hw2

Question 1. Compute the Fourier series of $f(x) = x^2, -L < x < L$.
Solution: $\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{L}$.

Sol Since it is an even function there are only cos terms in the Fourier series.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos_n$$

$$= \frac{1}{2L}\int_{-L}^{L} x^2 dx + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L} dx\right)\cos\frac{n\pi x}{L}$$

$$= \frac{1}{3}L^2 + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L x^2\cos\frac{n\pi x}{L}dx\right)\cos\frac{n\pi x}{L}$$

$$\int_0^L x^2\cos\frac{n\pi x}{L}dx = \left[x^2\left(\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right)\right]_0^L - \frac{2L}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx$$

$$\int_0^L x\sin\frac{n\pi x}{L}dx = \left[x\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx$$

$$= \frac{L^2}{(n\pi)^2}\int_0^{n\pi}\cos t\, dt = 0$$

$$\Rightarrow \int_0^L x\sin\frac{n\pi x}{L}dx = \frac{-L^2}{n\pi}(-1)^n$$

$$\Rightarrow \int_0^L x^2\cos\frac{n\pi x}{L}dx = 0 - \frac{2L}{n\pi}\left(\frac{-L^2}{n\pi}(-1)^n\right) = \frac{2L^3}{n^2\pi^2}(-1)^n$$

$$\Rightarrow f(x) = \frac{1}{3}L^2 + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{L}$$

Question 2. Compute the Fourier series of $f(x) = e^x, -L < x < L$.

Solution: $\frac{\sinh L}{L}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n\pi x/L) - (n\pi/L)\sin(n\pi x/L)}{1+(n\pi/L)^2}\right]$.

Sol $f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos_n + B_n \sin_n)$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} e^x\,dx = \frac{1}{2L}(e^L - e^{-L}) = \frac{\sinh L}{L}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} e^x \cos\frac{n\pi x}{L}\,dx$$

$$I_{\cos} = \int_{-L}^{L} \overset{f}{e^x}\, \overset{g'}{\cos\frac{n\pi x}{L}}\,dx = \underbrace{\left[\frac{L}{n\pi}\overset{g}{\sin\frac{n\pi x}{L}}\, \overset{f}{e^x}\right]_{-L}^{L}}_{=0} - \frac{L}{n\pi} I_{\sin}$$

$$I_{\sin} = \int_{0}^{L} \overset{f}{e^x}\, \overset{g'}{\sin\frac{n\pi x}{L}}\,dx = \underbrace{\left[\frac{-L}{n\pi}\overset{g}{\cos\frac{n\pi x}{L}}\, \overset{f}{e^x}\right]_{-L}^{L}}_{= \frac{-L}{n\pi}(-1)^n(e^L - e^{-L})} + \frac{L}{n\pi} I_{\cos}$$

$$\Rightarrow I_{\sin} = \frac{-L}{n\pi}(-1)^n\, 2\sinh(L) + \frac{L}{n\pi}\left(-\frac{L}{n\pi}\right) I_{\sin}$$

$$\Rightarrow I_{\sin} = \frac{\frac{-L}{n\pi}(-1)^n\, 2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}$$

$$\Rightarrow I_{\cos} = -\frac{L}{n\pi} I_{\sin} = \frac{\left(\frac{L}{n\pi}\right)^2(-1)^n\, 2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}$$

$$\Rightarrow f(x) = \frac{\sinh L}{L} + \sum_{n=1}^{\infty}\left(\frac{1}{L}\,\frac{\left(\frac{L}{n\pi}\right)^2(-1)^n\, 2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}\cos_n + \frac{1}{L}\,\frac{\frac{-L}{n\pi}(-1)^n\, 2\sinh L}{1+\left(\frac{L}{n\pi}\right)^2}\sin_n\right)$$

$$= \frac{\sinh L}{L}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\left(\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L} - \frac{L}{n\pi}\sin\frac{n\pi x}{L}}{1+\left(\frac{L}{n\pi}\right)^2}\right)$$

(multiplying $\left(\frac{n\pi}{L}\right)^2$ on nominator & denominator)

$$= \frac{\sinh L}{L}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\cos\frac{n\pi x}{L} - \frac{n\pi}{L}\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2 + 1}\right)$$

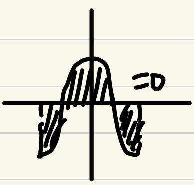
Question 3. Compute the Fourier series of $f(x) = \sin^2 2x, -\pi < x < \pi$.
Solution: $\frac{1}{2} - \frac{1}{2}\cos 4x$.

Sol $f(x) = \sin^2 2x = \frac{1-\cos 4x}{2}$, $-\pi < x < \pi$ is an even function, thus only has even terms.

$\Rightarrow f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos n$ by Fourier cosine series on $(-\pi, \pi)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\cos 4x}{2}\,dx = \frac{1}{16\pi}\int_{-\pi}^{\pi}(1-\cos 4x)\,d(4x)$$
$$= \frac{1}{16\pi}\left[4x - \sin 4x\right]_{-\pi}^{\pi} = \frac{8\pi}{16\pi} = \frac{1}{2}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos 4x - 1}{2}\cos nx\,dx$$
$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 4x\cos nx\,dx}_{} - \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\cos nx\,dx}_{=0}$$
$$= \int_{-1}^{1}\cos 4\pi x\cos n\pi x\,d(\pi x)$$
$= 0$ unless $n=4$, by orthogonality on $[-\pi, \pi]$



$$\int_{-\pi}^{\pi}\cos^2 4x\,dx = \int_{-\pi}^{\pi}\frac{\cos 8x - 1}{2}\,dx$$
$$= \underbrace{\frac{1}{16}\int_{-\pi}^{\pi}\cos t\,dt}_{=0} - \frac{1}{2}\int_{-\pi}^{\pi} 1\,dx = -\frac{1}{2}(2\pi)$$

$\Rightarrow A_n = -\frac{1}{2}$ for $n=4$ and $A_n = 0$ for $n \neq 4$

$\Rightarrow f(x) = \frac{1}{2} - \frac{1}{2}\cos\frac{4\pi}{\pi}x = \frac{1}{2} - \frac{1}{2}\cos 4x = \frac{1-\cos 4x}{2}$

is the Fourier series of f.

(which is itself since f is trigonometric)

Question 4. Prove the orthogonality relations

1) $\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0, & n \neq m \text{ or } n = m = 0, \\ L, & n = m \neq 0. \end{cases}$

2) $\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0$ for all n, m.

Solution: Recall $\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$ and $\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$. From these relations, we can derive the trigonometric identities: $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$, $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$, and $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$. By using these identities, we can carry out the integrals in the orthogonality relations.

(1) Pf ① let $n \neq m \in \mathbb{N}$ be arbitrary

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \int_{-L}^{L}\frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx$$

by product-to-sum identity

$$= \frac{1}{2}\frac{L}{(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi}\cos\frac{(n-m)\pi x}{L}\,d\frac{(n-m)\pi x}{L} - \frac{1}{2}\frac{L}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi}\cos\frac{(n+m)\pi x}{L}\,d\frac{(n+m)\pi x}{2}$$

$$= \frac{1}{2(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi}\cos t\,dt - \frac{1}{2}\frac{L}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi}\cos t\,dt$$

Since the upper and lower limit is integer multiples of π, the integrals evaluate to 0

Thus $\int_{-L}^{L}\sin_n\sin_m dx = 0$

② If $n = m \neq 0 \Rightarrow \int_{-L}^{L}\sin_n\sin_m dx = \int_{-L}^{L}\sin_n^2 dx = \frac{1}{2}\int_{-L}^{L}(1-\cos\frac{2n\pi x}{L})dx = \frac{1}{2}\int_{-L}^{L}1\,dx = \frac{2L}{2} = L$

③ If $n = m = 0 \Rightarrow \int_{-L}^{L}\sin_n\sin_m dx = \int_{-L}^{L}0\,dx = 0$

This finishes the proof that $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0, & n\neq m \text{ or } n=m=0 \\ L, & n=m\neq 0\end{cases}$ □

(2) Pf let $n, m \in \mathbb{N}\cup\{0\}$ be arbitrary

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \int_{-L}^{L}\frac{1}{2}\left(\sin\frac{(n-m)\pi x}{L} - \sin\frac{(n+m)\pi x}{L}\right)dx$$

by product-to-sum identity

$$= \frac{1}{2}\int_{-L}^{L} \sin\frac{(n-m)\pi}{L}x\,dx - \frac{1}{2}\int_{-L}^{L} \sin\frac{(n+m)\pi x}{L}\,dx$$

both functions are odd, so the intergrals from $-L$ to L evaluate to 0.

Thus for all $n, m \in \mathbb{N} \cup \{0\}$. $\int_{-L}^{L} \sin_n \cos_m \,dx = 0$ $\square$

Question 5. Which of the following functions are even, odd, or neither? Explain the reason.

1) $f(x) = x^3 - 3x$,
2) $f(x) = x^2 + 4$,
3) $f(x) = \cos 3x$,
4) $f(x) = x^3 - 3x^2$.

Solution: 1): odd; 2), 3): even; 4): neither.

(1) $f(-x) = -x^3 + 3x = -f(-x) \Rightarrow$ odd

(2) $f(-x) = x^2 + 4 = f(x) \Rightarrow$ even

(3) $f(-x) = \cos(-3x) = \cos 3x = f(x) \Rightarrow$ even

(4) $f(-x) = -x^3 - 3x^2 \neq f(x)$ on domain (equal iff $x = 0$) $\Rightarrow$ neither

Question 6.

1) Find the Fourier sine series for $f(x) = e^x, 0 < x < L$.
2) Find the Fourier cosine series for $f(x) = e^x, 0 < x < L$.

Solution: 1) We obtain the Fourier sine series $\frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\left[\frac{1-e^L(-1)^n}{1+(n\pi/L)^2}\right]\sin\frac{n\pi x}{L}$ by either directly apply the B_n formula or multiply e^x by $\sin\frac{n\pi x}{L}$ and then apply the orthogonality 2) We obtain the Fourier cosine series $\frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^L - 1}{1+(n\pi/L)^2}\right]\cos\frac{n\pi x}{L}$ by either directly apply the A_n formula or multiply e^x by $\cos\frac{n\pi x}{L}$ and then apply the orthogonality.

Sol $\quad I_{\cos_n} = \int_0^L e^x \cos\frac{n\pi x}{L}\,dx = \underbrace{\left[\frac{L}{n\pi}\sin\frac{n\pi x}{L} e^x\right]_0^L}_{=0} - \frac{L}{n\pi} I_{\sin_n}$

$$I_{\sin_n} = \int_0^L e^x \sin\frac{n\pi x}{L}\,dx = \underbrace{\left[\frac{-L}{n\pi}\cos\frac{n\pi x}{L} e^x\right]_0^L}_{= \frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi}} + \frac{L}{n\pi} I_{\cos_n}$$

$$\Rightarrow I_{\sin_n} = \frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi} + \frac{L}{n\pi}\left(-\frac{L}{n\pi}\right) I_{\sin_n}$$

$$\Rightarrow I_{\sin_n} = \frac{\frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi}}{1 + \left(\frac{L}{n\pi}\right)^2}$$

$$\Rightarrow I_{\cos_n} = -\frac{L}{n\pi} I_{\sin_n} = \frac{\left(\frac{L}{n\pi}\right)^2 (-1)^n e^L - \left(\frac{L}{n\pi}\right)^2}{1+\left(\frac{L}{n\pi}\right)^2}$$

$$A_0 = \frac{1}{L}\int_0^L e^x\,dx = \frac{e^L-1}{L}$$

$\Rightarrow$ Fourier sine series $\underbrace{f_{\sin}(x)} = A_0 + \sum_{n=1}^{\infty} A_n \cos_n$

$$= \frac{e^L-1}{L} + \sum_{n=1}^{\infty} \frac{2}{L} I_{\cos_n} \cos\frac{n\pi x}{L}$$

$$= \frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \frac{\left(\frac{L}{n\pi}\right)^2 (-1)^n e^L - \left(\frac{L}{n\pi}\right)^2}{1+\left(\frac{L}{n\pi}\right)^2} \cos\frac{n\pi x}{L}$$

$$\underline{= \frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \frac{(-1)^n e^L - 1}{1+\left(\frac{L}{n\pi}\right)^2} \cos\frac{n\pi x}{L}} \quad , x \in (0, L)$$

Fourier cosine series $\underbrace{f_{\cos}(x)} = \sum_{n=1}^{\infty} B_n \sin_n$

$$= \frac{2}{L}\sum_{n=1}^{\infty} I_{\sin_n} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}\sum_{n=1}^{\infty} \frac{-\frac{n\pi}{L}(-1)^n e^L + \frac{n\pi}{L}}{\left(\frac{n\pi}{L}\right)^2+1}$$

$$\underline{= \frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\, \frac{1-(-1)^n e^L}{\left(\frac{n\pi}{L}\right)^2+1}} \quad , x \in (0, L)$$