Def Series of functions

if $(f_k: A\to\mathbb{R})_{k\in\mathbb{N}}$ is a seq. of functions,

那么称 $\boxed{(\sum_{k=1}^{n} f_k)_{n\in\mathbb{N}}}$ 为 seq. of partial sums of (f_k)

记 $\boxed{\sum f_k}$ 或 $\boxed{\sum_{k=1}^{\infty} f_k}$ 为 the infinite series determined by (f_k)

① 如果 $\forall x\in B\subseteq A$, $\lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)$ exists (即 $\sum_{k=1}^{\infty} f_k(x)$ conv.)
(iff)
或者 $\exists f: B\to\mathbb{R}$ s.t. $(\sum_{k=1}^{n} f_k)_{n\in\mathbb{N}} \to f$ ptwise.

则称 $\boxed{\sum f_k \text{ converges on } B}$ $(B\subseteq A)$

② 如果 $\exists f: B\to\mathbb{R}$ s.t. $(\sum_{k=1}^{n} f_k)_{n\in\mathbb{N}} \to f$ uniformly,

则称 $\boxed{\sum f_k \text{ converges. uniformly on } B.}$

③ 如果 $\forall x\in B\subseteq A$, $\lim_{n\to\infty}\sum_{k=1}^{n} |f_k(x)|$ exists (即 $\sum_{k=1}^{\infty} |f_k(x)|$ conv.)
(iff)
或者 $\exists f: B\to\mathbb{R}$ s.t. $(\sum_{k=1}^{n} |f_k|)_{n\in\mathbb{N}} \to f$ ptwise.

则称 $\boxed{\sum f_k \text{ converges absolutely on } B.}$

(同理有 $\sum f_k$ conv. unif. absly. on B)

Thm①

(1) If f_k is ctn. on A for each $k\in\mathbb{N}$
且 $\sum f_k \to S$ unif. on A.
$\Longrightarrow S$ is ctn on A

(2) If f_k is ctn. on $[a,b]$ for each $k\in\mathbb{N}$
且 $\sum f_k \to S$ unif. on $[a,b]$
$\Longrightarrow S$ is intble on $[a,b]$ 且 $\boxed{\int_a^b S = \sum \int_a^b f_k}$

(3) If $f_k\in C^1$ on $[a,b]$ for each $k\in\mathbb{N}$
且 $\sum f_k \to S$ on $[a,b]$ (不用unif.)
且 $\sum f_k'$ conv. unif. on $[a,b]$
$\Longrightarrow f\in C^1$ on $[a,b]$, 且 $\boxed{S' = \sum f_k'}$

*(3) (Stronger version)

If $f_k\in C^1$ on $[a,b]$ for each $k\in\mathbb{N}$
且 $\exists x_0\in[a,b]$ s.t. $\sum f_k(x_0)$ conv.
且 $\sum f_k'$ conv. unif. on $[a,b]$
$\Longrightarrow \sum f_k$ $\boxed{\text{conv. unif.}}$ to some $S\in C^1$
且 $\boxed{S' = \sum f_k'}$

Pf (1) Since $\forall k$, f_k ctn.
$\Longrightarrow (\sum_{k=1}^{n} f_k)_{n\in\mathbb{N}}$ 中, 每项都ctn
又 $(\sum_{k=1}^{n} f_k) \to S$ unif.
$\Longrightarrow S$ ctn

(2)(3)同理.
只需通过 ctn, diffble, intble 的 linearity, 把 (f_k) 的性质加到 $(\sum_{k=1}^{n} f_k)_{n\in\mathbb{N}}$ 上即可.

Power Series

Def Power series centered at c with coefficient (a_n)

For a seq. (a_n) in $\mathbb{R}$, the power series centered at c with coeff (a_n) is the series of functions $\boxed{\sum_{n=0}^{\infty} a_n (x-c)^n}$

note: the partial sums of a power series are polynomials

Custom: $\forall x\neq 0,\ 0^x = 0$
$\forall x,\ x^0 = 1 \quad (0^0 = 1)$

Note: 下面的定理中我们将全部把 power series center at 0
但实际上center并不重要, center at 任何位置, 同样的 Thms 都成立.

Thm② Cauchy-Hadamard Thm

Given a power series $\sum a_n x^n$,
let $\rho = \limsup |a_n|^{\frac{1}{n}}$

$\Longrightarrow \sum_{n=0}^{\infty} a_n x^n$ conv. absly. when $|x|\rho < 1$,
div. when $|x|\rho > 1$

因而 the radius of convergence $\boxed{R = \frac{1}{\rho}}$

Pf Let $x\in\mathbb{R}$
Suppose $|x|\cdot\limsup|a_n|^{\frac{1}{n}} < r < 1$
$\Longrightarrow$ For all but finitely many n, $|x||a_n|^{\frac{1}{n}} \le r$
(至多有限个 $|x||a_n| > r$) $\Longrightarrow |a_n x^n| \le r^n$
Since $0<r<1$, $\sum r^n$ conv.
因而 $\sum |a_n x^n|$ conv. by the Comparison Test.
On the other hand, if $|x|\rho > r > 1$, then for infly many n, $|a_n x^n| > r^n > 1$, 因而 $\sum a_n x^n$ div. by the n^{th} term test.

* the set of all x for which $\sum a_n (x-c)^n$ conv. is an interval, 称为 interval of convergence

* rad. of conv. 并不能 imply interval of conv.
因为我们知道除了 $(c-R, c+R)$ 之外，$c+R$ 和 $c-R$ 这两端也可能在其中，因而需单独判断.

<u>Fact (hw)</u>
如果 $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ 存在，那么 $\frac{1}{L}$ 为 rad. of. conv. of $\sum a_n x^n$

(通常这是 best way to find R，但是它弱于 $\limsup |a_n|^{\frac{1}{n}}$，因为 $\limsup |a_n|^{\frac{1}{n}}$ 总存在 (including $\pm\infty$)，而 $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ 未必存在)

<u>ex</u> (1) $\sum_{n=1}^{\infty} \frac{1}{n!}x^n$ (recall: if

$a_n = \frac{1}{n!} \Longrightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$

$\rho = 0$, $R = \infty$. 因而它 conv. for all $x \in \mathbb{R}$.

(in fact, $\forall x \in \mathbb{R}$, $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$ by Taylor)

(2) $\sum_{n=0}^{\infty} x^n$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow \rho = R = 1$

Since $\sum_{n=0}^{\infty} x^n$ div. for $x = \pm 1 \Longrightarrow$ int. of conv. is $(-1,1)$

且 $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for each $x \in (-1,1)$

(3) $\sum_{n=0}^{\infty}\left(\frac{1}{n}\right)x^n$

$a_n = \frac{1}{n} \Longrightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

So $\rho = R = 1$

Note: for $x = 1$, $\sum\frac{x^n}{n}$ is harmonic series $\Rightarrow$ div.
for $x = -1$, $\sum\frac{x^n}{n}$ is alternating harmonic series $\Rightarrow$ conv.

So int. of conv. is $[-1,1)$

(4) $\sum_{n=0}^{\infty}\frac{1}{n^2}x^n$

$a_n = \frac{1}{n^2} \Rightarrow \rho = R = 1$

$\sum_{n=0}^{\infty}\frac{1}{n^2}$ and $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ conv.

So int. of conv. is $[-1,1]$

(5) $\sum_{n=0}^{\infty} n!\,x^n$

$a_n = n!$ $\rho = \infty \Rightarrow R = 0$

So it div for all $x \in \mathbb{R}$

Goal: to show that if $\sum a_n x^n$ conv. on $(-R,R)$, then it conv. unily. on $[-K,K]$ for every $0 \le K < R$.

<u>Thm③</u> <u>Weierstrass M-Test.</u>

Let $f_k : A \to \mathbb{R}$ be a seq. of functions, M_k be a seq. in $\mathbb{R}$ s.t. $\boxed{|f_k(x)| \le M_k}$ for all $k \in \mathbb{N}$ and $x \in A$

If $\sum M_k < \infty$, then $\sum f_k$ conv. $\boxed{\text{unily. and absly.}}$ on A.

Weierstrass M-Test：即为每个 f_k 找到一个 bound，来用 bounds 构成一个 seq. if series of bounds conv，那么自然 $\sum M_k$ 也 conv.

Pf let $g_n(x) = \sum_{k=1}^{n} f_k(x)$ (seq. of partial sums)

let $\varepsilon > 0$.

Since $\sum M_k < \infty$, $\sum M_k$ satisfies the Cauchy Criterion

$\Rightarrow$ We can fix N s.t.

$$\left(\forall N \le m \le n, \left|\sum_{k=m+1}^{n} M_k\right| < \varepsilon\right)$$

Then $\forall x \in A$ and $n \ge m \ge N$

$$|g_n(x) - g_m(x)| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \le \sum_{k=m+1}^{n}|f_k(x)| \le \sum_{k=m+1}^{n} M_k < \varepsilon$$

因而 (g_n) is uniformly Cauchy.

即 $\sum f_k$ conv. uniformly on A.

Furthermore, the calculation shows that $\sum |f_k|$ is uniformly Cauchy, hence uniformly conv.

因而 $\sum f_k$ conv. unily and absly on A. □

<u>Corollary</u> 对于 power series $\sum a_n x^n$,

Write 其 rad. of. conv. as R

$\Longrightarrow \forall \boxed{0 \le K < R}$, $\sum a_n x^n$ conv. unily to a ctn. function on $\boxed{[-K,K]}$

idea：对于 $x \in [-K,K]$，K^n 就是 x^n 的一个 bound.

<u>Pf</u> let $0 \le K < R$.

So $\sum |a_n| K^n < \infty$ 且 $\boxed{|a_n x^n| < |a_n| K^n}$ for all $x \in [-K,K]$

So $\sum a_n x^n$ conv. unily. on $[-K,K]$ by Weierstrass M-test.

Using $M_n = |a_n| K^n$

Since each partial sum $\sum_{k=0}^{n} a_n x^k$ is ctn, the uniform limit is ctn. □

Corollary

If the rad. of conv. of the power series $\sum a_n x^n$ is R, then $f(x)=\sum a_n x^n$ is ctn. on $(-R,R)$

Directly follows from the previous corollary (by uni. conv.)

Thm Abel's thm

① 如果 power series $\sum_{k=1}^{\infty} a_k x^k$ 在 x_0 处 conv.

$\Longrightarrow$ 它一定在 $(-x_0, x_0)$ 上全都 uni. conv.

如果 $\sum_{k=1}^{\infty} a_k x^k$ 在 x_0 处 div.

$\Longrightarrow$ 它一定在 $(-\infty,-|x_0|)\cup(|x_0|,+\infty)$ 上都 div.

② 令 power series 的 rad. of conv 为 R

则如果 $\sum a_k x^k$ 在 R 处 conv., 则一定在 R 处左 ctn.

~ 在 $-R$ 处 conv., 则一定在 $-R$ 处右 ctn.

Pf 提一下. 不证 (略)

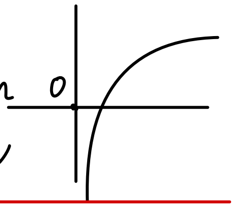
Note the convergence of $\sum a_n x^n$ on its interval of convergence may not be uniform !!

ex $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$ conv. to $\ln x$ on $(0,2]$

(0 上也 conv, 但不 conv to $\ln x$)

然而这个 convergence 并不 uniform (unbounded),

因为:



Fact A uniform limit of uniformly ctn functions is uniformly ctn

Thm term-by-term integration & Differentiation of Power series

Let $\sum a_n x^n$ be a power series with rad. of conv. $R>0$.

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ for all $x\in(-R,R)$

$\Longrightarrow$ (i) $\forall [a,b]\subseteq(-R,R)$, f is intble on $[a,b]$

且 $\boxed{\int_a^b f=\sum_{n=0}^{\infty}\int_a^b a_n x^n}$

(ii) the rad. of conv. of $\sum n a_n x^{n-1}$ is R,

f is diffble on $(-R,R)$,

且 $\forall x\in(-R,R)$, 都有 $\boxed{f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}}$

Pf sketch

(i) follows from integrability of polynomials and the fact: convergence of $\sum a_n x^n$ on $[a,b]$ is uniform.

(ii) follows from our previous results, once we can show that the rad. of conv. of $\sum n a_n x^{n-1}$ is R.

To see this, let $t\neq 0$ be arbitrary.

Note: $\limsup\left|\frac{n}{t}a_n\right|^{\frac{1}{n}}=\lim\left|\frac{n}{t}\right|^{\frac{1}{n}}\limsup|a_n|^{\frac{1}{n}}$

(recall: if $\lim a_n=L$, $\limsup(a_n b_n)=L\limsup b_n$)

因而 $\sum \frac{n}{t}a_n x^n$ 也有 rad. of conv. R

So $\forall t\neq 0$, $\sum n a_n t^{n-1}=\sum\frac{n}{t}a_n t^n$, conv. if $|t|<R$ and div. if $|t|>R$ as desired.

$\square$

ex if $f\in C^\infty$

then we can attempt to approximate f near c using taylor polynomials.

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$T(x)=\lim_{n\to\infty}P_n(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

where the domain is the int. of conv. of T.

In the following we have convergence on $\mathbb{R}$.

Def

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$e^{\pi i} + 1 = 0.$$

$$\int \cos x^2 \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{4n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} x^{4n+1}$$

Remarks

The Taylor expansion of f may not converge to f at $x=a$ even if it converges at $x=a$

---

ex $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\Rightarrow f \in C^\infty$ on $\mathbb{R}$ 且 $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$

Note that: $T(x)$ of f converges everywhere

但 converges to f itself only at $x=0$

(如果 $f \in C^\infty$ 且 $T(x) \to f$ piecewise for all $x \in dom(f)$)

则称 f 为一个 real analytic function

i.e. $f \in C^\omega$ ($C^\omega \subseteq C^\infty$)