

Math 451 Midterm Exam

Spring 2024

You have 110 minutes to complete this exam. You may not use notes, textbooks, or electronic devices of any kind. Write your answers clearly on the exam itself in the space provided for you. Circle your answers where appropriate.

Name Qinlin Fan

Problem 1 17/18

Problem 2 17/17

Problem 3 20/21

Problem 4 19/23

Problem 5 18/21

Total Score 91



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Problem 1: Definitions (18 points)

Write clear, precise definitions or statements of the following italicized terms or phrases.

(a) the sequence (a_n) in \mathbb{R} *converges* to the real number L

if For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ whenever $n \geq N$

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(b) $f: X \rightarrow Y$ is an *injective* (or *one-to-one*) function

if $x_1 \neq x_2$ whenever $f(x_1) = f(x_2)$

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(c) the real number m is the *infimum* of the set $A \subseteq \mathbb{R}$

if m is a lower bound of A and for all $x > m$, x is not a lower bound of A .

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(d) the subset V of the metric space X with metric d is *open*

if for all $x \in V$, there exists ^{some} open neighborhood $U_\varepsilon(x)$
(i.e. $\{y \mid d(y, x) < \varepsilon\}$) s.t. $U_\varepsilon(x) \subseteq V$.

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(e) State the *Completeness Axiom* for \mathbb{R}

for all $A \subseteq \mathbb{R}$, if A is bounded above then $\sup(A) \in \mathbb{R}$

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↑
nonempty

(f) State the *Bolzano-Weierstrass Theorem* for sequences of real numbers

Every bounded sequence in \mathbb{R} has a convergent subsequence.

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Problem 2: Short Answer (17 points)

- (a) (5 pts) Can an ordered field have a positive element ϵ such that $\epsilon < \frac{1}{n}$ for every natural number n ? Does \mathbb{R} have any such elements? Briefly explain.

An ordered field can have such element, ~~for example~~ but \mathbb{R} does not have such element since \mathbb{R} is an Archimedean ordered field.

which means that for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ st. $\frac{1}{n} < \epsilon$ ✓

- (b) (4 pts) Briefly explain how you know that some real numbers are transcendental.

By how we have proved: \mathbb{Q} is countable and we know that

\mathbb{R} is uncountable. ~~if all real numbers are algebraic, then \mathbb{R} is countable~~ ~~so there must exist $x \in \mathbb{R}$ st. $x \notin \mathbb{Q}$~~ ~~if all real numbers are algebraic, then \mathbb{R} is countable~~

$\mathbb{R} \subseteq \mathbb{Q}$ which contradicts. (if $\mathbb{R} \subseteq \mathbb{Q} \Rightarrow \mathbb{R} \subseteq \mathbb{Q}$) ✓

- (c) (4 pts) Can a sequence of negative numbers converge to a positive number? Justify your answer either with an example or with a short argument.

No ✓

Proof. Let (a_n) be a sequence of negative numbers.

Assume for contradiction that (a_n) converges to a positive number L .

Then consider $\epsilon = L - 0 = L$

By our assumption, $\exists N \in \mathbb{N}$ st. $\forall n \geq N, |a_n - L| < \epsilon \Rightarrow -a_n + L < \epsilon \Rightarrow a_n > L - \epsilon = 0$

- (d) (4 pts) In each part, give a concrete example of a set with the given properties, or else write IMPOSSIBLE if no such set exists. No justification necessary.

- (i) An infinite subset of \mathbb{R} that has no limit points in \mathbb{R} .

\mathbb{N} ✓

\mathbb{N} ✓

contradicts with a_n being negative ✓

- (ii) A countable subset of \mathbb{R} that has uncountably many limit points in \mathbb{R} .

\mathbb{Q} ✓



Problem 3: Computational Problems (21 points)

- (a) (4 pts) Determine the following limit using the limit laws and any particular limits from lecture or the text. You may simply carry out the steps in the computation *without* citing what laws you are using, but you should show some work and circle your final answer.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(2n)^{1/n}(n^2+1)}{\pi n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\pi} \cdot \lim_{n \rightarrow \infty} (n^2+1)^{1/n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\pi} \cdot \lim_{n \rightarrow \infty} (1 + \frac{1}{n^2})^{1/n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} \\ &= \frac{1}{\pi} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{\pi} \end{aligned}$$

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- (b) (5 pts) Let (a_n) be a sequence of real numbers such that $a_1 = \frac{1}{2}$ and for all $n \in \mathbb{N}$, $a_{n+1} = \frac{(n+1)^2}{n(n+2)} \cdot a_n$. Carefully prove by induction that $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Proof we prove by induction on $n \in \mathbb{N}$
Base case: $n=1$

$$a_1 = \frac{1}{2} = \frac{1}{1+1} = \frac{1}{2}, \text{ the statement holds true.}$$

Inductive step:

$$\text{Assume for } n=k \in \mathbb{N}, a_k = \frac{k}{k+1}$$

Then ~~$a_{k+1} = \frac{k+1}{k+2}$~~

(no need to use k here)

$$a_{n+1} = \frac{(n+1)^2}{n(n+2)} a_n = \frac{(k+1)^2}{k(k+2)} \cdot \frac{k}{k+1} = \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1} = \frac{(n+1)}{(n+1)+1}$$

the statement holds true

This finishes the proof that $a_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

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Problem 3 (Continued)

- (c) (4 pts) Find the supremum and the infimum in $\mathbb{R} \cup \{\pm\infty\}$ of each of the following sets of real numbers. If the sup or inf is $+\infty$ or $-\infty$, say so instead of writing "DNE." *No justification needed.*

(i) $\bigcap_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, \frac{n^2 + 1}{n^2}\right)$ $\text{supremum} = 1$
 $\text{infimum} = 1$ ✓

(ii) $\bigcup_{n \in \mathbb{N}} \left(-\frac{1}{n}, n + \frac{1}{n}\right)$ $\text{infimum} = -1$,
 $\text{supremum} = +\infty$ ✓

- (d) (4 pts) Find the min / max of the given set as indicated, if it exists; if it does not exist, write DNE. *No justification needed.*

(i) $\min \{|x - y| : x, y \in \mathbb{R} \text{ and } x \neq y\}$ DNE ✓

(ii) $\max [0, \sqrt{2}] \cap \mathbb{Q}$ ~~$\sqrt{2}$~~ DNE ✓

(iii) $\max [0, \sqrt{2}] \setminus \mathbb{Q}$ $= \sqrt{2}$ ✓

(iv) $\max \bigcap_{n \in \mathbb{N}} \left(-n, \frac{1}{n}\right]$ DNE ✗

- (e) (4 pts) Find the lim sup and lim inf in $\mathbb{R} \cup \{\pm\infty\}$ of the given sequences. If the lim sup or lim inf is $+\infty$ or $-\infty$, say so instead of writing "DNE." *No justification needed.*

- (i) the sequence $(a_n)_{n \in \mathbb{N}}$ whose n th term is $a_n = (-1)^n + \frac{1}{n}$ for each $n \in \mathbb{N}$.

$\limsup (a_n) = 1$, $\liminf (a_n) = -1$ ✓

- (ii) the sequence $(b_n)_{n \in \mathbb{N}}$ whose n th term is $b_n = \left(1 + \frac{1}{n}\right)^{n(-1)^n}$ for each $n \in \mathbb{N}$.

$\limsup (b_n) = e$
 $\liminf (b_n) = \frac{1}{e}$ ✓



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Problem 4: Limits (23 points)

- (a) (3 pts) Give a precise statement in terms of ϵ and δ of the fact that $\lim_{x \rightarrow 2} \frac{1}{x-1} = 1$.

For all $\epsilon > 0$, there exists $\delta > 0$ such that $\left| \frac{1}{x-1} - 1 \right| < \epsilon$ whenever $0 < |x-2| < \delta$

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- (b) (6 pts) Give a direct proof using ϵ and δ of the fact that $\lim_{x \rightarrow 2} \frac{1}{x-1} = 1$.

Proof Let $\epsilon > 0$

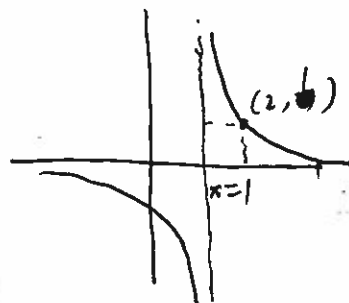
Consider $\delta = \frac{1}{2} \left(\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon} \right)$

note that $\frac{1}{1-\epsilon} - 1 < \delta < 1 - \frac{1}{1+\epsilon}$

what if $\epsilon = 1$?

We want $\delta > 0$ such that whenever $0 < |x-2| < \delta$, $\frac{1}{x-1} \in (1-\epsilon, 1+\epsilon)$

we can take $\delta = 1 - \frac{1}{1+\epsilon}$, then $\delta < 1 - \frac{1}{1+\epsilon}$ and $\delta > 1 - \frac{1}{1-\epsilon}$
 $\Rightarrow \frac{1}{1+\delta} > 1-\epsilon$, $\frac{1}{1-\delta} < 1+\epsilon$ then if $0 < |x-2| < \delta \Rightarrow x \in (2-\delta, 2+\delta) \Rightarrow$



but this could be negative (eg. if $\epsilon = \frac{1}{2}$)

- (c) (7 pts) Let $a, L \in \mathbb{R}$. Prove that for any functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$, if $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = L$, then $\lim_{x \rightarrow a} g(f(x)) = L$.

Proof. Let $\epsilon > 0$

Since $\lim_{x \rightarrow \infty} g(x) = L$, there exists $N \in \mathbb{R}$ st.

for all $x \geq N$, $|g(x) - L| < \epsilon$ ①

(fix such N) Since $\lim_{x \rightarrow a} f(x) = \infty$, there exists $\delta \in \mathbb{R}$ st. $f(x) \geq N$

whenever $0 < |x-a| < \delta$

Take the same δ , combining ① and ② we have

$$|g(f(x)) - L| < \epsilon \text{ whenever } 0 < |x-a| < \delta$$

This finishes the proof that $\lim_{x \rightarrow a} g(f(x)) = L$

good

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1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes the need for transparency and accountability in financial reporting.

2. The second part of the document outlines the various methods and techniques used to collect and analyze data. It includes a detailed description of the experimental procedures and the statistical tools employed.

3. The third part of the document presents the results of the study, showing the trends and patterns observed in the data. It includes several tables and figures that illustrate the findings in a clear and concise manner.

4. The fourth part of the document discusses the implications of the findings and provides recommendations for future research. It highlights the areas that need further exploration and the potential applications of the study.

5. The fifth part of the document concludes the study, summarizing the key points and reiterating the importance of the research. It expresses the hope that the findings will contribute to the understanding of the subject matter.

6. The sixth part of the document provides a list of references, citing the works of other researchers in the field. It ensures that all sources used in the study are properly acknowledged.

7. The seventh part of the document is a brief summary of the entire study, providing a quick overview of the objectives, methods, results, and conclusions.

Problem 4 (Continued)

- (d) (7 pts) Suppose (a_n) and (b_n) are sequences of real numbers. Prove that if (a_n) converges and $\lim b_n = \infty$, then $\lim(a_n + b_n) = \infty$.

Proof. Let $M > 0$.

~~(b_n) is~~ Since (a_n) converges, ~~(a_n)~~ is bounded ✓

So $k_1 < a_n < k_2$ for ~~some~~ all $n \in \mathbb{N}$, for some $k_1, k_2 \in \mathbb{R}$

Since $\lim b_n = \infty$, there exists $N \in \mathbb{N}$ s.t. $b_n > M - k_1$ for all $n \geq N$

Then $a_n + b_n > (M - k_1) + k_1 = M$ for all $n \geq N$

This finishes the proof that $\lim(a_n + b_n) = \infty$ ✓

Problem 5: General Proofs (21 points)

- (a) (7 pts) Prove that every convergent sequence of real numbers is Cauchy.

Proof Let (a_n) be a convergent sequence in \mathbb{R} . Write $\lim_{n \rightarrow \infty} a_n = L$ ✓

Let $\varepsilon > 0$ ✓

Since (a_n) converges, there exists $N \in \mathbb{N}$ such that

for all $n \geq N$, $|a_n - L| < \frac{\varepsilon}{2}$ ✓ Fix such N .

~~Let $m, n \geq N$~~ Take arbitrary $m, n \in \mathbb{N}$ such that $m, n \geq N$

Then $|a_n - L| < \frac{\varepsilon}{2}$, $|a_m - L| < \frac{\varepsilon}{2}$

$\Rightarrow a_n \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$, $a_m \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$

\Downarrow
 $-a_m \in (-\frac{\varepsilon}{2} - L, \frac{\varepsilon}{2} - L)$

So $a_n - a_m < \varepsilon$, $a_n + a_m > -\varepsilon$

\Rightarrow $|a_n - a_m| < \varepsilon$

This finishes the proof that every convergent sequence of real numbers is Cauchy. ✓
Since ε, m, n are arbitrary,
 $\bigwedge_{n \in \mathbb{N}}$



Problem 5 (Continued)

- (b) (7 pts) Let (a_n) be an increasing sequence of real numbers and (b_n) a decreasing sequence of real numbers such that $a_m \leq b_n$ for all $m, n \in \mathbb{N}$. Prove that

$$\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset.$$

Since $a_m \leq b_n$ for all $m, n \in \mathbb{N}$, $a_m \leq b_1$ for all $m \in \mathbb{N}$
and $b_n \geq a_1$ for all $n \in \mathbb{N}$.

~~Since~~ This means that (a_n) is bounded above and (b_n) is bounded below

Since (a_n) is increasing and (b_n) is decreasing,

$a_n \geq a_1$ and $b_n \leq b_1$ for all $n \in \mathbb{N}$

Therefore (a_n) is bounded below and (b_n) is bounded above

By 100, (a_n) and (b_n) are bounded, together with the fact that they are monotone, this shows (a_n) and (b_n) are convergent. write $\lim(a_n) = a$ and $\lim(b_n) = b$

Since $a_n \leq b_n$ for all $n \in \mathbb{N} \Rightarrow \lim(a_n) \leq \lim(b_n) \Rightarrow a \leq b$

- (c) (7 pts) Prove that if the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then for every subset $A \subseteq \mathbb{R}$,

$$f[\text{cl}(A)] \subseteq \text{cl}(f[A]).$$

(Recall that $\text{cl}(A)$ is the closure of A , defined by $\text{cl}(A) = A \cup A'$ where A' is the set of all limit points of A .)

Proof Since f is continuous, f is continuous on $A \subseteq \text{dom}(f)$ and thus continuous on every point a in A

So for $\forall a \in A$, either $\lim_{x \rightarrow a} f(x) = f(a)$, or a is an isolated point (i.e. $a \in A \setminus A'$)

Since $a \leq b$, $[a, b] \neq \emptyset$

\Rightarrow for all $n \in \mathbb{N}$, $[a, b] \subseteq [a_n, b_n]$

so $[a, b] \subseteq \bigcap_{n \in \mathbb{N}} [a_n, b_n]$

This finishes the proof

Let $a \in \text{cl}(A)$ be arbitrary

Case 1 $a \in A' \setminus A \Rightarrow \lim_{x \rightarrow a} f(x) \in \text{cl}(f[A])$

Case 2 $a \in A \Rightarrow$ then either $\lim_{x \rightarrow a} f(x) = f(a)$ or $a \in A \setminus A'$

(you're making this case more complicated than it needs to be) if $\lim_{x \rightarrow a} f(x) = f(a) \Rightarrow \lim_{x \rightarrow a} f(x) \in f[A] \subseteq \text{cl}(f[A])$
if a is an isolated point, then

