现在我们可以证明两个非常重要的Thms. EVT, IVT

Thm① Extreme Value Thm ✡✡

if $A \subseteq \mathbb{R}$ ($A \neq \emptyset$) is closed and bounded (compact)

and $f: A \to \mathbb{R}$ is ctn.

$\Rightarrow$ f is bounded and $\exists x_0, y_0 \in A$ s.t.

$f(x_0) \le f(x) \le f(y_0)$ for all $x \in A$

即: compact $A \subseteq \mathbb{R}$上的ctn.函数必有两端极值.

Pf Let $M = \sup\{f(x) \mid x \in A\} \in \mathbb{R} \cup \{+\infty\}$ (will prove it's impossible)

Let (x_n) be a seq. in A s.t. $\lim_{n\to\infty} f(x_n) = M$

Since (x_n) is ① bounded, $\exists$ conv. subseq. $\boxed{(x_{n_k}) \to y_0}$

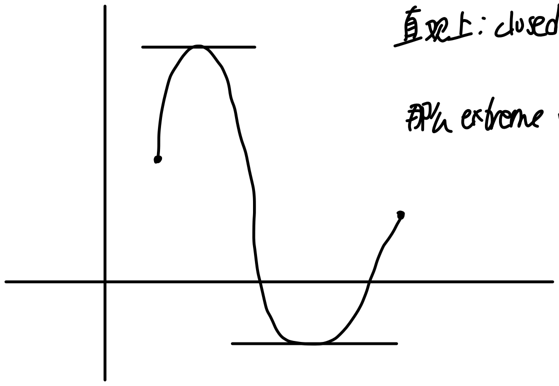
Since A is ② closed $\Rightarrow$ $\boxed{y_0 \in A}$ ✡

Since f is ③ ctn. at y_0, $\lim_{k\to\infty} f(x_{n_k}) = f(y_0)$

$M = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f(y_0) \in \mathbb{R}$

So f is bounded above and $f(y_0) = \max\{f(x) \mid x \in A\}$

(Dually, there is $x_0 \in A$ s.t. $f(x_0) = \min\{f(x) \mid x \in A\}$) □



直观上: closed + bounded A + ctn f

那么 extreme value 一定存在

Thm② Intermediate Value Thm ✡

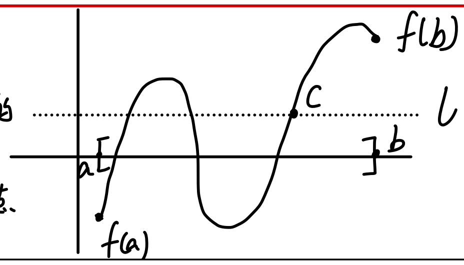
If $f: [a,b] \to \mathbb{R}$ is ctn

$\Rightarrow$ $\forall L$ between $f(a)$ and $f(b)$, ($f(a) < L < f(b)$ or $f(b) < L < f(a)$)

$\exists c \in [a,b]$ s.t. $f(c) = L$

即: 一个区间上的ctn函数 f, 其 $im(f)$ 一定覆盖了 $[f(a), f(b)]$

这是非常显然的. 因为从$f(a)$到$f(b)$的连续路径一定经过了$[f(a), f(b)]$中所有点.



Pf WLOG say $f(a) < L < f(b)$

Let $S = \{x \in [a,b] \mid f(x) \le L\}$

So S is nonempty and bounded above

let $c = \sup(S)$, then $a < c < b$ by continuity of f

let (s_n) be a seq. in S conv. to c

For each $n \in \mathbb{N}$ let $t_n = \min\{c + \frac{1}{n}, b\}$

So $t_n \to c$ and $c < t_n$ for all $n \in \mathbb{N}$

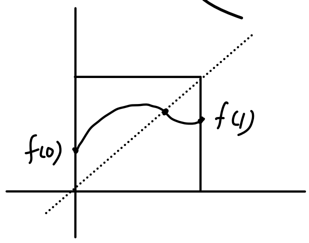
$\Rightarrow f(s_n) \le L < f(t_n)$ for all $n \in \mathbb{N}$

But by continuity of f, $\lim_{n\to\infty} f(s_n) = f(c) = \lim_{n\to\infty} f(t_n)$

So $L = f(c)$ □

ex If $f: [0,1] \to [0,1]$ is ctn. then f has a fixed pt. (i.e. a pt. $x_0 \in [0,1]$ s.t. $f(x_0) = x_0$)

(pf by graph)



Pf if $f(0) = 0$ or $f(1) = 1$ $\Rightarrow$ fixed pt. found.

因而只需考虑 when $f(0) > 0$ & $f(1) < 1$

let $g(x) = x - f(x)$

于是 g is ctn on $[0,1]$ 且 $g(0) < 0$, $g(1) > 0$

因而 by IVT $\Rightarrow$ $\exists x_0 \in (0,1)$ s.t. $g(x_0) = 0$

$\Rightarrow f(x_0) = g(x_0) - 0 = x_0$

Corollary① 如果 $I \subseteq \mathbb{R}$ 为一个 interval

且 $f: I \to \mathbb{R}$ ctn

$\Rightarrow$ 那么 $f[I]$ 也是一个 interval

Pf let $y_1 < y_2 \in f[I]$

fix $x_1, x_2 \in I$ s.t. $f(x_1) = y_1$, $f(x_2) = y_2$

(WLOG assume $x_1 \le x_2$) 那么 $f \restriction [x_1, x_2]$ 为 ctn 的

$\Rightarrow$ by IVT, $\forall L \in (y_1, y_2)$, $\exists x_0$ between x_1 and x_2

s.t. $f(x_0) = L$ $\Rightarrow$ $L \in f[I]$

因而 $f[I]$ 为一个 interval.

Corollary② 如果 I 是个 closed + bounded 的 interval $[a,b]$

且 $f: I \to \mathbb{R}$ ctn

$\Rightarrow$ 那么 $f[I]$ 也是个 closed + bounded interval

Pf let $f: [a,b] \to \mathbb{R}$ be ctn

By EVT, $\exists x_0, x_1 \in [a,b]$ s.t.

$\forall x \in [a,b]$, $\underbrace{m = f(x_0)}_{(\min)} \le f(x) \le \underbrace{f(x_1) = M}_{(\max)}$

Then $\operatorname{ran}(f) = [m, M]$ by IVT

(or Corollary ①)