(1) Do Challenge Problem (14) from HW 2: if (a_n) is a sequence in $\mathbb{R}$ and $\lim(a_{n+1} - a_n) = 0$, must (a_n) converge? Justify your answer.

No.

Counterexample. $a_n = \sqrt{n}$

So $\lim_{n\to\infty}(a_{n+1} - a_n) = \lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$

$$= \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \infty$

(2) Let (a_n) be a sequence in $\mathbb{R}$, and let $S \subseteq \mathbb{R}$ be its set of real subsequential limits. Prove that S is closed.

Proof Let $c \in S'$ be arbitrary

(WTS: $c \in S$, that is, there exists a subsequence of $\{a_n\}$ that converges to c)

Below we will construct a subsequent $\{b_m\}$ of $\{a_n\}$ that converges to c

Let $m \in \mathbb{N}$

Since $c \in S'$, $V_{\frac{1}{2m}}(c) \cap S \setminus \{c\} \neq \emptyset$

let $x \in V_{\frac{1}{2m}}(c) \cap S \setminus \{c\}$, then $x \in S$ and $|x - c| \leq \frac{1}{2m}$

So there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ s.t. $\{a_{n_k}\} \to x$ as $k \to \infty$

($n \xmapsto{g} n_k$ is monotonely increasing)

So $\exists K \in \mathbb{N}$ s.t. $\forall k \geq K$, $|a_{n_k} - x| \leq \frac{1}{2m}$

Construction (recursively), if $m=1$, we choose a_{n_K} as b_m

if $m \geq 1$, then $b_{m-1} = a_{n_{k_0}}$ for some $k_0 \in \mathbb{N}$

then we take $k = \max\{K, k_0\} + 1$

and choose a_{n_k} as b_m

Then $|b_m - c| \leq |b_m - x| + |x - c| \leq \frac{1}{m}$

Note that $\boxed{\{b_m\} \text{ is a subsequence of } \{a_n\}}$ since every term of $\{b_n\}$ is some term of $\{a_n\}$ with increasing index

Proof of $\lim b_m = c$

Now let $\varepsilon > 0 \Rightarrow \varepsilon > \frac{1}{n}$ for some $n \in \mathbb{N}$

Consider $N = n+1 \Rightarrow$ by our construction of $\{b_m\}$, $|b_m - c| \leq \frac{1}{m+1}$ for all $m \geq N$

So $\lim_{m\to\infty} b_m = c$

Here we have proved: c is a subsequential limit of $\{a_n\}$ $\Rightarrow c \in S$

Since c is arbitrary, $S' \subseteq S$

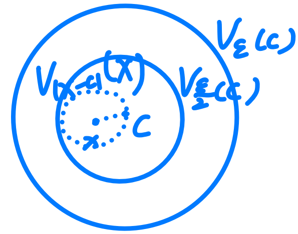
Therefore S is closed

(3) Given $A \subseteq \mathbb{R}$, write A' for the set of all limit points of A and define the *closure* of A to be the set $\text{cl}(A) = A \cup A'$.

(a) Prove that A' is closed.

(b) Prove that $\text{cl}(A)$ is closed.

(c) Prove that $\text{cl}(A)$ is the "smallest" closed set containing A, in the sense that $\text{cl}(A) \subseteq F$ for every closed set F containing A.



Proof

(a) Let $c \in (A')^c$

So c is not a limit point of A

Then for some $\varepsilon > 0$, $V_\varepsilon(c) \cap A \setminus \{c\} = \emptyset$

Let $x \in V_{\frac{\varepsilon}{2}}(c)$ be arbitrary (we have $|x-c| < \frac{\varepsilon}{2}$)

Consider $V_{|x-c|}(x)$, we have $V_{|x-c|}(x) \cap A = \emptyset$

So $x \notin A'$ which implies $x \in (A')^c$

Since x is arbitrary $\Longrightarrow V_{\frac{\varepsilon}{2}}(c) \subseteq (A')^c$

Since c is arbitrary $\Longrightarrow (A')^c$ is open

Therefore A' is closed

(b) Let $c \in (\text{cl}(A))^c$

So $c \notin A$ and $c \notin A'$

We can fix $\varepsilon > 0$ s.t. $V_\varepsilon(c) \cap A \setminus \{c\} = \emptyset$

Since $c \notin A \Longrightarrow V_\varepsilon(c) \cap A = \emptyset$

Let $x \in V_{\frac{\varepsilon}{2}}(c) \Longrightarrow x \notin A$ and $V_{\frac{\varepsilon}{2}}(x) \subseteq V_\varepsilon(c)$

so $V_{\frac{\varepsilon}{2}}(x) \cap A = \emptyset \Longrightarrow x \notin A'$

So $x \notin cl(A) \Rightarrow x \in (cl(A))^c$

Since x is arbitrary $\Rightarrow V_{\frac{\varepsilon}{2}}(c) \subseteq (cl(A))^c$

Since c is arbitrary $\Rightarrow (cl(A))^c$ is open

Therefore $cl(A)$ is closed.

(c) Let F be a closed set s.t. $A \subseteq F$

Let $a \in A'$ be arbitrary.

Let (a_n) be a sequence in A that converges to a

Since $A \subseteq F \Rightarrow \lim a_n = a \in F$

Since a is arbitrary $\Rightarrow A' \subseteq F$

So $cl(A) = A' \cup A \subseteq F$

Since F is arbitrary, this finishes the proof that $cl(A)$ is the smallest closed set containing A.

(4) (a) Prove explicitly using the ϵ/δ definition that $\lim_{x \to 2} x^3 = 8$.

(b) Given $\epsilon > 0$, find the *largest* $\delta > 0$ such that $|x^3 - 8| < \epsilon$ whenever $|x - 2| < \delta$.

(c) Prove explicitly using the ϵ/δ definition that $\lim_{x \to 4} \sqrt{x} = 2$.

(d) Given $\epsilon > 0$, find the *largest* $\delta > 0$ such that $|\sqrt{x} - 2| < \epsilon$ whenever $|x - 4| < \delta$.

(a) Proof

Let $\varepsilon > 0$

$|x^3 - 8| = |x^3 - 2^3| = |x-2||x^2+2x+4|$

For $1 < x < 3$, $|x^2+2x+4| = |(x+1)^2+3| \in \underbrace{[3, 19)}$

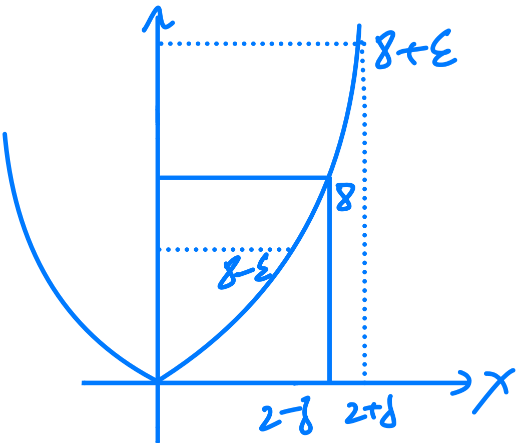
So consider $\delta = \min\{1, \frac{\varepsilon}{19}\}$

Then for $0 < |x-2| < \delta \Rightarrow$

$$|x^3-8| = |x-2||x^2+2x+4| < \delta \cdot 19 < \varepsilon$$

Since ε is arbitrary, this finishes the proof that $\lim_{x \to 2} x^3 = 8$

(b)



For any $\varepsilon > 0$,

We want: $(2-\delta)^3 \geq 8-\varepsilon$, $(2+\delta)^3 \leq 8+\varepsilon$

$2-\delta \geq \sqrt[3]{8-\varepsilon}$ $\quad$ $2+\delta \leq \sqrt[3]{8+\varepsilon}$

$\delta \leq 2-\sqrt[3]{8-\varepsilon}$ and $\delta \leq \sqrt[3]{8+\varepsilon}-2$

So the largest δ is $\min(2-\sqrt[3]{8-\varepsilon}, \sqrt[3]{8+\varepsilon}-2)$

$$= \underbrace{\sqrt[3]{8+\varepsilon}-2}$$

(c) <u>Proof</u>

Let $\varepsilon > 0$

$$|\sqrt{x}-2| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \frac{|x-4|}{|\sqrt{x}+2|}$$

where $|\sqrt{x}+2| \geq 2$

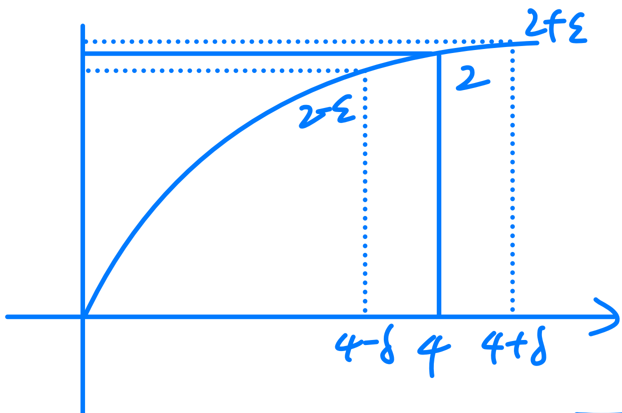
So consider $\delta = \varepsilon$

Then suppose $0 < |x-4| < \delta$

$$\Rightarrow |\sqrt{x}-2| = \frac{|x-4|}{|\sqrt{x}+2|} = \frac{\delta}{|\sqrt{x}+2|} \leq \frac{\delta}{2} < \varepsilon$$

Since ε is arbitrary, this finishes the proof that $\lim_{x\to4}\sqrt{x} = 2$

(4)



For any $\varepsilon > 0$, we want: $\sqrt{4+\delta} \leq 2+\varepsilon$, $\sqrt{4-\delta} \geq 2-\varepsilon$

$$\Rightarrow \delta \leq (2+\varepsilon)^2-4, \quad \delta \leq (2-\varepsilon)^2+4$$

So $\delta \leq \min\{(2+\varepsilon)^2-4, (2-\varepsilon)^2+4\} = (2+\varepsilon)^2-4$

So the largest δ is $\underline{(2+\varepsilon)^2-4}$

(5) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, suppose that $a \in \mathbb{R}$ is a limit point of $A \cap (a, \infty)$, and suppose $\lim_{x \to a^+} f(x) = \infty$. Also let $c \in \mathbb{R}$, let $g : (c, \infty) \to \mathbb{R}$ be a function, and suppose that $\lim_{x \to \infty} g(x) = L \in \mathbb{R}$. Prove that $\lim_{x \to a^+} (g \circ f)(x) = L$.

Proof Let $\varepsilon > 0$

Then $\exists N \in \mathbb{R}$ s.t.

$|g(x) - L| < \varepsilon$ whenever $x \geq N$

And $\exists \delta > 0$ s.t. $f(x) \geq N$ whenever $0 < x - a < \delta$

since $\lim_{x \to a^+} f(x) = \infty$

Then fix that δ

Let $a < x < a + \delta \implies$

$f(x) \geq N \implies |g(f(x)) - L| < \varepsilon$

Since x, ε is arbitrary, this finishes the proof that $\lim_{x \to a^+} (g \circ f)(x) = L$. $\square$

(6) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, let $a \in \mathbb{R}$, and suppose $\lim_{x \to a} f(x) = b$ and $\lim_{x \to b} g(x) = L$. Show by example that L need not be the limit of $g \circ f$ as $x \to a$.

example

Consider $f(x) = \begin{cases} 0, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$

So $\lim_{x \to 1} f(x) = 0$

$g(x) = \begin{cases} 2, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$

So $L = \lim_{x \to 0} g(x) = 0$

But $g(f(x)) = \begin{cases} 2, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1, \end{cases}$

$\lim_{x \to 1} g(f(x)) = 2 \neq 0$

(7) Prove that for any sequence (a_n) of nonzero real numbers, $\limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

Hint: Let $L > \limsup \left|\frac{a_{n+1}}{a_n}\right|$ be arbitrary; then there is N such that $\left|\frac{a_{n+1}}{a_n}\right| < L$ for all $n \geq N$; now use the fact that for any $n > N$,

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|$$

as a first step towards showing that $\limsup |a_n|^{1/n} \leq L$.

Proof Let $L > \limsup \left|\frac{a_{n+1}}{a_n}\right|$ be arbitrary

Then $\exists N \in \mathbb{N}$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < L$ whenever $n \geq N$

Let $n \geq N$ be arbitrary

So $|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|$

$< L^{n-N} |a_N|$

So $|a_n|^{\frac{1}{n}} < L^{\frac{n-N}{n}} |a_N|^{\frac{1}{n}} = L \cdot \left(\sqrt[n]{L^{-N} |a_N|}\right)$

Note that $L^{-N}|a_N|$ is a constant

So $\lim_{n\to\infty} \sqrt[n]{L^{-N}|a_N|} = 1$

Then $\limsup L\cdot(\sqrt[n]{L^{-N}|a_N|}) = \lim_{n\to\infty} L(\sqrt[n]{L^{-N}|a_N|}) = L$

Since $\forall n \geq N$, $|a_n|^{\frac{1}{n}} < L\cdot(\sqrt[n]{L^{-N}|a_N|})$,

$\Rightarrow \limsup |a_n|^{\frac{1}{n}} < \limsup (L\sqrt[n]{L^{-N}|a_N|}) = L$

Above shows that for any $L > \limsup\left|\frac{a_{n+1}}{a_n}\right|$,

we have $L > \limsup |a_n|^{\frac{1}{n}}$

Therefore we can conclude that $\limsup |a_n|^{\frac{1}{n}} \leq \limsup\left|\frac{a_{n+1}}{a_n}\right|$

(8) Let $A \subseteq \mathbb{R}$, suppose $a \in A \cap A'$, and let $f : A \to \mathbb{R}$ be a function. Prove that if $f(a) > 0$ and f is continuous at a, then there is $\epsilon > 0$ such that f is positive and bounded on $A \cap V_\epsilon(a)$.

Proof Since f is continuous at A,

there exists $\varepsilon > 0$ s.t. $|f(a) - f(x)| < f(a)$ whenever $|a - x| < \varepsilon$ and $x \in A$

So $0 < f(x) < 2f(a)$ whenever $x \in V_\varepsilon(a) \cap A$

(Since $a \in A'$, $V_\delta(a) \cap A \setminus \{a\} \neq \phi$)

We can conclude that f is positive and bounded on $A \cap V_\varepsilon(a)$.

$\square$

(9) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. Prove that if $f(x) = g(x)$ for all $x \in \mathbb{Q}$, then $f = g$.

Proof Let $a \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary

Let $\varepsilon > 0$ be arbitrary

Since f is continuous on $\mathbb{R}$, $\exists \delta > 0$ s.t.

$$|f(x) - f(a)| < \frac{\varepsilon}{2} \text{ whenever } x \in V_\delta(a)$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ s.t. $q \in V_\delta(a)$

So $|f(a) - f(q)| < \frac{\varepsilon}{2}$

Similarly we have $|g(q) - g(a)| < \frac{\varepsilon}{2}$

Since $q \in \mathbb{Q} \Rightarrow f(q) = g(q)$

So $|f(a) - g(a)| \leq |f(a) - f(q)| + |f(q) - g(a)| < \varepsilon$

Since ε is arbitrary, we have $f(a) = g(a)$

Since $a \in \mathbb{R} \setminus \mathbb{Q}$ is arbitrary, we have

$$f(x) = g(x) \text{ for } x \in \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$$

Therefore $f = g$.

$\square$

(10) Prove that if $A \subseteq \mathbb{R}$ is not closed, then there is an unbounded continuous function $f : A \to \mathbb{R}$.

Proof

Suppose $A \subseteq \mathbb{R}$ is not close

then there exists some $c \in A'$ s.t. $c \notin A$

We then consider the function

$f: A \to \mathbb{R}$ defined by $f(x) = \frac{1}{|x-c|}$

This function is well defined since $c \notin A \Rightarrow x-c \neq 0$

And it is continuous since $f_1(x) = \frac{1}{x-c}$ is rational,

is continuous everywhere and $f_2(x) = |x|$ is also

continuous everywhere, so $f(x) = f_2 \circ f_1(x)$ is

continuous everywhere.

Let $M \in \mathbb{N}$ be arbitrary

Since $c \in A'$, $V_{\frac{1}{M}}(c) \cap A \setminus \{x\} \neq \emptyset$

So there is some $x \in A$ s.t. $0 < |x-c| < \frac{1}{M}$

Then $c - \frac{1}{M} < x < c + \frac{1}{M} \Rightarrow |\frac{1}{x-c}| > M$

Since M is arbitrary, we have proved that

f is unbounded

Therefore for all $A \subseteq \mathbb{R}$

that is not closed, we can find $f: A \to \mathbb{R}$

s.t. f is unbounded and continuous.

$\square$

(11) Using only the definitions of continuity and open set, prove that for any function $f : \mathbb{R} \to \mathbb{R}$, f is continuous if and only if $f^{-1}[V]$ is open for every open set $V \subseteq \mathbb{R}$.

Proof ($\Rightarrow$ direction)

Suppose f is continuous.

Let $V \subseteq \mathbb{R}$ be an open set, so

$$f^{-1}[V] = \{x \in \mathbb{R} \mid f(x) \in V\}$$

Let $x \in f^{-1}[V]$, so $f(x) \in V$

Since V is open, there is some $\varepsilon > 0$ s.t. $V_\varepsilon(f(x)) \subseteq V$

Since f is continuous, there exists $\delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$

So $\forall y \in V_\delta(x)$, $f(y) \in V_\varepsilon(f(x)) \subseteq V \Rightarrow y \in f^{-1}[V]$

Therefore $V_\delta(x) \subseteq f^{-1}[V]$

Since x is arbitrary, this shows that $f^{-1}[V]$ is open

Since V is arbitrary, it is proved that $f^{-1}[V]$ is open for every open set $V \subseteq \mathbb{R}$ if f is continuous

($\Leftarrow$ direction):

Suppose $f^{-1}[V]$ is open for every open set $V \subseteq \mathbb{R}$

Let $x \in \mathbb{R}$ be arbitrary

Let $\varepsilon > 0$

Consider the set $V = \{y \in \mathbb{R} \mid |f(x) - y| < \varepsilon\} = V_\varepsilon(f(x))$

Then $f^{-1}[V]$ is open

Since $|f(x)-f(x)|=0<\varepsilon \Longrightarrow x \in V$

So $\exists \delta>0$ s.t. $V_\delta(x) \subseteq f^{-1}[V]$

i.e. for all $a \in \mathbb{R}$ s.t. $|x-a|<\delta$, we have $|f(a)-f(x)|<\varepsilon$

So $f(x)$ is continuous at a

Since $a \in \mathbb{R}$ is arbitrary $\Longrightarrow f$ is continuous.

$\square$

(12) Suppose $A \subseteq \mathbb{R}$ is closed, and let $f: A \to \mathbb{R}$ be a continuous function. Prove that there is a continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $g \restriction A = f$.

(13) Suppose $\{U_i : i \in I\}$ is a family of nonempty open sets in $\mathbb{R}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Prove that I is countable.

(14) Let $d(x, y) = |x - y|$ be the usual metric on $\mathbb{R}$, and let $\mathcal{T}$ be the metric topology on $\mathbb{R}$ generated by d, so $\mathcal{T}$ consists of all open subsets of $\mathbb{R}$.[3]

(a) Prove that for any subset $A \subseteq \mathbb{R}$, A is open if and only if A can be expressed as a union of countably many open intervals in $\mathbb{R}$.

(b) Is it true that every open set A in $\mathbb{R}$ can be expressed as a union of open intervals *with rational endpoints*? Either give a counterexample if not, or else briefly explain how your proof in (a) could be modified in order to prove this stronger result.