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Math 454: Boundary Value Problems for Partial Differential Equations
Due: September 10, 2024

Homework 2

Question 1. Compute the Fourier series of $f(x) = x^2, -L < x < L$.
Solution: $\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{L}$.

Question 2. Compute the Fourier series of $f(x) = e^x, -L < x < L$.
Solution: $\frac{\sinh L}{L}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n\pi x/L) - (n\pi/L)\sin(n\pi x/L)}{1+(n\pi/L)^2}\right]$.

Question 3. Compute the Fourier series of $f(x) = \sin^2 2x, -\pi < x < \pi$.
Solution: $\frac{1}{2} - \frac{1}{2}\cos 4x$.

Question 4. Prove the orthogonality relations

1) $\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0, & n \neq m \ or\ n = m = 0, \\ L, & n = m \neq 0. \end{cases}$
2) $\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0$ for all n, m.

Solution: Recall $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ and $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. From these relations, we can derive the trigonometric identities: $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)], \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$, and $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$. By using these identities, we can carry out the integrals in the orthogonality relations.

Question 5. Which of the following functions are even, odd, or neither? Explain the reason.

1) $f(x) = x^3 - 3x$,
2) $f(x) = x^2 + 4$,
3) $f(x) = \cos 3x$,
4) $f(x) = x^3 - 3x^2$.

Solution: 1): odd; 2), 3): even; 4): neither.

Question 6.

1) Find the Fourier sine series for $f(x) = e^x, 0 < x < L$.
2) Find the Fourier cosine series for $f(x) = e^x, 0 < x < L$.

Solution: 1) We obtain the Fourier sine series $\frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\left[\frac{1-e^L(-1)^n}{1+(n\pi/L)^2}\right]\sin\frac{n\pi x}{L}$ by either directly apply the B_n formula or multiply e^x by $\sin\frac{n\pi x}{L}$ and then apply the orthogonality 2) We obtain the Fourier cosine series $\frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^L - 1}{1+(n\pi/L)^2}\right]\cos\frac{n\pi x}{L}$ by either directly apply the A_n formula or multiply e^x by $\cos\frac{n\pi x}{L}$ and then apply the orthogonality.