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Math 454: Boundary Value Problems for Partial Differential Equations
Due: September 17, 2024

Homework 3

Question 1. Use the complex form to find the Fourier series of $f(x) = e^x, -L < x < L$.
Solution. $e^x = \sum_{n=-\infty}^{\infty}(-1)^n \frac{L+in\pi}{L^2+n^2\pi^2}(\sinh L)\exp\left(\frac{in\pi}{L}x\right)$

Question 2. Derive the following formula

1) Let $0 < r < 1, f(x) = 1/\left(1 - re^{ix}\right), -\pi < x < \pi$. Find the complex Fourier series of f
2) Let $0 \leq r < 1$. Use the Fourier series in 1) to derive

$$\frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = 1 + \sum_{n=1}^{\infty} r^n \cos nx,$$
$$\frac{r\sin x}{1 + r^2 - 2r\cos x} = \sum_{n=1}^{\infty} r^n \sin nx.$$

Solution. 1) Expand f as a power series in r. The answer is $f(x) = \sum_{n=0}^{\infty} r^n e^{inx}$. 2) Use Euler's formula and consider the real part and imaginary part.

Question 3. Find the mean square error for the Fourier series of the function $f(x) = 1$ for $0 < x < \pi, f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$. Then, show that $\sigma_N^2 = O\left(N^{-1}\right)$ as $N \to \infty$.
Solution. $\sigma_N^2 = \frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{[(-1)^n-1]^2}{n^2}$. To show $O\left(N^{-1}\right)$, define $n = 2m - 1$ and replace the summation with $\sum_{m=(N+2)/2}^{\infty}$ or $\sum_{m=(N+3)/2}^{\infty}$ depending on if N is even or odd. Then use integrals to estimate the sum.

Question 4. Find the mean square error for the Fourier series of $f(x) = x^2, -\pi \leq x \leq \pi$. Then, show that $\sigma_N^2 = O\left(N^{-3}\right)$ as $N \to \infty$.
Solution. $\sigma_N^2 = 8\sum_{n=N+1}^{\infty}\frac{1}{n^4}$.

Question 5. Write out Parseval's theorem for the Fourier series of

1) $f(x) = 1$ for $0 < x < \pi, f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$,
2) $f(x) = x^2, -\pi \leq x \leq \pi$.

Solution. 1) $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$, and 2) $\pi^4/90 = 1 + \frac{1}{16} + \frac{1}{81} + \cdots$.

Question 6. Prove the following Parseval's theorem for complex, cosine and sine Fourier coefficients.

1) $f(x) = \sum_{n=-\infty}^{\infty}\alpha_n e^{in\pi x/L}$ implies that

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2 dx = \sum_{n=-\infty}^{\infty}|\alpha_n|^2.$$

2) $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$ implies that

$$\frac{1}{L}\int_0^L f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2.$$

3) $f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ implies that

$$\frac{1}{L}\int_0^L f(x)^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2.$$

Solution. Try to mimic the proof of Parseval's theorem for Fourier series.

Question 7. Let us solve the heat equation in the slab $0 < z < L$:

$$\begin{cases} u_t = Ku_{zz} & 0 < z < L, t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0, \\ u(z,0) = 1 & 0 < z < L, \end{cases}$$

where $K > 0$ is the thermal conductivity.

1) Find the separated solution depending on λ.
2) Find the general solution which satisfies the boundary conditions.
3) Find the particular solution which satisfies the initial and boundary conditions.

Solution. 1) For $\lambda > 0, u = (A\cos\sqrt{\lambda}z + B\sin\sqrt{\lambda}z)e^{-\lambda Kt}$, for $\lambda = 0,\ u = (Az + B)$, for $\lambda < 0, u = \left(Ae^{\sqrt{-\lambda}z} + Be^{-\sqrt{-\lambda}z}\right)e^{-\lambda Kt}$. 2) $u = \sum_{n=1}^{\infty} A_n \sin(n\pi z/L)e^{-(n\pi/L)^2Kt}$. 3) $u = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi z}{L}e^{-(n\pi/L)^2Kt}$.