Recall last time:

Parseval's thm: $\|f\|^2 = \|A_0\|^2 + \sum_{n=1}^{\infty}\|A_n\cos_n + B_n\sin_n\|^2 = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2+B_n^2)$

$\downarrow$

$\frac{1}{2L}\int_{-L}^{L}|f|^2$

mean square error: $\sigma_N^2 = \|f-f_N\| = \frac{1}{2L}\int_{-L}^{L}|f-f_N|^2$

$\Rightarrow \sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2+B_n^2)$ by Parseval's thm

Today:

(1) Parseval's Thm for complex Fourier series $f(x) = \sum_{n\in\mathbb{Z}}\alpha_n e^{\frac{in\pi x}{L}}$

$\|f\|^2 = \sum_{n\in\mathbb{N}}\|\alpha_n\|^2$ (by inner product $\langle f,g\rangle = \frac{1}{2L}\int_{-L}^{L} f\bar{g}$)

即 $\frac{1}{2L}\int_{-L}^{L}|f|^2 = \sum_{n\in\mathbb{Z}}|\alpha_n|^2$

计算Pf $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \frac{1}{2L}\int_{-L}^{L}f\bar{f} = \frac{1}{2L}\int_{-L}^{L}\Big(\sum_n \alpha_n \underbrace{e_n}_{e^{\frac{in\pi x}{L}}}\Big)\Big(\sum_m \bar{\alpha}_m e_{-m}\Big)$

$= \frac{1}{2L}\sum_{n,m}\alpha_n\bar{\alpha}_m\int_{-L}^{L}e_n e_{-m}$

$= \sum_{n,m}\alpha_n\bar{\alpha}_m\delta_{nm} = \sum_{n\in\mathbb{Z}}|\alpha_n|^2$

Thm Parseval for cosine, sine

cos: $f(x) = A_0 + \sum_{n=1}^{\infty}A_n\cos_n$, $\|f\|^2 = \|A_0\|^2 + \sum_{n=1}^{\infty}\|A_n\|^2$

即 $\frac{1}{2L}\int_{-L}^{L}|f|^2 = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2$

sin: $f(x) = \sum_{n=1}^{\infty}B_n\sin_n$, $\frac{1}{2L}\int_{-L}^{L}|f|^2 = \frac{1}{2}\sum_{n=1}^{\infty}B_n^2$ (computational pf in hw)

today (2): PDE in rectangular coordinates

ex 一个ODE boundary value problem

$u''=1$, $x\in[0,1]$

$u(0)=0$, $u(1)=0$

Sol 1. General sol: $u(x) = \frac{1}{2}x^2 + Ax + B$

2. Match with B.C.

$u(0)=0 \Rightarrow B=0$

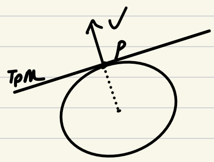
$u(1)=0 \Rightarrow A=-\frac{1}{2}$ $\Rightarrow u(x) = \frac{1}{2}x(x-1)$

ODE的BV problem 即确定常数(范围)

Now for PDE:

Def Normal vector

for $p\in M$ a manifold,
p的normal vector 是和
p在M上的 tangent space T_pM
垂直的 vector n



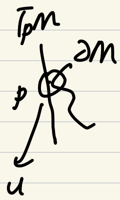
note: 如果M embedded into $\mathbb{R}^d$, $\dim(\partial M) = d-1$

take $p\in\partial M$, $\dim(T_pM) = \dim(M)$

令∂M在p附近的表示方程为 $F(x_1,\dots,x_d)=0$ $\Rightarrow$

$p\in U = \left\{\begin{bmatrix}x_1\\ \vdots\\ x_d\end{bmatrix} : F(x_1,\dots,x_d)=0\right\} \subseteq \partial M$

$\Rightarrow$ 可取 $\vec{n_p} = \nabla F(p)$ 得到 normal vector $\vec{u_p}$



Specially, for hyperplane $U = \left\{\begin{bmatrix}x_1\\ \vdots\\ x_d\end{bmatrix} : a_1x_1+\dots+a_dx_d=0\right\}$

$\forall p\in U$, $\vec{n_p} = \nabla F(p) = \begin{bmatrix}a_1\\ \vdots\\ a_d\end{bmatrix}$

$\left(\dim(\vec{n_p}) + \dim(T_pM) = \dim(\mathbb{R}^d) = d\right)$

Def Boundary condition

Given PDE $F[u]=0$ defined on a domain R

本课only consider 以下三种 boundary conditions:

1. Dirichlet boundary condition

已知 $u=g(x)$, $x\in\partial R$ (边界点的值)

2. Neumann boundary condition

已知 $n_x\nabla u(x) = g(x)$, $x\in\partial R$ (边界点的normal derivative)

3. Robin boundary condition

(以上两种的一个linear comb)

已知 $a(x)u + b(x)n_x\nabla u(x) = g(x)$, $x\in\partial R$

note: $n_x\nabla u = n_1\frac{\partial}{\partial x_1}u + \dots + n_d\frac{\partial}{\partial x_d}u$

$= D_{n_x}u$

$n\nabla u$ 就等于 u 在 n方向上的 directional derivative.

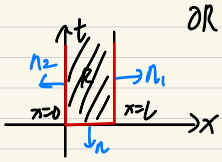
ex heat equation

$u_t = Ku_{xx}$, defined on $R = \{(x,t) \mid x\in(0,L), t\in(0,\infty)\}$

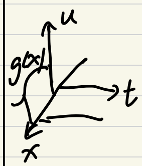
$\partial R = \{(0,t)\mid t>0\} \cup \{(L,t)\mid t>0\} \cup \{(x,0)\mid x\in(0,L)\}$

如果我们在这上面 impose 个 Robin BC

这上面 impose 个 Dirichlet BC



则这个 PDE BVP 的形式即:



$\partial_t u = K\partial_{xx}u$, $t>0$, $0<x<L$ (PDE)

$u(x,0) = g(x)$, $x\in(0,L)$, $t=0$ (Dirichlet's BC)

$a(x,t)u + b(x,t)\underbrace{\vec{n}\cdot\nabla u}_{D_n u(x,t)} = g(x,t)$, $x=0,L$, $t>0$ (Robin's BC)

note: $\nabla u = \begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{pmatrix}$

On $x=0$, 可取 $\vec{n_1} = \begin{pmatrix}-1\\ 0\end{pmatrix} \Rightarrow$ ③转为 $a(0,t)u - b(0,t)\frac{\partial}{\partial x}u = g(0,t)$

On $x=L$, 可取 $\vec{n_2} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow$ ②转为 $a(L,t)u + b(L,t)\frac{\partial}{\partial x}u = g(0,t)$

用 $a(t), b(t), g(t)$ 来表示 $a(0,t)$, $b(0,t)$ 和 $g(0,t)$

$\tilde{a}(t), \tilde{b}(t), \tilde{g}(t)$ 来表示 $a(L,t)$, $b(L,t)$ 和 $g(L,t)$

$$
\frac{\partial}{\partial t}u = k\frac{\partial^2}{\partial x^2}u,
$$

$$
\Rightarrow \begin{cases} a(t)u - b(t)\frac{\partial}{\partial x}u = g(t), \\ \tilde{a}(t)u + \tilde{b}(t)\frac{\partial}{\partial x}u = \tilde{g}(t), \\ u = f(x), \end{cases}
$$

Def 如果 $a(t)=a$, $b(t)=b$, $\tilde{a}(t)=\tilde{a}$, $\tilde{b}(t)=\tilde{b}$ for some consts $a, b, \tilde{a}, \tilde{b}$

且 $\tilde{g}(t)=g(t)=0$

则称这个PDE BV方程组 homogeneous

$$
\frac{\partial}{\partial t}u = k\frac{\partial^2}{\partial x^2}u,
$$

$$
\Rightarrow \begin{cases} au - b\frac{\partial}{\partial x}u = 0, \\ \tilde{a}u + \tilde{b}\frac{\partial}{\partial x}u = 0, \\ u = f(x), \end{cases}
$$

对于 homogeneous 的PDE BV方程组

可取某个 α, β 使得 $a=\cos\alpha$, $b=L\sin\alpha$, $\tilde{a}=\cos\beta$, $\tilde{b}=L\sin\beta$

$$
\frac{\partial}{\partial t}u = k\frac{\partial^2}{\partial x^2}u,
$$

$$
\Rightarrow \begin{cases} u\cos\alpha - \frac{\partial}{\partial x}u\,L\sin\alpha = 0, \\ u\cos\beta + \frac{\partial}{\partial x}u\,L\sin\beta = 0, \\ u = f(x), \end{cases}
$$

ex simple case where $\alpha=\beta=0 \Rightarrow u(0,t)=u(L,t)=0, \forall t.$