Fourier Series
$\left(\sin_n: \sin\frac{n\pi x}{L},\ \cos_n: \cos\frac{n\pi x}{L},\ e_n: e^{\frac{in\pi x}{L}}\right)$

$f(x)=A_0+\sum_{n=1}^{\infty}(A_n\cos_n+B_n\sin_n)$, $x\in(-L,L)$

$A_0=\frac{1}{2L}\int_{-L}^{L}f$, $A_n=\frac{1}{L}\int_{-L}^{L}f\cos_n$, $B=\frac{1}{L}\int_{-L}^{L}f\sin_n$

Fourier Series Complex form
$f(x)=\sum_{n\in\mathbb{Z}}\alpha_n e^{\frac{in\pi x}{L}}$, $x\in(-L,L)$

$\alpha_n=\frac{1}{2L}\int_{-L}^{L}fe^{\frac{in\pi x}{L}}$

Fourier cosine
$f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos_n$, $x\in(0,L)$

$A_0=\frac{1}{L}\int_0^L f$, $A_n=\frac{2}{L}\int_0^L f\cos_n$

(even extension, 形式不变)

$f_E(x)=A_0+\sum_{n=1}^{\infty}A_n\cos_n$, $x\in(-L,L)$

$A_0=\frac{1}{L}\int_0^L f$, $A_n=\frac{2}{L}\int_0^L f\cos_n$

Fourier sine
$f(x)=\sum_{n=1}^{\infty}B_n\sin_n$, $x\in(0,L)$

$B_n=\frac{2}{L}\int_0^L f\sin_n$

(odd extension, 形式不变)

$f_O(x)=\sum_{n=1}^{\infty}B_n\sin_n$, $x\in(-L,L)$;

$B_n=\frac{2}{L}\int_0^L f\sin_n$

Orthogonality
$\forall m,n$,

$\int_{-L}^{L}\cos_n\cos_m=\begin{cases}2L, & m=n=0\\ L\delta_{mn}, & 否\end{cases}$ $\quad\int_0^L\cos_n\cos_m=\begin{cases}L, & m=n=0\\ \frac{L}{2}\delta_{mn}, & 否\end{cases}$

$\int_{-L}^{L}\sin_n\sin_m=\begin{cases}0, & m=n=0\\ L\delta_{mn}, & 否\end{cases}$ $\quad\int_0^L\sin_n\sin_m=\begin{cases}0, & m=n=0\\ \frac{L}{2}\delta_{mn}, & 否\end{cases}$

$\int_{-L}^{L}\sin_n\cos_m=0$

$\int_{-L}^{L}e_n\overline{e_m}=2L\delta_{mn}$

ex compute the Fourier cosine series of $f(x)=x$, $x\in[0,1]$

① even extension

$f_E(x)=|x|$, $x\in[-1,1]$

Fourier of f_E:

$B_n=0$

$A_0=\frac{1}{2L}\int_{-L}^{L}|x|\,dx=\frac{1}{L}\int_0^L x\,dx=\frac{L}{2}$

$A_n=\frac{1}{L}\int_{-L}^{L}|x|\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L \underbrace{x\cos\frac{n\pi x}{L}}\,dx$ $\quad\left(\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right)'$

$=\frac{2}{L}\left[x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L-\frac{2}{L}\frac{L}{n\pi}\int_0^L\sin\frac{n\pi x}{L}dx$

$=-\frac{2}{n\pi}\int_0^L\sin\frac{n\pi x}{L}dx=-\frac{2}{n\pi}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L$

$=-\frac{2L}{n^2\pi^2}(1-\cos n\pi)=-\frac{2L}{n^2\pi^2}(1-(-1)^n)$

$\Rightarrow f(x)=\frac{L}{2}-\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{L}$

② $A_0=\frac{1}{L}\int_0^L x\,dx=\frac{L}{2}$

$A_n=\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx$

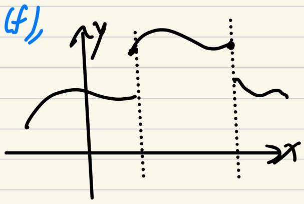
$=-\frac{2L}{n^2\pi^2}(1-(-1)^n)$

Def Piecewise continuous function

For at most finitely many $c\in dom(f)$,

$\lim_{x\to c^-}f(x)\neq\lim_{x\to c^+}f(x)$

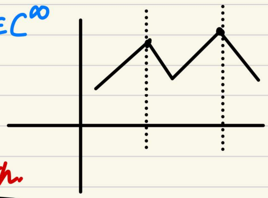
then call f a pw. ctn. func.



Def Piecewise smooth function

即可以把 $dom(f)$ divide 成 finitely many intervals $\{I_n\}$

s.t. $\forall n$, $f[I_n]\in C^\infty$



Rmk pw ctn 即只有finite点上不 ctn

pw smooth 即只有 finite点上不 smooth.

Def convergence

Given Fourier Series $f(x)=A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}+B_n\sin\frac{n\pi x}{L}\right)$

for $\boxed{\text{Partial sum}}$ $f_N(x)=A_0+\sum_{n=1}^{N}\left(A_n\cos\frac{n\pi x}{L}+B_n\sin\frac{n\pi x}{L}\right)$

: if $\lim_{N\to\infty}f_N(x)=f(x)$, say the Fourier series is $\boxed{\text{convergent at } x}$

: if $\lim_{N\to\infty}\max_{x\in[-L,L]}|f_N(x)-f(x)|=0$, say the FS is $\boxed{\text{uniformly convergent}}$ $\boxed{\text{on } [-L,L]}$

Thm Convergence thm

if $f(x)$, $x\in(-L,L)$ pw smooth

令 $F(x)$ 为其 Fourier series

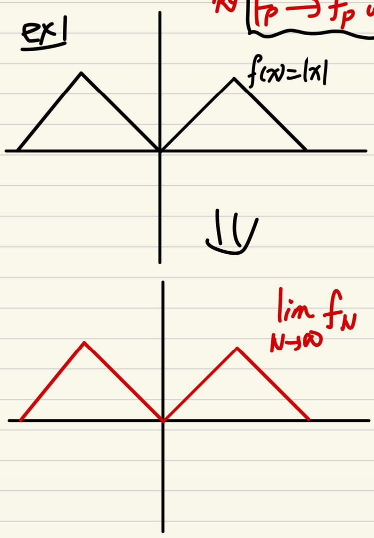
用 f_p, F_p 表示把这两个函数 extend 为 $2L$-periodic function on $\mathbb{R}$

$\Rightarrow \forall x_0\in\mathbb{R}$, $\boxed{F_p(x_0)=\frac{1}{2}\left(\overline{f_p}(x_0)+\underline{f_p}(x_0)\right)}$ (即若 f_p 在 x_0 处 ctn 则 $F_p(x_0)\to f_p$ 否则 F_p 在其上下极限中点)

特别地, if $f(-L)=f(L)$ 且 f ctn on $(-L,L)$

则 $\boxed{F_p\to f_p \text{ uniformly.}}$

ex1



ex2