## Def Series

If $(a_k)_{k\in\mathbb{N}}$ 为一个 seq. in $\mathbb{R}$

称 $S_n=\sum_{k=1}^{n}a_n$ 为它的 $\boxed{n^{th}\text{ partial sum.}}$

$(S_n)$ 为 its $\boxed{\text{sequence of partial sums.}}$

我们使用 $\boxed{\sum_{k=1}^{\infty}a_k}$ 这一符号定义 the infinite series determined by $(a_k)$:

- 称它 converges if $\lim_{n\to\infty}\sum_{k=1}^{n}a_k=l$ informal notation: $\Sigma a_k<\infty$
- 称它 diverges, otherwise

note: $\sum_{k=1}^{\infty}a_k$ 代表一个 limit 而非 algebraic operation

ex (1) The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to $+\infty$

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\dots+\frac{1}{16}\right)+\dots$$
$$\geq\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots=\sum_{n=1}^{\infty}\frac{1}{2}=\infty$$

(2) Given $a,r\in\mathbb{R}$ and $m\in\mathbb{Z}$, $\sum_{k=m}^{\infty}ar^k$ is called a geometric series.

Claim If $r\neq1$, the partial sums of $\sum_{k=m}^{\infty}ar^k$ is
$$\sum_{k=m}^{n}ar^k=a\frac{r^m-r^{n+1}}{1-r}$$
因而 $\sum_{k=m}^{\infty}ar^k=\lim_{n\to\infty}\left(a\frac{r^m-r^{n+1}}{1-r}\right)=\begin{cases}\frac{ar^m}{1-r}, & \text{if } |r|<1\\ \text{DNE}, & \text{if } |r|\geq1\end{cases}$

Pf For $m\leq n$
$$(1-r)\sum_{k=m}^{n}ar^k=a\left[(r^m+\dots+r^n)-(r^{m+1}+\dots+r^{n+1})\right]$$
$$\Rightarrow\sum_{k=m}^{n}ar^k=a\frac{r^m-r^{n+1}}{1-r}$$

(3) Given $p\in\mathbb{R}$, a series of the form $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p$ is called a p-series

Claim a p-series conv. iff $p>1$

Pf Case1 $p\leq1\Rightarrow n^p\leq n\Rightarrow\frac{1}{n^p}\geq\frac{1}{n}$
$$\Rightarrow\Sigma\frac{1}{n^p}\geq\Sigma\frac{1}{n}=\infty$$

Case2 $p>1\Rightarrow$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\left(\frac{1}{4^p}+\dots+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\dots+\frac{1}{15^p}\right)+\dots$$
$$\leq1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\frac{16}{16^p}+\dots$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^n=\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}}<\infty$$

例如: $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$

$\sum_{n=1}^{\infty}\frac{1}{n^3}$ no nice formula

(4) $\sum_{n=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$ (telescoping)

(5) The alternating harmonic series
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

Let $S_n=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\Rightarrow(S_{2n})\nearrow\ (S_{2n+1})\searrow$

Thus $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\sup\{S_{2n}\}=\inf\{S_{2n+1}\}=\ln2$

Thm Suppose $\Sigma a_n$ and $\Sigma b_n$ conv.
Let $c\in\mathbb{R}$

$\Rightarrow$ (i) $\Sigma ca_n=c\Sigma a_n$

(ii) $\Sigma(a_n+b_n)=\Sigma a_n+\Sigma b_n$

note: $\Sigma a_nb_n\neq(\Sigma a_n)(\Sigma b_n)$

Test1 Cauchy Criterion for convergence

Let $\Sigma a_k$ be a series, $(S_n)$ 为它的 seq. of partial sums.

$\Rightarrow$ $\boxed{\Sigma a_k\text{ conv. iff }(S_n)\text{ is Cauchy.}}$ ☆

Recall: Cauchy seq. means $\forall\varepsilon>0,\exists N\in\mathbb{N}$ s.t $\boxed{|S_n-S_m|}<\varepsilon$ whenever $N\leq m\leq n$

$$=\left|\sum_{k=m+1}^{n}a_k\right|$$

Pf 课后

**Test 2 The $n^{th}$ term test** contrapositive 很有用, $(a_n)\not\to 0$ then $\sum a_k$ div.

一个显然的结论: if $\sum a_k$ conv. $\Rightarrow$ $(a_k)\to 0$

(但反之converse显然不成立)

Pf $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty}(S_k - S_{k-1}) = \lim\limits_{k\to\infty} S_k - \lim\limits_{k\to\infty} S_{k-1} = 0$

**Test 3 The comparison test**

Let $(a_n)$ be a seq. of nonnegative numbers

$(b_n)$ be any seq.

$\Rightarrow$ (i) if $\sum a_n$ conv. 且 $\boxed{|b_n| \le a_n \text{ for all } n}$

$\Rightarrow \sum b_n$ conv.

(ii) if $\sum a_n = \infty$ and $\boxed{b_n \ge a_n \text{ for all } n}$

$\Rightarrow \sum b_n = \infty$

Pf 以 $(S_n)$ 和 $(t_n)$ 分别表示 $\sum a_k$ 和 $\sum b_k$ 的 seq. of partial sums.

(i) If $|b_n| \le a_n$ for all $n$

$\Rightarrow |t_n - t_m| = \left|\sum\limits_{k=m+1}^{n} b_k\right| \le \sum\limits_{k=m+1}^{n} |b_k| \le \sum\limits_{k=m+1}^{n} a_k = |S_n - S_m|$

So $\sum b_k$ 也满足 Cauchy Criterion $\Rightarrow$ conv.

(ii) 易得

* **Fact** Comparison test 可忽略 finite tail

即 $\sum b_n$ conv. iff $\exists N \in \mathbb{N}$ s.t. $|b_n| \le a_n$ for all $n \ge N$

(of course then the limit is different)

ex $\sum\limits_{n=2}^{\infty} \frac{\sin(n)}{n^2 \ln(n)}$ conv. by the comparison test with $\sum \frac{1}{n^2}$ since for all sufficiently large $n$, $\left|\frac{\sin(n)}{n^2 \ln(n)}\right| \le \frac{1}{n^2}$

**Def Absolute convergence**

The series $\sum\limits_{k=1}^{\infty} a_k$ $\boxed{\text{converges absolutely}}$ if $\sum\limits_{k=1}^{\infty} |a_k|$ converges.

**Thm** Absolute convergence 是比 convergence 更强的条件.

if $\sum a_n$ conv. absly $\Rightarrow \sum a_n$ conv.

Pf Suppose $\sum |a_k|$ conv.

Then for all $m \le n$, $|S_n - S_m| = \left|\sum\limits_{k=m+1}^{n} a_k\right| \le \sum\limits_{k=m+1}^{n} |a_k|$

So $\sum a_k$ satisfies Cauchy Criterion if $\sum |a_k|$ does.

**Def Conditional convergence**

A conv. series that does not conv. absly is said to converge conditionally

ex $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ conv. condly.

**Disturbing Fact**

一个 condly. conv. series 可以 be made to conv. to 任何一个数 by "reordering" its terms

condly. conv. 的 series 在reorder后会改变 limit, 并且通过reorder我们甚至可以让它conv. to 任何一个实数.

ex $\sum \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{4} \ldots$

只要把小的terms 放到前面, 然后在超过$\sqrt{2}$ 时放上前面大的 terms 作为 compensate, 最后会 conv. to $\sqrt{2}$

$\forall a \in \mathbb{R}$, 我们都可以apply这一个过程, 使得 reorder 后的 series conv. to $a$, 只需要调整 compensate 的项和频率就可以了.

但是我们希望一个 series 中的项的次序最好是可以 reorder 的.

**Reassuring Fact** Absly. conv. 的 series 是 closed under reordering 的

即: if $\sum a_k$ conv. absly

$\Rightarrow$ 令 $f: \mathbb{N}\to\mathbb{N}$ 为任意bijection, 都有 $\boxed{\sum\limits_{k=1}^{\infty} a_{f(k)} = \sum\limits_{k=1}^{\infty} a_k}$

Pf

首先我们claim: 对于 conv. $\sum a_n$, 如果 $a_k \ge 0$ for all $k \in \mathbb{N}$, 那么 $\forall$ bij $f: \mathbb{N}\to\mathbb{N}$, $\sum a_{f(k)} = \sum a_k$ ①

这直观. 回忆我们刚才的"改变condly conv. 的 series 的limit"的方法, 必须通过移动负项作为"compensate"来实现, 没有正负项交替则不可能改变limit

这是易证的 因为这一条件下, $\left(\sum\limits_{k=1}^{n} a_k\right)_{n\in\mathbb{N}}$ 和 $\left(\sum\limits_{k=1}^{n} a_{f(k)}\right)_{n\in\mathbb{N}}$ 都为 increasing sequence 且有相同的 sup. 因而它们 limit 相同.

而后我们使用claim① 来证明 general case:

Suppose $\sum a_n$ conv. absly.

Let $b_n = \begin{cases} a_n, & \text{if } a_n \ge 0 \\ 0, & \text{otherwise} \end{cases}$, $c_n = \begin{cases} |a_n|, & \text{if } a_n < 0 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow \boxed{a_n = b_n - c_n \text{ for all } n \in \mathbb{N}}$

由于 $\sum a_n$ conv. absly $\Rightarrow \sum |a_n|$ conv.

Note: $|b_k| \le a_k$, $|c_k| \le a_k$ for all $k$

于是 by Comparison test, $\sum b_n$ 和 $\sum c_n$ conv. absly.

$\sum a_n = \sum (b_n - c_n) = \sum b_n - \sum c_n$

由于 $b_k, c_k \ge 0$ for all $k \in \mathbb{N}$, we apply claim ①:

$\sum b_n = \sum b_{f(n)}$, $\sum c_n = \sum c_{f(n)}$

因而 $\sum a_n = \sum b_{f(n)} - \sum c_{f(n)} = \sum (b_{f(n)} - c_{f(n)})$
$= \sum a_{f(n)}$ □

## Test 4 Root test

Let $(a_n)$ be a seq. in $\mathbb{R}$; let $\boxed{p = \limsup |a_n|^{\frac{1}{n}}}$

$\Rightarrow$ (i) if $p<1 \Rightarrow \boxed{\sum a_n \text{ conv. absly.}}$

(ii) if $|a_n| \ge 1$ for $\boxed{\text{infly many } n}$ (which happens when $p \ge 1$)

$\Rightarrow \boxed{\sum a_n \text{ diverges}}$

*Note: $L = \limsup (a_n) \iff \forall \varepsilon > 0$, $\{n \in \mathbb{N} \mid a_n > L+\varepsilon\}$ is finite and $\{n \in \mathbb{N} \mid a_n > L-\varepsilon\}$ is infinite

Pf (i) Assuming $p < 1$

Fix $p \le r < 1$ & $N \in \mathbb{N}$ s.t. $|a_n|^{\frac{1}{n}} \le r$ (by limsup 的性质)

$\Rightarrow |a_n| \le r^n$ for all $n \ge N$

Since $0 \le r < 1 \Rightarrow \sum r^n$ conv.

$\Rightarrow \sum |a_n|$ conv. by Comparison Test

(ii) if $|a_n| \ge 1$ for infly many $n$, then $(a_n) \not\to 0$

So $\sum a_n$ div. by the $n^{th}$ term test

## Test 5 Ratio Test

Let $(a_n)$ be a seq. of nonzero numbers.

(i) 如果 $\boxed{\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1} \Rightarrow \sum a_n$ $\boxed{\text{conv. absly.}}$

(ii) 如果 $\boxed{\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1} \Rightarrow \sum a_n$ $\boxed{\text{div.}}$

Pf This follows from the Root Test and the fact that

$$\boxed{\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{\frac{1}{n}} \le \limsup |a_n|^{\frac{1}{n}} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|}$$

(proved in hw4⑤)

## Remarks

1. root test implies ratio test.
   root test 通常比 ratio test 更强.
2. root & ratio test 在 $\limsup |a_n|^{\frac{1}{n}} = 1$, $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$ 时是 inconclusive 的
   (ex: $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$ ...)
3. 如果 $|a_n|^{\frac{1}{n}} \to r$ 或 $\left|\frac{a_{n+1}}{a_n}\right| \to r$, 则 $\sum a_n$ absly conv when $r<1$, div when $r>1$.

   *note: root test 和 ratio test $r<1$ imply 的结果 不止 convergence 而且是 abs convergence.
4. whenever the root test is inconclusive, the ratio test 一定也 inconclusive. (因而不用再试)

## Test 6 Alternating Series Test

If $(a_k)$ is a decreasing① seq. of positive② numbers conv. to 0, 那么 $\boxed{\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ conv.}}$

Pf Assume the hypothesis, for each $n \in \mathbb{N}$ let

$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$

Since $S_{2n} = \overset{(+)}{(a_1 - a_2)} + \overset{(+)}{(a_3 - a_4)} + \overset{(+)}{(a_5 - a_6)} + \dots$

$= a_1 + \underset{(-)}{(-a_2 + a_3)} + \underset{(-)}{(-a_4 + a_5)} + \underset{(-)}{(-a_6 + a_7)} + \dots$

因而 $S_{2n}$ 是 increasing 且 bounded above by $a_1$ 的

$\Rightarrow \boxed{(S_{2n}) \text{ conv.}}$ say $(S_{2n}) \to L$

现在我们证明: $(S_{2n+1})$ 也 $\to L$.

Let $\varepsilon > 0$. Fix $N \in \mathbb{N}$ s.t. $|S_{2n} - L| < \frac{\varepsilon}{2}$ and $|a_{2n+1}| < \frac{\varepsilon}{2}$ whenever $n \ge N$.

$\Rightarrow \forall n \ge N$, $|S_{2n+1} - L| = |S_{2n} + a_{2n+1} - L|$
$\le |S_{2n} - L| + |a_{2n+1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

因而 $(S_{2n+1}) \to L$

$(S_{2n}) \to L$ 且 $(S_{2n+1}) \to L \Rightarrow (S_n) \to L$

因而 $\sum a_k = L$.

## Test 7 Integral test

Let $f$ be positive且 decreasing 的 function on $[1,\infty)$

$\Rightarrow \sum_{k=1}^{\infty} f(k)$ conv. iff the improper integral $\underbrace{\int_1^{\infty} f(x)\,dx}$ conv.

$\Downarrow$

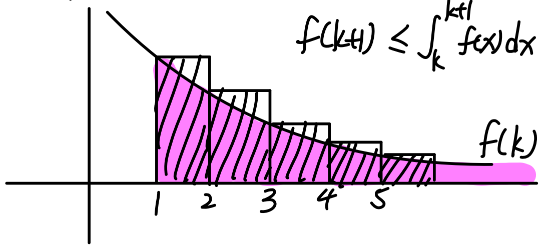
$= \lim_{b\to\infty} \int_1^b f(x)\,dx$

note: 此处 我们还没有严谨定义过 integral

因而 integral test 的证明以后再证

但其实很直观.

$$f(k+1) \leq \int_k^{k+1} f(x)\,dx \leq f(k)$$




## Numerical Series Summary

7个 test

(1) Cauchy Criterion: $\sum a_k$ conv. iff $(a_n)$ is Cauchy

(2) $n^{th}$ root Test: $(a_n) \not\to 0 \Rightarrow \sum a_k$ div.

(3) Comparison Test: $|b_n| \leq a_n \Rightarrow$ 如果 $\sum a_k$ conv 则 $\sum b_k$ conv.

absly (4) Root Test: $\limsup |a_n|^{\frac{1}{n}}$

absly (5) Ratio Test: $\limsup / \inf \left|\frac{a_{n+1}}{a_n}\right| \ldots$

(6) Alternating Series Test: positive $\downarrow$ 的 $(a_n)$ 的 alternating 一定 conv.

(7) Integral Test: $\sum_{k=1}^{n} f(k)$ conv. iff $\int_1^{\infty} f(x)\,dx$ conv. for positive $\downarrow f$.

Abs Conv. $\Rightarrow$ Conv

Abs Conv. is closed under reordering, condly. conv. 则不是