10/24/23

Function composition

Pigeonhole principle (抽屉原理)

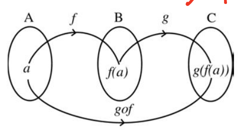
Lec 16: Composition Proofs & Pigeonhole Principle -- ANSWERS

recap: $g \circ f(x) = g(f(x))$

① Theorem 1: f, g 满射且 $\exists\, g \circ f(x)$, 则 $g \circ f(x)$ 满射

Proof: f, g onto $\rightarrow g \circ f$ onto

Simple logic: every point in C can be reached by some point in B, and every point in B can be reached by some point in A, so every point in C can be reached by some point in A.



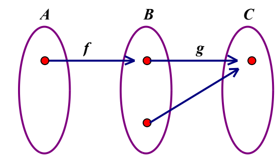
- $f: A \rightarrow B$ is onto, then by definition
  - $\forall b \in B\ \exists a \in A\ [f(a) = b]$
- $g: B \rightarrow C$ is onto, then by defn
  - $\forall c \in C\ \exists b \in B\ [g(b) = c]$
- Consider any element $c_0 \in C$
  - g is onto so there must be some element b_0 with $g(b_0) = c_0$.
  - f is onto so there must be some element a_0 with $f(a_0) = b_0$.
  - So $(g \circ f)(a_0) = g(f(a_0)) = g(b_0) = c_0$.
  - So $\exists a \in A$ such that $(g \circ f)(a) = c_0$.
- $\forall c \in C\ \exists a \in A\ [(g \circ f)(a) = c]$
- So $g \circ f$ is onto (by definition of onto)

②

Proof: $g, g \circ f$ onto $\not\rightarrow$ f onto

- Define functions f and g as follows:

Simple logic: Every point in C can be reached by some point in B and A, but there's no "every point" in B all along.



- Every element in C can be reached from B (g is onto)
- Every element in C can be reached from A ($g \circ f$ is onto)
- There is an element in B that cannot be reached from A (f is not onto)

ex: How many cards must you be dealt from the standard 52 card deck to guarantee 2 cards of the same unit?

4 ✗
5 ✓

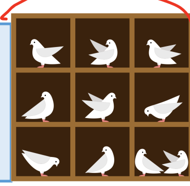
Pigeonhole Principle

③

***Generalized* Pigeonhole Principle**

- If there are P pigeons in H pigeonholes, then there must be at least $\left\lceil \frac{P}{H} \right\rceil$ pigeons in some hole.

④ $\left\lceil \frac{P}{H} \right\rceil$: ceil $\frac{P}{H}$ (向上取整)



$\left\lfloor \frac{P}{H} \right\rfloor$: floor $\frac{P}{H}$ (向下取整)

1. How many cards do you need to draw from a standard deck to guarantee that you get 2 cards with the same suit? **4+1= 5** Holes = suits (4) Pigeons = cards
2. How many people to guarantee that 2 people have the same birth month? 12+1 = **13** Holes = birth months (12) Pigeons = people
3. How many people to guarantee that **5** people have the same birth month? 4(12)+1 = **49** Holes = birth months (12) Pigeons = people
4. Do two EECS 203 students have the same 2-letter initials? 985 students total #holes = (26 options for 1st initial) * (26 options for last initial) = 676 26*26+1 = **677** Holes = 2-letter initials Pigeons = students (985)

eg: $\left\lceil \frac{49}{12} \right\rceil = 5$

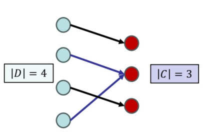
$\left\lfloor \frac{49}{12} \right\rfloor = 4$

Officially, the pigeonhole principle is a statement about **injective functions**.

⑤ **The Pigeonhole Principle:**

If $f : D \rightarrow C$ is a function, and $|D| > |C|$, then f is not an injection.

that is, there are distinct $d_1, d_2 \in D$ with $f(d_1) = f(d_2)$



P = a set of 13 people
M = the 12 months

Function $birthMonth : P \rightarrow M$ maps people to the month they were born

By **the pigeonhole principle**, $birthMonth$ is not an injection.
There are distinct $p_1, p_2 \in P$ with $birthMonth(p_1) = birthMonth(p_2)$

Prove or Disprove: Exists x, y with Sum of 9

Prove or disprove:

For all sets $S \subseteq \{1,2, \dots, 8\}$, if $|S| \geq 5$, then there exist $x, y \in S$ with $x + y = 9$.

Pair off the elements of S like this.

- There are 4 **holes** (pairs)
- There are 5 **pigeons** (elements of S)
- By the pigeonhole principle, S contains two elements x, y in the same pair.
- These two elements have $x + y = 9$.

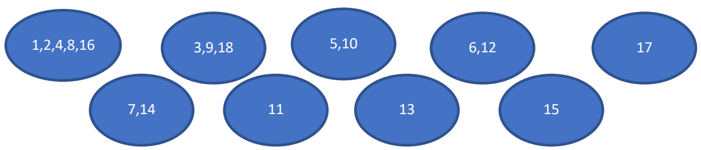
1 8 | 2 7 | 3 6 | 4 5

Prove : Exists distinct x, y where $x|y$

Prove:

For all sets $S \subseteq \{1,2,\ldots,18\}$, if $|S| \geq 10$, then there exist **distinct** $x, y \in S$ with $x|y$.

- **Holes:** Consider the partition of $\{1,2,\ldots,18\}$ represented below. There are 9 parts.
  - Note: This is not the only partition that would work. Any partition where each element in the set divides each larger element in the set would work.
- **Pigeons:** Elements in S. There are $|S| \geq 10$ pigeons.
- By the **pigeonhole principle** there exist two distinct elements $x, y \in S$ in the same part.
- By our choice of parts, we have $x|y$ or $y|x$.



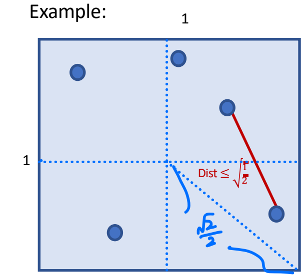
Point Placement

Prove:

For all ways of placing 5 points in a 1×1 square, there exist two points at distance $\leq \sqrt{\frac{1}{2}}$

- **Pigeons =** points. There are 5 pigeons.
- **Holes =** subsquares. There are 4 holes.
- So, **by the pigeonhole principle**, there is a subsquare with at least 2 points in it
  - Find distance using $a^2 + b^2 = c^2$
  - distance $\leq \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$

Example:



eg: $7 \times 158730 = 1111110$

Decimal Digits

Prove:

For every integer n, there exists a multiple of n that has only 0s and 1s in its decimal expansion.

1
11
111
1,111
11,111
111,111
1,111,111
11,111,111
⋮

- **(Holes)** A number can have n possible values (mod n)
- **(Pigeons)** Consider the first $n+1$ integers written with just 1s: $\{1, 11, 111, \ldots\}$
- **By the pigeonhole principle**, there exist distinct numbers x, y both written with only $1's$, that have the same value $(mod\ n)$. That is, $x \equiv y\ (mod\ n)$.
- Consider the number $x - y$ (WLOG, assume $x > y$)
  - Since $x \equiv y\ (mod\ n)$, we know that $x - y$ is a multiple of n
  - Since x and y are both written with all 1s, then $x - y$ has only 1s and 0s in its decimal expansion
  - So the number $x - y$ satisfies the proposition.

Party Pals

There is a party with $n \geq 2$ people. Each pair of people either chats, or doesn't chat.
Prove: there are two people at the party who chat with the same number of people.

We will use a proof by cases. Either there is someone who chats with 0 others, or everyone at the party chats with at least 1 other.

Case 1: Assume there is a person who chats with 0 others.

- **(Pigeons)** There are n people at the party
- **(Holes)** Everyone at the party chats with $\{0, \ldots, n-2\}$ others. Nobody chats with themselves, nor with the person who chats with 0 others.
- There are n pigeons and $n-1$ holes, so by the **Pigeonhole Principle**, there are two people who chat with the same number of other people.

Case 2: Assume everyone at the party chats with at least 1 other.

- **(Pigeons)** There are n people at the party.
- **(Holes)** Everyone at the party chats with $\{1, \ldots, n-1\}$ others. Nobody chats with themselves, and we assume at least 1 chat
- There are n pigeons and $n-1$ holes, so by the **Pigeonhole Principle**, there are two people who chat with the same number of other people.

New Property Composition Proof

- Definition: $f: \mathbb{R} \to \mathbb{R}$ is "even" if
  - $\forall x, f(-x) = f(x)$
- Definition: $f: \mathbb{R} \to \mathbb{R}$ is "odd" if
  - $\forall x, f(-x) = -f(x)$

Reminder: You don't need to memorize these specific properties

- Consider arbitrary functions $f, g: \mathbb{R} \to \mathbb{R}$
- Suppose f is even and g is odd. Prove that $f \circ g$ is even
- Goal: prove $\forall x, (f \circ g)(-x) = (f \circ g)(x)$
  - Consider an arbitrary x
  - $(f \circ g)(-x) = f(g(-x))$
  - $= f(-g(x))$ (g is odd)
  - $= f(g(x))$ (f is even)
  - $= (f \circ g)(x)$
- $\forall x, (f \circ g)(-x) = (f \circ g)(x)$, so $f \circ g$ is even!

Stable Subsequences [Erdos-Szekeres theorem]

Let S be a sequence of $n^2 + 1$ distinct integers. Prove that S contains an **increasing** or **decreasing** subsequence of length $n + 1$.

Proof.

- Label each number in S with a pair of numbers (i, d) where:
  - i : the length of the longest **increasing** subsequence *that ends at that number.*
  - d : the length of the longest **decreasing** subsequence *that ends at that number.*
- **Seeking contradiction**, assume that the max label size is $(\boldsymbol{n}, \boldsymbol{n})$.
  - By **pigeonhole principle**, there are two numbers with the same label $(\boldsymbol{i}, \boldsymbol{j})$.
    - Whether the second number is **larger or smaller**, then we get a **contradiction**: labels can't be exactly the same.

possible number of pairs:
$n \cdot n = n^2 < n^2+1$

so

5	0	4	9	8	1	3	7	2	6
(1,1)	(1,2)	(2,2)	(3,1)	and so on		(i, j)	$(\boldsymbol{i+1}, j)$	$(i, j+1)$	

i) If the next num is smaller, then we can build a dereasing sequence of length j+1 by adding it to the j-seq ending at the num
ii) larger, i+1 increasing.

Subset Sums

Prove:

For all sets $S \subseteq \{1,2,\ldots,100\}$ with $|S| = 10$, there exist two distinct subsets $A, B \subseteq S$ where the sum of the elements in A is equal to the sum of the elements in B.

- **(Pigeons)** Subsets of S.
- There are $|P(S)| = 2^{10} = 1024$ pigeons.

>900 (∵不可能1024个独立value)

$(1+10) \cdot 10/2 = 55$

(Actually: $(91+100) \cdot 10/2 = 955$)

- **(Holes)** Possible Sums.
- Every subset of S sums to a number that is at least 0, and at most $10 \cdot 100 = 1000$

Note: we could get a tighter upper bounds on our subset sums by thinking harder. But we don't need to!

- So, **by the pigeonhole principle**, there exist two distinct subsets of S that have the same sum.