Homework

Math 451: Spring 2024

Homework 1: Due Tuesday, May 14

(1) For each statement about sets given below, either *prove* the statement if it is true for all sets, or else give a *counterexample* using specific sets if it is false.

(a) $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

(b) $(A \cup B) \setminus C \supseteq A \cup (B \setminus C)$.

(c) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

(d) $A \subseteq B$ if and only if $A \cap B = A$.

(2) For each $n \in \mathbb{N}$, let $A_n = \{nk \,:\, k \in \mathbb{N}\}$.

(a) What is $A_2 \cap A_3$?

(b) Determine (i.e., give simple descriptions of) the sets $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$.

(3) (a) Guess a formula for $1 + 3 + \cdots + (2n-1)$ by evaluating the sum for $n = 1$, 2, 3, and 4. (For $n = 1$, the sum is simply 1).

(b) Prove that your formula is correct using mathematical induction.

(4) Determine for which integers the inequality $2^n > n^2$ is true, and prove your claim by induction.

(5) For each of the subsets of $\mathbb{R}$ given in (a) – (x) below, state (i) whether or not the set is bounded above; (ii) whether or not it is bounded below; (iii) what the supremum is (if it exists); and what the infimum is (if it exists). You may write all your answers on one line, with no justification needed, as in the answer for (a) given below:

"Bounded below but not above; inf = 1."

(a) $\mathbb{N}$

(b) $[0, 1)$

(c) $\{2, 7\}$

(d) $\{\pi, e\}$

(e) $\{\frac{1}{n} \,:\, n \in \mathbb{N}\}$

(f) $\{0\}$

(g) $[0, 1] \cup [2, 3]$

(h) $\bigcup_{n=1}^{\infty} [2n, 2n+1]$

(i) $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1 + \frac{1}{n}]$

(j) $\{1 - \frac{1}{3^n} \,:\, n \in \mathbb{N}\}$

(k) $\{n + \frac{(-1)^n}{n} \,:\, n \in \mathbb{N}\}$

(l) $\{r \in \mathbb{Q} \,:\, r < 2\}$

(m) $\{r \in \mathbb{Q} \,:\, r^2 < 4\}$

(n) $\{r \in \mathbb{Q} \,:\, r^2 < 2\}$

(o) $\{x \in \mathbb{R} \,:\, x < 0\}$

(p) $\{1, \frac{\pi}{3}, \pi^2, 10\}$

(q) $\{0, 1, 2, 4, 8, 16\}$

(r) $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

(s) $\{\frac{1}{n} \,:\, n \in \mathbb{N} \text{ and } n \text{ is prime}\}$

(t) $\{x \in \mathbb{R} \,:\, x^3 < 8\}$

(u) $\{x^2 \,:\, x \in \mathbb{R}\}$

(v) $\{\cos(\frac{n\pi}{3}) \,:\, n \in \mathbb{N}\}$

(w) $\bigcup_{n=1}^{\infty}\{\frac{k}{n} : k \in \mathbb{N}\}$ (x) $\bigcap_{n=1}^{\infty}\{\frac{k}{n} : k \in \mathbb{N}\}$

(6) The complex numbers form a *field*; that is, the algebraic structure $(\mathbb{C}, +, \cdot, 0, 1)$ satisfies our Axioms 1–9. In fact, $\mathbb{C}$ also satisfies a version of the Completeness Axiom, so that $\mathbb{C}$ is a *complete* field. Prove, however, that it is impossible to define a linear order relation $<$ on $\mathbb{C}$ that makes $\mathbb{C}$ an *ordered* field; i.e., it is impossible to define a linear order relation $<$ on $\mathbb{C}$ that satisfies Axioms (13) and (14). [HINT: argue by contradiction. The *only* things you are allowed to use without proof are the ordered field axioms and the results in the handout "Elementary Properties of Real Numbers," which hold in any ordered field.]

(7) (a) Let $a, b \in \mathbb{R}$. Show that if $a \leq c$ for every $c > b$, then $a \leq b$.
(b) Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$, and suppose L is an upper bound of A. Show that $L = \sup A$ if and only if for every $\epsilon > 0$ there is $a \in A$ such that $L - \epsilon < a \leq L$.

(8) Let S and T be nonempty bounded subsets of $\mathbb{R}$.
(a) Prove that $\inf S \leq \sup S$.
(b) Supposing that $S \subseteq T$, put the four numbers $\sup S$, $\inf S$, $\sup T$, $\inf T$ in order (with respect to $\leq$), and prove your claims.
(c) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

(9) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B = \{a+b : a \in A \text{ and } b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$.

(10) Prove that $\mathbb{R} \setminus \mathbb{Q}$ is *dense* in $\mathbb{R}$ in the sense that for every pair of real numbers a and b, if $a < b$ then there exists an irrational number r such that $a < r < b$.

A set $A \subseteq \mathbb{R}$ is *discrete* if for every $a \in A$ there is $\epsilon > 0$ such that $V_\epsilon(a) \cap A = \{a\}$, where $V_\epsilon(a) = (a - \epsilon,\, a + \epsilon)$ is the open interval of radius ϵ centered at a.

(11) (a) Prove that every finite subset of $\mathbb{R}$ is discrete.
(b) Either prove the following if it is true, or else give a counterexample if it is false: if $A \subseteq \mathbb{R}$ is discrete, then there is $\epsilon > 0$ such that $|a - b| \geq \epsilon$ for every pair of distinct elements $a, b \in A$.

(12) OPTIONAL CHALLENGE PROBLEM.[1] For $A, B \subseteq \mathbb{R}$, let $AB = \{ab : a \in A \text{ and } b \in B\}$. Find a simple expression for $\sup(AB)$ in the case where A and B are nonempty and bounded, and prove your result.

[1]These may come up every now and then; you don't have to do them, and they will not be graded.