(1) Suppose $\{U_i : i \in I\}$ is a family of nonempty open sets in $\mathbb{R}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Prove that I is countable.

Proof Let $i \in I$ be arbitrary

Let $x \in U_i$, then by definition, $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq U_i$

And by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $q \in \mathbb{Q}$ s.t. $q \in V_\varepsilon(x) \subseteq U_i$

Therefore we can construct a function

$$f : \underline{I \to \mathbb{Q}}$$

<u>sending every $i \in I$ to a rational number in U_i</u>

Since $\forall i,j \in I$, $U_i \cap U_j = \emptyset \Rightarrow$ <u>f is injective</u>

So $I \lesssim \mathbb{Q} \Rightarrow I$ is countable $\square$

(2) In each of (a) – (d) below, determine whether the given continuous function is uniformly continuous on the given interval. Justify your answers.

(a) $y = x^3$ on $[0, 1]$

(b) $y = x^3$ on $(0, 1)$

(c) $y = x^3$ on $\mathbb{R}$

(d) $y = 1/x^3$ on $(0, 1]$

(a) uniformly continuous.

Because $y = x^3$ is continuous on $\mathbb{R}$ and $[0,1]$ is closed and bounded

(b) uniformly continuous.

Let $\varepsilon > 0$ and fix it.

Take $\delta = \frac{\varepsilon}{3}$.

Let $x, y \in (0,1)$ be arbitrary with $|x-y| < \delta$

Then $|x^3 - y^3| = |x-y||x^2+xy+y^2| < \frac{\varepsilon}{3} \cdot 3 = \varepsilon$

(c) not uniformly continuous.

Take $\varepsilon = 1$

Let $\delta > 0$ be arbitrary. Take $x = \sqrt{\frac{\varepsilon}{\delta}}$

$$(x+\tfrac{\delta}{3})^3 - x^3 = \frac{\delta}{3}\left|(x+\tfrac{\delta}{3})^2 + (x+\tfrac{\delta}{3})x + x^2\right| > \frac{\delta}{3} \cdot 3x^2$$

$$= \delta x^2 = \varepsilon$$

So not uniformly continuous.

(d) not uniformly continuous.

Take $\varepsilon = 1$

Let $\delta > 0$ be arbitrary, Take $x = \min\{1 - \frac{\delta}{3}, \sqrt{\frac{\varepsilon}{\delta}}\}$.

Then $\left|(\frac{1}{x})^3 - (\frac{1}{x+\frac{\delta}{3}})^3\right| = \dfrac{(x+\frac{\delta}{3})^3 - x^3}{x^3 (x+\frac{\delta}{3})^3}$

So for the same as (c), $(x+\frac{\delta}{3})^3 - x^3 > \frac{\delta}{3} \cdot 3x^2 = \varepsilon$

So $\left|(\frac{1}{x})^3 - (\frac{1}{x+\frac{\delta}{3}})^3\right| > \varepsilon$ and $x^3 (x+\frac{\delta}{3})^3 \leq 1$

(3) Prove that if there is $a > 0$ such that the continuous function $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[a, \infty)$, then f is uniformly continuous.

---

hw 5 ① 在$[0,\infty)$上的ctn函数只要在某个$[a,\infty)$上uni. ctn，一定整体上uni. ctn.

（闭方向上的tail不影响整体的uni. ctn）

Proof Suppose f is continuous and is uniformly continuous on $[a,\infty)$ for some $a>0$

Since $[0,a]$ is close, f is uniformly continuous on $[0,a]$

Let $\varepsilon>0$

Then there is some $\delta_1>0$ s.t. $|f(x)-f(y)|<\frac{\varepsilon}{2}$ whenever $|x-y|<\delta_1$ with $x,y\in[0,a]$

and some $\delta_2>0$ s.t. $|f(x)-f(y)|<\frac{\varepsilon}{2}$ whenever $|x-y|<\delta_2$ with $x,y\in[a,\infty)$

We take $\delta=\min(\delta_1,\delta_2)$

Now let $x,y\in[0,\infty)$ be arbitrary with $|x-y|<\delta$

There are three cases in total.

Case 1 $x,y\in[0,a]$. Since $|x-y|<\delta_1 \Rightarrow$
$|f(x)-f(y)|<\frac{\varepsilon}{2}<\varepsilon$

Case 2 $x,y\in[a,\infty)$. Since $|x-y|<\delta_2 \Rightarrow$
$|f(x)-f(y)|<\frac{\varepsilon}{2}<\varepsilon$

Case 3 One of x,y is in $[0,a]$ and the other is in $[a,\infty)$

WLOG assume $x\in[0,a]$, $y\in[a,\infty)$

Since $|x-y|<\delta \Rightarrow |x-a|=a-x<\delta$, $|y-a|=y-a<\delta$

So $|f(x)-f(a)|<\frac{\varepsilon}{2}$, $|f(a)-f(y)|<\frac{\varepsilon}{2}$

$\Rightarrow |f(x)-f(y)|\leq|f(x)-f(a)|+|f(a)-f(y)|<\varepsilon$

Since $|f(x)-f(y)|<\varepsilon$ in every case and ε is arbitrary, we have proved f is uniformly continuous. $\square$

(4) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a continuous function, and suppose $a \in A' \setminus A$. Further suppose that there is $\epsilon > 0$ such that f is uniformly continuous on $V_\epsilon(a) \cap A$.

(a) Prove that for any two sequences (a_n) and (b_n) in A that converge to a, we have $\lim f(a_n) = \lim f(b_n)$.

(b) Prove that there is a continuous function $g : A \cup \{a\} \to \mathbb{R}$ such that $g \restriction A = f$.

(We describe this by saying that "f extends continuously to $A \cup \{a\}$.")

hw 5 ②: ctn function f 如果在某个 limit pt. a 处局部 uni. ctn, 则可以 将 ctnity 延伸到 dom(f) ∪ {a} by $g(a) = \lim_{x \to a} f(a)$

像这种不行

(a) Proof Assume the hypothesis.

Write $\lim f(a_n) = L$

Let $\varepsilon_2 > 0$ be arbitrary.

Since $(a_n), (b_n) \to a$, $\exists N \in \mathbb{N}$ s.t.

$|a_n - a| < \varepsilon$ and $|b_n - a| < \varepsilon$ whenever $n \geq N_1$

$\Rightarrow a_n, b_n \in V_\varepsilon(a) \cap A$ whenever $n \geq N_1$

Since f is uniformly continuous on $V_\varepsilon(a) \cap A$,

$\exists \delta > 0$ s.t. $|f(b_n) - f(a_n)| < \frac{\varepsilon_2}{2}$ whenever $|a_n - b_n| < \delta$

Since $\lim(a_n) = \lim(b_n)$, $\exists N_2 \in \mathbb{N}$ s.t. $|a_n - b_n| < \delta$ whenever $n \geq N_2$

$\Rightarrow |f(b_n) - f(a_n)| < \frac{\varepsilon_2}{2}$ whenever $n \geq N_2$

And since $\lim f(a_n) = L$, $\exists N_3 \in \mathbb{N}$ s.t. $|f(a_n) - L| < \frac{\varepsilon_2}{2}$ whenever $n \geq N_3$

Take $N = \max\{N_1, N_2, N_3\}$

then $|f(b_n)-f(a_n)| < |f(b_n)-f(a_n)| + |f(a_n)-L|$

$< \frac{\varepsilon_2}{2}+\frac{\varepsilon_2}{2} = \varepsilon_2$ whenever $n \geq N$

Since ε_2 is arbitrary, it proves that

$\lim f(a_n) = \lim f(b_n)$ $\square$

(b) let $g(x) = \begin{cases} f(x), & x \in A \\ \lim\limits_{x\to a} f(x), & x = a \end{cases}$

Then $\underbrace{g|A = f}$

Now we show that g is continuous.

Since $a \in A'$ and $dom(g) = A \cup \{a\}$

$\Longrightarrow a \in (dom(g))'$

Also, $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} f(x) = g(a)$, so g is continuous at a

Since g is continuous at a and on A, g is continuous on $dom(g)$, so g is continuous.

(5) Show that "a composition of uniformly continuous functions is uniformly continuous." That is, prove that if $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are uniformly continuous, where $\text{ran}(f) \subseteq B$, then $g \circ f : A \to \mathbb{R}$ is uniformly continuous.

hw5 ③ 两个 uni. ctn. 函数的 composition 仍是 uni. ctn. 的

Proof Let $\varepsilon > 0$

Fix $\delta_1 > 0$ s.t. $|g(a) - g(b)| < \varepsilon$ whenever $|a-b| < \delta_1$ where $a, b \in B$

Fix $\delta_2 > 0$ s.t. $|f(x) - f(y)| < \delta_1$ whenever $|x-y| < \delta_2$ where $x, y \in A$

So for all $x, y \in A$ with $|x-y| < \delta_2$, $|f(x) - f(y)| < \delta_1$

$\Longrightarrow |g(f(x)) - g(f(y))| < \varepsilon$

i.e. $|g \circ f(x) - g \circ f(y)| < \varepsilon$

Since ε is arbitrary, this finishes the proof that $g \circ f$ is uniformly continuous.

6) Find the derivatives of the following functions using the definition of derivative:

(a) $y = 1/x$ (b) $y = x^3$

(a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$= \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h}$

$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$

(b) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} (3x^2+3xh+h^2)$

$= 3x^2+0 = 3x^2$

(7) Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ x^3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Find all points where f is continuous, and all points where f is differentiable. *(No justification needed.)*

f is continuous only at $x=0$

and differentiable only at $x=0$

(8) Show that if $|f(x)-f(y)| \le (x-y)^2$ for all $x, y \in \mathbb{R}$, then the function $f : \mathbb{R} \to \mathbb{R}$ must be a constant function.

Pf Assume the hypothesis

Let $y \in \mathbb{R}$ be arbitrary and fix it.

Let $x \in \mathbb{R}$ be arbitrary

Consider the function $g(x) = \frac{f(x)-f(y)}{x-y}$

Since $|f(x)-f(y)| \le (x-y)^2$ for all $x \in \mathbb{R}$,

$\Rightarrow$ $0 \le |g(x)| \le |x-y|$ for all $x \in \mathbb{R}$

So $\lim_{x\to y} |g(x)| \le \lim_{x \to y} |x-y| = 0$ by Squeeze Thm, implying $\lim_{x\to y} g(x) = 0$.

So $\lim_{x\to y} \frac{f(x)-f(y)}{x-y} = 0$, i.e. $f'(y) = 0$

Since y is arbitrary, it proves that f is a constant function. $\square$

(9) Prove that if f and g are differentiable on $\mathbb{R}$, $f(0) = g(0)$, and $f'(x) \le g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \le g(x)$ for all $x \ge 0$.

---

hws ④ 如果 $f(a) = g(a)$

且 $f'(x) \le g'(x)$ for all $x \ge a$

$\Rightarrow$ $f(x) \le g(x)$ for all $x \ge a$

Proof Let $h(x) = f(x) - g(x)$ for all $x \in \mathbb{R}$

So $h'(x) = f'(x) - g'(x) \leq 0$

So h is a decreasing function on $\mathbb{R}$

Since $h(0) = f(0) - g(0) = 0$,

for any $x \in \mathbb{R}$, we have $h(x) \leq 0$

So $f(x) - g(x) \leq 0$, $f(x) \leq g(x)$ for all $x \geq 0$. $\square$

(10) Let $a < b$. In each part below, either prove that the given statement is true for *all* such functions f, or else give a counterexample (and show your counterexample works) if it could be false for *some* such function f.

(a) If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then f is bounded on $[a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then f' is bounded on $[a, b]$.

(c) If $f : (a, b) \to \mathbb{R}$ is differentiable and bounded on (a, b), then f' is bounded on (a, b).

(d) If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and f' is bounded on (a, b), then f is bounded on (a, b).

hw 5 ⑤ f 在闭区间上 diffble 则一定 bounded 而 f' 则未必 bounded

f 在开区间上 diffble 且 f' bounded $\Rightarrow$ 则 f 一定 bounded

(a) True.

Pf Since f is differentiable on $[a,b]$, it is continuous on $[a,b]$ which is closed and bounded, by the extreme value thm, $\exists x_0, y_0 \in [a,b]$ s.t. $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a,b]$. So $\sup(f[a,b]) = f(y_0)$, $\inf(f[a,b]) = f(x_0)$

(b) False.

Counterexample $f(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \neq 0 \\ 0, & x = \end{cases}$

is differentiable on $[-1, 1]$

for $x=0$, $f'(x) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} x\sin(\frac{1}{x^2})$

$= 0$

for $x\neq 0$, $\underline{f'(x) = 2x\sin(\frac{1}{x^2}) - \frac{2\cos(\frac{1}{x^2})}{x}}$ which is unbounded since $2x\sin(\frac{1}{x^2}) \to 0$ as $x\to 0$ but $-\frac{2\cos(\frac{1}{x^2})}{x}$ is unbounded as $x\to 0$.

(for any $M>0$, we can always find $x>0$ s.t. $2\cos(\frac{1}{x^2})$ is close to 1 and $\frac{2\cos(\frac{1}{x^2})}{x} > M$)

(c) False.

By the same counterexample in (b), while we restrict $f(x)$ on $(-1, 1)$

(d) True.

Pf Assume the hypothesis.

So $|f'(x)| \leq M$ for some $M>0$ for all $x\in(a,b)$

let $a<m<n<b \Rightarrow f$ is differentiable on $[m,n]$

So by EVT, $\exists k\in[m,n]$ s.t. $f(k)\geq f(x)$ for all $x\in[m,n]$

Let $x \in (a,b)$ be arbitrary.

Since f is continuous and differentiable between x and k, $\exists c$ between x and k

s.t. $\underline{|f'(c)| = \frac{f(x)-f(k)}{x-k}}$

So $f(x) - f(k) \leq M(x-k)$

$f(x) \leq M(x-k) + f(k)$

So $M(a-k) + f(k) \leq f(x) \leq M(b-k) + f(k)$ for all $x \in (a,b)$ $\square$

11) Let (a, b) be a nonempty open interval in $\mathbb{R}$, and suppose the function $f : (a, b) \to \mathbb{R}$ is differentiable. In class, we stated the following facts:

(a) if $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b);

(b) if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b);

For each of (a) and (b), either prove the converse implication or give a counterexample to show that it does not hold.

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hw5 (b) $f'(x) \geq 0 \Longleftrightarrow f$ weakly $\nearrow$

$f(x) > 0 \implies f$ strictly $\nearrow$, $(\not\Leftarrow)$

(a) Converse: if f is increasing on (a,b) then $f'(x) \geq 0$ for all $x \in (a,b)$

<u>Pf</u> Assume the hypothesis.

Let $x \in (a,b)$ be arbitrary

So $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Suppose for contradiction that $f'(x) < 0$

Consider $\varepsilon = -\frac{f'(x)}{2}$

Since $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$, $\exists$ some $\delta > 0$ s.t.

$$\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right| < \varepsilon \quad \text{whenever } 0 < h < \delta$$

$$\Rightarrow -\frac{3f'(x)}{2} < \frac{f(x+h)-f(x)}{h} < \frac{-f'(x)}{2} < 0$$

But since $h > 0 \Rightarrow f(x+h) \geq f(x)$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} \geq 0 \Rightarrow \text{contradicts}$$

So $f'(x) \geq 0$

Since x is arbitrary, the converse is proved. $\square$

(b) Converse: if f is strictly increasing on (a,b) then $f'(x) > 0$ for all $x \in (a,b)$

Counterexample $f(x) = x^3$ is strictly increasing on $(0,1)$

But $f'(0) = 0$

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

hw5 ⑦ 如果 $\lim_{x\to\infty} f(x)$ 和 $\lim_{x\to\infty} f'(x)$ 都存在，则一定有 $\lim_{x\to\infty} f'(x)=0$

Proof Assume the hypothesis and suppose for contradiction that $\lim_{x\to\infty} f'(x) = M \neq 0$

Write $\lim_{x\to\infty} f(x) = L$

Let $0<\varepsilon<M$, then $\exists N_1 \in \mathbb{N}$ s.t. $|f(x)-L|<\varepsilon$ whenever $x \geq N_1$

And $\exists N_2 \in \mathbb{N}$ s.t. $|f'(x)-M|<\varepsilon$ whenever $x \geq N_2$

So $f'(x) \in (M-\varepsilon, M+\varepsilon)$ for all $x \geq N_2$

Take $N = \max\{N_1, N_2\}$, so $L-\varepsilon < f(N) < L+\varepsilon$ ①

and $f'(x) > M-\varepsilon$ for all $x \geq N$

Consider $x = N + \frac{2\varepsilon}{M-\varepsilon}$

By MVT, $\exists c \in (N, x)$ s.t. $\boxed{f'(c) = \frac{f(x)-f(N)}{x-N} > M-\varepsilon}$

So $f(x)-f(N) > (M-\varepsilon)\cdot(x-N)$

$= (M-\varepsilon)\left(\frac{2\varepsilon}{M-\varepsilon}\right) = 2\varepsilon$

$\Longrightarrow f(x) > L+\varepsilon$ ②

①② contradicts, which completes the proof of $\lim_{x\to\infty} f'(x) = 0$

$\square$

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$, and suppose that f is differentiable at a. For each of the following statements, either prove the statement if it must be true, or else give a counterexample (and show your counterexample works) if it could be false.

(a) If $f'(a) > 0$, then there is $\delta > 0$ such that $f(x) > f(a)$ for all $x \in (a, a+\delta)$.

(b) If $f'(a) > 0$, then there is $\delta > 0$ such that f is strictly increasing on $(a, a+\delta)$.

hw 5 ⑧

如果f在a处 diff'ble 且 $f'(a)>0$，则一定存在 $(a, a+\delta)$ 区间上，f是 strictly ↑的 (and locally)

(a) True

Proof Assume the hypothesis

Then $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} > 0$

Let $\varepsilon = \frac{1}{2} f'(a)$

$\Rightarrow \exists \delta > 0$ s.t. $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \varepsilon$ whenever $x \in (a, a+\delta)$

So $\frac{f(x)-f(a)}{x-a} > \frac{1}{2} f'(a)$ whenever $x \in (a, a+\delta)$

Since on $(a, a+\delta)$, $x-a>0 \Rightarrow f(x)-f(a) > 0$

i.e. $f(x) > f(a)$ whenever $x \in (a, a+\delta)$ $\square$

(b) True

Proof Use the same δ as in (a)

Let $a < x_1 < x_2 < a+\delta$ be arbitrary

So $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some $c \in (x_1, x_2)$ by MVT

By (a), $f'(c) > 0$, so $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$

Since $x_2 > x_1 \Rightarrow f(x_2) - f(x_1) > 0$ so $f(x_2) > f(x_1)$

Since x_1, x_2 is arbitrary, $x > y$ implies $f(x) > f(y)$ on $(a, a+\delta)$

This finishes the proof that f is strictly increasing on $(a, a+\delta)$

$\square$

Optional Challenge Problems:

(14) Let (a_n) be an increasing sequence of real numbers, and let (b_n) be a decreasing sequence of real numbers such that $a_m < b_n$ for all $m, n \in \mathbb{N}$. On the midterm exam you were asked to show that

$$\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset.$$

Is it necessarily true that

$$\bigcap_{n \in \mathbb{N}} (a_n, b_n) \neq \emptyset?$$

Either prove that this is true or else give a counterexample.

(15) Does there exist an open set $U \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq U$ and $\mathbb{R} \setminus U$ is uncountable?