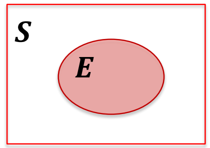
Lec 24: Bayes' Rule -- ANSWERS

Probability Recap:

- **Experiment**: Procedure that yields an outcome
- **Sample space**: Set of all possible outcomes
- **Event**: *subset* of the sample space

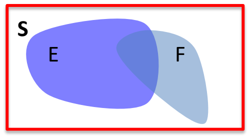


If S is a sample space of ***equally likely outcomes***, the **probability** of an event E is $p(E) = \frac{|E|}{|S|}$

The **conditional probability** of event E given event F is

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{|E \cap F|}{|F|}$$

if equally likely outcomes



Total Probability

Example: Kyla loves the ice cream shop. Each time she visits, she gets one of 3 flavors.

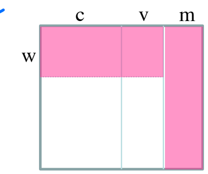
- Chocolate 50% of the time
- Vanilla 25% of the time
- Mango 25% of the time

When she gets mango, she always gets a waffle cone, but when she gets chocolate or vanilla, she only gets a waffle cone 1/3 of the time. Kyla just walked into the ice cream shop. What is the probability that she gets a waffle cone?

Solution: We need the *total* probability that she gets a waffle cone.

Pr[waffle] = Pr[waffle and chocolate] + Pr[waffle and vanilla] + Pr[waffle and mango]

$$\Pr[w] = \Pr[w|c]\Pr[c] + \Pr[w|v]\Pr[v] + \Pr[w|m]\Pr[m]$$
$$= \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4} + 1\cdot\frac{1}{4}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{4}$$
$$= \frac{1}{2}$$



Bayes' Rule = How to update your beliefs, given new Evidence

Example: Kyla loves the ice cream shop. Each time she visits, she gets one of 3 flavors.

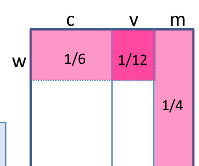
- Chocolate 50% of the time
- Vanilla 25% of the time
- Mango 25% of the time

Original beliefs: Pr[vanilla] = 1/4

When she gets mango, she always gets a waffle cone, but when she gets chocolate or vanilla, she only gets a waffle cone 1/3 of the time.

You see that Kyla has a waffle cone (**new evidence!**).
What is the [updated] probability that she got vanilla ice cream?

What proportion of waffle cones are from vanilla?



Updated probability: $\Pr[v|w] = \frac{\frac{1}{12}}{\frac{1}{6}+\frac{1}{12}+\frac{1}{4}} = \frac{1}{6}$

Congratulations! You've just used Bayes' Rule!

Bayes' Theorem

Suppose that $\boldsymbol{E}$ and $\boldsymbol{F}$ are events from a sample space $\boldsymbol{S}$ both with non-zero probabilities. Then

$$\boldsymbol{p(F|E)} = \frac{p(E|F)\,p(F)}{p(E)}$$

Alternative form of Bayes', with expanded denominator:

$$\boldsymbol{p(F|E)} = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

Denominator is the **total probability** of E

Helpful tips for Bayes':

- $p(E|F) + p(\bar{E}|F) = 1$
- $p(E|\bar{F}) + p(\bar{E}|\bar{F}) = 1$
- $p(E) + p(\bar{E}) = 1$
- $p(F) + p(\bar{F}) = 1$

Handout

Bayes' Rule = How to Update your Beliefs, Given New Evidence

Example: Two unlabeled coins, one fair, one biased with **p(H) = 2/3** and **p(T) = 1/3**. Pick one up a coin, c, and flip it. It comes up as heads. What should your new beliefs be about the probability that c is the weighted coin, given this new evidence?

One solution = table method:

Four Possibilities:	c = fair coin, $p = \frac{1}{2}$	c = weighted coin, $p = \frac{1}{2}$
c flips tails	$p(F)p(T\|F) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$	$p(W)p(T\|W) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$
c flips heads	$p(F)p(H\|F) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$	$p(W)p(H\|W) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

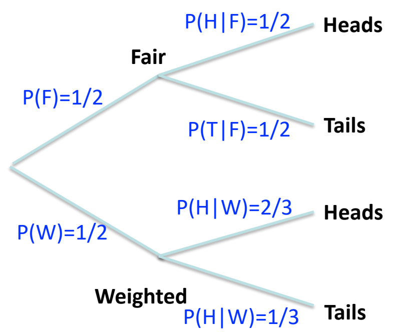
Frac of **remaining** probability on weighted coin:

$$\frac{1/3}{1/3 + 1/4} = \frac{4/12}{4/12 + 3/12} = 4/7$$

Interpretation: evidence of flipping heads increased your belief that you grabbed the weighted coin from $\frac{1}{2}$ to $\frac{4}{7}$

Weighted Coin Flip: Tree Method

- Fill in the probability of taking each choice along each edge
- Compute the probability of reaching each final node we care about
- Determine what fraction of the Heads portion was done with the Weighted coin



$$p(F \cap H) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$p(W \cap H) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

$$p(W|H) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}} = \frac{4}{7}$$

Example: Rare Disease Testing

$p(D) = \frac{1}{10{,}000}$

- $\frac{1}{10{,}000}$ people are afflicted with 203-itis, a rare disease that compels you to keep solving math problems.
- A test is developed: $p(Y|D) = 1$
  - when given to someone has the disease it always reports YES
  - when given to someone does not have the disease it has 1% false positive rate $p(Y|\bar{D}) = .01$
- A random person is screened and they test positive. What is the probability that they have the disease? $p(D|Y) = ?$

D = has disease
Y = "yes", tests positive

$$p(D|Y) = \frac{p(Y|D)\,p(D)}{p(Y)} = \frac{p(Y|D)\,p(D)}{p(Y|D)p(D) + p(Y|\bar{D})\,p(\bar{D})}$$

$$= \frac{1\cdot 1/10{,}000}{1\cdot 1/10{,}000 + (.01)\frac{9{,}999}{10{,}000}} \approx \boxed{\frac{1}{100}}$$

Example: Lost Cell Phone

- You're ***80% sure*** you left your cell phone in your bag. Inside your bag there are ***4 compartments*** that you could have left it in (with equal probability). You look in compartments ***#1, #2, #3 and do not find it***. What's the probability that you find it in compartment 4?

Solution 1, using Bayes Rule equation:
(other solution(s) in lecture slides)

Let B be the event that the phone is in the **B**ag,
and N be the event that it's **N**ot in compartments 1, 2, or 3.

$\Pr[B] = 0.8$
$\Pr[N|B] = 1/4$
We want $\Pr[B|N]$.

$$\Pr[B|N] = \frac{\Pr[N|B]\Pr[B]}{\Pr[N]} = \frac{\Pr[N|B]\Pr[B]}{\Pr[N|B]\Pr[B] + \Pr[N|not(B)]\Pr[not(B)]} = \frac{\frac{1}{4}\cdot\frac{4}{5}}{\frac{1}{4}\cdot\frac{4}{5} + 1\cdot\frac{1}{5}} = \frac{1}{2}$$