Practice Exam 1
QUESTIONS PACKET
EECS 203
Winter 2023

Name (ALL CAPS): ______________________________

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*****MAKE SURE YOU HAVE PROBLEMS 1 - 17 IN THIS BOOKLET.*****

General Instructions

You have 120 minutes to complete this exam. You should have two exam packets.

- **Questions Packet:** Contains ALL of the questions for this exam, worth 100 points total. There are 9 Multiple Choice questions (4 points each), 4 Short Response questions (26 points total), and 4 Free Response questions (38 points total). You may do scratch work on this part of the exam, but only work in the Answers Packet will be graded.
- **Answers Packet:** Write all of your answers in the Answers Packet, including your answers to multiple choice questions. **For free response questions, you must show your work! Answers alone will receive little or no credit.**
- You may bring **one** 8.5" by 11" note sheet, front and back, created by you.
- You may **NOT** use any other sources of information, including but not limited to electronic devices (including calculators), textbooks, or notes.
- After you complete the exam, sign the Honor Code Pledge on the front of the Answers Packet.
- You must turn in both parts of this exam.
- **You are not to discuss the exam until the solutions are published.**

Part A1: Single Answer Multiple Choice

Problem 1. (4 points)

Alia wants to give out books to 4 students. What is the minimum number of books she needs to hand out to guarantee that at least 1 student has at least 4 books?

(a) 13

(b) 14

(c) 15

(d) 16

(e) 17

Solution: a
If we had 12 books, we could give 3 to each student, however, thanks to the pigeonhole principle, if we had 13 books, at least one of the students would get at least $\lceil 13/4 \rceil = 4$.

Problem 2. (4 points)

Define $f : \mathbb{Z} \to \mathbb{Z}$ to be $f(x) = \frac{x}{2} + \frac{x^2}{2}$. Which of the following is true?

(a) f is not a function since $\frac{x}{2}$ is not an integer

(b) f is a bijective function

(c) f is one-to-one but not onto

(d) f is onto but not one-to-one

(e) f is neither one-to-one nor onto

Solution: (e)

f **is a function** since every input still gives an integer output. For even input $x \in \mathbb{Z}$, both $\frac{x}{2}$ and $\frac{x^2}{2}$ are integers. For odd $x \in \mathbb{Z}$, each fraction is 1/2 more than an integer, which combines to remain an integer. (for proof, we have $\frac{2k+1}{2} + \frac{(2k+1)^2}{2} = \frac{2k+1+4k^2+4k+1}{2} = \frac{4k^2+6k+2}{2} = 2k^2 + 3k + 1$ which is an integer.)
f is **not one-to-one** since f(-1) = 0 and f(0) = 0, among other counterexamples.
f is **not onto** since there is nothing that maps to the integer 2 (there are many other examples).

Problem 3. (4 points)

Matt has a very strict diet and only eats fish, rice, peas or beans on a given day. He has the following constraints:

- If he ate fish today, he must eat beans tomorrow.
- He will not eat rice two days in a row.

What is the recurrence relation for $M(n)$, where $M(n)$ is the number choices of meals Matt could have eaten over n days?

(a) $M(n) = 3M(n-1) + M(n-2)$

(b) $M(n) = 2M(n-1) + 2M(n-2)$

(c) $M(n) = 2M(n-1) + 3M(n-2) + M(n-3)$

(d) $M(n) = 2M(n-1) + M(n-2) + M(n-3)$

(e) $M(n) = 2M(n-1) + 3M(n-2)$

Solution: (c)
Consider the restrictions that follow after Matt eats each item on a given day.

- Case 1, Matt eats fish: Given the first restriction, Matt must eat beans tomorrow. Then, we can treat the day following that as the n-th day. Thus meaning there are $M(n-2)$ choices from this case.
- Case 2, Matt eats beans: Give that there are no forward restrictions, he can choose anything the next day, giving us $M(n-1)$ possibilities.
- Case 3, Matt eats peas: Similar to the previous case, since there are no restrictions, there are $M(n-1)$ possible choices.
- Case 4, Matt eats rice: Give the second restriction, we know the next day must be one of the three other options. This is where we would split into our previous cases again.
  - Case A: Matt eats beans or peas following the day he eats rice. Both of these would result in $M(n-2)$ choices since there are no more restrictions the following day. Giving us a total of $2M(n-2)$ for this case.
  - Case B: Matt eats fish following the day he eats rice. This would mean he would have to have the very specific 3 day pattern of rice, then fish, then beans given the first restriction. This would give us $M(n-3)$ choices.

Then, summing up all of our cases would yield us a total that matches (c).

Part A2: Multiple Answer Multiple Choice

Problem 4. (4 points)

Given that $11 \leq x \leq 14$, which of the following **could** be true?

(a) $x \equiv 2 \pmod 4$

(b) $x \equiv 6 \pmod 7$

(c) $x \equiv 3 \pmod 8$

(d) $x \equiv 24 \pmod{31}$

(e) $x \equiv 13 \pmod{77}$

Solution:
a, b, c, e

(a) True. $x \equiv 14 \pmod 4$, $11 \leq 14 \leq 14$

(b) True. $x \equiv 13 \pmod 7$, $11 \leq 13 \leq 14$

(c) True. $x \equiv 11 \pmod 8$, $11 \leq 11 \leq 14$

(d) False. In this case, x can only be equivalent to numbers less than or equal to -7, or greater than or equal to 24. Either of these options put you outside of the range.

(e) True. $x \equiv 13 \pmod{77}$, $11 \leq 13 \leq 14$

Problem 5. (4 points)

Let $f(x) = 2x+1$. Which of the following domain/codomain pairs would make f a bijection?

(a) $\mathbb{R} \to \mathbb{R}$

(b) $\mathbb{Z} \to \mathbb{Z}$

(c) $\mathbb{R}^+ \to \mathbb{R}^+$

(d) $\mathbb{Q} \to \mathbb{Q}$

(e) $\mathbb{N} \to \mathbb{N}$

Solution: a,d

(b) Option b is not onto because for integer inputs, f only outputs odd integers. So none of the even integers in the codomain are mapped to by the function.

(c) Real numbers in the range (0, 1] do not have a preimage in the domain. Hence the function would not be onto.

(e) The function would not be onto for the same reason as option b. None of the even natural numbers will have a preimage in the domain.

Problem 6. (4 points)

Which of the following are countably infinite?

(a) The real solutions to the equation $\sin x = 0$

(b) $\mathbb{Z} - (\mathbb{Z} \times \mathbb{Z})$

(c) $\mathbb{Z} - \mathbb{R}$

(d) $\{e, \pi\} \times \mathbb{N}$

(e) $\mathbb{R} - \{x | x \in \mathbb{R} \land (|x| > 10)\}$

Solution: (a), (b), (d)

(a) The solutions will be $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$. We can find a bijection between the natural numbers with odd numbers n mapping to $-\frac{(n+1)}{2}\pi$ and even numbers n mapping to $\frac{n}{2}\pi$. Thus it is countably infinite. (note that since we know $\mathbb{Z}$ is countably infinite too, we could even make a bijection between this set and $\mathbb{Z}$, with simply $n\pi$).

(b) Since $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are disjoint, $\mathbb{Z} - (\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$ which we know is countably infinite

(c) Since $\mathbb{Z}$ is a subset of $\mathbb{R}$, $\mathbb{Z} - \mathbb{R} = \emptyset$ which is finite

(d) This is countably infinite since we can map odd numbers to $(e, \frac{(n+1)}{2})$ and even numbers n mapping to $(\pi, \frac{n}{2})$. Thus it is countably infinite

(e) This is the set of all real numbers whose absolute value is less than or equal to 10, which is uncountably infinite

Problem 7. (4 points)

Suppose that $f \circ g$ is bijective, where $g : A \to B$ and $f : B \to C$. Which of the statements below **must** be true?

(a) f is onto

(b) g is onto

(c) f is one-to-one

(d) g is one-to-one

(e) f is one-to-one if and only if g is onto

Solution: (a)(d)(e)

(a) True. $f(x)$ must always reach every element of set C for $f \circ g(x)$ to be bijective. For any element $c \in C$, if $f \circ g$ is bijective, then some element (call it $a \in A$) makes $(f \circ g)(a) = c$. This means $f(g(a)) = c$, providing a element of B that maps to c: $g(a)$. Thus f is onto.

(b) False. If g is not onto, some elements of B will not be mapped to. As long as the image of those elements in C is still covered by other elements of B, we will be fine. In particular, consider the example where $A = \{1\}, B = \{2, 3\}, C = \{4\}$, $g(1) = 2$, and $f(x) = 4$ (the only possibility). The composite $f \circ g$ is bijective; however, g is not onto.

(c) False. If f is not one-to-one, some elements of B may map to the same elements of C as others. As long as only one of each of these bunches is mapped to from A, we are fine. In the example described in (b), the composite $f \circ g$ is bijective, but f is not one-to-one.

(d) True. Suppose $g(x) = g(y)$. Applying the function f to both sides tells us that $f(g(x)) = f(g(y))$. Because $f \circ g$ is one-to-one, we know that $x = y$. We assumed $g(x) = g(y)$ and concluded that $x = y$, so g must be one-to-one.

(e) True. If $g(x)$ is onto (and one-to-one, see part (d)), then there is a one-to-one correspondence between sets A and B. As $f(g(x))$ is also bijective, there is a one-to-one correspondence between sets A and C. $f(x)$, which takes input from set B and outputs to set C, must further be bijective and thus one-to-one. The same is true vice versa.

Problem 8. (4 points)

Which are the following pairs of Base Cases and Inductive Steps are sufficient to prove that $\forall n \in \mathbb{Z}^+,\ P(n)$?

(a) Base Case: $P(1)$; Inductive Step: $\forall k \in \mathbb{Z}^+, P(k) \rightarrow P(k+1)$

(b) Base Case: $P(0)$; Inductive Step: $\forall k \in \mathbb{Z}^+, P(k) \rightarrow P(k+1)$

(c) Base Case: $P(1)$; Inductive Step: $\forall k \in \mathbb{Z}^+, [P(1) \wedge \cdots \wedge P(k)] \rightarrow P(k)$

(d) Base Case: $P(0)$; Inductive Step: $\forall k \in \mathbb{Z}^+, P(k-1) \rightarrow P(k)$

(e) Base Cases: $P(1)$, $P(2)$, and $P(3)$; Inductive Step: $\forall k \in \mathbb{Z}^+, P(k) \rightarrow P(3k)$

Solution: a, d

(a) This is correct, since it is a standard way of applying induction. Proving the Base Case and Inductive Step are true allows us to conclude $P(k)$ is true for all positive integers k.

(b) This is incorrect, because the Inductive Step is only defined for positive integers k, so the smallest statement proved in the inductive step is $P(1) \rightarrow P(2)$. We therefore do not know if $P(0) \rightarrow P(1)$, so we cannot conclude that $P(1)$ is true. Therefore, we cannot conclude whether $P(k)$ is true for all positive integers k.

(c) This is incorrect, because the left side of the implication in the Inductive Step includes $P(k)$, and the right side is also $P(k)$. Therefore, $P(k)$ needs to be true for $P(k)$ to be true, which is circular logic.

(d) This is correct, because when $k = 1$ is plugged into the Inductive Step, we get $P(0) \rightarrow P(1)$. We can then use the Inductive Step on the other positive integers.

(e) This is incorrect, because we do not know from these Bases Cases and Inductive Step that $P(4)$ is true. Therefore, we do not know whether $P(k)$ is true for all positive integers k. This will only prove $P(k)$ for powers of 3 and numbers that are twice a power of 3.

Problem 9. (4 points)

Which of the following are possible cardinalities for sets A and B given that $|A - B| = 8$?

(a) $|A| = 20$ and $|B| = 10$

(b) $|A| = 9$ and $|B| = 0$

(c) $|A| = 8$ and $|B| = 8$

(d) $|A| = 10$ and $|B| = 20$

(e) $|A| = 5$ and $|B| = 3$

Solution: c, d

(a) With these cardinalities, the smallest possible cardinality of $A - B$ would be 10 (occurring if every element of B was in A), which is more than 8.

(b) With these cardinalities, the smallest possible (and only) cardinality of $A - B$ would be 9, which is more than 8

(c) If B did not share any elements with A (in other words $A \cap B = \emptyset$), then the cardinality of $A - B$ would be 8

(d) If B has exactly two elements that were also in A, and no other elements that were in both sets, then the cardinality of $A - B$ would be 8.

(e) The biggest possible cardinality of $A - B$ could be 5 (occurring A and B did not share any elements), which is less than 8

Problem 10. (4 points) Which of the following sets are **not countably infinite**?
Note: $\mathbb{Z}^-$ is the set of negative integers.

(a) $\mathbb{Q} \times \mathbb{Q}$

(b) $(\mathbb{Z} - \mathbb{N}) - \mathbb{Z}^-$

(c) $\mathbb{N} \cup \{\pi\}$

(d) $\mathbb{R} - \mathbb{Z}$

(e) $\mathbb{Q} - \mathbb{Z}$

Solution: b, d

a) $\mathbb{Q}$ is countably infinite, and the cartesian product of two countably infinite sets is also countably infinite (this is similar to how we showed $\mathbb{Q}$ is countable in the first place, by viewing it as pairs of integers and laying them out in a grid).

b) $\mathbb{N}$ has all the positive integers and 0, and $\mathbb{Z}^-$ has all the negative integers, so when we remove both of these from $\mathbb{Z}$, the set of all integers, we are left with nothing, the empty set, which is not countably infinite. (it is countable, as it is finite, but not countably **infinite**)

c) The set $\{\pi\}$ has 1 element, so adding a single element to $\mathbb{N}$ will keep its cardinality $\aleph_0$, so it is still countably infinite.

d) $\mathbb{R}$ is uncountable, so if we remove only countably infinitely many things (the size of $\mathbb{Z}$), we will still have an uncountable set.

e) This set only contains rational numbers, so it can't be uncountable (larger than $\mathbb{Q}$). After removing the integers, we still have infinitely many rational numbers left.

Problem 11. (4 points)

Let P be a predicate over $\mathbb{N}$ that is not always true, and such that $\forall n, P(n+1) \rightarrow P(n)$. Which of the following **must** be true?

(a) If $P(0)$, then $P(203)$.

(b) If $P(203)$, then $P(0)$.

(c) There is some $c \in \mathbb{N}$ such that $P(c)$ is true but $P(c+1)$ is false.

(d) There is some $c \in \mathbb{N}$ such that $P(n)$ is true if and only if $n < c$.

Solution: b, d

(a) We're given that P is not always true, and the truth value of $P(0)$ doesn't tell us anything about the successive predicates. Then, $P(203)$ is not necessarily true.

(b) This follows from the same ideas as induction:
we can see that $P(203) \rightarrow P(202) \rightarrow \cdots \rightarrow P(0)$.

(c) Consider the case where $P(n)$ is always false.

(d) There must be a largest value x such that $P(x)$ is true, as otherwise, P would be always true. Thanks to the given implies statement, every value less than $x+1$ must be true. In the case that P is always false (so there is no largest x that makes $P(x)$ true because there is no value that makes it true), we can use $c = 0$.

Part B: Short Answer

If there are intermediate steps involved, show your work and/or include justification **in the answer packet** to get full credit.

Problem 12. (7 points)

Suppose you want to prove the following claim using strong induction:

Claim: $P(n) \;\; \forall n \geq 5$.

(a) Fill in the blanks to complete the inductive step below.
Note: Write all your answers in the Answer Packet. Nothing written below will be graded.

Inductive Step:
Let $k \geq$ ___.
Assume $P(j)$ is true for all _______ $\leq j \leq$ _______.
Since $P(k-4)$, ... [*specific deductions omitted*] ..., then $P(k)$.

(b) Given the inductive step in Part (a), for which values of n will $P(n)$ need to be proven using base cases?

Solution:

(a) *Inductive Step:*
Let $k \geq \mathbf{9}$.
Assume $P(j)$ is true for all $\mathbf{5} \leq j \leq \mathbf{k-1}$.
Since $P(k-4)$, then $P(k)$.

For the inductive hypothesis, any range of j that includes $j = k-4$ and does not allow $j \geq k$ or $j < 5$ is correct and should get full credit.

(b) $n = 5, 6, 7, 8$. These values are not covered by the inductive step, but are part of the claim, and thus need to be proven with base cases.

Grading Guidelines:

(a) [+5 total]
[+2] Correct minimum value for k, such as **9** or max(base case) + 1
[+3] Correct range for j: includes $k-4$ for all k's and excludes k.
[+1.5] Correct lower bound and includes k-4, but incorrect upper bound
[+1.5] Incorrect lower bound but correct upper bound

(b)

[+2] Completely correct: $n = 5, 6, 7, 8$
[+1] More or less than the number of base cases required for student's IH
[+1] Base cases does not start from 5, but right number of base cases.
[-0.1] If the number of base cases is more than 4, but is still appropriate for the answer in part a.

Common mistakes:

1. $k \geq 5$
   P(5), P(6), P(7) and P(8) cannot be proven in the inductive step since, in order to prove for example P(5), you need to use $P(k-4) = P(1)$. However, we are not given in the question that P(1) is true.

2. $5 \leq j \leq k$
   Since we are trying to prove P(k), we cannot asssume that P(k) is true.

3. Base cases n = 1, 2, 3, 4
   We are not sure that P(1) ... P(4) is true. We only have to prove $P(5)$ and up

4. Base cases n = 5, 6, 7, ...
   If the answer ends in ellipses, even though it starts at 5, no points were given (even if you start from 5), since this defeats the point of induction.

5. Note that we did accept answers that had a tighter range than 5 to $k-1$. As long as your range included $k-4$ and doesn't go below 5 or above $k-1$, you can get full credit.

Problem 13. (6 points)

Let $g : A \to B$ and $f : B \to C$.
Prove that if f and g are both one-to-one, then $f \circ g$ is one-to-one.

Solution:
Notes: (not required for full credit)
Since $f : B \to C$ is one-to-one, we have $\forall b_1, b_2 \in B\,[(f(b_1) = f(b_2)) \to (b_1 = b_2)]$, and since $g : A \to B$ is one-to-one, we have $\forall a_1, a_2 \in A\,[(g(a_1) = g(a_2)) \to (a_1 = a_2)]$.

We want to show that $f \circ g : A \to C$ is one-to-one, i.e.,
$\forall a_1, a_2 \in A\,[((f \circ g)(a_1) = (f \circ g)(a_2)) \to (a_1 = a_2)]$.

Proof:
Let a_1, a_2 be arbitrary elements of A, and suppose $(f \circ g)(a_1) = (f \circ g)(a_2)$.
Then we have:

$$\begin{aligned}
(f \circ g)(a_1) &= (f \circ g)(a_2) && \\
f(g(a_1)) &= f(g(a_2)) && \text{(defn of function composition)} \\
g(a_1) &= g(a_2) && \textbf{because } f \textbf{ is one-to-one} \\
a_1 &= a_2 && \textbf{because } g \textbf{ is one-to-one}
\end{aligned}$$

Thus, $f \circ g$ is one-to-one.

Note that justifications above in boldface are required for full credit.

NOTE: There were many ways to word/structure a full-credit proof. Below are three examples.

Alternate Solution I: proving contrapositive of "$f \circ g$ is one-to-one": $(a_1 \neq a_2) \rightarrow ((f \circ g)(a_1) \neq (f \circ g)(a_2))$
Let a_1, a_2 be arbitrary elements of A, and suppose $a_1 \neq a_2$.
Then we have:

$$\begin{aligned}
a_1 &\neq a_2 && \\
g(a_1) &\neq g(a_2) && \textbf{because } g \textbf{ is one-to-one} \\
f(g(a_1)) &\neq f(g(a_2)) && \textbf{because } f \textbf{ is one-to-one} \\
(f \circ g)(a_1) &\neq (f \circ g)(a_2) && \text{(def. of function composition)}
\end{aligned}$$

Thus, $f \circ g$ is one-to-one.
Alternate Solution II: using proof by contradiction
Assume f and g are one-to-one, and $f \circ g$ is not, meaning that there exist $a_1, a_2 \in A$ such that $(f \circ g)(a_1) = (f \circ g)(a_2)$, but $a_1 \neq a_2$.
Then we have:

$$\begin{aligned}
(f \circ g)(a_1) &= (f \circ g)(a_2) && \\
f(g(a_1)) &= f(g(a_2)) && \text{(defn of function composition)} \\
g(a_1) &= g(a_2) && \textbf{because } f \textbf{ is one-to-one} \\
a_1 &= a_2 && \textbf{because } g \textbf{ is one-to-one}
\end{aligned}$$

This contradicts our assumption that $a_1 \neq a_2$. Therefore, $f \circ g$ is one-to-one.

Alternate Solution II: using proof by contradiction
Assume f and g are one-to-one, and $f \circ g$ is not, meaning that there exist $a_1, a_2 \in A$ such that $(f \circ g)(a_1) = (f \circ g)(a_2)$, but $a_1 \neq a_2$. This can happen in two ways:

1. $g(a_1) = g(a_2) = b$ where $b \in B$, $f(b) = c$ where $c \in C$
2. $g(a_1) = b_1, g(a_2) = b_2$, where $b_1, b_2 \in B$, and $f(b_1) = f(b_2) = c$ where $c \in C$. Note that $b_1 \neq b_2$ because g is one-to-one.

Possibility 1 does not work as it contradicts the assumption that g is one-to-one. Possibility 2 does not work as it contradicts the assumption that f is one-to-one. Therefore, $f \circ g$ must be one-to-one.

Grading Guidelines:

- +1 Correct premise to prove that $f \circ g$ is one-to-one (e.g., "assume that $(f \circ g)(a_1) = (f \circ g)(a_2)$" or "assume $a_1 \neq a_2$")
- +1 Correct conclusion to prove that $f \circ g$ is one-to-one (e.g., "$a_1 = a_2$" or "$(f \circ g)(a_1) \neq (f \circ g)(a_2)$")
- +1 Demonstrates understanding of function composition, i.e., $(f \circ g)(x) = f(g(x))$
- +1.5 Correctly applies property that f is one-to-one
- +1.5 Correctly applies property that g is one-to-one

Common mistakes:

- Using one example where f and g are one-to-one, and showing that $f \circ g$ is one-to-one. Having one example shows that there exists a situation that it can happen, but it does not prove it for arbitrary graphs f and g.
- "Proving" that $f \circ g$ is one to one because f and g are both one-to-one without actually following a one-to-one proof.
- Using cardinality to "prove" that fog is one-to-one. Just because $|A| \leq |C|$ doesn't guarantee that a function from A to C is one-to-one. For example, the function $f : \{1, 2\} \to \{3, 4, 5\}$ with $f(x) = 3$ is not one-to-one.
- Mixing up domains within functions, for instance, using f(b) and g(b), when they should be in different domains.

Part C: Free Response

Problem 14. (10 points)

Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ with $f(x) = (x-5)^2$.

(a) Prove or disprove that f is onto.

(b) Prove or disprove that f is one-to-one.

Solution:

(a) f is onto.
Proof:
Let $y \in \mathbb{R}^+ \cup \{0\}$. Consider $x = \sqrt{y} + 5$.
$f(x) = f(\sqrt{y}+5) = \left((\sqrt{y}+5)-5\right)^2 = (\sqrt{y})^2 = y$. Thus, f is onto.

(Not required for full credit, but included for clarity and completeness: note that $x > 0$ for any y, so x is in the domain of our function f.)

(b) f is not one-to-one.
Disproof by counterexample: Consider $a = 4$ and $b = 6$.
$f(a) = (4-5)^2 = (-1)^2 = 1$, and $f(b) = (6-5)^2 = 1^2 = 1$. So $f(a) = f(b)$ but $a \neq b$, thus f is not one-to-one.

Note: there are many possible a, b values that will work as a counterexample.

Grading Guidelines: [10 points]
Part a: [5 points]
+1 Circle "Prove"
+1 Pick arbitrary y in the codomain
+1 Give a value of x in the domain
+2 Show that $f(x) = y$
Part b: [5 points]
+1 Circle "Disprove"
+2 Give a counterexample consisting of two distinct values x_1 and x_2 in the domain
+2 Demonstrate that $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, and this disproves the claim
+1.5 (Partial Credit) Attempt to prove by citing that the square function is invertible on the positive reals, but forget that $x-5$ can be negative. The proof must be otherwise fully correct and clearly explain that the reasoning for inverting the square is the non-negative domain of f.

Common Mistakes:
Part a:

- Not choosing an arbitrary y. Every $\forall$ proof requires that we utilize an arbitrary value, and in this case, we needed to assume an arbitrary y value to encapsulate the entire codomain.

- Using the wrong codomain for the arbitrary y. Many students either used just $\mathbb{R}$ or $\mathbb{R}^+$ as the codomain. Whether it went unnoticed or it was mistaken as the domain instead, noting the correct codomain for any onto proof is important in order to have the arbitrary prove for the entire codomain.

- Using the term positive instead of non-negative. Similar to the previous bulletpoint, it's important to note the addition of 0 to the positive reals, making the values of the codomain non-negative rather than positive.

Part b:

- Attempting to prove and taking the square root of both sides of $f(x) = f(y)$. This was *very* prevalent in our solutions so we made a separate rubric item giving partial credit for this mistake. Notably, we can't take the square root of the function $(x-5)^2$ because we don't know if $(x-5)$ is non-negative, given our domain. For example, when we plug in $x = 1$, we get $(x-5)^2 = (-4)^2 = 16$, and taking the square root of this gives 4, not -4, which suggests the x input was 9, not 1.

- Attempting to disprove using contradiction instead of by counterexample. As is standard with all $\forall$ disproves, all we want is a singular example that disproves it. So in the case of this question, we were looking for a value of x and y such that $f(x) = f(y) \wedge x \neq y$. This would then disprove one to one because it breaks the mathematical definition of it: $\forall x, y(f(x) = f(y) \rightarrow x = y)$. This is a recurring mistake throughout many different questions so note that disproving any for all statement *only* requires a single counterexample.

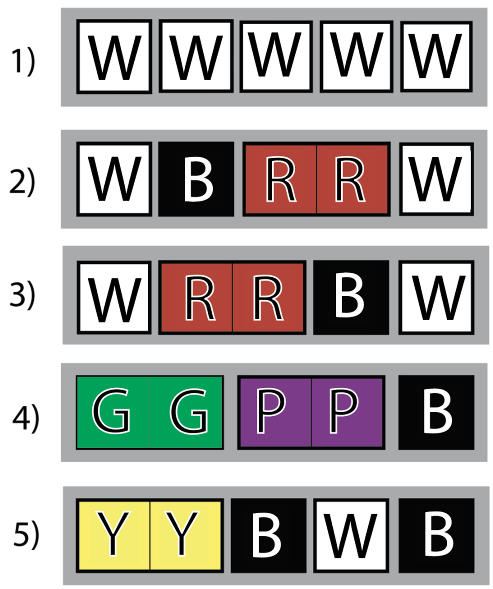
- Using 0 as one of the values for the counterexample. Note that 0 only shows up in our codomain, *not* our domain, meaning that we can't choose 0 as a counterexample value.

Problem 15. (10 points)

1-D Tetris

Let a_n be the number of ways to cover a $1 \times n$ rectangle with 1×1 tiles that are black or white and 1×2 tiles that are red, yellow, green or purple.

Here are 5 examples of the many possible ways to tile a $1 \times n$ board when $n = 5$:



a) Compute a_n for $n = 1, 2$

b) Find a recurrence relation for a_n
Reminder: You must explain your recurrence relation

c) What are the base cases necessary to solve this recurrence and what are their values?
Note: To get full credit for this part, you must use the minimal number of base cases.

Solution:

a)

$$a_1 = 2 \qquad \text{(B or W)}$$
$$a_2 = 2 \cdot 2 + 4 = 8 \qquad \text{(two 1x1 tiles or one 1x2 tile)}$$

b) $a_n = 2a_{n-1} + 4a_{n-2}$
Since each $1 \times n$ tile can be obtained by adding a 1×1 tile to a $1 \times (n-1)$ rectangle, or by adding a 1×2 tile to a $1 \times (n-2)$ rectangle. All $1 \times n$ tilings can be obtained in one of these ways, and every tiling can only be done by one of these (based on what the last tile is). This means we can simply add these counts with no inclusion-exclusion needed.

c) We need 2 base cases to solve this recurrence. We can either use $a_0 = 1$ and $a_1 = 2$ or $a_1 = 2$ and $a_2 = 8$.

Grading Guidelines:
Part a:
+1.5 Each Correctly computed value of n for $n = 1, 2$
Part b:
+2.5 Correct term of $2a_{n-1}$ with proper coefficient
+2.5 Correct term of $4a_{n-2}$ with proper coefficient
+1 (Partial) Either term with correct subscript, but incorrect coefficient.
-1 Any extra values of a_{n-x}
Part c:
+1 Correctly identifying 2 base cases.
+1 Correct values associated with base cases.

Common mistakes:

a) Using 3 options for colored tile instead of 4: this causes errors in the calculations for a_2.

b) Unnecessary cases: we only need to handle cases for ending in Black/White or Color Tile, and don't need cases for every combination

c) Using a_0, a_1, a_2 for part c) would not get full points because we are looking for the **minimum** number of base cases to solve this recurrence relation.

d) Claiming $a_0 = 0$. In fact, $a_0 = 1$ because there is only 1 way to tile a 1 x 0 size rectangle - by placing no tiles.

Problem 16. (10 points)

Prove using induction that $n^2 < 10^n$ for all integers $n \geq 3$.

Note: Every inequality in your inductive step should be supported by one of the following:

1. The inductive hypothesis (IH)
2. $k^i < k^j$ when $i < j$ because $k > 1$ (e.g., $k^2 < k^4$)
3. $c \leq k$ when $c \leq 3$ because $k \geq 3$ (e.g., $2 \leq k$)
4. For any three integers a, b, c: $(c > 0 \wedge 0 < a < b) \rightarrow ac < bc$
5. For any two integers x, y and some positive constant k : $(x < y \rightarrow kx < ky)$

Solution:

Inductive Step:
Let $k \geq 3$. Assume the inductive hypothesis, that $k^2 < 10^k$.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&= k^2 + 2k + k^0 \\
&< k^2 + k \cdot k + k^0 && 2 < k \\
&< k^2 + k \cdot k + k^2 && k^0 < k^2 \\
&= k^2 + k^2 + k^2 \\
&= 3k^2 \\
&< 3 \cdot 10^k && \text{by IH} \\
&< 10 \cdot 10^k && \text{by step 5} \\
&= 10^{k+1}.
\end{aligned}$$

Base Case: $n = 3$
$n^2 = 3^2 = 9 < 1000 = 10^3 = 10^n$.

Alternate 1
Let $k \geq 3$. Assume the inductive hypothesis, that $k^2 < 10^k$.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< k^2 + 2k + k && k > 1 \\
&= k^2 + 3k \\
&\leq k^2 + k \cdot k && k \geq 3 \\
&= k^2 + k^2 \\
&= 2k^2 \\
&< 2 \cdot 10^k && \text{by IH} \\
&< 10 \cdot 10^k && \text{by step 5} \\
&= 10^{k+1}.
\end{aligned}$$

Alternate 2 *This approach starts from RHS, and end at LHS.*

Let $k \geq 3$. Assume the inductive hypothesis, that $k^2 < 10^k$.

$$\begin{aligned}
10^{k+1} &= 10 \cdot 10^k \\
&> 10 \cdot k^2 && \text{by IH} \\
&= k^2 + 9k^2 \\
&> k^2 + 9k && k^2 > k \\
&= k^2 + 2k + 7k \\
&> k^2 + 2k + 7 && k > 1 \\
&> k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}$$

Alternate 3
Let $k \geq 3$. Assume the inductive hypothesis, that $k^2 < 10^k$.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&= k^2\left(1 + \frac{2}{k} + \frac{1}{k^2}\right) && k \geq 3, \text{ so } \frac{2}{k} < 1 \text{ and } \frac{1}{k^2} < 1 \\
&< k^2 \cdot 3 \\
&= 3k^2 \\
&< 3 \cdot 10^k && \text{by IH} \\
&< 10 \cdot 10^k && \text{by step 5} \\
&= 10^{k+1}.
\end{aligned}$$

Grading Guidelines:
+1 Base Case: Has the correct number of base cases for the induction approach used
+1 Base Case: Correct work for the base case ($3^2 = 9 < 1000 = 10^3$)
+1 Inductive Hypothesis: Assumes a valid inductive hypothesis
+1 Inductive Hypothesis: Attempts to apply inductive hypothesis
+1.5 Inductive Hypothesis: Application of IH produces a logical next step towards correct conclusion (in other words, you correctly apply the IH following from the previous step and proceed in a direction that leads to solving the problem)
+1.5 Inductive Step: Correct "want to show" or correct conclusion ($(k+1)^2 < 10^{k+1}$ or equivalent)
+2.0 Inductive Step: Correct and justified algebra pre-IH
+1.0 Inductive Step: Correct and justified algebra post IH
-1.0 Major algebraic error or lack of justification
-0.5 Minor algebraic error or lack of justification

Note: if you started from what you were trying to prove ($(k+1)^2 < 10^{k+1}$), you lost 2 algebra points (but algebra points are never negative, so if you only got 1 algebra point then you end up with 0 algebra points, not -1)

Note: you should have received deductions for algebra errors in particular parts of the proof *only if* the corresponding part of the proof was marked as correct. If you did not receive points for that portion of the proof and also received a deduction, *submit a regrade request.*

Common mistakes:

- Claiming $10^k + 10^k = 10^{k+1}$, $3 \cdot 10^k = 10^{k+1}$, or something similar. In order for this to be an equality, you must have $10 \cdot 10^k$.
- Beginning with the inequality you want to show, then reaching another true inequality. Logically, this corresponds to $P(k) \to (P(k+1) \to T)$ rather than $P(k) \to P(k+1)$, which is what we want. This does not apply if you continually rewrite the desired right hand side of the inequality after each step without transforming it, although we strongly discourage this as well.
- Dropping terms without justification or by calling them "negligible." This isn't sufficient justification – you should explicitly say which term in the next step your negligible term is less than.
- Assuming in the inductive hypothesis $k^2 < 10^k$ for *all* integers k rather than *some* arbitrary integer k. This assumes the proof.
- Starting the inductive step with the inductive hypothesis and attempting to transform the inequality into the desired conclusion. While is this logically sound, it is extremely difficult to do correctly for inequality proofs in particular, and generally considered bad style besides.
- Incorrectly evaluating exponents in the base case. While we generally didn't take off points if it just seemed like a calculation error, let this serve as a warning for counting problems: don't bother simplifying your answers, because you are more likely to get it wrong than you'd think.
- Writing the framework for the inductive proof without filling in the algebra. This explains why many students did not receive full credit, but writing that framework isn't a mistake. We strongly encourage this as a way to demonstrate your understanding of induction even if you aren't able to do this particular problem!

Problem 17. (9 points)

Prove that the following claim is true:

$$(A - B) \cup (\overline{A} - \overline{B}) \subseteq (A \cup B) - (A \cap B)$$

Solution: This question could be approached in two ways:

- as a containment proof, by showing all elements of $(A - B) \cup (\overline{A} - \overline{B})$ are also elements of $(A \cup B) - (A \cap B)$.
- as a set equivalence proof, by using set identities to show that one of the sets is equal to the other. Note that this proves *more* than was asked, as it proves both sides are subsets of the other. In this particular case, the two sets are equal, so that is possible, but in general, a subset proof may require the other method if they are not guaranteed to be equal.

There are two **mutually exclusive** rubrics for this question. One is for containment proofs, and the other is for set equivalence proofs. This means every solution was graded under only one of these rubrics.

Some solutions used elements of both techniques. Complete proofs that did this correctly got credit for corresponding items on the best fitting rubric. However, for example, the first half of a containment proof and the first half of a set equivalence proof are not worth full credit; only one of the two can be graded.

Containment Solution (Concise and Perfectly Acceptable):
Take arbitrary $x \in (A - B) \cup (\overline{A} - \overline{B})$
$\Rightarrow x \in (A - B)$ or $x \in (\overline{A} - \overline{B})$ by def of union
We consider each of these cases:

Case 1: $x \in A - B$
$\Rightarrow x \in A$ and $x \notin B$ by def of set minus
$x \in A \Rightarrow x \in A \cup B$ by def of union
$x \notin B \Rightarrow x \notin A \cap B$ by def of intersection
$\Rightarrow x \in (A \cup B) - (A \cap B)$ by def of set minus

Case 2: $x \in \overline{A} - \overline{B}$
$\Rightarrow x \in \overline{A}$ and $x \notin \overline{B}$ by def of set minus
$\Rightarrow x \notin A$ and $x \in B$ by def of set complement
$x \in B \Rightarrow x \in A \cup B$ by def of union
$x \notin A \Rightarrow x \notin A \cap B$ by def of intersection
$\Rightarrow x \in (A \cup B) - (A \cap B)$ by def of set minus

Thus, $x \in (A - B) \cup (\overline{A} - \overline{B})$ implies $x \in (A \cup B) - (A \cap B)$
$\Rightarrow (A - B) \cup (\overline{A} - \overline{B}) \subseteq (A \cup B) - (A \cap B)$ by def of subset

Containment Solution (With More Explanation):
Let $x \in (A - B) \cup (\overline{A} - \overline{B})$ be chosen arbitrarily. This means x could be any element of that set. In other words, that's the only fact we know about x until we use it to prove more.

By definition of union, we know $x \in A - B$ or $x \in \overline{A} - \overline{B}$. To make use of this "or" statement, we proceed by cases.

Case 1: $x \in A - B$
The definition of set difference tells us that $x \in A$, but $x \notin B$. Because $x \in A$, we know that $x \in A \cup B$. Because $x \notin B$, we know that $x \notin A \cap B$. Thus, $x \in (A \cup B) - (A \cap B)$.

Case 2: $x \in \overline{A} - \overline{B}$
The definition of set difference tells us that $x \in \overline{A}$ and $x \notin \overline{B}$. By the definition of complement, $x \notin A$ and $x \in B$. Because $x \in B$, we know that $x \in A \cup B$. Because $x \notin A$, we know that $x \notin A \cap B$. Thus, $x \in (A \cup B) - (A \cap B)$.

In both cases, we showed that $x \in (A \cup B) - (A \cap B)$.

Since whenever x is an element of $(A - B) \cup (\overline{A} - \overline{B})$, it's an element of $(A \cup B) - (A \cap B)$, we conclude that $(A - B) \cup (\overline{A} - \overline{B}) \subseteq (A \cup B) - (A \cap B)$.

Set Equivalence Solution:

$$
\begin{aligned}
&(A - B) \cup (\overline{A} - \overline{B}) \\
&\quad = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{\overline{B}}) && \text{(Definition of Set-Minus)} \\
&\quad = (A \cap \overline{B}) \cup (\overline{A} \cap B) && \text{(Complementation Law)} \\
&\quad = (A \cup (\overline{A} \cap B)) \cap (\overline{B} \cup (\overline{A} \cap B)) && \text{(Distributive Law)} \\
&\quad = ((A \cup \overline{A}) \cap (A \cup B)) \cap ((\overline{B} \cup \overline{A}) \cap (\overline{B} \cup B)) && \text{(Distributive Law)} \\
&\quad = (U \cap (A \cup B)) \cap ((\overline{B} \cup \overline{A}) \cap U) && \text{(Complement Law)} \\
&\quad = (A \cup B) \cap (\overline{B} \cup \overline{A}) && \text{(Identity Law)} \\
&\quad = (A \cup B) \cap (\overline{B \cap A}) && \text{(De Morgan's Law)} \\
&\quad = (A \cup B) - (B \cap A) && \text{(Definition of Set-Minus)}
\end{aligned}
$$

Common Mistakes:

- Generally describing the sets instead of giving a formal proof. In order to prove a subset, you need to take an arbitrary element in the left-hand side, and then show it's a member of the right-hand side. Pictures (i.e. Venn diagrams), specific examples, or descriptions **are not proofs**.

- Working both sides of an or-statement or union instead of splitting into cases. Many students identified the fact that if $x \in (A - B) \cup (\overline{A} - \overline{B})$, then $x \in A - B$ **or** $x \in \overline{A} - \overline{B}$, and then proceeded by saying "If $x \in A - B$ then... If $x \in \overline{A} - \overline{B}$ then..." The correct way to proceed with the proof is to split into two separate cases and handle them separately.

- Concluding that $\overline{A} - \overline{B}$ is equal to $B - A$ without justification. This is a substantial leap that you would need to prove to receive full credit. An example proof is given below

$$\begin{aligned}
\overline{A} - \overline{B} &= \overline{A} \cap \overline{\overline{B}} && \text{(Definition of Set-Minus)} \\
&= \overline{A} \cap B && \text{(Complementation Law)} \\
&= B \cap \overline{A} && \text{(Commutative Law)} \\
&= B - A && \text{(Definition of Set-Minus)}
\end{aligned}$$

- Mixing logical notation with set notation. Many submissions stated things like "$(A - B) \cup (\overline{A} - \overline{B})$ is the same as $(x \in A \land x \notin B) \cup (y \in A \land y \notin B)$. Firstly, x and y need to be properly introduced before using them in a proof. Here, it is ambiguous what x and y are. Second of all, $\cup$ can only be applied to sets. "$x \in A \land x \notin B$" is a proposition about the variable x, and is either true or false, so we can't apply set operations to this.

- Claiming that if $x \in A - B$, then $x \in A - (A \cap B)$ without justification. We can show this is true as follows:

$$\begin{aligned}
A - (A \cap B) &= A \cap \overline{A \cap B} && \text{(Definition of Set-Minus)} \\
&= A \cap (\overline{A} \cup \overline{B}) && \text{(De Morgan's Law)} \\
&= (A \cap \overline{A}) \cup (A \cap \overline{B}) && \text{(Distributive Law)} \\
&= \emptyset \cup (A \cap \overline{B}) && \text{(Complement Law)} \\
&= A \cap \overline{B} && \text{(Identity Law)} \\
&= A - B && \text{(Definition of Set-Minus)}
\end{aligned}$$

- Claiming that $A - (A \cap B) \cup B - (A \cap B) = (A \cup B) - (A \cap B)$ without justification. If you wanted to show this you would need to apply the definition of set minus and use the distributive law.

Note on Grading:
You were either graded according to the containment guidelines below, or according to the set equivalence proof. You cannot receive rubric items from both. Generally speaking, we graded your solution according to the rubric on which you would have scored higher. Note our standard policy is to grade the solution that scores worse, but we decided that for this particular problem, that didn't lead to accurate assessment.

Containment Grading Guidelines:

+1 Takes an arbitrary element x of $(A - B) \cup (\overline{A} - \overline{B})$
+1 States "$x \in (A - B)$ **or** $x \in \overline{A} - \overline{B}$" (or immediately breaks into those two cases)
+1 Ending both cases with same conclusion

Case 1 (2.5 Points): $x \in A - B$
+1 States that $x \in A$ and $x \notin B$
+0.5 States $x \notin A \cap B$ (must have said $x \notin B$)
+0.5 States $x \in A \cup B$ (must have said $x \in A$)
+0.5 Concludes $x \in (A \cup B) - (A \cap B)$ (must have said previous two items)

Case 2 (3.5 Points): $x \in \overline{A} - \overline{B}$
+1 States that $x \in \overline{A}$ and $x \notin \overline{B}$
+1 States that $x \notin A$ and $x \in B$
+0.5 States $x \in A \cup B$ (must have said $x \in B$)
+0.5 States $x \notin A \cap B$ (must have said $x \notin A$)
+0.5 Concludes $x \in (A \cup B) - (A \cap B)$ (must have said previous two items)

Other:
-1 bad notation (e.g. mixes up subset with element of)

Set Equivalence Grading Guidelines:
+1.5 applies definition of set minus
+2.5 applies distributive law
+2 applies complement/domination law
+2 applies DeMorgan's law
+1.5 sufficient justification/law citations for the reader to understand the proof
-1 one incorrect step
-2 multiple incorrect steps

Problem 18. (8 points) Positive integers and differences

Given 10 different positive integers that have value at most 14, show that at least four pairs of them have the same positive difference.

Solution: This is a pigeonhole principle problem.

Our pigeons are the number of pairs that we can make. We can find this by adding up all distinct pairs, giving us that the number of pigeons is $9+8+7+...+1 = 45$. This decreasing sum comes from the idea that the pair (x_1, x_2) is the same as the pair (x_2, x_1), since we are only looking at its positive difference. Thus, we are counting up the pairs in which the first value is less than the right: $(x_1, x_2), (x_1, x_3)...(x_1, x_{10}), (x_2, x_3), (x_2, x_4), ...(x_9, x_{10})$, where x_n represents the n-th lowest positive integer out of the 10 we are using. This can also be represented by $\binom{10}{2}$, which we will learn more about in the counting section of this course.

Our holes are the number of possible positive differences we can have. The number of holes is $14 - 1 = 13$. By the generalized pigeonhole principle, there are at least $\lceil \frac{45}{13} \rceil \approx \lceil 3.5 \rceil = 4$ pairs with the same positive difference.

Grading Guidelines:
+3 Correct pigeons and number of pigeons
+3 Correct pigeonholes and number of pigeonholes
+2 Correct final value

Common Mistakes:

- Treating each number as the pigeons rather than the pairs. Because we wanted to ensure that there are at least 4 **pairs** with the same positive difference, we want our pigeons to be the pairs.
- Using minimum difference = 0 instead of 1. Since all 10 selected integers are distinct, the minimum difference is 1.
- Listing out 4 pairs of numbers that have the same positive differences. Because we are picking 10 out of the 14 integers, we need to ensure that with any 10 integers selected, there will be at least 4 pairs of numbers that will have the same positive differences.