(1) Prove that if the function f is continuous on $[a, b]$, then there is $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

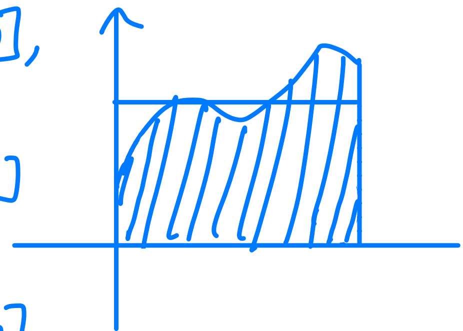
Proof Since f is continuous on $[a,b]$,

by extreme value theorem,

there exist $x_1, x_2 \in [a,b]$

s.t. $f(x_1) \le f(x) \le f(x_2)$

for all $x \in [a,b]$



And since f is continuous on $[a,b]$

$\Longrightarrow$ f is integrable on $[a,b]$

So $\int_a^b f(x_1)dx \le \int_a^b f(x)dx \le \int_a^b f(x_2)dx$

by monotonicity of integration

i.e. $(b-a)f(x_1) \le \int_a^b f(x)dx \le (b-a)f(x_2)$

$\Longrightarrow$ $f(x_1) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(x_2)$

By continuity of f between x_1 and x_2 and IVT

$\exists\, c \in [x_1, x_2]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

$\square$

(2) (a) Let $f : [a,b] \to \mathbb{R}$ be nonnegative and continuous. Prove that if $f(x) > 0$ for some $x \in [a,b]$, then $\int_a^b f > 0$.

(b) Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions such that $f(x) \le g(x)$ for all $x \in [a,b]$. Prove that if $\int_a^b f = \int_a^b g$, then $f = g$.

(a) Proof

Assume the hypothesis. Let $x_0 \in [a,b]$ s.t. $f(x_0) > 0$

Since f is continuous on $[a,b]$,

there exist $\varepsilon > 0$ s.t. for all $x \in V_\varepsilon(x_0) \cap [a,b]$,

$f(x) > 0$ (by hw 4. problem 8)

Note that $V_\varepsilon(x_0) \cap [a,b] \neq \emptyset$ is an interval

Fix a closed interval $[c,d] \subseteq V_\varepsilon(x_0) \cap [a,b]$

Since f is continuous on $[a,b]$, it is integrable on $[c,d]$ and $\int_c^d f > 0$ since $\int_c^d f = (d-c) f(x_0)$ for some $x_0 \in [c,d]$, by problem (1), and $f(x_0) > 0$

Since $f(x) \geq 0$ for all $x \in [a,b]$. $\int_a^c f \geq 0$ and $\int_d^b f \geq 0$

Therefore $\int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f > 0$ $\square$

(b) Assume the hypothesis and suppose for contradiction that

$\int_a^b f = \int_a^b g$ but $f \neq g$

Consider the function $g(x) - f(x)$ is nonnegative and continuous

Since $f(x) \le g(x)$ for all $x \in [a,b]$, $f \neq g$ implies $f(c) < g(c)$ for some $c \in [a,b]$,

So by (a), $\int_a^b (g-f) > 0$ i.e. $\int_a^b g > \int_a^b f$, contradicts with $\int_a^b f = \int_a^b g$.

This proves that if $\int_a^b f = \int_a^b g$ then $f = g$. $\square$

(3) (a) Prove that if the function f is integrable on $[a,b]$, then there is $c \in [a,b]$ such that $\int_a^c f = \int_c^b f$.

(b) Give an example to show that in part (a) we may not have $c \in (a,b)$.

(a) Proof

Since f is (Riemann) integrable on $[a,b]$,

by Fundamental Theorem of Calculus,

$F(x) = \int_a^x f(x)\,dx$ is continuous on $[a,b]$

Since $0 = F(a) < \frac{F(a)+F(b)}{2} < F(b) = \int_a^b f$

by IVT, there exists $c \in [a,b]$ s.t. $F(c) = \frac{F(a)+F(b)}{2}$

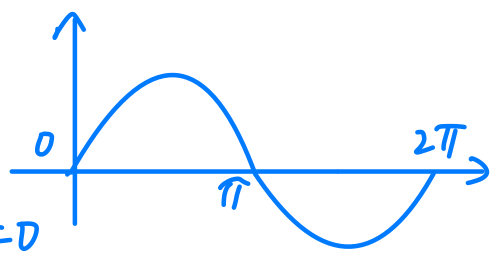
So $\int_a^c f = \int_c^b f = \frac{\int_a^b f}{2}$ since $\int_a^c f + \int_c^b f = \int_a^b f$

$\square$

(b) Consider $a=0$, $b=2\pi$

$f = \sin(x)$ is continuous on $[a,b]$, thus integrable

Consider $c=a$, $\int_a^c f = \int_c^b f = 0$



(4) Compute the following limits:

(a) $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$ (b) $\lim_{h\to 0}\int_3^{3+h} e^{t^2}\,dt$

(a) Since e^{t^2} is continuous at $t=0$, by Fundemental Theorem of Calculus ②, $F(x)=\int_0^x e^{t^2}dt$ is differentiable at $x=0$

And $\lim_{x\to 0}\frac{\int_0^x e^{t^2}dt}{x} = \lim_{x\to 0}\frac{\int_0^x e^{t^2}dt - \int_0^0 e^{t^2}dt}{x-0}$

$=\lim_{x\to 0}\frac{F(x)-F(0)}{x-0} = F'(0) = \frac{d}{dx}\left(\int_0^x e^{t^2}dt\right)\Big|_{x=0}$

$= e^{x^2}\big|_{x=0}$ by Fundemental Theorem of Calculus ②

$=1$

(b) $\lim_{h\to 0}\int_3^{3+h} e^{t^2}dt = \lim_{h\to 0}\left(\frac{\int_3^{3+h} e^{t^2}dt - 0}{h-0}\cdot h\right)$

$=\left(\frac{d}{dx}\left(\int_3^{3+h} e^{t^2}dt\right)\right)\left(\lim_{h\to 0} h\right)$ since e^{t^2} is continuous at 0.

$=\left(e^{(3+h)^2}\big|_{h=0}\right)\cdot\left(\lim_{h\to 0} h\right)$ by Fundemental Theorem of Calculus ②

$= e^{9}\cdot 0 = 0$

(5) For all $x \geq 0$ and $n \in \mathbb{N}$, let $f_n(x) = \dfrac{x^n}{1+x^n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Show that for all $0 < b < 1$, f_n converges uniformly on $[0, b]$.

(c) Does f_n converge uniformly on $[0, 1]$? Prove your claim.

(a) $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x^n}{1+x^n} = \lim_{n\to\infty} \left(1 - \dfrac{1}{1+x^n}\right)$

for $x \in [0,1)$, $\lim_{n\to\infty} \left(1 - \dfrac{1}{1+x^n}\right) = 0$ since $\lim_{n\to\infty} x^n = 0$

for $x = 1$, $\lim_{n\to\infty} \left(1 - \dfrac{1}{1+x^n}\right) = \dfrac{1}{2}$ since $\lim_{n\to\infty} x^n = 1$

for $x > 1$, $\lim_{n\to\infty} \left(1 - \dfrac{1}{1+x^n}\right) = 1$ since $\lim_{n\to\infty} \dfrac{1}{1+x^n} = 0$

So $f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ \frac{1}{2}, & \text{if } x = 1 \\ 1, & \text{if } x > 1 \end{cases}$

(b) Take arbitrary b s.t. $b \in (0,1)$

let $\varepsilon > 0$.

Note that for all $0 \leq x \leq b < 1$, $x^n < b^n$ for all $n \in \mathbb{N}$

$\Rightarrow 1 > \dfrac{1}{1+x^n} > \dfrac{1}{1+b^n}$

Since $\lim_{n\to\infty} \dfrac{1}{1+b^n} = 1$,

we can fix $N \in \mathbb{N}$ s.t. $\left|\dfrac{1}{1+b^n} - 1\right| < \varepsilon$

So $\left|1 - \dfrac{1}{1+x^n} - 0\right| = 1 - \dfrac{1}{1+x^n} < 1 - \dfrac{1}{1+b^n} < \varepsilon$ for all $x \in [0,b]$

Since ε is arbitrary, this proves that

for all $0 < b < 1$, f_n converges uniformly on $[a,b]$ $\square$

(c) (f_n) does not uniformly converges to f on $[0,1]$

Take $\varepsilon = \frac{1}{4}$

let $n \in \mathbb{N}$ be arbitrary

Since $\lim_{x \to 1^-} |f_n(x) - f(x)| = \lim_{x \to 1^-} \frac{x^n}{1+x^n} = \frac{1}{2}$,

$\exists \delta > 0$ s.t. for all $1 > x > 1-\delta$, $|f_n(x) - \frac{1}{2}| < \varepsilon$

$\Rightarrow f_n(x) \in (\frac{1}{4}, \frac{3}{4})$

So take $x_0 \in (1-\delta, 1)$, $|f_n(x) - f(x)| = f_n(x) - 0 > \frac{1}{4} = \varepsilon$

This shows that for all $n \in \mathbb{N}$, $\exists x \in [0,1]$ s.t. $|f_n(x) - f(x)| > \varepsilon$

So f_n does not converge uniformly on $[0,1]$ $\square$

(6) Prove that if (f_n) is a sequence of uniformly continuous functions on the interval (a, b) such that $f_n \to f$ uniformly on (a, b), then f is also uniformly continuous on (a, b).

Proof Assume the hypothesis.

let $\varepsilon > 0$ be arbitrary.

So $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ whenever $n \geq N$, for all $x \in (a,b)$

and $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$ whenever $|x-y| < \delta$ for some $\delta > 0$

let $x, y \in (a,b)$ s.t. $|x-y| < \delta$

$\Longrightarrow |f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

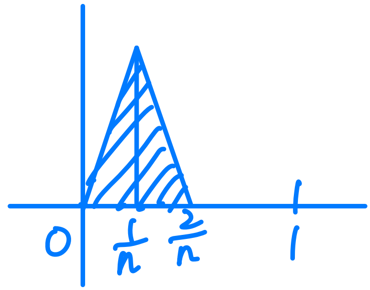
This finishes the proof. $\square$

(7) Give an example of a sequence (f_n) of continuous functions from $[0,1]$ to $\mathbb{R}$ that converges pointwise but *not* uniformly to a continuous limit function $f : [0,1] \to \mathbb{R}$.

ex Consider:

on $x \in [0,1]$:

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0 \le x \le \frac{1}{n} \\ 2n - n^2 x, & \text{if } \frac{1}{n} < x < \frac{2}{n} \\ 0, & \text{if } \frac{2}{n} \le x \end{cases}$$



So $(f_n) \longrightarrow f(x) = 0,\ x \in [0,1]$ pointwisely

since $\forall x \in [0,1],\ \lim_{n \to \infty} f_n(x) = 0$

But take $\varepsilon = 1$

Let $n \in \mathbb{N}$ be arbitrary

Consider $x = \frac{1}{n} \Rightarrow f_n(x) = n \ge 1$

$\Rightarrow |f_n(x) - f(x)| \ge 1 = \varepsilon$

So the convergence is not uniform, though f_n is continuous for each $n \in \mathbb{N}$ and f is also continuous.

Pf Assume the hypothesis.

Let $\varepsilon > 0$.

Fix $N \in \mathbb{N}$ s.t. $|f_m'(x) - f_n'(x)| < \frac{\varepsilon}{2(b-a)}$ for all $x \in [0,1]$ and $|f_n(a) - f_m(a)| < \frac{\varepsilon}{2}$ whenever $n, m \geq N$ (by the uniform convergence of (f_n') on $[0,1]$ and convergence of $(f_n(a))$)

Let $x \in [0,1]$ be arbitrary.

Let $m, n \geq N$ be arbitrary

$$\Rightarrow \int_a^x f_n'(t)\,dt = f_n(x) - f_n(a)$$

$$\int_a^x f_m'(t)\,dt = f_m(x) - f_m(a)$$

by FTC.

$$\begin{aligned}\Rightarrow |f_n(x) - f_m(x)| &= \left|f_m(a) - f_n(a) + \int_a^x (f_m'(t) - f_n'(t))\,dt\right| \\ &\leq |f_m(a) - f_n(a)| + \left|\int_a^x (f_m'(t) - f_n'(t))\,dt\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}|x-a| < \varepsilon\end{aligned}$$

Therefore (f_n) is uniformly Cauchy, thus uniformly convergent for all $x \in [0,1]$.

(9) A function $f : [a, b] \to \mathbb{R}$ is called a *step function* if there is a partition $\mathcal{P} = (x_k)_{k=0}^n$ of $[a, b]$ such that f is constant on (x_{k-1}, x_k) for each k. Prove that for every continuous function $f : [a, b] \to \mathbb{R}$, there is a sequence $f_n : [a, b] \to \mathbb{R}$ of step functions such that $f_n(x) \le f(x)$ for all $x \in [a, b]$ and $f_n \to f$ uniformly on $[a, b]$.

Let continuous $f: [a,b] \to \mathbb{R}$ be arbitrary.

<u>Construction</u> For each $n \in \mathbb{N}$:

Let $P_n = \{x_0, x_1, \ldots, x_n\}$ where $x_k = a + k\frac{b-a}{n}$

and define $f_n(x) = \inf\limits_{x \in [x_{k-1}, x_k]} f(x)$, if $x \in [x_{k-1}, x_k]$

for each $0 \le k \le n$

Now we prove that $(f_n) \to f$ uniformly.

<u>Proof</u> Let $\varepsilon > 0$

Since f is continuous on $[a,b]$, it is uniformly continuous

So $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$

Fix this δ and fix $N \in \mathbb{N}$ s.t. $\frac{b-a}{N} < \delta$

$\Rightarrow \forall n \ge N, \frac{b-a}{n} < \delta$

So for all $n \ge N$ and $x \in [a,b]$,

$x \in [x_{k-1}, x_k]$ for some $0 \le k \le n$

So $f_n(x) = \inf\limits_{y \in [x_{k-1}, x_k]} f(y) = \min\limits_{y \in [x_{k-1}, x_k]} f(y)$ by EVT,

So $|f_n(x) - f(x)| = |f(x_0) - f(x)|$ for some $x_0 \in [x_{k-1}, x_k]$

$< \varepsilon$ since $|x - x_0| < \frac{b-a}{n} < \delta$

Since ε is arbitrary, this finishes the proof. $\square$

(10) Suppose $\sum c_n x^n$ is a power series such that $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L > 0$. Prove that $\sum c_n x^n$ converges for all $x \in (-R, R)$ and diverges for all $x \in \mathbb{R} \setminus [-R, R]$, where $R = \frac{1}{L}$.

Proof ① Let $-R < x < R = \frac{1}{L}$ and fix x

$$\Rightarrow |x| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n} \cdot x\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{c_{n+1} x^{n+1}}{c_n x^n}\right| < 1$$

So by ratio test of numerical sequence, $\sum c_n x^n$ converges (absolutely) for x

Since $x \in (-R, R)$ is arbitrary $\Rightarrow$ $\sum c_n x^n$ converges for all $x \in (-R, R)$

② Likewise, for $|x| > R$,

$$\lim_{n\to\infty} \left|\frac{c_{n+1} x^{n+1}}{c_n x^n}\right| > 1,$$

so by ratio test of numerical sequence, $\sum c_n x^n$ diverges for all $x \in \mathbb{R} \setminus [-R, R]$. $\square$

(11) For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum n^2 x^n$ (b) $\sum \left(\frac{2^n}{n^2}\right) x^n$ (c) $\sum \left(\frac{2^n}{n!}\right) x^n$

(a) $\lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| = \lim_{n\to\infty} \left|1 + \frac{2}{n} + \frac{1}{n^2}\right| = 1$

So by problem (10), the radius of convergence is 1

For $x = 1$, $\sum n^2 x^n = \sum n^2 = \infty$, diverge

For $x=-1$, $\sum n^2 x^n = \sum_{n=1}^{\infty} (-1)^n n^2 = \sum_{k=1}^{\infty} (2k)^2 - (2k-1)^2$

$= \sum_{k=1}^{\infty} 4k-1 = \infty$, diverges.

So the interval of convergence is $(-1,1)$

(b) $\lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)^2}{2^n/n^2} \right| = \lim_{n\to\infty} \left| \frac{2(n^2+2n+1)}{n^2} \right| = \lim_{n\to\infty} \left| 2+\frac{2}{n}+\frac{1}{n^2} \right|$

$=2$

So the radius of convergence is $\frac{1}{2}$

For $x=\frac{1}{2}$, $\sum \frac{2^n}{n^2} x^n = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$, converges.

For $x=-\frac{1}{2}$, $\sum \frac{2^n}{n^2} x^n = \sum \frac{(-1)^n}{n^2}$, converges by the alternating series test.

So the interval of convergence is $[-\frac{1}{2},\frac{1}{2}]$

(c) $\lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)!}{2^n/n!} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$

So the radius of convergence is ∞

the interval of convergence is $\mathbb{R}$.

(12) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$.

(a) Prove by induction that for all $n \in \mathbb{N}$ and $x \neq 0$, $f^{(n)}(x)$ exists and has the form $f^{(n)}(x) = p(\frac{1}{x})f(x)$ where p is a polynomial.

(b) Show that for every polynomial p, $\lim_{x\to 0} p(\frac{1}{x})f(x) = 0$. *Remark: you may freely use without proof the fact from calculus that* $\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$ *for every polynomial* p.

(c) Show by induction that $f^{(n)}(0)$ exists and is equal to 0 for all integers $n \geq 0$.

(d) Give an example of a C^∞ function g whose Taylor series expansion about 0 converges to g for all $x \leq 0$ and converges but *not* to g for all $x > 0$. *(No justification needed.)*

a) Proof We prove this by induction on $n \in \mathbb{N}$.

Base Case $n=1$, $f'(x) = (e^{-1/x^2})(-x^{-2})'$

$= (e^{-1/x^2})(2x^{-3})$

$= 2(\frac{1}{x})^3 f(x)$, the statement holds.

Inductive Step

Assume the statement holds true for n

So $f^{(n)}(x) = p(\frac{1}{x})f(x)$ for some polynomial of $\frac{1}{x}$:

$p(\frac{1}{x}) = \sum_{k=1}^{q} c_k (\frac{1}{x})^k$ where $q \in \mathbb{N}$ and $c_1, \dots, c_k$ are constants.

Then for $n+1$:

$f^{(n+1)}(x) = (f^{(n)}(x))' = f'(x)\, p(\frac{1}{x}) + f(x)\, p'(\frac{1}{x})$

$= f(x)\, p(\frac{1}{x}) + f(x) \sum_{k=1}^{q} -q c_k (\frac{1}{x})^{k+1}$

$= f(x)\left(p(\frac{1}{x}) + \sum_{k=1}^{q} -q c_k (\frac{1}{x})^{k+1}\right)$

is the product of $f(x)$ and a polynomial of $\frac{1}{x}$.

This finishes the proof by induction. $\square$

(b) Let p be an arbitrary polynomial of $\frac{1}{x}$

So $p(\frac{1}{x}) = \sum_{k=1}^{q} C_k (\frac{1}{x})^k$ where $q \in \mathbb{N}$ and $C_1, \dots, C_k$ are constants.

$$\lim_{x \to 0} p(\tfrac{1}{x}) f(x) = \lim_{x \to 0} \frac{\sum_{k=1}^{q} C_k (\frac{1}{x})^k}{e^{x^2}} = \sum_{k=1}^{q} \lim_{x \to 0} C_k \frac{e^{-x^2}}{x^k}$$

For each k, $\lim_{x \to 0} C_k \frac{e^{-x^2}}{x^k} = C_k \lim_{x \to 0} \frac{(-2x) e^{-x^2}}{x^{k-1}}$

$= C_k \lim_{x \to 0} \frac{(-2x)^k e^{-x^2}}{1}$ by L'Hopital's Rule (apply k times)

$= C_k \left(\lim_{x \to 0} (-2x)^k \right) \left(\lim_{x \to 0} (\frac{1}{e})^{x^2} \right)$

$= 0 \cdot 1 = 0$

So $\lim_{x \to 0} p(\frac{1}{x}) f(x) = \sum_{k=1}^{q} 0 = 0$

Since p is arbitrary, it proves that $\lim_{x \to 0} p(\frac{1}{x}) f(x) = 0$ for every polynomial p.

(c) We prove that $f^{(n)}(0)$ exists and $f^{(n)}(0) = 0$ for each $n \in \mathbb{N}$ by induction on n.

Base case $n=1$: $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} f(x) \cdot \frac{1}{x} = 0$ by (b), so the statement holds

Inductive step: Assume the statement holds for n

Then for $n+1$

$$f^{(n+1)}(0) = \frac{f^{(n)}(x) - f^{(n)}(0)}{x-0} = \frac{1}{x}\cdot f^{(n)}(x) = \lim_{x\to 0}\left(\frac{1}{x}\cdot p\left(\frac{1}{x}\right)\right) f(x)$$

for some polynomial p, $=0$ by (b)

This finishes the proof that $f^{(n)}(0)$ exists and $=0$ for all $n\in\mathbb{N}$

(d) $g(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases}$

(13) (a) For each $n \in \mathbb{N}$, define the function $f_n : (-1,1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -x - 2^{-n-1} & \text{if } -1 < x < 2^{-n} \\ 2^{n-1}x^2 & \text{if } -2^{-n} \leq x \leq 2^{-n} \\ x - 2^{-n-1} & \text{if } 2^{-n} < x < 1 \end{cases}$$

Show that each f_n is differentiable on $(-1,1)$, and that (f_n) converges uniformly to t
absolute value function on $(-1,1)$.

(b) For each $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = \dfrac{\sin(nx)}{n}$. Show that $(g$
converges uniformly on $\mathbb{R}$ to a differentiable function whose derivative is *not* $\lim_{n\to\infty} g_n'$

(14) Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$, and for each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \setminus \{q_n\} \to \mathbb{R}$ be the function defined by

$$f_n(x) \;=\; 4^{-n} \sin\left(\frac{1}{x - q_n}\right).$$

For each $x \in \mathbb{R} \setminus \mathbb{Q}$, let $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

(a) Prove that for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges. Thus $\mathrm{dom}(f) = \mathbb{R} \setminus \mathbb{Q}$.

(b) Prove that f is continuous.

(c) Prove that for every $q \in \mathbb{Q}$, $\lim_{x \to q} f(x)$ does not exist. (cf: HW 6, #17)