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Math 454: Boundary Value Problems for Partial Differential Equations
Due: September 24, 2024

Homework 4

Question 1. Find the steady-state (time independent) solution of the heat equation $u_t = Ku_{zz}$ in the slab $0 < z < L$, with boundary conditions $[u_z - h(u - T_0)](0) = 0$ and $[u_z + h(u - T_1)](L) = 0$. Assume that K, h, T_0, T_1 are all positive constants.
Solution. $u(x, y, z) = U(z) = \frac{T_1(1+hz)+T_0[1+h(L-z)]}{2+hL}$.

Question 2. Solve the initial-value problem $u_t = Ku_{zz}(K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
Solution. $u(z, t) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi z}{L} \exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$.

Question 3. Solve the initial-value problem $u_t = Ku_{zz}(K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u_z(0, t) = u_z(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
Solution. $u(z, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi z/L]}{(2n-1)^2} \exp\left[-\frac{(2n-1)^2\pi^2 Kt}{L^2}\right]$.

Question 4. Let $\varphi_1 = 1, \varphi_2 = x, \varphi_3 = x^2$ on the interval $0 \le x \le 1$. Compute the following quantities

1) $\langle \varphi_1, \varphi_2 \rangle$,
2) $\langle \varphi_1, \varphi_3 \rangle$,
3) $\|\varphi_1 - \varphi_2\|^2$,
4) $\|\varphi_1 + 3\varphi_2\|^2$.

Solution. 1) 1/2, 2) 1/3, 3) 1/3, 4) 7.

Question 5. Check if the following operator is symmetric on its domain with respect to given inner product.

1) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
2) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
3) $A = \frac{d}{dx}$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)xdx$.

Solution. 1) True, 2) False, 3) False. For 1) try integration by parts as in the class. For 2), 3) try to find counter-examples.

Question 6. Convert the following ODE into Sturm-Liouville form and write the $s(x)$, $\rho(x)$ and $q(x)$ functions.

1) $y'' + 2xy' + \lambda y = 0$.
2) $x^2y'' + xy' + (\lambda x^2 - 1)y = 0$.
3) $y'' + \frac{1}{x}y' + \lambda y = 0$.

Solution. 1) $(e^{x^2}y')' + \lambda e^{x^2}y = 0$, $s(x) = e^{x^2}$, $\rho(x) = e^{x^2}$ and $q(x) = 0$. 2) $(xy')' + (\lambda x - \frac{1}{x})y = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = \frac{1}{x}$. 3) $(xy')' + \lambda xy = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = 0$.