Def A, B are two sets. A function (or a mapping) f is a correspondence $A \to B$; for each $x \in A$, f associates to an element $f(x) \in B$. A is called the domain of f

Def $f: A \to B$ a mapping of A into B. If $E \subset A$, $f(E)$ $\{y \in B : \exists x \in A, f(x) = y\}$ is called the image of E (under f). $f(A)$ is called the range of f.

We call f is onto if $f(A) = B$

If $E \subset B$, $f^{-1}(E) = \{x \in A : f(x) \in E\}$. If $y \in B$, $f^{-1}(y) = \{x \in A : f(x) = y\}$.

We call f is 1-1 if for $\forall y \in f(A)$, $f^{-1}(y)$ has exactly 1 element.

or for any $x_1 \neq x_2 \in A$, $f(x_1) \neq f(x_2)$.

Metric spaces

Def A set X, whose elements we shall call pts, is said to be a metric space if with any two pts, there is a nonnegative real number $d(p,q)$, called the distance from p to q such that (a) $d(p,q) > 0$ if $p \neq q$ (b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(q,r)$ for any $r \in X$. (Triangle inequality)

Such function $d: X \times X \longrightarrow X$ is called a distance function (or metric).

Example $\mathbb{R}^k$, $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$.

Rmk: Every subset E of a metric space X is a metric space in its own right, with the same distance function.

Def For $a, b \in \mathbb{R}$, $a<b$, $(a,b) = \{x \in \mathbb{R} \mid a<x<b\}$ called segment.

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ called interval.

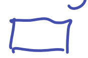
$(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}$ called half-open intervals.

$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$

For $a_i < b_i$, $i = 1, 2, \cdots, k$, the set of all pts $\vec{x}$ such that $a_i < x < b_i$ is called k-cell.

1-cell = interval

2-cell = rectangle



The open (closed) ball B centered at $\vec{x}$ with radius r is $\{y \in \mathbb{R}^k : |y-x| < r\}$.

$|y-x| \le r$

Def X = Metric space

(a) A neighborhood (nbd) of p is a set $N_r(p) = \{q \in X : d(p,q) < r\}$. ← radius

(b) A point p is called a limit pt of the set E if every nbd of p contains a $E \ni q \neq p$.

(c) $p \in E$ is an isolated point of E if p is not a limit pt.

(d) E is closed if every limit pt of E is in E.

(e) $p \in E$ is an interior pt if $\exists$ a nbd $N \ni p$ s.t. $N \subset E$.

(f) E is open if every $p \in E$ is an interior pt.

(g) The complement E^c of E is $\{p \in X : p \notin E\}$.

(h) E is bounded if $\exists M > 0$ and a point q such that $d(p,q) < M$ for $p \in E$.

(i) E is dense in X if every point $p \in X$ is a limit point of E, or a pt of E.

<u>Example</u> (a,b) = nbd in $\mathbb{R}^1$  = nbd in $\mathbb{R}^2$.

<u>Thm</u> Every nbd is open.

<u>Proof</u> 

<u>Thm</u> If p is a limit pt of E, then any nbd of p contains infinitely many pts of E.

<u>Thm</u> $\{E_\alpha\}$ is a collection of sets. Then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$.

<u>Proof</u> If $x \notin \bigcup_\alpha E_\alpha$, $x \notin E_\alpha$. If $x \notin E_\alpha$, then $x \in \left(\bigcup_\alpha E_\alpha\right)^c$. $\boxtimes$

<u>Thm</u> A set E is open iff E^c is closed.

<u>Proof</u> Assume E is open. If x is a limit pt of E^c, and $x \in E$, then $\exists$ nbd $N \ni x$ in E containing no pts of E^c. x cannot be a limit pt of E^c.

Assume E^c is closed and $x \in E$. Then x is not a limit pt of E. Then $\exists$ nbd $N \ni x$ is contained in E.

<u>Thm</u> (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

(c) For any finite collection opens $\{G_\alpha\}$, $\bigcap_\alpha G_\alpha$ is open

(d) (closed $\{F_\alpha\}$) ($\bigcup F_\alpha$ is closed)

<u>Def</u> The closure of is $\bar{E} = E \cup E'$ where E' = set of limit pts.

<u>Thm</u> (a) $\bar{E}$ is closed (b) E is closed iff $\bar{E}$ is closed.

<u>Thm</u> $E \subseteq \mathbb{R}^1$ non-empty bounded above. Then $\sup y \in \bar{E}$.

Example: Subsets in $\mathbb{R}^2$

	Closed	Open	Bound
$x^2+y^2<1$	✗	✓	✓
$x^2+y^2\leq 1$	✓	✗	✓
non-empty Finite set	✓	✗	✓
$\mathbb{Z}\times\{0\}$	✓	✗	✗
$\mathbb{R}^2$	✓	✓	✗
$\{(\frac{1}{n},0): n=1,2,3,\dots\}$	✗	✗	✓
$\{(x,0): x\in(a,b)\}$	✗		✓