Question 1. Find the steady-state (time independent) solution of the heat equation $u_t = Ku_{zz}$ in the slab $0 < z < L$, with boundary conditions $[u_z - h(u - T_0)](0) = 0$ and $[u_z + h(u - T_1)](L) = 0$. Assume that K, h, T_0, T_1 are all positive constants.

Solution. $u(x, y, z) = U(z) = \frac{T_1(1+hz)+T_0[1+h(L-z)]}{2+hL}$.

Sol Time independent $\Rightarrow u_t = 0, \forall t \Rightarrow Ku_{zz} = 0$

So $u(z) = Az + B$ for some A, B

$\Rightarrow u_z = A$

When $z = 0$, $u = B$

$\Rightarrow \underline{A - h(B - T_0) = 0}$ ①

When $z = L$, $u = LA + B$

$\Rightarrow \underline{A + h(LA + B - T_1) = 0}$ ②

Combining ①② we have

$$(2+hL)A - h(T_1 - T_0) = 0 \Rightarrow A = \frac{h(T_1 - T_0)}{2+hL}$$

$$\Rightarrow \frac{h(T_1 - T_0)}{2+hL} + hT_0 = hB \Rightarrow B = \frac{T_1 - T_0 + (2+hL)T_0}{2+hL}$$

$$= \frac{T_0 + T_1 + T_0 hL}{2+hL}$$

$$\Rightarrow u = \frac{h(T_1 - T_0)z + (T_0 + T_1 + T_0 hL)}{2+hL} = \frac{(1+hz)T_1 + (1+hL-hz)T_0}{2+hL}$$

Question 2. Solve the initial-value problem $u_t = Ku_{zz} (K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.

Solution. $u(z, t) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi z}{L} \exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$.

Sol Let $u = Z(z)T(t) = Ku_{zz}$

$\Rightarrow ZT' = KTZ''$

$$\frac{T'}{KT} = \frac{Z''}{Z} = -\lambda \Rightarrow \begin{cases} T' + \lambda KT = 0 \Rightarrow T = Ce^{-\lambda Kt}, \ t > 0 \\ Z'' + \lambda Z = 0 \end{cases}$$

$$\Rightarrow Z = \begin{cases} A\cos(\sqrt{\lambda} z) + B\sin(\sqrt{\lambda} z), & \lambda > 0 \\ Az + B, & \lambda = 0 \\ Ae^{\sqrt{-\lambda} z} + Be^{-\sqrt{-\lambda} z}, & \lambda < 0 \end{cases}$$

For $\lambda < 0$:

$$u(0,t)=0 \Rightarrow A+B=0$$
$$u(L,t)=0 \Rightarrow A=B=0 \Rightarrow u=0 \Rightarrow$$ contradicts with initial condition

For $\lambda = 0$. $A, B = 0 \Rightarrow u = 0 \Rightarrow$ contradicts with initial condition

For $\lambda > 0$:

$$u(0,t)=0 \Rightarrow A=0$$
$$u(L,t)=0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$
$$\Rightarrow u = \sum_{n=0}^{\infty} B_n \sin_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Since $u(z,0)=z$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L z \sin\frac{n\pi z}{L} dz = -\frac{2}{n\pi}\int_0^L z \, d\left(\cos\frac{n\pi z}{L}\right)$$
$$= \left[-\frac{2}{n\pi} z \cos\frac{n\pi z}{L}\right]_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi z}{L} dz$$
$$= \frac{2L}{-n\pi}(-1)^n - 0 + \underbrace{\left[\dots \sin\frac{n\pi z}{L}\right]_0^L}_{=0}$$
$$= -\frac{2L}{n\pi}(-1)^n$$

$\Rightarrow$ Above all, the solution to this BVT is

$$u(z,t) = \sum_{n=1}^{\infty} \frac{-2L}{n\pi}(-1)^n \sin\frac{n\pi z}{L} e^{-kt\left(\frac{n\pi}{L}\right)^2}$$

Question 3. Solve the initial-value problem $u_t = Ku_{zz} (K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u_z(0,t) = u_z(L,t) = 0$ and the initial condition $u(z,0) = z$, $0 < z < L$.
Solution. $u(z,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi z/L]}{(2n-1)^2} \exp\left[-\frac{(2n-1)^2\pi^2 Kt}{L^2}\right]$.

Sol General (separated) solution solved before:

$$u = \begin{cases} \left(A\cos(\sqrt{\lambda} z) + B\sin(\sqrt{\lambda} z)\right)e^{-k\lambda t}, & \lambda > 0 \\ Az + B, & \lambda = 0 \\ \left(Ae^{\sqrt{-\lambda} z} + Be^{-\sqrt{-\lambda} z}\right)e^{-k\lambda t}, & \lambda > 0 \end{cases}$$

for $\lambda > 0$:

$$u_z(0,t)=0 \Rightarrow B\sqrt{\lambda}e^{-\lambda kt} = 0 \ \forall t \Rightarrow B=0$$
$$u_z(L,t)=0 \Rightarrow A\sqrt{\lambda}\sin\sqrt{\lambda} z \, e^{-\lambda kt} = 0 \ (\forall t)$$
$$\Rightarrow \sqrt{\lambda} A \sin(L\sqrt{\lambda}) \cdot e^{-\lambda kt} = 0 \ \forall t \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

For $\lambda=0$, $u_z(0,t)=0, \forall z \Rightarrow A=0$
$u(z,0)=z, \forall z \Rightarrow A=1$ $\Rightarrow$ $\underline{\text{contradicts, impossible}}$

For $\lambda<0$, $u_z(z,t)=\left(A\sqrt{-\lambda}e^{\sqrt{-\lambda}z}-B\sqrt{-\lambda}e^{-\sqrt{-\lambda}z}\right)e^{-k\lambda t}$

$$u_z(0,t)=\sqrt{-\lambda}A-\sqrt{-\lambda}B=0 \quad \forall t \Rightarrow A=B$$

$$u(z,0)=0 \quad \forall z \Rightarrow Ae^{\sqrt{-\lambda}z}+Ae^{-\sqrt{-\lambda}z}=0 \quad \forall z \Rightarrow A=0$$

$\Rightarrow u=0 \ \forall z,t \Rightarrow$ $\underline{\text{contradicts with initial condition}}$

Thus only the $\lambda>0$ case holds, with $u(z,0)=\sum_{n=0}^{\infty} A_n \cos n = z$

We calculate the Fourier cosine series:

$$A_n=\frac{2}{L}\int_0^L z\cos\frac{n\pi z}{L}\,dz \qquad A_0=\frac{1}{L}\int_0^L x\,dx=\frac{L}{2}$$

$$=\frac{2}{n\pi}\int_0^L z\,d\sin\left(\frac{n\pi z}{L}\right)$$

$$=\underbrace{\frac{2}{n\pi}\left[z\sin\frac{n\pi z}{L}\right]_0^L}_{=0}-\frac{2}{n\pi}\int_0^L\sin\frac{n\pi z}{L}\,dz$$

$$=\frac{2L}{n^2\pi^2}\left[\cos\frac{n\pi z}{L}\right]_0^L=\frac{2L}{n^2\pi^2}\left((-1)^n-1\right)$$

Thus
$$u(z,t)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left((-1)^n-1\right)\exp\left(-\frac{n^2\pi^2kt}{L^2}\right)$$

$$=\frac{L}{2}+\frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos\frac{(2m-1)\pi z}{L}}{(2m-1)^2}\exp\left(-\frac{(2m-1)^2\pi^2kt}{L^2}\right)$$

Question 4. Let $\varphi_1 = 1, \varphi_2 = x, \varphi_3 = x^2$ on the interval $0 \le x \le 1$. Compute the following quantities

1) $\langle \varphi_1, \varphi_2 \rangle$,
2) $\langle \varphi_1, \varphi_3 \rangle$,
3) $\|\varphi_1 - \varphi_2\|^2$,
4) $\|\varphi_1 + 3\varphi_2\|^2$.

Solution. 1) 1/2, 2) 1/3, 3) 1/3, 4) 7.

1) $\langle \varphi_1, \varphi_2 \rangle = \int_0^L \varphi_1(x)\varphi_2(x)\,dx = \int_0^L x\,dx = \frac{1}{2}$

2) $\langle \varphi_1, \varphi_3 \rangle = \int_0^L \varphi_1(x)\varphi_3(x)\,dx = \int_0^1 x^2\,dx = [\frac{1}{3}x^3]_0^1 = \frac{1}{3}$

3) $\|\varphi_1 - \varphi_2\|^2 = \int_0^1 (x-1)^2\,dx = [\frac{1}{3}x^3 - x^2 + x]_0^1 = \frac{1}{3}$

4) $\|\varphi_1 + 3\varphi_2\|^2 = \int_0^L (3x+1)^2\,dx = [3x^3 + 3x^2 + x]_0^1 = 7$

Question 5. Check if the following operator is symmetric on its domain with respect to given inner product.

1) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
2) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
3) $A = \frac{d}{dx}$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)x\,dx$.

Solution. 1) True, 2) False, 3) False. For 1) try integration by parts as in the class. For 2), 3) try to find counter-examples.

1) True

$A\varphi = -\frac{d^2\varphi}{dx^2} + \varphi$, $A\psi = -\frac{d^2\psi}{dx^2} + \psi$

$\Rightarrow \langle A\varphi, \psi \rangle = -\int_0^L \frac{d^2\varphi}{dx^2}\psi\,dx + \int_0^L \varphi\psi\,dx$

$= -\int_0^L \psi\, d\left(\frac{d\varphi(x)}{dx}\right) + \int_0^L \varphi\psi\,dx$

$= \left[-\psi(x)\frac{d\varphi(x)}{dx}\right]_{x=0}^{x=L} + \int_0^L \frac{d\varphi}{dx}\,d\psi + \int_0^L \varphi\psi\,dx$

$= \underbrace{-\psi(L)\frac{d\varphi}{dx}(L) + \psi(0)\frac{d\varphi}{dx}(0)}_{=0 \text{ since } \psi(0)=\psi(L)=0} + \int_0^L \frac{d\varphi}{dx}\frac{d\psi}{dx}\,dx + \int_0^L \varphi\psi\,dx$

$\langle \varphi, A\psi \rangle = -\int_0^L \frac{d^2\psi}{dx^2}\varphi\,dx + \int_0^L \psi\varphi\,dx$

$= -\int_0^L \varphi\, d\left(\frac{d\psi}{dx}\right) + \int_0^L \varphi\psi\,dx$

$$= \underbrace{\left[-\varphi(x)\frac{d\psi}{dx}\right]_{x=0}^{x=L}}_{=0 \text{ since } \varphi(0)=\varphi(L)=0} + \int_0^L \frac{d\psi}{dx}\frac{d\varphi}{dx}dx + \int_0^L \varphi\psi\, dx$$

$$= \int_0^L \frac{d\psi}{dx}\frac{d\varphi}{dx}dx + \int_0^L \varphi\psi\, dx = \langle A\varphi, \psi\rangle$$

Thus it is symmetric.

2) False

By calculation in 1) we already have

$$\langle \varphi, A\psi\rangle - \langle A\varphi, \psi\rangle = -\varphi(L)\frac{d\psi}{dx}(L) + \varphi(0)\frac{d\psi}{dx}(0) - \left(-\psi(L)\frac{d\varphi}{dx}(L) + \psi(0)\frac{d\varphi}{dx}(0)\right)$$

$$= -\varphi(L)\frac{d\psi}{dx}(L) + \psi(L)\frac{d\varphi}{dx}(L) \quad \text{since } \varphi(0)=\psi(0)=0$$

Consider $\varphi(x)=x^2,\ \psi(x)=x$

$$\Rightarrow \langle \varphi, A\psi\rangle - \langle A\varphi, \psi\rangle = -L^2 + 2L^2 = L^2 \neq 0, \text{ so } \langle \varphi, A\psi\rangle \neq \langle A\varphi, \psi\rangle.$$

3) False

$$\langle A\varphi, \psi\rangle = \int_0^L (x\varphi'\psi)dx, \quad \langle \varphi, A\psi\rangle = \int_0^L (x\psi'\varphi)dx$$

Consider $L=\pi$, $\varphi(x)=\sin x$, $\psi(x)=\sin 2x$ which satisfies the domain

$$\langle A\varphi, \psi\rangle = \int_0^\pi x\cos x \sin 2x\, dx = \frac{2}{3}\pi$$

$$\langle \varphi, A\psi\rangle = \int_0^\pi 2x\cos 2x \sin x\, dx = -\frac{2}{3}\pi$$

This counterex. suffices to show that A is not symmetric

Question 6. Convert the following ODE into Sturm-Liouville form and write the $s(x)$, $\rho(x)$ and $q(x)$ functions.

1) $y'' + 2xy' + \lambda y = 0$.
2) $x^2y'' + xy' + (\lambda x^2 - 1)y = 0$.
3) $y'' + \frac{1}{x}y' + \lambda y = 0$.

Solution. 1) $(e^{x^2}y')' + \lambda e^{x^2}y = 0$, $s(x) = e^{x^2}$, $\rho(x) = e^{x^2}$ and $q(x) = 0$. 2) $(xy')' + (\lambda x - \frac{1}{x})y = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = \frac{1}{x}$. 3) $(xy')' + \lambda xy = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = 0$.

Recall Sturm-Liouville form: $\frac{d}{dx}(s(x)\frac{dy}{dx}) + (\lambda\rho(x) - q(x))y = 0$

1) Note that $\frac{d}{dx}(e^{x^2}y') = e^{x^2}y'' + 2xe^{x^2}y'$

So we set $s(x) = e^{x^2}$

Mutiply the ODE by e^{x^2} on both sides $\Rightarrow e^{x^2}y'' + 2xe^{x^2}y' + \lambda e^{x^2}y = 0$

Then the ODE becomes $\frac{d}{dx}(e^{x^2}\frac{dy}{dx}) + \lambda e^{x^2}y = 0$

$\Rightarrow \rho(x) = e^{x^2}, q(x) = 0$

2) Dividing both sides by $x \Rightarrow xy'' + y' + (\lambda x - \frac{1}{x})y = 0$

Note that $\frac{d}{dx}(xy') = xy'' + y'$

So set $s(x) = x$

So the ODE becomes $\frac{d}{dx}(xy') + (\lambda x - \frac{1}{x})y = 0$

So $\rho(x) = x$, $q(x) = \frac{1}{x}$

3) multiply both sides by $x \Rightarrow xy'' + y' + \lambda xy = 0$

Note that $\frac{d}{dx}(xy') = xy'' + y'$

Then the ODE becomes $\frac{d}{dx}(xy') + \lambda xy = 0$

$\Rightarrow \rho(x) = x, q(x) = 0$