1. (10 pts) Prove, using the definition of a least upper bound, that a subset E in an ordered set S cannot have two different least upper bounds. *Hint: Suppose that both a and b were both least upper bounds of a set S. Show that $a = b$.*

<u>Proof</u>. Let S be an ordered set and E be an arbitrary subset of S.

Suppose $a = \sup S$ and $b = \sup S$.

Without loss of generally, assume $b < a$, seeking contradiction.

Since $a = \sup S$, by definition of least upper bound, any element in S that is less then a is not an upper bound of E.

Therefore b is not an upper bound of E

Since $b = \sup E$, contradicts.

Hence $b \geq a$.

Similarly, assume $a < b$, we will come to that a is not an upper bound of S which contradicts with $a = \sup E$. So $a \geq b$

Since $b \geq a$ and $b \leq a$, b must equal to a.

$\square$

2. (10 pts) Find the least upper bound and the greatest lower bound, if they exist, of the following subsets of $\mathbb{Q}$. Also decide which sets have a maximum or minimum. Recall that the maximum (resp. minimum) of an order set is the largest (resp. smallest) element in it.

(a) $\{1/n : n \in \mathbb{N}\}$.

(b) $\{1/n : n \in \mathbb{Z} \text{ and } n \neq 0\}$.

(c) $\{x : x = 0 \text{ or } x = 1/n \text{ for some } n \in \mathbb{N}\}$.

(d) $\{1/n + (-1)^n : n \in \mathbb{N}\}$.

(a) LUB: 1

GLB: 0

maximum: 1

minimum; does not exist

(* this answer uses definition that $0 \notin \mathbb{N}$)

Justification.

Denote the set by E.

For any $n \in \mathbb{N}$ greater than 1, $\frac{1}{n} < \frac{1}{1} = 1$

Therefore 1 is an upper bound of E,

Since $1 = \frac{1}{1} \in E$ 1 is also an maximum.

Let u be an arbitrary upper bound of E with $u < 1$

$\Rightarrow 1 \in E$ is greater than u, contradicts

therefore by definition of LUB, $1 = \sup E$

Since $\forall n \in \mathbb{N}, \frac{1}{n} > 0$, 0 is an lower bound of E

Let $w > 0$ an arbitrary

By the Archimedean property

we know $\exists$ some $n \in \mathbb{N}$ s.t. $nw > 1 \Rightarrow w > \frac{1}{n}$

Therefore any $w>0$ is not an lower bound of E

so $\underbrace{0 = \inf E}$

$\underbrace{\text{E has not minimal}}$ since if we assume some $n \in \mathbb{N}$ s.t. $\frac{1}{n}$ is the minimal of E, by taking $n+1$, $\frac{1}{n+1}$ is small since $(\frac{1}{n} - \frac{1}{n+1}) > 0$. so contradicts $\Rightarrow$ no minimal.

(b) LUB: 1

GLB: -1

maximum: 1

minimum: -1

Justification.

Denote the set as S.

For any $n<0$ with $n \in \mathbb{N}$ $\frac{1}{n}<0<1$, $1 \in S$

Therefore exactly the same as (a) we conclude that $\underbrace{\sup E = 1}$, $\underbrace{\text{maximum is also } 1}$.

For any $n>0$ with $n \in \mathbb{N}$, $\frac{1}{n} > 0 > -1$, $-1 \in S$

Therefore to find the minimal and $\inf E$ is to find the minimal and GLB of

$$\{\frac{1}{n} : n \in \mathbb{Z}^-\} = \{-\frac{1}{n} : n \in \mathbb{N}\}$$

Note that the process is the same as finding the maximum and LUB of the set in (a)

So the minimal and $\inf E$ is -1.

(c) LUB: 1
GLB: 0
maximum: 1
minimum: 0

Justification,

Denote the set as E and the set in (a) as E_a,
Note that $E \setminus E_a = \{0\}$.
And $0 < 1$, $1 \in E_a$
So the LUB and maximum of E is 1, same as E_a.
Let $e \in E$ be arbitrary,
if $e \neq 0$, then $e = \frac{1}{n}$ for some $n \in \mathbb{N}$,
$e > 0$.
Since $0 \in E$, 0 is minimal of E
and 0 is a lower bound of E
Let $f > 0$ be arbitrary, Note $0 \in E$ and $0 < f$
so f is not a lower bound of E
Therefore $0 = \inf E$.

(d) LUB: $\frac{3}{2}$

GLB: -1

minimum: does not exist

maximum: $\frac{3}{2}$

Justification.

Denote the set as E

$E = \{0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, \dots\}$

Note that when n is odd, $(-1)^n = -1$,
when n is even, $(-1)^n = 1$

Therefore we divide it into two subsets.

$\Rightarrow E = \{\frac{1}{n} - 1 : n \in \mathbb{N} \text{ and } n \text{ is odd}\} \cup \{\frac{1}{n} + 1 : n \in \mathbb{N} \text{ and } n \text{ is even}\}$

We denote these two sets as E_1, E_2 respectfully

(Since $n \in \mathbb{N}$ can only be even or odd, there is no third case.)

Since for any $n \in \mathbb{N}$ and n is odd, $\frac{1}{n} - 1 > -1$ and $\frac{1}{n} - 1 \leq 0$

for any $n \in \mathbb{N}$ and n is even,
$\frac{1}{n} + 1 > 1$ and $\frac{1}{n} + 1 \leq \frac{3}{2}$,

Any elements of E_2 is greater than all elements of E_1

Therefore, The problem is to find

the maximal and LUB
of E_2 and minimal and GLB of E_1
It is exactly the same logic as (a)
Therefore we have $\sup E = \sup E_2 = \frac{3}{2}$
$\inf E = \inf E_1 = -1$
maximum $= \frac{3}{2}$
minimum does not exist
since minimum of E_1 does not exist
(same as (a))

3. (10 pts) Suppose that A and B are two nonempty subsets of an ordered set S such that $x \leq y$ for all $x \in A$ and $y \in B$.
   (a) Prove that $\sup A \leq y$ for all y in B.
   (b) Prove that $\sup A \leq \inf B$.

(a) Let b be an arbitrary element of B.
Since $x \leq y$ for all $x \in A$ and $y \in B$,
$x \leq b$ for all $x \in A$
By definition of upper bound, b is an upper bound of A
Then by definition of least upper bound,
$\sup A \leq a$.
Since a is arbitrary, we can conclude that
for all $y \in B$, $\sup A \leq y$. $\square$

(b) Assume for sake of contradiction that $\sup A > \inf B$.

By definition of greatest lower bound, $\sup A$ is not a lower bound of B.

Therefore, there exists some $b \in B$ such that $b < \sup A$

Then by definition of least upper bound, b is not an upper bound of A.

Therefore, there exists some $a \in A$ such that $a > b$.

This contradicts with $x \leq y$ for all $x \in A$ and $y \in B$.

Hence we have proved that $\sup A \leq \inf B$. $\square$