

- (1) For each statement about sets given below, either *prove* the statement if it is true for all sets, or else give a *counterexample* using specific sets if it is false.
- $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.
 - $(A \cup B) \setminus C \supseteq A \cup (B \setminus C)$.
 - $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.
 - $A \subseteq B$ if and only if $A \cap B = A$.

(a) Proof

Assume $x \in (A \cup B) \setminus C$

So $(x \in A \text{ or } x \in B)$, and $x \notin C$

$\Rightarrow (x \in A \text{ but } x \notin C) \text{ or } (x \in B \text{ but } x \notin C)$

This contains $x \in A$ or $(x \in B \text{ but } x \notin C)$

So $x \in A \cup (B \setminus C)$

Therefore $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

(b) Counterexample:

$$A = \{1, 2, 3, 4, 5\}, B = \{1, 2, 3\}, C = \{1, 2, 3, 4, 5\}$$

$$5 \in A \Rightarrow 5 \in A \cup (B \setminus C)$$

$$\text{But } 5 \notin (A \cup B) \setminus C$$

$$\text{So } (A \cup B) \setminus C \not\subseteq A \cup (B \setminus C)$$

(c) Counterexample.

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{1, 2, 3, 4, 5\}$$

$$C = \{1, 2, 3\}$$

$$4 \in A \setminus C \Rightarrow 4 \in (A \setminus B) \cup (A \setminus C)$$

But $4 \notin A \setminus (B \cup C)$

$$\text{So } A \setminus (B \cup C) \neq (A \setminus B) \cup (A \setminus C)$$

(d) Proof

Assume $A \subseteq B$ \Rightarrow if $x \in A$, then $x \in B$ \emptyset

Take $x \in A \cap B$, so $x \in A$

Take $x \in A \Rightarrow x \in B$ by \emptyset , so $x \in A \cap B$

$$\Rightarrow \text{Therefore } A \subseteq A \cap B, A \cap B \subseteq A \Rightarrow \underline{A = A \cap B}$$

Assume $A = A \cap B$

Fix $a \in A$ $\Rightarrow a \in A \cap B \Rightarrow a \in A$ and $a \in B$

So $a \in A$ implies $a \in B$, therefore $A \subseteq B$

Then we have proved $A = A \cap B$ iff $A \subseteq B$

(2) For each $n \in \mathbb{N}$, let $A_n = \{nk : k \in \mathbb{N}\}$.

(a) What is $A_2 \cap A_3$?

(b) Determine (i.e., give simple descriptions of) the sets $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$.

(a) $A_2 = \{2k | k \in \mathbb{N}\}$

$A_3 = \{3k | k \in \mathbb{N}\}$

So $x \in A_2 \cap A_3$ iff $2|x$ and $3|x$ (and $x \in \mathbb{N}$)

$\Leftrightarrow \gcd(2,3) | x$ (and $x \in \mathbb{N}$)
i.e. $6|x$ (and $x \in \mathbb{N}$)

So $A_2 \cap A_3 = \{6k | k \in \mathbb{N}\}$

(b) $\bigcup_{n=2}^{\infty} A_n = \{x \in \mathbb{N} : 2|x \text{ or } 3|x \text{ or } \dots\}$

= $\{x \in \mathbb{N} \text{ s.t. } x \text{ is some multiple of any natural number that } \geq 2\}$

= $\{x \in \mathbb{N} | x \geq 2\}$

$\bigcap_{n=2}^{\infty} A_n = \{x \in \mathbb{N} : 2|x \text{ and } 3|x \text{ and } \dots\}$

= $\{x \in \mathbb{N} \text{ s.t. } x \text{ has all natural numbers that } \geq 2 \text{ as factors}\} = \underline{\phi}$

- (3) (a) Guess a formula for $1 + 3 + \dots + (2n - 1)$ by evaluating the sum for $n = 1, 2, 3$, and 4.
 (For $n = 1$, the sum is simply 1).
 (b) Prove that your formula is correct using mathematical induction.

(a) $1 + 3 + \dots + (2n - 1)$

$$= \underbrace{1+2n-1}_{2n} + \underbrace{3+2(n-1)-1}_{2n} + \dots \\ = \frac{1}{2} \cdot 2n = \underline{\underline{n^2}}$$

(b) Proof by induction on n

Base case: $n=1, \sum_{k=1}^1 (2k-1) = 1 = n^2$

Inductive step: assume for $n=k, \sum_{k=1}^n (2k-1) = k^2$

Then for $n=k+1$

$$\begin{aligned} \sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^n (2k-1) + 2(4k+1) - 1 \\ &= k^2 + 2k + 1 = \underline{\underline{(k+1)^2}} \end{aligned}$$

This finishes the proof that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^n (2k-1) = k^2$$

.

- (4) Determine for which integers the inequality $2^n > n^2$ is true, and prove your claim by induction.

Pf. $n=0$ or $n \geq 5$

Case 1: $n=0$. $2^0=1$, $0^2=0$, $\Rightarrow 2^0 > 0^2$

Case 2: $n \geq 5$

We prove this by induction on n

Base case: $n=5$, $2^5=32$, $5^2=25 \Rightarrow 2^5 > 5^2$

Inductive step: assume for $n=k$ (where $k \in \mathbb{N}$,

$$\underline{k \geq 5}, \underline{2^k > k^2}$$

Then $2^{k+1} = 2 \cdot 2^k = 2^{k+2^k}$, $(k+1)^2 = k^2 + 2k + 1$

Note that $k^2 - (k+1)^2 = (k-2)k - 1$

Since $k \geq 5$, $k-2 \geq 3$, so $(k-2)k - 1 \geq 14 > 0$

Therefore $k^2 > (k+1)^2$

$$\text{So } \underline{2^{k+1}} = \underline{2^{k+2^k}} > \underline{2^k + k^2} > \underline{2^k + 2k + 1} > k^2 + 2k + 1$$

This finishes the proof that

for all integer $n \geq 5$, $\underline{2^n > n^2} (= \underline{(k+1)^2})$

(5) For each of the subsets of \mathbb{R} given in (a) – (x) below, state (i) whether or not the set is bounded above; (ii) whether or not it is bounded below; (iii) what the supremum is (if it exists); and what the infimum is (if it exists). You may write all your answers on one line, with no justification needed, as in the answer for (a) given below:

“Bounded below but not above; $\inf = 1$.”

- | | |
|---|--|
| (a) \mathbb{N} | (l) $\{r \in \mathbb{Q} : r < 2\}$ |
| (b) $[0, 1]$ | (m) $\{r \in \mathbb{Q} : r^2 < 4\}$ |
| (c) $\{2, 7\}$ | (n) $\{r \in \mathbb{Q} : r^2 < 2\}$ |
| (d) $\{\pi, e\}$ | (o) $\{x \in \mathbb{R} : x < 0\}$ |
| (e) $\{\frac{1}{n} : n \in \mathbb{N}\}$ | (p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$ |
| (f) $\{0\}$ | (q) $\{0, 1, 2, 4, 8, 16\}$ |
| (g) $[0, 1] \cup [2, 3]$ | (r) $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ |
| (h) $\bigcup_{n=1}^{\infty} [2n, 2n+1]$ | (s) $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$ |
| (i) $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1 + \frac{1}{n}]$ | (t) $\{x \in \mathbb{R} : x^3 < 8\}$ |
| (j) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$ | (u) $\{x^2 : x \in \mathbb{R}\}$ |
| (k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$ | (v) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$ |
| (w) $\bigcup_{n=1}^{\infty} \{\frac{k}{n} : k \in \mathbb{N}\}$ | (x) $\bigcap_{n=1}^{\infty} \{\frac{k}{n} : k \in \mathbb{N}\}$ |

(a) Bounded below but not above (l) Bounded above but not below
 $\inf = 1$ $\sup = 2$

(b) Bounded below and above (m) Bounded below and above
 $\inf = 0$, $\sup = 1$ $\inf = -2$, $\sup = 2$

(c) Bounded below and above (n) Bounded below and above
 $\inf = 2$, $\sup = 7$ $\inf = -\sqrt{2}$, $\sup = \sqrt{2}$

(d) Bounded below and above (o) Bounded above but not below
 $\inf = e$, $\sup = \pi$ $\sup = 0$

(e) Bounded below and above (p) Bounded below and above
 $\inf = 0$, $\sup = 1$ $\inf = 1$, $\sup = 10$

(f) Bounded below and above
 $\inf = \sup = 0$

(g) Bounded below and above
 $\inf = 0, \sup = 3$

(h) Bounded below but not above
 $\inf = 2$

(i) Bounded below and above
 $\inf = 0, \sup = 1$

(j) Bounded below and above
 $\inf = \frac{2}{3}, \sup = 1$

(k) Bounded below but not above
 $\inf = 0$

(l) Bounded below but not above
 $\inf = 0$

(q) Bounded below and above
 $\inf = 0, \sup = 16$

(r) Bounded below and above
 $\inf = \sup = 1$

(s) Bounded below and above
 $\inf = 0, \sup = \frac{1}{2}$

(t) Bounded above but not below
 $\sup = 2$

(u) Bounded below but not above
 $\inf = 0$

(v) Bounded below and above
 $\inf = -1, \sup = 1$

(x) Bounded below but not above
 $\inf = 1$

(6) The complex numbers form a *field*; that is, the algebraic structure $(\mathbb{C}, +, \cdot, 0, 1)$ satisfies our Axioms 1–9. In fact, \mathbb{C} also satisfies a version of the Completeness Axiom, so that \mathbb{C} is a *complete field*. Prove, however, that it is impossible to define a linear order relation $<$ on \mathbb{C} that makes \mathbb{C} an *ordered field*; i.e., it is impossible to define a linear order relation $<$ on \mathbb{C} that satisfies Axioms (13) and (14). [HINT: argue by contradiction. The *only* things you are allowed to use without proof are the ordered field axioms and the results in the handout “Elementary Properties of Real Numbers,” which hold in any ordered field.]

hw 1 ① \mathbb{C} 上不可以 define linear relation

PP: \mathbb{C} 无法成为 ordered field

Proof. Assume for contradiction that a linear relation $<$ is defined on \mathbb{C} such that axiom 13–14 stands.
that is: $\forall x, y, z \in \mathbb{C}$, if $x < y$ then $z+x < z+y$
if $x < y$ and $z > 0$ then $x \cdot z < y \cdot z$

Case 1: we define $i > 0$, so by axiom 14:
by multiplying i on both sides, $i \cdot i > 0 \cdot i$

$$\Rightarrow \underline{-1 > 0},$$

then multiply -1 on both sides $\Rightarrow \underline{1 > 0}$

$$\text{So by axiom 13, } \underline{-1+1 > 0+1} = 0$$

This contradicts with the definition of -1 that $-1+1=0$

Case 2: Define $i = 0 \Rightarrow i^2 = -1 = 0$ by axiom 4

$$\Rightarrow 1 = -(-1) = 0 \Rightarrow \text{contradicts with axiom 6.}$$

Case 3 : define $i < 0$

then $i = -a$ for some $a \in \mathbb{C}$ s.t. $a > 0$

$$\text{So } i^2 = (-a)(-a) = (-1)(-1)a^2 = a^2 > 0$$

$$\text{So } -1 > 0 \quad (\text{by axiom 5}) \quad (\text{by axiom 14})$$

Then by the same reason in case 1, it contradicts with the definition of -1 that $-1 + 1 = 0$ by axiom 4

Since in all cases, the assumption of linear order contradicts with the properties of \mathbb{C} . therefore we have proved that it is impossible to define a linear relation on \mathbb{C} s.t. axiom B-14 holds.

- (7) (a) Let $a, b \in \mathbb{R}$. Show that if $a \leq c$ for every $c > b$, then $a \leq b$.
- (b) Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$, and suppose L is an upper bound of A . Show that $L = \sup A$ if and only if for every $\epsilon > 0$ there is $a \in A$ such that $L - \epsilon < a \leq L$.

hw 1 ② 判定 $\sup A$ 的方法：减去任意 $\epsilon > 0$, 都存在 $a \in A$ 在 $L - \epsilon$ 和 L 之间

(a) suppose $a > b$ for contradiction

Then by the density of \mathbb{Q} in \mathbb{R} , $\exists q \in \mathbb{Q}$
s.t. $a > q > b$

Then by the given conditions, $a \leq q$, which

So $a \leq b$

contradicts with $a > b$

(b) One direction:

Assume $L = \sup A$ WTS: $\forall \varepsilon > 0, \exists a \in A$ st.

We prove this by assuming for contradiction $L - \varepsilon < a \leq L$
that for some $\varepsilon > 0$, $\nexists a \in A$ st. $L - \varepsilon < a \leq L$ \textcircled{D}

Since $L = \sup A$, by definition of supremum,

$\nexists a \in A$ st. $a > L$ \textcircled{D}

Therefore combining \textcircled{D} , $\nexists a \in A$ st. $a > L - \varepsilon$

So $L - \varepsilon$ is an upper bound of A

This contradicts with the definition of supreme

since $L - \varepsilon < L$, $L - \varepsilon$ can not be

The other direction an upper bound of A.

Assume $\forall \varepsilon > 0, \exists a \in A$ st $L - \varepsilon < a \leq L$

WTS: $L = \sup A$ (given the condition that L
is an upper bound of A)

let M be an arbitrary upper bound of A.

Suppose if $M < L$, then $\exists a \in A$ st. $M < a \leq L$
 \Rightarrow contradicts with M being

Therefore $M \geq L$. an upper bound of A

Since M is arbitrarily selected by definition $L = \sup A$

This completes the proof that $L = \sup A$ iff $\forall \varepsilon > 0, \exists a \in A$
s.t. $L - \varepsilon < a \leq L$

(8) Let S and T be nonempty bounded subsets of \mathbb{R} .

(a) Prove that $\inf S \leq \sup S$.

(b) Supposing that $S \subseteq T$, put the four numbers $\sup S$, $\inf S$, $\sup T$, $\inf T$ in order (with respect to \leq), and prove your claims.

(c) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

hw 1 ③
 ~~$\sup(S \cup T) =$~~

(a) Proof Take arbitrary $s \in S$ $\max\{\sup S, \sup T\}$

So by definition of upper and lower bound,
 $\inf S \leq s$, $\sup S \geq s$

So $\inf S \leq \sup S$ by the transitivity of linear order and equivalence relation.

(b) $\inf T \leq \inf S \leq \sup S \leq \sup T$

Proof by (a) we have proved $\inf S \leq \sup S$

So it suffices to prove that $\inf T \leq \inf S$
and $\sup S \leq \sup T$

let $s \in S$, since $S \subseteq T \Rightarrow S \subseteq T$

therefore if $a > t$ for all $t \in T$

then also, $a > s$ for all $s \in S$

This shows that any upper bound of T is also an upper bound of S .

Dually, any lower bound of T is also a lower bound of S

Therefore $\{\text{lower bounds of } T\} \subseteq \{\text{lower bounds of } S\}$

and $\{\text{upper bounds of } T\} \subseteq \{\text{upper bounds of } S\}$

Since $\inf S = \max \{\text{lower bounds of } S\}$

and $\inf T \in \{\text{lower bounds of } T\} \subseteq \{\text{lower bounds of } S\}$

$$\Rightarrow \underline{\inf S \geq \inf T}$$

Dually, $\sup S = \min \{\text{upper bounds of } S\}$

and $\sup T \in \{\text{upper bounds of } T\} \subseteq \{\text{upper bounds of } S\}$

$$\Rightarrow \underline{\sup T \leq \sup S}$$

This finishes the proof that $\inf T \leq \inf S \leq \sup S \leq \sup T$

(c) We first claim: $\underline{\max(\sup S, \sup T)}$ is
an upper bound of $S \cup T$

Let x be an arbitrary element of $S \cup T$

Then $x \in S$ or $x \in T$

If $x \in S$, then $x \leq \sup S$ by definition

$$\text{so } x \leq \max(\sup S, \sup T)$$

If $x \in T$, then $x \leq \sup T$ by definition

$$\text{so } x \leq \max(\sup S, \sup T)$$

Since in both cases, $\pi \leq \max(\sup S, \sup T)$ and π is arbitrary, we have proved that $\max(\sup S, \sup T)$ is an upper bound of $S \cup T$.

Then we prove that any upper bound of $S \cup T$
 $\geq \max(\sup S, \sup T)$

Let b be an arbitrary upper bound of $S \cup T$.

So b is an upper bound of both S and T .

Assume for contradiction that $b < \max(\sup S, \sup T)$.

Then b is less than at least one of $\sup S, \sup T$.

Without loss of generality, suppose $b < \sup S$.

Then b is not a upper bound of S , by definition

This contradicts with the fact that b is an upper bound of S .

Therefore $b \geq \max(\sup S, \sup T)$

This finishes the proof that $\sup(S \cup T) = \max(\sup S, \sup T)$

(9) Let A and B be nonempty bounded subsets of \mathbb{R} , and let $A+B = \{a+b : a \in A \text{ and } b \in B\}$.

Prove that $\sup(A+B) = \sup A + \sup B$.

Proof ① First we claim: $\sup A + \sup B$ is an upper bound of $A+B$

Let $a+b$ be an arbitrary element of $A+B$ ($a \in A, b \in B$)

Then $\sup A > a$, $\sup B > b$ by definition

So $\sup A + \sup B > a + \sup B > a + b$ by axiom 13

Since $a+b$ is arbitrary, we have proved that

$\sup A + \sup B$ is an upper bound of $A+B$

This implies: $\sup(A+B) \leq \sup A + \sup B$

② Now we show $\sup A + \sup B \leq \sup(A+B)$

Assume for contradiction: $\sup(A+B) < \sup A + \sup B$

$\Rightarrow \sup(A+B) = \sup A + \sup B - \varepsilon$ for some $\varepsilon > 0$

$$\sup(A+B) = (\sup A - \frac{\varepsilon}{2}) + (\sup B - \frac{\varepsilon}{2})$$

By definition of supreme, $(\sup A - \frac{\varepsilon}{2})$ is not an upper bound of $A \Rightarrow \exists$ some $a_0 \in A$ s.t. $a_0 > \sup A - \frac{\varepsilon}{2}$
Similarly, then \exists some $b_0 \in B$ s.t. $b_0 > \sup B - \frac{\varepsilon}{2}$

$\Rightarrow a_0 + b_0 \in A+B$, but $a_0 + b_0 > \sup(A+B)$ \Rightarrow contradiction

So $\sup A + \sup B \leq \sup(A+B)$

By (1)②, we have proved $\sup A + \sup B = \sup(A+B)$

- (10) Prove that $\mathbb{R} \setminus \mathbb{Q}$ is *dense* in \mathbb{R} in the sense that for every pair of real numbers a and b , if $a < b$ then there exists an irrational number r such that $a < r < b$.

Proof. Take arbitrary $a, b \in \mathbb{R}$ st $a < b$,

Then $b = a + \varepsilon$ for some $\varepsilon \in \mathbb{R}$ st $\varepsilon > 0$

By the Archimedean property of \mathbb{R} ,

there exists $n \in \mathbb{N}$ st. $n\varepsilon > 1$ $\Rightarrow \varepsilon > \frac{1}{n}$

So consider $\varepsilon' = \frac{1}{n\sqrt{2}} = \underbrace{\frac{1}{n} \cdot \frac{\sqrt{2}}{2}}$

which is an irrational number
since $\frac{1}{n} \in \mathbb{Q}$ and $\frac{\sqrt{2}}{2}$ is irrational

Also, $\varepsilon' < \varepsilon$ since $\frac{\sqrt{2}}{2} < 1$,

by axiom 14, $\frac{1}{n} \cdot \frac{\sqrt{2}}{2} < \frac{1}{n} \cdot 1 = \frac{1}{n}$

Therefore $a < a + \varepsilon' < \underbrace{a + \varepsilon}_{(=b)}$

Then we have proved that $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} .

A set $A \subseteq \mathbb{R}$ is *discrete* if for every $a \in A$ there is $\epsilon > 0$ such that $V_\epsilon(a) \cap A = \{a\}$, where $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ is the open interval of radius ϵ centered at a .

(11) (a) Prove that every finite subset of \mathbb{R} is discrete.

(b) Either prove the following if it is true, or else give a counterexample if it is false: if $A \subseteq \mathbb{R}$ is discrete, then there is $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$.

hw 1 Q finite $A \subseteq \mathbb{R}$ is discrete

(a) Proof.

Let $A \subseteq \mathbb{R}$ be an arbitrary finite set and consider: $B = \{\|a - x\| \mid x \in A\}$ $a \in A$ be arbitrary

This is a finite set since A is finite,

\Rightarrow So B has a smallest element

Let $\epsilon = \min(B)$, then $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$
where ϵ is the distance of a from its nearest element in X

$\Rightarrow V_\epsilon(a) \cap A = \{a\}$

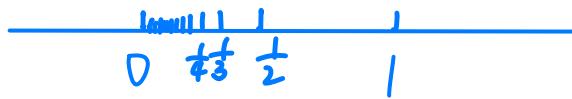
Since a is arbitrary, A is discrete

Since A is arbitrary we have proved that every finite subset of \mathbb{R} is discrete

(b) False

Consider the following counterexample:

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$



This is a discrete set since for any element

$\frac{t}{n} \in A$, consider $\varepsilon = \frac{1}{nt}$

$$\text{So } V_\varepsilon\left(\frac{t}{n}\right) = \left(\frac{1}{nt}, \frac{2}{n} - \frac{1}{nt}\right) \Rightarrow V_\varepsilon\left(\frac{t}{n}\right) \cap A = \underline{\{\frac{t}{n}\}}$$

But such ε does not exist on A

Since if so, then $\frac{1}{n} - \frac{1}{nt} > \varepsilon$ for all $n \in \mathbb{N}$

$$\varepsilon < \frac{1}{n(nt)} \text{ for all } n \in \mathbb{N}$$

$$\Rightarrow n(nt)\varepsilon < 1 \text{ for all } n \in \mathbb{N}$$

which contradicts with the

Archimedean property of \mathbb{R}

(12) OPTIONAL CHALLENGE PROBLEM.¹ For $A, B \subseteq \mathbb{R}$, let $AB = \{ab : a \in A \text{ and } b \in B\}$.

Find a simple expression for $\sup(AB)$ in the case where A and B are nonempty and bounded, and prove your result.

$$\sup(AB) = \max \{ \inf A \cdot \inf B, \inf A \cdot \sup B, \sup A \cdot \inf B, \sup A \cdot \sup B \}$$

