Ch6 Differentiation

Def Derivative

Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$

let $a\in (A\cap A')$ 比ctn更严格:必须既$\in \mathrm{dom}f$ 又$\in(\mathrm{dom}f)'$

Define the derivative of f at a:

$$\boxed{f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}}$$

如果 let $x=a+h$ 则有 $h=x-a$, then

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

如果 $f'(a)$ exists, 则称 f is differentiable at a

使用 a 为 variable, 我们得到了 the derivative as a function.

$$\boxed{f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}}$$

with $\mathrm{dom}(f')=\{x\in\mathrm{dom}(f)\mid f \text{ is diffble at } x\}$

Def 如果 on $B\subseteq\mathrm{dom}(f)$, $\forall x\in B$ 都有 f diffble at x
则称 f is differentiable on B

我们都知道 derivative 的 geometrical meaning:

the slope of the line tangent to the graph of $y=f(x)$ at point $(a, f(a))$

我们称 $L(x)=f(a)+f'(a)(x-a)$ 为 the linear approximation of f near $x=a$.

Thm① If f is diffble at a, 则 f is ctn. at a
(diffble 是比 ctn 更强的条件:不仅连续,且在a点平滑)

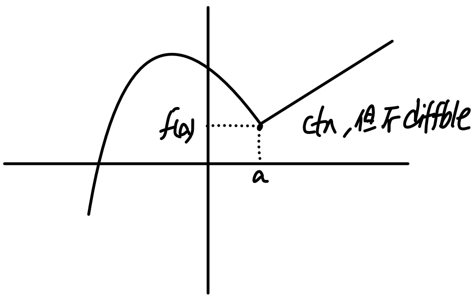
Pf Suppose $f'(a)$ exists, 则 $a\in(\mathrm{dom}(f))'$

Then $$\boxed{\lim_{x\to a}f(x)}=\lim_{x\to a}\left[f(a)+\frac{f(x)-f(a)}{x-a}(x-a)\right]$$
$$=f(a)+\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)$$
$$=f(a)+f'(a)\cdot 0=\boxed{f(a)}$$

Since $a\in(\mathrm{dom}f)'$, $\lim_{x\to a}f(x)=f(a)$ implies ctn

因而 diffbility $\Rightarrow$ ctnity

但 $\not\Leftarrow$



Thm② Linearity of the derivative

Suppose f,g are diffble at a; $\forall c\in\mathbb{R}$.

$\Rightarrow$ cf, $f+g$ 在 a 处 diffble

且 $(cf)'(a)=cf'(a)$, $(f+g)'(a)=f'(a)+g'(a)$

即: $\frac{d}{dx}$ 是一个 linear operator

Pf $$(cf)'(a)=\lim_{h\to0}\frac{cf(a+h)-cf(a)}{h}=c\lim_{h\to0}\frac{f(a+h)-f(a)}{h}=cf'(a)$$

$$(f+g)'(a)=\lim_{h\to0}\frac{(f+g)(a+h)-(f+g)(a)}{h}=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to0}\frac{g(a+h)-g(a)}{h}=f'(a)+g'(a)$$

Thm③ Product Rule

if f,g diffble at a, 则 fg diffble at a 且

$(fg)'(a)=f'(a)g(a)+f(a)g'(a)$

Pf $$(fg)'(a)=\lim_{x\to h}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}$$
$$=\lim_{h\to0}\frac{(f(a+h)-f(a))g(a+h)+f(a)(g(a+h)-g(a))}{h}$$
$$=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\underbrace{\lim_{h\to0}g(a+h)}_{=g(a)\text{ since diffble}\Rightarrow\text{ctn}}+\lim_{h\to0}f(a)\lim_{h\to0}\frac{g(a+h)-g(a)}{h}$$
$$=f'(a)g(a)+f(a)g'(a)$$
□

Thm④ Quotient rule

if f,g diffble at a 且 $g(a)\neq0$

则 f/g diffble at a 且

$$\left(\frac{f}{g}\right)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$$

Pf 相似于 product rule.

Notation 习惯

$$f'(x) = \frac{d}{dx}(f), \quad f'(a) = \frac{dy}{dx}\Big|_{x=a}(f)$$

$$f''(x) = \frac{d^2y}{dx^2}, \ f'''(x), \ f^{(4)}(x), \ldots$$

ex $p(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial

then $p'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$

Pf By induction on n

$$\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x \cdot x^n) = \frac{d}{dx}(x)\cdot x^n + x \cdot \frac{d}{dx}(x^n)$$
$$= x^n + x \cdot n x^{n-1} = (n+1)x^n$$

Fact $\forall p \in \mathbb{R}$, $\frac{d}{dx}(x^p) = p \cdot x^{p-1}$

$\forall a \in \mathbb{R}$, $\frac{d}{dx}(a^x) = (\ln a) a^x$

(specially $\frac{d}{dx}(e^x) = e^x$)

$\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$

Thm⑤ Chain Rule

如果 f 在 a 处 diff'ble 且 g 在 $f(a)$ 处 diff'ble

$\Rightarrow$ $g\circ f$ 在 a 处 diff'ble，且 $\boxed{(g\circ f)'(a) = g'(f(a))\cdot f'(a)}$

Pf 我们定义以下函数：

$\varphi: \mathrm{dom}(g) \to \mathbb{R}$

by
$$\varphi(u) = \begin{cases} \dfrac{g(u)-g(f(a))}{u-f(a)}, & \text{if } u \in \mathrm{dom}(g)\setminus\{f(a)\} \\ g'(f(a)), & \text{if } u = f(a) \end{cases}$$

$$\left(= \lim_{u\to f(a)} \frac{g(u)-g(f(a))}{u-f(a)} \right)$$

于是 $\varphi(u)(u-f(a)) = g(u) - g(f(a))$ for all $u \in \mathrm{dom}(g)$

且易证，φ 在 $f(a)$ 处 ctn.

因而 $g'(f(a)) = \varphi(f(a)) = \varphi(\lim_{x\to a} f(x)) = \lim_{x\to a}\varphi(f(x))$

$$f'(a)\cdot g'(f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}\varphi(f(x))$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\frac{g(f(x))-g(f(a))}{f(x)-f(a)} = \lim_{x\to a}\frac{g\circ f(x) - g\circ f(a)}{x-a} = (g\circ f)'(a)$$

□

这一证明的核心在于构建了一个函数 φ，
用来模拟用 tangent line 逼近 $g(f(a))$ 的行为
并通过 g 的 diff'bility 来说明 φ 在 $g(f(a))$ 处的 ctnity，
从而利用 ctnity 的复合性质来用 φ 替代 $g'(f(a))$，以把 $g\circ f$ 的 expression 带入最后的 lim 形式中.

使用 derivative laws 和定义的 exs

(1) $$\frac{d}{dx}\sqrt{x} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{h}}\right)$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(2) $f(x) = |x|$ is diff'ble everywhere except $x = 0$

(3) $\frac{d}{dx}(e^{x^3}\sin x^2) = 3x^2 e^{x^3}\sin x^2 + 2x e^{x^3}\cos(x^2)$

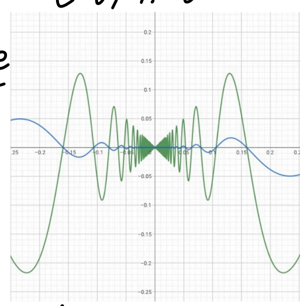
(4) $\lim_{x\to 4}\frac{x^{\frac{3}{2}}-\sqrt{x}-6}{x-4}$?

Let $f(x) = x^{\frac{3}{2}} - \sqrt{x} \Rightarrow f(4) = 6$, $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

So $\lim_{x\to 4}\frac{x^{\frac{3}{2}}-\sqrt{x}-6}{x-4} = f'(4) = \frac{11}{4}$

Fact derivative function 未必 ctn.

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x = 0 \end{cases}, \quad g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x = 0 \end{cases}$$



我们知道：f, g ctn everywhere

For $x \neq 0$,

$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$

$g'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

而 for $x = 0$,

$f'(0) = \lim_{x\to 0}\frac{x\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0}\sin\frac{1}{x}$ DNE

$g'(0) = \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0}x\sin\frac{1}{x} = 0$

但 $\lim_{x\to 0} g'(x) = f(0)$ DNE

因而 derivatives 不一定 ctn.

Def C^n

Given $n \in \mathbb{N}$, the function $f \in C^n$ (n-times continuously differentiable) on open set $U \subseteq \mathbb{R}$ if $\boxed{f^{(n)} \text{ exists}}$ 且 ctn on U