hw3

Question 1. Use the complex form to find the Fourier series of $f(x) = e^x, -L < x < L$.
Solution. $e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{L+in\pi}{L^2+n^2\pi^2} (\sinh L) \exp\left(\frac{in\pi}{L} x\right)$

Fourier series: $F(x) = \sum_{n\in\mathbb{Z}} \alpha_n e^{\frac{in\pi x}{L}}, \ -L<x<L$

where
$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} e^x e^{-\frac{in\pi x}{L}}\,dx$$
$$= \frac{1}{2L}\int_{-L}^{L} e^{(1-\frac{in\pi}{L})x}\,dx$$
$$= \frac{1}{2L}\frac{1}{1-\frac{in\pi}{L}}\int_{-L}^{L} e^{(1-\frac{in\pi}{L})x}\,d\left((1-\tfrac{in\pi}{L})x\right)$$
$$= \frac{1}{2L-2in\pi}\left(e^{L-in\pi} - e^{-L+in\pi}\right) \quad (\text{note: } e^{in\pi} = (-1)^n)$$
$$= \frac{(-1)^n}{2L-2in\pi}(e^L - e^{-L}) = \frac{(-1)^n}{2L-2in\pi} 2\sinh(L)$$
$$= \frac{(-1)^n}{L-in\pi}\sinh(L)$$
$$= (-1)^n \frac{L+in\pi}{L^2+n^2\pi^2}\sinh(L)$$

Therefore $f_c(x) = \sum_{n\in\mathbb{Z}} (-1)^n \frac{L+in\pi}{L^2+n^2\pi^2}\sinh(L)\exp\left(\frac{in\pi}{L}x\right)$

Question 2. Derive the following formula

1) Let $0 < r < 1$, $f(x) = 1/\left(1 - re^{ix}\right), -\pi < x < \pi$. Find the complex Fourier series of f
2) Let $0 \le r < 1$. Use the Fourier series in 1) to derive

$$\frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = 1 + \sum_{n=1}^{\infty} r^n \cos nx,$$
$$\frac{r\sin x}{1 + r^2 - 2r\cos x} = \sum_{n=1}^{\infty} r^n \sin nx.$$

Solution. 1) Expand f as a power series in r. The answer is $f(x) = \sum_{n=0}^{\infty} r^n e^{inx}$. 2) Use Euler's formula and consider the real part and imaginary part.

(1) note that $f(x) = \frac{1}{1-re^{ix}} = \sum_{k=0}^{\infty}(re^{ix})^n = \sum_{k=0}^{\infty} r^n e^{inx}$ by geometric series

(Here we can apply geometric series since $|re^{ix}| = |r||\cos x + i\sin x| < 1$)

the Fourier series $F(x) = \sum_{n\in\mathbb{Z}} d_n e^{\frac{in\pi x}{\pi}} = \sum_{n\in\mathbb{Z}} d_n e^{inx}$

where $d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=0}^{\infty}(re^{ix})^k e^{-inx}\,dx$

By the uniform convergence of integrand,

$$d_n = \frac{1}{2\pi}\sum_{k=0}^{\infty} \underbrace{\int_{-\pi}^{\pi} r^k e^{i(k-n)x}\,dx}_{} = 2\pi\delta_{kn} r^k \text{ by orthogonality}$$

Since $n \ge 0$, $d_n = \frac{2\pi}{2\pi} r^n = r^n$ for $n \ge 0$ and $d_n = 0$ for $n < 0$

$$\Rightarrow F(x) = \sum_{n\in\mathbb{Z}} d_n e^{inx} = \sum_{n=0}^{\infty} r^n e^{inx}$$

(2) Notice that for f in (1);

$$f(x) = \frac{1}{1-re^{ix}} = \frac{1}{(1-r\cos x) + ir\sin x} = \frac{1 - r\cos x + ir\sin x}{(1-r\cos x)^2 + r^2\sin^2 x}$$
$$= \frac{1 - r\cos x}{1 - 2r\cos x + r^2} + i\frac{r\sin x}{1 - 2r\cos x + r^2}$$

$$\text{Also } f(x) = \sum_{n=0}^{\infty} r^n e^{inx} = \sum_{n=0}^{\infty} r^n \cos nx + i r^n \sin nx$$

$$= \sum_{n=0}^{\infty} r^n \cos nx + i \sum_{n=0}^{\infty} r^n \sin nx$$

$$\Longrightarrow \frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = \sum_{n=0}^{\infty} r^n \cos nx = 1 + \sum_{n=1}^{\infty} r^n \cos nx$$

$$\frac{r\sin x}{1 + r^2 - 2r\cos x} = \sum_{n=0}^{\infty} r^n \sin nx = \sum_{n=1}^{\infty} r^n \sin nx$$

Question 3. Find the mean square error for the Fourier series of the function $f(x) = 1$ for $0 < x < \pi$, $f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$. Then, show that $\sigma_N^2 = O(N^{-1})$ as $N \to \infty$.

Solution. $\sigma_N^2 = \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{[(-1)^n - 1]^2}{n^2}$. To show $O(N^{-1})$, define $n = 2m - 1$ and replace the summation with $\sum_{m=(N+2)/2}^{\infty}$ or $\sum_{m=(N+3)/2}^{\infty}$ depending on if N is even or odd. Then use integrals to estimate the sum.

Sol Note that this is an odd function on $(-\pi, \pi)$,

so $A_0 = A_n = 0, \ \forall n$

$$B_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin nx \, dx + \int_0^{\pi} \sin nx \, dx\right)$$

$$= \frac{1}{n\pi}\left(\cos nx \Big|_{-\pi}^{0} - \cos nx \Big|_0^{\pi}\right)$$

$$= \frac{1}{n\pi}\left(1 - (-1)^n - (-1)^n + 1\right) = \frac{2}{n\pi}(1 - (-1)^n)$$

$$\text{So for } N \in \mathbb{N}, \ \sigma_N^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} \left(\frac{2}{n\pi}(1 - (-1)^n)\right)^2$$

$$= \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{(1 - (-1)^n)^2}{n^2}$$

$$= \frac{2}{\pi^2}\left(\sum_{m=\frac{N+2}{2}}^{\infty} \frac{(1 - \underbrace{(-1)^{2m-1}}_{\text{always} = (-1)^{N+1}})^2}{(2m-1)^2} + \sum_{m=\frac{N+3}{2}}^{\infty} \frac{(1 - \underbrace{(-1)^{2m-1}}_{\text{always} = (-1)^{N+2}})^2}{(2m-1)}\right)$$

$$= \frac{2}{\pi^2} \sum_{m=\frac{N+2}{2}}^{\infty} \frac{2}{(2m-1)^2} \text{ if } N \text{ is even}$$

$$\text{or } \frac{2}{\pi^2} \sum_{m=\frac{N+3}{2}}^{\infty} \frac{2}{(2m-1)^2} \text{ if } N \text{ is odd}$$

In whatever circumstance, we have

$$\sigma_N^2 \le \frac{2}{\pi^2}\int_{\frac{N}{2}}^{\infty} \frac{2}{(2m-1)^2}\,dm = \frac{2}{\pi^2}\int_{N-1}^{\infty} \frac{1}{(2m-1)^2}\,d(2m-1)$$

$$= \frac{2}{\pi^2}\left[-\frac{1}{x}\right]_{N-1}^{\infty} = \frac{2}{\pi^2}\frac{1}{N+1} \in O(N^{-1})$$

Question 4. Find the mean square error for the Fourier series of $f(x) = x^2, -\pi \le x \le \pi$. Then, show that $\sigma_N^2 = O(N^{-3})$ as $N \to \infty$.
Solution. $\sigma_N^2 = 8\sum_{n=N+1}^{\infty} \frac{1}{n^4}$.

Sol Since f is an even function, $B_n = 0$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{n\pi}\int_0^{\pi} x^2\, d\sin(nx)$$

$$= \left[\frac{2}{n\pi}x^2\sin(nx)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi}\sin(nx)\,x\,dx$$

$$= \frac{4}{n^2\pi}\int_0^{\pi} x\, d(\cos nx)$$

$$= \left[\frac{4}{n^2\pi}x\cos nx\right]_0^{\pi} - \underbrace{\frac{4}{n^2\pi}\int_0^{\pi}\cos(nx)}_{0}\,dx$$

$$= \frac{4}{n^2}(-1)^n$$

So $\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\frac{4}{n^4} = \frac{1}{8}\sum_{n=N+1}^{\infty}\frac{1}{n^4} \le \frac{1}{8}\int_N^{\infty}\frac{1}{n^4}\,dn$ by integral test

$$= \frac{1}{8}\left[\frac{n^{-3}}{-3}\right]_N^{\infty}$$

$$= \frac{1}{24N^3} \in O(N^{-3})$$

Question 5. Write out Parseval's theorem for the Fourier series of

1) $f(x) = 1$ for $0 < x < \pi$, $f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$,
2) $f(x) = x^2$, $-\pi \le x \le \pi$.

Solution. 1) $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$, and 2) $\pi^4/90 = 1 + \frac{1}{16} + \frac{1}{81} + \cdots$.

Sol. (1) Since f is odd $\Rightarrow A_0, A_n = 0 \ (\forall n)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2$$

$$\Leftrightarrow \quad 1 = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\right)^2$$

$$\Leftrightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

(2) f is even $\Rightarrow B_n = 0, \forall n$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\cdot\frac{2}{3}\pi^2 = \frac{1}{3}\pi^2$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos n\pi = \frac{4}{n^2}(-1)^n \text{ by the last problem}$$

$$\Rightarrow A_n^2 = \frac{16}{n^4}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |x^2|^2\,dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

$$\Leftrightarrow \frac{1}{2\pi}\frac{2}{5}\pi^5 = \frac{1}{9}\pi^4 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Leftrightarrow \left(\frac{1}{5} - \frac{1}{9}\right)\pi^4 = \sum_{n=1}^{\infty}\frac{8}{n^4}$$

$$\Leftrightarrow \frac{4}{45}\pi^4 = \sum_{n=1}^{\infty}\frac{8}{n^4}$$

$$\Leftrightarrow \frac{1}{90}\pi^4 = \sum_{n=1}^{\infty}\frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \dots$$

Question 6. Prove the following Parseval's theorem for complex, cosine and sine Fourier coefficients.

1) $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}$ implies that

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 .$$

2) $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$ implies that

$$\frac{1}{L}\int_{0}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 .$$

3) $f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ implies that

$$\frac{1}{L}\int_{0}^{L} f(x)^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2 .$$

Solution. Try to mimic the proof of Parseval's theorem for Fourier series.

(1) Pf $\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \frac{1}{2L}\int_{-L}^{L} f(x)\overline{f(x)}dx$

$$= \frac{1}{2L}\int_{-L}^{L} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \alpha_n \bar{\alpha}_m e^{\frac{i(n-m)\pi x}{L}} dx$$

As we have proved: $\int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}} dx = 2L\delta_{nm}$, $\neq 0$ iff $n=m$

$$\Rightarrow \frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \qquad \square$$

(2) $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$

$$|f(x)|^2 = A_0^2 + 2A_0\sum_{n=1}^{\infty} A_n \cos_n + \left(\sum_{n=1}^{\infty} A_n \cos_n\right)^2$$

$$\Rightarrow \frac{1}{L}\int_0^L |f(x)|^2 dx = \underbrace{\frac{1}{L}\int_0^L A_0^2 dx}_{=A_0^2} + \left(\sum_{n=1}^{\infty} \frac{2}{L} A_0 \underbrace{\int_0^L \cos\frac{n\pi x}{L} dx}_{=0}\right)$$

$$+ \underbrace{\frac{1}{L}\int_0^L \left(\sum_{n=1}^{\infty} A_n \cos_n\right)\left(\sum_{n=1}^{\infty} A_m \cos_m\right) dx}_{=\frac{1}{2}\delta_{mn},\ \neq 0 \text{ iff } n=m}$$

$$\Rightarrow \frac{1}{L}\int_0^L |f(x)|^2 dx = A_0^2 + \frac{1}{L}\sum_{n=1}^{\infty} A_n^2 \underbrace{\int_0^L \cos_n^2 dx}_{} = \frac{1}{2}L$$

$$\Rightarrow \frac{1}{L}\int_0^L |f(x)|^2 dx = A_0^2 + \sum_{n=1}^{\infty} A_n^2$$

$\square$

$$(3) \quad \frac{1}{L}\int_0^L |f(x)^2| dx = \frac{1}{L}\int_0^L \left(\sum_{n=1}^{\infty} B_n \sin_n\right)^2 dx$$

$$= \frac{1}{L}\underbrace{\int_0^L \left(\sum_{n=1}^{\infty} B_n \sin_n\right)\left(\sum_{m=1}^{\infty} B_m \sin_m\right) dx}_{=\frac{L}{2}\delta_{nm},\ \neq 0 \text{ iff } n=m}$$

$$= \frac{1}{L}\sum_{n=1}^{\infty} B_n^2 \int_0^L \sin_n \sin_m dx = \frac{1}{2}L$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n^2$$

$\square$

Question 7. Let us solve the heat equation in the slab $0 < z < L$:

$$\begin{cases} u_t = K u_{zz} & 0 < z < L, t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0, \\ u(z,0) = 1 & 0 < z < L, \end{cases}$$

where $K > 0$ is the thermal conductivity.

1) Find the separated solution depending on λ.
2) Find the general solution which satisfies the boundary conditions.
3) Find the particular solution which satisfies the initial and boundary conditions.

Solution. 1) For $\lambda > 0, u = (A\cos\sqrt{\lambda}z + B\sin\sqrt{\lambda}z)e^{-\lambda Kt}$, for $\lambda = 0$, $u = (Az + B)$, for $\lambda < 0, u = \left(Ae^{\sqrt{-\lambda}z} + Be^{-\sqrt{-\lambda}z}\right)e^{-\lambda Kt}$. 2) $u = \sum_{n=1}^{\infty} A_n \sin(n\pi z/L)e^{-(n\pi/L)^2 Kt}$. 3) $u = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$.

(1) Suppose $u(z,t) = Z(z)\,T(t)$ for the separated solutions

$$\Rightarrow ZT' = KTZ''$$

$$\frac{T'}{KT} = \frac{Z''}{Z} = -\lambda$$

$$\Rightarrow T' = -K\lambda T, \quad Z'' = -\lambda Z$$

$$\Rightarrow T(t) = Ce^{-K\lambda t} \qquad \Rightarrow Z = \begin{cases} A_1\cos\sqrt{\lambda}z + A_2\sin\sqrt{\lambda}z, & \lambda > 0 \\ A_1 z + A_2, & \lambda = 0 \\ A_1 e^{\sqrt{-\lambda}z} + A_2 e^{-\sqrt{-\lambda}z}, & \lambda < 0 \end{cases}$$

Thus the general separated solutions are

$$u(x,t)=\begin{cases} e^{-k\lambda t}(A_1\cos(\sqrt{\lambda}x)+A_2\sin(\sqrt{\lambda}x)), & \lambda>0 \\ A_1x+A_2, & \lambda=0 \\ e^{-k\lambda t}(A_1e^{\sqrt{-\lambda}x}+A_2e^{-\sqrt{-\lambda}x}), & \lambda<0 \end{cases}$$

(2) For $\lambda>0$: $A_1=0,\ A_2\sin(\sqrt{\lambda}L)e^{-k\lambda t}=0$

$$\Rightarrow \sin(\sqrt{\lambda}L)=0$$

$$\Rightarrow \sqrt{\lambda}L=n\pi \Rightarrow \lambda=\left(\frac{n\pi}{L}\right)^2$$

So $u=\sum_{n=1}^{\infty}A_n\sin_n e^{-(\frac{n\pi}{L})^2kt}$ in this case

for $\lambda=0 \Rightarrow A_1=A_2=0 \Rightarrow u=0\ \forall x,t.$

for $\lambda<0$, we have $\forall t,\ A_1e^{-k\lambda t}=0\Rightarrow A_1=0$

$\forall t,\ A_2\sinh\sqrt{-\lambda}L\cdot e^{-k\lambda t}=0\Rightarrow A_2=0$

by boundary conditions

$\Rightarrow u=0\ \forall x,t$ in this case

Therefore overall, $u=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}e^{-(\frac{n\pi}{L})^2kt}$, $0<x<L,\ t>0$

is the general solution satisfying the boundary conditions

(3) $u(x,0)=\sum_{n=1}^{\infty}A_n\sin_n=1$

Since there is no A_0, we calculate the Fourier sine coeffs to get the A_n, $n\in\mathbb{N}$ s.t $\underline{\sum_{n=1}^{\infty}A_n\sin_n=1}$

$$\Rightarrow A_n=\frac{2}{L}\int_0^L\sin\frac{n\pi x}{L}dx$$

$$=\left[-\frac{2}{L}\cos\frac{n\pi x}{L}\right]_0^L\cdot\frac{L}{n\pi}=\frac{-2}{n\pi}((-1)^n-1)=\frac{2}{n\pi}(1-(-1)^n)$$

Therefore $u(x,t)=\frac{2}{n\pi}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\frac{n\pi x}{L}e^{-(\frac{n\pi}{L})^2kt}$, $0<x<L,\ t>0$

is the solution satisfying boundary and initial conditions