permutations
combinations
multinomial coefficients

11/14/23

Permutation

Arrange k of n objects:
1. line all people up : $n!$
2. Don't care about the $(n-k)$ unselected people: divide by $(n-k)!$ (by product rule)

$\Rightarrow P(n,k) = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$

Counting Blitz: Strings of English Letters

- How many strings of 8 English letters are there
  - 1) That contain no vowels, if letters can be repeated? 21^8
  - 2) That contain no vowels, if letters cannot be repeated? $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$
  - 3) That start with a vowel, if letters can be repeated? $5 \cdot 26^7$
  - 4) That contain at least one vowel, if letters can be repeated? $26^8 - 21^8$
  - 5) That contain exactly one vowel, if letters can be repeated? $8 \cdot 5 \cdot 21^7$
  - 6) That contain exactly 2 vowels, letters can be repeated? $\frac{8 \cdot 7}{2} \cdot 5^2 \cdot 21^6$
  - 7) That contain exactly 2 vowels, not consecutive, letters can be repeated?

$$\frac{(2 \cdot 6 + 6 \cdot 5)}{2} \cdot 5^2 \cdot 21^6$$

L22 Handout: Permutations and Combinations -- ANSWERS

$n!$ (read: n factorial) = $n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

Permutations

$$P(n,k) = \frac{n!}{(n-k)!}$$

- # ways to select a sequence of k things from a set of size n
- Order matters? Yes

Ex: Group of n people. Pick k of them and line them up for a picture.

Combinations

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

- # ways to select a set of k things from a set of size n
- Order matters? No

Ex: Group of n people.
Choose k of them to get ice cream.

Combination

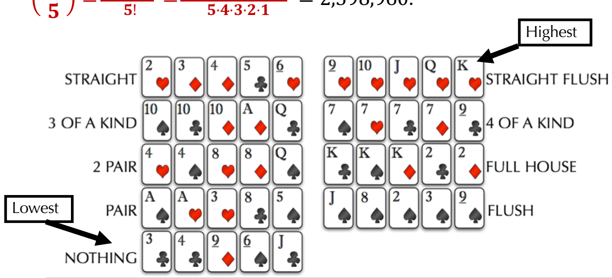
Choose k out of n objects:
1. $P(n,k)$
2. Don't care about the order of k selected either: divide by $k!$,

$\Rightarrow \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$

Poker Hands

- A deck consists of 52 cards (13 ***ranks*** in 4 ***suits***).
- How many ***5-card hands*** are there?

$$\binom{52}{5} = \frac{P(52,5)}{5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960.$$



Poker Hands

- How many ways are there to make a pair (and nothing better)?
- **Stage 1:** pick the rank of the pair
  - 13 options.
- **Stage 2**: pick the two suits of the pair
  - $\binom{4}{2} = 6$ ways to pick 2 suits.
- **Stage 3**: pick the 3rd card of a ***different*** rank: 48 options.
- **Stage 4**: pick the 4th card of a ***different*** rank (than either of previous two): 44 options.
- **Stage 5**: pick the 5th card of a ***different*** rank (than any of previous three): 40 options.

This process could pick each hand in 3! different ways
3! ways to order cards selected in last 3 steps

Answer: $13 \cdot \binom{4}{2} \cdot \frac{48 \cdot 44 \cdot 40}{3!}$

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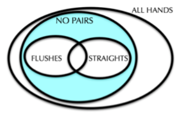
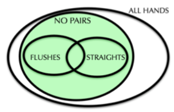
(# nothing) = (# no pairs) − (# straight and flushes)
= (# no pairs) − [(# straight) + (# flushes) − (# straight flushes)]

Exercise: How many ways to make nothing?

- We're counting hands:
  - (1) without pairs
  - (2) that also do not contain straights or flushes



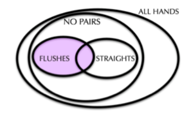
Counting hands without pairs



- **Solution 1:**
- Stage 1: pick any card. 52 choices.
- Stage 2: pick any card of a new rank. 48 choices.
- Stage 3: pick any card of a new rank. 44 choices.
- Stage 4: pick any card of a new rank. 40 choices.
- Stage 5: pick any card of a new rank. 36 choices.
- Observation: every hand could have been picked in 5! ways.
- $\frac{52\cdot 48\cdot 44\cdot 40\cdot 36}{5!}$ hands with no pairs.

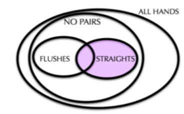
- **Solution 2:**
- Stage 1: Pick 5 ranks $\binom{13}{5}$ choices
- Stage 2: Pick a suit for each rank. 4^5 choices
  - (suit for highest, then 2nd highest, ...)
- $\binom{13}{5}\cdot 4^5$ hands with no pairs.

Counting flushes



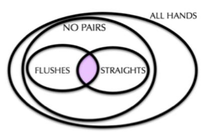
- Stage 1: pick a suit. 4 choices.
- Stage 2: pick a set of 5 cards in that suit. $\binom{13}{5} = \frac{13!}{8!5!}$
- Every hand is picked in exactly one way: $4\binom{13}{5}$ flushes.

Counting straights



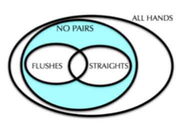
- Stage 1: pick the lowest rank. 10 choices {A,2,3,...,10}.
- Stage 2: pick the suit of the lowest card. 4 choices.
- ...
- Stage 6: pick the suit of the highest card. 4 choices.
- Each hand picked in exactly one way. $10\cdot 4^5$ straights.

Counting straight flushes



Stage 1: pick the suit. 4 choices.
Stage 2: pick the lowest rank. 10 choices.
(Once stages 1 and 2 are done all 5 cards are determined. There is nothing more to decide.)
$4\cdot 10$ straight flushes.

#Ways to make "nothing"



= (#without pairs) - (#flushes) - (#straights) + (#straightflushes)

$$= 4^5\binom{13}{5} - 10\cdot 4^5 - 4\binom{13}{5} + 4\cdot 10$$

Generalized Permutations/Combinations

How many anagrams are there of **MISSISSIPPI**?

Method 1: Place groups of letters

Stage 1: place the **M**. $\binom{11}{1}$ choices.

Stage 2: place the **I**s. $\binom{10}{4}$ choices

Stage 3: place the **S**s. $\binom{6}{4}$ choices

Stage 4: place the **P**s. $\binom{2}{2}$ choices

One answer: $\binom{11}{1}\cdot\binom{10}{4}\cdot\binom{6}{4}\cdot\binom{2}{2}$

Method 2: Division rule

There are 11! ways to place the letters initially

There are 4! ways to scramble the I's, which doesn't change the permutation
$I_1SSI_2PPI_3SSI_4M = I_3SSI_4PPI_2SSI_1M$

There are 4! ways to scramble the S's, which doesn't change the permutation

There are 2! ways to scramble the P's, which doesn't change the permutation

Equivalent Answer: $\frac{11!}{4!4!2!}$

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since we can swap same letter's order

Combinatorial Proofs vs. Algebraic Proofs

- ***Algebraic proof*** of identity: $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

- ***Combinatorial proof***. Count the same thing in two different ways.
  - **Story**: We have n faculty members and need to form a committee of k of them
  - **LHS**: There are $\binom{n}{k}$ ways to choose the k people on the committee.
  - **RHS**: On the other hand, this is the same as picking $n-k$ faculty members to ***not*** serve on the committee. There are $\binom{n}{n-k}$ ways to choose the $n-k$ non-members.
  - These count the same thing! So $\binom{n}{k} = \binom{n}{n-k}$.

Combinatorial Proofs vs. Algebraic Proofs

- **Pascal's Identity**: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- **Combinatorial proof.**
- **Story**: There are $n+1$ people, n freshmen and 1 sophomore, and you are forming a team of k people.
- **LHS**: Put everyone together into 1 group and pick k from it.
  - There are $\binom{n+1}{k}$ ways to do this.
- **RHS**: Consider separately the cases where the sophomore is or is not on the team:
  - Case 1: sophomore ***is not*** the team.
    - Pick the k team members from the n freshmen: $\binom{n}{k}$ ways to do do this.
  - Case 2: sophomore ***is*** on the team.
    - Put the sophomore on the team
    - Then, pick the other $k-1$ team members from the n freshmen: $\binom{n}{k-1}$ ways
  - **(Sum Rule)** Total ways for RHS = $\binom{n}{k} + \binom{n}{k-1}$
- These count the same thing! So they are equal.