

Recall that for any sets X and Y , we write $X \preceq Y$ if there exists an injective function from X to Y , and $X \approx Y$ if there exists a bijective function from X to Y . By the Cantor-Schroder-Bernstein Theorem, $X \approx Y$ if and only if $X \preceq Y$ and $Y \preceq X$.

$$h_wz \text{ (1)} \quad \left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$$

- (1) Use induction and the triangle inequality for real numbers to prove that for all $n \in \mathbb{N}$ and for all $a_1, \dots, a_n \in \mathbb{R}$,

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|.$$

Proof. We prove it by induction on $n \in \mathbb{N}$.

Base case: $n = 1$, $\left| \sum_{k=1}^1 a_k \right| = |a_1| = \sum_{k=1}^1 |a_k|$

the claim holds true.

Inductive step: assume the inequality holds true for all $a_1, \dots, a_n \in \mathbb{R}$,
for $n = 1, 2, \dots, j$

Then $\left| \sum_{k=1}^{j+1} a_k \right| = \left| \sum_{k=1}^j a_k + a_{j+1} \right|$

By inductive hypothesis when $n = j$,

$$\left| \sum_{k=1}^{j+1} a_k \right| = \left| \sum_{k=1}^j a_k + a_{j+1} \right| \leq \left| \sum_{k=1}^j a_k \right| + |a_{j+1}| \quad \text{(1)}$$

By inductive hypothesis when $n = j$,

$$\left| \sum_{k=1}^j a_k \right| \leq \sum_{k=1}^j |a_k| \quad \text{(2)}$$

Combining (1)(2), we get $\left| \sum_{k=1}^{j+1} a_k \right| \leq \sum_{k=1}^{j+1} |a_k|$

This finishes the proof of the inequality by induction

(2) Let $A \subseteq \mathbb{R}$ be bounded, let $c \in \mathbb{R}$, and write $cA = \{ca : a \in A\}$. Find an expression for $\sup(cA)$, and prove your claim. Then state (but do not prove) the “dual” claim for $\inf(cA)$.

$$\sup(cA) = \begin{cases} c\sup A, & c \geq 0 \\ c\inf A, & c < 0 \end{cases} \quad \inf(cA) = \begin{cases} c\inf A, & c \geq 0 \\ c\sup A, & c < 0 \end{cases}$$

Proof. 1° let $c > 0$, take arbitrary $a \in A$, so ca is an arbitrary element of cA

Since $\sup A \geq a$, $c\sup A \geq ca$

So $c\sup A$ is an upper bound of cA ①

let cb be an upper bound of cA

then $cb \geq ca$ for all $a \in A$

Since $c > 0 \Rightarrow b \geq a$, so b is an upper

bound of $A \Rightarrow b \geq \sup A \Rightarrow cb \geq c\sup A$

By ①②, $c\sup A = \sup(cA)$ when $c > 0$ ②

2° let $c = 0$, then $cA = \{0\}$, so $c\sup A = 0 = \sup(cA)$

3° let $c < 0$, take arbitrary $a \in A$, so ca is an arbitrary element of cA

Since $\inf A \leq a$, $c\inf A \geq ca$

So $c\inf A$ is an upper bound of cA ③

let cb be an upper bound of cA

then $cb \geq ca$ for all $a \in A$

Since $c < 0 \Rightarrow b \leq a$, so b is a lower

bound of $A \Rightarrow b \leq \inf A \Rightarrow cb \geq c\inf A$ ④

By ③④, $c\inf A = \sup(cA)$ when $c < 0$

□

- (3) Let X , Y , and Z be any sets, and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions.
- (a) Prove that if f and g are injective, then $g \circ f$ is injective.
- (b) Prove that \leq is reflexive and transitive; that is, prove that $X \leq X$ and that if $X \leq Y$ and $Y \leq Z$, then $X \leq Z$. *hw2 ② \leq is a partial order*

(a) Proof Suppose f and g are injective

Let $g \circ f(a) = g \circ f(b)$, i.e. $g(f(a)) = g(f(b))$
where $a, b \in \text{dom}(g \circ f)$

Since g is injective $\Rightarrow f(a) = f(b)$

Then since f is injective $\Rightarrow a = b$

Therefore for arbitrary $a, b \in \text{dom}(g \circ f)$,

$$g \circ f(a) = g \circ f(b) \Rightarrow a = b$$

So by definition, $g \circ f$ is injective

□

(b) Proof Let X be an arbitrary set

Consider $f : X \rightarrow X$ defined by $f(x) = x$

f is injective by the uniqueness of any element in a set.

So $X \leq X$ by definition, therefore \leq is reflexive.

Let $X \leq Y$ and $Y \leq Z$

So there exists some injective function $f : X \rightarrow Y$

and $g : Y \rightarrow Z$

By (a) we know $g \circ f : X \rightarrow Z$ is injective,

so $X \leq Z$, therefore \leq is transitive

□

(4) Let A and B be any nonempty sets.

(a) Prove that if $A \subseteq B$ then $A \preceq B$.

(b) Prove that there is an injective function from A to B if and only if there is a surjective function from B to A .

hw 2 (3) $A \subseteq B \Rightarrow A \preceq B$

Proof (a) Assume $A \subseteq B$, so $\forall x \in A, x \in B$

\Rightarrow consider $f: A \rightarrow B$ defined by $f(x) = x$

f is injective by the uniqueness of any element in a set.

Therefore $A \preceq B$ by definition

(b) ① Suppose $f: A \rightarrow B$ is injective

Let $a \in A$ be arbitrary

Then consider $g: B \rightarrow A$ defined by

$$g(x) = \begin{cases} b \in f^{-1}(\{x\}), & \text{if } x \in \text{ran}(f) \\ a, & \text{if } x \notin \text{ran}(f) \end{cases}$$

This function is well-defined since f is injective,
there is only one element in $f^{-1}(\{x\})$ for any $x \in B$

So $\text{ran}(g) = A \Rightarrow g$ is surjective

② Suppose $g: B \rightarrow A$ is surjective

Then for any $a \in A$, exists some $b \in B$ s.t. $f(b) = a$

i.e. $g^{-1}(\{a\}) \neq \emptyset$ for any $a \in A$

$f: A \rightarrow B$ sending every $a \in A$ to some $b \in g^{-1}(\{a\})$

The well-definedness of f is guaranteed by $g^{-1}(\{a\}) \neq \emptyset$

And f is injective since by the well-definedness of g ,
for any $a \in A$

$\forall a_1, a_2 \in A, g^{-1}(\{a_1\}) \cap g^{-1}(\{a_2\}) = \emptyset$, so every $a \in A$ is mapped to a

By ①② we have finished the if-and-only-if proof distinct $b \in B$

□

- (5) (a) Prove that if A is an infinite set and A_0 is a finite subset of A , then $A \approx A \setminus A_0$.
 (b) Prove that if A is an uncountable set and A_0 is a countable subset of A , then $A \approx A \setminus A_0$.

(a) Proof ① first we construct $f: A \setminus A_0 \rightarrow A$

defined by $f(a) = a$

It is injective by the uniqueness of any element in a set.

So $A \setminus A_0 \leq A$

② $A_0 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for some $\alpha_1, \alpha_2, \dots, \alpha_n \in A$

since A_0 is finite

hw 2 ④

Since A is infinite and A_0 is finite
 infinite set | $\Rightarrow A \setminus A_0$ is infinite, (otherwise $|A \setminus A_0| = m \in \mathbb{N}$,
 finite set so $|A| \leq m+n \in \mathbb{N}$,
 contradicting the fact that A is infinite)

unctb set | So no matter $A \setminus A_0$ is countable or uncountable, we can take a countable
 ctb set subset $A_1 \subseteq A \setminus A_0$ and denote $A_1 = \{y_1, y_2, \dots\}$

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Now we construct $f: A \rightarrow A \setminus A_0$

defined by $f(x) = \begin{cases} x, & \text{if } x \in A \setminus A_0 \setminus A_1 \\ y_{2k}, & x = \alpha_k \text{ for some } k \in \mathbb{N} \\ y_{2k-1}, & x = \alpha_k \text{ for some } k \in \mathbb{N}. \end{cases}$

$f(x)$ is well-defined since $(A \setminus A_0 \setminus A_1) \cup A_0 \cup A_1 = A$

and $f(x)$ is injective since $(A \setminus A_0 \setminus A_1) \cap A_0 \cap A_1 = \emptyset$

So $A \leq A \setminus A_0$

Then by Cantor-Schröder-Bernstein Thm,
 $A \approx A \setminus A_0$ □

(b) Proof. Since A_0 is countable, $\underline{A_0 = \{\bar{z}_1, \bar{z}_2, \dots\}}$ for some $\bar{z}_1, \bar{z}_2, \dots \in A$
 Take countably infinite subset $\underline{A_1 = \{y_1, y_2, \dots\}} \subseteq A \setminus A_0$

We construct $f: A \rightarrow A \setminus A_0$ defined by

$$f(x) = \begin{cases} x, & x \in (A \setminus A_0) \setminus A_1 \\ y_{2k}, & x = \bar{z}_k \text{ for some } k \in \mathbb{N} \\ y_{2k-1}, & x = y_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then f is well-defined since $(A \setminus A_0 \setminus A_1) \cup A_0 \cup A_1 = A$

f is injective since $(A \setminus A_0 \setminus A_1) \cap A_0 \cap A_1 = \emptyset$

f is surjective since for any $a \in A \setminus A_0$,

$$a \in (A \setminus A_0) \setminus A_1 \text{ or } a \in A_1$$

And in both cases $\exists x \text{ s.t. } f(x) = a$

So f is bijective

Therefore $A \simeq A \setminus A_0$

□

(6) (a) Prove that $\overline{\mathbb{Q}}$ is countable. Conclude that there are uncountably many transcendental real numbers.

(b) Prove that for all $a, b \in \mathbb{R}$, if $a < b$ then there are uncountably many transcendental numbers in the interval (a, b) .

hw 2 ⑤ $\overline{\mathbb{Q}}$ is cba, BNR \mathbb{Q}

(a) let A_k be the set of all roots of unctb polynomials with rational numbers as coefficients with k terms

So by definition, $\overline{\mathbb{Q}} = \bigcup_{k \in \mathbb{N}} A_k$

Let k be an arbitrary natural number

Let $A_{k,q}$ where $q = (\frac{b_1}{a_1}, \frac{b_2}{a_2}, \dots, \frac{b_k}{a_k}) \in \mathbb{Q}^k$

be the polynomial with $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \dots, \frac{b_k}{a_k}$ as coefficients

So $A_k = \bigcup_{q \in \mathbb{Q}^k} A_{k,q}$

Since $\overline{\mathbb{Q}} \subseteq \mathbb{C}$, by the fundamental theorem of algebra, $A_{k,q}$ has at most k roots

So for each $q \in \mathbb{Q}^k$, $A_{k,q}$ is finite.

By the thm on lecture 4, since \mathbb{Q} is countable,
 $\Rightarrow \mathbb{Q}^k$ is countable, so $A_k = \bigcup_{q \in \mathbb{Q}^k} A_{k,q}$ is countable

Therefore for each $k \in \mathbb{N}$, A_k is countable

And since \mathbb{N} is countable

We have proved that $\overline{\mathbb{Q}} = \bigcup_{k \in \mathbb{N}} A_k$ is countable

Conclusion: Since $\mathbb{C} \cong \mathbb{R}^2$ is uncountable, \square

and $\overline{\mathbb{Q}}$ is countable, by problem (b),

$\mathbb{C} \setminus \overline{\mathbb{Q}}$ is uncountable ($\mathbb{C} \setminus \overline{\mathbb{Q}} \approx \mathbb{C}$)

which indicates that there are uncountably many transcendental numbers.

(b) Let $a, b \in \mathbb{R}$ be arbitrary with $a < b$

Let \mathbb{Q}_0 be the set of all algebraic numbers in (a, b)

So $\mathbb{Q}_0 \subseteq \overline{\mathbb{Q}}$, since $\overline{\mathbb{Q}}$ is countable, \mathbb{Q}_0 is countable

Since we have proved that (a, b) is uncountable,

$(a, b) \setminus \mathbb{Q}_0$ is uncountable by problem 5(b)

So there are uncountably many transcendental numbers in (a, b)

\square

(7) Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions from \mathbb{R} to \mathbb{R} .

hw2(1) $\mathbb{R} \subset P(\mathbb{R}) \subset \mathbb{R}^{\mathbb{R}}$

(a) Prove that $P(\mathbb{R}) \leq \mathbb{R}^{\mathbb{R}}$.

(b) Prove that there is no surjective function from \mathbb{R} to $\mathbb{R}^{\mathbb{R}}$. (This shows that there are strictly more functions from \mathbb{R} to \mathbb{R} than there are real numbers.)

(a) Let $A \in P(\mathbb{R})$ be an arbitrary element
Proof. so $A \subseteq \mathbb{R}$

Consider the function $f_A : \mathbb{R} \rightarrow \mathbb{R}$

defined by $f_A(x) = \begin{cases} x, & x \in A \\ 0, & x \notin A \end{cases}$

Then we consider $\varphi : P(\mathbb{R}) \rightarrow \mathbb{R}^{\mathbb{R}}$ defined by

defined by $\varphi(A) = f_A(x)$ for each $A \in P(\mathbb{R})$

φ is injective since if $\varphi(A) = \varphi(B)$, then

$$f_A(x) = f_B(x) \Rightarrow \begin{array}{l} \forall x \in A, x \in B \Rightarrow A \subseteq B \\ \forall x \in B, x \in A \Rightarrow B \subseteq A \end{array} \Rightarrow A = B$$

Therefore $P(\mathbb{R}) \leq \mathbb{R}^{\mathbb{R}}$

□

(b) Assume for contradiction that there exist some
Proof surjective function from \mathbb{R} to $\mathbb{R}^{\mathbb{R}}$

Then by problem 4(b), \exists some injection function from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{R} \Rightarrow \mathbb{R}^{\mathbb{R}} \leq \mathbb{R}$. Together with $P(\mathbb{R}) \leq \mathbb{R}^{\mathbb{R}}$,
 $P(\mathbb{R}) \leq \mathbb{R}$ by problem 3(b).

So there exists some surjective
function from \mathbb{R} to $P(\mathbb{R})$, which contradicts with
Cantor's theorem. So by contradiction, \nexists surjective function
from \mathbb{R} to $\mathbb{R}^{\mathbb{R}}$

□

(8) Use the definition of convergence directly to prove that the following sequences converge to the given limits:

$$(a) \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

$$(b) \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

(a) let $\epsilon > 0$

proof Take $N > \frac{1}{\epsilon}$ (by Archimedean property of \mathbb{R})

$$\text{So } \epsilon > \frac{1}{N}$$

let $n \geq N$, then

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Since ϵ is arbitrary, we have proved $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$

(b) let $\epsilon > 0$

$$\text{Take } N > \frac{1}{\epsilon} - 1$$

$$\Rightarrow N+1 > \frac{1}{\epsilon} \Rightarrow \epsilon > \frac{1}{N+1}$$

$$\text{let } n \geq N, \text{ then } \left| \frac{n}{N+1} - 1 \right| = \frac{1}{N+1} \leq \frac{1}{n+1} < \epsilon$$

$$\text{Since } \epsilon \text{ is arbitrary, we have proved } \lim_{n \rightarrow \infty} \frac{n}{N+1} = 1$$

(9) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} . Prove that if $\lim_{n \rightarrow \infty} a_n = L \in \mathbb{R}$, then $\lim_{n \rightarrow \infty} |a_n| = |L|$.

Proof Let $\varepsilon > 0$ \text{Inv 2 } \textcircled{1} \lim(a_n) = L \Rightarrow \lim(|a_n|) = |L|,
Fix $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ whenever $n \geq N$
Since $(|a_n| - |L|)^2 = a_n^2 - 2(a_n L) + L^2 \lim(a_n^k) = L^k$
 $|a_n - L|^2 = a_n^2 - 2a_n L + L^2$
 $\Rightarrow (|a_n| - |L|)^2 \leq |a_n - L|^2$
 $\Rightarrow ||a_n| - |L|| \leq |a_n - L| < \varepsilon$
Since ε is arbitrary, $\lim_{n \rightarrow \infty} |a_n| = |L|$ □

(10) Prove that for every $n \in \mathbb{N}$ and sequence (a_k) in \mathbb{R} , if (a_k) converges to L then $\lim a_k^n = L^n$

Proof We prove it by induction on n

Base case: $n=1$, $\lim_{k \rightarrow \infty} a_k = L = L^1$

Inductive step: assume $n=k$, $\lim_{k \rightarrow \infty} a_k^k = L^k$

$$\text{then } \lim_{k \rightarrow \infty} a_k^{n+1} = \lim_{k \rightarrow \infty} a_k^k \cdot a_k$$

$$= \lim_{k \rightarrow \infty} a_k^k \cdot \lim_{k \rightarrow \infty} a_k \text{ by limit law}$$

$$= L^k \cdot L = L^{k+1}$$

This finishes the proof that for all $n \in \mathbb{N}$,

if (a_k) converges to L then $\lim a_k^n = L^n$

□

- (11) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} , and for each $n \in \mathbb{N}$ let $s_n = a_{n+1} - a_n$. Prove that if (a_n) converges, then (s_n) converges to zero.

Proof Since (a_n) converges, $\lim a_n = l$ for some $l \in \mathbb{R}$.
Let $\epsilon > 0$. Fix $N \in \mathbb{N}$ s.t. $|a_n - l| < \frac{\epsilon}{2}$ whenever $n \geq N$.
So $|s_n - 0| = |a_{n+1} - a_n|$.
Since $l - \frac{1}{2}\epsilon < a_{n+1} < l + \frac{1}{2}\epsilon$,
 $l - \frac{1}{2}\epsilon < a_n < l + \frac{1}{2}\epsilon$,
 $\Rightarrow 0 < |a_{n+1} - a_n| < \frac{1}{2}\epsilon - (-\frac{1}{2}\epsilon) = \epsilon$.
So $|s_n - 0| < \epsilon$.
Since ϵ is arbitrary, $\lim_{n \rightarrow \infty} s_n = 0$. \square

- (12) Let S be a bounded nonempty subset of \mathbb{R} . Show that there is a sequence in S that converges to $\sup S$.

hw2 Q8 S bounded $\Rightarrow \exists (a_n) \rightarrow \sup S$ in S

Proof Consider $b_n = \sup S - \frac{1}{n}$, $n \in \mathbb{N}$.
By the definition of supremum, for any $n \in \mathbb{N}$,
 b_n is not an upper bound of S .
So for each $n \in \mathbb{N}$, there exists some $a > b_n$ where $a \in S$. We take one such a as a_n for each b_n .
(For sure, same a can be taken repeatedly).
Then (a_n) is a seq. in S .

Let $\varepsilon > 0$

Take $N \in \mathbb{N}$ s.t. $N\varepsilon > 1 \left(\Rightarrow \frac{1}{N} < \varepsilon\right)$

Then for any $n \geq N$,

$$|b_n - \sup S| = \frac{1}{n} < \frac{1}{N} < \varepsilon$$

Since $a_n > b_n$ and $a_n < \sup S$

$$|a_n - \sup S| < |b_n - \sup S| \underset{\sim}{<} \varepsilon$$

So $\lim_{n \rightarrow \infty} a_n = \sup S$ since ε is arbitrary

□

Optional Challenge Problems:

- (13) (a) Prove that $[0, 1]$ cannot be expressed as the union of an indexed family of open intervals.
- (b) Prove that $(0, 1)$ cannot be expressed as the intersection of an indexed family of closed intervals.
- (14) Is the converse of Problem (11) true? That is, if (a_n) is a sequence in \mathbb{R} and $\lim(a_{n+1} - a_n) = 0$, must (a_n) converge?

