· Lecture 1. What are real numbers?

We will denote by $\mathbb{Z}$ the set of integers:

$$\{\cdots -2, -1, 0, 1, 2, 3, \cdots\}$$

Denote by $\mathbb{Q}$ the set of rational numbers:

$$\left\{\frac{p}{q} : p \in \mathbb{Z},\ q \text{ non-zero integer}\right\}$$

The rational number is not inadequate for the purpose of analysis.

- There is a no rational number r such that $r^2 = 2$. $\Rightarrow$ have to introduce irrational numbers (can be expressed as infinite decimal expansions)

- The solution can be "approximated" by the sequence of finite decimals

1, 1.4, 1.41, 1.414,

why is there a number whose square is 2?

But what is the meaning of "approximate"? $\Rightarrow$ Have to define $\sqrt{2}$.

$\Rightarrow$ Construction of "Real number system".

- Define $A = \{p \in \mathbb{Q} : p^2 < 2\}$, $B = \{p \in \mathbb{Q} : p^2 > 2\}$.

A is bounded above but contains no largest number

B is bounded below but contains no smallest number. $\Rightarrow$ The "rational number system" has gaps.

Thm There exists an ordered field $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield, satisfying the least-upper-bound property

Ordered sets (A necessary notion)

Read as less than

Def: S = set. An order on S is a relation "$<$" such that

① If $x \in S$ and $y \in S$, the one and only one of the following

$$x<y \qquad x=y \qquad y<x$$

holds.

② If $x, y, z \in S$, if $x<y$, $y<z$, then $x<y<z$.

Rmk (Remark) $y>x \Leftrightarrow x<y$, $x \leq y \Leftrightarrow x<y$ or $x=y$ etc.

The pair $(S,<)$ is called an ordered set. Example: $(\mathbb{Q},<)$ $x<y$ iff $y-x$ is positive.

There exists such that

Def. $E \subset S$ a subset of ordered set. If $\exists \beta \in S$ s.t. $x \leq \beta$ for all $x \in E$, we say E is bounded above, and β is an upper bound of E.

Replace $\leq$ by $\geq$ to define lower bound.

Def Suppose S is an ordered set, $E \subset S$ bounded above. We say α is a least upper bound or supremum of E if ① α is an upper bound of E ② if $\gamma < \alpha$ then γ is not an upper bound of E.

denote by $\sup E$

Greatest lower bound can be defined similarly. $\inf E$

Example: $E = \{p \in \mathbb{Q} : p = \frac{1}{n}\}$ $\quad \inf E = 0 \notin E$

Def An ordered set S is said to have the least-upper-bound property if for any non-empty $E \subset S$ bounded above, the $\sup E$ exists in S.

Thm Suppose S is an ordered set w/ the least-upper-bound property, B is a non-empty subset which is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S and $\alpha = \inf B$. / Proof If $\gamma \in L$, $\gamma \leq x \in B$

$\sup L \leq x \Rightarrow \sup L$ is a lower bound of B. $\sup L < x \Rightarrow \gamma < x \ \forall \gamma \in L$

x is not a lower bound.