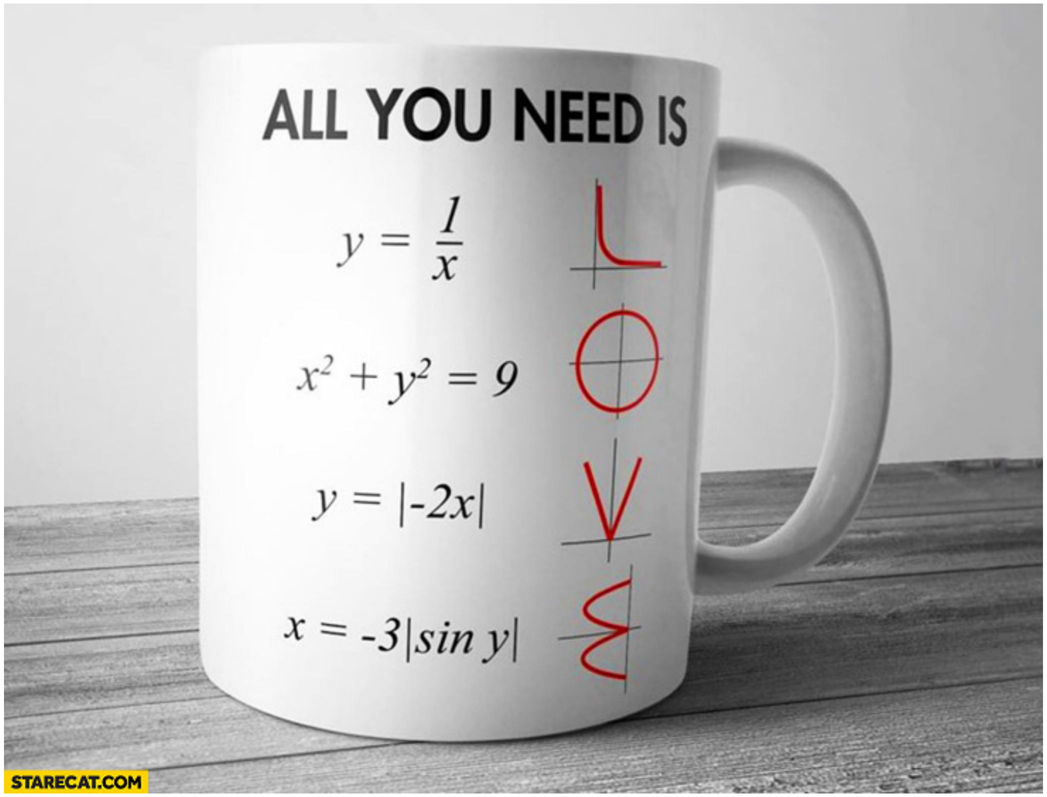
Functions



EECS 203: Discrete Mathematics

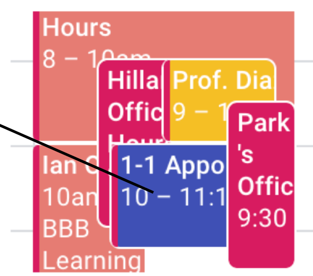
Lecture 15

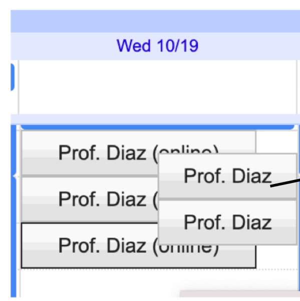
Reminder: Individual Appointments Available

As of 8 am, there are open appointment slots **today 4-5** and **tomorrow morning**

- Individual appointments available with faculty
  - Over the week or two
  - If you want to discuss your exam, your progress in the course, or anything else in a 1-on-1 format
  - See 203 Office Hour calendar for times and info
    - Look for events titled **"1-1 Appointment with [...]"**

1-on-1 appointments marked in BLUE on OH calendar. Click the link to go to the appointments page





On the appointments page, click a time that works for you, Book the appointment, then Save

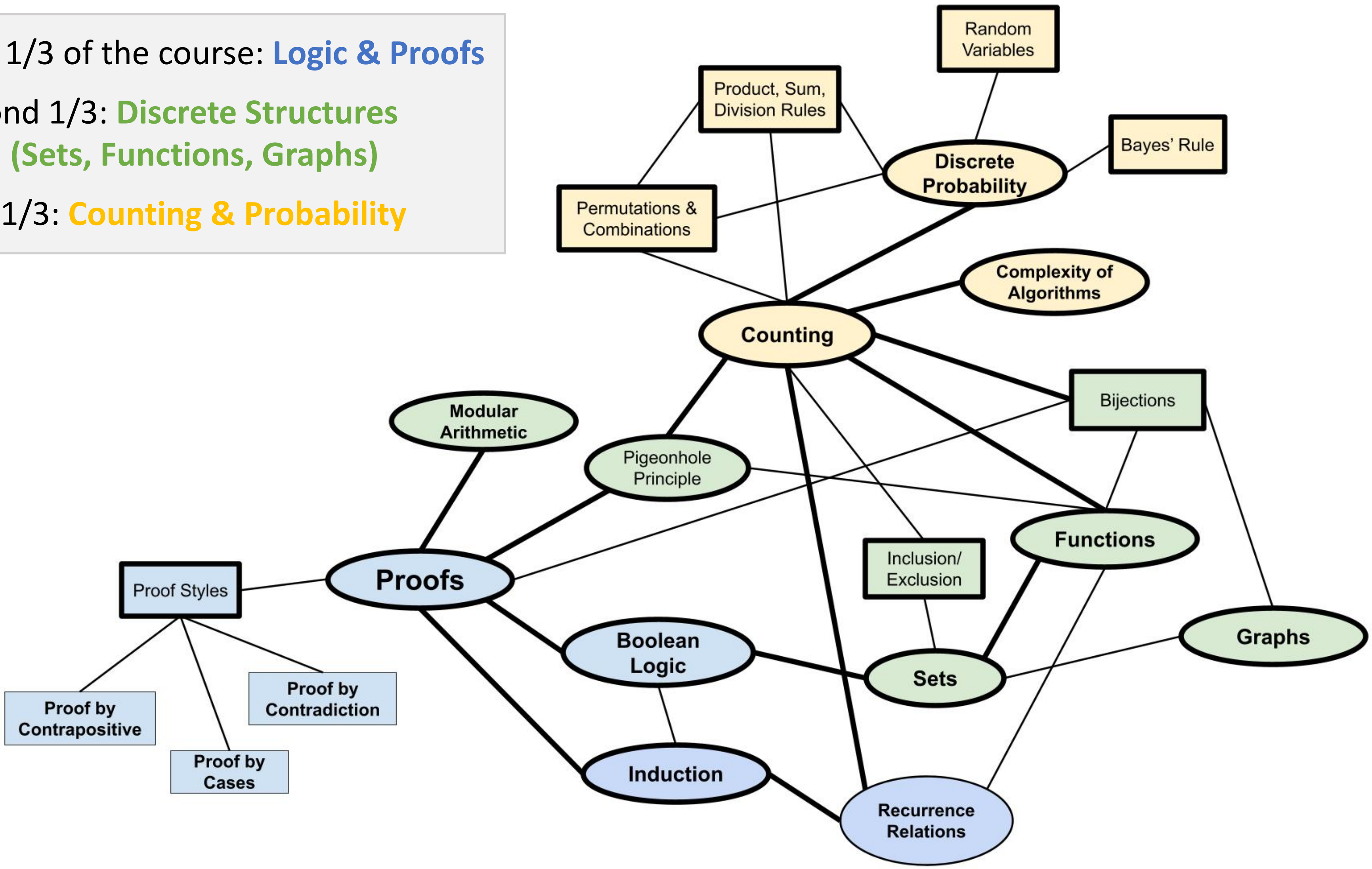
- Note: a C or better in 203 is needed to count towards a CS/CE/DS major or minor
  - If you are concerned, we have resources and advice we can share. We can discuss your individual situation in a 1-1 appt.

Fall Break due date shifts

- HW 6 / Check-in 6 due FRIDAY
- HW 4 Regrades open until tonight, 11:59pm
- Exam 1 Regrades open through Sunday, 11:59pm

The "Big Picture" of 203: How does this all fit together?

- First 1/3 of the course: **Logic & Proofs**
- Second 1/3: **Discrete Structures (Sets, Functions, Graphs)**
- Last 1/3: **Counting & Probability**



Learning Objectives: Functions & Properties

After today's lecture (and this week's readings, discussion & homework), you should be able to:

- **Know Technical vocab**: mapping, function, domain, codomain, range, onto, one-to-one, bijection, inverse function, function composition

- Function definition
  - Identify whether there is a function satisfying a given input/output mapping

- Function properties
  - Determine the properties of a given function: onto, one-to-one, bijection
  - Find inverses of bijections
  - Prove that a given function is or isn't onto, and is or isn't one-to-one

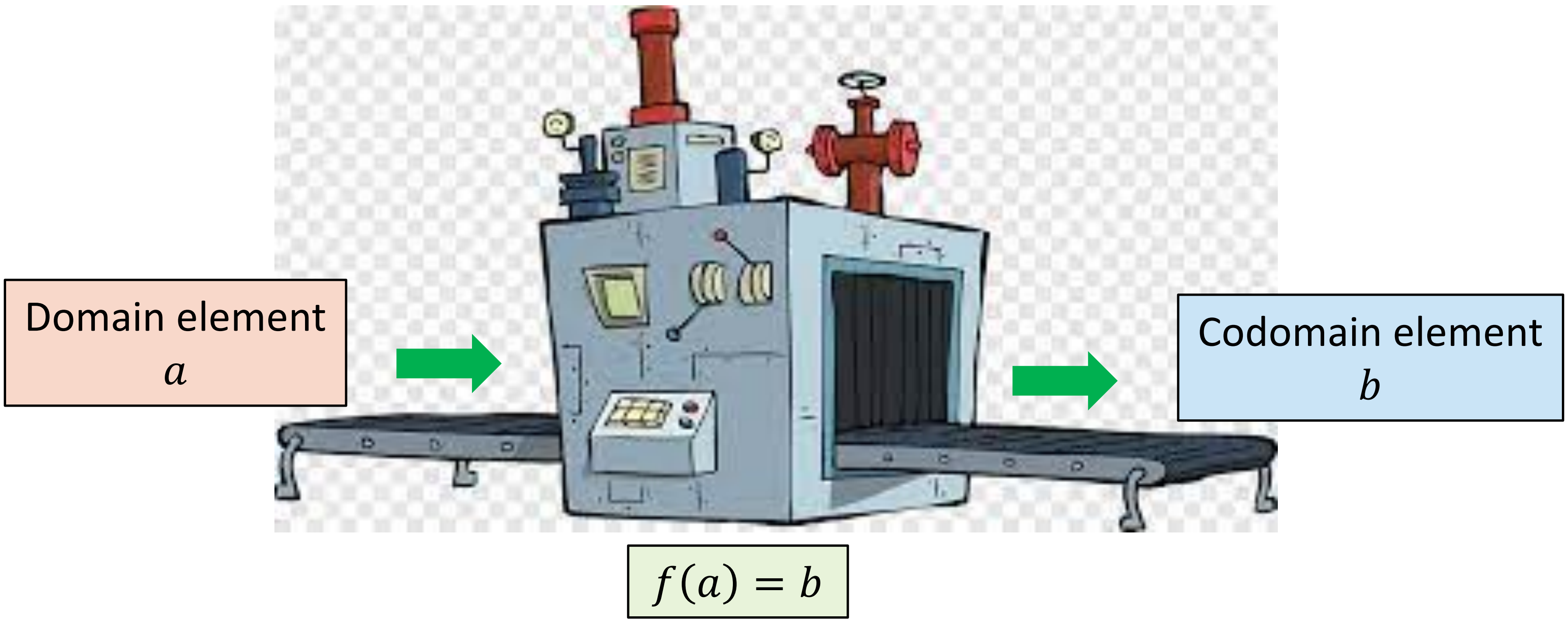
- Operations with Functions
  - Determine whether a function has an inverse, and if so, find the inverse function.
  - Find the composite function of f with g (mapping, domain & codomain)
  - Prove properties of composite functions (next time)

Outline

- **What is a function?**
  - **Definition, Domain, Codomain, Range**
  - Recognizing (non)-functions
- Function properties
  - Onto, One-to-one
  - Proofs of properties
- Bijections and Inverses
- Function Composition

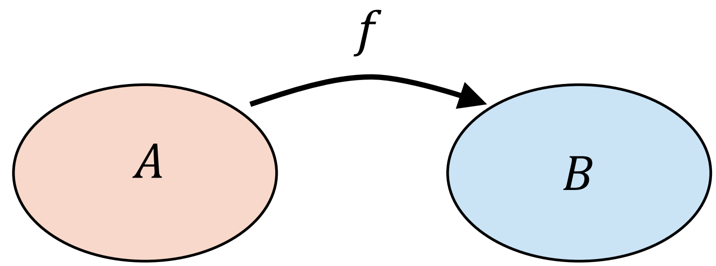
Functions

- A **function** is a **machine** where:
  - you can input an element from one given set (the "**domain**"),
  - and it outputs an element from another given set (the "**codomain**")



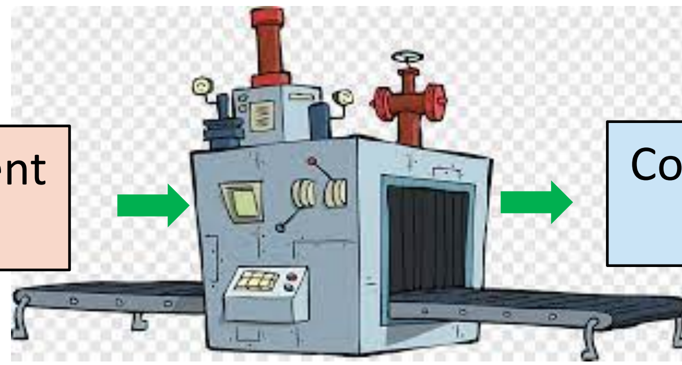
- For a function f, the output of domain element a is written $f(a)$
- The output is **exactly one** codomain element! Never 0, never ≥ 2
- But it **is** possible that two domain elements give the same codomain element as output

Writing Functions



- **Notation**: $f : A \to B$
  - Means that f is a function with **domain A** and **codomain B**

Domain element
$a \in A$



Codomain element
$b \in B$

Two ways to write a function:

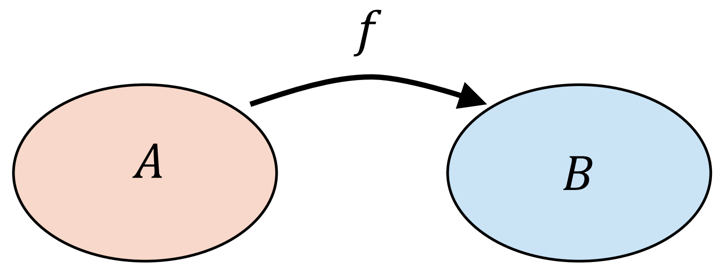
(1) Define the transformation

$$f : \mathbb{Z} \to \mathbb{N}, \boldsymbol{f(x) = x^2 + 2}$$

Domain is $\mathbb{Z}$ (you can input any integer)
Codomain is $\mathbb{N}$ (always returns a natural number)

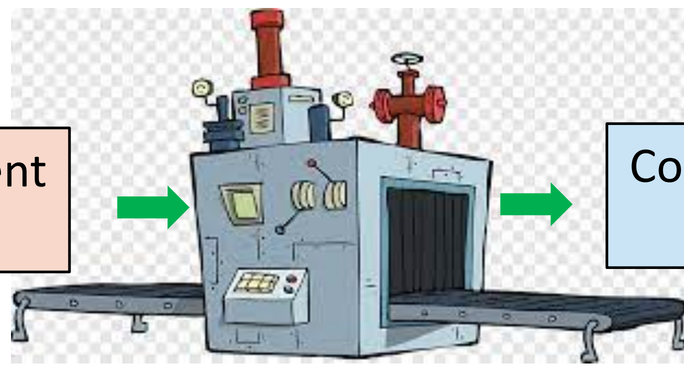
…
$f(-1) = 3$
$f(0) = 2$
$f(1) = 3$
$f(2) = 6$
…

Writing Functions



- **Notation**: $f : A \to B$
  - Means that f is a function with **domain A** and **codomain B**

Domain element $a \in A$

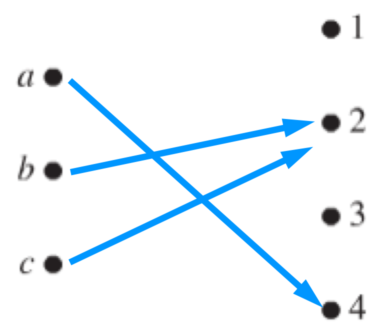


Codomain element $b \in B$

Two ways to write a function:

(2) (For **finite** domains) explicitly give the entire input/output mapping

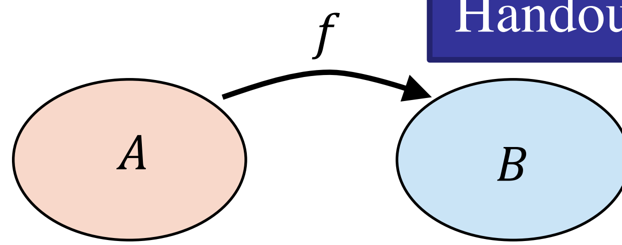
$$f : \{a, b, c\} \to \{1, 2, 3, 4\}$$



$f(a) = 4$
$f(b) = 2$
$f(c) = 2$

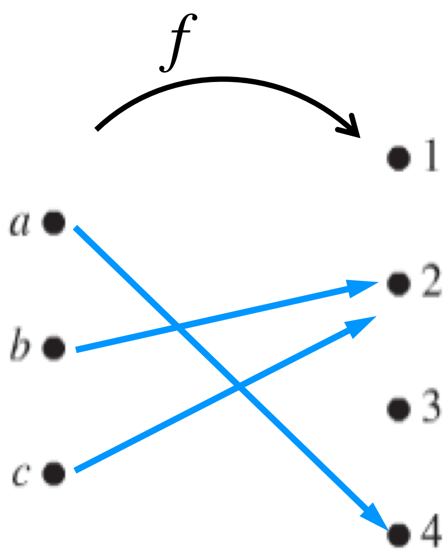
Lec 15: Functions & Properties

Handout



- A **function** $\boldsymbol{f} : \boldsymbol{A} \rightarrow \boldsymbol{B}$ is a machine where you input an element from a given set (the "________________") and it outputs an element from another given set (the "________________").

- **Key Requirements** (to be a function):
  - For every input, the function produces _______________ output
  - A mapping not necessarily satisfying these requirements is a **relation**.



Domain = ________________

Codomain = ______________

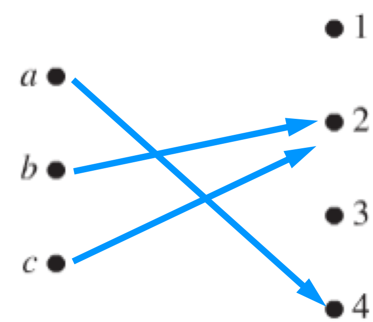
Range = ________________

Range vs. Codomain

$$f : D \rightarrow C$$

- Every element of the **domain** maps to exactly one element of the **codomain**.
- It's ok if some elements of the **codomain** are not mapped to, or are mapped to several times.
- **Range**: The set of elements in the **codomain** that **do** get mapped to at least once.

f:

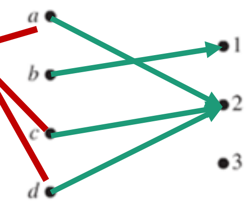


domain: {a, b, c}

codomain: {1, 2, 3, 4}
range: {2, 4}

Image and Preimage

Given a function $f: A \to B$



- For $x \in A$, the **image** of x is value of $f(x)$
  - Because f is a function, there is exactly 1 image of a value
- For $y \in B$, a **preimage** of y is a value x where $f(x) = y$
  - Note there may be 0 or multiple preimages for a given y

- Sometimes rather than using a single value for an input, we use a **set** of values: $f(\{a, b, c\})$
  - This returns the **set** of output values:
    - $f(\{a, b, c\}) = \{f(a), f(b), f(c)\} = \{2,1,2\} = \{1,2\}$
    - We also call $\{1,2\}$ the **image** of $\{a, b, c\}$

- We use the inverse notation for a **preimage** function (even if a function doesn't have an inverse,):
  - $f^{-1}(S) = \{x \mid f(x) \in S\}$, the set of all x that map to something in S
  - $f^{-1}(\{1,2\}) = \{x \mid f(x) \in \{1,2\}\} = \{a, b, c, d\}$

Example

- The **floor** function of x, written $\lfloor x \rfloor$, maps x to the largest integer smaller than or equal to x.
  - Example: $\lfloor 3.2 \rfloor = 3, \;\; \lfloor -2.1 \rfloor = -3, \;\; \lfloor 5 \rfloor = 5$
  - Say **Domain** = **Codomain** = $\mathbb{R}$
  - What's the **range** of this function?

a. $\mathbb{R}$

b. $\mathbb{R}^+$

c. $\mathbb{N}$

d. $\mathbb{Z}$

e. $\mathbb{Z}^+$

Example

- The **floor** function of x, written $\lfloor x \rfloor$, maps x to the largest integer smaller than or equal to x.
  - Example: $\lfloor 3.2 \rfloor = 3, \quad \lfloor -2.1 \rfloor = -3, \quad \lfloor 5 \rfloor = 5$
  - Say **Domain** = **Codomain** = $\mathbb{R}$
  - What's the **range** of this function?

a. $\mathbb{R}$

b. $\mathbb{R}^+$

c. $\mathbb{N}$

d. $\mathbb{Z}$

e. $\mathbb{Z}^+$

Outline

- What is a function?
  - Definition, Domain, Codomain, Range
  - **Recognizing (non)-functions**
- Function properties
  - Onto, One-to-one
  - Proofs of properties
- Bijections and Inverses
- Function Composition

Is That A Function?

One type of problem we'll see:

- Given a **description of input/output pairs**, is there a function with that particular input/output mapping?
  - Every **domain** element paired with exactly one **codomain** element → yes
  - Some **domain** element paired with 0 or ≥ 2 **codomain** elements → no

Is there a function $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = y$ iff $x = y^2$?

No: this "function" would need to have $f(1) = 1$ and $f(1) = -1$.

Handout

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and

   getChild(x) = y iff y is the child of x?

   A. Yes
   B. No

2. Is there a function getMother, with domain/codomain both the set of all people, and

   getMother(x) = y iff y is the mother of x?

   A. Yes
   B. No

3. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   $f(x) = y$ iff $x = y^2$?

   A. Yes
   B. No

4. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   $f(x) = y$ iff $y = 1/x^2$

   A. Yes
   B. No

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and

   getChild(x) = y iff y is the child of x?

   A. Yes
   B. No

- Some people x have 0 children (getChild(x) no mapping)
- Some people have $\geq$ 2 children (getChild(x) multiply mapped)

2. Is there a function getMother, with domain/codomain both the set of all people, and

   getMother(x) = y iff y is the mother of x?

   A. Yes
   B. No

3. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   $f(x) = y$ iff $x = y^2$?

   A. Yes
   B. No

4. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   f(x) = y iff $y = 1/x^2$

   A. Yes
   B. No

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and

   getChild(x) = y iff y is the child of x?

   A. Yes
   B. No

- Some people x have 0 children (getChild(x) no mapping)
- Some people have $\geq$ 2 children (getChild(x) multiply mapped)

2. Is there a function getMother, with domain/codomain both the set of all people, and

   getMother(x) = y iff y is the mother of x?

   A. Yes
   B. No

- Not everyone has exactly 1 mother (even if you limit it to “biological”)
- Nature is very rarely discrete, and there are many ways this line blurs

3. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   $f(x) = y$ iff $x = y^2$?

   A. Yes
   B. No

4. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

   f(x) = y iff y = $1/x^2$

   A. Yes
   B. No

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and

getChild(x) = y iff y is the child of x?

A. Yes
B. No

- Some people x have 0 children (getChild(x) no mapping)
- Some people have $\geq$ 2 children (getChild(x) multiply mapped)

2. Is there a function getMother, with domain/codomain both the set of all people, and

getMother(x) = y iff y is the mother of x?

A. Yes
B. No

- Not everyone has exactly 1 mother (even if you limit it to "biological")
- Nature is very rarely discrete, and there are many ways this line blurs

3. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

$$f(x) = y \text{ iff } x = y^2?$$

A. Yes
B. No

- Some x have 0 real square roots ($f(-1)$ no mapping)
- Some x have 2 real square roots ($f(4)$ multiply mapped, to 2 and -2)

4. Is there a function f: $\mathbb{R} \rightarrow \mathbb{R}$ with

f(x) = y iff y = $1/x^2$

A. Yes
B. No

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and

   getChild(x) = y iff y is the child of x?

A. Yes
B. No

- Some people x have 0 children (getChild(x) no mapping)
- Some people have $\geq$ 2 children (getChild(x) multiply mapped)

2. Is there a function getMother, with domain/codomain both the set of all people, and

   getMother(x) = y iff y is the mother of x?

A. Yes
B. No

- Not everyone has exactly 1 mother (even if you limit it to "biological")
- Nature is very rarely discrete, and there are many ways this line blurs

3. Is there a function f: $\mathbb{R} \to \mathbb{R}$ with

   $f(x) = y$ iff $x = y^2$?

A. Yes
B. No

- Some x have 0 real square roots ($f(-1)$ no mapping)
- Some x have 2 real square roots ($f(4)$ multiply mapped, to 2 and -2)

4. Is there a function f: $\mathbb{R} \to \mathbb{R}$ with

   f(x) = y iff y = $1/x^2$

A. Yes
B. No

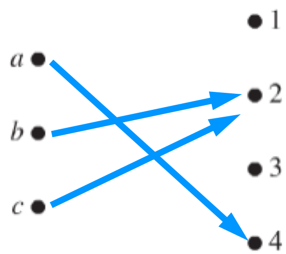
- $x = 0$ has no mapping

Outline

- What is a function?
  - Definition, Domain, Codomain, Range
  - **Recognizing (non)-functions**
- **Function properties**
  - **Onto, One-to-one**
  - Proofs of properties
- Bijections and Inverses
- Function Composition

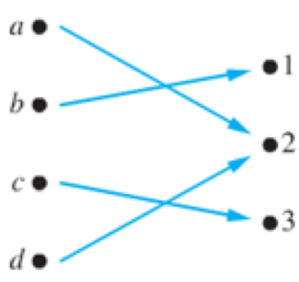
Function Properties

- Every function **must** map each domain element to exactly one codomain element, no matter what.

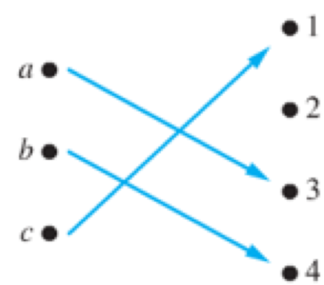


Each domain element (point on the left) has exactly one outgoing arrow

- Functions that **just so happen** to have related properties for their ***codomain*** have special names:



"Onto" means that: all codomain points have at least one incoming arrow



"1-to-1" means that: all codomain points have at most one incoming arrow

More useful in proofs: no 2 domain points map to the same codomain point

Onto

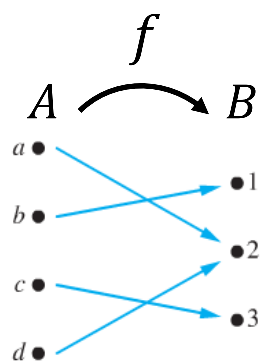
Also called **"surjective"**

For a function $f: A \to B$, **"f is onto"** means that every codomain element has **at least one** domain element that maps to it.

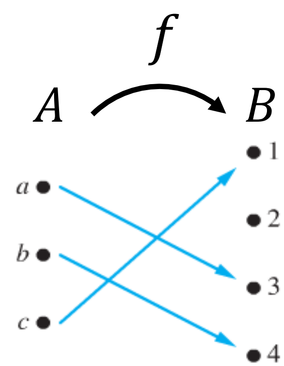
$$\forall b \in B \qquad \exists a \in A \qquad f(a) = b$$

Example: Mapping of guests to rooms in a full hotel

"f is onto" means the same thing as "range(f) = codomain(f)"



onto



Not onto (nothing maps to $2 \in B$)

Not onto:

$range(f) = \{1,3,4\}$

$codomain(f) = \{1,2,3,4\}$

$range(f) \neq codomain(f)$

One-to-one

Also called **"injective"**

For a function $f: A \to B$, **"f is 1-to-1"** means that

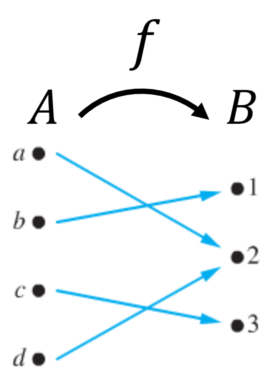
no two (different) domain elements map to the same thing

$$\neg \quad \exists a_1, a_2 \in A, \quad a_1 \neq a_2 \land \quad f(a_1) = f(a_2)$$

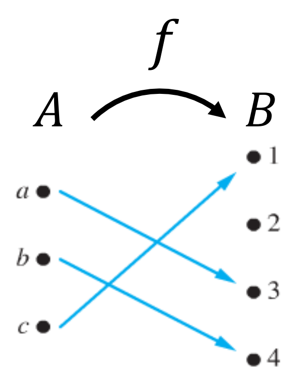
$$\equiv \quad \forall a_1, a_2 \in A \quad \left[\left(f(a_1) = f(a_2)\right) \to (a_1 = a_2)\right]$$

Far better for proofs

Example: Mapping of students to seats in a [possibly non-full] classroom



Not 1-to-1
($f(a) = 2$ and $f(d) = 2$)



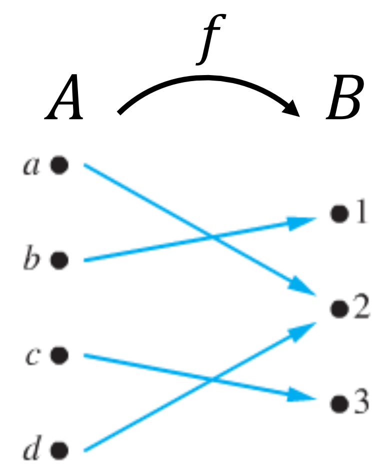
1-to-1

Onto, One-to-one, Bijections, Inverse Functions

Handout

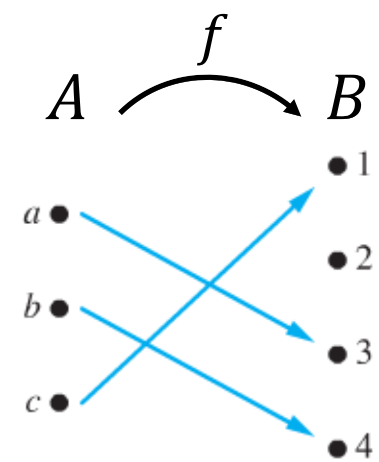
Given a function $f: A \to B$

- f is **onto** iff



onto

- f is **one-to-one** iff



one-to-one

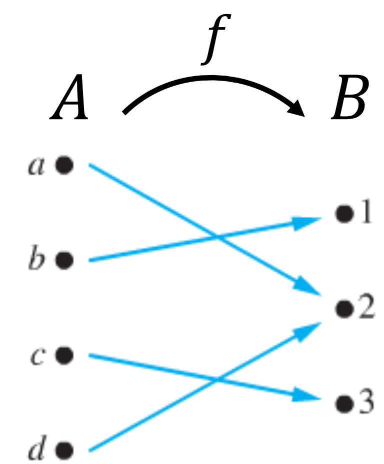
- f is a **bijection** iff

- f has an inverse f^{-1} iff f is ____________
  - $f^{-1}(b) = a \quad \leftrightarrow \quad f(\text{\_\_\_\_}) = \text{\_\_\_\_}$

Onto, One-to-one, Bijections, Inverse Functions

Handout

Given a function $f: A \to B$



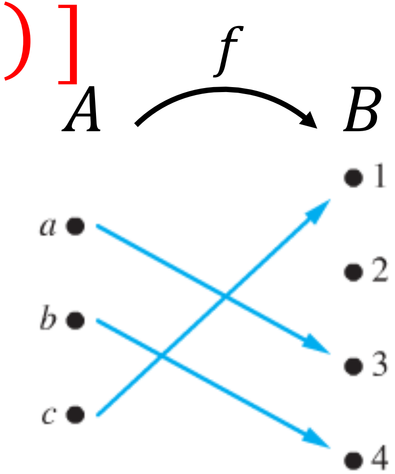
onto

- f is **onto** iff

$$\forall b \in B, \exists a \in A \ [\, f(a) = b \,]$$

- f is **one-to-one** iff

$$\forall a_1, a_2 \in A \ \left[\left(f(a_1) = f(a_2) \right) \to (a_1 = a_2) \right]$$



one-to-one

- f is a **bijection** iff

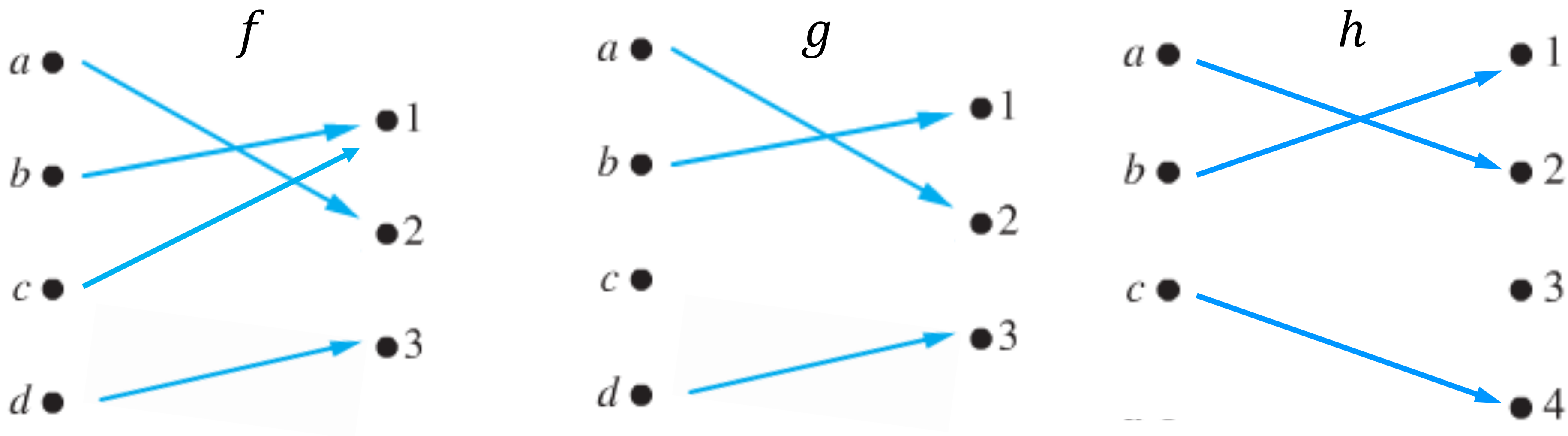
- f^{-1} exists iff

  - $f^{-1}(b) = a \ \leftrightarrow \ f(\quad) =$

Exercise: One-to-one and Onto

Handout

1. For each of f, g, and h: is it a function? If so, which properties does it have?



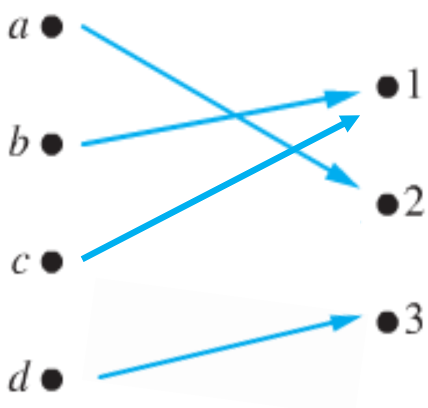
2. Draw a function that is
   a) Onto and one-to-one
   b) Neither onto nor one-to-one

Exercise: One-to-one and Onto

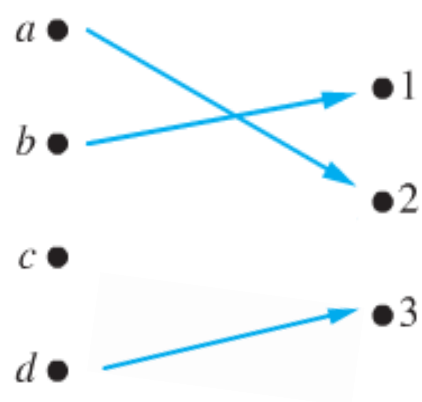
Handout

Which of these are functions? If functions, which are 1-to-1? Which are onto?

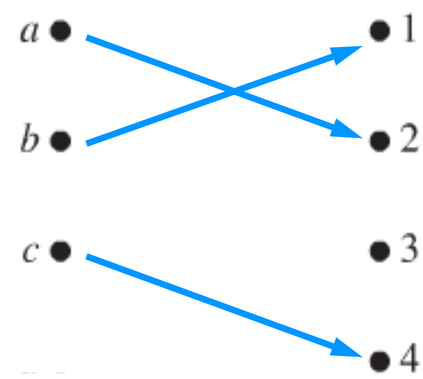
f



g



h

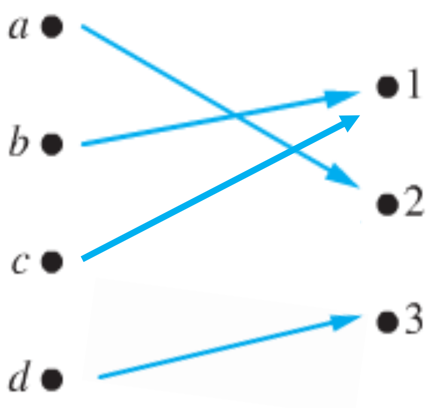


Exercise: One-to-one and Onto

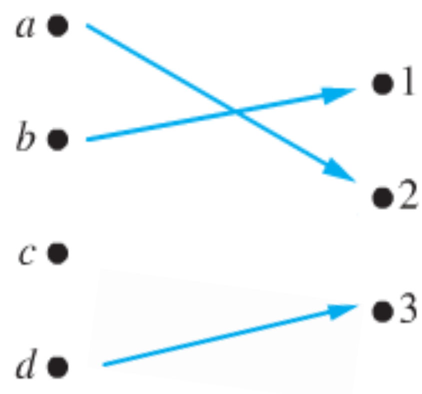
Handout

Which of these are functions? If functions, which are 1-to-1? Which are onto?

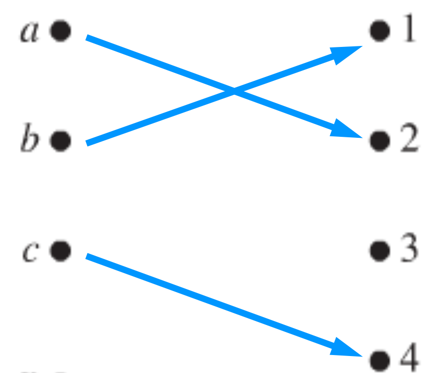
f



g



h



Function
Yes Onto
Not 1-to-1: $f(b) = f(c) = 1$

Not a function
$f(c)$ undefined

Function
Not onto: nothing maps to 3
Yes 1-to-1

Properties Practice

- Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x + 4$.

Which describes the function f?

A. Both one-to-one and onto
B. Onto but not one-to-one
C. Not onto, but one-to-one
D. Neither onto nor one-to-one

Properties Practice

- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 2x + 4$.

Which describes the function f?

A. **Both one-to-one and onto**
B. Onto but not one-to-one
C. Not onto, but one-to-one
D. Neither onto nor one-to-one

Properties Practice

- Let $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = 2x + 4$.

Which describes the function f?

A. Both one-to-one and onto
B. Onto but not one-to-one
C. Not onto, but one-to-one
D. Neither onto nor one-to-one

Properties Practice

- Let $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = 2x + 4$.

Which describes the function f?

A. Both one-to-one and onto

B. Onto but not one-to-one

C. Not onto, but one-to-one

D. Neither onto nor one-to-one

(nothing maps to 1, 3, 5…)

…but how could we **prove** these properties?

Outline

- What is a function?
  - Definition, Domain, Codomain, Range
  - **Recognizing (non)-functions**
- Function properties
  - Onto, One-to-one
  - **Proofs of properties**
- Bijections and Inverses
- Function Composition

Proof: $f(x)$ is one-to-one

Handout

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4.$

Prove: f is one-to-one.

Goal: Prove logical expression

____

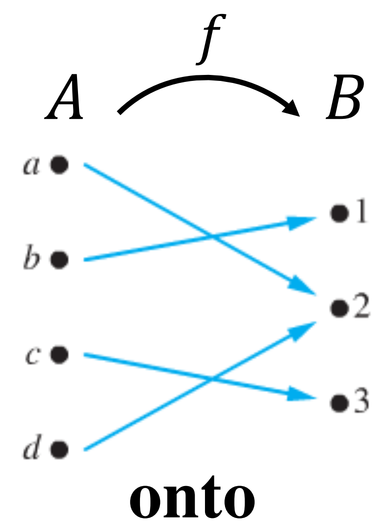
Onto, One-to-one, Bijections, Inverse Functions

Handout

Given a function $f: A \to B$

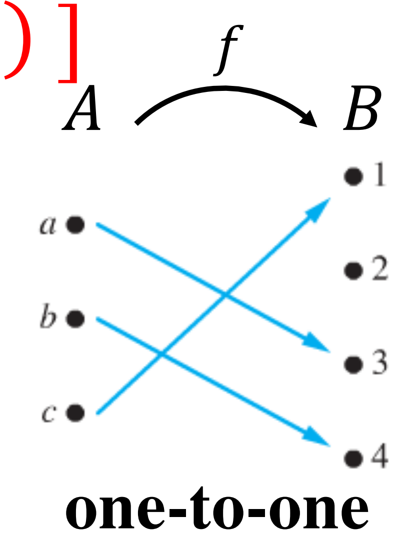
- f is **onto** iff

  $\forall b \in B, \exists a \in A \ [\, f(a) = b \,]$



- f is **one-to-one** iff

$\forall a_1, a_2 \in A \ \left[\left(f(a_1) = f(a_2) \right) \to (a_1 = a_2) \right]$



- f is a **bijection** iff

- $\boldsymbol{f^{-1}}$ exists iff

  - $f^{-1}(b) = a \leftrightarrow \quad f(\quad) =$

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \rightarrow \mathbb{R},\ f(x) = 2x + 4$.
Prove: f is one-to-one.

Goal: Prove logical expression
"For all $a_1, a_2 \in \mathbb{R}$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$."

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x + 4$.

Prove: f is one-to-one.

Goal: Prove logical expression

"For all $a_1, a_2 \in \mathbb{R}$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$."

- Let a_1, a_2 be arbitrary real numbers.
- Assume that $f(a_1) = f(a_2)$.

- So $a_1 = a_2$.

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$.
Prove: f is one-to-one.

Goal: Prove logical expression
"For all $a_1, a_2 \in \mathbb{R}$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$."

- Let a_1, a_2 be arbitrary real numbers.
- Assume that $f(a_1) = f(a_2)$.

$$2a_1 + 4 = 2a_2 + 4.$$
$$2a_1 = 2a_2$$
$$a_1 = a_2$$

- So $a_1 = a_2$.
- Thus, f is one-to-one.

Proof: $f(x)$ is onto

Handout

Let $f: \mathbb{R} \rightarrow \mathbb{R},\ f(x) = 2x + 4.$

Prove: f is onto.

Goal: Prove expression

____

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$.

Prove: f is onto.

Goal: Prove expression

"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \rightarrow \mathbb{R},\ f(x) = 2x + 4.$

Prove: f is onto.

Goal: Prove expression

"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

- Let b be an arbitrary real number.

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$.
Prove: f is onto.

Goal: Prove expression
"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

- Let b be an arbitrary real number.
- Consider $a = \cdots$

To prove "there exists," we should name **some particular value** for a (possibly depending on b).

But which value? We need to think more about **why** the proposition is true before continuing.

Proof: $f(x)$ is onto

Let $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4.$

Prove: f is onto.

Goal: Prove expression

"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

- Let b be an arbitrary real number.
- Consider $a = \cdots$
- (on hold)

Side work: Which choice of x maps to a given y?

$$f(a) = b$$
$$2a + 4 = b$$
$$2a = b - 4$$
$$\boldsymbol{a = \frac{b}{2} - 2}$$

This side work is **not part of your proof**, and (if comfortable) you could jump straight to the choice $a = \frac{b}{2} - 2$. But be careful!

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4.$

Prove: f is onto.

Goal: Prove expression

"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

- Let b be an arbitrary real number.
- Consider $a = \frac{b}{2} - 2$

(a quotient/difference of real numbers is real)

Side work: Which choice of a maps to a given b?

$$f(a) = b$$
$$2a + 4 = b$$
$$2a = b - 4$$
$$\boldsymbol{a = \frac{b}{2} - 2}$$

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4.$
Prove: f is onto.

Goal: Prove expression
"For all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$."

- Let b be an arbitrary real number.
- Consider $a = \frac{b}{2} - 2$ (a quotient/difference of real numbers is real)
  - Note that a is a real number, and thus in the domain of f
- So $2a = b - 4$
- So $2a + 4 = b$
- So $f(a) = b$. *Done!*

Prove or Disprove: $f(x)$ is onto

Handout

Let $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1$.

Prove or Disprove (circle one): f is onto.

We will try to prove the expression:

____

Prove or Disprove: $f(x)$ is onto

Let $f\colon \mathbb{Z} \rightarrow \mathbb{Z},\ f(x) = x^3 + 1.$

Prove or Disprove: f is onto.

What do you think? Should we prove or disprove?

Possible side work:

"Onto" = every codomain element is mapped to.

Let's try a few values:

- $f(-1) = (-1)^3 + 1 = 0$
- $f(0) = 0^3 + 1 = 1$
- $f(1) = 1^3 + 1 = 2$
- $f(2) = 2^3 + 1 = 9$
- $f(3) = 3^3 + 1 = 28$
- …

Looks like a gap here
Nothing will map to these values
Probably disprove

Reminder: **this is not a (dis)proof!** Just building intuition on whether to prove/disprove.

Disprove: $f(x)$ is onto

Let $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1.$

Disprove: f is onto.

Reminder: onto means
"$\forall b \in \mathbb{Z}, \exists a \in \mathbb{Z}$ such that $f(a) = b$."

So (DeMorgan): not onto means
"$\exists b \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}$, we have $f(a) \neq b$."

Prove **there exists** by naming a specific value for b.
Our previous side work suggested that $b = 3$ is a good choice for b...

Disprove: $f(x)$ is onto

Let $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1.$
Disprove: f is onto.

So (DeMorgan): not onto means
"$\exists b \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}$, we have $f(a) \neq b$."

- Consider $b = 3$.
- Let a be an arbitrary integer.
- *Now what? How do we prove $f(a) \neq b$?*
- *Hint: it's usually much easier to work with an **equality**…*

Disprove: $f(x)$ is onto

Let $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1.$
Disprove: f is onto.

So (DeMorgan): not onto means
"$\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, we have $f(x) \neq y$."

- Consider $b = 3.$
- Let a be an arbitrary integer.
- **Seeking contradiction**, assume that $f(a) = b.$
- So $a^3 + 1 = 3$
- So $a^3 = 2$
- So $a = 2^{\frac{1}{3}}$
- This contradicts that $a \in \mathbb{Z}$ (since $2^{\frac{1}{3}}$ is not an integer), which completes the contradiction. So $f(a) \neq b.$
- In other words the only a that solves $f(a) = 3$ is **not in the domain of $\boldsymbol{f}$**. Therefore, f is not onto.

Outline

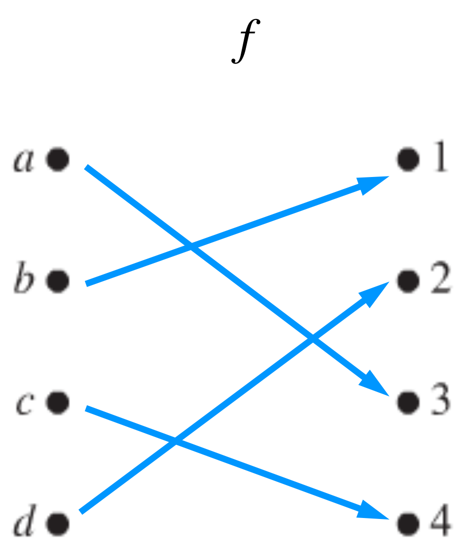
- What is a function?
  - Definition, Domain, Codomain, Range
  - **Recognizing (non)-functions**
- Function properties
  - Onto, One-to-one
  - Proofs of properties
- **Bijections and Inverses**
- Function Composition

Bijections

For a function $f: A \to B$, **"f is a bijection"** means:

- f is both onto (surjective) and 1-to-1 (injective)
- Equivalently, every codomain element $b \in B$ has **exactly** one domain element $a \in A$ that maps to it.

- Confusingly, some resources (like the textbook) sometimes call a bijection a *one-to-one correspondence*.
- We will try to only say "bijection" but be careful if you come across this phrase in the wild!



bijection

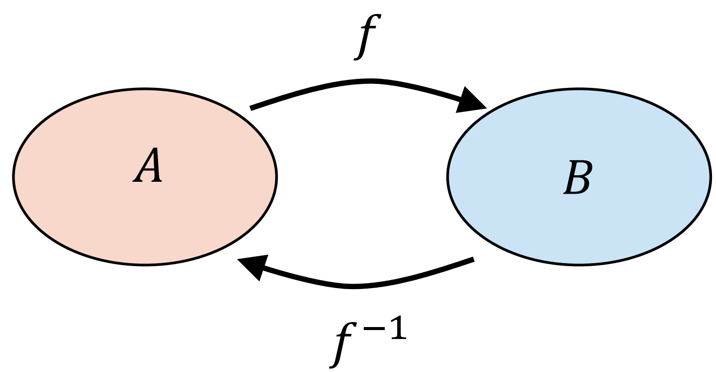
Bijections are important mainly because they have two special properties:

- Bijections have **"inverses"** *covered next*
- If there exists a bijection $f: A \to B$, then $\boldsymbol{|A| = |B|}$.
  much more on this in the third part of the course!

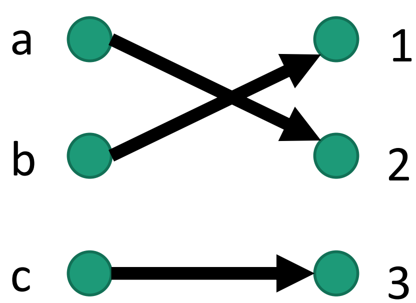
Bijections Have Inverses

Bijections f have "**inverse functions**," written f^{-1}.

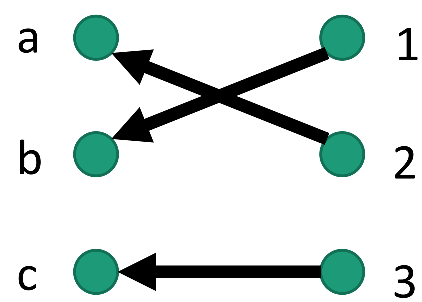
- For $f: A \to B$, the inverse is $f^{-1}: B \to A$ where $f(a) = b$ if and only if $f^{-1}(b) = a$
- **(In a dots-and-arrows diagram)** just reverse the direction of the arrows



Bijection $f: \{a, b, c\} \to \{1,2,3\}$



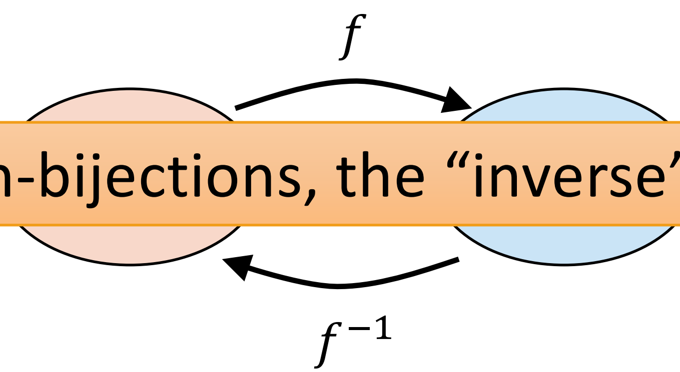
Inverse $f^{-1}: \{1,2,3\} \to \{a, b, c\}$



Bijections Have Inverses

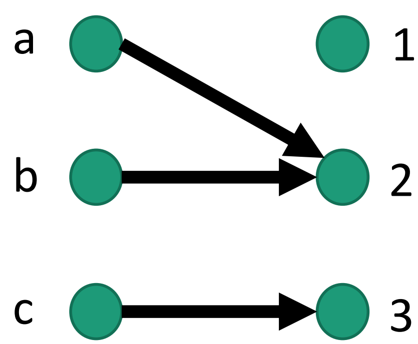
Bijections f have "**inverse functions**," written f^{-1}.

- For $f: A \to B$, the inverse is $f^{-1}: B \to A$ where $f(a) = b$ if and only if $f^{-1}(b) = a$
- **(In a dots-and-arrows diagram)** just reverse the direction of the arrows

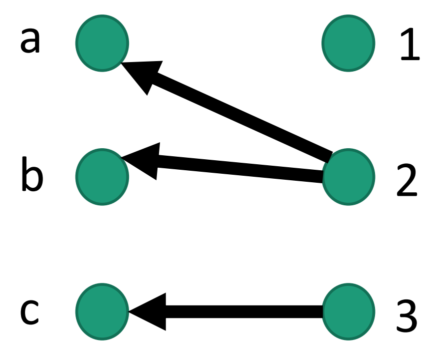


Caution: for non-bijections, the "inverse" is not a function!

Non-Bijection $f: \{a, b, c\} \to \{1,2,3\}$



Not a function



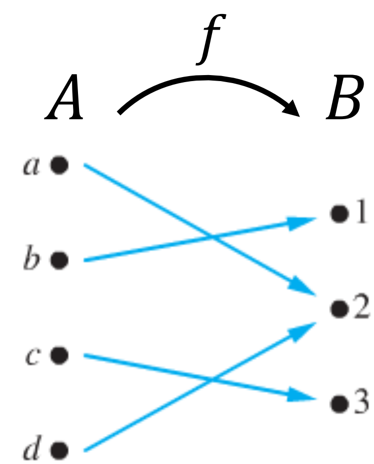
Onto, One-to-one, Bijections, Inverse Functions

Handout

Given a function $f: A \to B$

- f is **onto** iff

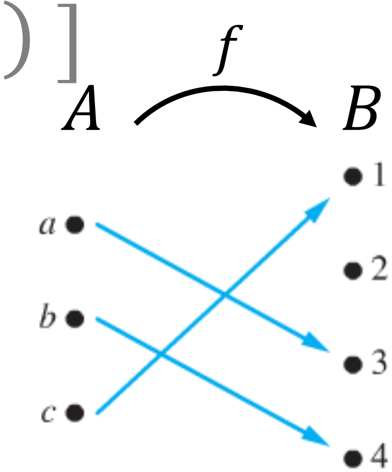
  $\forall b \in B, \exists a \in A \ [\, f(a) = b \,]$



onto

- f is **one-to-one** iff

$\forall a_1, a_2 \in A \ \ [\, \big(f(a_1) = f(a_2)\big) \to (a_1 = a_2) \,]$



one-to-one

- f is a **bijection** iff

- $\boldsymbol{f^{-1}}$ exists iff

  - $f^{-1}(b) = a \leftrightarrow \quad f(\quad) =$

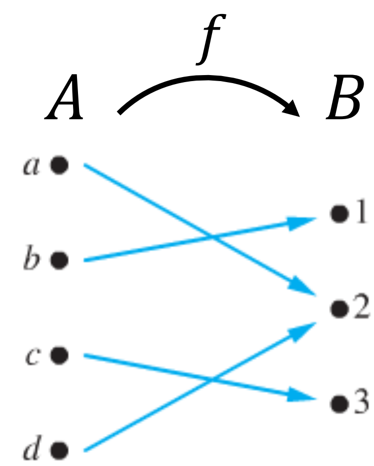
Onto, One-to-one, Bijections, Inverse Functions

Handout

Given a function $f: A \to B$

- f is **onto** iff

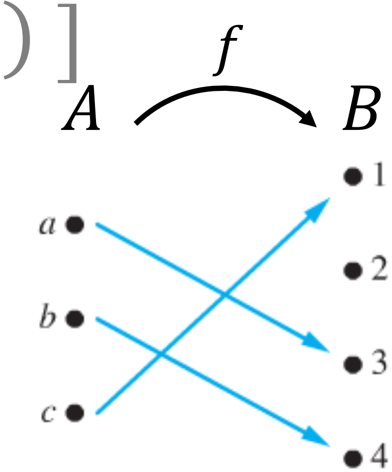
  $\forall b \in B, \exists a \in A \; [\, f(a) = b \,]$



onto

- f is **one-to-one** iff

$\forall a_1, a_2 \in A \;\; [\, \big(f(a_1) = f(a_2)\big) \to (a_1 = a_2) \,]$



one-to-one

- f is a **bijection** iff

  it's onto and 1-1

- $\boldsymbol{f^{-1}}$ exists iff f is a bijection

  - $f^{-1}(b) = a \;\leftrightarrow\; f(a) = b$

Inverses and Composition

Handout

$f:\{a,b,c\} \to \{1,2,3\}$ $\quad$ $g:\{1,2,3\} \to \{x,y,z\}$

Which of these exist?

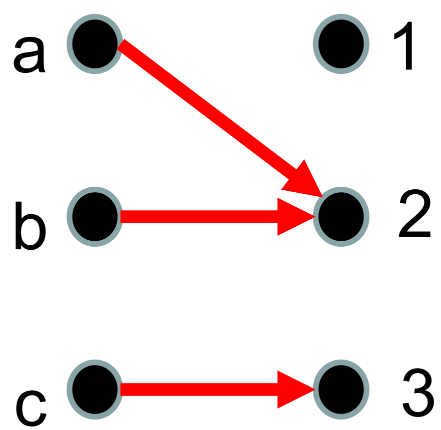
1. f^{-1}
2. g^{-1}
3. $f \circ g$
4. $g \circ f$

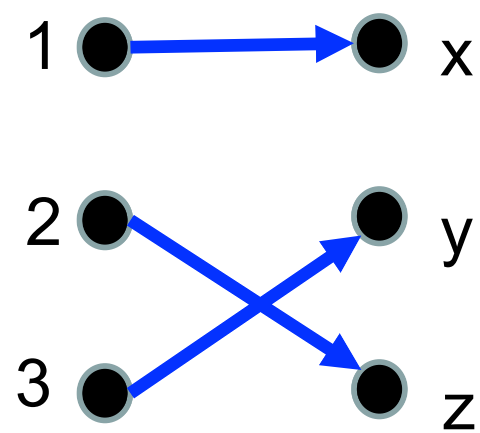
Inverses and Composition

Handout

$f: \{a, b, c\} \rightarrow \{1,2,3\}$

$g: \{1,2,3\} \rightarrow \{x, y, z\}$





Which of these exist?

1. f^{-1} - NO (f is not a bijection)
2. g^{-1} - YES (g is a bijection)

Inverses Example

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = 9 - 3x$. What is f^{-1}?

(f is indeed a bijection, so f^{-1} exists, although we won't prove this part here)

Inverses Example

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = 9 - 3x$. What is f^{-1}?

(f is indeed a bijection, so f^{-1} exists, although we won't prove this part here)

We are looking for the function $f^{-1}: \mathbb{R} \to \mathbb{R}$ where, for x, y:

- **Given that** $f(x) = 9 - 3x = y$,
- **We can find** $f^{-1}(y) = x$.

Starting point: we have $9 - 3x = y$.

Inverses Example

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = 9 - 3x$. What is f^{-1}?

(f is indeed a bijection, so f^{-1} exists, although we won't prove this part here)

We are looking for the function $f^{-1}: \mathbb{R} \to \mathbb{R}$ where, for x, y:

- **Given that** $f(x) = 9 - 3x = y$,
- **We can find** $f^{-1}(y) = x$.

Starting point: we have $9 - 3x = y$.

Ending point: so [something depending on y] = x

Inverses Example

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = 9 - 3x$. What is f^{-1}?

(f is indeed a bijection, so f^{-1} exists, although we won't prove this part here)

We are looking for the function $f^{-1}: \mathbb{R} \to \mathbb{R}$ where, for x, y:

- **Given that** $f(x) = 9 - 3x = y$,
- **We can find** $f^{-1}(y) = x$.

Starting point: we have $9 - 3x = y$.

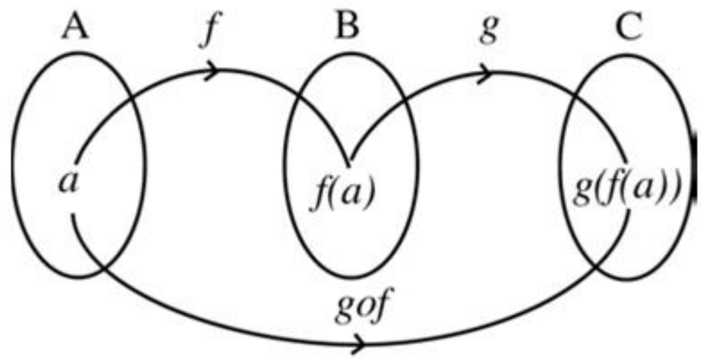
- So $9 - y = 3x$
- So $3 - \frac{y}{3} = x$

$$\boxed{f^{-1}(y) = 3 - \frac{y}{3}}$$

Outline

- What is a function?
  - Definition, Domain, Codomain, Range
  - **Recognizing (non)-functions**
- Function properties
  - Onto, One-to-one
  - Proofs of properties
- Bijections and Inverses
- **Function Composition**

Composition of Functions

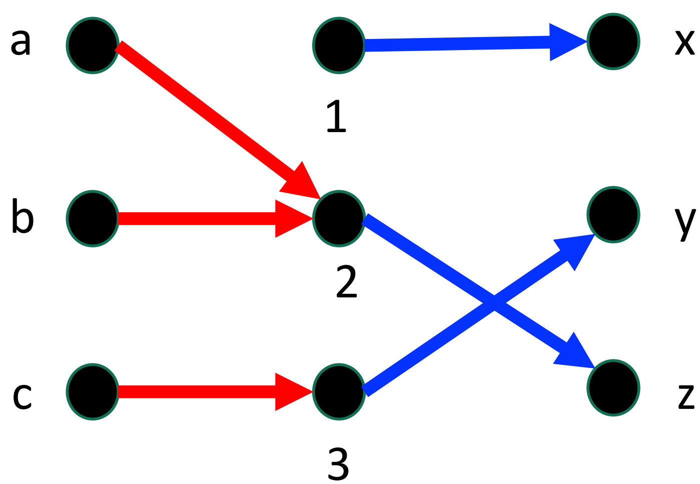


Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f: \{a, b, c\} \rightarrow \{1,2,3\}$

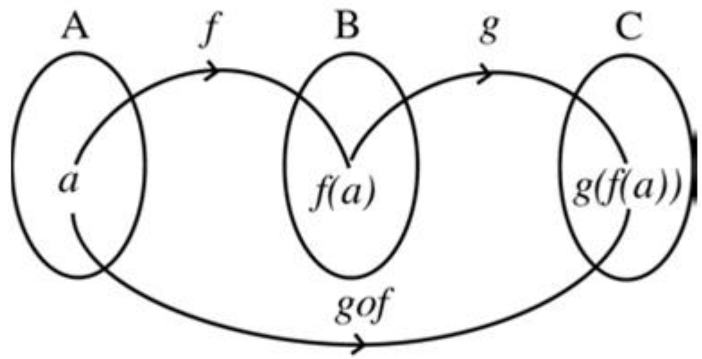
$g: \{1,2,3\} \rightarrow \{x, y, z\}$



Definition:

$$(g \circ f)(k) = g(f(k))$$

Composition of Functions

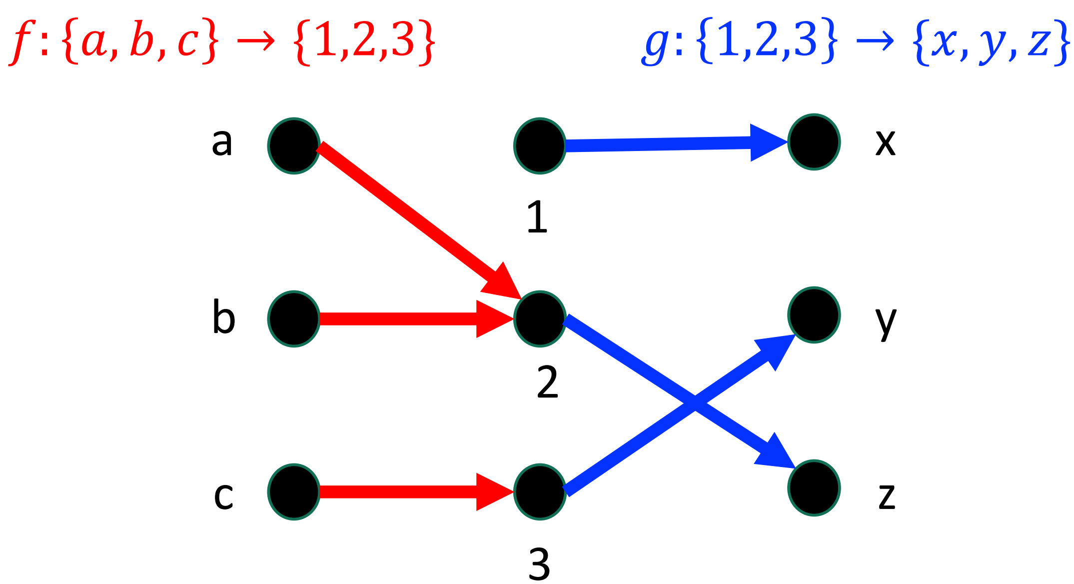


Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

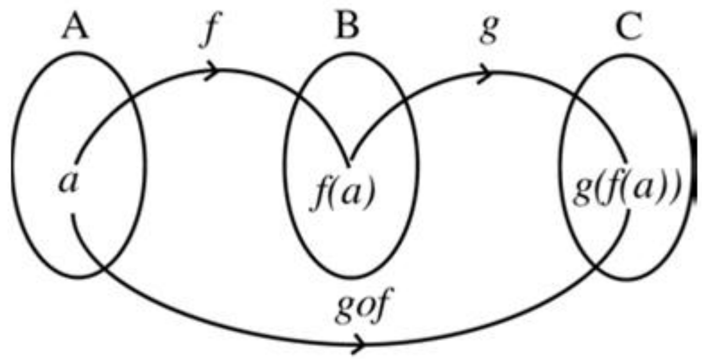
Definition:

$$(g \circ f)(k) = g(f(k))$$



$(g \circ f)(a) = ?$

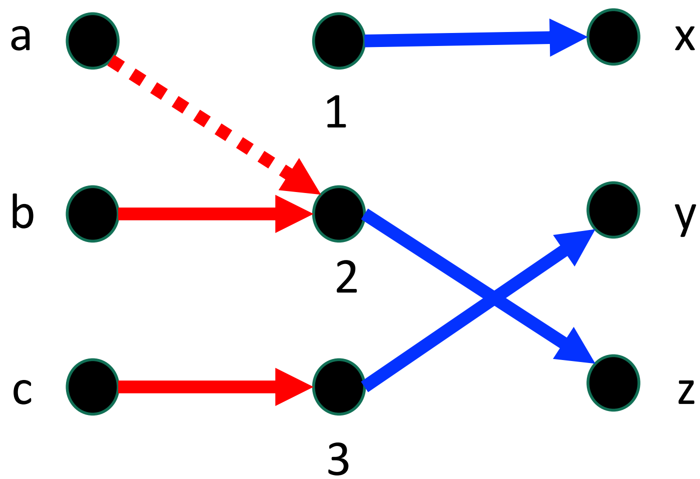
Composition of Functions



Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f : \{a, b, c\} \rightarrow \{1,2,3\}$ $g : \{1,2,3\} \rightarrow \{x, y, z\}$

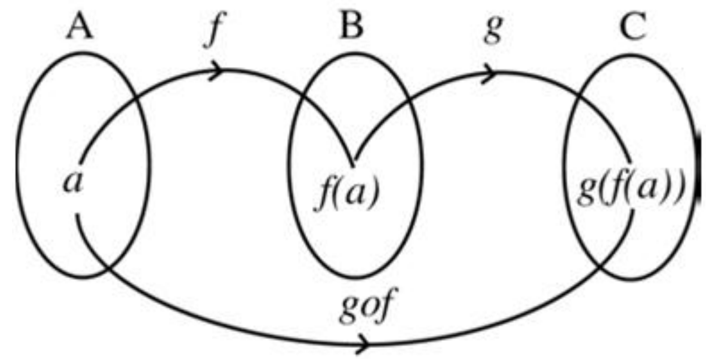


Definition:

$$(g \circ f)(k) = g(f(k))$$

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(2)\end{aligned}$$

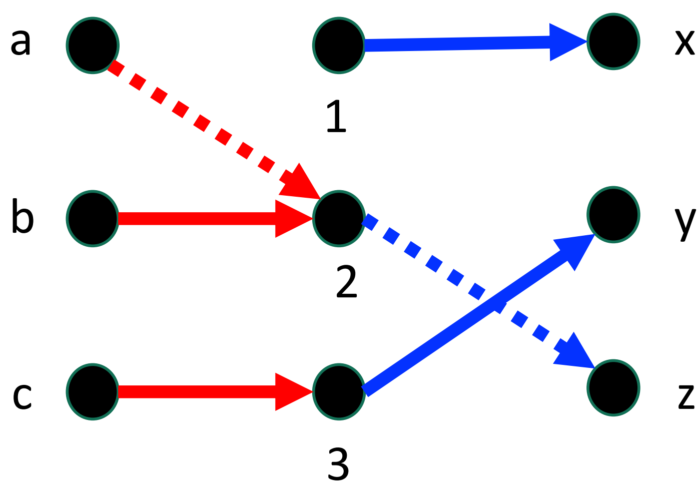
Composition of Functions



Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f: \{a, b, c\} \rightarrow \{1,2,3\}$ $g: \{1,2,3\} \rightarrow \{x, y, z\}$

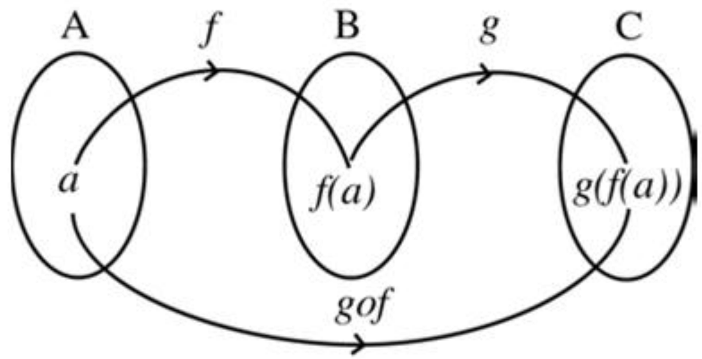


Definition:

$$(g \circ f)(k) = g(f(k))$$

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(2) \\ &= z\end{aligned}$$

Composition of Functions

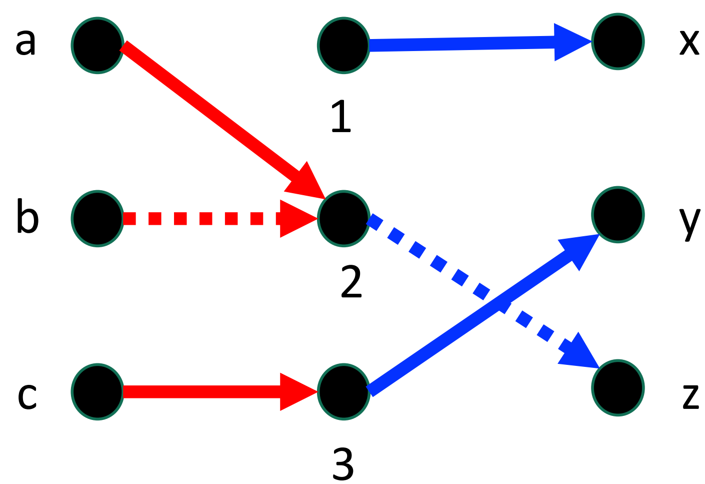


Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f: \{a, b, c\} \to \{1,2,3\}$

$g: \{1,2,3\} \to \{x, y, z\}$



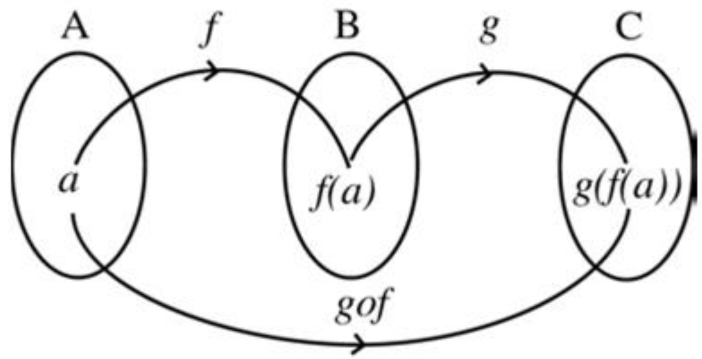
Definition:

$$(g \circ f)(k) = g(f(k))$$

$(g \circ f)(a) = z$

$(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{b}) = \boldsymbol{z}$

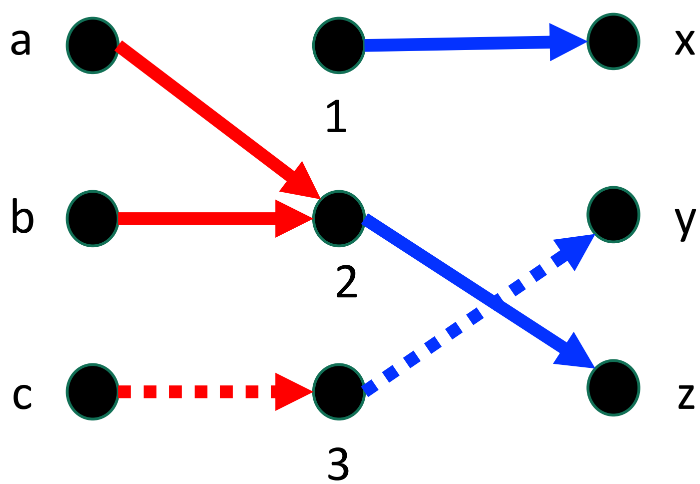
Composition of Functions



Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f: \{a, b, c\} \to \{1,2,3\}$ $\qquad$ $g: \{1,2,3\} \to \{x, y, z\}$

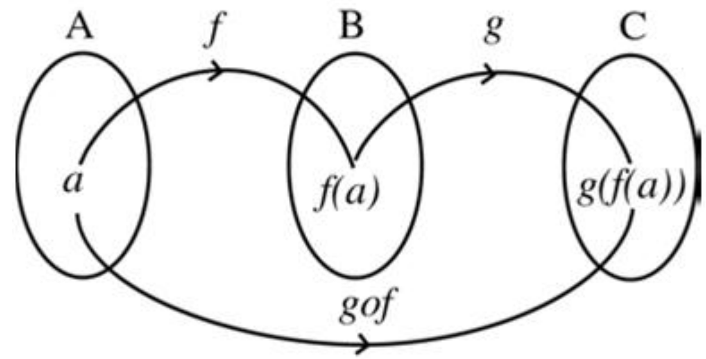


Definition:

$$(g \circ f)(k) = g(f(k))$$

$(g \circ f)(a) = z$
$(g \circ f)(b) = z$
$\boldsymbol{(g \circ f)(c) = y}$

Composition of Functions

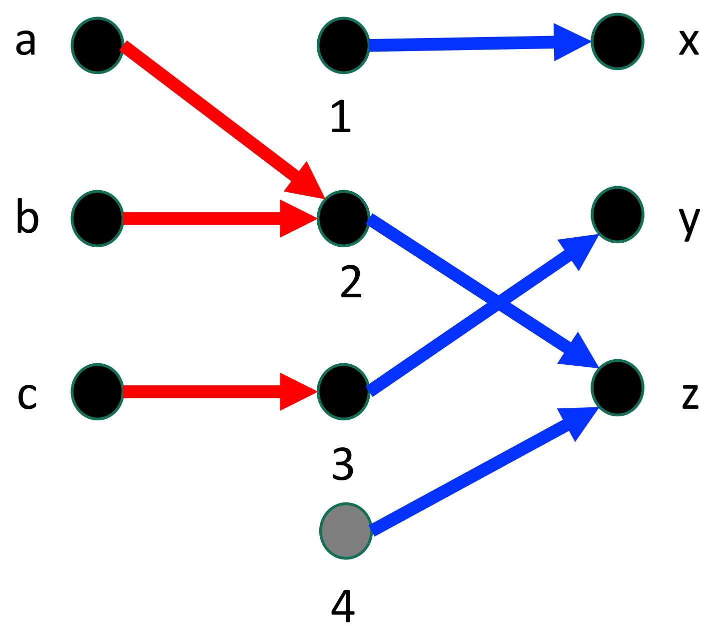


Two functions f, g can be **composed** into a third function when the codomain of f is the domain of g.

Composed function written $g \circ f$

$f: \{a, b, c\} \rightarrow \{1,2,3\}$

$g: \{1,2,3,4\} \rightarrow \{x, y, z\}$



Definition:

$$(g \circ f)(k) = g(f(k))$$

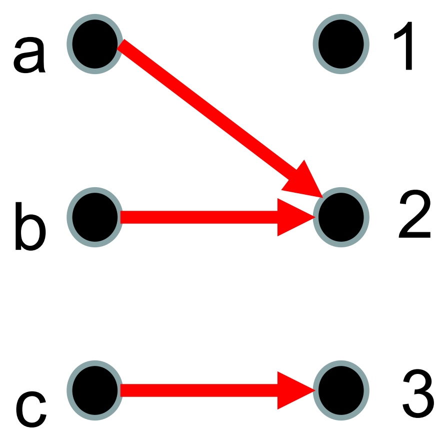
$(g \circ f)(a) = z$
$(g \circ f)(b) = z$
$(g \circ f)(c) = y$

Note: we can also compose $g \circ f$ when $codom(f) \subseteq dom(g)$.
(The "extra" domain elements of g won't affect $g \circ f$.)

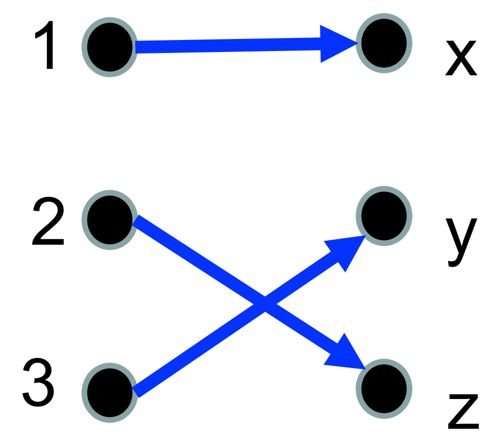
Inverses and Composition

Handout

$f\colon \{a, b, c\} \to \{1,2,3\}$



$g\colon \{1,2,3\} \to \{x, y, z\}$



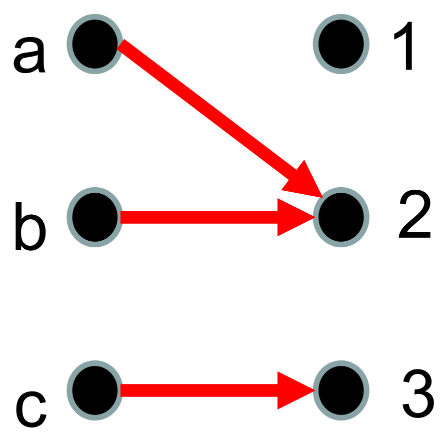
Which of these exist?

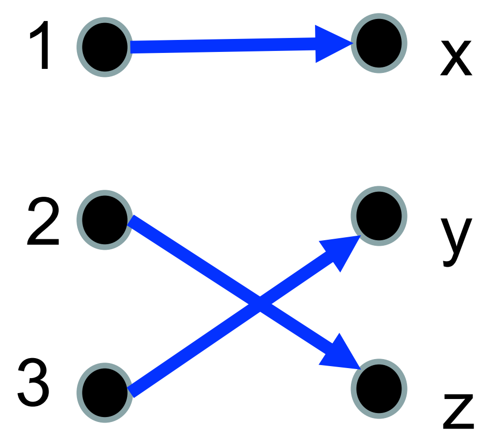
1. f^{-1}
2. g^{-1}
3. $f \circ g$
4. $g \circ f$

Inverses and Composition

$f: \{a, b, c\} \rightarrow \{1,2,3\}$

$g: \{1,2,3\} \rightarrow \{x, y, z\}$





Which of these exist?

1. f^{-1} - NO (f is not a bijection)
2. g^{-1} - YES (g is a bijection)
3. $\mathrm{f} \circ \mathrm{g}$ - NO ($codom(g)$ not subset of $dom(f)$)
4. $\mathrm{g} \circ \mathrm{f}$ - YES ($codom(f) \subseteq dom(g)$)

Exercise

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = 3x + 1$

What is $(f \circ g)(x)$?

Reminder: $(f \circ g)(x)$ is defined as $f(g(x))$.

Exercise

$f: \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$

$g: \mathbb{R} \to \mathbb{R}, g(x) = 3x + 1$

What is $(f \circ g)(x)$?

Reminder: $(f \circ g)(x)$ is defined as $f(g(x))$.

$$
\begin{aligned}
f\big(g(x)\big) &= f(3x + 1) \\
&= 2(3x + 1) + 3 \\
&= 6x + 5
\end{aligned}
$$

Caution:

Order matters! $(f \circ g)(x)$ is not the same as $(g \circ f)(x)$

Here, $(g \circ f)(x) = g(2x + 3) = 3(2x + 3) + 1 = 6x + 7$

Wrapup

- We'll revisit functions throughout this course

- The properties of **onto**, **one-to-one**, and **invertibility** are important in:
  - Counting (later this term)
  - Hashing, Cryptography, Error-correcting codes, Computational Geometry, …