EECS 203: Discrete Mathematics
Fall 2023
Homework 3

Due **Thursday, Sept. 21**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $6 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Division Tradition [18 points]

Let the domain of discourse be positive integers. The notation $a \mid b$ means "a divides b," or more formally: "there exists an integer q such that $b = a \cdot q$." Prove the following statements:

(a) For all k, n, if $k \mid n$, then $k^2 \mid n^2$.

(b) For all k, n_1, n_2, if $k \mid n_1$ and $k \mid n_2$, then k *divides*$(n_1 + n_2)$.

(c) For all a, b, c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution:

(a) Let k be an arbitrary integer.
Since $k \mid n$, let $n = p \cdot k$, p is an integer. then $n^2 = p^2 \cdot k^2$.
Since p is an integer, p^2 is also an integer.
Therefore we have proved that $k^2 \mid n^2$.

(b) Let k be an arbitrary integer.
Since $k \mid n_1$, let $n_1 = p \cdot k$, p is an integer.
Since $k \mid n_2$, let $n_2 = q \cdot k$, q is an integer.
then $(n_1 + n_2) = (p + q) \cdot k$.
Since p, q are integers, $(p + q)$ is an integer.
Therefore we have proved that $k \mid (n_1 + n_2)$.

(c) Let a be an arbitrary integer.
Since $a \mid b$, let $b = p \cdot a$, p is an integer.
Since $b \mid c$, let $c = q \cdot b$, q is an integer.
then $c = (p \cdot q) \cdot a$.
Since p, q are integers, $(p \cdot q)$ is an integer.
Therefore we have proved that $a \mid c$.

2. Even Stevens Rerun [15 points]

Let n be an integer. Using only the definitions of even and odd, prove that these statements are equivalent:

(i) $n + 1$ is odd
(ii) $5n - 7$ is odd
(iii) n^2 is even

Hint: You can prove that these three statements are equivalent in a circular way by showing (i) $\to$ (ii), (ii) $\to$ (iii), and (iii) $\to$ (i).

Solution:

(a) (i) $\to$ (ii)
Let $(n+1)$ be an arbitrary odd integer. There exist integer k such that $(n+1) = 2k+1$.
Then $5n - 7 = 5(n+1) - 12 = 10k - 7 = 2(5k-4) + 1$.
Since k is an integer, $5k-4$ is an integer, $5n-7 = 2(5k-4)+1$ is an odd integer.
Therefore we have proved that (i) $\to$ (ii).

(b) (ii) $\to$ (iii)
Let $(5n-7)$ be an arbitrary odd integer. There exist integer k such that $(5n-7) = 2k+1$.
Then

$$\begin{aligned}(5n-7)^2 = 25n^2 - 70n + 49 &= 4k^2 + 4k + 1 \\ 25n^2 &= 4k^2 + 4k + 70n - 48\end{aligned}$$

Since k is an integer, $(2k^2 + 2k + 35 - 24)$ is an integer, and since $(4k^2 + 4k + 70n - 48) = 2(2k^2 + 2k + 35n - 24)$, $25n^2$ is even.
And since n^2 is an integer, for the odd integer 25, n^2 must be an even integer to make their product $25n^2$ an even integer.
Therefore we have proved that (ii) $\to$ (iii).

(c) (iii) $\to$ (i)
To prove "If n^2 is even, then $n+1$ is odd", we can prove its contrapositive, that is, "If $n+1$ is even, then n^2 is odd."
let $n+1$ be an arbitrary even integer, then there exist integer k such that $n+1 = 2k$.
Then $n^2 = (2k-1)^2 = 4k^2 - 4k + 1$.
Since k is an integer, $4k^2 - 4k = 2(2k^2 - 2k)$ is an even integer, and therefore $n^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$ is an odd integer.
Then we have proved (iii) $\to$ (i) by proving its contrapositive.

(d) Therefore, (i) $\to$ (ii) $\to$ (iii) $\to$ (i)...... We can deduce that (i) $\longleftrightarrow$ (ii) $\longleftrightarrow$ (iii)

3. Even the Odds [15 points]

Prove or disprove: there exists an integer n where n is even and $n^2 + 4$ is odd.

Solution:

We can disprove it by proving its negation: "There does not exists an integer n where n is even and $n^2 + 4$ is odd."
Let n be an arbitrary odd integer. Then there exist an integer k such that $n = 2k$.
Then $n^2 + 4 = (2k)^2 + 4 = 4k^2 + 4 = 2(2k^2 + 2)$.
since k is an integer, $(2k^2 + 2)$ is an integer, therefore $2(2k^2 + 2)$ must be an odd integer.
So when n is even, $n^2 + 4$ can not be odd.

4. To Prove or Not to Prove [18 points]

Prove or disprove the following statements where the domain of discourse is all integers:

(a) For all x there exists a y such that $x^2 + y = 2$.

(b) For all y there exists an x such that $2x - y = 8$.

(c) There exists an x such that for all y, $\frac{y}{x} = y$.

(d) There exists a y such that for all x, $x^2 + y = 2$.

Solution:

(a) Prove it.
Let x be an arbitrary integer.
Since every integer has its square that is an integer, x^2 is an integer.
Let $y = 2 - x^2$. Then y is an integer
Then we have y such that $x^2 + y = 2$.

(b) Disprove it.
We will prove its negation: "Therefore there exists an y that in the domain of all integers, for all x, $2x - y \neq 8$." For all y there exists an x such that $2x - y = 8$.
Consider $y = 1$.
If there exists an x such that $2x - y = 8$, then $x = \frac{8+y}{2} = 4.5$, which is not an integer.
Therefore there exists an y that in the domain of all integers, for all x, $2x - y \neq 8$.

(c) Prove it.
Consider $x = 1$. Let y be an arbitrary integer.
Then $\frac{y}{x} = y$.
Therefore when $x = 1$, for all y, $\dfrac{y}{x} = y$.

(d) Disprove it.
We will prove its negation: "There does not exist a y such that all x, $x^2 + y = 2$."
Seeking a contradiction, assume that for integer y, $x^2 + y = 2$ for all x.
Then when $x = 1$, y should be $2 - x^2 = 1$.
When $x = 2$, y should be $2 - x^2 = -2$.
This completes the contradiction since y can not be 1 and -2 at the same time.
Therefore we have proved that there do not exist a y such that all x, $x^2 + y = 2$.

5. What's Your Rationale? [20 points]

Prove that for any rational number and any irrational number, there exists an irrational number between them. You may assume the following without proof:

- The sum of a rational and irrational is irrational.
- The product of a nonzero rational and irrational is irrational.

Solution:

Let x be an arbitrary rational number, and y be an arbitrary irrational number.
Without Loss of Generality (due to symmetry), assume $x < y$.
We always have $\frac{x+y}{2}$:
$\frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} = y$.
$\frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} > \frac{x}{2} + \frac{x}{2} = x$.
Then we know that $x < \frac{x+y}{2} < y$.
Since x is rational, y is irrational, and the sum of a rational and irrational is irrational, $\frac{x+y}{2}$ is irrational.
Therefore we have proved that for any rational number x and any irrational number y, there exists an irrational number between them.

6. Contra > 0 [14 points]

Let x be an integer. Prove that if $6x + 2$ is negative, then x is negative using

(a) a proof by contrapositive.

(b) a direct proof.

Solution:

(a) We will prove it by proving its contrapositive: "Let x be an integer. If x is not negative, then $6x + 2$ is not negative".
Let x be an arbitrary integer in the domain that $x \geq 0$.
Then $6x + 2 \geq 2 > 0$, is not negative.
Therefore we have proved the original proposition by proving its contrapositive.

(b) Let $6x + 2$ be an arbitrary integer in the domain that $6x + 2 < 0$.
Then $6x < -2$, $x < -\frac{1}{3} < 0$.
Therefore we have proved that if $6x + 2$ is negative, then x is negative.