Homework 6: Due Tuesday, June 18, at 11:59pm, on Gradescope

(1) Let (a_n) and (b_n) be bounded sequences in $\mathbb{R}$, and suppose $\lim a_n = A > 0$. Show that $\limsup(a_n b_n) = A \cdot \limsup(b_n)$. *Hint: recall that* $\limsup(s_n)$ *is the largest subsequential limit of* (s_n).

(2) (a) For each $n \in \mathbb{N}$, find the nth derivative of the function $y = x^n$, and prove by induction that your claim is correct.

(b) For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^n & \text{if } x \geq 0; \\ -x^n & \text{if } x < 0. \end{cases}$$

Show that f_{n+1} is n-times differentiable but not $(n+1)$-times differentiable.

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Prove by induction that for all $n \geq 0$, if $f : \mathbb{R} \to \mathbb{R}$ is $(n+1)$-times differentiable and $f^{(n+1)}(x) = 0$ for all $x \in \mathbb{R}$, then f is a polynomial of degree at most n.

(4) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, but $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\epsilon}}$ converges for all $\epsilon > 0$.

(5) Show that $\displaystyle\sum \frac{(-1)^n}{n^{1+1/n}}$ converges conditionally.

(6) Given an example of a sequence (a_n) of positive real numbers converging to zero such that $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. Prove that your example works.

(7) In each of the following, determine whether the given series converges or diverges, and justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{1/n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} (\cos \pi n) \ln \left(1 + \frac{1}{n}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$

(8) Let (a_k) and (b_k) be sequences of real numbers. Prove that if the series $\sum a_k^2$ and $\sum b_k^2$ both converge, then $\sum a_k b_k$ converges absolutely.

[Hint: use the Cauchy-Schwarz inequality, which says that $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$ for all $n \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$, where $\cdot$ is the usual dot product and $\| \ \|$ is the Euclidean metric on $\mathbb{R}^n$].

In the problems below, integrable *means* Riemann integrable*, or, equivalently,* Darboux integrable. *In each problem concerning integration, you may choose whether to use the Riemann theory or the Darboux theory in solving the problems. Throughout problems (12) – (16), assume* $a < b$.

(9) Show that if the function f is integrable on $[a, b]$, then f is integrable on every subinterval $[c, d] \subseteq [a, b]$.

(10) Suppose the function f is integrable on $[a, b]$. Show that for every infinite subset $S \subseteq [a, b]$, there is a function $g : [a, b] \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in [a, b] \setminus S$ and g is not integrable. [cf: 7.1.3 in text.]

(11) Show directly that if the bounded function $f : [a, b] \to \mathbb{R}$ is continuous everywhere in $[a, b]$ except possibly at one point $x_0 \in (a, b)$, then f is integrable on $[a, b]$.

(12) Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove that there is $x_0 \in (a, b)$ such that $f(x_0) = g(x_0)$.

Optional Challenge Problems:

(13) Suppose that for each $n \in \mathbb{N}$, V_n is an open subset of $\mathbb{R}$. For each n define the function $f_n : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_n(x) = \begin{cases} 0 & \text{if } x \in V_n \\ 2^{-n} & \text{if } x \in \mathbb{Q} \setminus V_n \\ -2^{-n} & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \setminus V_n). \end{cases}$$

Then define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that for all $a \in \mathbb{R}$, f is continuous at a if and only if $a \in \bigcap_{n \in \mathbb{N}} V_n$.