Homework 2: Due Tuesday, May 21, at 11:59pm, on Gradescope

Recall that for any sets X and Y, we write $X \preceq Y$ if there exists an injective function from X to Y, and $X \approx Y$ if there exists a bijective function from X to Y. By the Cantor-Schroder-Bernstein Theorem, $X \approx Y$ if and only if $X \preceq Y$ and $Y \preceq X$.

(1) Use induction and the triangle inequality for real numbers to prove that for all $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n \in \mathbb{R}$,
$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|.$$
(2) Let $A \subseteq \mathbb{R}$ be bounded, let $c \in \mathbb{R}$, and write $cA = \{ca : a \in A\}$. Find an expression for $\sup(cA)$, and prove your claim. Then state (but do not prove) the "dual" claim for $\inf(cA)$.

(3) Let X, Y, and Z be any sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.
  (a) Prove that if f and g are injective, then $g \circ f$ is injective.
  (b) Prove that $\preceq$ is reflexive and transitive; that is, prove that $X \preceq X$ and that if $X \preceq Y$ and $Y \preceq Z$, then $X \preceq Z$.

(4) Let A and B be any nonempty sets.
  (a) Prove that if $A \subseteq B$ then $A \preceq B$.
  (b) Prove that there is an injective function from A to B if and only if there is a surjective function from B to A.

(5) (a) Prove that if A is an infinite set and A_0 is a finite subset of A, then $A \approx A \setminus A_0$.
  (b) Prove that if A is an uncountable set and A_0 is a countable subset of A, then $A \approx A \setminus A_0$.

(6) (a) Prove that $\overline{\mathbb{Q}}$ is countable. Conclude that there are uncountably many transcendental real numbers.
  (b) Prove that for all $a, b \in \mathbb{R}$, if $a < b$ then there are uncountably many transcendental numbers in the interval (a, b).

(7) Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.
  (a) Prove that $\mathcal{P}(\mathbb{R}) \preceq \mathbb{R}^{\mathbb{R}}$.
  (b) Prove that there is no surjective function from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{R}}$. (This shows that there are strictly more functions from $\mathbb{R}$ to $\mathbb{R}$ than there are real numbers.)

(8) Use the definition of convergence directly to prove that the following sequences converge to the given limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ (b) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$

(9) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Prove that if $\lim\limits_{n\to\infty} a_n = L \in \mathbb{R}$, then $\lim\limits_{n\to\infty} |a_n| = |L|$.

(10) Prove that for every $n \in \mathbb{N}$ and sequence (a_k) in $\mathbb{R}$, if (a_k) converges to L then $\lim a_k^n = L^n$.

(11) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and for each $n \in \mathbb{N}$ let $s_n = a_{n+1} - a_n$. Prove that if (a_n) converges, then (s_n) converges to zero.

(12) Let S be a bounded nonempty subset of $\mathbb{R}$. Show that there is a sequence in S that converges to $\sup S$.

Optional Challenge Problems:

(13) (a) Prove that $[0, 1]$ cannot be expressed as the union of an indexed family of open intervals.
  (b) Prove that $(0, 1)$ cannot be expressed as the intersection of an indexed family of closed intervals.

(14) Is the converse of Problem (11) true? That is, if (a_n) is a sequence in $\mathbb{R}$ and $\lim(a_{n+1} - a_n) = 0$, must (a_n) converge?