Recall: ① metric function 是满足: (i) $d(x,y)\geq 0$ 且仅在 $x=y$ 时 $=$
(ii) symmetric (iii) triangular ineq. 的 function

② set X with a metric d defined on it: (X,d)
为一个 metric space

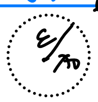
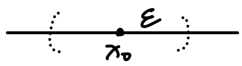
③ 如果 metric space (X,d) 上 every Cauchy seq. 都 conv 于 some $L\in X$，则 (X,d) 为一个 complete metric space

一些常见的 metric:

1. $d(x,y)=|x-y|$ is a metric on $\mathbb{R}$
2. $d(\vec{x},\vec{y})=\|\vec{x}-\vec{y}\|=\sqrt{(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})}$ is the Euclidean metric on $\mathbb{R}^n$
3. $d(\vec{x},\vec{y})=\sum_{i=1}^{n}|x_i-y_i|$ is the taxi-cab metric on $\mathbb{R}^n$
4. $d(a+bi,c+di)=\sqrt{(a-c)^2+(b-d)^2}$ is a metric on $\mathbb{C}$

Def Open-neighborhood

Let (X,d) be a metric space, $x_0\in X$
则 $V_\varepsilon(x_0)=\{x\in X \mid d(x,x_0)<\varepsilon\}$
被称为 an $\boxed{\text{open nbh about } x_0 \text{ of radius } \varepsilon}$

in $\mathbb{R}^2$:  in $\mathbb{R}$: 

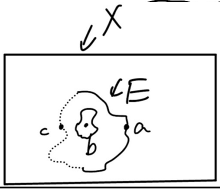
Def A set U in the metric space X 是 open 的, if $\boxed{\forall x\in U}$, $\boxed{\exists\varepsilon>0 \text{ s.t. } V_\varepsilon(x)\subseteq U}$ 即: interior pt.
$F\subseteq X$ is closed iff $X\setminus F$ is open.

ex. $\boxed{\emptyset \text{ and } X \text{ is both open and closed in } X}$ ☆
$\mathbb{R}$ is closed but not open in $\mathbb{C}$.

(2. Rudin's version)

Def⑤ interior point (内点)　　E 的所有 int pts: 写作 $\boxed{\text{int}(E)}$
p 为 $E\subset X$ 的一个 interior point
if $(\exists N_r(p))$, s.t. $\boxed{N\subset E}$

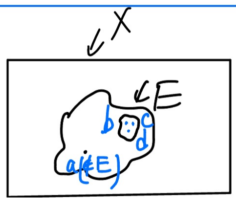
也就是说: $p\in E$ 是 p 为 E 的 interior point 的 necessary condition, 但并不 sufficient;
(内点一定在E中) p 不仅要 $\in E$, 还要有周围的一片邻域也在 E 中
(因而 $\boxed{\text{isolated points 一定不是 interior points}}$) ☆



c, b 都不是 interior point
b 是 isolated 的, a 在闭边缘
c 是 interior point (c 也在边缘但是在开的边缘)

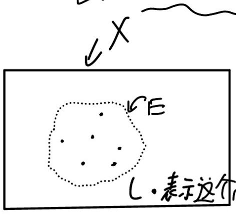
note: $\forall E$, $\text{int}(E)\subseteq E$ ☆
(因而 Rudin's 版 def of open set:
U 为一个 open set if $\boxed{\text{int}(U)=U}$)

Def③ isolated point (孤立点)　($p\notin E'$)　即 p 存在一个邻域其没有 p 以外的 E 点
如果 $p\in E$ 但 p 却不是 E 的 limit point
那么 p 被称为 E 的一个 isolated point

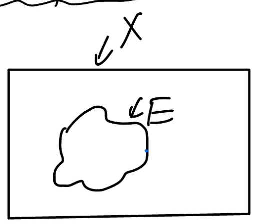


a: limit point, 尽管 $a\notin E$
b, c, d: 在 E 中, 但是周围是空的
⟹ isolated point (形象)

因而: 有 isolated points 一定不是 open set.



open set ✓

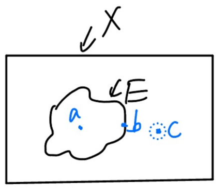


not open ✗
(闭边缘上的点非 interior)

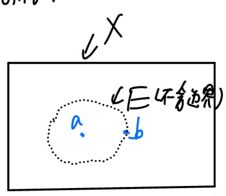
Def② Limit point (极限点)

$p\in X$ 为 set E 的一个 limit point if:
p 的任意 Neighborhood 都包含一点 $q\in X$
s.t. $q\neq p$, $q\in E$

E 在 X 中的所有 limit pts: 写作 $\boxed{E'}$ ☆
$E'\cup E$ 被称为 E 的 $\boxed{\text{closure (闭包)}}$, 写作 $\boxed{\bar{E}}$ ☆



a, b 是 limit point: 以任何半径扩开, 都能含入 E 中点.
c 不是 limit point: $r<d(b,c)$ 时, $N_r(c)\cap E=\{c\}$



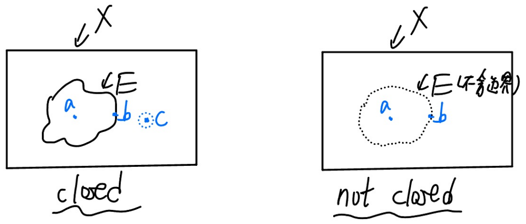
a, b 是 limit point: 以任何半径扩开, 都能含入 E 中点.
尽管 $a\in E$ 而 $b\notin E$

而Rudin的 closed set 是这样定义的

Def④ closed set (闭集)

E is closed if 每个极限点都 $\in E$

(即 $E' \subseteq E$) ☆



我们的 closed set 则定义为(其complement 为 open set)

但是 by Thm 2.23, 这两个定义 equivalent.

Thm 2.23 E is open iff E^c is closed

(而我们也有一个定理表明了 rudin 里定义的意思)

Thm A set $F \subseteq \mathbb{R}$ is closed iff $\forall$ convergent seq. (a_n) of points in F, $\lim(a_n) \in F$

Pf. Suppose F is closed, so $U = \mathbb{R} \setminus F$ is open.
Let (a_n) be any convergent seq. in $\mathbb{R}$ st. $\lim(a_n) = l \in U$.
Since U is open, we can fix $\varepsilon > 0$ st. $V_\varepsilon(l) \subseteq U$.
Since $a_n \to l$, there exists $N \in \mathbb{N}$ st. $a_n \in V_\varepsilon(l) \subseteq U$ for all $n \geq N$.
So (a_n) is not a seq. in F.
This shows that every convergent seq in F must have it limit in F. ($\Rightarrow$)
Now suppose F is not closed, then $U = X \setminus F$ is not open ☆.
So $\exists x_0 \in U$ s.t. $\forall \varepsilon > 0$, $(x_0-\varepsilon, x_0+\varepsilon) \cap F = \emptyset$
Then fix that x_0, for every $n \in \mathbb{N}$ let $a_n \in (x_0-\varepsilon, x_0+\varepsilon) \cap F$
Then (a_n) is a convergent seq. in F whose limit does not belong to F.

至此:穿插完了本课的 point-set topology 和 rudin 的补充部分.
我们现在继续回到$\mathbb{R}$, 看一些 facts.

Fact① in $\mathbb{R}$, the "open nbh's" $V_\varepsilon(x_0)$ is exactly the open interval $(x-\varepsilon, x+\varepsilon)$

Fact② 一个 nonempty set $U \subseteq \mathbb{R}$ is open iff $\forall x \in U$, $\exists a<b$ s.t. $x \in (a,b) \subseteq U$
(实际上就是定义在$\mathbb{R}$下的具体解释)

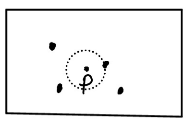
Fact③ $\mathbb{R}$中, closed interval 是 closed set.

Fact④ $\mathbb{R}$中, 任意 finite set 都 closed ☆

实际上 fact④ 可推广至任意 metric space 因为:

Thm 2.20 E的任意 limit point p 的任意 nbh 都有 inf many pts of E
(因而: 一个 finite set 是没有 limit pts 的)

Pf. (by contradiction)
令p为E的一个 limit pt, $N_r(p)$ 为其一个 nbh
suppose $N_r(p)$ 只有 finitely many pts
即 $N \cap E = \{q_1, \dots, q_n\}$
并令 $\boxed{x = \min_{1 \leq k \leq n} d(p, q_k)}$ (所有点中和P最近的距离)
$\Rightarrow N_x(p) \cap E = \{p\}$
因而 p 不是一个 lim pt (contradicts)



finite point set 无 lim pts, 因而 $E' = \emptyset \subseteq E$, 因而是 closed 的

Fact⑤ $\mathbb{R}$中, U为 open set iff U为a countable union of open intervals (必须ctb unions, 因为 unctbly many个 open interval是可以 union出 closed set 的)

这个性质的意思是: $\mathbb{R}$上的 open set是很简单粗暴的, 就是 a union of open intervals.

3. 现在我们把 convergence 的概念推广至 general metric space

Def. convergence in general metric space.

A seq. (x_n) in X converges to $L \in X$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\left(d(x_n, L) < \varepsilon \text{ whenever } n \geq N \right)$

Def complete metric space

A metric space X is complete if every Cauchy seq. in X converges. (to some $L \in X$)

Fact: $\mathbb{R}^n$, $\mathbb{C}$ is complete, $\mathbb{Q}$ is not complete.

Def Bounded set

$S \subseteq X$ is bounded if $\exists M > 0$ s.t. $\forall x, y \in X$, $d(x,y) \leq M$.

Generalized BW Thm

Every bounded seq. in $\boxed{\text{a complete metric}}$ space has a convergent subseq.