Math 451 Final Exam

Spring 2024

You have two hours to complete this exam. You may not use notes, textbooks, or electronic devices of any kind. Write your answers clearly on the exam itself in the space provided for you. Circle your answers where appropriate. Academic dishonesty on this exam will result in a score of zero.

*** Please read all instructions carefully before working each problem. ***

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Total Score: 92

Problem 1: (15 points)

(a) Precisely define what it means for the function f to be *Riemann integrable* on the interval $[a, b]$. (3pts)

There is an $L \in \mathbb{R}$ s.t. for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. ($\forall$ tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ s.t. $\|\dot{\mathcal{P}}\| < \delta$, $|S(f, \dot{\mathcal{P}}) - L| < \varepsilon$)

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where a tagged partition is a finite point set $\{x_0 = a, x_1, \ldots, x_n = b\} \subset [a,b]$ together with a tag in every $[x_{k-1}, x_k]$ $t_k =$

$\|\dot{\mathcal{P}}\| = \max\{ x_{k+1} - x_k \mid 0 \le k \le n\}$

(b) Define what it means for f to be *Darboux integrable* on the interval $[a, b]$, making sure to define any notation or terminology in your answer that is specifically related to the Darboux integral. (3pts)

f is Darboux integrable on $[a,b]$ if $\underline{U(f) = L(f)}$

where $U(f) = \inf\{U(f, \mathcal{P}) \mid \mathcal{P}$ is a partition of $[a,b]\}$

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$L(f) = \sup\{L(f, \mathcal{P}) \mid \mathcal{P}$ is a partition of $[a,b]\}$

(for a partition $\mathcal{P}$, $U(f, \mathcal{P}) = \sum_{k=1}^{n} \sup_{I_k}(f) \cdot \Delta x_k$

$L(f, \mathcal{P}) = \sum_{k=1}^{n} \inf_{I_k}(f) \cdot \Delta x_k$)

(c) For all $x \in \mathbb{R}$, let $g(x) = \int_0^x (18t - 6t^2)\,dt$. Find $g(1)$ and $g'(1)$, and circle your answers. (4pts)

$g(1) = \int_0^1 (18t - 6t^2) = 9t^2 - 2t^3 \Big]_0^1 = 7$ since $(9t^2 - 2t^3)' = 18t - 6t^2$ on $[0,1]$, by FTC ①

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$g'(1) = 18t - 6t^2 \Big|_{t=1} = 12$, since $\overset{f(t)=}{18t - 6t^2}$ is continuous on $t = 1$, by FTC ②

(d) Circle all the infinite series listed below that converge, and cross out those that diverge. (5pts)

$\cancel{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}$ $\quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ $\quad \cancel{\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{1/n}}$ $\quad \cancel{\sum_{n=2}^{\infty} \frac{1}{n \ln n}}$

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Problem 2: (14 points)

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$. Prove that if f is differentiable at a, then f is continuous at a. (5pts)

<u>Proof</u> Suppose f is differentiable at a

So $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L$ for some $L \in \mathbb{R}$

Since $a \in \mathbb{R}'$, we can apply limit law here, $\lim_{x\to a} f(x)$ exists, ?

So $\boxed{\lim_{x\to a} f(x) = L \lim_{x\to a}(x-a) + f(a)} = 0 + f(a) = f(a)$

This implies that f is continuous at a. $\square$

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Can you provide a bit more work to justify why this is true?

(b) Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove in detail that if $f(x) \geq 0$ for all $x \in [a,b]$ and $f(x) > 0$ for some $x \in [a,b]$, then $\int_a^b f(x)\,dx > 0$. (6pts)

<u>Proof</u> Assume the hypothesis, let $f(c) > 0$ for $c \in [a,b] \cap \mathrm{dom}(f)$

Then by continuity of f, $\exists\, \delta > 0$ s.t. $\forall x \in V_\delta(c)$, $|f(x)-f(c)| < f(c)$

which means that $f(x) > 0$ on $V_\delta(c)$ ✓

Note that since $V_\delta(c)$ is an interval with at least one end open ($\ni f(c)$?), so we can take closed interval $[e,g] \subseteq V_\delta(c)$

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Then By EVT, $\exists\, c_0 \in [e, g]$ s.t. $f(c_0) \leq f(x)$ for all $x \in [e,g]$

Since $[e,g]$ is a closed interval

So $\int_e^g f \geq \int_e^g f(c_0) = (g-e) f(c_0) > 0$ by monotonicity of integral

Therefore $\int_a^b f = \int_a^e f + \int_e^g f + \int_g^b f > 0$ (since $\int_a^e f, \int_g^b f \geq 0$) ✓

(c) State (but do not prove) the Mean Value Theorem, including all hypotheses. (3pts)

MVT: Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b)

then there exists some point $c \in (a,b)$ s.t.

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$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$ ✓

Problem 3: (17 points)

For (a) and (b), let $f_n(x) = (1 - |x|)^n$ for all $x \in (-1, 1)$ and $n \in \mathbb{N}$.

(a) Find the pointwise limit f of the sequence of functions (f_n) on the interval $(-1, 1)$. (3pts)

For all $x \in (-1,1)$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (1-|x|)^n = 0$ since $0<|x|<1 \Rightarrow 0<1-|x|<1$

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So it is the constant function $f(x)=0,\ x\in(-1,1)$

(limit at $x=0$ is 1)

(b) Does (f_n) converge to f uniformly on $(-1, 1)$? Answer YES or NO, and justify your answer. (3pts)

No

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Consider $\varepsilon = \frac{1}{2}$

let $N>2$ be arbitrary

what is n?

Consider $x = \log_N \frac{1}{2} \in (-1,0)$? $\Rightarrow (1-|x|^n) - 0 > \varepsilon$

(I don't think this works)

For (c) and (d), find the intervals of convergence (i.e., the domains) of the given power series. (4pts each)

(c) $\sum \left(\frac{1}{(n+1)^2 \cdot 2^n} \right) x^n$

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+2)^2\cdot 2^{n+1}}}{\frac{1}{(n+1)^2\cdot 2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{2(n+2)^2}\right| = \frac{1}{2} \Rightarrow R=2$$

For $x=2$, $\sum \frac{1}{(n+1)^2 2^n}\cdot x^n = \sum \frac{1}{(n+1)^2}$ converges by p-series

$x=-2 \Rightarrow \sum (-1)^n \frac{1}{(n+1)^2}$ converges by alternating series test

So interval of convergence is $[-2,2]$

(d) $\sum \left(\frac{2^n}{n \cdot 3^n} \right) x^n$

4

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\,\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\left(1-\frac{1}{n+1}\right)\right| = \frac{2}{3} \Rightarrow R=\frac{3}{2}$$

For $x=\frac{3}{2}$, $\sum\left(\frac{2^n}{n\cdot 3^n}\right)x^n = \sum \frac{1}{n}$ diverges

For $x=-\frac{3}{2}$, $\sum \frac{2^n}{n\cdot 3^n}x^n = \sum (-1)^n\frac{1}{n}$ converges by alternating series test

So interval of convergence is $[-\frac{3}{2},\frac{3}{2})$

(e) List *all* points in the domains you found in (c) and (d) for which the convergence is *conditional*: write NONE if there are no such points. Circle your answer(s). (3pts)

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(c): NONE

(d): $x=-\frac{3}{2}$

Problem 4: (20 points)

For each of (a) – (e), determine whether or not an object with the given property actually exists. Write YES or NO, and then briefly justify your answer either with an *explicit example* or a brief argument.

(a) An uncountable collection of nonempty open intervals in $\mathbb{R}$ that are all disjoint from each other.

4

No

Since every nonempty open interval in $\mathbb{R}$ must contain a rational by density of $\mathbb{Q}$ in $\mathbb{R}$,
we can select a rational for every interval
Since $\mathbb{Q}$ is countable, this collection is countable

(explain where you are using the fact that the intervals are disjoint)

(b) A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ for which $f'(0)$ exists and is negative.

4

No.

$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ $\lim_{h\to 0} \frac{f(h) - f(0)}{h}$ exists

Since f is strictly increasing, for all $h>0$, $f(h) > f(0) \Rightarrow \frac{f(h)-f(0)}{h} > 0$

$\Rightarrow \lim_{h\to 0} \frac{f(h)-f(0)}{h} \geq 0$

$\Rightarrow$ untradicts

(c) A bounded continuous function $f : (0,1) \to \mathbb{R}$ that cannot be extended continuously to $[0,1]$.

1

No

By taking $g(x) = \begin{cases} \lim_{x\to 1} f(x), & x=1 \\ f(x), & x \in (0,1) \\ \lim_{x\to 0} f(x), & x=0 \end{cases}$, $g : [0,1] \to \mathbb{R}$ with $g|_{(0,1)} = f$

(but these limits need not exist)

is always continuous since $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - \lim_{x\to 0} f(x)| < \delta$ whenever $|x-0| = x < \delta$
by definition of limit.

(d) A bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) \neq x$.

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No

Take $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in \mathbb{R}$

let $g(x) = f(x) - x \Rightarrow g(M) \geq 0$, $g(-M) \leq 0$

$\Rightarrow$ By IVT, $\exists c \in [-M, M]$ s.t. $g(c) = 0$. i.e. $f(c) = c$.

(e) A Riemann integrable function $f : [0,1] \to \mathbb{R}$ such that $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ has no convergent subsequence.

4

No

Since f is Riemann integrable, it is bounded

So $(f(\frac{1}{n}))_{n\in\mathbb{N}}$ is a bounded sequence

Then by BW Thm, there exists a convergent subsequence

Problem 5: (17 points)

(a) Suppose $I \subseteq \mathbb{R}$ is a bounded nondegenerate interval, and $f : I \to \mathbb{R}$ a differentiable function. In each part below, determine whether the given statement *must be true*. Clearly write "T" if it must be true, or "F" otherwise. *No justification necessary.* (10pts)

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- F $f[I]$ is bounded.
- T $f[I]$ is an interval.
- T If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.
- F If f is increasing on I, then $f'(x) > 0$ for all $x \in I$.
- T For any $c \in I$, if $f'(c) > 0$ then there is $\delta > 0$ such that f is increasing on $V_\delta(c) \cap I$.
- T If $c = \min(I)$ and $f'(c) \le 0$, then c is a local maximum of f on I.
- F f' is continuous on I.
- F f is uniformly continuous on I.
- T If f' takes on both positive and negative values on I, then $f'(x) = 0$ for some $x \in I$.
- T If f' is bounded on I, then f is also bounded on I.

(b) Let $A \subseteq \mathbb{R}$, and suppose (f_n) and (g_n) are sequences of bounded functions on A that converge uniformly on A to the functions f and g, respectively. Prove that $(f_n g_n) \to fg$ uniformly on A. (7pts)

Proof Take M_1, M_2 s.t. $|f_n(x)| \le M_1$ and $|g_n(x)| \le M_2$ for all $n \in \mathbb{N}$ and $x \in A$,

why should the same bounds work respectively uniformly for all n?

let $\varepsilon > 0$

Then $\exists N \in \mathbb{N}$ s.t. $\forall x \in A$ $|f_n(x) - f(x)| < \min\left\{\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3M_2}\right\}$ and $|g_n(x) - g(x)| < \min\left\{\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3M_1}\right\}$ whenever $n \ge N$

So $|f_n(x)g_n(x) - f(x)g(x)| = |(f_n(x) - f(x))(g_n(x) - g(x)) + f(x)g_n(x) + f_n(x)g(x) - 2f(x)g(x)|$

$\le |(f_n(x) - f(x))(g_n(x) - g(x))| + |f(x)(g_n x - g(x))| + |g(x)(f_n(x) - f(x))|$

$< \left(\sqrt{\frac{\varepsilon}{3}}\right)^2 + M_1 \frac{\varepsilon}{3M_1} + M_2 \frac{\varepsilon}{3M_2}$

$= \varepsilon$ whenever $n \ge N$, for all $x \in A$

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Therefore $(f_n g_n) \to fg$ uniformly on A

D

Problem 6: (17 points)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that $0 < L = \sup\{f(x) \ : \ x \in \mathbb{R}\} < \infty$.

(a) Prove that the set $P = \{x \in \mathbb{R} \ : \ f(x) > 0\}$ is uncountable. (6pts)

Proof Since $\sup\{f(x) \mid x \in \mathbb{R}\} = L > 0$

take $0 < \varepsilon < L \implies L - \varepsilon > 0$

By definition of supremum, $L-\varepsilon$ is not an upper bound of $\{f(x) \mid x \in \mathbb{R}\}$

Therefore $\exists x_0 \in \mathbb{R}$ s.t. $f(x_0) > L - \varepsilon > 0$ ✓

what ε are you using to get this δ?

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Since f is continuous on $\mathbb{R}$, $\exists \delta > 0$ s.t. $\underbrace{V_\delta(x_0)}_{\forall x \in}$, $f(x) > 0$ ✓

Note that $V_\delta(x_0)$ is an nondegenerate interval in $\mathbb{R}$, ✓

So it contains uncountably many points.

So $P \supseteq V_\delta(x_0)$ is uncountable ✓ □

(b) Prove that the set $C = \{x \in \mathbb{R} \ : \ f(x) = 0\}$ is closed. (5pts)

Proof We prove this by showing $C^c = \{x \in \mathbb{R} : f(x) \neq 0\}$ is open ✓

let $a \in C^c \implies f(a) > 0$ or $f(a) < 0$ ✓

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Since f is continuous $\implies \exists \delta_1 > 0$ s.t. $|f(x) - f(a)| < |f(a)|$ whenever $|x-a| < \delta_1$

for $f(a) > 0$ (ok) $\implies 0 < f(x) < 2f(a) \implies x \in C^c$

$\implies V_{\delta_1}(a) \subseteq C^c$

for $f(a) < 0$, $\exists \delta_2 > 0$ s.t. $|f(x) - f(a)| < -f(a)$ whenever $|x-a| < \delta_2$

$\implies -2f(a) < f(x) < 0 \implies x \in C^c \implies V_{\delta_2}(a) \subseteq C^c$

Therefore C^c is open

So $C = (C^c)^c$ is closed ✓ □

(c) Must the function $g(x) = \int_0^x f(t)\,dt$ be uniformly continuous on $\mathbb{R}$? Justify your answer. (6pts)

$g(x) = \int_0^x f(t)\,dt$ must be uniformly continuous on $\mathbb{R}$

Proof Since $f(t)$ is continuous on $\mathbb{R}$, by FTC(2), $g(x)$ is differentiable on $\mathbb{R}$

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and $g'(x) = f(x)$ for all $x \in \mathbb{R}$ $(g(x) \in C^1)$ ✓

let $\varepsilon > 0$. Since $f(x)$ is bounded, take $M > 0$ s.t. $|f(x)| \le M$ for all $x \in \mathbb{R}$

Take $\delta = \frac{\varepsilon}{M}$.

let $x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$. WLOG suppose $x < y$

$\implies g(x) - g(y) = \int_x^y f(t)\,dt < \int_x^y \sup\{f(t) \mid t \in \mathbb{R}\}\,dt = |x-y| M < \frac{\varepsilon}{M} \cdot M = \varepsilon$

This prove that g is uniformly continuous on $\mathbb{R}$ ✓

□