Def Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$, $c\in A$

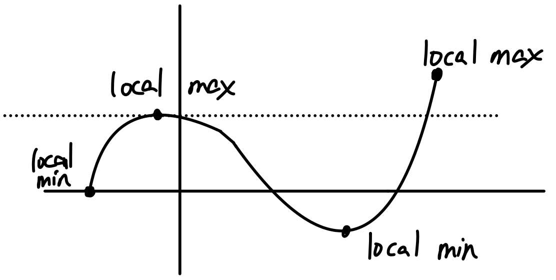
如果$\exists V_\varepsilon(c)$ s.t. $f(x)\le f(c)$ for all $x\in V_\varepsilon(c)\cap \mathrm{dom}(f)$

则称 c 为一个 local maximum point of f 并称

$f(c)$ 为一个 local max value of f

And dually: local minimum point 与 local min value

它们统称: local extreme point 与 local extrema



Key Lemma Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$, $c\in A\cap A'$

Suppose f is diff'ble at c

$\Rightarrow$ (i) if $f'(c)>0$, 则 $\exists\delta>0$ s.t.

$\forall x,y\in V_\delta(c)\cap A$, $x<c<y$ implies $f(x)<f(c)<f(y)$

(ii) ((i)的 dual)

if $f'(c)<0$ 则 $\exists\delta<0$ s.t. $\forall x,y\in V_\delta(c)\cap A$

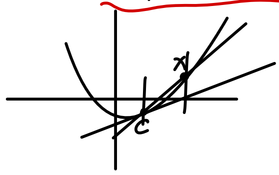
$x<c<y$ implies $f(x)<f(c)<f(y)$

这两条lemma讲的事情是: 如果$f'(c)\neq 0$, 那么f在c的某个 open nbh 中必然 strict monotone.

Pf Assume the hypothesis, and suppose $f'(c)>0$

Let $\varepsilon=\frac{f'(c)}{2}$. Fix $\delta>0$ s.t. $\forall x\in A$,

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon \text{ whenever } 0<|x-c|<\delta$$



因而 $0<\frac{f'(c)}{2}<\frac{f(x)-f(c)}{x-c}<\frac{3f'(c)}{2}$

Let $x,y\in V_\delta(c)\cap A$ and suppose $x<c<y$

$\Rightarrow$ Since $x<c$ 且 $\frac{f(x)-f(c)}{x-c}>0$, we have $f(x)<f(c)$

同理, Since $c<y$ 且 $\frac{f(y)-f(c)}{y-c}>0$, we have $f(c)<f(y)$

(i) □

(ii): dual.

Corollary① Fermat's Thm

Suppose f is defined on an open nbh of c.

如果 c 为 f 的一个 local extreme pt. of f 且 $f'(c)$ 存在

$\Rightarrow$ 则一定有 $f'(c)=0$

Pf Directly follows from Key Lemma.

如果 $f'(c)\neq 0$ $(>,<)$ 则 c 无法是一个 local extreme pt..

Corollary② Rolle's Thm

If f is ① ctn on $[a,b]$ 且 ② diff'ble on (a,b)

且 ③ $f(a)=f(b)$

$\Rightarrow$ 则一定有 some pt. $c\in(a,b)$ s.t. $f'(c)=0$

(很直观. 如果两端平齐, 那么要么 const function, 否则一定有 extreme pt. 于是 by Fermat's Thm 这些点上 f'为0.)

Pf By EVT, $\exists x_0,y_0\in[a,b]$ s.t. $\forall x\in[a,b]$,

$$\underset{(\min)}{f(x_0)}\le f(x)\le \underset{(\max)}{f(y_0)}$$

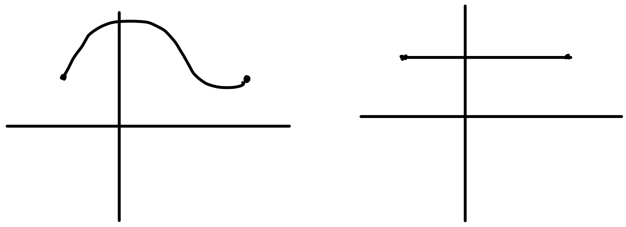
(by Fermat)

如果 $f(x_0)<f(a)$ $\Rightarrow$ x_0 为 local min $\Rightarrow$ $f'(x_0)=0$

如果 $f(y_0)>f(a)$ $\Rightarrow$ y_0 为 local max $\Rightarrow$ $f'(y_0)=0$

如果两者都不是 $\Rightarrow$ f 在 $[a,b]$ 上 const $\Rightarrow$

$\forall c\in[a,b]$, $f'(c)=0$ □



Corollary③ Mean-Value Thm

If f is ① ctn on $[a,b]$ and ② diff'ble on (a,b)

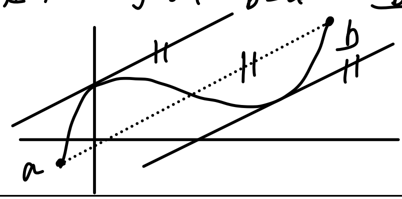
$\Rightarrow$ $\exists c\in(a,b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$

Pf let $g(x)=f(x)-\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$

那么 $g(x)$ 也是在 $[a,b]$ 上 ctn 且在 (a,b) 上 diff'ble 的

by Rolle's Thm, $\exists c\in(a,b)$ s.t. $g'(c)=0$

于是 $f'(c)=g'(c)+\frac{f(b)-f(a)}{b-a}=\frac{f(b)-f(a)}{b-a}$ □



Corollary④ (区间上 $f'=0 \Longleftrightarrow$ 区间上 f const)

If f is diffble on (a,b)
且 $\forall x\in(a,b)$, $f'(x)=0$
$\Longrightarrow$ 则 f 在 (a,b) 上 constant

Pf Assume f 非 const
则 $\exists\, x\neq y$ s.t. $f(x)\neq f(y)$
$\Longrightarrow \dfrac{f(x)-f(y)}{x-y}\neq 0$, contradicting MVT □

Corollary⑤ (I 上 $f'=g' \Longrightarrow f$ 和 g 平移重合)

If f 和 g 在 (a,b) 上 diffble 且
$f'\restriction(a,b) = g'\restriction(a,b)$
$\Longrightarrow$ 则 $\exists\, c\in\mathbb{R}$ s.t. $f=g+c$ on (a,b)

Pf Apply corollary ④ to $f-g\restriction(a,b)$

Def Increasing (weakly) 只在 interval 上 defined

function f is increasing on the interval I
if: $\forall x,y\in I$, $x<y$ implies $f(x)\le f(y)$
Dually: decreasing. (strictly: $f(x)<f(y)$)

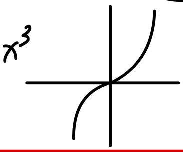
f is (weakly) increasing or decreasing on I
统称为 monotone on I

Corollary⑥ (Increasing/decreasing test)

If f is diffble on (a,b) 则
(i) if $\forall x\in(a,b)$ 有 $f'(x)\ge 0 \Longrightarrow$ 则 $f\nearrow$ on (a,b)
And dually...
(ii) if $\forall x\in(a,b)$ 有 $f'(x)>0 \Longrightarrow$ 则 f monotonely $\nearrow$ on (a,b)
And dually...

* note: (i) 是一个 iff statement, 但 (ii) 不是

Consider $y=x^3$



$x<y \Rightarrow x^3<y^3$
But $f'(0)=0$

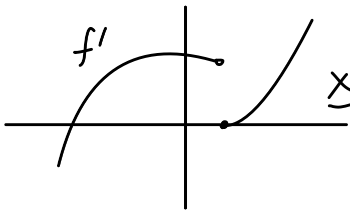
Pf Suppose $f'(x)\ge 0$ for all $x\in(a,b)$
$\Longrightarrow$ By MVT, $\forall x<y$ in (a,b), $\exists\, c\in(a,b)$ s.t.
$\dfrac{f(y)-f(x)}{y-x} = f'(c)\ge 0 \Longrightarrow f(x)\le f(y)$

Corollary⑦ The first derivative test

Let $c\in\mathbb{R}$ and suppose f is ctn on $V_\varepsilon(c)$
for some $\varepsilon>0$, 且 diffble on $(c-\varepsilon,c)$ 和 $(c,c+\varepsilon)$
(i) If $f'>0$ on $(c-\varepsilon,c)$ and $f'<0$ on $(c,c+\varepsilon)$
则 c 一定是 f 的一个 local max ($f'(c)$ 一定 $=0$)
(dually we have local min)

Pf Let $x\in(c-\varepsilon,c)$
By MVT $\Longrightarrow \exists\, t\in(x,c)$ s.t. $\dfrac{f(x)-f(c)}{x-c} = f'(t)>0$
$\Longrightarrow f(x)<f(c)$
Similarly, let $y\in(c,c+\varepsilon)$
By MVT $\Longrightarrow \exists\, t\in(c,y)$ s.t. $\dfrac{f(y)-f(c)}{y-c} = f'(t)<0$
因而 c 为 local max of f. $\Longrightarrow f(y)<f(c)$
□

Recall: 即使 f diffble, f' 也不一定 ctn. eg: $y=x^2\sin\frac{1}{x}$
但是 f' 一定满足 IVT 的 conclusion.
(不可能有 jump/infinite disctnity)



Thm⑧ Darboux's Thm: 闭区间上 diffble function 会经历首尾间所有 slopes ($[f'(a),f'(b)]\subseteq \mathrm{im}(f')$)

If f is $\boxed{\text{diffble on } [a,b]}$ 且 $\boxed{L \text{ 在 } f'(a) \text{ 和 } f'(b) \text{ 之间}}$
$\Longrightarrow$ 则有 $\boxed{c\in(a,b) \text{ s.t. } f'(c)=L}$

Pf WLOG suppose $f'(a)<L<f'(b)$
let $g(x)=f(x)-Lx$ for all $x\in[a,b]$
$\Longrightarrow \boxed{g'(a)<0<g'(b)}$ 且 g 在 $[a,b]$ 上 ctn.
By EVT, g 在 $[a,b]$ 上有某个 min. val. pt. c
Since $g'(a)<0$ and $g'(b)>0 \Longrightarrow c\in(a,b)$
$\Longrightarrow g'(c)=0$ by Fermat's Thm,
$\Longrightarrow f'(c)=g'(c)+L=L$