## Subsequence

$(\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

$\xrightarrow{\text{some subseq}} (\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{17}, \dots)$

**Def** Let $s: \mathbb{N} \to \mathbb{R}$ be a seq.
S的一个subseq. 是一个 $s\circ g: \mathbb{N} \to \mathbb{R}$
where $g: \mathbb{N} \to \mathbb{N}$ 是 strictly increasing 的
(即: $m<n \Rightarrow g(m)<g(n)$)

通常,我们使用 $\boxed{(s_{n_k}: k\in\mathbb{N})}$ 来表示一个 subseq. of $\boxed{(s_n: n\in\mathbb{N})}$
这里 $\boxed{k \overset{g}{\mapsto} n_k}$ 就是一个 strictly increasing 的函数

人话:把小index map 到更大(或等于)的 index

比如: $\underbrace{1 \overset{g}{\mapsto} 2}_{(n_1)}$ $\underbrace{2 \overset{g}{\mapsto} 4}_{(n_2)}$ $\underbrace{3 \overset{g}{\mapsto} 6}_{(n_3)}$

ex(1) $s_n = (-1)^n$
即 $(s_n) = (-1, 1, -1, 1, \dots)$
Let $g: \mathbb{N} \to \mathbb{N}$ defined by $g(n) = 2n$ strictly ↑ 也可写: $(s_{2n}: n\in\mathbb{N})$
So $s\circ g = (s_{n_k}: k\in\mathbb{N})$ where $n_k = 2k$ for all $k$
Note: $\lim(s_{2n}) = 1$, $\lim(s_n)$ DNE

ex(2) Let $s_n = \sin(\frac{n\pi}{2})$
$\Longrightarrow s_n = (1, 0, -1, 0, 1, 0, -1, 0, \dots)$
Then $\lim(s_{2n}) = 0$, $\lim(s_{4n+3}) = 1$, $\lim(s_{4n+1}) = -1$
$\lim(s_n)$ DNE

ex(3) every subseq. of $(\frac{1}{n})_{n\in\mathbb{N}}$ conv to 0

ex(4) every tail of $(a_n)$ is a subseq. of $(a_n)$

**Thm ①** ☆☆ conv seq 任意 subseq conv 于同一点.
$(a_n)$ conv $\longrightarrow L$ iff
every subseq. of it conv $\longrightarrow L$

直观: conv. 那么任意 subseq 既然 infinite, 一定也收敛于这一点

Pf $\xleftarrow{\text{back}}$: trivial, 因为 $(a_n)$ 是自己的 subseq.
$\xrightarrow{\text{forward}}$: Let $\varepsilon>0$. Fix $N$ s.t. $\forall n\geq N, |a_n - L| < \varepsilon$
Then $|s_{n_k} - L| < \varepsilon$ whenever $k \geq N$
因为 $k \overset{g}{\mapsto} n_k$ 是 strictly ↑ 的 ☆

**Remark**: still true when $(s_n) \to \infty$

**Lemma**: every seq. has a monotone subseq.
☆☆

声明: increasing (↑) $\Leftrightarrow$ weakly increasing (↑)
$\Leftrightarrow$ non-decreasing
(↓)
means: $(m>n \longrightarrow s_m \geq s_n)$
strictly increasing (↑)
means: $(m>n \longrightarrow s_m > s_n)$

Pf Let $(s_n)$ be a seq.
(def) $\boxed{\text{称 term } s_n \text{ 为 dominant 的, if } s_n > s_m \text{ for all } m>n}$

Case ①: 如果 $(s_n)$ 有 infly many dominant terms
$\Longrightarrow (s_k: s_k \text{ is dominant})$ 为 $\boxed{(s_n)\text{的一个monotone (decreasing) subseq.}}$ ☆

Case ②: 如果 $(s_n)$ 只有 finitely many dominant terms
$\Longrightarrow$ fix $n_1 \in \mathbb{N}$ st. $\forall n \geq n_1$, $s_n$ is not dominant
即 $n_1$ 起, 任意项 $s_n$ 后面总有更大的项 ☆
于是可继续 inductively 取 $n_2 < n_3 < n_4 < \dots$, 因为 ☆
每项都不 dominant, 于是 $\boxed{(s_{n_k})\text{为}(s_n)\text{的一个 increasing subseq.}}$

**Corollary** ☆☆☆☆ Bolzano–Weierstrass Thm
Every bounded seq. in $\mathbb{R}$ has a $\boxed{\text{convergent subseq.}}$

Pf. (simple.) 因为 $(s_n)$ bounded
因而 every subseq. bounded $+$ $\Rightarrow$ conv.
又 by Lemma, $(s_n)$ 有 monotone subseq.

ex $(\sin k)_{k\in\mathbb{N}}$ 是 bounded between $[1, -1]$ 的
于是可以找到一个 $(\sin k)_{k\in\mathbb{N}}$ 的 convergent subseq.
(神奇,但是想想图很合理)

**Def** Subsequential limit
$(s_n)$ in $\mathbb{R}$ 的任意一个 subseq. 的 limit 都被称为 $(s_n)$ 的一个 subsequential limit.

**Thm** ☆ 对于任意 bounded seq. $(s_n)$ in $\mathbb{R}$,
令 $S$ 为 $\boxed{\text{所有 subseq. limits}}$ 的集合,
则 $\Longrightarrow$ (i) $S \neq \emptyset$  $S$ 一定非空, by BW: $(s_n)$ 一定有个 conv. subseq.
显然 (ii) if $\lim(s_n) = L \in \mathbb{R}$, 则 $S = \{L\}$ directly follows from Thm ①.
(iii) $\limsup(s_n) = \max(S)$, $\liminf(s_n) = \min(S)$

Pf of (iii): limsup 是所有 subseq. lim. 中最大的

(dually) liminf 是所有 subseq. lim. 中最小的:

Assume $\limsup(s_n) = l$

For each $k \in \mathbb{N}$, choose $m_k$ s.t. $\left|\sup\{s_i \mid i > m_k\} - l\right| < \frac{1}{k}$

Then: $|s_{n_k} - l| < \frac{2}{k}$, $\Rightarrow (s_{n_k}) \to l$

因而 $l \in S$

Suppose $M > l$

Consider $\varepsilon = \frac{M-l}{2}$, can find $N \in \mathbb{N}$ s.t. $\sup\{s_k \mid k \geq N\} < l + \varepsilon$

$\Rightarrow \forall n \geq N$, $s_n < l + \varepsilon = M - \varepsilon$

$\Rightarrow M \notin S$

因而 $l = \max(S)$, Dually $\liminf(s_n) = \min(S)$

Remark: The previous thm holds for even unbounded seq., 只要允许 $\pm\infty$ as subseq. limit.

ex1 $s_n = n^{(-1)^n} = (1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \dots)$

$\Rightarrow S = \{0, +\infty\}$: $\limsup(s_n) = +\infty$ $\liminf(s_n) = 0$

ex2 let $(s_n)$ be a seq. in $[0,1)$ 且

$\inf\{s_n\} = 0$, $\sup\{s_n\} = 1$

则 $\{1\} \subseteq S \subseteq [0,1]$

1 一定在 S 中, 因为 $\sup\{s_n\} = 1$ 但 $(s_n)$ 是在 $[0,1)$ 中的

因而 $s_n$ 一定是 get close to 1 infinitely times 的

(如果停下了, 则没有 $\sup(s_n) = 1$)

即 $\limsup(s_n) = 1$, 因而 $\max(S) = 1$

但是由于 $[0,1)$ 中 0 是 closed 的 可以有 $n \in \mathbb{N}$ 使 $s_n = 0$,

且只有 finite 次, 不影响 $\inf\{s_n\} = 0$.

ex3 if $a: \mathbb{N} \to \mathbb{Q}$ 为 surj 的, then $S = \mathbb{R} \cup \{\pm\infty\}$

Pf $\forall r \in \mathbb{R}$, $\exists$ Cauchy seq. $(s_n)$, $(s_n) \to r$

Since $a: \mathbb{N} \to \mathbb{Q}$ is surj.

$\forall n \in \mathbb{N}$, $s_n = a_m$ for some $m \in \mathbb{N}$

$(a_n) = (\dots, \dots, \dots, \dots, \dots)$

$(s_n) = (\dots, \dots, \dots, \dots, \dots)$

于是取 $(a_n)$ 的 subseq. $(a_m: a_m = s_n \text{ for some } n \in \mathbb{N})$

而又可以对 $(a_m)$ 取一个 monotone subseq $(a_k)$

Since $(s_n)$ is bounded $\Rightarrow (a_m)$ is bounded $\Rightarrow (a_k)$ is bounded

因而 $(a_k)$ monotone 且 bounded $\Rightarrow$ conv. to $r$