Homework 4: Due Tuesday, June 4, at 11:59pm, on Gradescope

(1) Do Challenge Problem (14) from HW 2: if (a_n) is a sequence in $\mathbb{R}$ and $\lim(a_{n+1} - a_n) = 0$, must (a_n) converge? Justify your answer.

(2) Let (a_n) be a sequence in $\mathbb{R}$, and let $S \subseteq \mathbb{R}$ be its set of real subsequential limits. Prove that S is closed.

(3) Given $A \subseteq \mathbb{R}$, write A' for the set of all limit points of A and define the *closure* of A to be the set $\mathrm{cl}(A) = A \cup A'$.
   (a) Prove that A' is closed.
   (b) Prove that $\mathrm{cl}(A)$ is closed.
   (c) Prove that $\mathrm{cl}(A)$ is the "smallest" closed set containing A, in the sense that $\mathrm{cl}(A) \subseteq F$ for every closed set F containing A.

(4) (a) Prove explicitly using the ϵ/δ definition that $\lim\limits_{x\to 2} x^3 = 8$.
   (b) Given $\epsilon > 0$, find the *largest* $\delta > 0$ such that $|x^3 - 8| < \epsilon$ whenever $|x - 2| < \delta$.
   (c) Prove explicitly using the ϵ/δ definition that $\lim\limits_{x\to 4} \sqrt{x} = 2$.
   (d) Given $\epsilon > 0$, find the *largest* $\delta > 0$ such that $|\sqrt{x} - 2| < \epsilon$ whenever $|x - 4| < \delta$.

(5) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, suppose that $a \in \mathbb{R}$ is a limit point of $A \cap (a, \infty)$, and suppose $\lim\limits_{x\to a^+} f(x) = \infty$. Also let $c \in \mathbb{R}$, let $g : (c, \infty) \to \mathbb{R}$ be a function, and suppose that $\lim\limits_{x\to\infty} g(x) = L \in \mathbb{R}$. Prove that $\lim\limits_{x\to a^+} (g \circ f)(x) = L$.

(6) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, let $a \in \mathbb{R}$, and suppose $\lim\limits_{x\to a} f(x) = b$ and $\lim\limits_{x\to b} g(x) = L$. Show by example that L need not be the limit of $g \circ f$ as $x \to a$.

(7) Prove that for any sequence (a_n) of nonzero real numbers, $\limsup |a_n|^{1/n} \leq \limsup \left|\dfrac{a_{n+1}}{a_n}\right|$.

Hint: Let $L > \limsup \left|\frac{a_{n+1}}{a_n}\right|$ be arbitrary; then there is N such that $\left|\frac{a_{n+1}}{a_n}\right| < L$ for all $n \geq N$; now use the fact that for any $n > N$,

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|$$

as a first step towards showing that $\limsup |a_n|^{1/n} \leq L$.

(8) Let $A \subseteq \mathbb{R}$, suppose $a \in A \cap A'$, and let $f : A \to \mathbb{R}$ be a function. Prove that if $f(a) > 0$ and f is continuous at a, then there is $\epsilon > 0$ such that f is positive and bounded on $A \cap V_\epsilon(a)$.

(9) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Prove that if $f(x) = g(x)$ for all $x \in \mathbb{Q}$, then $f = g$.

(10) Prove that if $A \subseteq \mathbb{R}$ is not closed, then there is an unbounded continuous function $f : A \to \mathbb{R}$.

(11) Using only the definitions of continuity and open set, prove that for any function $f : \mathbb{R} \to \mathbb{R}$, f is continuous if and only if $f^{-1}[V]$ is open for every open set $V \subseteq \mathbb{R}$.

Optional Challenge Problems:

(12) Suppose $A \subseteq \mathbb{R}$ is closed, and let $f : A \to \mathbb{R}$ be a continuous function. Prove that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g \restriction A = f$.

(13) Suppose $\{U_i : i \in I\}$ is a family of nonempty open sets in $\mathbb{R}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Prove that I is countable.

(14) Let $d(x, y) = |x - y|$ be the usual metric on $\mathbb{R}$, and let $\mathcal{T}$ be the metric topology on $\mathbb{R}$ generated by d, so $\mathcal{T}$ consists of all open subsets of $\mathbb{R}$.[3]

(a) Prove that for any subset $A \subseteq \mathbb{R}$, A is open if and only if A can be expressed as a union of countably many open intervals in $\mathbb{R}$.

(b) Is it true that every open set A in $\mathbb{R}$ can be expressed as a union of open intervals *with rational endpoints*? Either give a counterexample if not, or else briefly explain how your proof in (a) could be modified in order to prove this stronger result.

[3]Recall that, by definition, the set $U \subseteq \mathbb{R}$ belongs to $\mathcal{T}$ (ie, is open) if for all $x \in U$ there is $\epsilon > 0$ such that $V_\epsilon(x) \subseteq U$.