SOME PROOFS INVOLVING CARDINALITY

Theorem 1 (Cantor-Schröder-Bernstein)**.** *Let X and Y be sets, and suppose there exist injective functions $f : X \to Y$ and $g : Y \to X$. Then there exists a bijective function $h : X \to Y$.*

Proof. Suppose the functions $f : X \to Y$ and $g : Y \to X$ are injective. Define the function $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\varphi(A) \;=\; X \setminus \Big(g\big[Y \setminus f[A]\big]\Big).$$

Define the sequence $(A_n)_{n\in\omega}$ of subsets of X inductively by setting $A_0 = \emptyset$ and then for each $n \geq 0$ letting $A_{n+1} = \varphi(A_n)$. Let $A = \cup_n A_n$. We show that the function $h : X \to Y$ defined by

$$h(x) \;=\; \begin{cases} f(x) & \text{if } x \in A; \\ g^{-1}(x) & \text{if } x \in X \setminus A \end{cases}$$

is bijective. To see this, note that by De Morgan's laws and the fact that unions and intersections are preserved under forward images by injective functions, we have

$$\begin{aligned} \varphi(A) \;=\; \varphi(\cup_n A_n) \;&=\; X \setminus g\big[Y \setminus f[\cup_n A_n]\big] \\ &=\; X \setminus g\big[Y \setminus \cup_n f[A_n]\big] \\ &=\; X \setminus g\big[\cap_n \left(Y \setminus f[A_n]\right)\big] \\ &=\; X \setminus \big(\cap_n g\big[Y \setminus f[A_n]\big]\big) \\ &=\; \cup_n \big(X \setminus g\big[Y \setminus f[A_n]\big]\big) \;=\; \cup_n \varphi(A_n) \;=\; \cup_n A_{n+1} \;=\; A. \end{aligned}$$

This means that $X \setminus A = g\big[Y \setminus f[A]\big]$, which implies that h is bijective. □

Theorem 2. *If $A_1, \ldots, A_n$ are countable sets, then $A_1 \times \cdots \times A_n$ is countable.*

Proof. Suppose A_i is countable for each $1 \leq i \leq n$. Then there exist injective functions $f_i : A_i \to \mathbb{N}$. Define $f : A_1 \times \cdots \times A_n \to \mathbb{N}$ by

$$f(a_1, \ldots, a_n) \;=\; \prod_{i=1}^{n} p_i^{f_i(a_i)},$$

where p_i is the ith prime number. Then f is injective by the Fundamental Theorem of Arithmetic. This shows that $A_1 \times \cdots \times A_n$ is countable. □

Theorem 3. *Let $\{A_i : i \in I\}$ be an indexed family of sets. If I is countable and if A_i is countable for each $i \in I$, then $\bigcup_{i \in I} A_i$ is countable.*

Proof. Let $f : \mathbb{N} \to I$ be a surjection, and for each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to A_{f(n)}$ be a surjection. We know that $\mathbb{N} \times \mathbb{N}$ is countable, so let $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be surjective, and write $h(n) = (n_1, n_2) \in \mathbb{N} \times \mathbb{N}$. Now define $g : \mathbb{N} \to \bigcup_{i \in I} A_i$ by $g(n) = f_{n_1}(n_2)$. Then g is surjective, so $\bigcup_{i \in I} A_i$ is countable. $\square$