Practice Exam 1
QUESTIONS PACKET
EECS 203
Fall 2023

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

8-Digit UMID: ______________________________

*****MAKE SURE YOU HAVE PROBLEMS 1 - 16 IN THIS BOOKLET.*****

General Instructions

You have 120 minutes to complete this exam. You should have two exam packets.

- **Questions Packet:** Contains ALL the questions for this exam, worth 90 points total:
  - 5 Single Answer Multiple Choice questions (4 points each),
  - 4 Multiple Answer Multiple Choice questions (4 points each),
  - 2 Short Answer questions (6 points each), and
  - 5 Free Response questions (8 or 9 points each)

  Questions Packet is for scratch work only. Work in this packet will not be graded.
- **Answers Packet:** Write all of your answers in the Answers Packet, including your answers to multiple choice questions.

  For free response questions, you must show your work! Answers alone will receive little or no credit.
- You may bring **one** 8.5" by 11" note sheet, front and back, created by you.
- You may **NOT** use any other sources of information, including but not limited to electronic devices (including calculators), textbooks, or notes.
- After you complete the exam, sign the Honor Code on the front of the Answers Packet.

- You must turn in both parts of this exam.
- **You are not to discuss the exam until the solutions are published.**

Part A1: Single Answer Multiple Choice

Problem 1. (4 points)

How many rows does the truth table for $(p \vee q) \wedge (r \rightarrow (\neg r \wedge q))$ have?

(a) 4

(b) 5

(c) 8

(d) 16

(e) 32

Solution: (c)
There are 3 unique elementary propositions: p, q, and r, so we need 1 row for each of the $2^3 = 8$ combinations of truth values for all 3.

Problem 2. (4 points)

Consider the proposition

$$(s \vee \neg b) \rightarrow h$$

where:

- s is "Shubh overslept"
- b is "The busses are on time"
- h is "Shubh is late to office hours"

Which of the following is a correct translation of the given proposition?

(a) If Shubh overslept or the busses are not on time, then Shubh is late to office hours

(b) If the busses are not on time and Shubh overslept, then Shubh is late to office hours

(c) If Shubh is late to office hours, then either they overslept or the busses are not on time

(d) Shubh is late to office hours if and only if they overslept or the busses are not on time

(e) If Shubh overslept, then the busses are not on time and they are late to office hours

Solution: a: This contains the propositions s and $\neg b$ connected by an "or," all on the left side of an implication, with the conclusion being the proposition h.
The logical translations of the other four answers (which are not logically equivalent to the given) are listed below:

- b: $(\neg b \wedge s) \to h$
- c: $h \to (s \vee \neg b)$
- d: $h \leftrightarrow (s \vee \neg b)$
- e: $s \to (\neg b \wedge h)$

Problem 3. (4 points)

Which of the following is the correct assumption to start a proof by **contradiction** for the statement:

"If n^2 is even, n is also even."

"Seeking a contradiction, assume ..."

(a) n^2 is odd and n is even.

(b) n^2 is even and n is odd.

(c) if n is odd, then n^2 is odd.

(d) if n^2 is even, then n is odd.

(e) if n^2 is even, then n is even.

Solution: (b)

(a) This would be the negation of the converse statement ("If n^2 is odd, n is also odd."), not the statement given in the question.

(b) This is the correct negation of "If n^2 is even, n is also even."

(c) This answer choice is the contrapositive of the statement. If we were doing a proof by contrapositive, we would prove this statement, but this is not what we assume for a proof by contradiction.

(d) This is not the negation of the given statement, since if n^2 were odd, both this statement and the statement given in the question would be true.

(e) This is just the given statement, rather than its negation. Since we are seeking contradiction, we would assume the negation rather than the statement itself.

Problem 4. (4 points)

Let $P(x, y)$ mean "person x can work on day y". Which of the following statements has the same meaning as this sentence:

"No one can work every day."

(a) $\exists x \neg \exists y \, P(x, y)$

(b) $\neg \forall x \exists y \, P(x, y)$

(c) $\neg \exists x \forall y \, P(x, y)$

(d) $\forall x \forall y \, \neg P(x, y)$

(e) $\exists x \forall y \, \neg P(x, y)$

Solution: c

"No one can work every day" is equivalent to "there does not exist someone who can work every day"". "There does not exist someone" translates to $\neg \exists x$ and "everyday" translates to $\forall y$

(a) Translation: "There exists someone who cannot work on any day." / "There exists someone where there is no one single day on which they can work"

(b) Translation: "Not everyone can work on some days" / "Someone cannot work on every day"

(c) Translation: "There does not exist someone who can work every day"

(d) Translation: "Everyone cannot work on any possible day" / "For everyone, there is no one single day on which they can work"

(e) Translation: "There exists someone who cannot work on any day." / "There exists someone where there is no one single day on which they can work"

Problem 5. (4 points)

Define the following predicates:

- $O(x, y)$ means person x ordered pizza y
- $D(x, y)$ means person x delivered pizza y

Which is an equivalent translation to the following: Some delivery-person delivered every pizza that they themselves ordered.

(a) $\exists d \forall p[O(d, p) \rightarrow D(d, p)]$

(b) $\exists d \forall p[O(d, p) \wedge D(d, p)]$

(c) $\forall p \exists d[O(d, p) \rightarrow D(d, p)]$

(d) $\forall p \exists d[O(d, p) \wedge D(d, p)]$

Solution: (a)
The ordering of predicates must go in the order of $\exists d \forall p$ because we are talking about a *singular* delivery person. Thus, quantifying the exists first scopes it down as such. Had we done otherwise, It would be with the phrasing "All pizzas were delivered by some delivery person", which no longer requires for it to be the same person.
We must use an implication for this statement because we the statement "delivered every pizza" to only apply to those pizzas that the delivery person also delivered, meaning we must use an implication.

Part A2: Multiple Answer Multiple Choice

Problem 6. (4 points)

Let the domain of x and y be the **non-zero integers**. Which of the following are true?

(a) $\exists x \exists y \, [x^2 + y^2 = 3]$

(b) $\forall x \exists y \, [(y < 0) \rightarrow (y < x^2)]$

(c) $\forall x \forall y \, [(y < 0) \rightarrow (y < x^2)]$

(d) $\forall y \exists x \left[\frac{x}{y} = 1\right]$

(e) $\exists x \forall y \left[\frac{x}{y} = 1\right]$

Solution: b, c, d

(a) Incorrect, there are no combination of perfect squares that add up to 3

(b) Correct. See justification for (c)

(c) Correct. The square of any non-zero integer will be positive, so no matter what x we pick, if y is negative, it will definitely be less than x^2.
Answer (b) says something similar, but says $\exists y$ instead of $\forall y$. We know this is true for all y, so there will definitely exist one: pick literally any value. Of note, this could be a sensible choice, like $y = -3$, but in fact, we can do something very strange. We have an implies statement immediately inside an exists, where the premise of the implies can be affected by the exists variable. These rarely make reasonable sense, as we can just pick a y value that fails that, such as $y = 2$, and it makes the implies statement true.

(d) Correct, every nonzero integer has a number (itself) that upon being divided equals 1.

(e) Incorrect, there is no integer that when divided by every integer equals 1

Problem 7. (4 points)

Which of the following statements are equivalent to the **negation** of the proposition below?

$$\exists x \forall y \, [P(x, y) \rightarrow (x \neq y)]$$

(a) $\forall x \exists y \, [P(x,y) \land (x = y)]$

(b) $\forall x \exists y \, [\neg P(x,y) \to (x = y)]$

(c) $\forall x \neg \forall y \, [P(x,y) \to (x \neq y)]$

(d) $\exists x \forall y \, [P(x,y) \land (x = y)]$

(e) $\exists x \forall y \, [\neg P(x,y) \land (x \neq y)]$

Solution: a, c
We can rewrite the negation as follows:

$$\neg \exists x \forall y \, [P(x,y) \to (x \neq y)]$$

$$\begin{aligned}
&\equiv \forall x \neg \forall y \, [P(x,y) \to (x \neq y)] \\
&\equiv \forall x \exists y \neg \, [P(x,y) \to (x \neq y)] \\
&\equiv \forall x \exists y \neg \, [\neg P(x,y) \lor (x \neq y)] \\
&\equiv \forall x \exists y \, [P(x,y) \land (x = y)]
\end{aligned}$$

(a) Correct. This is the final line shown above

(b) Incorrect. This is similar to if you incorrectly "distributed" the negation in the third line across the implies. This is not the negation of the implies, but the inverse of it.

(c) Correct. This is the second line shown above.

(d) Incorrect. However, this is similar to the last line, but the quantifiers are not properly flipped.

(e) Incorrect. This does not match any of the lines above.

Problem 8. (4 points)

Which of the following are tautologies?

(a) $p \vee q$

(b) $p \wedge \neg p$

(c) $p \vee \neg p$

(d) $(p \to q) \vee (p \wedge \neg q)$

(e) $p \vee (p \to q)$

Solution: c, d, e

(a) $p \vee q$ can be false for values of p, q both false.

(b) This is definitionally a contradiction, it is always false, as any assignment of p yields false. Another way to think about it is that we will always have $T \wedge F$ which is false.

(c) Being the negation of (b), it follows a similar logic and is definitionally true as it will always reduce to $T \vee F$ which is always true.

(d) Simplifying: $(p \to q) \vee (p \wedge \neg q)$ to $(\neg p \vee q) \vee (p \wedge \neg q)$. Let $r = (p \wedge \neg q)$. Then $\neg r = \neg(p \wedge \neg q) = (\neg p \vee q)$. Thus we have $\neg r \vee r$ which as shown in (c) is always true.

(e) Simplifying: $p \vee (p \to q)$ to $p \vee \neg p \vee q$. As $p \vee \neg p$ is true, we have $T \vee q$ which simplifies to T.

Problem 9. (4 points)

Which of these are true over the domain of discourse $\mathbb{R}$? For this, we will define $\frac{0}{0} = 0$, while anything else divided by 0 is undefined, and therefore not equal to 0.

(a) $\forall x \exists y (\frac{x}{y} = 0)$

(b) $\exists x \forall y (\frac{x}{y} = 0)$

(c) $\exists x \exists y (\frac{x}{y} = 0)$

(d) $\exists y \forall x (\frac{x}{y} = 0)$

(e) $\forall y \exists x (\frac{x}{y} = 0)$

Solution: (b), (c), (e)

(a) False. Take $x = 1$ for example.

(b) True. When $x = 0$, all answers will $= 0$, given the definition of $\frac{0}{0}$ from the question itself

(c) Similar to previous answer, take when $x = 0 \land y = 1$

(d) False. No value will always yield zero when placed in the denomenator of a fraction.

(e) True. No matter what y is, if $x = 0$, the result will be 0.

Part B: Short Answer

Problem 10. (6 points)

Prove or disprove the following statement:

$$\forall x \exists y \, [y > x]$$

Note: The domain for x and y is **integers.**

Solution: The statement is true. Thus, we will attempt to prove directly.
Let x be an arbitrary element of the domain. Since x is an integer, $y = x + 1$ is also an integer, but is larger than it. Since x is an arbitary integer, this shows that $\forall x \exists y$ such that $y = x + 1 > x$.

Alternate Solution (Proof by Contradiction):
Seeking a contradiction, assume:

$$\exists x \forall y \, [y \leq x]$$

In order for this to be true, there must be some integer x that is greater than or equal to every integer y. Since integers are infinite, however, for any integer x, we can find some y that is greater than x by setting y = x + 1. This is true for any arbitrary integer x, since x + 1 must also be an integer and $x + 1 > x$. Therefore, the negation must be false, and the original claim must be true.

Grading Guidelines:
+2 Select prove
+2 Consider an arbitrary integer (for all statement within domain)
+2 State $y = x + 1 > x$, or equivalent

Partial Credit:
+1 Attempts to prove (going beyond restating the premise in English)
+1 Attempts to show pattern on how to select larger value of y (example considering multiple pairs of x,y)

Alternate Solution (Proof by Contradiction):
+2 Selects Prove
+1 Correctly negates original statement and assumes

$$\exists x \forall y \, [y <= x]$$

+1 considers an arbitrary integer
+2 Correct argument for disproving negation

Common Mistakes

- Attempting a proof by example by assigning x and y specific values and stating those values work or don't.
- Assigned y to be infinity. Infinity is not an integer.
- Assigning y to 2x or 2x+1. If x is negative then y is not strictly greater.
- In a proof by contradiction, assuming an incorrect negation where y is strictly less than x and not less than or equal to.

Problem 11. (6 points)

Using a proof by contradiction, prove that if $3n^2 + 3$ is even, then n is odd.
Note: You cannot use the lemmas "even + odd = odd", "even · even= even", etc. without proving it.

Solution:
Proof by Contrapositive
We will prove the contrapositive: If n is even, then $3n^2 + 3$ is odd.
Assume that n is even. Then there is an integer k so that $n = 2k$. Therefore, $3n^2 + 3 = 3(2k)^2 + 3 = 12k^2 + 3 = 2(6k^2 + 1) + 1$. Letting $l = 6k^2 + 1$, we see that $3n^2 + 3 = 2l + 1$. Thus, $3n^2 + 3$ is odd and the claim holds. Note that there were valid ways to do proof by contrapositive involving cases, but cases were not necessary.

Alternate Solution: Proof by Contradiction
Seeking contradiction, assume: $3n^2 + 3$ is **even** and n is **even**. By definition of even, $n = 2k$ and thus if we plug it into $3n^2 + 3$, we get $3(2k)^2 + 3$. This becomes $3(4k^2) + 2 + 1$ which becomes $2(3(2k^2) + 1) + 1$. $3(2k^2) + 1$ can be represented by the integer j. So, $3n^2 + 3$ can be represented as $2j + 1$, and thus is odd, which contradicts our assumption

that its even. This completes our contradiction.

Grading Guidelines:
Contrapositive
+1 Writing the correct contrapositive
+2 Correctly applying even/odd definition to n
+2 Correctly applying even/odd definition to $3n^2 + 3$
+1 Concluding that $3n^2 + 3$ is odd
Contradiction
+1 Assuming $3n^2 + 3$ is even for contradiction
+2 Correctly applying even/odd definition to n
+2 Correctly applying even/odd definition to $3n^2 + 3$
+1 Concluding that $3n^2 + 3$ is odd and deriving a contradiction
Deductions
-0.25 For minor transcription or arithmetic errors
-0.5 For using the even/odd lemma's that were disallowed in the exam (even + even = even, etc).
5.0/6.0 maximum given for attempting a direct proof while trying to use the converse of even/odd lemmas.

Common Mistakes

- Proving the converse. In this case the converse is "if n is odd, then $3n^2 + 3$ is even. The converse is **not** equivalent to the original statement.

- Proving the inverse. In this case the inverse is "if $3n^2 + 3$ is odd, then n is even. The inverse **not** equivalent to the original statement.

- Reusing the same variable name for different expressions. For example, if you let $n = 2k$ for some integer k, then show $3n^2 + 3 = 2(6k^2 + 1)$, you should **not** rewrite this as $3n^2 + 3 = 2k$ since we already defined k, and this would imply $k = 6k^2 + 1$. Instead you should define a new variable such as ℓ. For example, "$3n^2 + 3 = 2\ell$ with $\ell = 6k^2 + 1$"

- Similarly, some submissions attempted to prove a lemma like "even + odd = odd" by considering an arbitrary even integer $a = 2k$ for some integer k and an arbitrary odd integer $b = 2k + 1$ for some integer k and showing $a + b$ is odd. The issue is that both expressions uses the same variable k, so this implies that a and b are consecutive integers ($b = a + 1$). This therefore does **not** prove the statement for arbitrary integers. Instead, you need to make sure each expression uses a different variable. For example: "consider an arbitrary even integer $a = 2k$ for some k and an arbitrary odd integer $b = 2j + 1$ for some integer j"

- Attempting a direct proof by using the converse of even/odd lemmas. For example, one may attempt to claim that since $3n^2 + 3$ is even and 3 is odd, because odd + odd = odd, $3n^2 + 3$ is odd. However, this is **not** valid logic. The lemma generally tells us that if a is even and b is odd, then $a + b$ is odd. However, this does **not** directly tell us that if $a+b$ is odd and b is odd then a is odd. So this is an additional lemma that would need to be proved.

Part C: Free Response

Problem 12. (8 points)

p	q	r	s	t	w
T	T	T	T	T	F
T	T	F	T	F	F
T	F	T	T	T	F
T	F	F	T	F	F
F	T	T	T	T	F
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Use the truth table for the compound propositions s, t, and w given above to answer the following questions.

(a) Is $(s \wedge w) \vee t$ a tautology? Briefly explain your answer.

(b) For each unknown proposition, s, t, and w:

- Find an expression for the proposition as a compound proposition using p, q, and/or r.
- You may use **only** $\wedge$, $\vee$, $\neg$, and parentheses in each expression.
- You may use p, q, and r **at most once** in each expression.

Solution:

(a) No, it is not a tautology: The compound proposition evaluates to false when $p \equiv T$, $q \equiv T$, and $w \equiv F$, and so s is True, t is False, w is False.

(b) $s \equiv p \vee r \vee (\neg q)$
$t \equiv \neg p \vee r$
$w \equiv \neg p \wedge (\neg q \vee \neg r)$

To get this solution, there are a few tactics we can utilize. Beginning by examining s and looking at the truth table. We notice that it is only false when $p \equiv F$, $q \equiv T$, and $r \equiv F$, thus, avoiding any of these three assignments is good enough, and thus we get $s \equiv p \vee \neg q \vee r$.

Next, examine the second column t. Again, identifying the conditions that make the statement false or true can help. For example, since the bottom half of the table is true, we see that $\neg p$ is good enough to make this statement true. However, more ways to make it true exist, and we see that the true values line up exactly with the true values for r, and so we see that we must have $\neg p$ or r to make t true, giving a final answer of $\neg p \vee r$.

Finally, examine the last column w. Again, identifying the conditions that make the statement false or true can help. By examining the table, we can see that $p \equiv T$ makes this statement false, and so we must have $\neg p$. However, because $\neg p$ alone is not enough, we must pair it with a $\wedge$. What exactly do we need? Well, by the table, we see that given $\neg p$, we also can't have $q \wedge r$, as both q and r being true will make the statement false. Thus, by combining these two facts, we get that $w \equiv \neg p \wedge \neg(q \wedge r)$.

Grading Guidelines:
Part a
+1 Correctly choosing not a tautology
+1 Finding a counterexample

Part b
+2 For each correct proposition for s, t and w

Common Mistakes:

(a) Considering all combinations of (T/F, T/F, T/F) as possible values for s, t, w. The only valid combinations are on the table given.

(b) Using $\rightarrow$ in part (b).

(c) Saying p instead of $\neg p$ in part (b).

Problem 13. (9 points)

Prove or disprove each of the following statements.

Note: If you use a specific irrational number in your proof/disproof, you do **not** need to prove that it is irrational. You can simply state that it is.

(a) Prove or disprove: For all rational numbers x and irrational numbers y, their sum $x+y$ is irrational.

(b) Prove or disprove: For all irrational numbers x and y, their difference $x-y$ is irrational.

Solution:

(a) Seeking contradiction, assume there exists some rational number x and irrational number y such that their sum $x+y$ is rational. Since x is a rational number, $x = \frac{a}{b}$ for integers a and b (with $b \neq 0$). Similarly, $x + y = \frac{c}{d}$ for integers c and d (with $d \neq 0$). Then,

$$y = (x+y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}$$

Since $bc-ad$ is an integer and bd is a non-zero integer, y is a rational number, which contradicts our original assumption. Thus, we have proven by contradiction that for all rational numbers x and irrational numbers y, their sum $x+y$ is irrational.

(b) We can disprove this statement with a counterexample: Let $x = y = \sqrt{2}$. Then $x - y = \sqrt{2} - \sqrt{2} = 0$. 0 is a rational number because it can be expressed in the form $\frac{0}{1}$ (since 0 and 1 are integers).

Grading Guidelines:

part a: (6 points)

+0.5: Selects prove

+1.5: Correct assumption for proof by contradiction (x rational, y irrational, and $x+y$ rational)

+1.0: Correctly interprets x, y, and $x+y$ using definition of rational and/or irrational based on assumption

+1.0: Correct algebra to combine terms into single term

+1.0: Correctly uses definition of rational to show term is rational

+1.0: Reaches contradiction

part b: (3 points)

+0.5: Selects disprove

+0.5: Gives specific irrational counterexample for x and y (with intent to disprove)

+2.0: Shows difference of given irrational numbers is rational

Common Mistakes:

- Attempting a direct proof: the definition of irrational is that there we cannot express it as $\frac{a}{b}$ for *any* integers a, b. This is almost impossible to prove, but it is much easier to prove that something *is* rational because you just need one pair of integers.

- When doing a direct proof, students would often let $x = \frac{a}{b}$, then get that $x + y = \frac{a+by}{b}$. They would then claim that since $a + by$ is not an integer, then $x + y$ is irrational. Even if we correctly proved $a + by$ is not an integer, this only proves that $x + y$ can't be expressed as an integer over the same denominator as x, but not that it can't be expressed as an integer over *any* integer denominator.

- Incorrectly negating the quantifiers: We didn't take off points for this (as long as the proof itself worked correctly), but the negation of the statement we are trying to prove is that "there exist a rational x and irrational y such that $x + y$ is rational". Many students instead negated it as "for all rational numbers x and irrational numbers y, $x + y$ is rational". This is a much stronger statement which is much easier to disprove.

- Relatedly, students would often attempt to disprove the negation by providing specific x and/or y and deriving a contradiction. This would work to disprove a for-all statement, but since the negation of the statement we want to prove is an exists statement, we must disprove it with arbitrary elements. The intuition here is that we are only assuming that some such x and y exist, but we don't know which one it is, so we have to derive the contradiction for any arbitrary x and y

- Similar to above, many students would (correctly) let x be arbitrary then (incorrectly) pick a specific y. Since the same quantifier is on both of them, we should do the same thing with both of them in the proof.

- Expressing an irrational number as an integer over 0: this probably arises from incorrectly negating the definition of rational. A number over 0 isn't well-defined at all, but irrational numbers are.

- Using a generic example in part b: the given statement is a for-all, so disproving it amounts to proving an exists statement. Using arbitrary values is not sufficient to prove an exists statement, because we also need to show that the domain is non-empty. If your proof was otherwise valid, you would still get most of the points.

Problem 14. (9 points)

Use the definitions of "even" and "odd" to prove the following:

If either x is odd or y is even, then $y^2(x-1)$ is even.

Note: You cannot use the lemmas "even + odd = odd", "even · even= even", etc. without proving it.

Solution:
We will prove the contrapositive statement: If x is odd or y is even, $y^2(x-1)$ is even
Assume x is odd or y is even

Case 1: x is odd
If x is odd, then $x = 2k+1$, for some integer k
$y^2(x-1) = y^2(2k+1-1) = 2(ky^2)$, thus even

Case 2: y is even
If y is even, then $y = 2j$, for some integer j
$y^2(x-1) = (2j)^2(x-1) = 4j^2(x-1) = 2(2j^2(x-1))$, thus even

Since both cases conclude $y^2(x-1)$ is even, we know the contrapositive holds.

Alternate Solution: We use a proof by cases. There are four combinations of even/odd x and y:

Case 1: x is odd and y is odd
If x is odd, then $x = 2k+1$, for some integer k
If y is odd, then $y = 2j+1$, for some integer j
$y^2(x-1) = (2j+1)^2(2k+1-1) = (4j^2+4j+1)2(k) = 2(4j^2k+4jk+k)$.
Since $(4j^2k+4jk+k)$ is an integer, $y^2(x-1)$ is even

Case 2: x is odd and y is even
If x is odd, then $x = 2k+1$, for some integer k
If y is odd, then $y = 2j$, for some integer j
$y^2(x-1) = (2j)^2(2k+1-1) = (4j^2)2(k) = 2(4j^2k)$.
Since $(4j^2k)$ is an integer, $y^2(x-1)$ is even

Case 3: x is even and y is odd
If x is even, then $x = 2k$, for some integer k
If y is odd, then $y = 2j+1$, for some integer j
$y^2(x-1) = (2j+1)^2(2k-1) = (4j^2+4j+1)(2k+1) = 2(4j^2k+4jk+k)+(4j^2+4j+1) = 2(4j^2k+4jk+k+2j^2+2j)+1$.

Thus $y^2(x-1)$ is odd

Case 4: x is even and y is even
If x is even, then $x = 2k$, for some integer k
If y is even, then $y = 2j$, for some integer j
$y^2(x-1) = (2j)^2(2k-1) = (4j^2)(2k+1) = 2(2j^2)(2k+1)$
Thus $y^2(x-1)$ is even

Since the only case where $y^2(x-1)$ is odd is when x is even and y is odd, we can conclude that if $y^2(x-1)$ is odd, then x is even and y is odd

Grading Guidelines:
Contrapositive:
+2 correct contrapositive statement
+3 correct split of cases
+2 correct definitions of even/odd
+2 correct mathematical computation

Cases:
+3 correct split of cases
+2 correct definitions of even/odd
+2 correct mathematical computation
+2 identifying that case 3 is the only case in which the statement can be odd.

Common Mistakes:

- Writing the contrapositive incorrectly, for example, "if x is odd and y is even, then $y^2(x-1)$ is even
- Using the same variable k for even/odd definition for x and y
- Using XOR instead of inclusive or i.e. only considering the cases where *only* x is odd or *only* y is even (rather than both).

Problem 15. (9 points)

Prove the following statement, and state which proof method you are using. The domain for a and b is the set of all **integers**.

"If $a+b > -1$ and $ab > 0$, then a and b are both positive."

Solution: We will give an example of proof by cases. There are four possible cases for a and b: both positive, both negative, one positive one negative, and at least one zero.

- **Case 1:** If both are positive, then $a+b > 0 > -1$ and $ab > 0$, so both conditions are satisfied, making the implication $T \to T \equiv T$.
- **Case 2:** If both are negative, then since a and b are integers, we have $a \leq -1$ and $b \leq -1$, leading to $a+b \leq -2$; thus, $a+b > -1$ is violated, making the implication $F \to F \equiv T$
- **Case 3:** If one is positive and one is negative, then we have $ab < 0$ and $ab > 0$ is violated, making the implication $F \to F \equiv T$
- **Case 4:** If at least of them is zero, then we have $ab = 0$ and $ab > 0$ is violated, making the implication $F \to F \equiv T$

In all 4 cases, we have proved the implication, so it must be true

Alternate Solution:
We will prove the contrapositive: "If at least 1 of a and b are nonpositive, then $a+b \leq -1$ or $ab \leq 0$." Assume one of our variables is nonpositive, without loss of generality, let $a \leq 0$. If $a = 0$, $ab = 0 \leq 0$, proving one of the possible conclusions. Otherwise$a < 0$, and we will examine 3 cases based on what b is:

- $b > 0$: The product of a positive and negative is negative, so this means $ab < 0$
- $b = 0$: $ab = 0 \leq 0$
- $b < 0$: Because a and b are integers, $a \leq -1$ and $b \leq -1$. Putting these together gives $a+b \leq -2 \leq -1$

In every case, we proved $ab \leq 0$ or $a+b \leq -1$, so we proved the contrapositive of the desired statement, which is equivalent to what we wanted. **Grading Guidelines**:

Cases:
+1 correct split of cases
+2 correct justification per case
Contrapositive:

+1 correct contrapositive statement
+2 either using WLOG or having more correct total cases when not using WLOG
+2 per correct case for b

Common mistakes:

- If a is negative, that does not mean that you write $-a$. This expression is the negation of a, which would be positive.

Problem 16. (9 points)

Prove or disprove that for all integers x and y:

If $x - 2xy$ is even, then one of the variables is even and the other is odd.

Hint: You may find it easier to consider the contrapositive of the given statement.

If you choose to prove, you may find it helpful to use the following 6 properties about odd and even numbers. For this question only, you may use them without proving them.

- Odd + Odd = Even
- Odd + Even = Odd
- Even + Even = Even
- Odd $\times$ Odd = Odd
- Odd $\times$ Even = Even
- Even $\times$ Even = Even

Solution:

Solution 0: Direct Proof by Counterexample

Scratch work on the side: We observe that $2xy$ has to be even $\forall x, y \in \mathbb{Z}$, so let's try adding that to the statement we know something about. Then we have $x = (x - 2xy) + 2xy$. If $x - 2xy$ is even, then (because Even + Even = Even) x must be even. Likewise, if $x - 2xy$ is odd, then (because Odd + Even = Odd) x must be odd. Therefore $x - 2xy$ is even iff x is even. However, the claim says that if $x - 2xy$ is even then x & y must have different parities. These two statements don't seem to agree (we can violate the "different parities" clause, but still fulfill the "$x - 2xy$" condition, with an even x & even y) so we will proceed with a disproof.

Disproof by counterexample: We will show that there exist $x, y \in \mathbb{Z}$ such that $x - 2xy$ is even, but the two variables have the same parity. We noticed that in order for $x - 2xy$ to be even, we need x to be even, so let's try making x even and y also even.
Our attempt at a counterexample is $x = y = 2$. This means $x - 2xy = 2 - 2 \cdot 2 \cdot 2 = 2 - 8 = -6$. This is even, but we don't have an odd variable, so it is indeed a counterexample.

You could also do this by considering arbitrary even integers. Let x_0 and y_0 be two integers that are both even, i.e. $x_0 = 2k_x$ and $y_0 = 2k_y$, where $k_x, k_y \in \mathbb{Z}$. Then plugging these in, $x_0 - 2x_0y_0 = 2k_x - 2(2k_x)(2k_y) = 2(k_x - 4k_xk_y)$, so $x - 2xy$ is even for this choice of x and y. However, x_0 and y_0 are both even, so have the same parity. Rather than proving there is some pair of even numbers that work as a counterexample, this proves that all pairs of even numbers would work. Because we know that there are plenty of pairs of even numbers to choose from, there will definitely be a counterexample, so the statement is false.

Solution 1: Proof by Counterexample, using Contrapositive

Contrapositive: There are different ways to get the contrapositive of this claim. Here are some suggestions:

1. The English-to-logic translation of the proposition states that

$$\forall x, y \in \mathbb{Z},\ \text{Even}(x - 2xy) \to \big[\text{Even}(x) \oplus \text{Even}(y)\big]$$

   We take the contrapositive of the implication:

$$\forall x, y \in \mathbb{Z},\ \big[\text{Even}(x) \leftrightarrow \text{Even}(y)\big] \to \text{Odd}(x - 2xy)$$

2. If we don't want to think about negating $\oplus$ we can define a new predicate in two variables, over the domain of integers.

   Let $P(x, y)$: integers x and y have the same parity (both even or both odd).

   Then, the proposition states that

$$\forall x, y \in \mathbb{Z},\ \text{Even}(x - 2xy) \to \neg P(x, y)$$

   We take the contrapositive of the implication:

$$\forall x, y \in \mathbb{Z},\ P(x, y) \to \text{Odd}(x - 2xy)$$

3. Another option is to use nested quantifiers:

$$\forall x, y \in \mathbb{Z},\ \text{Even}(x - 2xy) \to \Big[\big(\exists a \in \{x, y\}, \text{Even}(a)\big) \wedge \big(\exists b \in \{x, y\}, \text{Odd}(b)\big)\Big]$$

   We take the contrapositive of the implication:

$$\forall x, y \in \mathbb{Z},\ \Big[\big(\forall a \in \{x, y\}, \text{Odd}(a)\big) \vee \big(\forall b \in \{x, y\}, \text{Even}(b)\big)\Big] \to \text{Odd}(x - 2xy)$$

4. As yet another way, we could write “one of the variables is even” as Even(x)$\vee$Even(y) and similar for “the other is odd”. This means the original statement is

$$\forall x, y \in \mathbb{Z}, \text{Even}(x - 2xy) \rightarrow [(\text{Even}(x) \vee \text{Even}(y)) \wedge (\text{Odd}(x) \vee \text{Odd}(y))]$$

   If we take the contrapositive of this implies statement and apply De Morgan's law to bring the negation all the way inside, noting that Even and Odd are negations of each other, we get

$$\forall x, y \in \mathbb{Z}, [(\text{Odd}(x) \wedge \text{Odd}(y)) \vee (\text{Even}(x) \wedge \text{Even}(y))] \rightarrow \text{Odd}(x - 2xy)$$

So our contrapositive states that for all integers x and y, if the variables are both even or both odd then $x - 2xy$ must be odd.

Disproof by counterexample: At this point, we need to show that there are $x, b \in \mathbb{Z}$ that have the same parity, but $x - 2y$ is even. This is exactly what we wanted in the previous solution, so the rest of the proof should look exactly the same as the ”Disproof by counterexample” section above.

Solution 2: Prove the Negation Directly

Using the predicate $P(x, y)$ as defined above, our claim p is

$$p \equiv \forall x, y \in \mathbb{Z},\ \text{Even}(x - 2xy) \rightarrow \neg P(x, y)$$

Then the negation $\neg p$ is

$$\neg p \equiv \exists x, y \in \mathbb{Z},\ \text{Even}(x - 2xy) \wedge P(x, y)$$

Observe that if we let $x = y = 2$, then $x - 2xy = -6 = 2(-3)$ is even but the variables have the same parity (both even). Any specific or general counterexample of the form x even, y even works here.

Therefore, there exist integers x and y such that $x - 2xy$ is even and the variables have the same parity.

Therefore, $\neg p \equiv T$, so p is false & we have disproved the claim.

Grading Guidelines [8 points]

disproof of original claim:

+4: specific or general counterexample [item 2]
+5: explains why named counterexample is valid [item 3]
+2: attempted counterexample, but incorrect x, y [item 4]

disproof of contrapositive:

+2: correct contrapositive [item 7]
+1: partially correct contrapositive (attempts something in the form of $\neg q \rightarrow \neg p$ but incorrectly negates "one even, one odd") [item 8]
+3: specific or general counterexample [item 5]
+4 explains why named counterexample is valid [item 3]
+6: incorrect contrapositive, but correct proof/disproof of stated contrapositive (assuming it doesn't fit into the previous rubric items) [item 9]

deductions:

-1: minor mistake (poor wording leading to incorrect statement, etc) [item 11]
-2: correct proof/disproof, but incorrect framing of proof/disproof method [item 12]

Common mistakes

- Many students incorrectly negated "one variable is odd and the other is even". The most common incorrect negation was "one variable is even or the other is odd". This stems from attempting to apply DeMorgan's law under an incorrect interpretation of the original statement. However, note that this statement is poorly defined–it appears to actually cover all cases for the parities of x and y. As such, many students would proceed to incorrectly interpret their incorrect contrapositive, leading to an incorrect proof/disproof of it. See solution 1 for a detailed explanation of how to correctly take the contrapositive.

- After taking the correct contrapositive, many students broke the premise of the new implication into two cases: both variables even and both variables odd. They would then (correctly) notice that when both variables are even, $x - 2xy$ is even, and when both are odd, $x - 2xy$ is odd. However, students would often incorrectly conclude that because one case works, the implication in general is true. However, both cases of the premise must lead to the conclusion for the implication to be proven true. Here's a logical equivalence proof of that, where p is "x and y even",

q is "x and y odd", and r is "x - 2xy odd":

$$\begin{aligned}(p \vee q) \to r &\equiv \neg(p \vee q) \vee r \\ &= (\neg p \wedge \neg q) \vee r \\ &= (\neg p \vee r) \wedge (\neg q \vee r) \\ &= (p \to r) \wedge (q \to r)\end{aligned}$$

- Many students started or ended their proof saying they were doing a "disproof by cases." Disproof by cases is not really a proof method. For this problem, you only need to show one case where the implication doesn't hold, which is called a proof by counterexample. Alternatively, you may need to show that the implication doesn't hold in all cases, which is proving the negation by cases (this strategy doesn't apply to this problem since there are cases where the implication does hold)

- Some students ended their proof by saying something along the lines of "because we could not prove every case, the claim is disproven" or "because the two cases are different, the claim is disproven." Both of these are slightly incorrect ways of concluding the proof because a failure to prove a statement does not necessarily mean that the claim is disproven (it may just mean we haven't found the proof yet). A more correct statement would be "because the case where x, y are both even serves as a counterexample, the claim is disproven"

- Many students correctly proved that $x - 2xy$ even $\to x$ even. While this statement is true, and typically involved similar work to prove it, proving a separate statement does not necessary imply that the original statement was false. You would need to then separately prove that the statement you proved and the original statement cannot both be true, which students rarely did.

- Many students would declare x, y as "arbitrary" integers of some parity, but then share constants between applications of definition of even/odd. For example, they may say $x = 2k$ & $y = 2k + 1$ to obtain some quadratic expression in k. A correct application would give "$x = 2k_x$ & $y = 2k_y + 1$" where k_x & k_y are potentially different arbitrary integers. Some students did the same with the definition of even when constructing a counterexample, writing something like "let $x = 2k, y = 2k$". While it seems like the intention was typically for x and y to be arbitrary even integers, this actually declares x and y as the *same* even integer. Conveniently, because we only need one counterexample to disprove the statement, this mistake still leads to a valid disproof, though it is important to be conscious of anyway.

- As a minor technical detail, when attempting to show a case where the implication is false, we need to show that such a case exists. In this problem, you could assume without proof that a pair of two even integers exists, but it is a problem with "disproof by cases" as a general strategy

Scratch paper. **Nothing written on this page will be graded.**

Scratch paper. **Nothing written on this page will be graded.**

Scratch paper. **Nothing written on this page will be graded.**