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Math 454: Boundary Value Problems for Partial Differential Equations
Due: September 3, 2024

Homework 1

Question 1. Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.

1) $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$
2) $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$
3) $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$
4) $u_{xx} + xu_{yy} = 0$

Solution: 1) hyperbolic, 2) elliptic, 3) parabolic, 4) elliptic if $x > 0$ and hyperbolic if $x < 0$.

Question 2. Prove the following claims.

1) Prove the formulas $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
2) Use Euler's formula to prove $\sin(x+y) = \sin x \cos y+\cos x \sin y$ and $\cos(x+y) = \cos x \cos y-\sin x \sin y$ similarly.
3) Use Euler's formula to prove that all the real functions in $C_+e^{ix} + C_-e^{-ix}$ is of the form $A \cos x + B \sin x$.

Solution: 1) $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x-e^{-x}}{2}\frac{e^y+e^{-y}}{2} + \frac{e^x+e^{-x}}{2}\frac{e^y-e^{-y}}{2} = \sinh(x + y)$. $\cosh x \cosh y + \sinh x \sinh y = \frac{e^x+e^{-x}}{2}\frac{e^y+e^{-y}}{2} + \frac{e^x-e^2}{2}\frac{e^y-e^{-y}}{2} = \cosh(x + y)$.

2) $\sin x \cos y + \cos x \sin y = \frac{e^{ix}-e^{-ix}}{2i}\frac{e^{iy}+e^{-iy}}{2} + \frac{e^{ix}+e^{-ix}}{2}\frac{e^{iy}-e^{-iy}}{2i} = \sin(x + y)$. $\cos x \cos y + \sin x \sin y = \frac{e^{ix}+e^{-ix}}{2}\frac{e^{iy}+e^{-iy}}{2} + \frac{e^{ix}-e^{-ix}}{2i}\frac{e^{iy}-e^{-iy}}{2i} = \cos(x + y)$.

Question 3. Find the general solutions.

1) $y' = ky(1 - y)$,
2) $xy' + 4y = x^2$,
3) $y'' + 4y' + 4y = 0$,
4) $y'' + 2y' - 15y = 0$.

c 1) $y(x) = 1/\left(1 + Ce^{-kx}\right)$, 2) $y(x) = x^2/6 + C/x^4$, 3) $y(x) = C_1e^{-2x} + C_2xe^{-2x}$, and 4) $y(x) = C_1e^{3x} + C_2e^{-5x}$.

Question 4. Find the separated equations satisfied by $X(x), Y(y)$ for the following PDEs.

1) $u_{xx} - 2u_{yy} = 0$,
2) $u_{xx} + u_{yy} + 2u_x = 0$,
3) $x^2u_{xx} - 2yu_y = 0$,
4) $u_{xx} + u_x + u_y - u = 0$.

Solution: 1) $X'' - 2\lambda X = 0, Y'' - \lambda Y = 0$, 2) $X'' + 2X' + \lambda X = 0, Y'' - \lambda Y = 0$, 3) $x^2X'' - \lambda X = 0, 2yY' - \lambda Y = 0$ 4) $X'' + X' - \lambda X = 0, Y' + (\lambda - 1)Y = 0$.

Question 5. Find the separated solutions of $u_{xx} + yu_y + u = 0$.
Solution:

$$u(x,y) = \begin{cases} \left(A_1e^{x\sqrt{\lambda}} + A_2e^{-x\sqrt{\lambda}}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda > 0, \\ (A_1 + A_2x)\left(1/|y|\right) & \text{for } \lambda = 0, \\ \left(A_1 \cos x\sqrt{-\lambda} + A_2 \sin x\sqrt{-\lambda}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda < 0. \end{cases}$$

Question 6. Find the separated solutions $u(x, y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L, y > 0$, that satisfy the boundary conditions $u(0, y) = 0, u(L, y) = 0$ and the boundedness condition $|u(x, y)| \leq M$ for $y > 0$, where M is a constant independent of (x, y).
Solution:

$$u(x, y) = C \sin \frac{n\pi x}{L} e^{-n\pi y/L} \quad (n = 1, 2, \ldots)$$

Question 7. Find the separated solutions $u(x, t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L, t > 0$, that satisfy the boundary conditions $u(0, t) = 0, u(L, t) = 0$.
Solution:

$$u(x, t) = \begin{cases} \left(A_1 e^{kx} + A_2 e^{-kx}\right) e^{k^2 t} & \text{for } \lambda = k^2, k > 0, \\ A_1 x + A_2 & \text{for } \lambda = 0, \\ \left(A_1 e^{ilx} + A_2 e^{-ilx}\right) e^{-l^2 t} & \text{for } \lambda = -l^2, l > 0. \end{cases}$$

The case $\lambda < 0$ satisfies the boundary conditions. We obtain

$$u(x, t) = C \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 t} \quad (n = 1, 2, \ldots).$$