Last time:

Convergence thm

For piecewise smooth $f:[-L,L]\to\mathbb{R}$

$$\lim_{N\to\infty} f_N(x) = \frac{1}{2}[f(x-)+f(x+)],\quad x\in(-L,L)$$

$$\lim_{N\to\infty} f_N(\pm L) = \frac{1}{2}[f(L)+f(-L)],\quad x=\pm L$$

Today (1): Parseval's Thm (类比 Pythagorean Thm)

Thm Generalized Pythagorean Thm

Let $(V,\langle\cdot,\cdot\rangle)$ be any inner product space

Let $v_1,v_2,\dots,v_n\in V$ 两两orthogonal (i.e. $\forall i\neq j,\ \langle v_i,v_j\rangle=0$)

$$\Rightarrow \left\|\sum_{i=1}^n v_i\right\|^2=\sum_{i=1}^n\|v_i\|^2$$

Thm Parseval's Thm

(Generalized pythagorean Thm 在

if f 在 $(-L,L)$ 上 pw smooth

Fourier expand $f=A_0+\sum_{n=1}^\infty(A_n\cos_n+B_n\sin_n)$

$$\Rightarrow \boxed{\frac{1}{2L}\int_{-L}^{L}f^2 = A_0^2+\frac{1}{2}\sum_{n=1}^\infty(A_n^2+B_n^2)}$$

def: mean square

即: take $\langle f,g\rangle=\frac{1}{2L}\int_{-L}^{L}fg$ on $L^2([-L,L])$

$$\Rightarrow \|f\|^2=\|A_0\|^2+\sum_n\|A_n\cos_n+B_n\sin_n\|^2$$

Pf $\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx$

$$=\frac{1}{2L}\int_{-L}^{L}\left(\sum_{n=0}^\infty(A_n\cos_n+B_n\sin_n)\right)\left(\sum_{m=0}^\infty(A_m\cos_m+B_m\sin_m)\right)dx$$

$$=\frac{1}{2L}\int_{-L}^{L}\sum_{n,m=0}^\infty\Big(A_nA_m\cos_n\cos_m+\underbrace{A_nB_m\cos_n\sin_m+B_nA_m\sin_n\cos_m}_{=0\text{ by orthogonality of }\sin/\cos}+B_nB_m\sin_n\sin_m\Big)$$

note
$\int_{-L}^{L}A_nA_m\cos_n\cos_m=0$ if $n\neq m$, $\int_{-L}^{L}B_nB_m\sin_n\sin_m=0$ if $n\neq m$

$$\text{So it}=\frac{1}{2L}\sum_{n=0}^\infty\int_{-L}^{L}A_n^2\cos^2\frac{n\pi x}{L}dx+\int_{-L}^{L}B_n^2\sin^2\frac{n\pi x}{L}dx$$

$$=\frac{1}{2L}\left(\int_{-L}^{L}A_0^2dx+\sum_{n=1}^\infty\Big(A_n^2\underbrace{\int_{-L}^{L}\cos_n^2dx}_{L}+B_n^2\underbrace{\int_{-L}^{L}\sin_n^2dx}_{L}\Big)\right)$$

$$=A_0^2+\frac{1}{2}\sum_{n=1}^\infty(A_n^2+B_n^2)$$

Def Mean Square Error

For $f_N(x)=A_0+\sum_{n=1}^N(A_n\cos_n+B_n\sin_n)$

$f(x)=A_0+\sum_{n=1}^\infty(A_n\cos_n+B_n\sin_n)$

Def Mean square error

$$\sigma_N^2=\frac{1}{2L}\int_{-L}^{L}|f(x)-f_N(x)|^2dx$$

即 $\|f(x)-f_N(x)\|^2$ on $L^2([-L,L])$

Prop. $\sigma_N^2=\frac{1}{2}\sum_{n=N+1}^\infty(A_n^2+B_n^2)$

Pf $f(x)-f_N(x)=\sum_{n=N+1}^\infty(A_n\cos_n+B_n\sin_n)$

(显然) By Parseval's, $\|f(x)-f_N(x)\|^2=\left\|\sum_{n=N+1}^\infty(A_n\cos_n+B_n\sin_n)\right\|^2$

$$=\frac{1}{2}\sum_{n=N+1}^\infty(A_n^2+B_n^2)\quad\square$$

虽然 $f(x)$ uncomputable

但 $f_N(x)$ for $N\in\mathbb{N}$ 是 computable

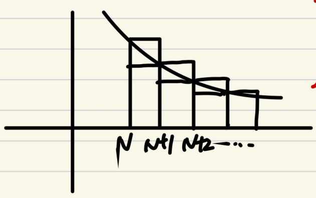
Question How large should N be to get $\sigma_N=0.001$?

ex Find σ_N^2 for $f(x)=x$

Sol $f(x)=\sum_{n=1}^\infty B_n\sin\frac{n\pi x}{L}$, $B_n=\frac{2L}{n\pi}(-1)^{n+1}$

$$\sigma_N^2=\frac{1}{2}\sum_{n=N+1}^\infty B_n^2=\frac{2L^2}{\pi^2}\sum_{n=N+1}^\infty\frac{1}{n^2}$$

Integral Test



For decreasing f:

$$\int_{N+1}^\infty f\le\sum_{n=N+1}^\infty f(n)\le\int_N^\infty f$$

Then
$$\underbrace{\int_{N+1}^\infty\frac{1}{x^2}dx}_{=\frac{1}{N+1}}\le\sum_{n=N+1}^\infty\frac{1}{n^2}\le\int_{N+1}^\infty\frac{1}{(x-1)^2}dx=\underbrace{\int_N^\infty\frac{1}{x^2}dx}_{=\frac{1}{N}}$$

$$\Rightarrow\sigma_N^2=\frac{2L^2}{\pi^2}\frac{1}{N}\left[1+O\left(\frac{1}{N}\right)\right]=O\left(\frac{1}{N}\right),\ N\to\infty$$

ex2 σ_{2N}^2 for $f(x)=|x|$, $-L<x<L$

$\int_{N+1}^\infty\frac{1}{(2x-3)^4}$, $\int_{N+1}^\infty\frac{1}{(2x-1)^4}$ 中间

$$\sigma_{2N}^2=\frac{1}{2}\sum_{n=2N+1}^\infty A_n^2=\frac{1}{2}\sum_{m=N+1}^\infty A_{2m-1}^2=\frac{8L^2}{\pi^4}\sum_{m=2N-1}^\infty\frac{1}{(2m-1)^4}$$

$$=\frac{L^2}{6\pi^4N^3}+O(N^{-4})=O(N^{-3})$$