Recall last time:

Parseval's thm: $\|f\|^2=\|A_0\|^2+\sum_{n=1}^{\infty}\|A_n\cos_n+B_n\sin_n\|^2=A_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(A_n^2+B_n^2)$

$\downarrow$

$\frac{1}{2L}\int_{-L}^{L}|f|^2$

mean square error: $\sigma_N^2=\|f-f_N\|=\frac{1}{2L}\int_{-L}^{L}|f-f_N|^2$

$\Rightarrow \sigma_N^2=\frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2+B_n^2)$ by Parseval's thm

Today:

(1) Parseval's Thm for complex Fourier series $f(x)=\sum_{n\in\mathbb{Z}}\alpha_n e^{\frac{in\pi x}{L}}$

$\|f\|^2=\sum_{n\in\mathbb{N}}\|\alpha_n\|^2$ (by inner product $\langle f,g\rangle=\frac{1}{2L}\int_{-L}^{L}f\bar{g}$)

即 $\frac{1}{2L}\int_{-L}^{L}|f|^2=\sum_{n\in\mathbb{Z}}|\alpha_n|^2$

计算 Pf $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx=\frac{1}{2L}\int_{-L}^{L}f\bar{f}=\frac{1}{2L}\int_{-L}^{L}\left(\sum_n\alpha_n \underset{e^{\frac{in\pi x}{L}}}{e_n}\right)\left(\sum_m\bar{\alpha}_m e_{-m}\right)$

$=\frac{1}{2L}\sum_{n,m}\alpha_n\bar{\alpha}_m\int_{-L}^{L}e_ne_{-m}$

$=\sum_{n,m}\alpha_n\bar{\alpha}_m\delta_{nm}=\sum_{n\in\mathbb{Z}}|\alpha_n|^2$

Thm Parseval for cosine, sine

cos: $f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos_n$, $\|f\|^2=\|A_0\|^2+\sum_{n=1}^{\infty}\|A_n\|^2$

即 $\frac{1}{2L}\int_{-L}^{L}|f|^2=A_0^2+\frac{1}{2}\sum_{n=1}^{\infty}A_n^2$

sin: $f(x)=\sum_{n=1}^{\infty}B_n\sin_n$, $\frac{1}{2L}\int_{-L}^{L}|f|^2=\frac{1}{2}\sum_{n=1}^{\infty}B_n^2$ (computational pf in hw)

today (2): PDE in rectangular coordinates

ex 一个 ODE boundary value problem

$u''=1$, $x\in[0,1]$

$u(0)=0$, $u(1)=0$

Sol 1. General sol: $u(x)=\frac{1}{2}x^2+Ax+B$

2. Match with B.C.

$u(0)=0\Rightarrow B=0$

$u(1)=0\Rightarrow A=-\frac{1}{2}$

$\Rightarrow u(x)=\frac{1}{2}x(x-1)$

ODE 的 BV problem 即确定常数 (范围)

Now for PDE:

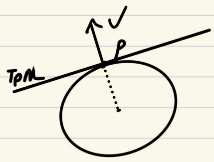
Def Normal vector

for $p\in M$ a manifold,

p 的 normal vector 是和

p 在 M 上的 tangent space T_pM

垂直的 vector n



note: 如果 M embedded into $\mathbb{R}^d$, $\dim(\partial M)=d-1$

take $p\in\partial M$, $\dim(T_pM)=\dim(M)$

令 ∂M 在 p 附近的表示方程为 $F(x_1,\dots,x_d)=0\Rightarrow$

$p\in U=\left\{\begin{bmatrix}x_1\\ \vdots\\ x_d\end{bmatrix}:F(x_1,\dots,x_d)=0\right\}\subseteq\partial M$

$\Rightarrow$ 可取 $\vec{n_p}=\nabla F(p)$ 得到 normal vector $\vec{n_p}$

Specially, for hyperplane $U=\left\{\begin{bmatrix}x_1\\ \vdots\\ x_d\end{bmatrix}:a_1x_1+\dots+a_dx_d=0\right\}$

$\forall p\in U$, $\vec{n_p}=\nabla F(p)=\begin{bmatrix}a_1\\ \vdots\\ a_d\end{bmatrix}$

$\left(\dim(\vec{n_p})+\dim(T_pM)=\dim(\mathbb{R}^d)=d\right)$

Def Boundary condition

Given PDE $F[u]=0$ defined on a domain R

本课 only consider 以下三种 boundary conditions:

1. Dirichlet boundary condition

已知 $u=g(x)$, $x\in\partial R$ (边界点的值)

2. Neumann boundary condition

已知 $n_x\nabla u(x)=g(x)$, $x\in\partial R$ (边界点的 normal derivative)

3. Robin boundary condition

(以上两种的一个 linear comb)

已知 $a(x)u+b(x)n_x\nabla u(x)=g(x)$, $x\in\partial R$

note: $n_x\nabla u=n_1\frac{\partial}{\partial x_1}u+\dots+n_d\frac{\partial}{\partial x_d}u$

$=D_{n_x}u$

$n\nabla u$ 就等于 u 在 n 方向上的 directional derivative.

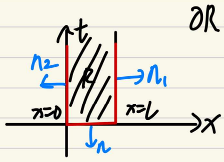
ex heat equation

$u_t=Ku_{xx}$, defined on $R=\{(x,t)\mid x\in(0,L),t\in(0,\infty)\}$

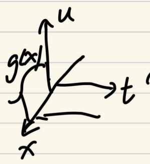
$\partial R=\{(0,t)\mid t>0\}\cup\{(L,t)\mid t>0\}\cup\{(x,0)\mid x\in(0,L)\}$

如果我们在这上面 impose 个 Robin BC

这上面 impose 个 Dirichlet BC



则这个 PDE BVP 的形式即:



$$\begin{cases}\partial_t u=K\partial_{xx}u,\ t>0,\ 0<x<L & \text{(PDE)}\\ u(x,0)=g(x),\ x\in(0,L),\ t=0 & \text{(Dirichlet's BC)}\\ a(x,t)u+b(x,t)\vec{n}\cdot\nabla u=g(x,t),\ x=0,L,\ t>0 & \text{(Robin's BC)}\end{cases}$$

$\vec{n}\cdot\nabla u$: $D_n u(x,t)$

note: $\nabla u=\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{pmatrix}$

On $x=0$, 可取 $\vec{n_1}=\begin{pmatrix}-1\\0\end{pmatrix}\Rightarrow$ ③转为 $a(0,t)u-b(0,t)\frac{\partial}{\partial x}u=g(0,t)$

On $x=L$, 可取 $\vec{n_2}=\begin{pmatrix}1\\0\end{pmatrix}\Rightarrow$ ③转为 $a(L,t)u+b(L,t)\frac{\partial}{\partial x}u=g(0,t)$

用 $a(t),b(t),g(t)$ 来表示 $a(0,t)$, $b(0,t)$ 和 $g(0,t)$

$\tilde{a}(t),\tilde{b}(t),\tilde{g}(t)$ 来表示 $a(L,t)$, $b(L,t)$ 和 $g(L,t)$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}u = K\frac{\partial^2}{\partial x^2}u, \\ a(t)u - b(t)\frac{\partial}{\partial x}u = g(t), \\ \tilde{a}(t)u + \tilde{b}(t)\frac{\partial}{\partial x}u = \tilde{g}(t), \\ u = f(x), \end{cases}$$

Def 如果 $a(t)=a$, $b(t)=b$, $\tilde{a}(t)=\tilde{a}$, $\tilde{b}(t)=\tilde{b}$ for some consts $a, b, \tilde{a}, \tilde{b}$

且 $\tilde{g}(t)=g(t)=0$

则称这个PDE BV方程组 homogeneous

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}u = K\frac{\partial^2}{\partial x^2}u, \\ au - b\frac{\partial}{\partial x}u = 0, \\ \tilde{a}u + \tilde{b}\frac{\partial}{\partial x}u = 0, \\ u = f(x), \end{cases}$$

对于 homogeneous 的PDE BV方程组

可取某个 α, β 使得 $a=\cos\alpha$, $b=L\sin\alpha$, $\tilde{a}=\cos\beta$, $\tilde{b}=L\sin\beta$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}u = K\frac{\partial^2}{\partial x^2}u, \\ u\cos\alpha - \frac{\partial}{\partial x}u\, L\sin\alpha = 0, \\ u\cos\beta + \frac{\partial}{\partial x}u\, L\sin\beta = 0, \\ u = f(x), \end{cases}$$

ex simple case where $\alpha=\beta=0 \Rightarrow u(0,t)=u(L,t)=0, \forall t.$