Review:

Archimedeen property (5条)

metric space (d: (i)正定 (ii)对称 (iii)三角)

$V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$ (the open ε-neighborhood of a)

Note: In any ordered field,

if $\boxed{\forall \varepsilon > 0, |a-b| < \varepsilon}$, then $a=b$ ☆

特别地, In $\boxed{\text{Archimedeon}}$ ordered fields,

if $\boxed{\forall n \in \mathbb{N}, \ |a-b| < \frac{1}{n}}$, then $a=b$

Note 2: 如果 $A \subseteq \mathbb{R}$ 有 max, 则 A 也有 sup 且 $\max A = \sup A$ ☆

今日 1.*

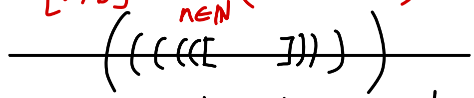
用 infly many 个 open intervals intersect 出 closed interval

用 infly many 个 closed intervals union 出 open interval

(记得 Rudin 2.24 中说: finite 个 open/closed sets intersect/union 出的 set 仍是 open/closed 的, 这里我们给出 $\mathbb{R}$ 中 infly many 个 sets 的反例) ***

(1) if $a<b$, then

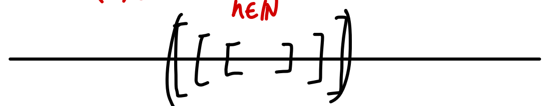
$$[a,b] = \bigcap_{n\in\mathbb{N}} \left(a-\frac{1}{n}, b+\frac{1}{n}\right)$$



the closed interval can be expressed as an intersection of open intervals.

(2) if $a<b$, then

$$(a,b) = \bigcup_{n\in\mathbb{N}} \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$$



open interval (a,b) can be expressed as a union of countably many closed intervals.

2. $\forall$ nonempty $A, B \subseteq \mathbb{R}$

(1) $\inf(A) \le \sup(A)$

(2) $\inf(A \cup B) = \min(\inf(A), \inf(B))$

(3) $\sup(A \cup B) = \max(\sup(A), \sup(B))$

(4) if $c>0$, then $\sup(cA) = c \sup(A)$

(5) $\sup(-A) = -\inf(A)$ ☆ (proved in hw1)

(6) $\sup(A+B) = \sup A + \sup B$ ☆ (proved in hw1)

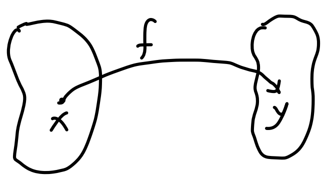
(7) $\sup(AB) \ne \sup(A)\sup(B)$ ☆

2. Functions

Def Function (rigorously)

A function $f: X \to Y$ is a subset $\boxed{f \subseteq X \times Y}$ ☆

s.t. $\forall x \in X \ \exists! y \in Y, (x,y) \in f$



$X = \text{dom}(f)$ $\qquad Y = \text{cod}(f)$

$\text{im}(f) = \text{ran}(f) = \{f(x) \mid x \in X\} \subseteq \text{cod}(f)$

$f[A] = \{f(x) \in \text{cod}(f) \mid x \in A_{(\subseteq \text{dom}(f))}\} \subseteq \text{cod}(f)$

$f^{-1}[B] = \{x \in \text{dom}(f) \mid f(x) \in B_{(\subseteq \text{cod}(f))}\} \subseteq \text{dom}(f)$

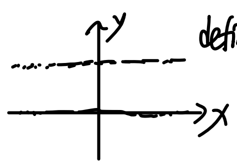
(1) squaring function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

(2) reciprocal function $g: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ define by $f(x) = \frac{1}{x}$

(3) supremum function $s: \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\pm\infty\}$ defined by $f(A) = \sup A$

(4) the harmonic function $h: \mathbb{N} \to \mathbb{R}$ defined by $h(n) = \frac{1}{n}$

(5) dirichlet's function $D: \mathbb{R} \to \mathbb{R}$



defined by $D(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q} \end{cases}$

Cardinality

Def set X is $\boxed{\text{finite}}$ if $\exists n \in \mathbb{N}$ s.t. X has n elements.

denoted: $|X| = n$

X is $\boxed{\text{infinite}}$ if $\exists$ inj $f: \mathbb{N} \to X$

Notation: write $X \preceq Y$ if $\exists$ inj $f: X \to Y$

$X \approx Y$ if $\exists$ bij $f: X \to Y$

Remark (hw) $X \preceq Y$ ($\exists$ inj $f: X \to Y$) iff $\exists$ surj $g: Y \to X$)

Thm Cantor–Schröder–Bernstein Thm

If $X \preceq Y$ and $Y \preceq X$ then $X \approx Y$

(pf: kind of hard)

Example: $\mathbb{N} \approx \mathbb{Z}$

since $f: \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} -\frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

is bijective

$\mathbb{N}$		$\mathbb{Z}$
1	$\to$	0
2	$\to$	1
3	$\to$	-1
4	$\to$	2
5	$\to$	-2
6	$\to$	3
$\vdots$		$\vdots$

Def X is countably infinite if $X \approx \mathbb{N}$

countable if $X \preceq \mathbb{N}$

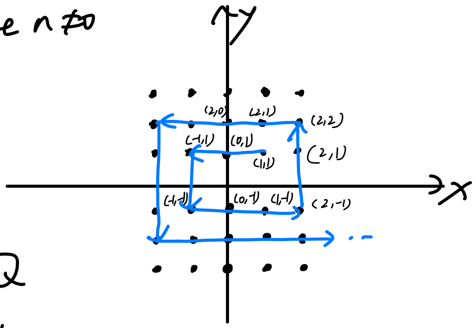
uncountable if X is not countable

i.e. $\exists$ inj $f: X \to \mathbb{N}$

or surj $g: \mathbb{N} \to X$

Thm $\mathbb{Q}$ is countable

Pf View rationals m/n as pairs $(m,n) \in \mathbb{Z}, \mathbb{Z}$ where $n \neq 0$



$\mathbb{N}$	$\longrightarrow$	$\mathbb{Q}$
1		1
2		0
3		-1
4		1
5		0
$\vdots$		$\vdots$
		$\frac{1}{2}$

Thm (Cantor) $\mathbb{R}$ is uncountable

Pf. We use the fact that every real num can be expressed as a decimal

(eg: $\pi = 3.1415926\ldots$)

It suffices to show that $(0,1)$ is uncountable.

We prove that no $f: \mathbb{N} \to (0,1)$ can be surj.

Let $f: \mathbb{N} \to (0,1)$ be any function

and for each $n \in \mathbb{N}$ we write

$$f(n) = 0.n_1 n_2 n_3 \ldots \in (0,1)$$

现考虑 $x = 0.d_1 d_2 d_3 \ldots \in (0,1)$

where d_n 是任意 $\neq n_n$ 的数

(对每个$n \in \mathbb{N}$ 都选取和$f(n)$ 的第n位不同 的digit)

$\Rightarrow$ $\forall n \in \mathbb{N}, x \neq f(n)$, so $x \notin \text{ran}(f)$

Since f is arbi, no function $f: \mathbb{N} \to (0,1)$ can be surj.

$\Rightarrow$ $(0,1)$ is unctb and so is $\mathbb{R}$.

Thm (Cantor) $\forall$ set X, $\nexists$ surj $f: X \to \mathcal{P}(X)$

即: $|\mathcal{P}(X)| > |X|$ for all X

Pf. Given $f: X \to \mathcal{P}(X)$,

consider $D = \{x \in X \mid x \notin f(x)\} \in \mathcal{P}(X)$

如果 f surj $\Rightarrow$ $\boxed{D = f(x_0) \text{ for some } x_0 \in X}$

Then; if $x_0 \in D \Rightarrow$ by def of D, $x_0 \notin D$

if $x_0 \notin D \Rightarrow$ by def of D, $x_0 \in D$

因而没有任何元素可以映射到 such D $\Rightarrow$ contradicts

(所有(像中无x)的x的集合) $\Rightarrow f$ 不可能 surj

(eg: $x_1 \mapsto \{x_2, x_3, x_{56}\}$)

Question 1: are there any cardinalities strictly larger than that of $\mathbb{R}$

Answer: $\mathbb{C} \approx \mathbb{R}^2$ (though $\mathbb{C} \cong \mathbb{R}^2$)

2. Are there any cardinalities strictly between $\mathbb{N}$ and $\mathbb{R}$? (Fact: $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$)

Answer: nobody knows

and the statement that there is no cardinality between $\mathbb{N}$ and $\mathbb{R}$ is called $\boxed{\text{continuum hypothesis}}$.

Thm If $A_1, \dots, A_n$ are ctbl sets, then $A_1 \times \dots \times A_n$ is ctbl ☆

i.e. finite product of ctbl sets is ctbl.

Pf. $A_1, A_2, \dots, A_n$

(简略版) $A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, \dots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, \dots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, \dots\}$

$A_4 = \{\dots \quad \dots \quad \dots \quad \dots\}$

Thm Let $\{A_i \mid i \in I\}$ be an indexed family of sets.

If: ① I 是 ctbl 的

② $\forall i \in I$, A_i 都是 ctbl 的

$\Rightarrow \bigcup_{i \in I} A_i$ 是 ctbl 的

Pf 同理

使用 ctbl union thm 来证明一些结论

$\forall a < b$, (a,b) 中有 unctbly many irrationals.

Pf. $(a,b) \cap \mathbb{Q}$ is ctbl $(\leq \mathbb{Q})$

if $(a,b) \cap (\mathbb{R} \setminus \mathbb{Q})$ is ctbl (for contradiction)

then $(a,b) = [(a,b) \cap \mathbb{Q}] \cup [(a,b) \cap (\mathbb{R} - \mathbb{Q})]$

would be ctbl $\Rightarrow$ contradicts.

(hw: $\overline{\mathbb{Q}}$ is ctbl, so there are unctbly many transcendental nums.)