Groupwork

1. Grade Groupwork 8

Using the solutions and Grading Guidelines, grade your Groupwork 8:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2	2/2	2/2	3/3	3/3	2/2							12 /12
Problem 3	5/5	5/5	3/3	0/3	2/4	5/5	5/5					25 /30
Total:												37 /42

Previous Groupwork 8(1): Divisibility by Seven [12 points]

In this question we will show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7. You are allowed to cross off zero digits.

For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number $2345 = 7 \cdot 335$.

We will label the digits of an arbitrary 7-digit number as

$$x_6 x_5 x_4 x_3 x_2 x_1 x_0.$$

(a) Prove that there exists some $i < 7$ such that either $x_i x_{i-1} \ldots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_j x_{j-1} \ldots x_0$ is congruent to it modulo 7.

(b) Use part (a) to prove that if there does not exist some $i < 7$ such that $x_i x_{i-1} \ldots x_0$ is divisible by 7, then there exists $7 > i > j \geq 0$ so that

$$\underbrace{x_i x_{i-1} \ldots x_{j+1} 0 \ldots 0}_{i+1 \text{ digits total}}$$

is divisible by 7.

(c) Prove the full claim. That is, show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7.

Solution:

That was excellent! Nice job!

(a) There are 7 numbers: $x_6x_5x_4x_3x_2x_1x_0$, $x_5x_4x_3x_2x_1x_0$, $\cdots$, x_0.
Use them as Pigeons.
The remainer of any number mod 7, if the number cannot be divided by 7, has 6 circumstances: 1,2,3,4,5,6
Use them as Holes.
Assume that all these 7 numbers is not divisible by 7, then there are 7 Pigeons in 6 holes, which means that there are at least 2 Pigeons in one whole.
This means that $i < 7$ and $j < i$ such that $= x_i x_{i-1} \ldots x_0$ and $x_j x_{j-1} \ldots x_0$ have the same remainer when divided by 7.
Mark the two numbers as a and b.
Then $b \equiv a \pmod 7$.
$(b - a) \equiv 0 \pmod 7$.
Then we have proved the statement: There exists some $i < 7$ such that either $x_i x_{i-1} \ldots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_j x_{j-1} \ldots x_0$ is congruent to it modulo 7.

+2

+2

+3

(b) If there does not exist some $i < 7$ such that $x_i x_{i-1} \ldots x_0$ is divisible by 7, then from (a) who know: there exists some $j < i$ such that $x_j x_{j-1} \ldots x_0$ is congruent to it.
i.e. $(x_i x_{i-1} \ldots x_0 - x_j x_{j-1} \ldots x_0) \equiv 0 \pmod 7$.
that is, $x_i x_{i-1} \cdots x_{j+1} 0 \cdots 0 \equiv 0 \pmod 7$.
$\therefore$ We have proved the statement. ✓

+3

(c) Given a 7-digit number $x_6x_5x_4x_3x_2x_1x_0$,
Case 1: $\exists i < 7$ such that $x_i x_{i-1} \ldots x_0$ is divisible by 7.
Then if $i = 6$, the whole digit is divisible by 7, else we can just cross out all digits from x_6 to x_{i+1}. The remaining digit $x_i x_{i-1} \ldots x_0$ is divisible by 7.

Case 2: $\nexists i < 7$ such that $x_i x_{i-1} \ldots x_0$ is divisible by 7.

Then through (b) we know: there exists some $j < i$ such that $x_i x_{i-1} \ldots x_{j+1} 0 \ldots 0$ is divisible by 7.
We know that $x_i x_{i-1} \ldots x_{j+1} 0 \ldots 0 = x_i x_{i-1} \ldots x_{j+1} \times 100 \cdots$
Then $x_i x_{i-1} \ldots x_{j+1} \times 100 \cdots \equiv 0 \pmod 7$. +2
Since $100 \cdots \not\equiv 0 \pmod 7$
There must be $x_i x_{i-1} \ldots x_{j+1} \equiv 0 \pmod 7$.
So we can rule out all digits before x_i and after x_j to get a number divisible by 7.

Previous Groupwork 8(2): A Powerful Proof [30 points]

In this question we will prove that for any set X, $|\mathcal{P}(X)| > |X|$ ($\mathcal{P}(X)$ is the power set of X). Note that while this is simple in the case where X is finite, things get more complicated when we allow X to be infinite. This proof covers all cases.

(a) Show that for all (possibly infinite) sets X, $|\mathcal{P}(X)| \geq |X|$.

(b) Let $g \colon X \to P(X)$ be an arbitrary function. Show that the set $D := \{a \in X \mid a \notin g(a)\}$ is not in the range of g.

(c) Explain why this shows that $|P(X)| \leq |X|$ is false and conclude the proof.

(d) Based on your conclusions above, are there uncountable sets "larger" than $\mathbb{R}$? Explain.

(b) is quite abstract and we lost some points, but that was almost right and we could do better!

Solution:

(a) We define $f : X \to \mathcal{P}(X)$: +5
The map of every element in X to a set in $\mathcal{P}(X)$ that only contains that element.
That is: for $X = \{a, b, c, \cdots\}$, $f(a) = \{a\}$, $f(b) = \{b\}, \cdots$. +5
Since for every element in X, we can find a subset of X that only contains that element, and every subset is unique, we know this is a one-to-one function.
$\therefore |X| \leq |\mathcal{P}(X)|$.

(b) The range of g is $\{g(a) | a \in X\}$.
D is a subset of X which contains all the elements that are not an element of its image. We know $D \in \mathcal{P}(X)$. +3
Assume $D \in range(X)$, then $\exists x \in X$ such that $g(x) = D$, which is an element of $\mathcal{P}$.
(1) If $x \in D$, then due to the definition of D, $x \notin g(x) = D$. (2) if $x \notin D$, then
This causes contradiction. +2 $x \notin g(D) \Rightarrow$ contradiction
$\therefore D$ is not in the range of g.
(4-2) (lack of case, -3)

(c) Since every element in D is also in X, it is a subset of X, so D is an element of $\mathcal{P}(X)$, which is in the codomain of g.
However, D is not in the range of g.
$\therefore$ the $range(g) < codom(g)$, there does not exist an onto function g from X to $\mathcal{P}(X)$.
$\therefore |\mathcal{P}(X)| \leq |X|$ is false, $|\mathcal{P}(X)| > |X|$.

+5

(d) From c, we can know that $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. This is a "larger" set than $\mathbb{R}$.

+5

2. Square the Cycle [15 points]

Prove that every n-node graph ($n \geq 3$) in which all nodes have degree at least $\lceil \sqrt{n} \rceil$ has a 3-cycle subgraph or a 4-cycle subgraph.

Hint: One useful concept is the neighborhood of a vertex; the neighborhood of $v \in V$ is the set $N(v) = \{u \in V : u \text{ is adjacent to } v\}$. We can also define the neighborhood of a set $A \subseteq V$:

$$N(A) = \{u \in V : u \text{ is adjacent to some } v \in A\}.$$

We recommend using a proof by contradiction, although this can also be done with a clever direct proof. Suppose a graph satisfying the above condition does not have a 3-cycle or 4-cycle. Fix a vertex $v \in V$. What can we say about the size of $N(v)$? What about $N(N(v))$?

Solution:

Seeking contradiction, assume:

$\exists$ n-node graph ($n \geq 3$) s.t. $\forall v \in V$, $\deg(v) \geq \lceil \sqrt{n} \rceil$, but it does not contain a 3-cycle or 4-cycle subgraph

Let v be an arbitrary node of it, v has $\geq \lceil \sqrt{n} \rceil$ neighbours.

$\Rightarrow$ each node in $N(v)$ also has $\geq \lceil \sqrt{n} \rceil$ deg.

$\Rightarrow |N(N(v))| \geq |N(v)| \cdot \lceil \sqrt{n} \rceil \geq \lceil \sqrt{n} \rceil \cdot \lceil \sqrt{n} \rceil \geq n$

This implies there are at least n distinct nodes in the graph, which contradicts the fact the graph has only n nodes.

$\therefore$ The graph must contain at least one cycle

$\therefore$ contradicts the assumption that it does not contain a 3-cycle or 4-cycle subgraph

QED

3. The Office Allocation [15 points]

Consider a new office building with n floors and k offices per floor in which you must assign $2nk$ people to work, each sharing an office with exactly one other person. Find a closed form solution for the number of ways there are to assign offices if from floor to floor the offices are distinguishable, but any two offices on a given floor are not.

Solution:

(1) To assign $2k \cdot n$ people to n floors, 2k people per floor

$$C(2nk, 2k) \cdot C(2(n-1)k, 2k) \cdot \ldots \cdot \underbrace{C(2k, 2k)}_{=1}$$

(2) To assign the 2k people per floor to k offices (not distinguishable)

(2a) $C(2k, 2) \cdot C(2(k-1), 2) \cdot \ldots \cdot \underbrace{C(2,2)}_{=1}$

(2b) Since k offices in a floor is not distinguishable,

$$\Rightarrow \frac{C(2k,2) \cdot C(2(k-1),2) \cdot \ldots \cdot C(2,2)}{P(k,k)}$$

(2c) Since there are n floors

$$\Rightarrow \left(\frac{C(2k,2) \cdot C(2(k-1),2) \cdot \ldots \cdot C(2,2)}{P(k,k)}\right)^n$$

(3) So in total:

$$\prod_{i=1}^{n} C(2ik, 2k) \cdot \left(\frac{\prod_{j=1}^{k} C(2j,2)}{k!}\right)^n$$