(1) For each statement about sets given below, either *prove* the statement if it is true for all sets, or else give a *counterexample* using specific sets if it is false.

(a) $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

(b) $(A \cup B) \setminus C \supseteq A \cup (B \setminus C)$.

(c) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

(d) $A \subseteq B$ if and only if $A \cap B = A$.

(a) Proof

Assume $x \in (A \cup B) \setminus C$

So $(x \in A$ or $x \in B)$, and $x \notin C$

$\Rightarrow$ $(x \in A$ but $x \notin C)$ or $(x \in B$ but $x \notin C)$

This contains $x \in A$ or $(x \in B$ but $x \in C)$

So $x \in A \cup (B \setminus C)$

Therefore $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

(b) Counterexample:

$A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3, 4, 5\}$

$5 \in A \Rightarrow 5 \in A \cup (B \setminus C)$

But $5 \notin (A \cup B) \setminus C$

So $(A \cup B) \setminus C \not\supseteq A \setminus (B \cup C)$

(c) <u>Counterexample</u>.

$A = \{1,2,3,4,5\}$

$B = \{1,2,3,4,5\}$

$C = \{1,2,3\}$

$4 \in A \setminus C \Rightarrow \underline{4 \in (A \setminus B) \cup (A \setminus C)}$

But $\underline{4 \notin A \setminus (B \cup C)}$

So $A \setminus (B \cup C) \neq (A \setminus B) \cup (A \cup C)$

(d) <u>Proof</u>

Assume $\underline{A \subseteq B} \Rightarrow \underline{\text{if } x \in A, \text{ then } x \in B}$ ①

Take $x \in A \cap B$, so $\underline{x \in A}$

Take $x \in A \Rightarrow x \in B$ by ①, so $\underline{x \in A \cap B}$

$\Rightarrow$ Therefore $A \subseteq A \cap B$, $A \cap B \subseteq A \Rightarrow \underline{A = A \cap B}$

Assume $\underline{A = A \cap B}$

Fix $a \in A \Rightarrow a \in A \cap B \Rightarrow a \in A$ and $a \in B$

So $a \in A$ implies $a \in B$, therefore $\underline{A \subseteq B}$

Then we have proved $\underline{A = A \cap B \text{ iff } A \subseteq B}$

(2) For each $n \in \mathbb{N}$, let $A_n = \{nk : k \in \mathbb{N}\}$.

(a) What is $A_2 \cap A_3$?

(b) Determine (i.e., give simple descriptions of) the sets $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$.

(a) $A_2 = \{2k \mid k \in \mathbb{N}\}$

$A_3 = \{3k \mid k \in \mathbb{N}\}$

So $x \in A_2 \cap A_3$ iff $\underline{2 \mid x \text{ and } 3 \mid x}$ (and $x \in \mathbb{N}$)

$\iff \gcd(2,3) \mid x$ (and $x \in \mathbb{N}$)

i.e. $6 \mid x$ (and $x \in \mathbb{N}$)

So $A_2 \cap A_3 = \underline{\{6k \mid k \in \mathbb{N}\}}$

(b) $\bigcup_{n=2}^{\infty} A_n = \{x \in \mathbb{N} : 2 \mid x \text{ or } 3 \mid x \text{ or} \ldots\}$

$= \{x \in \mathbb{N}$ s.t. x is some multiple of any natural number that $\geq 2\}$

$= \underline{\{x \in \mathbb{N} \mid x \geq 2\}}$

$\bigcap_{n=2}^{\infty} A_n = \{x \in \mathbb{N} : 2 \mid x \text{ and } 3 \mid x \text{ and} \ldots\}$

$= \{x \in \mathbb{N}$ s.t. x has all natural numbers that ≥ 2 as factors$\}$ $\underline{= \emptyset}$

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(3) (a) Guess a formula for $1+3+\cdots+(2n-1)$ by evaluating the sum for $n = 1, 2, 3$, and 4. (For $n = 1$, the sum is simply 1).

(b) Prove that your formula is correct using mathematical induction.

(a) $1+3+\dots+(2n-1)$

$$= \underbrace{1+2n-1}_{2n} + \underbrace{3+2(n-1)-1}_{2n} + \dots$$

$$= \frac{n}{2}\cdot 2n = \underline{n^2}$$

(b) Proof by induction on n

Base case: $n=1$, $\sum_{k=1}^{1}(2k-1) = 1 = n^2$

Inductive step: assume for $n=k$, $\sum_{k=1}^{n}(2k-1) = k^2$

Then for $n=k+1$

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + 2(k+1)-1$$

$$= k^2+2k+1 = \underline{(k+1)^2}$$

This finishes the proof that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}(2k-1) = k^2$$

(4) Determine for which integers the inequality $2^n > n^2$ is true, and prove your claim by induction.

Pf. $\underline{n=0 \text{ or } n \geq 5}$

Case 1: $n=0$. $2^n = 1$, $n^2 = 0$, $\Rightarrow 2^n > n^2$

Case 2: $n \geq 5$

We prove this by induction on n

Base case: $n=5$, $2^n = 32$, $n^2 = 25$ $\Rightarrow 2^n > n^2$

Inductive step: assume for $n=k$ $\underline{(\text{where } k \in \mathbb{N}, k \geq 5)}$, $2^n > n^2$

Then $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k$, $(k+1)^2 = k^2 + 2k + 1$

Note that $k^2 - (2k+1) = (k-2)k - 1$

Since $k \geq 5$, $k-2 \geq 3$, so $\underline{(k-2)k - 1 \geq 14 > 0}$

Therefore $k^2 > 2k+1$

So $\underline{2^{k+1}} = 2^k + 2^k > \underline{2^k + k^2 > 2^k + 2k+1}$
$> k^2 + 2k + 1$
$(= \underline{(k+1)^2})$

This finishes the proof that for all integer $n \geq 5$, $\underline{2^n > n^2}$

(5) For each of the subsets of $\mathbb{R}$ given in (a) – (x) below, state (i) whether or not the set is bounded above; (ii) whether or not it is bounded below; (iii) what the supremum is (if it exists); and what the infimum is (if it exists). You may write all your answers on one line, with no justification needed, as in the answer for (a) given below:

"Bounded below but not above; inf = 1."

(a) $\mathbb{N}$

(b) $[0, 1)$

(c) $\{2, 7\}$

(d) $\{\pi, e\}$

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(f) $\{0\}$

(g) $[0, 1] \cup [2, 3]$

(h) $\bigcup_{n=1}^{\infty} [2n, 2n + 1]$

(i) $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1 + \frac{1}{n}]$

(j) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

(l) $\{r \in \mathbb{Q} : r < 2\}$

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$

(n) $\{r \in \mathbb{Q} : r^2 < 2\}$

(o) $\{x \in \mathbb{R} : x < 0\}$

(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$

(q) $\{0, 1, 2, 4, 8, 16\}$

(r) $\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$

(s) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$

(t) $\{x \in \mathbb{R} : x^3 < 8\}$

(u) $\{x^2 : x \in \mathbb{R}\}$

(v) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

(w) $\bigcup_{n=1}^{\infty} \{\frac{k}{n} : k \in \mathbb{N}\}$

(x) $\bigcap_{n=1}^{\infty} \{\frac{k}{n} : k \in \mathbb{N}\}$

(a) Bounded below but not above
inf = 1

(b) Bounded below and above
inf = 0, sup = 1

(c) Bounded below and above
inf = 2, sup = 7

(d) Bounded below and above
inf = e, sup = π

(e) Bounded below and above
inf = 0, sup = 1

(l) Bounded above but not below
sup = 2

(m) Bounded below and above
inf = −2, sup = 2

(n) Bounded below and above
inf = $-\sqrt{2}$, sup = $\sqrt{2}$

(o) Bounded above but not below
sup = 0

(p) Bounded below and above
inf = 1, sup = 10

(f) Bounded below and above

inf = sup = 0

(g) Bounded below and above

inf = 0, sup = 3

(h) Bounded below but not above

inf = 2

(i) Bounded below and above

inf = 0, sup = 1

(j) Bounded below and above

inf = $\frac{2}{3}$ sup = 1

(k) Bounded below but not above

inf = 0

(w) Bounded below but not above

inf = 0

(q) Bounded below and above

inf = 0, sup = 16

(r) Bounded below and above

inf = sup = 1

(s) Bounded below and above

inf = 0, sup = $\frac{1}{2}$

(t) Bounded above but not below

sup = 2

(u) Bounded below but not above

inf = 0

(v) Bounded below and above

inf = -1, sup = 1

(x) Bounded below but not above

inf = 1

(6) The complex numbers form a *field*; that is, the algebraic structure $(\mathbb{C}, +, \cdot, 0, 1)$ satisfies our Axioms 1–9. In fact, $\mathbb{C}$ also satisfies a version of the Completeness Axiom, so that $\mathbb{C}$ is a *complete* field. Prove, however, that it is impossible to define a linear order relation $<$ on $\mathbb{C}$ that makes $\mathbb{C}$ an *ordered* field; i.e., it is impossible to define a linear order relation $<$ on $\mathbb{C}$ that satisfies Axioms (13) and (14). [HINT: argue by contradiction. The *only* things you are allowed to use without proof are the ordered field axioms and the results in the handout "Elementary Properties of Real Numbers," which hold in any ordered field.]

Proof. Assume for contradiction that a linear relation $<$ is defined on $\mathbb{C}$ such that axiom 13 – 14 stands.

that is: $\forall x, y, z \in \mathbb{C}$, if $x<y$ then $z+x<z+y$

if $x<y$ and $z>0$ then $x \cdot z < x \cdot y$

Case 1: We define $i>0$, so by axiom 14:

by multiplying i on both sides, $i \cdot i > 0 \cdot i$

$\Rightarrow -1 > 0$,

then multiply -1 on both sides $\Rightarrow 1 > 0$

So by axiom 13 $-1+1 > 0+1 > \underbrace{0+0}^{=0}$

This contradicts with the definition of -1 that $-1+1=0$ by axiom 4

Case 2: Define $i=0 \Rightarrow i^2 = -1 = 0$

$\Rightarrow 1 = -(-1) = 0 \Rightarrow$ contradicts with axiom 6.

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Case 3: define $i < 0$

then $i = -a$ for some $a \in \mathbb{C}$ s.t. $a > 0$

So $i^2 = (-a)\cdot(-a) = (-1)(-1)a^2 = a^2 > 0$

(by axiom 5) (by axiom 14)

So $-1 > 0$

Then by the same reason in case 1, it contradicts with the definition of -1 that $-1+1=0$ by axiom 4

Since in all cases, the assumption of linear order contradicts with the properties of $\mathbb{C}$. therefore we have proved that it is impossible to define a linear relation on $\mathbb{C}$ s.t. axiom 13–14 holds.

(7) (a) Let $a, b \in \mathbb{R}$. Show that if $a \leq c$ for every $c > b$, then $a \leq b$.

(b) Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$, and suppose L is an upper bound of A. Show that $L = \sup A$ if and only if for every $\epsilon > 0$ there is $a \in A$ such that $L - \epsilon < a \leq L$.

(a) suppose $a > b$ for contradiction

Then by the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q \in \mathbb{Q}$ s.t. $a > q > b$

Then by the given conditions, $a \leq q$ which contradicts with $a > q$

So $a \leq b$

(b) One direction:

Assume $L = \sup A$ WTS: $\forall \varepsilon > 0, \exists a \in A$ s.t. $L-\varepsilon < a \leq L$

We prove this by assuming for contradiction that for some $\varepsilon > 0$, $\nexists a \in A$ s.t. $L-\varepsilon < a \leq L$ ①

Since $L = \sup A$, by definition of supremum, $\nexists a \in A$ s.t. $a > L$ ②

Therefore combining ①②, $\nexists a \in A$ s.t. $a > L-\varepsilon$

So $L-\varepsilon$ is an upper bound of A

This contradicts with the definition of supreme since $L-\varepsilon < L$, $L-\varepsilon$ can not be an upper bound of A.

The other direction

Assume $\forall \varepsilon > 0, \exists a \in A$ st $L-\varepsilon < a \leq L$

WTS: $L = \sup A$ (given the condition that L is an upper bound of A)

Let M be an arbitrary upper bound of A.

Suppose if $M < L$, then $\exists a \in A$ s.t. $M < a < L$ $\Rightarrow$ contradicts with M being an upper bound of A

Therefore $M \geq L$.

Since M is arbitrarily selected, by definition $L = \sup A$

This completes the proof that $L = \sup A$ iff $\forall \varepsilon > 0, \exists a \in A$ s.t. $L-\varepsilon < a < L$

(8) Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Prove that $\inf S \leq \sup S$.

(b) Supposing that $S \subseteq T$, put the four numbers $\sup S$, $\inf S$, $\sup T$, $\inf T$ in order (with respect to $\leq$), and prove your claims.

(c) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

(a) Proof Take arbitrary $s \in S$

So by definition of upper and lower bound,

$\inf S \leq s$, $\sup S \geq s$

So $\inf S \leq \sup S$ by the transitivity of linear order and equivalence relation.

(b) $\inf T \leq \inf S \leq \sup S \leq \sup T$

Proof by (a) we have proved $\inf S \leq \sup S$

So it suffices to prove that $\inf T \leq \inf S$ and $\sup S \leq \sup T$

let $s \in S$, since $S \subseteq T \Rightarrow s \in T$

therefore if $a > t$ for all $t \in T$

then also, $a > s$ for all $s \in S$

This shows that any upper bound of T is also an upper bound of S.

Dually, any lower bound of T is also a lower bound of S

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Therefore {lower bounds of T} $\subseteq$ {lower bounds of S}

and {upper bounds of T} $\subseteq$ {upper bounds of S}

Since $\inf S = \max$ {lower bounds of S}

and $\inf T \in$ {lower bounds of T} $\subseteq$ {lower bounds of S}

$\Rightarrow \inf S \geqslant \inf T$

Dually, $\sup S = \min$ {upper bounds of S}

and $\sup T \in$ {upper bounds of T} $\subseteq$ {upper bounds of S}

$\Rightarrow \sup T \leqslant \sup S$

This finishes the proof that $\inf T \leq \inf S \leqslant \sup S \leqslant \sup T$

(c) We first claim: $\max(\sup S, \sup T)$ is an upper bound of $S \cup T$

Let x be an arbitrary element of $S \cup T$

Then $x \in S$ or $x \in T$

If $x \in S$, then $x \leqslant \sup S$ by by definition

so $x \leqslant \max(\sup S, \sup T)$

If $x \in T$, then $x \leqslant \sup T$ by by definition

so $x \leqslant \max(\sup S, \sup T)$

Since in both cases, $x \leq \max(\sup S, \sup T)$ and x is arbitrary, we have proved that $\max(\sup S, \sup T)$ is an upper bound of $S \cup T$

Then we prove that any upper bound of $S \cup T$ $\geq \max(\sup S, \sup T)$

Let b be an arbitrary upper bound of $S \cup T$

So b is an upper bound of both S and T.

Assume for contradiction that $b < \max(\sup S, \sup T)$

Then b is less than at least one of $\sup S$, $\sup T$

Without Loss of Generality, suppose $b < \sup S$

Then b is not a upper bound of S, by definition of supremum

This contradicts with the fact that b is an upper bound of S.

Therefore $b \geq \max(\sup S, \sup T)$

This finishes the proof that $\sup(S \cup T) = \max(\sup S, \sup T)$

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(9) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B = \{a+b : a \in A \text{ and } b \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$.

Proof ① First we claim: sup$A$ + sup$B$ is an upper bound of $A+B$

Let $a+b$ be an arbitrary element of $A+B$ ($a \in A$, $b \in B$)

Then $\sup A > a$, $\sup B > b$ by definition

So $\sup A + \sup B > a + \sup B > a+b$ by axiom B

Since $a+b$ is arbitrary, we have proved that $\sup A + \sup B$ is an upper bound of $A+B$

This implies: $\sup(A+B) \leq \sup A + \sup B$

② Now we show $\sup A + \sup B \leq \sup(A+B)$

Assume for contradiction: $\sup(A+B) < \sup A + \sup B$

$\Rightarrow \sup(A+B) = \sup A + \sup B - \varepsilon$ for some $\varepsilon > 0$

$\sup(A+B) = (\sup A - \frac{\varepsilon}{2}) + (\sup B - \frac{\varepsilon}{2})$

By definition of supreme, $(\sup A - \frac{\varepsilon}{2})$ is not an upper bound of $A \Rightarrow \exists$ some $a_0 \in A$ s.t. $a_0 > \sup A - \frac{\varepsilon}{2}$

Similary, then $\exists$ some $b_0 \in B$ s.t. $b_0 > \sup B - \frac{\varepsilon}{2}$

$\Rightarrow a_0 + b_0 \in A+B$, but $a_0 + b_0 > \sup(A+B) \Rightarrow$ contradicts

So $\sup A + \sup B \leq \sup(A+B)$

By ①②, we have proved $\sup A + \sup B = \sup(A+B)$

(10) Prove that $\mathbb{R} \setminus \mathbb{Q}$ is *dense* in $\mathbb{R}$ in the sense that for every pair of real numbers a and b, if $a < b$ then there exists an irrational number r such that $a < r < b$.

Proof. Take arbitrary $a, b \in \mathbb{R}$ s.t $a < b$.

Then $b = a + \varepsilon$ for some $\varepsilon \in \mathbb{R}$ s.t. $\varepsilon > 0$

By the Archimedean property of $\mathbb{R}$,

there exists $n \in \mathbb{N}$ s.t. $n\varepsilon > 1 \Rightarrow \varepsilon > \frac{1}{n}$

So consider $\varepsilon' = \frac{1}{n\sqrt{2}} = \frac{1}{n} \cdot \frac{\sqrt{2}}{2}$

which is an irrational number

since $\frac{1}{n} \in \mathbb{Q}$ and $\frac{\sqrt{2}}{2}$ is irrational

Also, $\varepsilon' < \varepsilon$ since $\frac{\sqrt{2}}{2} < 1$,

by axiom 14, $\frac{1}{n} \cdot \frac{\sqrt{2}}{2} < \frac{1}{n} \cdot 1 = \frac{1}{n}$

Therefore $a < a + \varepsilon' < \underbrace{a + \varepsilon}_{(=b)}$

Then we have proved that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

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A set $A \subseteq \mathbb{R}$ is *discrete* if for every $a \in A$ there is $\epsilon > 0$ such that $V_\epsilon(a) \cap A = \{a\}$, where $V_\epsilon(a) = (a - \epsilon,\, a + \epsilon)$ is the open interval of radius ϵ centered at a.

(11) (a) Prove that every finite subset of $\mathbb{R}$ is discrete.

(b) Either prove the following if it is true, or else give a counterexample if it is false: if $A \subseteq \mathbb{R}$ is discrete, then there is $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$.

(a) Proof.

Let $A \subseteq \mathbb{R}$ be an arbitrary finite set and $a \in A$ be arbitrary

Consider: $B = \{|a - x| \mid x \in A\}$

This is a finite set since A is finite

$\Rightarrow$ So B has a smallest element

Let $\varepsilon = \min(B)$, then $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$

where ε is the distance of a from its nearest element in X

$\Rightarrow V_\varepsilon(a) \cap A = \{a\}$

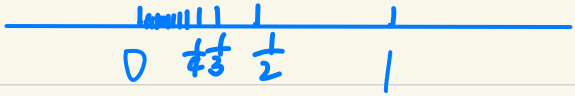
Since a is arbitrary, A is discrete

Since A is arbitrary we have proved that every finite subset of $\mathbb{R}$ is discrete

(b) False

Consider the following counterexample:

$A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$



This is a <u>discrete set</u> since for any element

$\frac{1}{n} \in A$, consider $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$

So $V_\varepsilon(\frac{1}{n}) = (\frac{1}{n+1}, \frac{2}{n} - \frac{1}{n+1}) \Rightarrow V_\varepsilon(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$

But such ε does not exist on A

Since if so, then $\frac{1}{n} - \frac{1}{n+1} > \varepsilon$ for all $n \in \mathbb{N}$

$\varepsilon < \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$

$\Rightarrow n(n+1)\varepsilon < 1$ for all $n \in \mathbb{N}$

which contradicts with the Archimedean property of $\mathbb{R}$

(12) OPTIONAL CHALLENGE PROBLEM.[1] For $A, B \subseteq \mathbb{R}$, let $AB = \{ab : a \in A \text{ and } b \in B\}$. Find a simple expression for $\sup(AB)$ in the case where A and B are nonempty and bounded, and prove your result.