

EECS 203: Discrete Mathematics  
Fall 2023  
Homework 5

Due **Thursday, October. 12**, 10:00 pm

No late homework accepted past midnight.

Number of Problems:  $7 + 2$

Total Points:  $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

# Individual Portion

## 1. Induction Construction [16 points]

Let  $P(n)$  be the statement that  $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$  whenever  $n$  is a positive integer. In this problem, we will prove this statement via weak induction.

- (a) What is the statement  $P(1)$ ?
- (b) Show that  $P(1)$  is true, which is the base case for our inductive step.
- (c) In the base case we prove  $P(1)$ ; what do you need to prove in the inductive step?
- (d) What is the inductive hypothesis for your proof?
- (e) Complete the inductive step, indicating where you used the inductive hypothesis.
- (f) Explain why this proof shows  $P(n)$  is true for all positive integers  $n$ .

### Solution:

- (a)  $P(1)$ :  $1 \cdot 1! = (1+1)! - 1$ .
- (b) For  $P(1)$ , LHS = 1  
RHS =  $2! - 1 = 2 \cdot 1 - 1 = 1$ .  
 $\therefore$  LHS = RHS.  $\therefore P(1)$  is true.
- (c) We need to prove that  $P(k) \rightarrow P(k+1)$  for any integer  $n$  which is  $\geq 1$ .
- (d) The inductive hypothesis: Assume  $P(k)$ :  $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ .
- (e) Let  $k$  be an arbitrary positive integer.  
Assume  $P(k)$ :  $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$   
Want to show:  $P(k+1)$ :  $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! = (n+1+1)! - 1$   
Using  $P(n)$  we know:  
$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(1 + n+1) + 1 \\ &= (n+1+1) \cdot (n+1) \cdot n \cdot (n-1) \cdots 1 - 1 \\ &= (n+1+1)! - 1 \end{aligned}$$

Thus  $P(k) \rightarrow P(k+1)$  for any integer  $n$  which is  $\geq 1$ .

(f) It is because that: (1)  $P(1)$ :  $1 \cdot 1! = (1 + 1)! - 1$ .  
 (2)  $P(k) \rightarrow P(k + 1)$  for any integer  $n$  which is  $\geq 1$ .  
 Therefore  $P(1) \rightarrow P(2) \rightarrow P(3) \cdots \rightarrow P(n)$   
 Where  $n$  can be any positive integer.  
 $\therefore$  Since we know (1) is true and (2) is true,  $P(n)$  is true for all positive integers  $n$ .

## 2. Base Two Blues [14 points]

Prove using mathematical induction that  $\log_2(n) < n$  for every positive integer  $n$ . You may assume that the base-2 logarithm function is strictly increasing on its domain.

**Fun Fact:**  $\log_b(n) < n$  is actually true for every positive real number  $n$  and arbitrary base  $b > 1$ , but we're asking you to prove this by induction for the special case where  $b = 2$  and  $n$  is a positive integer.

### Solution:

Let  $k$  be an arbitrary positive integer.

Assume  $P(k)$ :  $\log_2 k < k$

Want to show:  $P(k + 1)$ :  $\log_2 k + 1 < k + 1$

### Base Case:

$P(0)$ :  $\log_2 1 < 1$

Since  $\log_2 1 = 0 < 1$ , base case is true.

### Inductive Step:

Using the property of logarithm we know:

$$\begin{aligned}\log_2(k + 1) &= \log_2\left(\frac{k + 1}{k} \cdot k\right) \\ &= \log_2 \frac{k + 1}{k} + \log_2 k \\ &= \log_2\left(1 + \frac{1}{k}\right) + \log_2 k\end{aligned}$$

Since  $k$  is a positive integer,  $k \geq 1$ ,  $\frac{1}{k} \leq 1$ ,  $\frac{1}{k} + 1 \leq 2$

$\therefore \log_2\left(\frac{1}{k} + 1\right) \leq 1$ .

Using  $P(n)$  we know:  $\log_2 k < k$ .

And Since  $\log_2\left(\frac{1}{k} + 1\right) \leq 1$  and  $\log_2 k < k$

$\log_2(k + 1) = \log_2\left(1 + \frac{1}{k}\right) + \log_2 k < k + 1$ .

Then we have proved that  $P(K) \rightarrow P(k + 1)$  for all positive integer  $k$ .

Conclusion:  $\log_2(n) < n$  for every positive integer  $n$ .

### 3. Inductive Hypothe-six [15 points]

Prove by weak induction that 6 divides  $n^3 - n$  where  $n$  is a nonnegative integer. Don't include unneeded base cases.

**Solution:**

Let  $k$  be an arbitrary nonnegative integer.

Assume  $P(k)$ :  $6 \mid (k^3 - k)$

Want to show:  $P(k+1)$ :  $6 \mid [(k+1)^3 - (k+1)]$

**Base Case:**

$P(0)$ :  $6 \mid 0^3 - 0$

Since  $0^3 - 0 = 0$ ,  $6 \mid 0$ , base case is true.

**Inductive Step:**

Since  $P(k)$ :  $6 \mid (k^3 - k)$ , for some integer  $m$ ,  $6m = (k^3 - k)$

Then

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\&= k^3 + 3k^2 + 2k \\&= (k^3 - k) + 3k^2 + 3k \\&= 6m + 3(k^2 + k) &= 6m + 3k \cdot (k+1)\end{aligned}$$

Since  $k$  is an integer, and integers consist of alternating odd numbers and even numbers, one of  $k$  and  $k+1$  must be even. WLOG assume  $k$  is even, then for some integer  $p$ ,  $k = 2p$ .

Then  $(k+1)^3 - (k+1) = 6m + 3 \cdot 2p \cdot (k+1) = 6m + 6p \cdot (k+1) = 6[m + p(k+1)]$

Since  $p, m, k$  are integers,  $p(k+1)$  is an integer,  $[m + p(k+1)]$  is an integer.

Then  $6 \mid [m + p(k+1)]$ , i.e.  $6 \mid [(k+1)^3 - (k+1)]$ .

Therefore we have proved that  $P(k) \rightarrow P(k+1)$  for any nonnegative integer  $k$ .

Conclusion: 6 divides  $n^3 - n$  where  $n$  is a nonnegative integer.

## 4. Incorrect Strong Induction [14 points]

For each of the following **incorrect** strong induction proofs, note where the strong induction proof breaks down and is incorrect.

**Hint:** Consider where the inductive step breaks down.

- (a) Proving for every nonnegative integer  $n$ ,  $P(n): 3n = 0$ .

**Inductive Step:**

Assume that  $P(j): 3j = 0$  for all nonnegative integers  $j$  with  $0 \leq j \leq k$ . We wish to show  $P(k+1)$ . We will rewrite  $k+1 = a+b$  where  $a$  and  $b$  are nonnegative integers less than  $k+1$ . Thus,  $3 \cdot (k+1) = 3 \cdot (a+b) = 3a + 3b = 0 + 0 = 0$ , therefore  $P(k+1)$  is proven.

**Base Case:**  $P(0): 3 \cdot 0 = 0$

Since we have shown the basis step and the inductive step, we have proved for every nonnegative integer  $n$ ,  $P(n): 3n = 0$ .

- (b) Proving that every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

**Inductive Step:**

Assume we can form cent values of  $j$  cents for all  $3 \leq j \leq k$  using just 3-cent and 4-cent stamps. We wish to show we can form  $k+1$  cents using just 3-cent and 4-cent stamps. We can form a  $k+1$  cent value by replacing 1 3-cent stamp with 1 4-cent stamp or by replacing 2 4-cent stamps with 3 3-cent stamps.

**Base Case:**

We can form cent values of 3-cents using one 3-cent stamp and we can form cent values of 4-cents using one 4-cent stamp. This covers our two base cases.

Since we have shown the basis step and the inductive step, we have proved every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

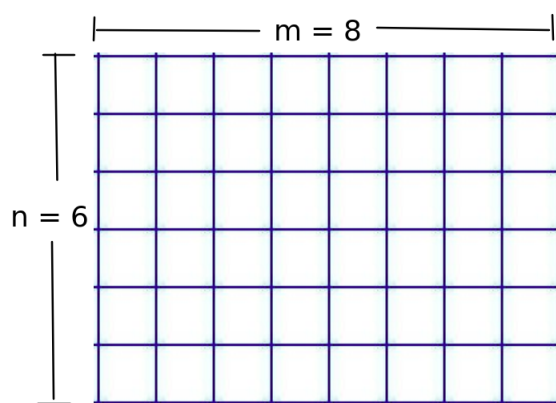
**Solution:**

- (a) The proof breaks down from the base case when we induce  $P(0) \rightarrow P(1)$ .  
At this point,  $k+1 = 1$ , but we cannot find two integers  $a$  and  $b$  that are less than 1 while they add up to 1. If  $a, b$  are less than 1 and nonnegative, then they can only be both 0. Then they add to 0 but not 1.  
Therefore the proof is incorrect.
- (b) The proof breaks down when we induce  $(P(3) \wedge P(4)) \rightarrow P(5)$ .  
The inductive step states that we can form a  $k+1$  cent value by replacing 1 3-cent

stamp with 1 4-cent or by replacing 2 4-cent stamps with 3 3-cent stamp. But at this point, we only have one 4-cent and can not apply the induction. Therefore the proof is incorrect.

## 5. Chopping Ice [15 points]

Claire doesn't have an ice tray, so she makes ice by freezing water into a rectangle and then dividing the rectangle into grid-aligned cells. She would like to divide her block of ice into  $n$  rows and  $m$  columns quickly, before the ice melts! See the image below for an example.



- (a) State the number of cuts Claire needs to make to divide her ice block into  $n \times m$  cells. One cut means splitting a single rectangle into two rectangles. In other words, you may NOT make a single cut across multiple pieces of ice. You may use  $n$  and/or  $m$  in your answer.
- (b) Prove your answer from part (a).

### Solution:

(a)  $(m - 1) \cdot (n - 1) = mn - m - n + 1$   
 $m, n$  are positive integers.

(b)

## 6. Pastry Recurrence [12 points]

A baker decorates a cookie in 2 minutes, a cupcake in 3 minutes, and a pie in 3 minutes. Let  $a_n$  denote the number of distinct ways the baker decorates pastries in exactly  $n$  minutes

for  $n \geq 0$  (where order matters).

- (a) Find a recurrence relation for  $a_n$ .
- (b) What are the initial conditions? Use the fewest initial conditions necessary.

**Solution:**

## 7. Raven's Wrestlers [14 points]

Raven has  $n$  weeks to build her wrestling figure collection. Every week, Raven buys one item to add to her collection. There are 4 different types of things she can buy: Figures, T-shirts for her wrestlers to wear, Weapons for them to fight with, or Display Stands to show them off on her shelves.

- Her shelves can fit 2 Stands nicely, so when she buys a Display Stand, she will always buy a second one the next week to finish the shelf. Additionally, the week after buying the second Stand, she will buy something other than a Display Stand (they aren't as exciting to buy)
- When she buys a Figure, she gets very excited about it and wants to buy a new T-shirt for it to wear the following week.

Let  $a_n$  represent the number of ways Raven can buy items across the  $n$  weeks (where  $n \geq 0$ )

- (a) Find a recurrence relation for  $a_n$ .
- (b) Which terms would need to be defined with initial conditions (no need to find the value, just which terms)

*Note 1:* Buying the same items in a different order counts as a different way of buying items. We treat all items in a category as identical.

*Note 2:* on week  $n$ , Raven will not buy a Figure (because she knows she will miss buying a T-shirt) or a Stand (what a sad way to end the collection). This information is not needed for the simplest solutions, but some alternate solutions may need to know this.

**Solution:**

# Groupwork

## 1. Grade Groupwork 4

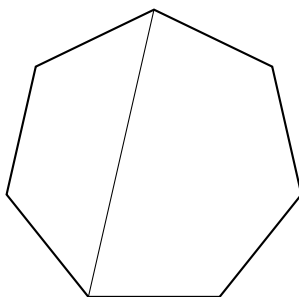
Using the solutions and Grading Guidelines, grade your Groupwork 4:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/20
Problem 3												/30
Total:												/50

## 2. Polly Gone [12 points]

A convex polygon is a 2D shape with all straight edges such that any line segment between any two non-adjacent vertices passes entirely through its interior (see example picture below). Show via induction that the sum of the interior angles of a convex polygon with  $n$  sides is  $(n - 2) \cdot 180^\circ$ . Don't include unneeded base cases.





**Hint 1:** It is helpful to know that a triangle's interior angles always sum to  $180^\circ$ . You may assume this is true for the problem.

**Hint 2:** In order to apply your inductive hypothesis to a convex polygon, you'll need to think of it in terms of smaller convex polygons. How can you make smaller polygons out of a big one?

**Solution:**

### 3. Running Recurrence [8 points]

An EECS 203 student goes to lecture everyday. On each day, she always chooses exactly one method of transportation, and always chooses to walk, bike, or take a bus. She also follows additional rules:

- She never walks two days in a row.
- If she takes the bus, she must have biked two days ago and walked a day ago.

Let  $a_n$  denote the number of ways she can go to EECS 203 lecture across  $n$  days for  $n \geq 0$ .

- Find a recurrence relation for  $a_n$ .
- What are the initial conditions? Use the fewest initial conditions necessary.

**Solution:**