Review

Def limit of function

$A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in A'$

称 $\lim\limits_{x\to c} f(x) = L$ if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x-c| < \delta$ and $x \in A$

因而 $\lim\limits_{x\to c} f(x)$ DNE if:

$\forall L \in \mathbb{R}, \exists \varepsilon > 0$ st $\forall \delta > 0$,

$(\exists x \in A$ st. $0 < |x-c| < \delta$ but $|f(x) - L| \geq \varepsilon)$

Fact: let $A = \{a_n \mid n \in \mathbb{N}\}$ be a set of a seq. in $\mathbb{R}$

$\Rightarrow \forall a \in A'$, a is a subseq. lim. of (a_n)

But backward direction is wrong

ex: const. seq.

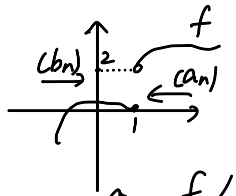
一个seq.构成的集合,任何lim pt.都是一个subseq.lim.

1. 使用 sequence 来判断 function 的 limit

review:

$\lim\limits_{x\to c} f(x) = L$ iff $\forall$ dom(f)$\setminus\{c\}$中 conv. to c 的 (a_n), $\lim\limits_{n\to\infty} f(a_n) = L$

ex

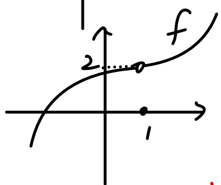


$(a_n), (b_n) \to 1$

$\lim f(a_n) = 2$, $\lim f(b_n) = 0$

因而 0,2 都不是 $x=1$ 处的 $\lim f(x)$

ex2



dom(f) $\setminus \{1\}$ 中任意 conv. to 1 的 seq, 其对应的 image 极限都为 2

因而 2 是 $x=1$ 处的 $\lim f(x)$

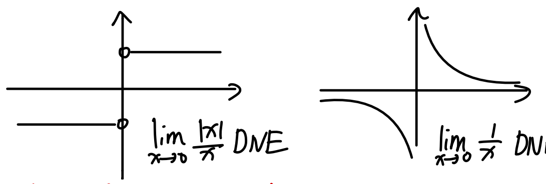
我们也可以发现,这一方法可检验一个function是否有limit 以及其值是否为limit.

Fact if $\exists$ seq. $(a_n) \in$ dom(f)$\setminus\{c\}$ s.t. $(a_n) \to c$ 但是 $f(a_n) \not\to L$ $\Rightarrow \lim\limits_{x\to c} f(x) \neq L$

Fact if $\exists$ seq. (a_n) in dom(f)$\setminus\{c\}$ st $(a_n) \to c$ 但 $f(a_n)$ diverges $\Rightarrow \lim\limits_{x\to c} f(x)$ DNE.

if $\exists$ seq. (a_n) & (b_n) in dom(f)$\setminus\{c\}$ st. (a_n) & $(b_n) \to c$ 但 $\lim f(a_n) \neq \lim f(b_n)$ $\Rightarrow \lim\limits_{x\to c} f(x)$ DNE

ex



2. limit of function 的更多形式

Def limit of function involving $\pm\infty$

① 写 $\lim\limits_{x\to c} f(x) = \infty$ if $\forall M > 0, \exists \delta > 0$ s.t. $f(x) > M$ whenever $x \in$ dom(f) and $0 < |x-c| < \delta$

即:无论M取多大,c点附近总有某个nbh的所有dom内的点比M更大

② 写 $\lim\limits_{x\to -\infty} f(x) = L$ if $\forall \varepsilon > 0, \exists N > 0$ s.t. (anything) $|f(x) - L| < \varepsilon$ whenever $x \in$ dom(f) and $x < -N$

即:无论ε取多小,总有一N使得对于所有小于-N的x, f(x)和L的dist都小于ε

③ 写 $\lim\limits_{x\to\infty} f(x) = \infty$ if $\forall M > 0, \exists N > 0$ s.t. $f(x) > M$ whenever $x > N$ and $x \in$ dom(f)

无论M取多大,总有一N使得对于所有小于-N的x, f(x)都比M大

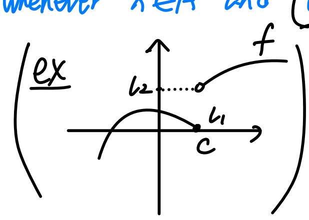
(and all dually)

Def One-side limit

Suppose $c \in \mathbb{R}$ is a lim. pt. of dom(f)$\cap (c, +\infty)$

如果 $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < x - c < \delta$

no abs value since from one side



则称 L 为 the limit of f(x) from the right, 写作 $\lim\limits_{x\to c^+} f(x) = L$

同理可define $\lim\limits_{x\to c^-} f(x) = L$ 及 $\lim\limits_{x\to c} f(x) = -\infty$ 等

$\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $f(x) < M$ whenever $0 < c - x < \delta$ ($x \in$ dom(f))

现在我们已define了:

$\lim\limits_{x\to \square} f(x) = \square$ — $L, +\infty, -\infty$

$c, c^-, c^+, +\infty, -\infty$

15 种 function limits

* Uniform definition of limit in terms of open sets.

Let $A \subseteq \mathbb{R}$; $f: A \to \mathbb{R}$; $c, l \in \mathbb{R} \cup \{\pm\infty\}$

Suppose $c \in A'$. ? (这里是一个 convention)

则称 $\lim\limits_{x\to c} f(x) = l$ if $\forall$ open nbh V of l

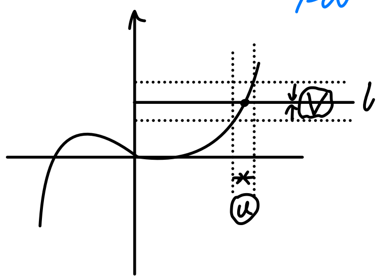
$\exists$ an open nbh U of c s.t.

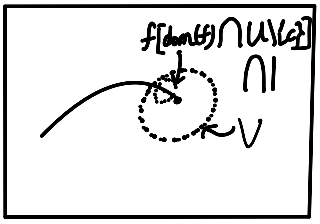
$$f[(A \cap U)\setminus\{c\}] \subseteq V$$

Conventions:

① 如果 A is bounded above/below, 我们说 $+\infty$/$-\infty$ $\in A'$ (lim. pt.)

② open nbhs of $+\infty$/$-\infty$ 的意思是 $(a, +\infty)$/$(-\infty, a)$





Thm $\lim\limits_{x\to c} f(x) = l$ iff $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x) = l$

(前提: c is a lim. pt. of both $(-\infty, c) \cap \mathrm{dom}(f)$ and $(c, +\infty) \cap \mathrm{dom}(f)$)

pf: ($\Rightarrow$) is immediate (by definition)

($\Leftarrow$): Let $\varepsilon > 0$. fix $\delta_1, \delta_2 > 0$ s.t.

$\forall x \in \mathrm{dom}(f)$, $|f(x) - l| < \varepsilon$ whenever

$0 < x - c < \delta_1$, $0 < c - x < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$,

$\Rightarrow \forall\ 0 < |x - c| < \delta$ and $x \neq c$, $|f(x) - l| < \varepsilon$

Thm ($A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in A'$)

TFAE (以下等价):

(i) $\lim\limits_{x\to c} f(x) = l$

(ii) $\lim\limits_{x\to c} (f(x) - l) = 0$

(iii) $\lim\limits_{x\to c} |f(x) - l| = 0$

(iv) $\lim\limits_{h\to 0} f(c+h) = l$ (换变量)

pf: exercise.