Question 1

Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem and use explicit computation to show that different eigenfunctions are orthogonal.

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = 0, \quad \phi'(L) = 0.$$

Solution.

$$\phi_n(x) = A\sin\left(\left(n - \frac{1}{2}\right)\frac{\pi x}{L}\right), \quad \lambda_n = \left(\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where A is an arbitrary constant. To compute $\int_0^L \phi_n(x)\phi_m(x)dx$, the following identity is useful:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)].$$

Case 1 $\lambda = 0 \Rightarrow$ the equation becomes $\varphi''(x) = 0$

$\Rightarrow \varphi(x) = C_1 x + C_2$

Apply the boundary conditions:

$\varphi(0) = 0 \Rightarrow C_2 = 0$

$\varphi'(x) = C_1 \Rightarrow \varphi'(L) = C_1 = 0$

So $\varphi(x) = 0$, not acceptable to be eigenfunction

Case 2 $\lambda = -\mu^2$ for some $\mu > 0$

The equation becomes $\varphi''(x) - \mu^2\varphi(x) = 0$

The general solution is $\varphi(x) = Ae^{\mu x} + Be^{-\mu x}$

$\varphi(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

$\varphi'(x) = A\mu e^{\mu x} - A\mu e^{-\mu x}$ $\varphi'(L) = 0 \Rightarrow A\mu(e^{\mu L} - e^{-\mu L}) = 0$

$\Rightarrow A = 0$

$\Rightarrow \varphi(x) = 0.$

Case 3 $\lambda = k^2$ for some $k>0$

The equation becomes $\varphi''(x) + k^2\varphi(x) = 0$

General sol: $\varphi(x) = A\cos(kx) + B\sin(kx)$

$\varphi(0) = 0 \Rightarrow C = 0$

$\varphi'(x) = Bk\cos(kx) \Rightarrow \varphi(L) = Bk\cos(kL) = 0$

$B = 0 \Rightarrow$ trivial sol

$B \neq 0 \Rightarrow \cos(kL) = 0,\ kL = (n-\frac{1}{2}\pi),\ n \in \mathbb{N}$

So eigenvalues are $\lambda_n = k_n^2 = \left((n-\frac{1}{2})\frac{\pi}{L}\right)^2$

corresponding eigenfunctions $\varphi_n(x) = B\sin(k_n x)$

$= B\sin\left((n-\frac{1}{2})\frac{\pi x}{L}\right)$

WTS: $\forall n \neq m,\ \int_0^L \varphi_n(x)\varphi_m(x)\,dx = 0$

$$I = \int_0^L \varphi_n(x)\varphi_m(x)\,dx = \int_0^L \sin\left((n-\tfrac{1}{2})\tfrac{\pi x}{L}\right)\sin\left((m-\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_0^L \left[\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m-1)\frac{\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left((n-m)\frac{\pi x}{L}\right)\right]_0^L - \frac{1}{2}\left[\frac{L}{(n+m-1)\pi}\sin\left((n+m-1)\frac{\pi x}{L}\right)\right]_0^L$$

$= 0$

Thus eigenfunctions are orthogonal over $[0,L]$

Question 2

Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem:

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi'(0) = 0, \quad \phi(L) = 0.$$

Solution.

$$\phi_n(x) = A\cos\left(\left(n - \frac{1}{2}\right)\frac{\pi x}{L}\right), \quad \lambda_n = \left(\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where A is an arbitrary constant.

Case 1 $\lambda = 0 \Rightarrow \varphi''(x) = 0 \Rightarrow \varphi(x) = C_1 x + C_2$

$\varphi'(0) = C_1 = 0 \quad \varphi(L) = C_2 = 0 \Rightarrow \varphi = 0$, trivial sol

Case 2 $\lambda = -\mu^2$ for some $\mu > 0$

$\varphi(x) = Ae^{\mu x} + Be^{-\mu x}$

$\varphi'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$. $\varphi'(0) = 0 \Rightarrow A = B$

$\varphi(L) = 0 \Rightarrow A(e^{\mu L} - e^{-\mu L}) = 0$

$L, \mu > 0 \Rightarrow A = 0 \Rightarrow$ trivial sol.

Case 3 $\lambda = k^2$ for some $k > 0$

$\varphi''(x) + k^2\varphi(x) = 0$

general sol: $\varphi(x) = A\cos(kx) + B\sin(kx)$

$\varphi'(x) = -Ak\sin(kx) + Bk\cos(kx)$, $\varphi'(0) = Bk = 0 \Rightarrow B = 0$

$\varphi(L) = A\cos(kL) = 0 \Rightarrow k = (n - \frac{1}{2})\frac{\pi}{L}$, $n \in \mathbb{N}$

So eigenvalues $\lambda_n = k_n^2 = ((n - \frac{1}{2})\frac{\pi}{L})^2$

eigenfunctions $\varphi_n(x) = A\cos(k_n x) = A\cos((n - \frac{1}{2})\frac{\pi x}{L})$

Question 3

Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem:

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).$$

Solution.

$$\phi_n(x) = A\cos\left(\frac{2n\pi x}{L}\right) + B\sin\left(\frac{2n\pi x}{L}\right), \quad \lambda_n = \left(\frac{2n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

and

$$\phi_0(x) = C, \quad \lambda_0 = 0,$$

where A, B, and C are arbitrary constants. (The orthogonality theorem can be extended to the case of periodic boundary conditions.)

Sol Case 1 $\lambda = 0 \Rightarrow \varphi''(x) = 0$

$\varphi(x) = C_1 x + C_2$

$\varphi(0) = \varphi(L) \Rightarrow C_1 L = 0 \Rightarrow C_1 = 0$

$\varphi'(0) = \varphi'(L) \Rightarrow C_2 = C_2$, nothing implied

There $\varphi(x) = C$, C constant

Case 2 $\lambda = -\mu^2$ for some $\mu > 0$

$\varphi(x) = Ae^{\mu x} + Be^{-\mu x}$

$\varphi(0) = \varphi(L) \Rightarrow A + B = Ae^{\mu L} + Be^{-\mu L}$ ①

$\varphi'(0) = \varphi'(L)$

$\Rightarrow \mu(A - B) = \mu(Ae^{\mu L} - Be^{-\mu L})$

$A - B = Ae^{\mu L} - Be^{-\mu L}$ ②

By ①② $\Rightarrow 2A(1 - e^{\mu L}) = 0 \Rightarrow A = 0$ since $\mu L > 0$

Similarly set $B=0$. So the solution is trivial

<u>Case 3</u> $\lambda = k^2$ for some $k>0$

$\varphi(x) = A\cos(kx) + B\sin(kx)$

$\varphi(0) = \varphi(L) \Rightarrow A = A\cos(kL) + B\sin(kL)$

$\underline{A(1-\cos(kL)) = B\sin(kL)}$ ①

$\varphi'(0) = \varphi'(L) \Rightarrow Bk = -Ak\sin(kL) + Bk\cos(kL)$

$\underline{B(1-\cos(kL)) = -A\sin(kL)}$ ②

By ①, $B = \dfrac{A(1-\cos(kL))}{\sin(kL)}$

$\Rightarrow \underbrace{(1-\cos(kL))^2}_{\text{always} \geq 0} = \underbrace{-(\sin(kL))^2}_{\text{always} \leq 0}$

So $1-\cos(kL) = \sin(kL) = 0$

$\Rightarrow k = \frac{2n\pi}{L},\ n \in \mathbb{N}$

So eigenvalues $\lambda_n = k_n^2 = \left(\frac{2n\pi}{L}\right)^2$

eigenfunctions $\varphi_n = A\cos\left(\frac{2n\pi x}{L}\right) + B\sin\left(\frac{2n\pi x}{L}\right)$, $n \in \mathbb{N}$

together with in case 1, $\lambda_0 = 0$

$\varphi_0(x) = C$

Question 4

Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0,t) = T_1$, $u_z(L,t) = \Phi_2$, and the initial condition $u(z,0) = T_3$, where K, T_1, Φ_2, and T_3 are positive constants.

Solution.

$$u(z,t) = T_1 + \Phi_2 z + \sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left(-\left(\left(n-\frac{1}{2}\right)\frac{\pi}{L}\right)^2 Kt\right),$$

where

$$A_n = \frac{2(T_3 - T_1)}{\left(n-\frac{1}{2}\right)\pi} - \frac{2L\Phi_2(-1)^{n+1}}{\left(n-\frac{1}{2}\right)^2\pi^2}.$$

$u(z,t) = v(z) + w(z,t)$

① Solve steady-state sol $v(z)$:

$v_{zz} = 0 \Rightarrow v(z) = C_1 z + C_2$

$v(0) = C_2 = T_1$

$v_z(L) = C_1 = \phi_2$

So $v(z) = \phi_2 z + T_1$

② Solve transient sol $w(z,t)$

$w_t = u_t - v_t = K(u_{zz} - v_{zz}) = Kw_{zz}$

$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0$

$w_z(L,t) = u_z(L,t) - v_z(L) = \phi_2 - \phi_2 = 0$

$w(z,0) = u(z,0) - v(z) = T_3 - (\phi_2 z + T_1) = \underline{T_3 - T_1 - \phi_2 z}$

Find separated sol for $w(z,t)$:

$w(z,t) = X(z)T(t)$

$X(z)T'(t) = KX''(z)T(t)$

$$\frac{T'(t)}{kT(t)} = \frac{X''(z)}{X(z)} = -\lambda$$

$$\Rightarrow T'(t) + k\lambda T(t) = 0, \quad X''(z) + \lambda X(z) = 0 \quad ②$$

$$\Downarrow$$

$$T(t) = e^{-k\lambda t}$$

with $X(0) = 0$, $X'(L) = 0$

Sol for ②: by previous problem,

only have sol when $\lambda = k^2$ for some $k > 0$

general sol: $X(z) = A\cos(kz) + B\sin(kz)$

$X(0) = 0 \Rightarrow A = 0$

$X'(L) = 0 \Rightarrow Bk\cos(kL) = 0 \Rightarrow k = (n - \frac{1}{2})\frac{\pi}{L}, \ n \in \mathbb{N}$

So $X_n(z) = B_n \sin((n - \frac{1}{2})\frac{\pi z}{L}), \ n \in \mathbb{N}$

This solves the transient sol:

$$w(z,t) = \sum_{n=1}^{\infty} B_n \sin\left((n - \tfrac{1}{2})\frac{\pi z}{L}\right) e^{-k((n-\frac{1}{2})\frac{\pi}{L})^2 t}$$

③ determine the coeffs B_n

$$w(z,0) = T_3 - T_1 - \phi_2 z = \sum_{n=1}^{\infty} B_n \sin\left((n - \tfrac{1}{2})\frac{\pi z}{L}\right)$$

Let $f(z) = (T_3 - T_1) - \phi_2 z$

Expand $f(z)$ by Fourier sine series:

$$f(z)=\sum_{n=1}^{\infty} B_n \sin(\alpha_n z)$$

where $\alpha_n=(n-\frac{1}{2})\frac{\pi}{L}$

$$\Rightarrow \int_0^L f(z)\sin(\alpha_m z)\,dz=\sum_{n=1}^{\infty} B_n \int_0^L \sin(\alpha_n z)\sin(\alpha_m z)\,dz$$

Since $\int_0^L \sin(\alpha_n z)\sin(\alpha_m z)\,dz=\frac{L}{2}\delta_{mn}$

we have $B_n=\frac{2}{L}\int_0^L f(z)\sin(\alpha_n z)\,dz$

$$=\frac{2}{L}\left[(T_3-T_1)\int_0^L \sin(\alpha_n z)\,dz-\phi_2\int_0^L z\sin(\alpha_n z)\,dz\right]$$

$$=\frac{2}{L}\left[(T_3-T_1)\frac{1}{\alpha_n}-\phi_2\frac{(-1)^{n+1}}{\alpha_n^2}\right]$$

$$=\frac{2(T_3-T_1)}{(n-\frac{1}{2})\pi}-\frac{2L\phi_2(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}$$

Therefore $T_1+\phi_2 z+\sum_{n=1}^{\infty}\left[\frac{2(T_3-T_1)}{(n-\frac{1}{2})\pi}-\frac{2L\phi_2(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}\right]\sin\left((n-\frac{1}{2})\frac{\pi z}{L}\right)e^{-k\left((n-\frac{1}{2})\frac{\pi}{L}\right)^2 t}$.

$n\in\mathbb{N}$

Question 5

Let us consider heat flow in a circular ring of circumference L.

1) Find all of the separated solutions of the heat equation $u_t = Ku_{zz}$ ($K > 0$) satisfying the periodic boundary conditions $u(0,t) = u(L,t)$ and $u_z(0,t) = u_z(L,t)$.

2) Solve the heat equation $u_t = Ku_{zz}$ ($K > 0$) satisfying the periodic boundary conditions $u(0,t) = u(L,t)$, and the initial conditions $u(z,0) = 100$ if $0 < z < L/2$ and $u(z,0) = 0$ if $L/2 < z < L$.

Solution. 1)

$$u_n(z,t) = \left(A_n \cos\frac{2n\pi z}{L} + B_n \sin\frac{2n\pi z}{L}\right)\exp\left(-\left(\frac{2n\pi}{L}\right)^2 Kt\right), \quad n = 0,1,2,\ldots$$

2)

$$u(z,t) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{2n\pi z}{L}\exp\left(-\left(\frac{2n\pi}{L}\right)^2 Kt\right).$$

1) Suppose $u(z,t) = X(z)T(t)$

$$\Rightarrow X(z)T'(t) = KX''(z)T(t)$$

$$\frac{T'}{KT} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow T(t) = T_0 e^{-K\lambda t}, \quad \underbrace{X''(z) + \lambda X(z) = 0}_{\text{with } X(0)=X(L),\ X'(0)=X'(L)} \ ②$$

Solve ②: by previous problem 3

② has non trivial sol only when $\lambda = k^2$ for some $k>0$

And sol is $X_n(z) = A_n\cos\left(\frac{2n\pi z}{L}\right) + B_n\sin\left(\frac{2n\pi z}{L}\right)$

Thus the separate sol is

$$u_n(z,t) = \left(A_n\cos\left(\frac{2n\pi z}{L}\right) + B_n\sin\left(\frac{2n\pi z}{L}\right)\right)e^{-K\left(\frac{2n\pi}{L}\right)^2 t}, \quad n\in\mathbb{N}$$

2) $u(z,0) = \begin{cases} 100, & 0 < z < \frac{L}{2} \\ 0, & \frac{L}{2} < z < L \end{cases}$

We expand $u(z,0)$ in terms of the eigenfunctions in 1)

$$u(z,0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi z}{L}\right) + b_n \sin\left(\frac{2n\pi z}{L}\right) \right)$$

$$a_0 = \frac{1}{L}\int_0^L u(z,0)\,dz = \frac{1}{L}\int_0^{\frac{L}{2}} 100\,dz + \int_{\frac{L}{2}}^{L} 0\,dz$$

$$= \frac{1}{L}\left(100 \cdot \frac{L}{2}\right) = 50$$

$$a_n = \frac{2}{L}\int_0^L u(z,0) \cos\left(\frac{2n\pi z}{L}\right) dz$$

$$= \frac{2}{L}\int_0^{\frac{L}{2}} 100 \cos\left(\frac{2n\pi z}{L}\right) dz)$$

$$= \frac{200}{L}\left[\frac{L}{2n\pi} \sin\left(\frac{2n\pi z}{L}\right)\right]_0^{\frac{L}{2}} = 0$$

$$b_n = \frac{2}{L}\int_0^L u(z,0) \sin\left(\frac{2n\pi z}{L}\right) dz = \frac{2}{L}\int_0^{\frac{L}{2}} 100 \sin\left(\frac{2n\pi z}{L}\right) dz$$

$$= \frac{200}{L}\left[-\frac{L}{2n\pi} \cos\left(\frac{2n\pi z}{L}\right)\right]_0^{\frac{L}{2}}$$

$$= -\frac{100}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= -\frac{100}{n\pi}\left((-1)^n - 1\right)$$

Therefore $u(z,t) = 50 + \sum_{n=1}^{\infty} \frac{100}{n\pi}\left(1 - (-1)^n\right) \sin\left(\frac{2n\pi z}{L}\right) e^{-k\left(\frac{2n\pi}{L}\right)^2 t}$