Key Lemma

Let $f: [a,b] \to \mathbb{R}$ be bounded with $|f(x)| \le B$ for all $x \in [a,b]$ and let $\boxed{Q \supseteq P = (x_k)_{k=0}^{n}}$ be partitions of $[a,b]$

i.e. Q 为 P 的一个 refinement.

Let $\boxed{J = \{k : Q \cap (x_{k-1}, x_k) \neq \emptyset\}}$

$\Rightarrow$ $L(f,P) \le L(f,Q)$ and

$$|L(f,P) - L(f,Q)| \le 2 \cdot |J| \cdot B \cdot \|P\|$$

以及 dually, $U(f,Q) \le U(f,P)$ and

$$|U(f,Q) - U(f,P)| \le 2 \cdot |J| \cdot B \cdot \|P\|$$

Key Lemma 陈述的事情是：对一个 partition 的 refinement 一定会让 upper sum 更小，lower sum 更大，即

$$L(f,P_1) \le L(f,P_1 \cup P_2) \le U(f,P_1 \cup P_2) \le U(f,P_1)$$

并且 refinement 后，L 和 U 的变化也有界，和 ① 函数界 ② 新加入的点数 ③ mesh 大小 都有关

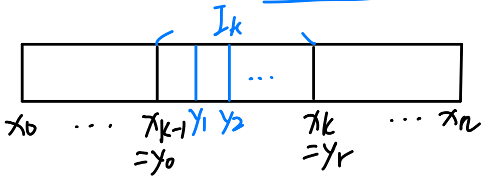
Pf Fix $k \in J$.

Let $y_0 < \dots < y_r$ be the partition pts. of Q in order that lies in $I_k = [x_{k-1}, x_k]$

So $y_0 = x_{k-1}$, $y_r = x_k$, $r \ge 2$

$$L(f, Q \cap I_k) = \sum_{i=1}^{r} \left(\inf_{[y_{i-1}, y_i]} f\right) \Delta y_i$$

$$\text{and } (\inf f[I_k]) \Delta x_k = \sum_{i=1}^{r} \inf f[I_k] \Delta y_i$$



因而 $0 \le L(f, Q \cap I_k) - (\inf_{I_k} f) \Delta x_k$

$$= \sum_{i=1}^{r} \left(\inf_{[y_{i-1}, y_i]} f - \inf_{I_k} f \right) \Delta y_i$$

$$\le \sum_{i=1}^{r} 2B \Delta y_i = 2B \Delta x_k \le 2B\|P\|$$

因为最坏的可能：本来在最高点 (B)，新增的区间全都包括了最低点 $(-B)$

Since this holds for all $k \in J$,

we have $0 \le L(f,Q) - L(f,P) = \sum_{k \in J} \left(L(f, Q \cap I_k) - (\inf_{I_k} f) \Delta x_k \right)$

因而 $0 \le L(f,Q) - L(f,P) \le \sum_{k \in J} 2B\|P\| = 2 \cdot |J| \cdot B \cdot \|P\|$ □

Thm① Rm intble $\Leftrightarrow$ Db intble

For a bounded function $f: [a,b] \to \mathbb{R}$, TFAE:

(1) $\boxed{f \text{ is Riemann intble on } [a,b]}$

(2) 对任意 $\varepsilon > 0$，都有 $\delta > 0$ 使得 $\forall$ tagged partition $\dot{P}$ and $\dot{Q}$ of $[a,b]$ s.t. $\|\dot{P}\|, \|\dot{Q}\| < \delta$，都有 $|S(f,\dot{P}) - S(f,\dot{Q})| < \varepsilon$

(3) 对任意 $\varepsilon > 0$，都有 $\delta > 0$ 使得 $\forall$ partition P of $[a,b]$ s.t. $\|P\| < \delta$，都有 $|U(f,P) - L(f,P)| < \varepsilon$

equiv

(4) $\boxed{f \text{ is Darboux intble on } [a,b].}$

(5) 对任意 $\varepsilon > 0$，都有某个 partition P of $[a,b]$, s.t. $|U(f,P) - L(f,P)| < \varepsilon$.

Pf 我们证明路线：

$(1) \to (2) \to (3) \to (1)$

$(3) \to (4) \to (5) \to (3)$

$(1) \Rightarrow (2)$：(自然得)

Assume f is Rieman intble on $[a,b]$

Let $\varepsilon > 0$

Fix $\delta > 0$ s.t. ($\forall \dot{P}$ s.t. $\|\dot{P}\| < \delta$，都有 $|S(f,\dot{P}) - L| < \frac{\varepsilon}{2}$)

Let $\|\dot{P}\|, \|\dot{Q}\|$ be arbitrary tagged partitions of $[a,b]$ with $\|\dot{P}\|, \|\dot{Q}\| < \delta$.

$$\Rightarrow |S(f,\dot{P}) - S(f,\dot{Q})| \le |S(f,\dot{P}) - L| + |S(f,\dot{Q}) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

$(2) \Rightarrow (3)$

这看起来很显然，实际上缺少的部分就是 $L(f,P)$ 和 $U(f,P)$ 并不一定是 rieman sums，因为 extrema 不一定存在。

但没关系，我们可以选取接近的 tag 来 ε 逼近 inf 和 sup。

let $\varepsilon > 0$

Fix $\delta > 0$ s.t. ($\forall \dot{P}, \dot{Q}$ of $[a,b]$ s.t. $\|\dot{P}\|, \|\dot{Q}\| < \delta$，都有 $|S(f,\dot{P}) - S(f,\dot{Q})| < \frac{\varepsilon}{2}$)

Let $P = (x_k)_{k=0}^{n}$ be arbitrary partition of $[a,b]$ s.t. $\|P\| < \delta$

For each $1 \le k \le n$, choose s_k, t_k s.t.

$$|f(s_k) - \inf_{I_k} f| < \frac{\varepsilon}{4(b-a)} \text{ 及 } |f(t_k) - \sup_{I_k} f| < \frac{\varepsilon}{4(b-a)}$$

其得到的两个 tagged partition 分别为

$$\dot{P}_s = \{(x_1, \dots, x_n), (s_1, \dots, s_n)\}, \quad \dot{P}_t = \{(x_1, \dots, x_n), (t_1, \dots, t_n)\}$$

于是 $|L(f,P)-S(f,\dot{P}_s)|<\frac{\varepsilon}{4}$, $|U(f,P)-S(f,\dot{P}_t)|<\frac{\varepsilon}{4}$

且 $|S(f,\dot{P}_s)-S(f,\dot{P}_t)|<\frac{\varepsilon}{2}$ as known

$\Longrightarrow |U(f,P)-L(f,P)|<\varepsilon$ □

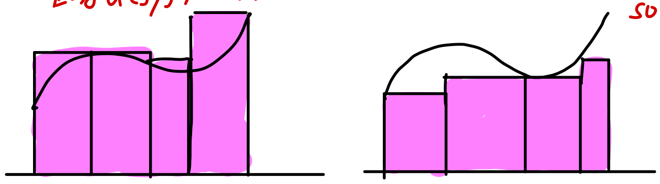
(3) $\Longrightarrow$ (4)

Immediately from the definition.

因为我们知道, $\forall P_1, P_2$, 都有 $U(f,P_1)\geq L(f,P_2)$

因而 $\boxed{\forall P,\ L(f,P)<U(f)}$ 且 $\boxed{U(f,P)>L(f)}$ 👍

因为 $U(f,P)$ 总是 $\geq$ area so far, 而 $L(f,P)$ 总是小于 area so far



因而 (3) 中我们可以忽略"δ"这个东西

它告诉我们的事情是:

$\Longrightarrow \forall\varepsilon>0$, 都有 P 使得 $L(f,P)<U(f,P)-\varepsilon$

$\Longrightarrow L(f,P)\leq L(f)\leq U(f)<U(f,P)-\varepsilon$

$\Longrightarrow |L(f)-U(f)|\leq|L(f,P)-U(f,P)|<\varepsilon$

This is how we define equal. 因而 $L(f)=U(f)$

$\Longrightarrow$ Darboux intble.

至此 我们已经证明了:

Riemann intble $\Longrightarrow$ Darboux intble. □

(3) $\Longrightarrow$ (1)

Assuming (3): 对任意 $\varepsilon>0$, 都有 $\delta>0$ 使得

$\forall$ partition P of $[a,b]$ s.t. $\|P\|<\delta$,

都有 $|U(f,P)-L(f,P)|<\varepsilon$

Then by 刚才的一条, f is DB intble on $[a,b]$.

Write $L=L(f)=U(f)$

Let $\varepsilon>0$.

Fix $\delta>0$ s.t ($|U(f,P)-L(f,P)|<\varepsilon$ whenever $\|P\|<\delta$)

$\Longrightarrow$ Then $\forall\dot{P}$ of $[a,b]$ with $\|\dot{P}\|<\delta$,

都有 $L-\varepsilon<L(f,P)\leq S(f,\dot{P})\leq U(f,P)<L+\varepsilon$

$\Longrightarrow |S(f,\dot{P})-L|<\varepsilon$ □

(4) $\Longrightarrow$ (5)

Assume (4): f is Darboux intble, 即 $L(f)=U(f)$

Let $\varepsilon>0$.

Fix P, Q of $[a,b]$ s.t. $|L(f,P)-L(f)|<\frac{\varepsilon}{2}$ 及

$|U(f,Q)-U(f)|<\frac{\varepsilon}{2}$.

(Recall: 之所以我们可以这么做, 是因为我们确定这样的 U(f, P) 是一定存在的. 因为对于一个集合 A, 如果 sup(A) = L, 那么任取一个 ε, A 中一定存在 (L−ε, L] 之间的元素, 否则 sup(A) <= L−ε, 矛盾.)

这个时候, 我们的 key Lemma 终于派上用处.

$\Longrightarrow L(f)-\frac{\varepsilon}{2}<L(f,P)\leq L(f,P\cup Q)\leq U(f,P\cup Q)$

$\leq U(f,Q)<U(f)+\frac{\varepsilon}{2}$

于是 $|U(f,P\cup Q)-L(f,P\cup Q)|<\varepsilon$ □

(5) $\Longrightarrow$ (3)

Assume (5): 对任意 $\varepsilon>0$, 都有某个 partition P of $[a,b]$,

s.t. $|U(f,P)-L(f,P)|<\varepsilon$.

WTS (3): 对任意 $\varepsilon>0$, 都有 $\delta>0$ 使得

$\forall$ partition P of $[a,b]$ s.t. $\|P\|<\delta$,

都有 $|U(f,P)-L(f,P)|<\varepsilon$

Let $\varepsilon>0$.

Fix $P_0=(z_k)_{k=0}^{m}$ of $[a,b]$ s.t.

$|U(f,P_0)-L(f,P_0)|<\frac{\varepsilon}{2}$

Fix B s.t. $|f(x)|\leq B$ for all $a\leq x\leq b$

Let $\delta=\frac{\varepsilon}{8mB}$. 函数界 准备使用 key lemma.

Let $P=(x_k)_{k=0}^{n}$ be arbi partition of $[a,b]$ s.t. $\|P\|<\delta$.

Let $Q=P\cup P_0 \Longrightarrow Q$ 为 P 的 refinement.

于是 By Lemma, $|L(f,Q)-L(f,P)|\leq 2mB\|P\|$. ①

$<2mB\delta\leq\frac{\varepsilon}{4}$

及同理 $|U(f,Q)-U(f,P)|\leq\frac{\varepsilon}{4}$ ②

且我们有 $L(f,P_0)\leq L(f,Q)\leq U(f,Q)\leq U(f,P_0)$

及 $|U(f,P_0)-L(f,P_0)|<\frac{\varepsilon}{2}$

$\Longrightarrow |U(f,Q)-L(f,Q)|<\frac{\varepsilon}{2}$ ③

Combining ①②③ $\Longrightarrow |U(f,P)-L(f,P)|<\varepsilon$. □

This finishes the proof of (1)—(5) being equivalent

包括了 the equivalence of Riemann and Darboux Integral

ex Let $f(x)=x^2$ on $[0,1]$ and for each n, let P_n be the regular partition with n intervals.

Then $U(f,P_n)=\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n}=\frac{1}{n^3}\sum_{k=1}^{n}k^2=\frac{2n^3+3n^2+n}{6n^3}$

and $L(f,P_n)=\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^2\frac{1}{n}=\frac{1}{n^3}\sum_{k=0}^{n-1}k^2=\frac{2(n-1)^3+3(n-1)^2+n-1}{6n^3}$

$\Rightarrow \lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}L(f,P_n)=\frac{1}{3}$

So $y=x^2$ is Db (hence Rm) intble on $[0,1]$ and $\int_0^1 x^2dx=\frac{1}{3}$

ex2 Let $D(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

Given any partition $P=(x_k)_{k=0}^{n}$ of $[0,1]$

$U(D,P)=\sum_{k=1}^{n}(\sup D[I_k])\Delta x_k=\sum_{k=1}^{n}1\Delta x_k=1$

$L(D,P)=\sum_{k=1}^{n}(\inf D[I_k])\Delta x_k=\sum_{k=1}^{n}0\Delta x_k=0$

$\Rightarrow U(D)=1,\ L(D)=0 \Rightarrow$ not Db (hence not Rm) intble

(But it is Lebesgue intble with $\int_0^1 D(x)dx=0$.

Thm② Linearity of Integration

If $f,g:[a,b]\to\mathbb{R}$ are Rm intble

let $c\in\mathbb{R}$

$\Rightarrow cf$ 和 $f+g$ 在 $[a,b]$ 上也 intble

并且 $\int_a^b cf=c\int_a^b f$, $\int_a^b(f+g)=\int_a^b f+\int_a^b g$

Pf follows from the fact that Riemann sums are linear.

$$\left(\begin{aligned}&\forall \dot{P},\ S(cf,\dot{P})=cS(f,\dot{P})\\ &S(f+g,\dot{P})=S(f,\dot{P})+S(g,\dot{P})\end{aligned}\right)$$

Thm③ Monotonically of Integration

If $f,g:[a,b]\to\mathbb{R}$ are Rm intble

且 $f(x)\le g(x)$ for all $x\in[a,b]$

$\Rightarrow \boxed{\int_a^b f\le\int_a^b g}$

Pf If $f(x)\le g(x)$ for all $x\in[a,b]$

$\Rightarrow U(f,P)\le U(g,P)$ for any P of $[a,b]$

因而 $U(f)\le U(g)$ □

Thm 4 Monotone $\Rightarrow$ Rm intble.

if $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$

$\Rightarrow f$ is Rm intble on $[a,b]$

Pf WLOG suppose f is increasing on $[a,b]$

let $\varepsilon>0$.

Let $P=(x_k)_{k=0}^{n}$ be arbitrary partition of $[a,b]$ with $\|P\|<\frac{\varepsilon}{f(b)-f(a)}$

$\Rightarrow U(f,P)-L(f,P)=\sum_{k=1}^{n}(\sup f[I_k]-\inf f[I_k])\Delta x_k$

$=\sum_{k=1}^{n}(f(x_k)-f(x_{k-1}))\Delta x_k$

$\le\sum_{k=1}^{n}(f(x_k)-f(x_{k-1}))\frac{\varepsilon}{f(b)-f(a)}$

$=(f(b)-f(a))\frac{\varepsilon}{f(b)-f(a)}=\varepsilon$ □