Seq & Series of functions

Def Pointwise convergence

(note: domain都相同)

令 $(f_n: A\to\mathbb{R})_{n\in\mathbb{N}}$ 为一个 seq. of functions.

称 (f_n) converges pointwise on A to the function $f: A\to\mathbb{R}$, 并写作 $\boxed{(f_n)\to f \text{ on } A}$

if: $\boxed{\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x\in A}$

seq. of functions 的 pointwise convergence 即: 对每一点 $x\in A$, $f_n(a)\to f(a)$. 即:

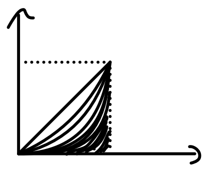
$\forall a\in A, \forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t. $(\forall n\ge N, |f_n(a)-f(a)|<\varepsilon)$

我们可以发现, pointwise convergence 是一个比较弱的条件. 因为 limit function f 并不一定保留这些 terms 的性质.

ex $f_n(x)=x^n$ on $[0,1]$

So $\forall x\in[0,1)$, $\lim_{n\to\infty} f(x) = 0$

而 for $x=1$, $\lim_{n\to\infty} f(x)=1$



因而 $(f_n)\to f(x) = \begin{cases} 0, & x\in[0,1) \\ 1, & x=1 \end{cases}$

我们发现 $\forall n\in\mathbb{N}$, $f_n(x)$ ctn且d'tible, 但 $f(x)$ 却甚至不ctn.

⟹ 因而 ptwise conv. 不 reserve continuity & differentiability

ex2 Write $\mathbb{Q}\cap[0,1] = \{q_n \mid n\in\mathbb{N}\}$ (排序任意,反正ctbl)

且 for each $n\in\mathbb{N}$, let

$$f_n(x) = \begin{cases} 1, & \text{if } x\in\{q_1,\dots,q_n\} \\ 0, & \text{otherwise} \end{cases}$$

$(f_n)\to D|[0,1]$ (Dirichlet's function)

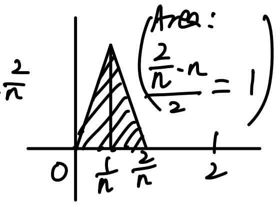
Note: $f_n(x)$ is Rm intble for all $n\in\mathbb{N}$

但 $D|[0,1]$ 并不 Rm intble.

⟹ 因而 ptwise conv. 不 reserve integrability.

ex3 on $x\in[0,2]$:

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0\le x\le \frac{1}{n} \\ 2n-n^2x, & \text{if } \frac{1}{n}<x<\frac{2}{n} \\ 0, & \text{if } \frac{2}{n}\le x \end{cases}$$



⟹ $\forall n\in\mathbb{N}$, $\int_0^2 f_n(x)dx = 1$

因而 $\lim_{n\to\infty}\int_0^2 f_n(x)dx = 1$

但 $(f_n)\to f(x)=0, x\in[0,2]$, $\int_0^2 f = 0$

$\lim_{n\to\infty}\int_0^2 f_n(x)dx \ne \int_0^2 f(x)dx$

⟹ 因而 ptwise conv. 不 reserve limit of integral

ex4 on $x\in\mathbb{R}$

Let $f_n(x) = \frac{\sin(2\pi n x)}{2\pi n}$

So $f_n'(x) = \cos(2\pi n x)$ for each n.

$f_n(x)\to f(x)=0, x\in\mathbb{R}$

$f_n'(0)=1$ for all $n\in\mathbb{N}$, 而 $f'(0)=0$

$\lim_{n\to\infty} f_n'(0)\ne f'(0)$

⟹ 因而 ptwise conv. 不 reserve limit of derivative

因而 pointwise limit can destroy everything: ctnty, d'tblity, intbility. 即便不 destroy 时也不 reserve the value of integral/derivative.

这是因为 ptwise conv. 是一个局部的逐点的性质而不是一个整体的性质: 是在每个点 $a\in A$ 上, $f_n(a)\to f(a)$, 最后的 f 是由每个 $x\in A$ 的 $\lim f_n(x)$ 拼凑出来的.

而我们如果想要 convergence 的定义保留函数整体的性质, 就不能使用这种逐点拼凑的定义, 而是需要一个更强的定义.

Def Uniform convergence

Let $(f_n: A\to\mathbb{R})_{n\in\mathbb{N}}$ be a seq. of functions.

称: (f_n) $\boxed{\text{converges uniformly}}$ on A to $f: A\to\mathbb{R}$

if: $\forall\varepsilon>0, \exists N\in\mathbb{N}$ s.t. $\boxed{(\forall x\in A \text{ 及 } n\ge N, \text{ 都有 } |f_n(x)-f(x)|<\varepsilon)}$

ptwise conv. 和 uni. conv. 的区别:

ptwise: $\forall x\in A, \forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $|f_n(x)-f(x)|<\varepsilon$ whenever $n\ge N$.

uniform: $\forall x\in A, \forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $|f_n(x)-f(x)|<\varepsilon$ whenever $n\ge N$.

ptwise是逐点各自用各自的 ε 来bound

uniform是一个 ε bound所有的 $x\in A$, 这就把A中所有点作为整体联系起来了.

Fact Uni. conv. implies ptwise. conv. (显然)

uni. conv. 是比 ptwise conv. 更强的性质.

Def Uniformly Cauchy

$f_n: A\to\mathbb{R}$ is uniformly Cauchy on A

if $\forall\varepsilon>0, \exists N$ s.t. $|f_n(x)-f_m(x)|<\varepsilon$ for all $x\in A$ and $m,n\ge N$

Thm ① Uni. conv. $\Longleftrightarrow$ uni. Cauchy

$(f_n : A \to \mathbb{R})_{n\in\mathbb{N}}$ conv. unily.

iff it is unily Cauchy on A

Pf (① $\Rightarrow$)

Suppose $(f_n) \to f$ unily on A

Let $\varepsilon > 0$

Fix N s.t. $\forall x \in A$ and $n \geq N$, $|f_n(x) - f(x)|$

$\Rightarrow \forall x \in A$ 及 $m, n \geq N$

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ □

(② $\Leftarrow$)

Suppose (f_n) is uni. Cauchy on A

$\Rightarrow \forall x \in A$, $(f_n(x))$ is Cauchy, so converges

Build $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$

Let $\varepsilon > 0$

Fix N s.t. $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ whenever $x \in A$ and $m, n \geq N$

$\Rightarrow \forall x \in A$ and $n \geq N$, $f_n(x) \in (f_N(x) - \frac{\varepsilon}{2}, f_N(x))$

$\subseteq [f_N(x) - \frac{\varepsilon}{2}, f_N(x) + \frac{\varepsilon}{2}]$

So $|f_n(x) - f(x)| < \varepsilon$ □

Thm ② A uniform limit of ctn. functions is ctn.

if $(f_n : A \to \mathbb{R}) \to f$ unily.

& f_n is ctn. at a for each $n \in \mathbb{N}$

$\Rightarrow f$ is ctn. at a.

i.e. $\lim_{x\to a} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to a} f_n(x)$

$\lim_{x\to a} f(x)$ $\qquad$ $f(a)$

Pf Let $a \in A$.

Assume $f_n(x)$ is ctn. at a for each $n \in \mathbb{N}$

Let $\varepsilon > 0$

By uniform convergence of $(f_n) \to f$ on A

Fix $N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ for all $x \in A$

By ctnity of f_N at a,

Fix $\delta > 0$ s.t. $\forall x \in A$, $|x - a| < \delta$ implies $|f_N(x) - f_N(a)| < \frac{\varepsilon}{3}$

$\Rightarrow \forall x \in A$ st $|x - a| < \delta$, we have

$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

Hence f is ctn at a. □

Thm ③ uniform limit of intble functions is intble

Suppose $(f_n : [a,b] \to \mathbb{R})_{n\in\mathbb{N}} \to f$ unily on $[a,b]$.

$\Rightarrow f$ is intble and $\boxed{\int_a^b \lim_{n\to\infty} f_n = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n}$

Pf Let $\varepsilon > 0$

Since $(f_n) \to f$ uniformly on $[a,b]$

$\Rightarrow (f_n)$ is uniformly Cauchy on $[a,b]$

So we can fix $N \in \mathbb{N}$ s.t.

$|f_m(x) - f_n(x)| < \frac{\varepsilon}{b-a}$ for all $x \in [a,b]$, $m, n \geq N$

$\Rightarrow \left|\int_a^b f_m - \int_a^b f_n\right| < \varepsilon$

Thus $(\int_a^b f_n)$ is Cauchy, so it conv.

Write $\lim_{n\to\infty} \int_a^b f_n = L$

接下来 we show: $\int_a^b f = L$

Let $\varepsilon > 0$

Fix n st $\left|\int_a^b f_n - L\right| < \frac{\varepsilon}{3}$ 且 $|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$ for all $x \in [a,b]$

并 fix partition $P = (x_k)_{k=0}^{m}$ of $[a,b]$ s.t.

$|U(f_n,P) - L(f_n,P)| < \frac{\varepsilon}{3}$

$\Rightarrow |U(f,P) - U(f_n,P)| < \sum_{k=1}^{m} (\sup f[I_k] - \sup f_n[I_k]) \Delta x_k$

$\leq \sum_{k=1}^{m} \frac{\varepsilon}{3(b-a)} \Delta x_k = \frac{\varepsilon}{3}$

$\Rightarrow |U(f,P) - L| \leq |U(f,P) - U(f_n,P)| + |U(f_n,P) - \int_a^b f_n| + |\int_a^b f_n - L|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

Likewise, $|L(f,P) - L| < \varepsilon$

Since ε is arbitrary, $\int_a^b f = L$. □

(这是 by: $\int_a^b f = L$ iff $\forall \varepsilon > 0$, 都 $\exists$ 某个 partition P of $[a,b]$ s.t. $|U(f,P) - L(f,P)| < \varepsilon$)

Thm ④ Uniform limit of a derivative seq. of a conv. C^1 seq. = derivative of limit of original seq.

Suppose $(f_n : [a,b] \to \mathbb{R})_{n\in\mathbb{N}}$ is a seq. of $\boxed{C^1 \text{ functions}}$

且 $\boxed{(f_n) \to f \text{ ptwisely}}$ on $[a,b]$.

且 $\boxed{(f_n') \text{ conv. unily}}$ on $[a,b]$

$\Rightarrow f \in C^1$, 且 $f' = (\lim f_n)' = \lim (f_n')$ on $[a,b]$

Pf Assume the ypothesis.

Write $f = \lim_{n\to\infty} F_n'$

① $F_n \in C^1$ 的用处

Since $\forall n,\ F_n \in C^1 \Rightarrow \forall n,\ F_n'$ is ctn., thus intble

Since (F_n') conv. unily $\Rightarrow \lim(F_n')$ is also ctn. and intble by Thm②③ on $[a,b]$

② (F_n') conv. unily 的用处

$\Rightarrow \forall x \in [a,b]$,

$$\int_a^x f = \int_a^x \lim_{n\to\infty} F_n' = \lim_{n\to\infty} \int_a^x F_n' = \lim_{n\to\infty} (F_n(x) - F_n(a)) \quad \text{by FTC ①}$$
$$= F(x) - F(a)$$

③ F_n conv. 的用处

$\Rightarrow F(x) = F(a) + \int_a^x f$ (for all $x \in [a,b]$)

Since f is ctn on $[a,b]$,

by FTC②: F is diffble on (a,b) and $F' = f$ □

Thm④的条件有点小多，我们其实有一个stronger version：条件更少，结论更多。

Thm④(Stronger) Uniform Convergence Derivative Thm

if $(f_n : [a,b] \to \mathbb{R})_{n\in\mathbb{N}}$, $\forall n,\ f_n \in C^1$，且至少有一点 $x_0 \in [a,b]$ 上 $(f_n(x_0))$ conv.

且 $(f_n') \to g$ unily.

$\Rightarrow (f_n) \to f$ unily.

where $f' = g$ on $[a,b]$

只要原 seq. 在一点 x_0 上conv.即可

Pf Let $\varepsilon > 0$.

Fix N s.t. $|f_m'(x) - f_n'(x)| < \frac{\varepsilon}{2(b-a)}$ for all $x \in [a,b]$

且 $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$ whenever $n,m \geq N$

Let $x \in [a,b]$ be arbitrary.

Let $m,n \geq N$ be arbitrary

$$\Rightarrow \int_{x_0}^x f_n'(t)\,dt = f_n(x) - f_n(x_0)$$

$$\int_{x_0}^x f_m'(t)\,dt = f_m(x) - f_m(x_0)$$

$$\Rightarrow |f_n(x) - f_m(x)| = \left|f_m(x_0) - f_n(x_0) + \int_{x_0}^x (f_m'(t) - f_n'(t))\,dt\right|$$
$$\leq |f_m(x_0) - f_n(x_0)| + \left|\int_{x_0}^x (f_m'(t) - f_n'(t))\,dt\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}|x - x_0| < \varepsilon$$

因而 $f(x)$ 是 uni. Cauchy 的 $\Rightarrow$ uni. conv.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_{x_0}^x f_n'(t)\,dt \quad \text{by FTC2}$$
$$= \int_{x_0}^x \lim_{n\to\infty} f_n'(t)\,dt$$
$$= \int_{x_0}^x g$$

So $f'(x) = g(x)$ by FTC2

Since $x \in [a,b]$ is arbitrary, we have $f' = g$ on $[a,b]$ □

Summarize:

① uniform limit of ctn (f_n) is also ctn.

② Under suitable conditions, $\int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n$

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\frac{d}{dx} f_n(x)\right)$$

由于 differentiation 及 integration 都可看作某种极限运算，uni. conv. 提供的条件实际上是极限运算次序的可交换性