Homework 3: Due Tuesday, May 28, at 11:59pm, on Gradescope

Definition: A sequence (a_n) of real numbers is *eventually constant* if there is $c \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $a_n = c$ for all $n \geq N$.

(1) Let (a_n) be a sequence in $\mathbb{R}$, and consider the bi-implication: "$\lim a_n = \infty \iff \lim \frac{1}{a_n} = 0$." For each direction of this implication, either prove that direction if it is true, or else give a counterexample if it is false.

(2) Let (a_n) and (b_n) be sequences of real numbers. Prove that if $\lim a_n = 0$ and (b_n) is bounded, then $\lim a_n b_n = 0$.

(3) Determine the limits (in $\mathbb{R} \cup \{\pm\infty\}$) of the following sequences, and prove your results:

(a) $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!}$ (b) $\displaystyle\lim_{n\to\infty} \frac{n^n}{n!}$ (c) $\displaystyle\lim_{n\to\infty} b_n$, where $b_1 = 2$ and $\displaystyle b_{n+1} = \frac{b_n^2 + 2}{2b_n}$

(4) Suppose A is a discrete[2] subset of $\mathbb{R}$, and let (a_n) be a convergent sequence of numbers in A. Prove that either (a_n) is eventually constant or $\lim a_n \notin A$.

(5) For each positive integer M, let $\mathbb{Q}_M$ be the set of all rational numbers m/n where $m, n \in \mathbb{Z}$ and $|m| \leq M$. Prove that for all $M \in \mathbb{N}$, every sequence of distinct numbers in $\mathbb{Q}_M$ converges.

(6) Let (a_n) and (b_n) be sequences of real numbers such that $a_n < b_n$ for all n.

(a) Show that if $\lim a_n = \infty$, then $\lim b_n = \infty$.

(b) Given an example to show that (a_n) and (b_n) could converge to the same real number.

(7) Let (a_n) be a sequence of positive real numbers. Show that if $\lim \frac{a_{n+1}}{a_n} = L > 1$, then $\lim a_n = \infty$.

(8) Find the lim sup and lim inf of the following sequences. (No justification is needed).

(a) $(a_n)_{n\geq 1}$, where $\displaystyle a_n = (-1)^{n+1} + \frac{(-1)^n}{n}$

(b) $(b_n)_{n\geq 1}$, where $b_n = \sin \frac{1}{n}$

(c) (c_n), where $c : \mathbb{N} \to \mathbb{Q}$ is any bijection

(d) (d_n), where $d_n = \ln n + \cos n$

(9) Let $a, b \in \mathbb{R}$ with $a < b$. Find the limit of the sequence (s_n) defined recursively by $s_1 = a$, $s_2 = b$, and for all $n \in \mathbb{N}$,

$$s_{n+2} = \frac{s_n + s_{n+1}}{2}.$$

Prove your claim.

(10) Give an example of a divergent sequence (a_n) in $\mathbb{R}$ with a convergent subsequence such that all convergent subsequences of (a_n) converge to the same limit.

(11) Let (a_n) and (b_n) be bounded sequences of positive real numbers.

(a) Show that $\limsup(a_n + b_n) \leq \limsup(a_n) + \limsup(b_n)$.

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[2] See HW 1 #11 for the definition of *discrete*.

(b) Give an example to show that $\limsup(a_n+b_n)$ might not equal $\limsup(a_n)+\limsup(b_n)$.

(c) Show that if (a_n) converges, then $\limsup(a_n + b_n) = \limsup(a_n) + \limsup(b_n)$.

(12) Prove that there exists a sequence (a_n) in $\mathbb{R}$ such that for every $r \in \mathbb{R}$ there is a subsequence of (a_n) that converges to r.

(13) Determine whether the following sets are open, closed, both, or neither (no justification needed):

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(b) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

(c) $\displaystyle\bigcup_{n\geq 1} \left[\frac{1}{n}, 3 - \frac{1}{n}\right]$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

(f) $\displaystyle\bigcap_{n\geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$

(14) Either prove the following if it is true, or else give a counterexample if it is false: if $A \subseteq \mathbb{R}$ is closed and discrete, then there is $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$. [cf: HW 1, #11(b)]

(15) Suppose the set $A \subseteq \mathbb{R}$ is infinite, bounded, and discrete. Prove that there is a convergent sequence in A whose limit is not in A.