Homework 5: Due Tuesday, June 11, at 11:59pm, on Gradescope

(1) Suppose $\{U_i : i \in I\}$ is a family of nonempty open sets in $\mathbb{R}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Prove that I is countable.

(2) In each of (a) – (d) below, determine whether the given continuous function is uniformly continuous on the given interval. Justify your answers.

(a) $y = x^3$ on $[0, 1]$

(b) $y = x^3$ on $(0, 1)$

(c) $y = x^3$ on $\mathbb{R}$

(d) $y = 1/x^3$ on $(0, 1]$

(3) Prove that if there is $a > 0$ such that the continuous function $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[a, \infty)$, then f is uniformly continuous.

(4) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous function, and suppose $a \in A' \setminus A$. Further suppose that there is $\epsilon > 0$ such that f is uniformly continuous on $V_\epsilon(a) \cap A$.

(a) Prove that for any two sequences (a_n) and (b_n) in A that converge to a, we have $\lim f(a_n) = \lim f(b_n)$.

(b) Prove that there is a continuous function $g : A \cup \{a\} \to \mathbb{R}$ such that $g \restriction A = f$. (We describe this by saying that "f extends continuously to $A \cup \{a\}$.")

(5) Show that "a composition of uniformly continuous functions is uniformly continuous." That is, prove that if $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are uniformly continuous, where $\mathrm{ran}(f) \subseteq B$, then $g \circ f : A \to \mathbb{R}$ is uniformly continuous.

(6) Find the derivatives of the following functions using the definition of derivative:

(a) $y = 1/x$ (b) $y = x^3$

(7) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ x^3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Find all points where f is continuous, and all points where f is differentiable. *(No justification needed.)*

(8) Show that if $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$, then the function $f : \mathbb{R} \to \mathbb{R}$ must be a constant function.

(9) Prove that if f and g are differentiable on $\mathbb{R}$, $f(0) = g(0)$, and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for all $x \geq 0$.

(10) Let $a < b$. In each part below, either prove that the given statement is true for *all* such functions f, or else give a counterexample (and show your counterexample works) if it could be false for *some* such function f.

(a) If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then f is bounded on $[a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then f' is bounded on $[a, b]$.

(c) If $f : (a, b) \to \mathbb{R}$ is differentiable and bounded on (a, b), then f' is bounded on (a, b).

(d) If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and f' is bounded on (a, b), then f is bounded on (a, b).

(11) Let (a, b) be a nonempty open interval in $\mathbb{R}$, and suppose the function $f : (a, b) \to \mathbb{R}$ is differentiable. In class, we stated the following facts:
(a) if $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b);
(b) if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b);
For each of (a) and (b), either prove the converse implication or give a counterexample to show that it does not hold.

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$, and suppose that f is differentiable at a. For each of the following statements, either prove the statement if it must be true, or else give a counterexample (and show your counterexample works) if it could be false.
(a) If $f'(a) > 0$, then there is $\delta > 0$ such that $f(x) > f(a)$ for all $x \in (a, a + \delta)$.
(b) If $f'(a) > 0$, then there is $\delta > 0$ such that f is strictly increasing on $(a, a + \delta)$.

Optional Challenge Problems:

(14) Let (a_n) be an increasing sequence of real numbers, and let (b_n) be a decreasing sequence of real numbers such that $a_m < b_n$ for all $m, n \in \mathbb{N}$. On the midterm exam you were asked to show that

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset.$$

Is it necessarily true that

$$\bigcap_{n\in\mathbb{N}} (a_n, b_n) \neq \emptyset?$$

Either prove that this is true or else give a counterexample.

(15) Does there exist an open set $U \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq U$ and $\mathbb{R} \setminus U$ is uncountable?