Spring 2024 Math 451: Practice Problems

(1) Prove that for all sets A and B, we have $A \subseteq B$ if and only if $A \cup B = B$.

(2) DeMorgan's Laws state that for all sets A, B, and C, we have $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. Choose one of these equations and prove it.

(3) DeMorgan's Laws also hold for indexed families of sets, even if the indexing family is infinite. For instance, let A be a set and suppose that B_n is a set for every $n \in \mathbb{N}$. Then we have

$$A \setminus \left(\bigcup_{n \in \mathbb{N}} B_n \right) = \bigcap_{n \in \mathbb{N}} (A \setminus B_n) \qquad \text{and} \qquad A \setminus \left(\bigcap_{n \in \mathbb{N}} B_n \right) = \bigcup_{n \in \mathbb{N}} (A \setminus B_n).$$

Prove whichever version you did not choose in (2).

(4) Let X and Y be sets, and let $f : X \to Y$ be a function. Prove that for all $A, B \subseteq X$ and $C, D \subseteq Y$, the following are true:

(a) $f[f^{-1}[C]] \subseteq C$

(b) $f^{-1}[f[A]] \supseteq A$

(c) $f[A \cup B] = f[A] \cup f[B]$

(d) $f[A \cap B] \subseteq f[A] \cap f[B]$

(e) $f[A \setminus B] \supseteq f[A] \setminus f[B]$

(f) $f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D]$

(g) $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$

(h) $f^{-1}[C \setminus D] = f^{-1}[C] \setminus f^{-1}[D]$

(5) Give conditions (on f) under which the containments in (a), (b), (d), and (e) from (4) above are in fact equalities.

(6) Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove the following:

(a) f is injective if and only if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

(b) f is surjective if and only if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

(c) f is bijective if and only if f is invertible.

(7) Let X, Y, and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following:

(a) If f and g are injective, then so is $g \circ f$.

(b) If f and g are surjective, then so is $g \circ f$.

(c) If f and g are bijective, then so is $g \circ f$.

(d) If $g \circ f$ is injective, then so is f.

(e) If $g \circ f$ is surjective, then so is g.

(8) Recall[1] Kuratowski's set-theoretic definition of ordered pair: $(a, b) := \{\{a\}, \{a, b\}\}$. Using this definition, prove that for all a, b, c, d we have $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

(9) Use induction to prove the following formulas:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

[1] From the *More Joy of Sets* handout.

(10) Prove that $\sqrt{3}$ is irrational without using the Rational Roots Theorem. Then show how the irrationality of $\sqrt{3}$ follows from the Rational Roots Theorem.

(11) Suppose $<$ is a linear order on the set X. Using nothing but the linear order axioms, prove that for all $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.

(12) Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$, and suppose that $b = \max A$ is the greatest element of A. Prove that $b = \sup A$.

(13) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below, and let L be the set of all lower bounds of A in $\mathbb{R}$. Prove that $\sup L = \inf A$.

(14) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below, and let $-A = \{-a : a \in A\}$. Prove that $\inf A = -\sup(-A)$.

(15) *Show[2] that if we were to drop the Distributive Law (Axiom 9) from the field axioms, we would no longer be able to prove that $0 \cdot x = 0$ for all x.

(16) Prove that for all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$.

(17) Let $a \in \mathbb{R}$ and let $\epsilon > 0$. Prove that for all $x, y \in V_\epsilon(a)$, we have $|x - y| < 2\epsilon$.

(18) A function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing [decreasing]* if for all $x, y \in \mathbb{R}$, $x < y$ implies $f(x) < f(y)$ $[f(x) > f(y)]$, and *strictly monotone* if f is either strictly increasing or strictly decreasing.
   (a) Prove that every strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is injective.
   (b) Show by example that a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ need not be bijective.
   (c) Show by example that a bijective function $f : \mathbb{R} \to \mathbb{R}$ need not be strictly monotone.

(19) Determine whether the given function is injective, surjective, both, or neither:
   (a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + |x|$.
   (b) The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x|x|$.
   (c) The function $h : \mathbb{R} \to (0, \infty)$ defined by $h(x) = e^x$.
   (d) The function $p : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = x + y$.
   (e) The function $m : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{3\}$ defined by $m(x) = \frac{3x+5}{x+2}$.
   (f) The function $s : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ defined by $s(n) = \{k \in \mathbb{N} : k \leq n\}$, where the set $\mathcal{P}(\mathbb{N})$ is called the *powerset* of $\mathbb{N}$ and is defined by $\mathcal{P}(\mathbb{N}) = \{A : A \subseteq \mathbb{N}\}$.

(20) For any sets X and Y and subset $R \subseteq X \times Y$, define $R^{-1} := \{(y, x) \in Y \times X : (x, y) \in R\}$. Prove that for any function $f : X \to Y$, the set f^{-1} is a function if and only if f is injective. Assuming f is injective, what is $\mathrm{dom}(f^{-1})$?

(21) For any sets X and Y, we define $X^Y = \{f : f \text{ is a function from } Y \text{ to } X\}$. Recalling that, as a set, $2 = \{0, 1\}$, show that for every set X we have $\mathcal{P}(X) \approx 2^X$.

(22) Define the function $f : 2^{\mathbb{N}} \to \mathbb{R}$ by $f(\alpha) = \displaystyle\sum_{k=1}^{\infty} \frac{2\alpha(k)}{3^k}$.
   (a) What does $\mathrm{ran}(f)$ look like? Try[3] to draw a picture.
   (b) Show that f is injective.
   (c) Show that $\mathcal{P}(\mathbb{N}) \preceq \mathbb{R}$.

(23) (a) Show that $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q})$.
   (b) Show that $\mathbb{R} \preceq \mathcal{P}(\mathbb{Q})$.
   (c) Show that $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$.

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[2]All these practice problems are optional, but ones with *'s are even more optional! (ie, don't worry if you don't know how to do them.)

[3]Hint: spend a few minutes reading about the *Cantor set* on Wikipedia!

(24) (a) Show that $2^{\mathbb{N}} \preceq \mathbb{N}^{\mathbb{N}}$.
(b) Show that $\mathbb{N}^{\mathbb{N}} \preceq 2^{\mathbb{N}}$.
(c) Show that $2^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$.

(25) A sequence $f : \mathbb{N} \to X$ is *eventually constant* if there is $x \in X$ and $N \in \mathbb{N}$ such that $f(n) = x$ for all $n \geq N$.
(a) Prove that there are only countably many eventually constant sequences in $2^{\mathbb{N}}$.
(b) How many eventually constant sequences are there in $\mathbb{N}^{\mathbb{N}}$?

(26) Evaluate the limits $\lim\limits_{n\to\infty} \frac{n^2}{2^n}$ and $\lim\limits_{n\to\infty} \frac{2^n}{n^2}$. (Don't bother proving your claims, but take a moment to consider how you would proceed.)

(27) Let (a_n) be a sequence in $\mathbb{R}$. Prove that if $\lim a_n = L \in \mathbb{R}$, then $\lim |a_n| = |L|$.

(28) Can a sequence of positive real numbers converge to a negative number? Can a sequence of positive real numbers converge to a number that is not positive? Justify your claims.

(29) Prove that if $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim a_n b_n = -\infty$.

(30) Prove that if $\lim a_n = L \in \mathbb{R}$ and $\lim b_n = \infty$, then $\lim(a_n - b_n) = -\infty$.

(31) Let (a_n) be a sequence in $\mathbb{R}$. Prove in detail that (a_n) converges *iff* some tail of (a_n) converges *iff* every tail of (a_n) converges. [Hint: there is a "logically efficient" way of proving these implications; can you find it?]

(32) *Let (a_n) and (b_n) be two sequences such that for all $n \in \mathbb{N}$, we have $a_n < b_n$ if n is even and $a_n > b_n$ if n is odd. Prove that if (a_n) and (b_n) both converge, then $\lim a_n = \lim b_n$.

(33) Prove that if $a_n \leq b_n$ for all n and $\lim a_n = \infty$, then also $\lim b_n = \infty$.

(34) In lecture we showed that if (a_n) and (b_n) are convergent sequences of real numbers for which $a_n \leq b_n$ for all n, then $\lim a_n \leq \lim b_n$. Can these nonstrict inequalities be replaced by strict ones? That is, if (a_n) and (b_n) are convergent sequences of real numbers for which $a_n < b_n$ for all n, does it necessarily follow that $\lim a_n < \lim b_n$?

(35) Prove the Squeeze Theorem directly using the definition of limit, but without using lim inf and lim sup. (The Squeeze Theorem says: if $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = L \in \mathbb{R}$, then $\lim s_n = L$.)

(36) Find the lim inf and lim sup of the sequences whose nth terms are given as follows:
(a) $2^{n(-1)^n}$
(b) $1 + (-1)^n(1 - \frac{1}{n})$
(c) $\sin\left(\frac{\pi n}{3}\right)\cos\left(\frac{\pi n}{4}\right)$

(37) Prove that for every sequence (a_n) in $\mathbb{R}$, if $\lim a_n = \infty$ then $\liminf a_n = \infty$ and $\limsup a_n = \infty$.

(38) Prove that for all $L \in \mathbb{R}$ and for every bounded sequence (a_n) in $\mathbb{R}$, $\limsup(a_n) = L$ if and only if for every $\epsilon > 0$ the set $\{n \in \mathbb{N} : a_n > L - \epsilon\}$ is infinite and $\{n \in \mathbb{N} : a_n > L + \epsilon\}$ is finite.

(39) Evaluate the following limits:
(a) $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^{2n}$
(b) $\lim\limits_{n\to\infty} \left(1 - \dfrac{1}{n}\right)^{n}$
(c) (s_n), where $s_1 = 2$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{3}{s_n}\right)$ for each $n \in \mathbb{N}$.

(40) Go back and prove (32), now that you know about subsequences.

(41) Suppose (a_n) is a bounded sequence in $\mathbb{R}$. Prove that (a_n) diverges if and only if (a_n) has two subsequences that converge to different limits.

(42) Prove that if $\lim\limits_{n\to\infty} a_n = L$, then for every bijection $\pi : \mathbb{N} \to \mathbb{N}$, $\lim\limits_{n\to\infty} a_{\pi(n)} = L$. (Is this still true if we replace the word *bijection* with *injection*? How about *surjection*?

(43) Let (a_n) be a sequence of real numbers. Prove that if every subsequence of (a_n) diverges, then for all $M > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \notin [-M, M]$.

(44) (a) Prove that if U and V are open subsets of $\mathbb{R}$, then $U \cap V$ is also open.
(b) Prove that if U are V are open subsets of $\mathbb{R}$, then $U \cup V$ is also open.

(45) Is the previous problem still true if you replace "open" with "closed"?

(46) Prove that if the subset $C \subseteq \mathbb{R}$ is closed and bounded, then every sequence (a_n) in C has a subsequence that converges to a limit in C.

(47) Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Prove that $a_k > 0$ for infinitely many k and $a_k < 0$ for infinitely many k.

(48) Suppose $a_k \geq 0$ for all k, and let $f : \mathbb{N} \to \mathbb{N}$ be any bijection. For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} a_{f(k)}$. Prove that $\sup\{s_n : n \in \mathbb{N}\} = \sup\{t_n : n \in \mathbb{N}\}$.

(49) Let $\sum_{k=1}^{\infty} a_k$ be an infinite series of real numbers, and let (t_k) be a strictly increasing sequence of natural numbers such that $t_1 = 1$. For each $n \in \mathbb{N}$ let $b_n = \displaystyle\sum_{k=t_n}^{t_{n+1}-1} a_k$. (Write out a simple example to understand what is going on here, and how $\sum a_k$ and $\sum b_n$ are related to each other.)
(a) Supposing that $\sum a_k$ converges, show that $\sum b_n$ also converges and that $\sum a_k = \sum b_n$.
(b) Show by example that $\sum b_n$ could converge even if $\sum a_k$ does not converge.

(50) Determine whether the following infinite series converge or diverge, with justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \sin(n)}{n^3 + 3}$
(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{e^n}$
(c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n \ln(n)^2}$
(d) $\displaystyle\sum_{n=1}^{\infty} \sin(n)$
(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n}$
(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 7}$
(g) $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi) \ln\left(1 + \frac{1}{n}\right)$
(h) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$
(i) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^5 - 2n^2 + 1}{4n^5 + 9n^4 + \sqrt{n}}\right)^n$
(j) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$
(k) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$
(l) $\displaystyle\sum_{n=1}^{\infty} \sin(e^{-n})$

(51) Of the series from the previous problem that converge, which ones (if any) converge conditionally?

(52) Find the limit points of the following subsets of $\mathbb{R}$:

(a) $\{0, 1\}$
(b) $(0, 1)$
(c) $[0, 1]$
(d) $\{m \pm \frac{1}{n} : m, n \in \mathbb{N}\}$
(e) $\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$
(f) $\{\frac{m}{n} : m \in \mathbb{Z}$ and $n = 2^k$ for some $k \in \mathbb{N}\}$
(g) the set of transcendental real numbers
(h) the set of partial sums of the harmonic series

(53) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. We call c a *closure point* of A if $c \in \text{cl}(A) = A \cup A'$, where A' is the set of all limit points of A.
(a) Show that c is a limit point of A iff there is a sequence (a_n) in $A \setminus \{c\}$ converging to c.
(b) Show that c is a closure point of A iff there is a sequence (a_n) in A converging to c.

(54) Let (a_n) be a sequence in $\mathbb{R}$ and let $A = \{a_n : n \in \mathbb{N}\}$.
(a) Show that every limit point of A is a subsequential liimit of (a_n).
(b) Show by example that not every subsequential limit of (a_n) need be a limit point of A.

(55) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, let $c \in \mathbb{R}$, and suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$. Prove directly, without using sequences, that $\lim_{x\to c} \big(f(x) + g(x)\big) = L + M$.

(56) State the precise ϵ/δ-style definitions of the following:

(a) $\lim_{x\to c^-} f(x) = +\infty$

(b) $\lim_{x\to\infty} f(x) = -\infty$

(57) Find the following limits[4]:

(a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$

(b) $\lim_{x\to 3} \dfrac{x^2 + x - 6}{x - 3}$

(c) $\lim_{x\to 3^-} \dfrac{x^2 + x - 6}{x - 3}$

(d) $\lim_{x\to 3} \dfrac{x^2 + x - 6}{(x - 3)^2}$

(e) $\lim_{x\to -\infty} e^x$

(f) $\lim_{x\to \infty} e^x$

(g) $\lim_{x\to 0^+} \ln x$

(h) $\lim_{x\to 1} \ln x$

(i) $\lim_{x\to \infty} \ln x$

(j) $\lim_{x\to 0} \dfrac{\sin x}{x}$

(k) $\lim_{x\to 0} \dfrac{\sin x}{|x|}$

(l) $\lim_{x\to 0^-} \dfrac{\sin x}{|x|}$

(m) $\lim_{x\to 0^+} \dfrac{\sin x}{|x|}$

(58) Define the function $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = \begin{cases} x^3 - 2x^2 & \text{if } x \in \mathbb{Q}; \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Find all points $c \in \mathbb{R}$ for which $\lim_{x\to c} f(x)$ exists, and for each such point c, find $\lim_{x\to c} f(x)$.

(59) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let $c \in \mathbb{R}$ be a limit point of A, and let $L \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(c + h) = L$.

(60) *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions, and let $a, b, L \in \mathbb{R}$. Assuming that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = L$, does it necessarily follow that $\lim_{x\to a} g(f(x)) = L$?

(61) Let (X, d) be a metric space and let $F \subseteq X$. Prove directly from the definitions that F is closed if and only if every limit point of F in X belongs to F.

(62) Let $A \subseteq \mathbb{R}$, let $a \in A$, and let $f : A \to \mathbb{R}$ be a function. Show that f is continuous at a if and only if for every open neighborhood V of $f(a)$ there is an open neighborhood U of a such that $f[U \cap A] \subseteq V$.

(63) Prove directly (using the ϵ/δ definition and without using sequences) that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at $a \in \mathbb{R}$, then also fg is continuous at a.

(64) Note that the square root function $f(x) = \sqrt{x}$ is continuous at $x = 9$. Given $0 < \epsilon < 1$, find the largest $\delta > 0$ such that $|\sqrt{x} - 3| < \epsilon$ whenever $|x - 9| < \delta$.

(65) A function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* if for all $x, y \in \mathbb{R}$, $x \le y$ implies $f(x) \le f(y)$, *decreasing* if for all $x, y \in \mathbb{R}$, $x \le y$ implies $f(x) \ge f(y)$, and *monotone* if it is either increasing or decreasing.

(a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone and f is discontinuous at $a \in \mathbb{R}$, then f has a jump discontinuity at a.

(b) Prove that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has at most countably many discontinuities.

(66) *Define *Thomae's function* $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ where } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ are coprime;} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

[4]This means find the limit, if it exists, or else state that the limit is equal to $+\infty$ or $-\infty$, if that is the case, or else state that the limit does not exist. Note that since $\pm\infty$ are not real numbers, cases where the limit is equal to $\pm\infty$ are cases where the limit "does not exist;" however, in these cases, the limit fails to exist in a particular way and it is better to say that the limit is $+\infty$ or $-\infty$ rather than just saying that it does not exist, since doing so provides more information. Note that, as always in this class, "∞" means the same thing as "$+\infty$."

(a) Prove that for every $x \in \mathbb{Q}$, T is discontinuous at x.
(b) Prove that for every $x \in \mathbb{R} \setminus \mathbb{Q}$, T is continuous at x.

(67) Give an example of a bounded continuous function on $(0,1)$ that is not uniformly continuous.

(68) In each part, determine whether the given function is uniformly continuous on the given set. (No rigorous proofs required.)

(a) $y = x^{3/5}$ on $\mathbb{R}$
(b) $y = x^{5/3}$ on $\mathbb{R}$
(c) $y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(d) $y = \ln x$ on $[1, \infty)$
(e) $y = \ln x$ on $(0, 1]$
(f) $y = \sin x^2$ on $\mathbb{R}$
(g) $y = (\sin x)^2$ on $\mathbb{R}$
(h) $y = e^{-x} \sin x^2$ on $[0, \infty)$

(69) Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Then f is called *Lipschitz continuous* if there is $K \geq 0$ such that for all $x, y \in A$ we have $|f(x) - f(y)| \leq K|x - y|$.
(a) Prove that every Lipschitz continuous function is uniformly continous.
(b) Show by example that not every uniformly continuous function is Lipschitz continuous.

(70) Suppose the function $f : (0,1) \to \mathbb{R}$ is continuous but not uniformly continuous. Show that at least one of the limits $\lim\limits_{x \to 0^+} f(x)$ or $\lim\limits_{x \to 1^-} f(x)$ does not exist.

(71) Let (a_n) be the sequence defined recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for all $n \in \mathbb{N}$. Show that (a_n) converges and prove that $\lim(a_n) < 2$.

(72) Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions, suppose f is bounded, and suppose $\lim\limits_{x \to -\infty} h(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = 0$. Prove directly that $\lim\limits_{x \to -\infty} [f(x) \cdot g(h(x))] = 0$.

(73) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that if $f(x) \neq 0$ for all $x \in [0,1]$, then there is $\epsilon > 0$ such that either $f(x) < -\epsilon$ for all $x \in [0,1]$ or $\epsilon < f(x)$ for all $x \in [0,1]$.

(74) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. We say that f *dominates* g if $g(x) < f(x)$ for all $x \in \mathbb{R}$. Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that neither one dominates the other, then $f(x) = g(x)$ for some $x \in \mathbb{R}$.

(75) Let $A \subseteq \mathbb{R}$, let $a \in A \cap A'$, and let $f : A \to \mathbb{R}$ be a function. Prove that f is differentiable at a if and only if there is a function $\varphi : A \to \mathbb{R}$ that is continuous at a and has the property that for all $x \in A$,

$$\varphi(x)(x - a) = f(x) - f(a).$$

(76) Find the derivative of the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) \;=\; \frac{e^{\sin x^2}\left(x^{2/5} - \sqrt{x^2 + 1}\right)}{\cos(\ln(x))\, e^{e^x}}.$$

(77) (a) Give an example of a function $f : (-1, 1) \to \mathbb{R}$ that is C^1 but not twice-differentiable.
(b) Give an example of a function $f : (-1, 1) \to \mathbb{R}$ that is twice-differentiable but not C^2.

(78) Suppose $f : (a, b) \to \mathbb{R}$ is differentiable. In the Increasing/Decreasing Test, we stated that:
(i) if $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b);
(ii) if $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b);
(iii) if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b);
(iv) if $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b);
For which of these statements is the converse true? Prove those that are true, and give counterexamples for those that can fail.

(79) Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Show that if f' is bounded on I, then f is uniformly continuous. Then show by example that the converse can fail.

(80) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let $a \in A \cap A'$, and suppose f is differentiable at a. Show the following:

(a) If $f'(a) > 0$, then there is $\delta > 0$ such that for all $x, y \in A$, if $a - \delta < x < a < y < a + \delta$ then $f(x) < f(a) < f(y)$.

(b) If $f'(a) < 0$, then there is $\delta > 0$ such that for all $x, y \in A$, if $a - \delta < x < a < y < a + \delta$ then $f(x) > f(a) > f(y)$.

(81) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, let $c \in \mathbb{R}$, and suppose $f'(c) > 0$.

(a) Prove that if f is C^1, then there is an open neighborhood of c on which f is injective.

(b) Show by example that the result in (a) can fail if we do not assume f is C^1.

(82) Let $f(x) = x - 12x^{1/3}$.

(a) Find the largest interval I containing 5 on which f is injective.

(b) Find $\left((f \restriction I)^{-1}\right)'(11)$.

(c) Find all points in the range of $f \restriction I$ at which $(f \restriction I)^{-1}$ is not differentiable.

(83) Prove directly from the definitions that for all $a < b$, the identity function $f(x) = x$ is Darboux integrable on $[a, b]$.

(84) Does there exist a function $f : [0, 1] \to \mathbb{R}$ such that $|f|$ is integrable on $[0, 1]$ but f is not?

(85) Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a nonnegative integrable function such that $f(x) > 0$ for some $x \in [a, b]$.

(a) Show by example that we could have $\int_a^b f = 0$.

(b) Prove that if f is continuous, then $\int_a^b f > 0$.

(86) Suppose the function $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Show by example that F' need not be integrable on $[a, b]$. *(This shows that the assumption of integrability in the statement of the FTOC cannot be removed.)*

(87) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Is f integrable on $[-1, 1]$? Prove your claim.

(88) Let $a < b$ and $c < d$ be real numbers, and suppose the functions $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ are integrable. Does it follow that $g \circ f$ is integrable? Either prove this or give a counterexample.

(89) *Does $\displaystyle\lim_{x \to 0} \left(\frac{1}{x} \int_0^x \sin(\tfrac{1}{t})\, dt \right)$ exist? If so, evaluate it.

(90) For all $x \geq 0$ and $n \in \mathbb{N}$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim_{n \to \infty} f_n(x)$.

(b) Determine whether (f_n) converges uniformly to f on $[0, 1]$.

(c) Determine whether (f_n) converges uniformly to f on $[0, \infty)$.

(91) Show that $\displaystyle\lim_{n \to \infty} \int_1^2 e^{-nx^2}\, dx = 0$.

(92) Find a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that:

(i) for each $n \in \mathbb{N}$, f_n is discontinuous at every point $x \in \mathbb{R}$; and

(ii) the sequence (f_n) converges uniformly to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

(93) For each $n \in \mathbb{N}$, define the function $f_n : (-1, 1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -x & \text{if } -1 < x < -2^{-n} \\ 2^{n-1}x^2 + 2^{-(n+1)} & \text{if } -2^{-n} \le x \le 2^{-n} \\ x & \text{if } 2^{-n} < x < 1 \end{cases}$$

Show that each f_n is differentiable on $(-1, 1)$, and that (f_n) converges uniformly to the absolute value function on $(-1, 1)$.

(94) For each $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = \dfrac{\sin(nx)}{n}$. Show that (g_n) converges uniformly on $\mathbb{R}$ to a differentiable function whose derivative is *not* $\lim\limits_{n\to\infty} g_n'$.

(95) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let (f_n) be a sequence of continuous functions from A to $\mathbb{R}$ that converges uniformly on A to f. Prove that for every $a \in A$ and sequence (x_n) in A that converges to a, we have $\lim\limits_{n\to\infty} f_n(x_n) = f(a)$.

(96) Find the intervals of convergence of the power series:

$$\text{(a)} \sum_{n=0}^{\infty} \left(\frac{(n!)^3}{(3n)!}\right) x^n \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{n^n}{n!} x^n \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{5^{n+1}}{\sqrt{n} \cdot 3^{2n}}\right) x^n$$

Can you find the interval of convergence in (c)?

(97) (a) Using the fact that $\frac{d}{dx} \ln x = \frac{1}{x}$ for all $x > 0$, calculate the Taylor Series of the natural log function centered at $x = 1$.

(b) Using the fact (which you may assume without proof) that the Taylor Series you found in part (a) converges to the natural log function on $(0, 2]$, calculate the limit of the alternating harmonic series.

(98) Let $A \subseteq \mathbb{R}$. Prove that A is open if and only if A can be expressed as a disjoint union of countably many open intervals.

(99) Let $V \subseteq \mathbb{R}$ be an open set, and write $V = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$, where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for all $m \neq n$.[5] Define the *measure* of V to be $\mu(V) = \sum_{n=1}^{\infty}(b_n - a_n)$. In the problems below, you may use without proof the (geometrically obvious) fact that for any open set $V \subseteq \mathbb{R}$ and sequence of open intervals (a_n, b_n) in $\mathbb{R}$, if $V \subseteq \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ then $\mu(V) \le \sum_{n=1}^{\infty}(b_n - a_n)$.

(a) Prove that for every $\epsilon > 0$, there is an open subset of $\mathbb{R}$ that contains $\mathbb{Q}$ and has measure less than ϵ.

(b) Does there exist an open set $V \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq V$ and $\mathbb{R} \setminus V$ is uncountable?

(100) *For each pair of real numbers $\alpha, \beta \in \mathbb{R}$, define the function $f_{\alpha,\beta} : [0, \infty) \to \mathbb{R}$ as follows: if $\alpha, \beta \ge 0$, then $f_{\alpha,\beta}(x) = x^\alpha \sin x^\beta$, and if $\alpha < 0$ or $\beta < 0$ then

$$f_{\alpha,\beta} = \begin{cases} x^\alpha \sin x^\beta & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

(Note that for some $\alpha, \beta \in \mathbb{R}$, including all $\alpha, \beta \ge 0$, we can also define $f_{\alpha,\beta}(x)$ for $x < 0$, but to avoid certain complications we will just work on $[0, \infty)$ in this problem.)

(a) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is continuous.

(b) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is differentiable.

(c) Determine the set of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha,\beta}$ is C^1.

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[5] Note: we might need to take $a_n = b_n$ for infinitely many n here.