hw 1

Question 1. Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.

1) $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$
2) $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$
3) $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$
4) $u_{xx} + xu_{yy} = 0$

Solution: 1) hyperbolic, 2) elliptic, 3) parabolic, 4) elliptic if $x > 0$ and hyperbolic if $x < 0$.

1) $\Delta = b^2 - 4ac = 9 - 4 = 5 > 0 \Rightarrow$ hyperbolic

2) $\Delta = b^2 - 4ac = 9 - 32 = -23 < 0 \Rightarrow$ elliptic

3) $\Delta = b^2 - 4ac = 4 - 4 = 0 \Rightarrow$ parabolic

4) $\Delta = b^2 - 4ac = -4x \Rightarrow$ elliptic for $x > 0$
and hyperbolic for $x < 0$
and parabolic for $x = 0$

Question 2. Prove the following claims.

1) Prove the formulas $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
2) Use Euler's formula to prove $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$ similarly.
3) Use Euler's formula to prove that all the real functions in $C_+e^{ix} + C_-e^{-ix}$ is of the form $A\cos x + B \sin x$.

Solution: 1) $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} = \sinh(x+y)$. $\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^2}{2}\frac{e^y - e^{-y}}{2} = \cosh(x+y)$.

2) $\sin x \cos y + \cos x \sin y = \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} - e^{-iy}}{2i} = \sin(x+y)$. $\cos x \cos y + \sin x \sin y = \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2i} = \cos(x+y)$.

Pf 1) $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} \cancel{+ e^{x-y}} \cancel{- e^{y-x}} - e^{-y-x} + e^{x+y} \cancel{- e^{x-y}} \cancel{+ e^{y-x}} - e^{-y-x}}{4}$

$= \frac{2e^{x+y} - 2e^{-y-x}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$

$$\cosh x \cosh y + \sinh x \sinh y = \frac{e^{x}+e^{-x}}{2}\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}-e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}$$

$$= \frac{e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-x-y} + e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-x-y}}{4}$$

$$= \frac{2e^{x+y}+2e^{-y-x}}{4} = \frac{e^{x+y}+e^{-(x+y)}}{2} = \cosh(x+y) \quad \square$$

2) $$\sin x \cos y + \cos x \sin y = \frac{e^{ix}-e^{-ix}}{2i}\frac{e^{iy}+e^{-iy}}{2} + \frac{e^{ix}+e^{-ix}}{2i}\frac{e^{iy}-e^{-iy}}{2}$$

$$= \frac{e^{i(x+y)} + \cancel{e^{i(x-y)}} - \cancel{e^{i(y-x)}} - e^{i(-x-y)} + e^{i(x+y)} - \cancel{e^{i(x-y)}} + \cancel{e^{i(y-x)}} - e^{i(-x-y)}}{4i}$$

$$= \frac{2e^{i(x+y)} - 2e^{i(-x-y)}}{4i} = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} = \sin(x+y)$$

$$\cos x \cos y - \sin x \sin y = \frac{e^{ix}+e^{-ix}}{2}\frac{e^{iy}+e^{-iy}}{2} - \frac{e^{ix}-e^{-ix}}{2i}\frac{e^{iy}-e^{-iy}}{2i}$$

$$= \frac{e^{i(x+y)} + \cancel{e^{i(x-y)}} + \cancel{e^{i(y-x)}} + e^{i(-x-y)} + e^{i(x+y)} - \cancel{e^{i(x-y)}} - \cancel{e^{i(y-x)}} + e^{i(-x-y)}}{4}$$

$$= \frac{2e^{i(x+y)}+2e^{-i(x+y)}}{4} = \frac{e^{i(x+y)}+e^{-i(x+y)}}{2} = \cos(x+y) \quad \square$$

3) Let $f: \mathbb{R} \to \mathbb{R}$ sending $x \mapsto C_+ e^{ix} + C_- e^{-ix}$ for some C_+, C_- be a real function

$$\Rightarrow f(x) = C_+(\cos x + i\sin x) + C_-(\cos(-x) + i\sin(-x))$$

$$= C_+ \cos x + C_+ i \sin x + C_- \cos x - C_- i \sin x$$

$$= (C_+ + C_-)\cos x + (iC_+ - iC_-)\sin x$$

Question 3. Find the general solutions.

1) $y' = ky(1-y)$,
2) $xy' + 4y = x^2$,
3) $y'' + 4y' + 4y = 0$,
4) $y'' + 2y' - 15y = 0$.

c 1) $y(x) = 1/\left(1 + Ce^{-kx}\right)$, 2) $y(x) = x^2/6 + C/x^4$, 3) $y(x) = C_1e^{-2x} + C_2xe^{-2x}$, and 4) $y(x) = C_1e^{3x} + C_2e^{-5x}$.

1) $\int \frac{dy}{y(1-y)} = \int k\,dx \Rightarrow \int \left(\frac{1}{y} - \frac{1}{1-y}\right)dy = \int k\,dx$

$\Rightarrow \ln\left|\frac{y}{1-y}\right| = kx + c \Rightarrow \frac{y}{1-y} = e^{kx+c} = Ce^{kx}$

$\Rightarrow \underline{y(x) = \frac{1}{1+Ce^{-kx}}}$ as general solution

2) $y' + \frac{4}{x}y = x$

integrating factor : $\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4$

$\Rightarrow x^4y' + 4x^3y = x^5 \Rightarrow \frac{d}{dx}(x^4y) = x^5 \Rightarrow x^4y = \frac{x^6}{6} + C$

$\Rightarrow \underline{y(x) = \frac{x^2}{6} + \frac{C}{x^4}}$

as general solution

3) characteristic equation: $r^2 + 4r + 4 = 0$

$\Rightarrow r_1 = r_2 = -2$

$\Rightarrow \underline{y(x) = (C_1 + C_2x)e^{-2x}}$ as general solution.

4) characteristic equation : $r^2 + 2r - 15 = 0$

$\Rightarrow r_1 = -3, r_2 = 5$

$\Rightarrow \underline{y(x) = C_1e^{3x} + C_2e^{-5x}}$ as general solution

Question 4. Find the separated equations satisfied by $X(x), Y(y)$ for the following PDEs.

1) $u_{xx} - 2u_{yy} = 0$,
2) $u_{xx} + u_{yy} + 2u_x = 0$,
3) $x^2 u_{xx} - 2yu_y = 0$,
4) $u_{xx} + u_x + u_y - u = 0$.

Solution: 1) $X'' - 2\lambda X = 0, Y'' - \lambda Y = 0$, 2) $X'' + 2X' + \lambda X = 0, Y'' - \lambda Y = 0$, 3) $x^2X'' - \lambda X = 0, 2yY' - \lambda Y = 0$ 4) $X'' + X' - \lambda X = 0, Y' + (\lambda - 1)Y = 0$.

1) Assume $u = X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial y^2} \Rightarrow YX'' = 2XY'' \Rightarrow \frac{X''}{2X} = \frac{Y''}{Y} = \lambda$$

$$\Rightarrow \underline{X'' - 2\lambda X = 0,\ Y'' - \lambda Y = 0}$$

2) Assume $u = X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0 \Rightarrow YX'' + XY'' + 2YX' = 0$$

$$\Rightarrow YX'' + 2YX' = -XY'' \Rightarrow \frac{X'' + 2X'}{X} = \frac{Y''}{-Y} = \lambda$$

$$\Rightarrow \underline{X'' + 2X' - \lambda X = 0,\ Y'' + \lambda Y = 0}$$

3) Assume $u = X(x)Y(y)$

$$x^2\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial u}{\partial y} = 0 \Rightarrow x^2YX'' - 2yXY' = 0 \Rightarrow \frac{x^2X''}{X} = \frac{2Y'}{Y} = \lambda$$

$$\Rightarrow \underline{x^2X'' - \lambda X = 0,\ 2Y' - \lambda Y = 0}$$

4) Assume $u = X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0 \Rightarrow YX'' + YX' + XY' - XY = 0$$

$$\Rightarrow \frac{X'' + X'}{X} = \frac{Y'}{Y} - 1 = \lambda$$

$$\Rightarrow \underline{X'' + X' - \lambda X = 0,\ Y' - \lambda Y = 0}$$

Question 5. Find the separated solutions of $u_{xx} + yu_y + u = 0$.
Solution:

$$u(x,y) = \begin{cases} \left(A_1 e^{x\sqrt{\lambda}} + A_2 e^{-x\sqrt{\lambda}}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda > 0, \\ (A_1 + A_2 x)\,(1/|y|) & \text{for } \lambda = 0, \\ \left(A_1 \cos x\sqrt{-\lambda} + A_2 \sin x\sqrt{-\lambda}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda < 0. \end{cases}$$

Sol $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial u}{\partial y} + u = 0$

Suppose $u = X(x)Y(y)$

The equation becomes $YX'' + yXY' + XY = 0$

$\Rightarrow YX'' = -yXY' - XY$

$\Rightarrow \frac{X''}{X} = -y\frac{Y'}{Y} - 1 = \lambda,\ \lambda \in \mathbb{R}$

$\Rightarrow \underline{X'' - \lambda X = 0}, \quad \underline{\frac{Y'}{Y} = -(\lambda+1)\frac{1}{y}}$

$\Rightarrow$ characteristic equation: $t^2 - \lambda = 0$

for $\lambda > 0 \Rightarrow t_{1,2} = \pm\sqrt{\lambda}$

for $\lambda = 0 \Rightarrow t_{1,2} = 0$

for $\lambda < 0$, $t_{1,2} = \sqrt{-\lambda}\, i$

$\Rightarrow \int \frac{1}{Y}\, dY' = -(\lambda+1)\int \frac{1}{y}\, dy$

$\ln|Y| = -(\lambda+1)\ln|y|$

$Y = |y|^{-(1+\lambda)}$

Therefore the separation solutions are:

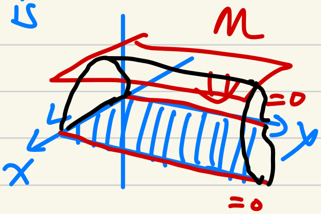
$$u(x,y) = \begin{cases} (A_1 e^{x\sqrt{\lambda}} + A_2 e^{-x\sqrt{\lambda}})\, y^{-|1+\lambda|}, & \lambda > 0 \\ (A_1 + A_2 x)\frac{1}{|y|}, & \lambda = 0 \\ (A_1 e^{ix\sqrt{-\lambda}} + A_2 e^{-ix\sqrt{-\lambda}})\, y^{-|1+\lambda|}, & \lambda < 0 \end{cases}$$

Question 6. Find the separated solutions $u(x, y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L, y > 0$, that satisfy the boundary conditions $u(0, y) = 0, u(L, y) = 0$ and the boundedness condition $|u(x, y)| \leq M$ for $y > 0$, where M is a constant independent of (x, y).

Solution:

$$u(x, y) = C \sin \frac{n\pi x}{L} e^{-n\pi y/L} \quad (n = 1, 2, \ldots)$$

The separated solutions of Laplace's equation is



$$u(x,y) = \begin{cases} (A_1 x + A_2)(B_1 y + B_2) & ① \\ (A_1 e^{ikx} + A_2 e^{-ikx})(B_1 e^{ky} + B_2 e^{-ky}),\ k>0 & ② \\ (A_1 e^{lx} + A_2 e^{-lx})(B_1 e^{ily} + B_2 e^{-ily}),\ l>0 & ③ \end{cases}$$

In case ①, to satisfy ($\forall y$, $u(0,y)=0$ and $u(L,y)=0$)

We have $A_1 = B_1 = 0$ and $A_2 B_2 = 0 \Rightarrow u = 0$

In case ②, $\forall y$ $u(0,y)$ and $u(L,y)=0$ implies $X(0)=0, X(L)=0$

we choose a different linearly independently solution form for X s.t. $X = A_1 \cos(kx) + A_2 \sin(kx)$

$\Rightarrow k = \frac{n\pi}{L}, n \in \mathbb{N} \Rightarrow X = A_2 \sin \frac{n\pi x}{L}, n \in \mathbb{N}$

Since $|u|$ is bounded and e^{ky} $(k>0)$ is unbounded for $y>0$

$\Rightarrow B_1 = 0 \Rightarrow Y = B_2 e^{-\frac{n\pi}{L} y}$

$\Rightarrow u = C \sin \frac{n\pi x}{L} e^{\frac{-n\pi y}{L}}, C \in \mathbb{R}, n \in \mathbb{N}$

In case ③, same as case ② we have $X(0)=0, X(L)=0$,

it only happens when $A_1 = A_2 = 0 \Rightarrow u = 0$

Hence overall, the separated solutions to the Laplace's equation for this boundary value problem is

$$u = C \sin \frac{n\pi x}{L} e^{\frac{-n\pi y}{L}}, C \in \mathbb{R}, n \in \mathbb{N}$$

Question 7. Find the separated solutions $u(x,t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L, t > 0$, that satisfy the boundary conditions $u(0,t) = 0, u(L,t) = 0$.

Solution:

$$u(x,t) = \begin{cases} \left(A_1 e^{kx} + A_2 e^{-kx}\right) e^{k^2 t} & \text{for } \lambda = k^2, k > 0, \\ A_1 x + A_2 & \text{for } \lambda = 0, \\ \left(A_1 e^{ilx} + A_2 e^{-ilx}\right) e^{-l^2 t} & \text{for } \lambda = -l^2, l > 0. \end{cases}$$

The case $\lambda < 0$ satisfies the boundary conditions. We obtain

$$u(x,t) = C \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 t} \quad (n = 1, 2, \ldots).$$

Sol First we look for the general separated solution:

Suppose $u = X(x)T(t)$

Then $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow XT' = TX'' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = \lambda$

$\Rightarrow X'' - \lambda X = 0, \quad T' - \lambda T = 0$

$\Downarrow \qquad \Downarrow$

$\Rightarrow$ characteristic equation : $x^2 - \lambda = 0$

for $\lambda > 0 \Rightarrow x_{1,2} = \pm\sqrt{\lambda}$

for $\lambda = 0 \Rightarrow x_{1,2} = 0$

for $\lambda < 0 \Rightarrow x_{1,2} = \sqrt{-\lambda}\, i$

$\int \frac{1}{T} dT = \int \lambda\, dt$

$\ln|T| = \lambda t + C$

$T = Ce^{\lambda t}$

So the general solutions are:

$$u(x,t) = \begin{cases} \left(A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x}\right) e^{\lambda t}, & \lambda > 0 \quad ① \\ A_1 x + A_2, \quad \lambda = 0 & ② \\ \left(A_1 e^{i\sqrt{-\lambda}x} + A_2 e^{i\sqrt{-\lambda}x}\right) e^{\lambda t}, & \lambda < 0 \quad ③ \end{cases}$$

In case ①, $(\forall t, u(0,t) = u(L,t) = 0)$ implies $X(0) = X(L) = 0$,

this can only be true when $A_1 = A_2 = 0 \Rightarrow u = 0$

In case ②, $(\forall t, u(0,t) = u(L,t) = 0)$ implies $A_1 = A_2 = 0 \Rightarrow u = 0$

In case ③, ($\forall t, u(0,t) = u(L,t) = 0$) implies $X(0) = X(L) = 0$

We choose a different linearly independently solution form for

X s.t. $X = A_1 \cos(\sqrt{-\lambda}x) + A_2 \sin(\sqrt{-\lambda}x)$

$$X(0) = X(L) = 0 \Longrightarrow A_1 = 0,\ \sqrt{-\lambda} = \frac{n\pi}{L},\ n \in \mathbb{N}$$

Hence overall, the separated solutions to the heat equation for this boundary value problem is

$$u = C \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad \text{where } C \in \mathbb{R},\ n \in \mathbb{N}$$