(所有 complete ordered field 都 $\cong \mathbb{R}$)

$\mathbb{R}$ is the unique complete ordered field.

$\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$.

$\mathbb{N}$: 没有 $+^{-1}$, $+_e$, $\times^{-1}$

$\cap$

$\mathbb{Z}$: 没有 $\times^{-1}$

$\cap$

$(\mathbb{Q}, +, \times, <)$ satisfies Axiom 1-14

so $\mathbb{Q}$ is an ordered field.

But $\mathbb{Q}$: algebraic deficiency (field with linear order)

there are algebraic equations using rational coefficients that has no rational roots

ex $x^2-2=0$ (Pythagoras: $\sqrt{2}$ is irrational)

Rational Roots Thm

Let $f(x)=\boxed{\sum_{k=0}^{n} a_k x^k}$ with $a_0 \neq 0$ and $\forall k,\ a_k \in \mathbb{Z}$

If $r=\frac{p}{q}$ is a root of $f(x)$, then $p \mid a_0$ and $q \mid a_n$

(p, q coprime, $q \neq 0$),

ex: $f(x)=x^2-2$

By the rational roots Thm,

if $r=\frac{p}{q}$ is a root of $f(x)$,

then $p \mid -2$, $q \mid 1 \Rightarrow r \in \{\pm 1, \pm 2\}$ (not roots)

so $f(x)$ has no root in $\mathbb{Q}$.

Pf of the Thm:

Assuming the hypothesis, then

$$f(r)=f\left(\frac{p}{q}\right)=\sum_{k=0}^{n} a_k\left(\frac{p}{q}\right)^k=0$$

multiplying by $q^n \Rightarrow \sum_{k=0}^{n} a_k p^k q^{n-k}=0$

$$\Rightarrow a_0 q^n = -\sum_{k=1}^{n} a_k p^k q^{n-k} = p\left(-\sum_{k=1}^{n} a_k p^{k-1} q^{n-k}\right) \in \mathbb{Z}$$

Same reason, $a_n p^n = q\left(-\sum_{k=0}^{n-1} p^k q^{n-1-k}\right) \in \mathbb{Z}$

(By FTArithmetic), since $(p,q)=1$ and $a_0, a_n \neq 0$

$\Rightarrow p \mid a_0,\ q \mid a_n$

Def A (complex) number z is algebraic 代数数

if z is a root of a polynomial with coeffs in $\mathbb{Q}$

otherwise, z is transcendental 超越数

ex $\sqrt{2}$ is root of $x^2-2 \Rightarrow$ algebraic

$\sqrt{2+\sqrt[3]{5}}$ is root of $x^6-6x^4+12x^2-13 \Rightarrow$ algebraic

$i=\sqrt{-1}$ is root of $x^2+1 \Rightarrow$ algebraic

(显然, $\forall q \in \mathbb{Q}$, q is algebraic since q is a root of $x-q=0$)

* π and e are transcendental (hard to prove)

The set of all algebraic number is denoted by $\overline{\mathbb{Q}}$, which is a field called the algebraic closure of $\mathbb{Q}$. 代数闭包

Def 一个 field $\mathbb{F}$ 被称为是 algebraically closed 的 代数闭的

if every polynomial of degree n with coeffs in $\mathbb{F}$ has n roots in $\mathbb{F}$ (counting multiplicities)

$\Rightarrow \overline{\mathbb{Q}}$ is algebraically closed

And FTAlgebra: $\mathbb{C}(=\overline{\mathbb{R}})$ is algebraically closed

($\overline{\mathbb{Q}}$ 中有: 一些 irr num, 一些 im num 和 $\mathbb{Q}$)

$\overline{\mathbb{Q}}$ algebraically closed, 但是仍有 geometric deficiency

见下: order theory

Order Theory

recall: ① 一个 irreflexive, transitive 的 order 称为 partial order (如 $\leq$)

② 一个 trichotomy 的 partial order 称为 linear order. (如 $<$)

Def upper bound

令 $<$ 为一个 linear relation on set X.

$A \subseteq X$, $b \in X$

如果 $\forall a \in A$, $a \le b$, 则称 b 为 A 的一个 upper bound

并称 A 是 bounded above in X 的

Def max (min), sup (inf)

$b = \max A$ (largest element of A)

if b 是 A 的一个 upper bound 且 $b \in A$

$b = \sup A$ if b 是 A 的 least upper bound

(supremum) $\forall$ UB of A as u, $u \ge b$

同理 we have: lower bound, bounded below, $\min A$, infimum ($\inf A$)

Def $A \subseteq X$ is bounded in X if A 在 X 中 both bounded above and below.

otherwise: unbounded above/below

Note: A 可以没有 max/min in X, 也可以没有 sup/inf in X. 但是

if $A \subseteq X$ has max in X, then $\max A$ is unique.

Pf. suppose $a, b = \max A$

then $a \in A$, $b \in A \Rightarrow a \le b$, $b \le a \Rightarrow a = b$

Def Interval (老概念)

Let $<$ be a linear relation on X

一个 interval on X, 指 $I \subseteq X$

s.t. $z \in I$ whenever $x, y \in I$ and $x < z < y$

(I中任意两点之间所有X中的点也在I中)

$[a,b] = \{x \in X \mid a \le x \le b\}$ closed interval

$(a,b] = \{x \in X \mid a < x \le b\}$ half-closed interval

⋮

Convention

$[a,\infty) = \{x \in X \mid x \ge a\}$

if A is not bounded above, $\sup A = +\infty$

if A is not bounded below, $\inf A = -\infty$

$\sup \emptyset = -\infty$ (They are not in $\mathbb{R}$)

$\inf \emptyset = +\infty$

ex (1) Every finite set $A \subseteq \mathbb{R}$ is bounded and has max, min.

(2) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are not bounded above in $\mathbb{R}$

(3) $\mathbb{N}$ is bounded below in $\mathbb{R}$

$\inf \mathbb{N} = 1$, $\{\text{all LB of } \mathbb{N} \text{ in } \mathbb{R}\} = (-\infty, 1]$

(4) $\inf (0,1) = \inf [0,1) = 0$

$\sup (0,1) = \sup (0,1] = 1$

(5) $\min (0,1)$ DNE, $\min [0,1] = 0$,

$\max (0,1)$ DNE, $\max [0,1] = 1$

(6) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

$\min A$ DNE, $\inf A = 0$, $\max A = 1 = \sup A$

Note: ① if x is UB of A in X, then $\forall y \ge x$ in X, y is UB of A}

(LB 同理)

② if $\exists \max A$, then $\exists \sup A$, 且 $\max A = \sup A$

LUB property

if $A \subseteq X$ is not empty $\Rightarrow \sup A \in X$

满足该 property 的 ordered set 被称为 具有 LUB property 的, 即 geometrically closed 的.

如果该 ordered set 是 field 则称其为 complete ordered field

(只有一个 complete ordered field: $\mathbb{R}$)

$\mathbb{Q}$ 和 $\bar{\mathbb{Q}}$ (代数数集) 都是有 geometric deficiency 的

$A = \{r \in \mathbb{Q} \mid r^2 < 2\} \Rightarrow \sup A = \sqrt{2} \notin \mathbb{Q}$.

而 $\mathbb{R}$ 是 complete ordered field (但有 algebraic deficiency)