EECS 203: Discrete Mathematics
Fall 2023
Homework 8

Due **Thursday, November 2**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total points: $100 + 42$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Why You Got a 12-Car Garage? [8 points]

Ashu recently acquired three 12-car garages, but he has no cars (yet).

(a) What is the minimum number of cars Ashu has to acquire in order to guarantee that at least one of the garages will have **more** than 6 cars in it? Justify your answer, including an explanation of why it is the minimum number.

(b) If the garages are all adjacent to one another, what is the minimum number of cars Ashu has to acquire in order to guarantee that the middle garage has more than 6 cars in it?

Solution:

(a) Pigeons: Cars, quantity $= n$
Holes: Garages, quantity $= 3$
According to Pigeonhole Principle, we want at least 6 cars in every hole, which means that $\lceil \frac{n}{3} \rceil > 6$.
$\therefore n > 18$.
$\therefore$ the minimum number of cars is 19.

(b) Consider the worst condition: the left and right garages are all full, while the middle garage only has 6 cars. They add up to $12 \times 2 + 6 = 30$ cars.
Now if we add another car, then it can only go to the middle, so there must be 7 cars after that.
If any one of the left and right garage is not full, then that new car can go to the left or right garage. That makes it possible to keep the middle garage with 6 cars.
$\therefore$ if and only if there are more than 30 cars, it is guaranteed that the middle garage has more than 6 cars in it.
$\therefore$ the minimum number of cars is 31.

2. Sum More Counting [14 points]

Consider the set of integers between 1 and 18, inclusive. What is the smallest integer n such that, for any subset $S \subseteq \{1, 2, \ldots, 18\}$ of size $|S| = n$, there are **distinct** integers $x, y \in S$ with $x + y = 18$? Prove that your answer is sufficient to guarantee this, and the minimum necessary number.

Note: To prove that your choice of n is smallest, you must also give an example of a set of size $|S| = n - 1$ that does not contain $x, y \in S$ with $x + y = 18$.

Solution:

The minimum necessary number n is 11.
Consider the partition of the set, shown as elements of a new set:
$\{(1,17),(2,16),(3,15),(4,14),(5,13),(6,12),(7,11),(8,10),9,18\}$
Every element either can form a sum of 18 in its pair or cannot form a sum of 18 with any element (which is 9 and 18).
$\therefore$ There is 10 Holes: partitions of the set, or elements of the new set.
Assume the minimum necessary number is n.
Then according to Pigeonhole Principle, $\lceil \frac{n}{10} \rceil > 1$. (We need at least one full combinition,)
$\therefore n > 10$, the minimum necessary number is 11.

Now we show the choice of 11 is smallest:
Consider the subset $\{9,10,11,12,13,14,15,16,17,18\}$ which has 10 elements, and do not have 2 elements that can sum up to 18

3. Set Sizes [12 points]

Determine which of these sets are finite, countably infinite, or uncountably infinite. Give a short (about 1 line) explanation for each part.

(a) $\{2,3\} \times \mathbb{N}$

(b) $(0,2) - \mathbb{Q}$

(c) $\{x \in \mathbb{R} \mid x^2 - 1 \leq 0\}$

(d) $\{x \in \mathbb{N} \mid x \leq 1000\}$

Solution:

(a) Countably infinite. For any $x \in \mathbb{N}$, We can map $(2,x)$ to an odd positive integer and $(3,x)$ to an even positive integer, alternately. By doing this we can find a 1-to-1 map from $\{2,3\} \times \mathbb{N}$ to $\mathbb{Z}^+$. Then we know $|\{2,3\} \times \mathbb{N}| \leq |\mathbb{Z}^+|$.

(b) Uncountably infinite. Even though there is a set difference from $\mathbb{Q}$, we can still use the table of digit to prove that by diagonalization, that is, to show there does not exist an onto function from $\mathbb{Z}^+$ to $(0,2) - \mathbb{Q}$. Then $|\mathbb{Z}^+| < |(0,2) - \mathbb{Q}|$.

(c) Uncountably infinite. $\{x \in \mathbb{R} \mid x^2 - 1 \leq 0\} = [-1,1]$. We can use the same diagonalization method as in (b) to prove that.

(d) Finite. $|\{x \in \mathbb{N} \mid x \leq 1000\}| = 1001$. It is just integers from 0 to 1000 inclusively.

4. Ready, set, count! [15 points]

Definition: $A \oplus B$ is the symmetric difference of the sets A and B, i.e. the set containing all elements which are in A or in B but not in both.

Provide two **uncountable** sets A and B such that $A \oplus B$ is

(a) finite.

(b) countably infinite.

(c) uncountably infinite.

Include in your justification a desciption of the set $A \oplus B$ without reference to the symmetric difference.

Solution:

(a) $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} = (0, 1]$.
$B = \{x \in \mathbb{R} \mid 0 \leq x < 1\} = [0, 1)$.
$A \oplus B = \{0, 1\}$. It is finite, with only two elements 0 and 1.

(b) $A = \mathbb{R} - \mathbb{Q}$.
$B = \mathbb{R}$.
These two sets are uncountable, which can be proved by table of digit(diagonalization).
Then $A - B = \mathbb{Q}$, which is countably infinite. This can be proved in the same way we prove $|\mathbb{Z}^+ \times \mathbb{Z}^+|$ is the same size as $|\mathbb{Z}^+|$.

(c) $A = \{x \in \mathbb{R} \mid 0 < x < 1\} = (0, 1)$.
$B = \{x \in \mathbb{R} \mid 1 \leq x < 2\} = [1, 2)$.
Then $A + B = \{x \in \mathbb{R} \mid 0 < x < 2\} = (0, 2)$.
We know that they are uncountably infinite in the same logic(diagonalization).

5. Corresponding Counts [18 points]

Prove that $|[0, 2]| = |(3, 6)|$.

For any functions that you name:

- Prove that the function is well-defined, i.e. that for any x in the domain of your function f, $f(x)$ lies in the codomain.

- Prove any function properties that you use (e.g. one-to-one, onto, etc).

Solution:

(a) Proof of $|[0,2]| \le |(3,6)|$:
We prove it by proving that $\exists$ a one-to-one function $f : [0,2] \to (3,6)$.
Consider $f(x) = \frac{x}{2} + 4$, $x \in [0,2]$.
For this function:

- Proof: It is well defined.
It is to prove that: (1) $\forall x \in [0,2], \exists y = f(x)$,where $3 \le y \le 6$.
(2) $\forall y_1, y_2 \in (3,6)$, if $[y_1 \ne y_2] \wedge [y_1 = f(x_1)] \wedge [y_2 = f(x_2)]$, then $x_1 \ne x_2$ (One x cannot map to multiple y).

  Proof of (1):
  Let a be an arbitrary real numbers in $[0,2]$.
  Since $a \ge 0$, $f(a) = \frac{a}{2} + 4 \ge 4 \ge 3$.
  Since $a \le 2$, $f(a) = \frac{a}{2} + 4 \le 5 \le 6$.
  $\therefore \forall x \in [0,2], \exists y = f(x)$, where $3 \le y \le 6$.

  Proof of (2):
  Let b_1, b_2 be arbitrary real numbers in $(3,6)$.
  Assume $[b_1 \ne b_2] \wedge [b_1 = f(a_1)] \wedge [b_2 = f(a_2)]$.
  Then $f(a_1) \ne f(a_2)$, $\frac{a_1}{2} + 4 \ne \frac{a_2}{2} + 4$, $\frac{a_1}{2} \ne \frac{a_2}{2}$, $a_1 \ne a_2$.
  $\therefore \forall y_1, y_2 \in (3,6)$, if $[y_1 \ne y_2] \wedge [y_1 = f(x_1)] \wedge [y_2 = f(x_2)]$, then $x_1 \ne x_2$. $\therefore$ $f(x)$ is well-defined.

- Proof: It is one-to-one.
It is to prove that $\forall a_1, a_2 \in [0,2]$, $[f(a_1) = f(a_2)] \to (a_1 = a_2)$.
Let a_1, a_2 be arbitrary real numbers in $[0,2]$.
Assume $f(a_1) = f(a_2)$.
Then $\frac{a_1}{2} + 4 = \frac{a_2}{2} + 4$, $\frac{a_1}{2} = \frac{a_2}{2}$, $a_1 = a_2$.
$\therefore f$ is one-to-one.
$\therefore |[0,2]| \le |(3,6)|$.

(b) Proof of $|(3,6)| \le |[0,2]|$:
We prove it by proving that $\exists$ a one-to-one function $f : (3,6) \to [0,2]$.
Consider $g(x) = \frac{x-3}{3}$, $x \in (3,6)$.
For this function:

- Proof: It is well defined.
It is to prove that: (1) $\forall x \in (3,6), \exists y = g(x)$,where $0 \le y \le 2$.
(2) $\forall y_1, y_2 \in [0,2]$, if $[y_1 \ne y_2] \wedge [y_1 = g(x_1)] \wedge [y_2 = g(x_2)]$, then $x_1 \ne x_2$ (One x cannot map to multiple y).

Proof of (1):
Let a be an arbitrary real numbers in $(3, 6)$.
Since $a > 3$, $g(a) = \frac{a-3}{3} > 0$, then≥ 0 ($> \rightarrow [> \vee =]$).
Since $a < 6$, $f(a) = \frac{a-3}{3} < 1 \leq 2$.
$\forall x \in (3, 6), \exists y = g(x)$,where $0 \leq y \leq 2$.

Proof of (2):
Let b_1, b_2 be arbitrary real numbers in $[0, 2]$.
Assume $[b_1 \neq b_2] \wedge [b_1 = g(a_1)] \wedge [b_2 = g(a_2)]$.
Then $g(a_1) \neq g(a_2)$, $\frac{a_1-3}{3} \neq \frac{a_2-3}{3}$, $a_1 - 3 \neq a_2 - 3$, $a_1 \neq a_2$.
$\therefore$ $\forall y_1, y_2 \in (3, 6)$, if $[y_1 \neq y_2] \wedge [y_1 = g(x_1)] \wedge [y_2 = g(x_2)]$, then $x_1 \neq x_2$. $\therefore$ $g(x)$ is well-defined.

- Proof: It is one-to-one.
  It is to prove that $\forall a_1, a_2 \in (3, 6)$, $[gf(a_1) = g(a_2)] \rightarrow (a_1 = a_2)$.
  Let a_1, a_2 be arbitrary real numbers in $(3, 6)$.
  Assume $g(a_1) = g(a_2)$.
  Then $\frac{a_1-3}{3} = \frac{a_2-3}{3}$, $a_1 - 3 = a_2 - 3$, $a_1 = a_2$.
  $\therefore$ g is one-to-one.
  $\therefore$ $|(3, 6)| \leq |[0, 2]|$.

(c) Due to Schroeder-Bernstein Theorem, since $|[0, 2]| \leq |(3, 6)|$ and $|(3, 6)| \leq |[0, 2]|$, $|(3, 6)| = |[0, 2]|$.

6. Composition Pr∘∘f [15 points]

Consider functions $g\colon A \rightarrow B$ and $f\colon B \rightarrow C$. **Prove or disprove** that if f and $f \circ g$ are one-to-one, then g is one-to-one.

Solution:

Proof of if f and $f \circ g$ are one-to-one, then g is one-to-one:

Assume f and $f \circ g$ are one-to-one.
According to Definition we know:
(1): $\forall y_1, y_2 \in B$, $[f(y_1) = f(y_2)] \leftrightarrow [y_1 = y_2]$.
(2): $\forall x_1, x_2 \in A$, $[f(g(x_1)) = f(g(x_2))] \leftrightarrow [x_1 = x_2]$.
We want to show that: $\forall x_1, x_2 \in A, [g(x_1) = g(x_2)] \rightarrow [x_1 = x_2]$.

Let a_1, a_2 be arbitrary real numbers in A. $b_1 = g(a_1)$, $b_2 = g(a_2)$.

Assume $b_1 = b_2$, that is, $g(a_1) = g(a_2) \in B$.
Then $f(b_1) = f(b_2)$.
Then $f(g(a_1)) = f(g(a_2))$.
Then from (2) we know: $a_1 = a_2$.
$\therefore$ We have proved that if f and $f \circ g$ are one-to-one, then g is one-to-one.

7. One Hit Wonder [18 points]

For this problem, we will define two new properties. Let S be a set and $f \colon S \to S$ be some function.

We say f is a *one hit wonder* if:

$$\forall x \in S\,[(f \circ f)(x) = f(x)].$$

Some examples of one-hit wonders from $\mathbb{R} \to \mathbb{R}$ are the absolute value function, the ceiling function, and the function which sends every number to 0.

We say f *does nothing* if:

$$\forall x \in S\,[f(x) = x].$$

(a) Prove that if f does nothing, then it is a one-hit wonder.

(b) Prove that if f is a one hit wonder and is one-to-one, then f does nothing.

(c) Prove that if f is a one hit wonder and is onto, then f does nothing.

Solution:

(a) Let a be an arbitrary element of S.
Assume f does nothing, then $f(a) = a \in S$.
Since $f(a) \in S$, then $\exists f(f(a))$, and $f(f(a)) = f(a) = a$.
Then $f \circ f(a) = f(a)$.
$\therefore$ We have proved that if f does nothing, then it is a one-hit wonder.

(b) Let a be an arbitrary element of S.
Assume f is a one hit wonder and is one-to-one.
Since f is a one hit wonder, $[(f \circ f)(a) = f(a)]$, $f(f(a)) = f(a)$.
Since f is one-to-one,$[f(f(a)) = f(a)] \to f(a) = a$.
$\therefore$ for any $x \in S$, $f(x) = x$, i.e. f does nothing.
$\therefore$ If f is a one hit wonder and is one-to-one, then f does nothing.

(c) Let b be an arbitrary element of S.
Assume f is a one hit wonder and is onto.
Since f is onto, $\exists a \in S, f(a) = b$.
Since f is a one hit wonder, $f(f(a)) = f(a) = b$.
Since $f(a) = b$ and $f(f(a)) = b$, $f(b) = b$.
$\therefore$ for any $x \in S$, $f(x) = x$, i.e. f does nothing.
$\therefore$ If f is a one hit wonder and is one-to-one, then f does nothing.