众所周知这是一个无 measure 的课，但是我们提一个重要的东西，告诉我们黎曼可积的函数的重要标准

Def zero-measure set

$A\subseteq\mathbb{R}$ has measure zero if:

$\forall \varepsilon>0, \exists$ a seq. of open intervals $((a_k,b_k))_{k\in\mathbb{N}}$

s.t. (i) $A\subseteq \bigcup_{k\in\mathbb{N}}(a_k,b_k)$

(ii) $\sum_{k=1}^{\infty}(b_k-a_k)<\varepsilon$

注：zero measure 的意义是：这个集合的 length 是 0. 这个集合可以是无限甚至 uncountable 的，但它的点以一种稀疏的方式分布. (eg: Cantor set, $|\mathcal{P}|=\aleph_1$, 但 zero measure)

Thm Lebesgue's Characterization of Integrability

$f:[a,b]\to\mathbb{R}$ is Rm intble $\boxed{\text{iff}}$

$\{x\in[a,b] \mid f \text{ is disctn at } x\}$ has measure zero

({f 的非连续点} 是零测的)

Fact 任意 ctbl 的 $A\subseteq\mathbb{R}$ 都 has measure zero

因而任意只有 ctbl 个点不 ctn 的 function 都 Riemann intble.

比如 Thomas function

Last time: mono functions are Rm intble.

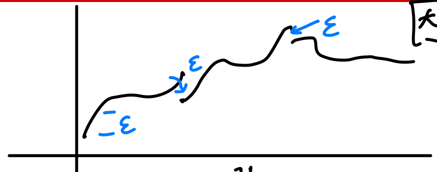
Lemma $g:[c,d]\to\mathbb{R}$

Suppose we have $\varepsilon,\delta>0$ s.t. (未必 ctn)

$(\forall x,y\in[c,d], |g(x)-g(y)|<\varepsilon$ whenever $|x-y|\le\delta)$

$\Rightarrow g$ is bounded, 且 $\boxed{\sup(g)-\inf(g)\le\left(\frac{d-c}{\delta}+1\right)\varepsilon}$

这一条 lemma 的意思是：即使不连续，只要左右 δ 内的点的值域差可以 ε-bound, 那么函数也一定 bounded 而且我们可以用 ε 和 δ 的大小来判断函数的上下界



Pf Let $x,y\in[c,d]$ with $x<y$.

Fix n (least) s.t. $\frac{d-c}{\delta}\le n \Rightarrow n<1+\frac{d-c}{\delta}$

For each $0\le k\le n$, let $z_k = x+k\frac{(y-x)}{n}$

So $|z_k-z_{k-1}|=\frac{y-x}{n}\le\frac{d-c}{n}\le\delta$ for each $1\le k\le n$

$\Rightarrow |g(x)-g(y)|=\left|\sum_{k=1}^n (g(z_k)-g(z_{k-1}))\right|$

$\le\sum_{k=1}^n|g(z_k)-g(z_{k-1})|<n\varepsilon<\left(1+\frac{d-c}{\delta}\right)\varepsilon$

Since x,y is arbitrary, the claim follows. □

Thm① Composition Thm

从 measure 上理解：$g\circ f$ 后，由于 g 是 ctn 的，g 保留了 f 的 ctn 和 disctn 的 pt, 因而 $g\circ f$ 的不连续点集仍是零测的.

Let $f:[a,b]\to\mathbb{R}$ is intble on $[a,b]$,

suppose $g:\mathbb{R}\to\mathbb{R}$ is ctn

$\Rightarrow g\circ f$ is intble on $[a,b]$

Remark: 两个 intble function 的 composition 未必 intble!! 但外层 ctn, 里层 intble 的 composition 则一定 intble.

Pf Since f is intble $\Rightarrow f$ is bounded

So fix a $\boxed{\text{closed bounded}}$ interval $\boxed{I\supseteq \operatorname{ran}(f)}$

$\Rightarrow g$ is uniformly ctn on I

Let $\varepsilon>0$, fix $\delta>0$ s.t.

$\left(\forall x,y\in I, |x-y|<\delta\Rightarrow|g(x)-g(y)|<\frac{\varepsilon}{2(b-a)}\right)$

Let $\mathcal{P}=(x_k)_{k=0}^n$ be a partition of $[a,b]$

s.t. $|U(f,\mathcal{P})-L(f,\mathcal{P})|<\delta(b-a)$

$\Rightarrow U(g\circ f,\mathcal{P})-L(g\circ f,\mathcal{P})=\sum_{k=1}^n\left(\sup_{I_k}(g\circ f)-\inf_{I_k}(g\circ f)\right)\Delta x_k$

$\le\sum_{k=1}^n\left[\frac1\delta\left(\sup_{I_k}f-\inf_{I_k}f\right)+1\right]\frac{\varepsilon}{2(b-a)}$ (by applying lemma on each subinterval $[\inf_{I_k}f,\sup_{I_k}f]$: for each $k\in\mathbb{N}$,

$\left(\sup_{I_k}g\circ f-\inf_{I_k}g\circ f\right)=\left(\sup_{y\in f[I_k]}g-\inf_{y\in f[I_k]}g\right)\le\frac{|f[I_k]|}{\delta}+1$)

$$=\frac{\varepsilon}{2\delta(b-a)}\underbrace{\sum_{k=1}^n\left(\sup_{I_k}f-\inf_{I_k}f\right)\Delta x_k}_{U(f,\mathcal{P})-L(f,\mathcal{P})}+\underbrace{\sum_{k=1}^n\frac{\varepsilon}{2(b-a)}\Delta x_k}_{\frac{\varepsilon}{2}}$$

$$<\frac{\varepsilon}{2\delta(b-a)}(\delta(b-a))+\frac{\varepsilon}{2}=\varepsilon \quad \square$$

Corollary 1.1 ctn functions are intble

Naturally. consider $g(x)=x$. $\Rightarrow g\circ f=f$ is intble.

Corollary 1.2 Product Thm

If f and g intble on $[a,b]$

$\Rightarrow fg$ is intble on $[a,b]$

Pf Consider $h(x)=x^2$ is ctn on $\mathbb{R}$

$\Rightarrow fg=\frac12\left((f+g)^2-f^2-g^2\right)$, so is intble on $[a,b]$ by the Composition Thm.

Additional Properties of Integral

Thm ②

If f is intble on $[a,b]$

$\Longrightarrow$ $|f|$ is also intble on $[a,b]$

且 $\left|\int_a^b f\right| \le \int_a^b |f|$

(directly follows from $|\Sigma a_n| \le \Sigma |a_n|$)

Pf Integrability of $|f|$ follows from integrability of f by the Composition Thm ($g(x)=|x|$ is ctn)

而 $-\int|f| = \int -|f| \le \int f \le \int |f|$

$\left|\int f\right| \le \left|\int |f|\right| = \int |f|$

Thm ③ Additivity of Integrals

Let $a<c<b$, $f:[a,b]\to\mathbb{R}$

$\Longrightarrow$ f is intble on $[a,b]$ iff f is intble on $[a,c]$ and $[c,b]$

且 $\int_a^b f = \int_a^c f + \int_c^b f$

Pf 对 $\forall [c,d] \subseteq [a,b]$,

取 $g(x) = \begin{cases} f(x), & \text{for all } x\in[c,d] \\ 0, & \text{for all } x\in[a,b]\setminus[c,d] \end{cases} = \boxed{f\cdot\chi_{[c,d]}}$

def: characteristic function of A
for $A\subseteq\mathbb{R}$, $\chi_A(x) = \begin{cases} 1, & x\in A \\ 0, & x\in\mathbb{R}\setminus A \end{cases}$

$\Longrightarrow$ 易证 $\boxed{\int_c^d f = \int_a^b g}$

$\left(\int_c^d f = \int_a^b f\cdot\chi_{[c,d]}\right)$

Thm thm follows.

Thm ④ 改变finitely many个pts不改变integral

If $f:[a,b]\to\mathbb{R}$ is intble on $[a,b]$

且 $g(x)=f(x)$ for all but finitely many $x\in[a,b]$

$\Longrightarrow$ g 在 $[a,b]$ 上也intble 且 $\boxed{\int_a^b f = \int_a^b g}$ 至多有限个点不同

Pf. 显然. 先证一个点: $\forall\varepsilon$, 都可选 $\delta>0$ 使 $\delta\cdot|f(x)-f(x_0)| < \frac{\varepsilon}{2}$
而后使用induction. (无法推广至ctbly many pts)

Fundamental Thm of Calculus

FTC ① Suppose $F:[a,b]\to\mathbb{R}$ is ctn on $[a,b]$ ①
且 diffble on (a,b) ②
并 Suppose F' is Rm intble on $[a,b]$ ③

$\Longrightarrow$ $\boxed{\int_a^b F'(x)\,dx = F(b)-F(a)}$ ☆

notation: $F(b)-F(a) = F(x)\Big|_a^b$

Proof

Let $\varepsilon>0$

fix a partition $\mathcal{P}=(x_k)_{k=0}^n$ of $[a,b]$

s.t. $|U(F',\mathcal{P}) - L(F',\mathcal{P})| < \varepsilon$

Using MVT, fix for each k a tag $t_k\in I_k$

s.t. $F'(t_k) = \dfrac{F(x_k)-F(x_{k-1})}{x_k - x_{k-1}}$

$\Longrightarrow$ $\boxed{F(b)-F(a)} = \sum_{k=1}^n (F(x_k)-F(x_{k-1})) = \sum_{k=1}^n F'(t_k)\Delta x_k$

$= S(F',\dot{\mathcal{P}})$

Therefore, $L(F',\mathcal{P}) \le \underset{=F(b)-F(b)}{S(F',\dot{\mathcal{P}})} \le U(F',\mathcal{P})$

又 $L(F',\mathcal{P}) \le \int_a^b F' \le \int U(F',\mathcal{P})$

$\Longrightarrow$ $\left|\int_a^b F' - (F(b)-F(a))\right| < \varepsilon$ $\square$

ex $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Fundamental Thm of Calculus

FTC ② Let $f:[a,b]\to\mathbb{R}$ be Rm intble

for $a\le x\le b$ let $\boxed{F(x) = \int_a^x f(t)\,dt}$

$\Longrightarrow$ F is (uniformly) ctn on $[a,b]$ ☆

且 if f is ctn at $x_0\in(a,b)$, then F is diffble at x_0 且 $F'(x_0) = f(x_0)$

① F is uniformly ctn on $[a,b]$

Pf Let $\varepsilon>0$

Fix B s.t. $|f(x)|\le B$ for all $x\in[a,b]$

令 $x,y\in[a,b]$ s.t. $0<x-y<\delta=\frac{\varepsilon}{B}$

$\Longrightarrow$ $|F(x)-F(y)| = \left|\int_y^x f(t)\,dt\right| \le \int_y^x |f(t)|\,dt$

$\le \int_y^x B\,dt = B(y-x) < B\delta = \varepsilon$

$\Longrightarrow$ F is (uniformly) ctn on $[a,b]$

② if f is ctn at x_0, then $F'(x_0) = f(x_0)$

Pf Let $x_0\in(a,b)$ 并 suppose f ctn at x_0

Note that $\forall x\in(a,b)$ st $x\ne x_0$,

都有 $\frac{F(x)-F(x_0)}{x-x_0}-\underbrace{f(x_0)}_{(\text{const})}=\frac{1}{x-x_0}\int_{x_0}^{x}(f(t)-f(x_0))dt$

let $\varepsilon>0$

By ctnity of f at x_0, fix $\delta>0$ s.t.

$|f(x)-f(x_0)|<\varepsilon$ whenever $|x-x_0|<\delta$

$\Longrightarrow$ for all $x\in V_\delta(x_0)$,

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|=\left|\frac{1}{x-x_0}\int_{x_0}^{x}(f(t)-f(x_0))dt\right|$$
$$\le\left|\frac{1}{x-x_0}\int_{x_0}^{x}\underbrace{|f(t)-f(x_0)|}_{<\varepsilon}dt\right|$$
$$<\frac{1}{x-x_0}\int_{x_0}^{x}\varepsilon\, dt=\varepsilon$$

Therefore $F'(x_0)=\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0)$ $\square$

Note: f 在 x_0 处 ctn 这一条件是很重要的，在 FTC2 中起关键作用.

ex ① $g(x)=\int_0^x\cos(t^2)dt$ is an antiderivative of $f(x)=\cos x^2$ on $\mathbb{R}$，因为 $f(x)$ 在 $\mathbb{R}$ 上 ctn

② $\frac{d}{dx}\int_0^x e^{t^2}dt=e^{x^2}$ (though generally the function 无法求出积分值)

③ $\frac{d}{dx}\int_0^{x^3}\sin t\, dt=\underbrace{\sin x^3}_{g'(f(x))}\cdot\underbrace{3x^2}_{f'(x)}$

$F(u)=\int_0^u\sin t\,dt$, $g(x)=x^3$

$F\circ g(x)=\frac{d}{dx}\int_0^{x^3}\sin t\,dt$

$F\circ g'(x)=F'(g(x))\cdot g'(x)=\sin x^3\cdot 3x^2$

$F'(g(x))=\sin(g(x))$, $g'(x)=3x^2$

(note: $\frac{d}{dx}\int_a^{g(x)}f(t)\,dt=f(g(x))\cdot g'(x)$

if f is Rm intble 且在 (a,b) 上 ctn

本质上是一样的，只是多了个复合函数求导.)

Remark

FTCs 主要陈述可以理解为："differentiation 和 integration 是 inverse operations"，但是:

① Derivatives 未必 integrable，比如 $f(x)=x^2\sin\frac{1}{x^2}$ is unbounded

② Indefinite integrals 未必是 antiderivatives，比如 Thomas (antiderivative 也未必存在) function 并没有 antiderivatives，而 integral 则为 constant zero

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实际上我们知道并不是每个函数都有 antiderivative 的：首先简单的函数才会在整个 R 上有单一的 antiderivative 的表达式，而一般的函数只有在某个区间上的 antiderivative，且更多函数并没有 antiderivative.

然而每个 differentiation Rule 都对应了一个 integral rule.

① $\frac{d}{dx}(x^r)=rx^{r-1}$, $\int x^r=\frac{x^{r+1}}{r+1}$

② Product rule $\longrightarrow$ Integration by Parts

③ Chain Rule $\longrightarrow$ Change of variable

Thm Integration by parts

If u 和 v 为 ctn functions on $[a,b]$，且在 (a,b) 上 ctn 且 u', v' 在 $[a,b]$ 上 intble

$\Longrightarrow \int_a^b u(x)v'(x)dx=u(x)v(x)\Big|_a^b-\int_a^b u'(x)v(x)dx$

$(\int u\,dv=uv-\int v\,du)$

Thm Change of variable

Suppose $u=f(x)$ 为一个 continuously diffable function on $\boxed{\text{open interval } J}$, $\boxed{\text{open interval } I\supseteq f[J]}$

则 if $\boxed{g \text{ 在 } I \text{ 上 ctn}}\Longrightarrow \forall a,b\in J$, $\boxed{\int_a^b g(f(x))f'(x)dx=\int_{f(a)}^{f(b)}g(u)du}$

Pf ① $\frac{d}{dx}u(x)v(x)=u'(x)v(x)+u(x)v'(x)$

$\Downarrow$ integrate both sides on $[a,b]$

$u(x)v(x)\Big|_a^b=\int_a^b u'(x)v(x)dx+\int_a^b u(x)v'(x)dx$

② $\int_a^b g(x)dx=G(b)-G(a)$

$(G\circ f(x))'=G'(f(x))\cdot f'(x)=g\circ f(x)\cdot f'(x)$

$\int_a^b g\circ f(x)f'(x)dx=(G\circ f(t))\Big|_a^b$ by FTC①

$=G(f(b))-G(f(a))$

$=\int_{f(a)}^{f(b)}g(u)\,du$