(1) Let (a_n) and (b_n) be bounded sequences in $\mathbb{R}$, and suppose $\lim a_n = A > 0$. Show that $\limsup(a_n b_n) = A \cdot \limsup(b_n)$. *Hint: recall that* $\limsup(s_n)$ *is the largest subsequential limit of* (s_n).

Pf. Let E denote the set of all subsequential limits of $(a_n b_n)$

So $\limsup(a_n b_n) = \max E$

Write $\limsup(b_n) = b$

Claim ① Ab is an upper bound for E.

Let $(a_{n_k} b_{n_k})$ be an arbitrary convergent subsequence of $(a_n b_n)$

So $\lim_{k\to\infty} a_{n_k} b_{n_k} \le \lim_{k\to\infty} a_{n_k} (\sup_{n\ge k} b_n) =$

$\left(\lim_{k\to\infty} a_{n_k}\right)\left(\lim_{k\to\infty} \sup_{n\ge k} b_n\right) = \left(\lim_{n\to\infty} a_n\right)\left(\limsup b_n\right) = Ab$

So Ab is an upper bound for E since $(a_n b_n)$ is arbitrary.

Claim ② $Ab \in E$

Consider (b_{n_m}) be a sebsequence of (b_n) s.t. $(b_{n_m}) \to b$.

So $\lim_{m\to\infty} a_{n_m} b_{n_m} = (\lim_{m\to\infty} a_{n_m})(\lim_{m\to\infty} b_{n_m}) = Ab \in E$

By claim ①②, $Ab = \max E$

Therefore $Ab = \limsup(a_n b_n)$

□

hw6 ① 如果 $(a_n) \to A > 0$, (b_n) is bounded

那么 $\limsup(a_n b_n) = A \limsup(b_n)$

(2) (a) For each $n \in \mathbb{N}$, find the nth derivative of the function $y = x^n$, and prove by induction that your claim is correct.

hw 6 ②

$$\boxed{\frac{d^n}{dx^n}(x^n) = n!}$$

(b) For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^n & \text{if } x \geq 0; \\ -x^n & \text{if } x < 0. \end{cases}$$

Show that f_{n+1} is n-times differentiable but not $(n+1)$-times differentiable.

(a) Proof the n^{th} derivative of $y=x^n$ is $y=n!$

We prove it by induction on $n \in \mathbb{N}$

Base case $n=1$, $\frac{d}{dx}(x) = 1 = 1!$

Inductive Step Assume the statement stands with n

Then $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) = \frac{d^n}{dx^n}\left(\frac{d}{dx}(x^n \cdot x)\right)$

$= \frac{d^n}{dx^n}\left(x^n \cdot 1 + nx^{n-1} \cdot x\right) = \frac{d^n}{dx^n}\left((n+1)x^n\right)$

$= (n+1)\frac{d^n}{dx^n}(x^n) = (n+1)n! = (n+1)!$

This finishes the proof of $\frac{d^n}{dx^n}(x^n) = n!$ for all $n \in \mathbb{N}$ by inductition

(b) For each $n \in \mathbb{N}$

$$f_{n+1}(x) = \begin{cases} x^{n+1}, & x \geq 0 \\ -x^n, & x < 0 \end{cases}$$

We have known that f_{n+1} is n-times differentiable on $\mathbb{R} \setminus \{0\}$ since it is rational respectively, with $f_{n+1}^{(n)}(x) = \begin{cases} (k+1)!\,x, & x > 0 \\ -(k+1)!\,x, & x < 0 \end{cases}$

So it suffices to prove that f_{n+1} is n-time differentiable but not $(n+1)$-time differentiable at $x=0$

$$\lim_{x\to 0^+}\frac{f_{n+1}^{(n-1)}(x)-f_{n+1}^{(n-1)}(0)}{x-0}=\lim_{x\to 0^+}\frac{\frac{(n+1)!}{2}x^2-0}{x}=0$$

$$=\lim_{x\to 0^-}\frac{\frac{(n+1)!}{2}x^2-0}{x}=\lim_{x\to 0^-}\frac{f_{n+1}^{(n-1)}(x)-f_{n+1}^{(n-1)}(0)}{x-0}$$

So $f^{(n)}(0)=0$ by definition

Therefore f_{n+1} is n-times differentiable on $\mathbb{R}$ ①

But $\displaystyle\lim_{x\to 0^+}\frac{f_{n+1}^{(n)}(x)-f_{n+1}^{(n)}(0)}{x-0}=\frac{\frac{(n+1)!}{2}x-0}{x-0}=\frac{(n+1)!}{2}>0$

But $\displaystyle\lim_{x\to 0^-}\frac{f_{n+1}^{(n)}(x)-f_{n+1}^{(n)}(0)}{x-0}=\frac{(n+1)!}{2}<0$

So $\displaystyle\lim_{x\to 0}\frac{f_{n+1}^{(n)}(x)-f_{n+1}^{(n)}(0)}{x-0}$ does not exists, which shows that f_{n+1} is not $(n+1)$-times differentiable at $x=0$

therefore f_{n+1} is not $(n+1)$-times differentiable, which finishes the proof. ② $\square$

(3) Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function. Prove by induction that for all $n\geq 0$, if $f:\mathbb{R}\to\mathbb{R}$ is $(n+1)$-times differentiable and $f^{(n+1)}(x)=0$ for all $x\in\mathbb{R}$, then f is a polynomial of degree at most n.

hw 6 ③ 如果 f 是 $(n+1)$-times diff'ble, 且 $f^{(n+1)}(x)=0$
那么 f 为一个 degree $\leq n$ 的 polynomial

Proof We prove by induction on $n \in \mathbb{N}$ that if $f: \mathbb{R} \to \mathbb{R}$ is $(n+1)$-times differentiable and $f^{(n+1)}(x)=0$ for all $x \in \mathbb{R}$, the f is polynomial of degree at most n.

Base Case: $n=0$. Assume the hypothesis
Since f is differentiable $\Rightarrow$ continuous and $f'(x)=0$ for all $x \in \mathbb{R}$, we have $f(x)=c$ for some $c \in \mathbb{R}$, which is a polynomial of degree 0.

Inductive Step. Assume the hypothesis and suppose the statement holds for n,
Then for $n+1$, since $f'^{(n+1)}(x)=0$ for all $x \in \mathbb{R}$ and f is $(n+1)$-times differentiable, we have $f'(x)$ is n-times differentiable and $(f')^{(n)}(x)=0$ for all $x \in \mathbb{R}$
So $f'(x)$ is a polynomial of degree at most n,
So $f'(x)=\sum_{k=1}^{n} t_k x^k$ for some $t_1, \ldots, t_n \in \mathbb{R}$
Therefore $f(x)=\sum_{k=1}^{n} \frac{t_k}{k+1} x^{k+1}$ for all $x \in \mathbb{R}$, which is a polynomial of degree at most $n+1$. Therefore the inductive step holds

This finishes the proof by induction $\square$

(4) Show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, but $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\epsilon}}$ converges for all $\epsilon > 0$.

Proof By the Integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\varepsilon}}$ converges

if and only if $\int_2^{\infty} \frac{1}{x(\ln x)^{1+\varepsilon}} dx$ converges

$\int_2^{\infty} \frac{1}{x(\ln x)^{1+\varepsilon}} dx = \int_{\ln 2}^{\infty} \frac{1}{u^{1+\varepsilon}} du$

1° For $\varepsilon = 0$ then $\int_{\ln 2}^{\infty} \frac{1}{u^{1+\varepsilon}} du = \int_{\ln 2}^{\infty} \frac{1}{u} du$

$= \lim_{u \to \infty} \ln u - \ln(\ln 2) \to \infty$, diverges,

Therefore $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

2° For $\varepsilon > 0$, then $\int_{\ln 2}^{\infty} u^{-(1+\varepsilon)} du = \lim_{u \to \infty} \left(\frac{1}{-\varepsilon} u^{-\varepsilon}\right) - \frac{1}{-\varepsilon}(\ln 2)$

$= \frac{1}{\varepsilon} \ln 2$, converges

Therefore $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\varepsilon}}$ converges for all $\varepsilon > 0$

(5) Show that $\sum \frac{(-1)^n}{n^{1+1/n}}$ converges conditionally.

Claim①. $\sum \frac{(-1)^n}{n^{1+\frac{1}{n}}}$ converges

Proof for all $m \geq n \in \mathbb{N}$, $1+\frac{1}{m} < 1+\frac{1}{n}$

So $\frac{1}{m^{1+\frac{1}{m}}} < \frac{1}{n^{1+\frac{1}{n}}}$

Therefore $\left(\frac{(-1)^n}{n^{1+\frac{1}{n}}}\right)$ is decreasing and positive

Also $\lim_{n\to\infty} \frac{1}{n^{1+\frac{1}{n}}} \leq \lim_{n\to\infty} \frac{1}{n} = 0$

So by the alternating series test, $\sum \frac{(-1)^n}{n^{1+\frac{1}{n}}}$ converges.

Claim②. $\sum \left|\frac{(-1)^n}{n^{1+\frac{1}{n}}}\right| = \sum \frac{1}{n^{1+\frac{1}{n}}}$ diverges.

Proof To prove claim ②, we introduce limit comparison test:

for series $\sum a_n$ and $\sum b_n$, if $a_n, b_n > 0$ for all $n \in \mathbb{N}$

and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof of limit comparison test:

Assume the hypothesis, and take $\varepsilon = \frac{c}{2}$

Then $\exists N \in \mathbb{N}$ s.t. $\left|\frac{a_n}{b_n} - c\right| < \varepsilon$ for all $n \geq N$

$\Rightarrow (c-\varepsilon) b_n < a_n < (c+\varepsilon) b_n$

$$b_n < \frac{1}{c-\varepsilon} a_n = \frac{2}{c} a_n \text{ and } a_n < \frac{3}{2} c\, b_n$$

for all $n \geq N$

So ignoring finite tail of all $n < N$, we have $\sum b_n$ converges if and only if $\sum a_n$ converges, through the comparison test

This proves the limit comparison test.

Now we look back to the series,

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^{1+\frac{1}{n}}}} = \lim_{n\to\infty} \frac{n^{1+\frac{1}{n}}}{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

So by limit comparison test, $\sum \frac{1}{n^{1+\frac{1}{n}}}$ diverges since $\sum \frac{1}{n}$ diverges.

This proves claim ②

By claim ① and claim ②, $\sum \frac{(-1)^n}{n^{1+\frac{1}{n}}}$ converges conditionally.

$\square$

(6) Given an example of a sequence (a_n) of positive real numbers converging to zero such that $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. Prove that your example works.

hw6 ④ 如果没有decreasing 这一条件, alternating series test 不起作用

ex $a_n = \begin{cases} \frac{1}{n} & \text{for all even } n \in \mathbb{N} \\ \frac{1}{2n+2} & \text{for all odd } n \in \mathbb{N} \end{cases}$

Proof Note that (a_n) is a sequence of positive real numbers with $\lim(a_{2n}) = \lim(a_{2n-1}) = 0$

So $\lim(a_n) = 0$.

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{k=1}^{\infty}(a_{2k}-a_{2k-1}) = \sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{2(2k-1)+2}\right)$$
$$= \sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{4k}\right) = \sum_{k=1}^{\infty}\frac{1}{4k} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k}$$

Therefore $\sum a_n$ diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series.

$\square$

(7) In each of the following, determine whether the given series converges or diverges, and justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{e^n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n}$ (e) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{n^2}}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n e^{1/n}$ (d) $\displaystyle\sum_{n=1}^{\infty}(\cos\pi n)\ln\left(1+\frac{1}{n}\right)$

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{e^n}$ diverges

Pf Write $a_n = \frac{n!}{e^n}$ for each $n \in \mathbb{N}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}}\right) = \lim_{n\to\infty}\frac{n+1}{e} > 1 \text{ (actually diverges to } \infty)$$

So by the ratio test, $\sum a_n$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n e^{\frac{1}{n}}$ diverges

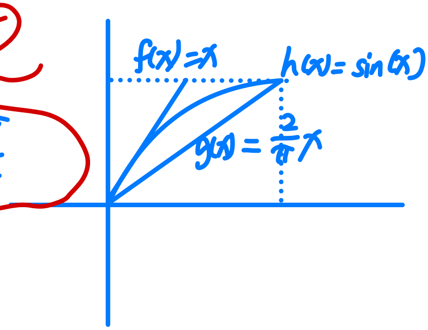
Pf Write $a_n = (-1)^n e^{\frac{1}{n}}$ for all $n \in \mathbb{N}$

So $\limsup a_n = 1$ and $\liminf a_n = -1$

$\Rightarrow \lim_{n\to\infty} a_n$ does not exist, so $\sum a_n$ diverges by the n^{th} term test.

(c) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges

hw 6 ⑤



Pf Note the inequality:

$\frac{2}{\pi}x \leq \sin x \leq x$ for all $0 \leq x \leq \frac{\pi}{2}$

$\Rightarrow \sin\frac{1}{x} \geq \frac{2}{\pi}\frac{1}{x}$ for all $x \geq \frac{2}{\pi}$

Since for all $n \in \mathbb{N}$, $n > \frac{2}{\pi}$,

we have $\sin(\frac{1}{n}) \geq \frac{2}{\pi}\frac{1}{n}$ for all $n \in \mathbb{N}$

Since $\sum \frac{2}{\pi}\frac{1}{n}$ diverges, by comparison test we have $\sum \sin(\frac{1}{n})$ diverges.

(d) $\sum_{n=1}^{\infty} \cos(\pi n)\ln(1+\frac{1}{n})$ converges

Pf Write $a_n = \ln(1+\frac{1}{n})$

So $\sum_{n=1}^{\infty} \cos(\pi n)\ln(1+\frac{1}{n}) = \sum_{n=1}^{\infty}(-1)^n a_n$

Then $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$.

And for all $m > n \in \mathbb{N}$, $\ln(1+\frac{1}{m}) < \ln(1+\frac{1}{n})$, so (a_n) is decreasing

Therefore by alternating series test, $\sum_{n=1}^{\infty}(-1)^n$ converges.

(e) $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$ diverges

Write $a_n = \frac{e^{n^2}}{n!}$ for each $n \in \mathbb{N}$

So $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|\frac{e^{2n+1}}{n+1}\right|$ note that $\frac{e^{2n+1}}{n+1}$ is unbounded above, so $\sum a_n$ diverges by ratio test.

(8) Let (a_k) and (b_k) be sequences of real numbers. Prove that if the series $\sum a_k^2$ and $\sum b_k^2$ both converge, then $\sum a_k b_k$ converges absolutely.
[Hint: use the Cauchy-Schwarz inequality, which says that $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$ for all $n \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$, where $\cdot$ is the usual dot product and $\| \ \|$ is the Euclidean metric on $\mathbb{R}^n$].

Proof. Assume the hypothesis.

Since $\sum a_k^2 = \sum |a_k|^2$, $\sum b_k^2 = \sum |b_k|^2$ converges,

$\sum |a_k|^2 = L_1$ $\sum |b_k|^2 = L_2$ for some L_1, L_2

By the Cauchy-Schwarz inequality,

$\left(\sum_{k=1}^{n} |a_k||b_k|\right)^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right)$ for all $n \in \mathbb{N}$,

So $\left(\sum_{k=1}^{n} |a_k||b_k|\right)$ is bounded above by $L_1 L_2$ and below by 0.

Hence the sequence of partial sum $\left(\sum_{k=1}^{n} |a_k||b_k|\right)$ is bounded and increasing, $\left(\sum_{k=1}^{n} |a_k b_k|\right)$ converges, so $\sum a_k b_k$ converges absolutely by definition.

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hw 6 ⑥ 如果 $\sum a_k^2$ 和 $\sum b_k^2$ converges,

则 $\sum a_k b_k$ conv. absly (by Cauchy-Schwarz)

In the problems below, integrable *means* Riemann integrable*, or, equivalently,* Darboux integrable. *In each problem concerning integration, you may choose whether to use the Riemann theory or the Darboux theory in solving the problems. Throughout problems (12) – (16), assume $a < b$.*

(9) Show that if the function f is integrable on $[a, b]$, then f is integrable on every subinterval $[c, d] \subseteq [a, b]$.

hw 6 ⑦ f 在[a,b]上 intble，则在[a,b]的每个sub interval 上 intble

Proof Assume the hypothesis.

Let $[c,d]$ be an arbitrary subinterval of $[a,b]$

let $\varepsilon > 0$.

Suppose for contradiction that f is not integrable on $[c,d]$

Since f is integrable, $\exists \delta > 0$ s.t. for all partition $\dot{P}, \dot{Q}$, if $\|\dot{P}\|, \|\dot{Q}\| < \delta$ then $|S(f,\dot{P}), S(f,\dot{Q})| < \varepsilon$ ①

Since f is not integrable on $[c,d]$, $\exists \dot{P}_0, \dot{Q}_0$ of $[c,d]$ s.t. $\|\dot{P}\|, \|\dot{Q}\| < \delta$ but $|S(f\restriction[c,d], \dot{P}_0) - S(f\restriction[c,d], \dot{Q}_0)| \geq \varepsilon$ ②

Now refine $\dot{P}_0, \dot{Q}_0$ to $[a,b]$ by taking $\frac{(b-a)-(d-c)}{\delta}+1$ regular extra points with the same tags, getting $\dot{P}, \dot{Q}$.

So we have $\dot{P} \setminus \dot{P}_0 = \dot{Q} \setminus \dot{Q}_0$ with $\|\dot{P} \setminus \dot{P}_0\| = \|\dot{Q} \setminus \dot{Q}_0\| < \delta$

$\Longrightarrow \|\dot{P}\|, \|\dot{Q}\| < \delta$ and

$S(f\restriction[a,c] \cup [b,d], \dot{P} \setminus \dot{P}_0) = S(f\restriction[a,c] \cup [b,d], \dot{Q} \setminus \dot{Q}_0)$

So $|S(f,\dot{P}) - S(f,\dot{Q})| = |S(f\restriction[c,d], \dot{P}_0) - S(f\restriction[c,d], \dot{Q}_0)| \geq \varepsilon$ ③ contradicting ①

So by contradiction, we have proved $[c,d]$ is integrable

(10) Suppose the function f is integrable on $[a, b]$. Show that for every infinite subset $S \subseteq [a, b]$, there is a function $g : [a, b] \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in [a, b] \setminus S$ and g is not integrable. [cf: 7.1.3 in text.]

Proof Let $S \subseteq [a,b]$ be arbitrary infinite subset

Since S is bounded and infinite $\Rightarrow$ S is not discrete

So $S \cap S' \neq \emptyset$, which means there is some point $a \in S$ s.t. for all $\varepsilon > 0$ $V_\varepsilon(a) \cap S \setminus \{a\} \neq \emptyset$

We define $g: [a,b] \to \mathbb{R}$ as below:

$$g(x) = \begin{cases} f(x), & x \in [a,b] \setminus S \\ \frac{1}{x-a}, & x \in S \end{cases}$$

Then we have: $g(x)$ is unbounded above

since taking arbitrary M, we can take $\varepsilon = \frac{1}{M}$,

$\Rightarrow$ there exists some $x \in V_\varepsilon(a) \cap S \setminus \{a\}$, so

so $g(x) = \frac{1}{x-a} > \frac{1}{\varepsilon} = M$

Since g is unbounded, g is not Riemann integrable.

hw 6 ⑧ 如果 f intble, 而 g 只有 finitely many 个 pt 上和 f 不一样, 那么 g 也 intble 且 $\int_a^b f = \int_a^b g$, 但不推广至 infinitely many 个 pt.

(11) Show directly that if the bounded function $f : [a, b] \to \mathbb{R}$ is continuous everywhere in $[a, b]$ except possibly at one point $x_0 \in (a, b)$, then f is integrable on $[a, b]$.

Proof Assume f is continuous everywhere except at c

Since f is bounded, there is some $B > 0$ s.t $-B \le f(x) \le B$ for all $x \in [a,b]$

Let $\varepsilon > 0$ be arbitrary.

So f is continuous on $[a, c)$ and $(c, b]$

Then there exists some $\delta_1 > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{6(c-a)}$ whenever $|x-y| < \delta_1$ for all $x, y \in [a, c)$

and some $\delta_2 > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{6(b-c)}$ whenever $|x-y| < \delta_2$ for all $x, y \in (c, b]$, by continuity.

Take $\boxed{\delta = \min\{\delta_1, \delta_2, \frac{\varepsilon}{6B}\}}$

Let $P = (x_0, \ldots, x_n)$ be an arbitrary partition of $[a,b]$ with $\|P\| < \delta$

Then $c \in I_{k_0} = [x_{k_0}, x_{k_0+1}]$ for some $k_0 \in \mathbb{N} \cup \{0\}$

For each $k = 0, \ldots, k_0 - 1$, by taking the midpoint x of I_k, we have: for any $y \in [x_k, x]$, $|f(x) - f(y)| < \frac{\varepsilon}{6(c-a)}$ and for any $y \in [x, x_{k+1}]$, $|f(x) - f(y)| < \frac{\varepsilon}{6(c-a)}$, so $|\sup_{I_k} f - \inf_{I_k} f| < \frac{\varepsilon}{3(c-a)}$

Then we have $\left| \sum_{k=0}^{k_0-1} (\sup_{I_k} f - \inf_{I_k} f) \Delta x_k \right| \le \sum_{k=0}^{k_0-1} |\sup_{I_k} f - \inf_{I_k} f| \Delta x_k$

$$< \frac{\varepsilon}{3(c-a)} \cdot (c-a) = \frac{\varepsilon}{3}$$

By the same reasoning,

$$\left| \sum_{k=k_0+1}^{n} \left(\sup_{I_k} - \inf_{I_k} f \right) \Delta x_k \right| < \frac{\varepsilon}{3(b-c)} \cdot (b-c) < \frac{\varepsilon}{3}$$

And for I_{k_0},

$$\left| \sup_{I_{k_0}} f - \inf_{I_{k_0}} f \right| \Delta x_0 < 2B \cdot \frac{\varepsilon}{6B} = \frac{\varepsilon}{3}$$

Hence $|U(f,P) - L(f,P)| = \left| \sum_{k=0}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) \Delta x_k \right|$

$$= \left| \sum_{k=0}^{k_0-1} \left(\sup_{I_k} f - \inf_{I_k} f \right) \Delta x_k + \left(\sup_{I_{k_0}} f - \inf_{I_{k_0}} f \right) \Delta x_0 + \sum_{k=k_0+1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) \Delta x_k \right|$$

$$\leq \left| \sum_{k=0}^{k_0-1} \left(\sup_{I_k} f - \inf_{I_k} f \right) \Delta x_k \right| + \left| \left(\sup_{I_{k_0}} f - \inf_{I_{k_0}} f \right) \Delta x_0 \right| + \left| \sum_{k=k_0+1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) \Delta x_k \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Since ε is arbitrary, f is Riemann integrable $\square$

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hw6 ⑨ 只有 finite 个点上不 ctn 的函数是 intble 的

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(12) Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove that there is $x_0 \in (a, b)$ such that $f(x_0) = g(x_0)$.

hw6 ⑮ 如果 $\int_a^b f = \int_a^b g$ 那么一定有 $x_0 \in [a,b]$ s.t. $f(x_0) = g(x_0)$

Proof Assume the hypothesis.

Suppose for contradiction that $f(x) > g(x)$ for all $x \in [a,b]$, then $U(f,P) > U(g,P)$ since for every subinterval I_k of $[a,b]$, $\underline{\sup_{I_k} f > \sup_{I_k} g}$

$\Rightarrow \int_a^b f(x)\,dx = U(f) > U(g) = \int_a^b g(x)\,dx$, contradicts

Therefore there exists $\underline{k_1 \in [a,b] \text{ s.t. } f(k_1) \leq g(k_1)}$

For the same reasoning, $\exists\, \underline{k_2 \in [a,b] \text{ s.t. } f(k_2) \geq g(k_2)}$

If $f(k_1) = g(k_1)$ or $f(k_2) = g(k_2)$, then the statement is true

If $f(k_1) \neq g(k_1)$ and $f(k_2) \neq g(k_2)$,

then $f(k_1) < g(k_1)$ and $f(k_2) > g(k_2)$

Let $h(x) = f(x) - g(x)$ defined on $[a,b]$,

So $h(x)$ is continuous

So $h(k_1) < 0$ and $h(k_2) > 0$, and

$[f(k_1) - g(k_1), f(k_2) - g(k_2)] \subseteq \operatorname{ran}(h)$

Then by intermediate value theorem, there exists some x_0 between k_1 and k_2 s.t. $h(x_0) = 0$

i.e. $f(x_0) = g(x_0)$ $\square$

Optional Challenge Problems:

(13) Suppose that for each $n \in \mathbb{N}$, V_n is an open subset of $\mathbb{R}$. For each n define the function $f_n : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_n(x) = \begin{cases} 0 & \text{if } x \in V_n \\ 2^{-n} & \text{if } x \in \mathbb{Q} \setminus V_n \\ -2^{-n} & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \setminus V_n). \end{cases}$$

Then define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that for all $a \in \mathbb{R}$, f is continuous at a if and only if $a \in \bigcap_{n \in \mathbb{N}} V_n$.