Numerical sequences and series.

<u>Def</u> A sequence (p_n) in a metric space X is convergent to a pt $p \in X$ if for $\forall \varepsilon > 0$, there $\exists N$ integer s.t. $n \geq N$ implies $d(p_n, p) < \varepsilon$.

Say p is a limit of (p_n). (p_n) convergent to p.

<u>Rmk</u> Depends on X. $(p_n = 1/n)$ convergent in $\mathbb{R}$ but not in $\mathbb{R}_{>0}$.

We call (p_n) is divergent if (p_n) is not convergent. (p_n) is bounded if the range is bounded.

Example (a) $\lim_{n\to\infty} s_n = 0$ if $s_n = 1/n \in \mathbb{R}$. has infinite range.

(b) $\lim_{n\to\infty} s_n = 1$ if $s_n = 1 \in \mathbb{R}$

<u>Thm</u> Let (p_n) be a sequence in a metric space X.

(a) (p_n) converges to $p \in X$ iff every nbd of p contains p_n for all but finitely many n.

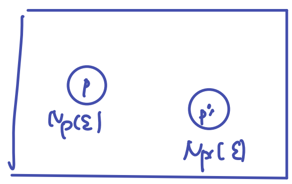
(b) If $p, p' \in X$ and (p_n) converges to p and p', then $p = p'$.

(c) If (p_n) converges then (p_n) bounded.

(d) If $E \subset X$ and if p is a limit pt of E, then there is a sequence (p_n) in E such that $p = \lim p_n$.

Proof (a) "$\Rightarrow$" Any nbd $N_p(r)$ of p, choose $\varepsilon < r$, there is N, $d(p, p_n) < \varepsilon$ for $n > N$. "$\Leftarrow$" For any $\varepsilon > 0$, only finitely many p_n not in $N_p(\varepsilon)$, i.e. $\exists N > 0$ $d(p_n, p) < \varepsilon$.

(b) If $p \neq p'$, $\exists N_p(\varepsilon)$ and $N_{p'}(\varepsilon)$, $N_p(\varepsilon) \cap N_{p'}(\varepsilon) = \emptyset$



but for n sufficiently large. $p_n \in N_p(\varepsilon)$, $p_n \in N_{p'}(\varepsilon)$

i.e. $p \in N_p(\varepsilon) \cap N_{p'}(\varepsilon) = \emptyset$, which is a contradiction.

(c) This is a collary of (a). Say $p_1, p_2, \cdots, p_k$ are those note in $N_p(\varepsilon)$. Then let $r = \max\{\varepsilon, d(p,p_1), \cdots, d(p,p_k)\}$. (p_n) is containd in $N_p(r)$.

(d) By def. $N_p(1/n)$ contains a point $p_n \in E$ $p_n \neq p$. $p_n \to p$.

We can study the algebraic relation for sequences on $\mathbb{R}$.

Thm Suppose (s_n) (t_n) are two real sequences and $\lim_{n\to\infty} s_n = s$ $\lim_{n\to\infty} t_n = t$

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} c s_n = cs$ for any constant c.

(c) $\lim_{n\to\infty} s_n \cdot t_n = st$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided that $s_n \neq 0$, and $s \neq 0$.

Proof (a) For $\varepsilon > 0$, there exists N_1, N_2

$$n > N_1 \Rightarrow |s_n - s| < \varepsilon/2$$

$$n > N_2 \Rightarrow |t_n - t| < \varepsilon/2$$

$$|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) For $\varepsilon > 0$, there exists N, $n \geq N \Rightarrow |s_n - s| < \varepsilon/c$ if $c \neq 0$.

i.e. $|cs_n - cs| < \varepsilon$

if $c = 0$, this is trivial.

(c) $(s_n t_n - st) = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$

For $\varepsilon > 0$, there is N, $n > N \Rightarrow |s_n - s| < \sqrt{\varepsilon}$

$|t_n - t| < \sqrt{\varepsilon}$

$\Rightarrow |(s_n - s)(t_n - t)| < \varepsilon$ i.e. $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$

$\lim (s_n t_n - st) = \lim (s_n - s)(t_n - t) + \lim s(t_n - t) + \lim t(s_n - s) = 0$

i.e. $\lim s_n t_n = st$

(d) Choosing m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \frac{1}{2}|s|$$

Given $\varepsilon>0$, there is $N>0$, $n>N \Rightarrow$ $|s_n - s| < \frac{1}{2}|s|^2 \cdot \varepsilon$.

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n \cdot s}\right| < \frac{1}{2}|s| \cdot \varepsilon \Big/ \frac{1}{2}|s|^2 = \varepsilon.$$

<u>Def</u> Given a sequence (p_n), consider subset of positive integers

$$n_1 < n_2 < n_3 < \cdots$$

The sequence (p_{n_i}) is called a subsequence (p_n). If (p_{n_i}) converges, its limit called subsequential limit of (p_n).

Clearly, (p_n) converges iff every subsequence of (p_n) converges.

<u>Thm</u> (a) If (p_n) is a sequence in a compact metric space X, then some subsequence converges to a point in X

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

<u>Proof</u> (a) Let E be the range of (p_n). If E is finite then $\exists\, n_1 < n_2 < \cdots$ such that $p_{n_1} = p_{n_2} = \cdots = p$.

If E is infinite, then E has a limit pt p (Thm 2.37) in Rudin. Choose $n_1 < n_2 < \cdots$ such that $d(p, p_{n_i}) < \frac{1}{i}$. Then (p_{n_i}) converges to p

$(p_{n_i}) \ldots p.$

Thm The subsequential limits of a sequence (p_n) in a metric space X form a closed subset.

Proof Let $E = \{$the subsequential limits of $(p_n)\}$ and p is a limit pt. We have to show that $p \in E$.

Choose n_1, $p_{n_1} \neq p$. Let $\delta = d(p_{n_1}, p)$. Haven chosen $p_{n_1}, p_{n_2}, \cdots, p_{n_{i-1}}$

There is a $x \in E$ such that $d(p_{n_1}, x) < \frac{\delta}{2^i}$

Since x is a subsequential limit, $\exists\, n_i > n_{i-1}$, $d(p_{n_i}, x) < \delta/2^i$

$$\therefore\ d(p, p_{n_i}) \leq d(p, x) + d(x, p_i) = \delta/2^{i-1}.$$

We see that $(p_{n_i}) \to p$. $\boxtimes$