## I. Review

$\overline{\mathbb{Q}}$ : algerbraically closed

$\mathbb{R}$ : geometrically closed

but $\overline{\mathbb{Q}} \not\subseteq \mathbb{R}$, $\mathbb{R} \not\subseteq \overline{\mathbb{Q}}$

Question: can we find a both-closed field?

Answer: yes. $\Rightarrow \mathbb{C}$

$\mathbb{C}$ is both algebraically and geometrically closed (topologically)

however: $\mathbb{C}$ is not an ordered field

(hw: impossible to define linear order on $\mathbb{C}$)

* Question: our completeness axiom of $\mathbb{R}$ is based on order: so how can $\mathbb{C}$ be geoly complete?

Answer: we can define the axiom without order $\Rightarrow$ $\boxed{\text{Cauchy seq.}}$ (next week)

### Dual ver. of completeness axiom

$\forall A \subseteq \mathbb{R}$ $(A \neq \emptyset)$

if $A$ is bounded below $\Rightarrow \exists \inf A \in \mathbb{R}$

Pf1 Assume the conditions.

Let $L = \{\text{all LBs of } A\}$

By completeness axiom $\Rightarrow \sup L \in \mathbb{R}$

WTS: $\sup L = \inf A$

Pf2 def $-A = \{-a \mid a \in A\}$

so $-A \neq \emptyset$ and since $A$ is bounded below, $-A$ is bounded above.

WTS: $\inf A = -\sup(-A)$

### Lec3. Useful facts ① 判断 sup 的方法/性质

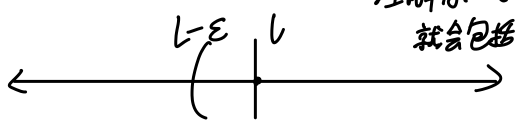
$\forall A \subseteq \mathbb{R}$ and $l \in \mathbb{R}$

if $l$ is an UB of $A$

$\Rightarrow l = \sup A$ iff

$\forall \varepsilon > 0, \exists a \in A$ s.t. $\boxed{l - \varepsilon < a \leq l}$

理解为: 只要下移一点点, 就会包括进




(Wrong) Def (by Newton/Lebniz): infinitesimal

$\varepsilon > 0$ is an infinitesimal if $(\forall n \in \mathbb{N},\ n\varepsilon < 1)$

Question: 存在 infinitesimal 吗?

Answer: depends on def of "$\mathbb{R}$"

acc. to our axiom 1-15 $\Rightarrow$ $\boxed{\text{no!}}$

现在我们 prove 它不存在, by Archimedean property of $\mathbb{R}$

### Lec3 Lemma. ordered field 包含 $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Let $\mathbb{F}$ be an order field ☆ (任意, 不止 $\mathbb{R}$)

then $\mathbb{F}$ must contain a copy of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$

Pf

$1_{\mathbb{F}} \Rightarrow 2_{\mathbb{F}} = 1_{\mathbb{F}} + 1_{\mathbb{F}}$

$3_{\mathbb{F}} = 1_{\mathbb{F}} + 1_{\mathbb{F}} + 1_{\mathbb{F}}$

$\vdots$ $\quad \mathbb{N}$

$0_{\mathbb{F}} \Rightarrow -1_{\mathbb{F}} = 0 - 1_{\mathbb{F}}$

$-2_{\mathbb{F}} = 0 - 1_{\mathbb{F}} - 1_{\mathbb{F}}$

$\vdots$ $\quad \mathbb{Z}$

$\left(\frac{p}{q}\right)_{\mathbb{F}} = \frac{p_{\mathbb{F}}}{q_{\mathbb{F}}}$ $\quad \mathbb{Q}$

## II. Archimedean properties (五条等价)

(1) $\forall x \in \mathbb{F}$, $\exists n \in \mathbb{N}$ s.t. $x < n$

显然等价

(2) $\forall x > 0$ in $\mathbb{F}$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$

(3) $\forall x \in \mathbb{F}$, $\exists!\, n \in \mathbb{Z}$ s.t. $n - 1 \leq x < n$

unique

(4) (Archimedean property) (Archimedean 性: "勺子填海")

$\forall x, y > 0 \in \mathbb{F}$, $\exists n \in \mathbb{N}$ s.t. $ny > x$

(4) $\Rightarrow$ (1) since can take $y=1$

But also (1) $\Rightarrow$ (4) since given $x,y>0$

by (1), can find $n>\frac{x}{y}$

(5) "density of $\mathbb{Q}$": $\forall x,y\in\mathbb{F}$ s.t. $x<y$

$\mathbb{Q}$在$\mathbb{F}$中稠密性; $\Rightarrow \exists r\in\mathbb{Q}$ s.t. $x<r<y$

任意两数间总有一个有理数)

(3)+(4) imply (5): $x<y\in\mathbb{F}\Rightarrow y-x>0$

$\Rightarrow$ by (4) $\exists n\in\mathbb{N}$, $n(y-x)>2$; By (3), fix $m\in\mathbb{Z}$

s.t. $m\le ny<m+1 \Rightarrow nx<m-1<ny\Rightarrow x<\frac{m-1}{n}<y$

满足 (1)–(5) properties (因为等价,所以其一即可)

的 ordered field 称为 archimedean 的.

$\mathbb{R}$, $\mathbb{Q}$都是 Archimedean 的

而也有不 Archimedean 的 ordered field: (因为没有 $\frac{1}{n}<\frac{1}{+\infty}$)

比如 ${}^*\mathbb{R}=\mathbb{R}\cup\{\pm\infty\}$ 不是 Archimedean 的

我们可以发现: 尽管每个 ordered field 都含了一个 $\mathbb{Q}$

且 $\mathbb{Q}$ 是 archimedean 的,

但却并不是每个 ordered field 都是 archimedean 的

多出的元素可以破坏 archimedean 性

其它例子: $\mathbb{R}(x)$ (all real functions); $p$-adic fields $\mathbb{Q}_p$

**Thm** $\mathbb{R}$是一个 Archimedean ordered field.

Pf. Suppose $x\in\mathbb{R}$ 且 $\nexists n\in\mathbb{N}$ s.t. $x<n$ (for contradiction)

$\Rightarrow x$ 为 $\mathbb{N}\subseteq\mathbb{R}$ 的一个 UB

$\Rightarrow$ By the completeness axiom, $\exists \sup\mathbb{N}\in\mathbb{R}$

Take $\sup\mathbb{N}-1\Rightarrow$ not an UB of $\mathbb{N}$

So $\exists m\in\mathbb{N}$ s.t. $\sup\mathbb{N}-1<n$

$\Rightarrow \sup\mathbb{N}<n+1$

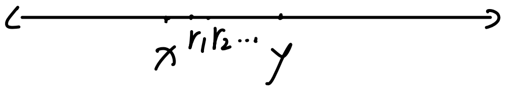
But $n+1\in\mathbb{N}\Rightarrow$ contradicts $y=\sup\mathbb{N}$ $\square$

尽管 infinitesimal 在 real line 上不存在,

there is a consistent and rigorous way to do calculus with infinitesimals (non-standard analysis)

The density of $\mathbb{Q}$ implies: $\forall x<y\in\mathbb{R}$,

$\exists$ inf many rational pts between $x,y$




Note: $\mathbb{R}\setminus\mathbb{Q}$ is also dense in $\mathbb{R}$ (hw1)

## III. Absolute value & metric space

(def 略)

Basic Properties of absolute value:

(i) $-|a|\le a\le|a|$

(ii) $|a|=\sqrt{a^2}$, $a^2=|a|^2$

(iii) $|ab|=|a||b|$

(iv) $|a+b|\le|a|+|b|$

$|a-b|\le|a|-|b|$ triangular ineqs.

$||a|-|b||\le|a-b|$

Pf of triangular ineqs.

$$|a+b|^2=(a+b)^2=a^2+2ab+b^2 \le a^2+2|ab|+b^2 = a^2+2|a||b|+b^2=(|a|+|b|)^2$$

So $|a+b|\le|a|+|b|$

extended triangular ineq.

$\forall a_1,\dots,a_n\in\mathbb{R}$, $\boxed{\left|\sum_{i=1}^n a_i\right|\le\sum_{i=1}^n|a_i|}$

**Def** Triangular properties, metric, metric space

对于一个集合$X$, 定义一个函数 $\boxed{d: X\times X\to\mathbb{R}}$

如果其满足:

(i) $d(a,b)\ge0$, and $d(a,b)=0$ iff $a=b$ (i): 正定

(ii) $d(a,b)=d(b,a)$ (ii): 对称

(iii) $d(a,b)+d(b,c)\ge d(a,c)$ (iii): 三角.

这三条被称为 triangular properties

则称 $d$ 为一个 metric on $X$ 度量

且称 $X$ 为一个 metric space 度量空间.

因而以 absolute value 作为 metric, $\mathbb{R}$ 是一个 metric space

**Thm** $\forall k \in \mathbb{Z}_{\geq 1}$, $\mathbb{R}^k$ is a metric space
by metric: $\boxed{d(\vec{x}-\vec{y}) = \|\vec{x}-\vec{y}\|}$ (norm)

(where $\mathbb{R}^k = \{\vec{x} = (x_1, x_2, \dots, x_k) \mid x_i \in \mathbb{R}, 1 \leq i \leq k\}$
It has an inner product: $\forall \vec{x}, \vec{y} \in \mathbb{R}^k$, $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$)

Pf of (i) 正定: $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} > 0$
if $x \neq y$ ($=0$ if $x = y$)

Pf of (ii) 对称: $\sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{k} (y_i - x_i)^2} = d(\vec{y}, \vec{x})$

Pf of (iii) 三角:

It first needs Schwartz inequality:

$\forall \vec{x}, \vec{y} \in \mathbb{R}^k$, $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ $(*)$

Pf of $*$:

$\forall \lambda \in \mathbb{R}$, $(\lambda\vec{x} - \vec{y})^2 \geq 0$

$\Rightarrow \lambda^2 \vec{x} \cdot \vec{x} - 2\lambda \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y} \geq 0$

$\Rightarrow \lambda^2 \|\vec{x}\|^2 - 2\lambda \vec{y} \cdot \vec{x} + \|\vec{y}\|^2 \geq 0$

Take $\lambda = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}$ if $x \neq 0$

$\Rightarrow \dfrac{(\vec{x} \cdot \vec{y})^2}{\|\vec{x}\|^2} - \dfrac{2(\vec{x} \cdot \vec{y})^2}{\|\vec{x}\|^2} + \|\vec{y}\|^2 \geq 0$

$\Rightarrow \dfrac{(\vec{x} \cdot \vec{y})^2}{\|\vec{x}\|^2} \leq \|\vec{y}\|^2$

$\Rightarrow \|\vec{x}\|^2 \|\vec{y}\|^2 \geq (\vec{x} \cdot \vec{y})^2$

$\Rightarrow \|\vec{x}\| \|\vec{y}\| \geq \vec{x} \cdot \vec{y}$

现在WTS: $\|\vec{x} - \vec{y}\| \leq \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}\|$

Let $\vec{a} = \vec{x} - \vec{y}$, $\vec{b} = \vec{x} - \vec{z}$, $\vec{c} = \vec{z} - \vec{y}$

$\Rightarrow \vec{a} = \vec{b} + \vec{c}$

$(\|\vec{b}\| + \|\vec{c}\|)^2 = \|\vec{b}\|^2 + 2\|\vec{b}\|\|\vec{c}\| + \|\vec{c}\|^2$

$\|\vec{b} + \vec{c}\|^2 = (\vec{b} + \vec{c})(\vec{b} + \vec{c}) = \|\vec{b}\|^2 + 2\vec{b} \cdot \vec{c} + \|\vec{c}\|^2$

By $*$: $\|\vec{b}\|\|\vec{c}\| \geq \vec{b} \cdot \vec{c}$

$\Rightarrow (\|\vec{b}\| + \|\vec{c}\|)^2 \geq \|\vec{b} + \vec{c}\|^2$

$\Rightarrow \|\vec{b}\| + \|\vec{c}\| \geq \|\vec{a}\|$

i.e. $\|\vec{x} - \vec{z}\| + \|\vec{z} + \vec{y}\| \geq \|\vec{x} - \vec{y}\|$