算.

(Rudin) Thm 3.3 Suppose $\{s_n\}$, $\{t_n\}$ are complex seq.

Let $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ 适用于$\mathbb{C}$

$\Rightarrow$ (a) $\lim_{n\to\infty} (s_n + t_n) = s+t$ (same for minus)

(b) $\forall c \in \mathbb{C}$, $\lim_{n\to\infty} c + s_n = c + s$, $\lim_{n\to\infty} c s_n = cs$

(c) $\boxed{\lim_{n\to\infty} s_n t_n = st}$ ☆

(d) if $\forall n \in \mathbb{N}$ $s_n \neq 0$, then $\boxed{\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}}$

Pf. take $\varepsilon > 0$

By def, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$|s_n - s| < \frac{\varepsilon}{2}$ whenever $n \geq N_1$;

$|t_n - t| < \frac{\varepsilon}{2}$ whenever $n \geq N_2$;

So consider $N = \max(N_1, N_2)$

$\Rightarrow |(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \varepsilon$

(a) $\square$

(b) trivial $\square$

Now prove (c):

$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$

☆ By (a)(b), $\lim_{n\to\infty}(s_n t_n - st) = \lim_{n\to\infty}(s_n - s)(t_n - t) + s\underbrace{\lim_{n\to\infty}(t_n - t)}_{=0} + t\underbrace{\lim_{n\to\infty}(s_n - s)}_{=0}$

$= \lim_{n\to\infty}(s_n - s)(t_n - t)$

let $\varepsilon > 0 \Rightarrow \exists N_1, N_2 \in \mathbb{N}$ 使

$|s_n - s| < \sqrt{\varepsilon}$ whenever $n \geq N_1$

$|t_n - t| < \sqrt{\varepsilon}$ whenever $n \geq N_2$

So take $N = \max(N_1, N_2)$,

$\Rightarrow |(s_n - s)(t_n - t)| = |s_n - s||t_n - t| < \varepsilon$ whenever $n \geq N$

$\Rightarrow \lim_{n\to\infty}(s_n t_n - st) = 0 \Rightarrow \lim_{n\to\infty} s_n t_n = st$

$\square$ (c)

Now we prove (d):

$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{|s_n s|} = \frac{|s - s_n|}{|s_n||s|}$

Since $\lim_{n\to\infty} s_n = s \Rightarrow \exists m \in \mathbb{N}$ s.t. $|s_n - s| < \frac{1}{2}|s|$ whenever $n \geq m$

$\Rightarrow$ suppose $s > 0 \Rightarrow s_n - s > -\frac{1}{2}s \Rightarrow s_n > \frac{1}{2}s$ $\Rightarrow |s_n| > \frac{1}{2}|s|$

suppose $s < 0 \Rightarrow s_n - s < -\frac{1}{2}s \Rightarrow s_n < \frac{1}{2}s$

whenever $n \geq m$

$\Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n||s|} < \frac{2}{|s|^2}|s_n - s|$ bound ①

并且, $\forall \varepsilon > 0$, $\boxed{\exists N > m}$ 使 $|s_n - s| < \frac{1}{2}|s|^2\varepsilon$ whenever $n \geq N$

$\Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| < \frac{2}{|s|^2}|s_n - s| < \frac{2}{|s|^2}\frac{1}{2}|s|^2\varepsilon$

$= \varepsilon$ bound ②

$\square$ (d)

lec 6. More limit laws

(e) if (a_n) converges, then $(|a_n|)$ 也 converge. (反之则不然, counterex: oscillating seq.)

(f) $\forall k \in \mathbb{N}$, $\lim_{n\to\infty}(a_n^k) = (\lim_{n\to\infty} a_n)^k$

(g) $\forall k \in \mathbb{N}$, $\lim_{n\to\infty}(a_n^{\frac{1}{k}}) = (\lim_{n\to\infty} a_n)^{\frac{1}{k}}$ (provided $\forall k$, $a_k \geq 0$)

(e)的proof：trivial

(f)的proof：$\lim_{n\to\infty} a_n^k = \lim_{n\to\infty} (a_n)(a_n)\dots(a_n) = a^k$ (trivial)

(g)的proof：首先引入一个定义

Def of real exponential (Terence 5.6.4)

$$x^{\frac{1}{n}} := \sup\{y\in\mathbb{R} \mid y\ge 0 \text{ and } y^n \le x\}$$

note: $x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \dots + y^{k-1})$

因而 $a_n - a = |a_n^{\frac{1}{k}} - a^{\frac{1}{k}}|\cdot(a_n^{\frac{k-1}{k}} + a_n^{\frac{k-2}{k}}a^{\frac{1}{k}} + \dots + a^{\frac{k-1}{k}})$

因而 $\forall\varepsilon>0$, $|a_n^{\frac{1}{k}} - a^{\frac{1}{k}}| = \frac{a_n - a}{(\cdots\cdots)}$ ($<\varepsilon$, bounded) ← is bounded.

(Rudin) Thm 3.4

(a) suppose $(\vec{x}_n)_{n\in\mathbb{N}} \in \mathbb{R}^k$

且 $\vec{x}_n = \begin{bmatrix}\alpha_{1,n}\\ \alpha_{2,n}\\ \vdots \\ \alpha_{k,n}\end{bmatrix}$ for some seq.s $(\alpha_{1,n})_{n\in\mathbb{N}}$, $(\alpha_{2,n})_{n\in\mathbb{N}}, \dots, (\alpha_{k,n})_{n\in\mathbb{N}}$

$\Rightarrow (x_n)_{n\in\mathbb{N}}$ converges to $\vec{x} = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_k\end{bmatrix}$ iff $\forall 1\le i\le k$, $\boxed{\lim_{n\to\infty} \alpha_{i,n} = \alpha_i}$

即：$\{\vec{x}_n\}_{n\in\mathbb{N}}$ 为一个seq. of real vector，其中每个entry都是一个seq. of real nums，那么 $\{\vec{x}_n\}_{n\in\mathbb{N}}$ 的limit 的每个entry 都是其对应 seq. 的limit

(b) suppose $\{\vec{x}_n\}, \{\vec{y}_n\}$ 为 $\mathbb{R}^k$上的 seq.s

$\{\beta_n\}$ 为 seq. of real nums.

且 $\lim_{n\to\infty}\vec{x}_n = \vec{x}$, $\lim_{n\to\infty}\vec{y}_n = \vec{y}$, $\lim_{n\to\infty}\beta_n = \beta$

$$\Rightarrow \quad \lim_{n\to\infty} \vec{x}_n + \vec{y}_n = x+y, \quad \lim_{n\to\infty} \vec{x}_n\cdot\vec{y}_n = \vec{x}\cdot\vec{y}, \quad \lim_{n\to\infty}\beta_n x_n = \beta x$$

Pf

one direction / $\|\vec{x}_n - \vec{x}\| = \left(\sum_{i=1}^{k}(\alpha_{i,n}-\alpha_i)^2\right)^{\frac{1}{2}}$

因而 $\forall i$, $|\alpha_{i,n} - \alpha_i| \le \|\vec{x}_n - \vec{x}\|$

因而 $|\alpha_{i,n}-\alpha_i|$ 是被 $\|\vec{x}_n - \vec{x}\|$ bounded 的

因而易得 $\left(\lim_{n\to\infty} x_n = x \Rightarrow \forall i, \lim_{n\to\infty}\alpha_{i,n} = \alpha_i\right)$ ①

other direction / Assume $\forall i$, $\lim_{n\to\infty}\alpha_{i,n} = \alpha_i$

于是 $\forall\varepsilon>0$, $\exists N_i$ s.t. $\forall n\ge N_i$, $|\alpha_{i,n} - \alpha_i| < \frac{\varepsilon}{\sqrt{k}}$

$\Rightarrow \|\vec{x}_n - \vec{x}\| = \left(\sum_{i=1}^{k}(\alpha_{i,n}-\alpha_i)^2\right)^{\frac{1}{2}} < (\varepsilon^2)^{\frac{1}{2}} = \varepsilon$

因而 $\left(\forall i, \lim_{n\to\infty}\alpha_{i,n} = \alpha_i \Rightarrow \lim_{n\to\infty} x_n = x\right)$ ②

(b) follows from (a) 及 Thm 3.3. □

因而现在我们apply limit rule to simply calculation：

$$\lim_{n\to\infty}\frac{3n+1}{4n-1} = \lim_{n\to\infty}\frac{3+\frac{1}{n}}{4-\frac{1}{n}} = \frac{3+\lim_{n\to\infty}\frac{1}{n}}{4-\lim_{n\to\infty}\frac{1}{n}} = \frac{3}{4}$$

Def. A $\boxed{\text{function}}$ $f: X\to\mathbb{R}$ 被称为$\boxed{\text{bounded}}$的 if $ran(f)$ 为一个bounded subset of $\mathbb{R}$

(same for: bounded above/below function)

Lec5. Thm ① 任意 convergent seq. of real nums is bounded

Pf. Suppose $(a_n) \to L$

Fix $N\in\mathbb{N}$ s.t. $|a_n - L| < 1$ whenever $n\ge N$

Let $M_1 = \min(L-1, \min\{a_k \mid k<N\})$

$M_2 = \max(L+1, \max\{a_k \mid k<N\})$

$\Rightarrow \forall k\in\mathbb{N}$, $M_1 \le a_k \le M_2$ for all $k\in\mathbb{N}$.

因而 (a_k) is bounded.

Limit as $n\to\infty$ of rational functions of n.

Let $f(x) = \frac{p(x)}{q(x)} = \boxed{\frac{a_m x^m + \dots + a_1 x + a_0}{b_k x^k + \dots + b_1 x + b_0}}$

where $a_m, b_k \ne 0$

then $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\frac{a_m n^m}{b_k n^k} = \begin{cases}\frac{a_m}{b_k}, & \text{if } m=k\\ 0, & \text{if } m<k\\ +\infty, & \text{if } m>k \text{ 且 } \frac{a_m}{b_k}>0\\ -\infty, & \text{if } m>k \text{ 且 } \frac{a_m}{b_k}<0\end{cases}$

pf directly follows from limit's law

L6(2): limits involving $\pm\infty$

Thm limit multiplication of $\pm\infty$

if $\lim a_n = +\infty$, $\lim b_n = L > 0$, 则 $\lim a_n b_n = +\infty$

if $\lim a_n = +\infty$, $\lim b_n = L < 0$, 则 $\lim a_n b_n = -\infty$

if $\lim a_n = -\infty$, $\lim b_n = L > 0$, 则 $\lim a_n b_n = -\infty$

if $\lim a_n = -\infty$, $\lim b_n = L < 0$, 则 $\lim a_n b_n = +\infty$

Same for $(+\infty)(+\infty)$, $(+\infty)(-\infty)$, $(-\infty)(-\infty)$, $(-\infty)(+\infty)$

Thm limit addition of $\pm\infty$

if $\lim a_n = \infty$ $(\pm)$ 且 (b_n) converges

$\Rightarrow \lim(a_n + b_n) = \infty$ $(\pm)$

exercise

if (a_n) is a seq. of reals $\Rightarrow$

(if positive $\Rightarrow$ $a_n \to +\infty$ iff $\frac{1}{a_n} \to 0$)

(if negative $\Rightarrow$ $a_n \to -\infty$ iff $\frac{1}{a_n} \to 0$)

L6(3): Monotone seq.s

Def. Monotone seq.s

A seq. (a_n) 称为 increasing 的, if $\forall n$, $a_n \le a_{n+1}$

称为 decreasing 的, if $\forall n$, $a_n \ge a_{n+1}$

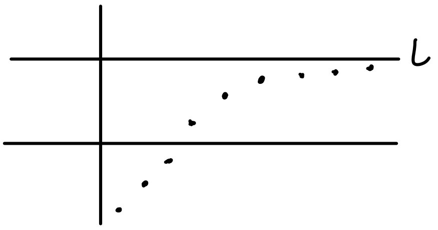
称为 monotone 的 if 它是 increasing 的 或 decreasing 的

Note Increasing seq. 必定 bounded below.

decreasing seq. 必定 bounded above. (因为 seq. 是有起点的)

Thm The monotone converge Thm ☆☆

every bounded monotonic seq. converges.



Pf. Suppose (a_n) is bounded and increasing.

Let $\boxed{L = \sup\{a_n\}}$

Let $\varepsilon > 0$. fix N s.t. $L - \varepsilon < a_N$

Since (a_n) is increasing,

$\forall n \ge N$, $L - \varepsilon < a_N \le a_n$

Therefore $\lim_{n\to\infty} a_n = L$

Dually, 可证: if (a_n) decreasing,

$\lim_{n\to\infty} a_n = \boxed{\inf\{a_n\}}$

L6(4): limsup and liminf

上极限 下极限

Def. limsup and liminf ☆

令 (a_n) 为一个 $\boxed{\text{bounded seq.}}$ in $\mathbb{R}$ $\Longrightarrow \sup\{a_n\}, \inf\{a_n\} \in \mathbb{R}$

note that: $\forall n \in \mathbb{N}$, $\sup\{a_k \mid k \ge n\} \ge \sup\{a_k \mid k \ge n+1\}$

令 $\boxed{u_n = \sup\{a_k \mid k \ge n\}}$ for each $n \in \mathbb{N}$.

于是 $(u_n)_{n\in\mathbb{N}}$ 是一个 bounded 且 decreasing 的 seq.

我们定义: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} u_n$ 即 $\lim_{n\to\infty} \sup\{a_k \mid k \ge n\}$

Similarly, 定义 $l_n = \inf\{a_k \mid k \ge n\}$

$(l_n)_{n\in\mathbb{N}}$ 是一个 bounded 且 increasing 的 seq.

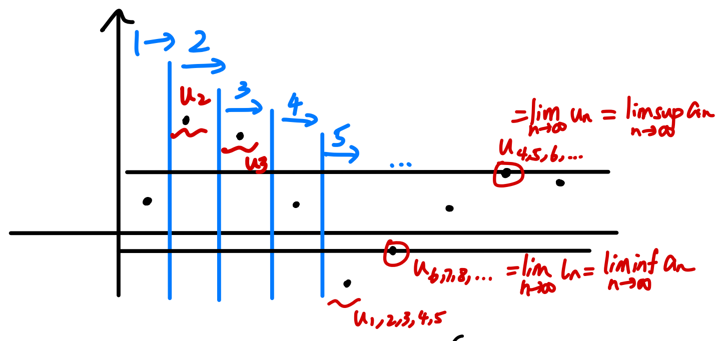
($\forall n \in \mathbb{N}$, $\inf\{a_k \mid k \ge n\} \le \sup\{a_k \mid k \ge n+1\}$

因为 if $A \subseteq B \subseteq \mathbb{R}$, 总是有 $\inf B \le \inf A \le \sup A \le \sup B$)

我们定义: $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} l_n$ 即 $\lim_{n\to\infty} \inf\{a_k \mid k \ge n\}$

note that:

$\inf\{a_n \mid n \in \mathbb{N}\} \le \liminf(a_n) \le \limsup(a_n) \le \sup\{a_n \mid n \in \mathbb{N}\}$



Intuition: limsup 是 the $\boxed{\text{largest num}}$ s.t. (liminf: smallest)

(a_n) gets arbitrarily close to, for $\boxed{\text{infinitely often 次}}$ ☆

即 $\forall \varepsilon > 0$, $L - \varepsilon < a_n$ for infly many n. (必须是 infly many, like 循环, 否则会在某个 u_n 被滤掉.)

Fact $l = \limsup(a_n)$ iff

$\forall \varepsilon > 0$, $\{n \in \mathbb{N} \mid a_n > l - \varepsilon\}$ is infinite

且 $\{n \in \mathbb{N} \mid a_n > l + \varepsilon\}$ is finite

ex. $\liminf (-1)^n = -1$, $\limsup (-1)^n = 1$

$\liminf ((-1)^n + \frac{1}{n}) = -1$, $\limsup ((-1)^n + \frac{1}{n}) = 1$

$\liminf (\sin(n)) = -1$, $\limsup (\sin(n)) = 1$

Def extend the definition of limsup, liminf to unbounded seq.

write $\liminf(a_n) = \underset{(+)}{-\infty}$ if $\inf\{a_n \mid n \in \mathbb{N}\} = \underset{(+)}{-\infty}$

and $\liminf(a_n) = +\infty$ if $\lim\limits_{n\to\infty} a_n = +\infty$ (可以很 oscillating)

(只可能发散至$+\infty$)

Same for limsup.

Thm Let (a_n) be a seq. of real nums.

$\Rightarrow$ (i) if (a_n) converges, then $\limsup a_n = \underbrace{\liminf a_n}_{= \lim a_n}$

(ii) if $\liminf a_n = \limsup a_n = l \in \mathbb{R}$, then $\lim a_n = l$

Pf. (i) Suppose $\lim a_n = l \in \mathbb{R}$

Let $\varepsilon > 0$, and fix N s.t. $\forall n \geq N$, $|a_n - l| < \varepsilon$

Then for any $n \geq N$, $\Downarrow$

$l - \varepsilon \leq \inf\{a_k \mid k \geq n\} \leq \sup\{a_k \mid k \geq n\} \leq l + \varepsilon$

因而

$\liminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$

(ii) Suppose $\liminf(a_n) = \limsup(a_n) = l \in \mathbb{R}$

Let $\varepsilon > 0$

Fix N_1, N_2 s.t. $|\inf\{a_k \mid k \geq n\} - l| < \varepsilon$ whenever $n \geq N_1$

及 $|\inf\{a_k \mid k \geq n\} - l| < \varepsilon$ whenever $n \geq N_2$

take $N = \max\{N_1, N_2\}$

then suppose $n \geq N (\geq N_1, N_2)$

$\Rightarrow$ $l - \varepsilon < \inf\{a_k \mid k \geq n\} \leq a_n \leq \sup\{a_k \mid k \geq n\}$

因而 $\lim a_n = l$

Remark: the thm extend to $\pm\infty$

$\lim a_n = +\infty \Leftrightarrow \liminf(a_n) = \limsup(a_n) = +\infty$