Homework 2: Due Tuesday, May 21, at 11:59pm, on Gradescope

Recall that for any sets X and Y, we write $X \preceq Y$ if there exists an injective function from X to Y, and $X \approx Y$ if there exists a bijective function from X to Y. By the Cantor-Schroder-Bernstein Theorem, $X \approx Y$ if and only if $X \preceq Y$ and $Y \preceq X$.

(1) Use induction and the triangle inequality for real numbers to prove that for all $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|.$$

Proof. We prove it by induction on $n \in \mathbb{N}$

Base case: $n=1$, $\left|\sum_{k=1}^{n} a_k\right| = |a_k| = \sum_{k=1}^{n} |a_k|$

the claim holds true.

Inductive step: assume the inequality holds true for all $a_1, \ldots, a_n \in \mathbb{R}$, for $n = 1, 2, \ldots, j$

Then $\left|\sum_{k=1}^{j+1} a_k\right| = \left|\sum_{k=1}^{j} a_k + a_{j+1}\right|$

By inductive hypothesis when $n=2$,

$$\left|\sum_{k=1}^{j+1} a_k\right| = \left|\sum_{k=1}^{j} a_k + a_{j+1}\right| \leq \left|\sum_{k=1}^{j} a_k\right| + |a_{j+1}| \quad ①$$

By inductive hypothesis when $n=j$,

$$\left|\sum_{k=1}^{j} a_k\right| \leq \sum_{k=1}^{j} |a_k| \quad ②$$

Combining ①②, we get $\left|\sum_{k=1}^{j+1} a_k\right| \leq \sum_{k=1}^{j+1} |a_k|$ $\square$

This finishes the proof of the inequality by induction

(2) Let $A \subseteq \mathbb{R}$ be bounded, let $c \in \mathbb{R}$, and write $cA = \{ca : a \in A\}$. Find an expression for $\sup(cA)$, and prove your claim. Then state (but do not prove) the "dual" claim for $\inf(cA)$.

$$\sup(cA) = \begin{cases} c\sup A, & c \geq 0 \\ c\inf A, & c < 0 \end{cases} \qquad \inf(cA) = \begin{cases} c\inf A, & c \geq 0 \\ c\sup A, & c < 0 \end{cases}$$

Proof. 1° Let $c > 0$, take arbitrary $a \in A$, so ca is an arbitrary element of cA

Since $\sup A \geq a$, $c\sup A \geq ca$

So $c\sup A$ is an upper bound of cA ①

Let cb be an upper bound of cA

then $cb \geq ca$ for all $a \in A$

Since $c > 0 \Rightarrow b \geq a$, so b is an upper bound of $A \Rightarrow b \geq \sup A \Rightarrow cb \geq c\sup A$ ②

By ①②, $\boxed{c\sup A = \sup(cA) \text{ when } c > 0}$

2° Let $c = 0$, then $cA = \{0\}$, so $\boxed{c\sup A = 0 = \sup(cA)}$

3° Let $c < 0$, take arbitrary $a \in A$, so ca is an arbitrary element of cA

Since $\inf A \leq a$, $c\inf A \geq ca$

So $c\inf A$ is an upper bound of cA ③

Let cb be an upper bound of cA

then $cb \geq ca$ for all $a \in A$

Since $c < 0 \Rightarrow b \leq a$, so b is a lower bound of $A \Rightarrow b \leq \inf A \Rightarrow cb \geq c\inf A$ ④

By ③④, $\boxed{c\inf(A) = \sup(cA) \text{ when } c < 0}$

$\square$

(3) Let X, Y, and Z be any sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Prove that if f and g are injective, then $g \circ f$ is injective.

(b) Prove that $\preceq$ is reflexive and transitive; that is, prove that $X \preceq X$ and that if $X \preceq Y$ and $Y \preceq Z$, then $X \preceq Z$.

(a) Proof Suppose f and g are injective

Let $g \circ f(a) = g \circ f(b)$, i.e. $g(f(a)) = g(f(b))$ where $a, b \in \operatorname{dom}(g \circ f)$

Since g is injective $\Rightarrow$ $f(a) = f(b)$

Then since f is injective $\Rightarrow$ $a = b$

Therefore for arbitrary $a, b \in \operatorname{dom}(g \circ f)$,

$g \circ f(a) = g \circ f(b) \Rightarrow a = b$

So by definition, $g \circ f$ is injective $\square$

(b) Proof Let X be an arbitrary set

Consider $f : X \to X$ defined by $f(x) = x$

f is injective by the uniqueness of any element in a set.

So $X \preceq X$ by definition, therefore $\preceq$ is reflexive.

Let $X \preceq Y$ and $Y \preceq Z$

So there exists some injective function $f : X \to Y$

and $g : Y \to Z$

By (a) we know $g \circ f : X \to Z$ is injective,

So $X \preceq Z$, therefore $\preceq$ is transitive $\square$

(4) Let A and B be any nonempty sets.

(a) Prove that if $A \subseteq B$ then $A \preceq B$.

(b) Prove that there is an injective function from A to B if and only if there is a surjective function from B to A.

Proof (a) Assume $A \subseteq B$, so $\forall x \in A, x \in B$

$\Rightarrow$ consider $f: A \to B$ defined by $f(x) = x$

f is injective by the uniqueness of any element in a set.

Therefore $A \preccurlyeq B$ by definition

(b) ①. Suppose $f: A \to B$ is injective

Let $a \in A$ be arbitrary

Then consider $g: B \to A$ defined by

$$g(x) = \begin{cases} b \in f^{-1}[\{x\}], & \text{if } x \in ran(f) \\ a, & \text{if } x \notin ran(f) \end{cases}$$

This function is well-defined since f is injective, there is only one element in $f^{-1}[\{x\}]$ for any $x \in B$

So $ran(g) = A \Rightarrow g$ is surjective

② Suppose $g: B \to A$ is surjective

Then for any $a \in A$, exists some $b \in B$ s.t. $f(b) = a$

i.e. $g^{-1}[\{a\}] \neq \emptyset$ for any $a \in A$

$f: A \to B$ sending every $a \in A$ to some $b \in g^{-1}[\{a\}]$

The well-definedness of f is guaranteed by $g^{-1}[\{a\}] \neq \emptyset$ for any $a \in A$

And f is injective since by the well-definedness of g, $\forall a_1, a_2 \in A$, $g^{-1}[\{a_1\}] \cap g^{-1}[\{a_2\}] = \emptyset$, so every $a \in A$ is mapped to a distinct $b \in B$

By ①② we have finished the if-and-only-if proof $\square$

(5) (a) Prove that if A is an infinite set and A_0 is a finite subset of A, then $A \approx A \setminus A_0$.

(b) Prove that if A is an uncountable set and A_0 is a countable subset of A, then $A \approx A \setminus A_0$.

(a) Proof ① First we construct $f: A \setminus A_0 \to A$ defined by $f(a) = a$

It is injective by the uniqueness of any element in a set.

So $\boxed{A \setminus A_0 \preceq A}$

② $A_0 = \{a_1, a_2, \dots, a_n\}$ for some $a_1, a_2, \dots, a_n \in A$ since A_0 is finite

Since A is infinite and A_0 is finite

$\Rightarrow$ $A \setminus A_0$ is infinite (otherwise $|A \setminus A_0| = m \in \mathbb{N}$, so $|A| \leq m + n \in \mathbb{N}$, contradicting the fact that A is infinite)

So no matter $A \setminus A_0$ is countable or uncountable, we can take a countable subset $A_1 \subseteq A \setminus A_0$ and denote $A_1 = \{y_1, y_2, \dots\}$

Now we construct $f. A \to A \setminus A_0$

defined by $f(x) = \begin{cases} x, & \text{if } x \in A \setminus A_0 \setminus A_1 \\ y_{2k}, & x = a_k \text{ for some } k \in \mathbb{N} \\ y_{2k-1}, & x = y_k \text{ for some } k \in \mathbb{N}. \end{cases}$

$f(x)$ is well-defined since $(A \setminus A_0 \setminus A_1) \cup A_0 \cup A_1 = A$

and $f(x)$ is injective since $(A \setminus A_0 \setminus A_1) \cap A_0 \cap A_1 = \emptyset$

So $\boxed{A \preceq A \setminus A_0}$ Then by Cantor-Schröder-Benstein Thm, $A \approx A \setminus A_0$ $\square$

(b) Proof. Since A_0 is countable, $A_0 = \{a_1, a_2, \dots\}$ for some $a_1, a_2, \dots \in A$

Take countably infinite subset $A_1 = \{y_1, y_2, \dots\} \subseteq A \setminus A_0$

We construct $f: A \to A \setminus A_0$ defined by

$$f(x) = \begin{cases} x, & x \in (A \setminus A_0) \setminus A_1 \\ y_{2k}, & x = a_k \text{ for some } k \in \mathbb{N} \\ y_{2k-1}, & x = y_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then f is well-defined since $(A \setminus A_0 \setminus A_1) \cup A_0 \cup A_1 = A$

f is injective since $(A \setminus A_0 \setminus A_1) \cap A_0 \cap A_1 = \emptyset$

f is surjective since for any $a \in A \setminus A_0$,

$a \in (A \setminus A_0) \setminus A_1$ or $a \in A_1$

And in both cases $\exists x$ s.t. $f(x) = a$

So f is bijective

Therefore $A \simeq A \setminus A_0$

$\square$

(6) (a) Prove that $\overline{\mathbb{Q}}$ is countable. Conclude that there are uncountably many transcendental real numbers.

(b) Prove that for all $a, b \in \mathbb{R}$, if $a < b$ then there are uncountably many transcendental numbers in the interval (a, b).

(a) Proof. Let $\boxed{A_k}$ be the set of all roots of polynomials with rational numbers as coefficients with k terms

So by definition, $\boxed{\overline{\mathbb{Q}} = \bigcup_{k \in \mathbb{N}} A_k}$

Let k be an arbitrary natural number

Let $A_{k,q}$ where $q = \left(\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}\right) \in \mathbb{Q}^k$ be the polynomial with $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}$ as coefficients

So $\boxed{A_k = \bigcup_{q \in \mathbb{Q}^k} A_{k,q}}$

Since $\overline{\mathbb{Q}} \subseteq \mathbb{C}$, by the fundamental theorem of algebra, $A_{k,q}$ has at most k roots

So for each $q \in \mathbb{Q}^k$, $A_{k,q}$ is finite.

By the thm on lecture 4, since $\mathbb{Q}$ is countable,

$\Rightarrow \mathbb{Q}^k$ is countable, so $\boxed{A_k = \bigcup_{q \in \mathbb{Q}^k} A_{k,q} \text{ is countable}}$

Therefore for each $k \in \mathbb{N}$, A_k is countable

And since $\mathbb{N}$ is countable

we has proved that $\boxed{\overline{\mathbb{Q}} = \bigcup_{k \in \mathbb{N}} A_n \text{ is countable}}$

$\square$

Conclusion: Since $\mathbb{C} \cong \mathbb{R}^2$ is uncountable,

and $\overline{\mathbb{Q}}$ is countable, by problems (b),

$\mathbb{C} \setminus \overline{\mathbb{Q}}$ is uncountable ($\mathbb{C} \setminus \overline{\mathbb{Q}} \approx \mathbb{C}$)

which indicates that there are uncountably many transcendental numbers.

(b) Let $a, b \in \mathbb{R}$ be arbitrary with $a < b$

Let $\mathbb{Q}_0$ be the set of all algebraic numbers in (a, b)

So $\mathbb{Q}_0 \subseteq \overline{\mathbb{Q}}$, since $\overline{\mathbb{Q}}$ is countable, $\mathbb{Q}_0$ is countable

Since we have proved that (a, b) is uncountable,

$(a, b) \setminus \mathbb{Q}_0$ is uncountable by problem 5(b)

So there are uncountably many transcendental numbers in (a, b)

$\square$

(7) Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) Prove that $\mathcal{P}(\mathbb{R}) \preceq \mathbb{R}^{\mathbb{R}}$.

(b) Prove that there is no surjective function from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{R}}$. (This shows that there are strictly more functions from $\mathbb{R}$ to $\mathbb{R}$ than there are real numbers.)

(a) Proof. Let $A \in P(\mathbb{R})$ be an arbitrary element so $A \subseteq \mathbb{R}$

Consider the function $f_A : \mathbb{R} \to \mathbb{R}$ defined by $f_A(x) = \begin{cases} x, & x \in A \\ 0, & x \notin A \end{cases}$

Then we consider $\varphi : P(\mathbb{R}) \to \mathbb{R}^{\mathbb{R}}$ defined by defined by $\boxed{\varphi(A) = f_A(x)}$ for each $A \in P(\mathbb{R})$

φ is injective since if $\varphi(A) = \varphi(B)$, then

$f_A(x) = f_B(x) \Rightarrow \begin{cases} \forall x \in A,\ x \in B \Rightarrow A \subseteq B \\ \forall x \in B,\ x \in A \Rightarrow B \subseteq A \end{cases} \Rightarrow A = B$

Therefore $P(\mathbb{R}) \preceq \mathbb{R}^{\mathbb{R}}$ $\square$

(b) Proof Assume for contradiction that there exist some surjective function from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{R}}$

Then by problem 4(b), $\exists$ some injection function from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{R} \Rightarrow \mathbb{R}^{\mathbb{R}} \preceq \mathbb{R}$. Together with $P(\mathbb{R}) \preceq \mathbb{R}^{\mathbb{R}}$, $P(\mathbb{R}) \preceq \mathbb{R}$ by problem 3(b).

So there exists some surjective function from $\mathbb{R}$ to $P(\mathbb{R})$, which contradicts with Cantor's theorem. So by contradiction, $\nexists$ surjective function from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{R}}$. $\square$

(8) Use the definition of convergence directly to prove that the following sequences converge to the given limits:

(a) $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ (b) $\lim_{n\to\infty} \frac{n}{n+1} = 1$

(a) proof

Let $\varepsilon > 0$

Take $N > \frac{1}{\varepsilon}$ (by Archimedean property of $\mathbb{R}$)

So $\varepsilon > \frac{1}{N}$

Let $n \geq N$, then

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Since ε is arbitrary, we have proved $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

(b) Let $\varepsilon > 0$

Take $N > \frac{1}{\varepsilon} - 1$

$\Rightarrow N+1 > \frac{1}{\varepsilon} \Rightarrow \varepsilon > \frac{1}{N+1}$

Let $n \geq N$, then $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} \leq \frac{1}{N+1} < \varepsilon$

Since ε is arbitrary, we have proved $\lim_{n\to\infty} \frac{n}{n+1} = 1$

(9) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Prove that if $\lim_{n\to\infty} a_n = L \in \mathbb{R}$, then $\lim_{n\to\infty} |a_n| = |L|$.

Proof Let $\varepsilon > 0$

Fix $N \in \mathbb{N}$ s.t $|a_n - L| < \varepsilon$ whenever $n \geq N$

Since $(|a_n| - |L|)^2 = a_n^2 - 2|a_n L| + L^2$

$|a_n - L|^2 = a_n^2 - 2a_n L + L^2$

$\Rightarrow (|a_n| - |L|)^2 \leq |a_n - L|^2$

$\Rightarrow ||a_n| - |L|| \leq |a_n - L| < \varepsilon$

Since ε is arbitrary, $\lim_{n\to\infty} |a_n| = |L|$ $\square$

(10) Prove that for every $n \in \mathbb{N}$ and sequence (a_k) in $\mathbb{R}$, if (a_k) converges to L then $\lim a_k^n = L^n$

Proof We prove it by induction on n

Base case: $n=1$, $\lim_{k\to\infty} a_k = L = L^1$

Inductive step: assume $n=k$, $\lim_{k\to\infty} a_k^n = L^n$

then $\lim_{k\to\infty} a_k^{n+1} = \lim_{k\to\infty} a_k^n \cdot a_k$

$= \lim_{k\to\infty} a_k^n \cdot \lim_{k\to\infty} a_k$ by limit law

$= L^n \cdot L = L^{n+1}$

This finishes the proof that for all $n \in \mathbb{N}$,

if (a_k) converges to L then $\lim a_k^n = L^n$

$\square$

(11) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and for each $n \in \mathbb{N}$ let $s_n = a_{n+1} - a_n$. Prove that if (a_n) converges, then (s_n) converges to zero.

Proof Since (a_n) converges, $\lim a_n = L$ for some $L \in \mathbb{R}$

Let $\varepsilon > 0$. Fix $N \in \mathbb{N}$ s.t. $|a_n - L| < \frac{\varepsilon}{2}$ whenever $n \geq N$

So $|s_n - 0| = |a_{n+1} - a_n|$

Since $L - \frac{1}{2}\varepsilon < a_{n+1} < L + \frac{1}{2}\varepsilon$

$L - \frac{1}{2}\varepsilon < a_n < L + \frac{1}{2}\varepsilon$,

$\Rightarrow 0 < |a_{n+1} - a_n| < \frac{1}{2}\varepsilon - (-\frac{1}{2}\varepsilon) = \varepsilon$

So $|s_n - 0| < \varepsilon$

Since ε is arbitrary, $\lim_{n\to\infty} s_n = 0$ □

(12) Let S be a bounded nonempty subset of $\mathbb{R}$. Show that there is a sequence in S that converges to $\sup S$.

Proof Conside $b_n = \sup S - \frac{1}{n}$, $n \in \mathbb{N}$

By the definition of supermum, for any $n \in \mathbb{N}$, b_n is not an upper bound of S

So for each $n \in \mathbb{N}$, there exists some $a > b_n$ where $a \in S$. We take one such a as a_n for each b_n

(For sure, same a can be taken repeatedly)

Then (a_n) is a seq. in S

Let $\varepsilon > 0$

Take $N \in \mathbb{N}$ s.t. $N\varepsilon > 1$ ($\Longrightarrow \frac{1}{N} < \varepsilon$)

Then for any $n \geq N$,

$|b_n - \sup S| = \frac{1}{n} < \frac{1}{N} < \varepsilon$

Since $a_n > b_n$ and $a_n < \sup S$

$|a_n - \sup S| < |b_n - \sup S| < \varepsilon$

So $\lim_{n\to\infty} a_n = \sup S$ since ε is arbitrary $\square$

Optional Challenge Problems:

(13) (a) Prove that $[0, 1]$ cannot be expressed as the union of an indexed family of open intervals.

(b) Prove that $(0, 1)$ cannot be expressed as the intersection of an indexed family of closed intervals.

(a)

(14) Is the converse of Problem (11) true? That is, if (a_n) is a sequence in $\mathbb{R}$ and $\lim(a_{n+1} - a_n) = 0$, must (a_n) converge?