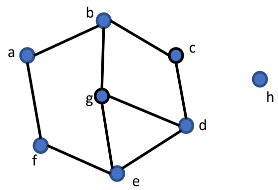
Terminology
1. shortest path 最短路
2. directed path 有向路径
3. weighted graph 加权图
4. bipartite 二分图
5. 2-colorable
6. Hamiltonian Cycle 哈密顿回路
7. Euler circuit. 欧拉回路

Lec 19: Graphs – Paths and Cycles -- ANSWERS

- ① The **length** of a path p is the number of edges in p.
- An (s,t) path p is ② a **shortest path** if there is no other (s,t) path of smaller length.
- The ③ **distance** between two nodes is the length of their shortest path(s).
  - Note: multiple shortest paths all have the same length
  - If no shortest (s,t) path, then $dist(s,t) = \infty$.



$dist(a,e) = 2$
$dist(b,d) = 2$
$dist(f,c) = 3$
$dist(e,e) = 1$
$dist(h,c) = \infty$

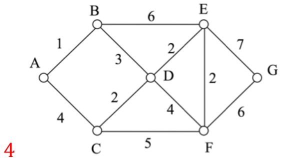
$dist(c,c) = 0$

2

written as: $G = (V, E, w)$
$w: E \to \mathbb{R}$ is a function that returns the weight of an edge.

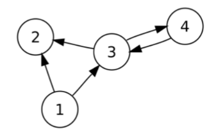
Alternate Graphs

- A ④ **weighted** graph is a graph with numbers (weights) attached to the edges.



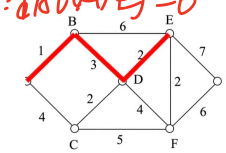
$w(A,B) = w(B,A) = 1$ $\quad dist(A,D) = 4$
$w(D,F) = w(F,D) = 4$ $\quad dist(A,E) = 6$
$w(E,G) = w(G,E) = 7$ $\quad dist(D,G) = 9$

- In a ⑤ **directed** graph, the edges are ordered pairs of distinct vertices.
  - Are (1,4) connected/not connected?
    Are (4,1) connected/not connected?



Shortest Paths in Weighted Graphs

eg: $dist(A,E) = 6$



- The length of a path in a **weighted** graph is the **sum** of its edge weights.
- Then, as before:
  - An (s,t) path p is a **shortest path** if there is no other (s,t) path of smaller length.
  - The **distance** from s to t is the length of the shortest (s,t) path(s).

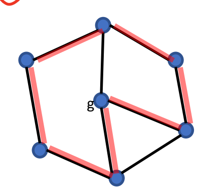
Hamiltonian Cycles

A ⑥ **Hamiltonian Cycle** in a graph G is a path that that contains every vertex exactly once, and returns where it started.

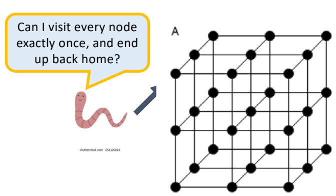
Equivalently: in an n-node graph, a **Hamiltonian Cycle** is a C_n subgraph.

⑦

Do these graphs have Hamiltonian Cycles?



yes



No (colorability argument in lecture)

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Bipartite Graphs

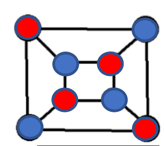
A graph $G = (V,E)$ is *bipartite* if there is a partition $V = A \cup B$ with $E \subseteq A \times B$.

⑧ less formally

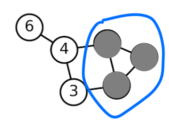
A graph $G = (V, E)$ is ***bipartite*** if we can partition of the vertices into 2 sets such that all edges go between the sets.

即：存在划分，使每条边的端点都是一红一蓝。

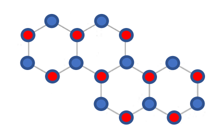
Which of these are bipartite?



Yes:
A = red nodes
B = blue nodes



No (has a C_3 subgraph)



Yes:
A = red nodes
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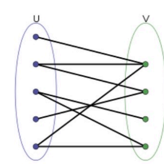

Bipartite (easy to verify from the drawing)

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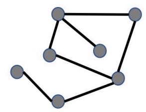
Part 1: bipartite ↔ 2-colorable

Sets A, B and colors (red, blue) are just two different ways to partition the nodes!
The constraint is the same: all edges go from one part to the other.

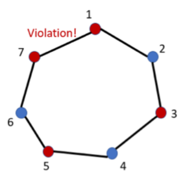


Bipartite Graph Characterization

Theorem: For any graph G, the following are equivalent:

- G is **bipartite**: we can partition nodes into sets A, B so that all edges go between A and B
- 9 G is **2-colorable**: we can assign colors "red" and "blue" to its vertices so that no edge has same-color endpoints
- 10 G has **no odd cycles**: for any odd integer k, G does not have C_k as a subgraph

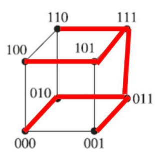
Part 2a: 2-colorable → no odd cycle subgraphs



Part 2a (contrapositive): G has an odd cycle subgraph → G is not 2-colorable

- Consider the odd cycle subgraph C_{2k+1}. Label its nodes $1, \dots, 2k+1$.
- Assume for contradiction that G is 2-colorable
- Assume without loss of generality that node 1 is colored red.
- Going clockwise around the circle: even nodes must be blue and odd nodes must be red.
- But then both endpoints of the edge $\{1, 2k+1\}$ are red. Contradiction!

Part 2b: G has NO odd cycle subgraph → G is 2-colorable

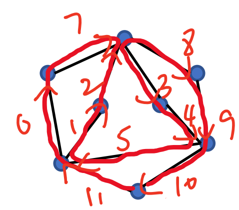


- Assume no odd cycles, we will build a 2-coloring as follows:
  - Pick any spanning tree T of G
  - Take a 2-coloring **of the spanning tree.** (Any tree is 2-colorable. Why?) (Start at some vertex as "root", and start 2-coloring)
  - **Claim:** This 2-coloring of the spanning tree **also** works for the entire graph! Why?

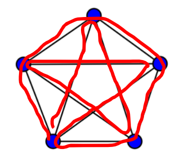
Euler Circuits

11 An **Euler Circuit** is a path in a graph that contains every edge exactly once, and that starts and ends at the same node node.

Can you find an Euler circuit in these graphs?



yes



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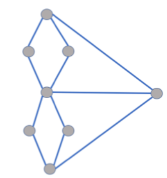
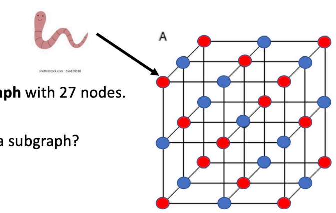
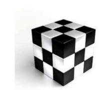


No (because there are nodes with odd degree node)

A Puzzle



- Model Wanda's cube as a **graph** with 27 nodes.
- Does this graph have C_{27} as a subgraph?
- The graph is **2-colorable**.
  - So Wanda's graph is bipartite
  - So it doesn't have any odd cycle subgraphs, including C_{27}!



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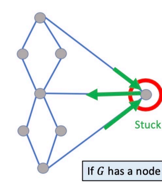

This node has degree 3.
Each time an Euler circuit enters the node, it needs to leave it along a new edge.

If G has a node of odd degree, then it does not have an Euler circuit.

Is that the "only reason?" If G has all even-degree nodes, must it have an Euler circuit?

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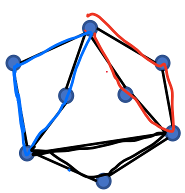
Euler's Theorem

12 Euler's Theorem
For a connected graph G, G has an Euler circuit if and only if all nodes in G have even degree.

Part 2: All nodes have even degree → G has an Euler circuit



Step 1: Partition the edges of G into cycle subgraphs

- Pick any node to start
- Start walking along arbitrary edges until you revisit a node
  - Even degree means you will never get stuck
- Use the loop in your path as one of your cycle subgraphs
- Remove its edges and repeat.
  - When you remove a cycle, all nodes still have even degree!

Step 2: splice together adjacent cycles into one big loop, i.e., into one big Euler circuit.

more formal proof → next page

Part 1: G has an Euler circuit → all nodes in G have even degree

Part 1 (contrapositive): G has a node of odd degree → G does not have an Euler circuit

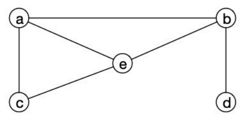
We just discussed this!

* ⑤ directed graph's degree

Degrees, Indegrees, and Outdegrees

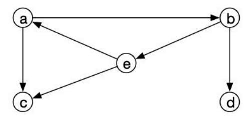
- The ***degree*** of a vertex in an **undirected** graph is its number of adjacent edges.
- In a **directed** graph, there are two different concepts of degree:
  - The ***out-degree*** of v is the number of edges **leaving** v
    - $outdeg(v) = |\{e \in E \mid v \text{ is the first node of } e\}|$
  - The ***in-degree*** of v is the number of edges **entering** v
    - $indeg(v) = |\{e \in E \mid v \text{ is the second node of } e\}|$

$\text{deg}(a) = 3, \text{deg}(c) = 2$



$\text{outdeg}(a) = 2, \text{outdeg}(c) = 0$

$\text{indeg}(a) = 1, \text{indeg}(c) = 2$



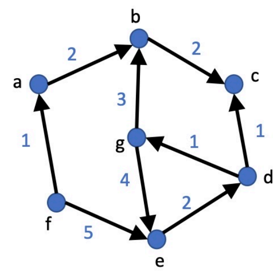
* ⑤ Directed Weighted Distance

Directed Weighted Distances

- **Directed** graphs can possibly also be **weighted**, too.
- Path length then defined using edge weights

$dist(d, g) = 1$
$dist(g, d) = 6$

$dist(e, b) = 3$
$dist(b, e) = \infty$

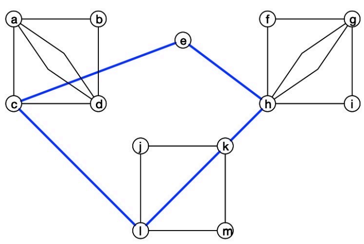


* ⑫ A formal proof

Euler's Theorem: A more formal proof sketch

For a connected graph G, G has an Euler circuit if and only if all nodes in G have even

Part 2: All nodes have even degree → G has an Euler circuit



- The graph $G = (V, E)$.
- The cycle $C = (V_C, E_C)$.
- The graph $G' = (V, E -$

- The graph G' has fewer
- Let's try to apply induct

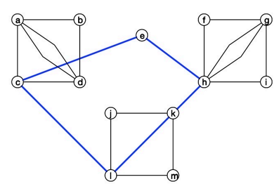
Q: What can we say about the degrees in the graph $G' = (V, E - E_C)$?
Q: Is G' necessarily connected ?

The Proof Sketch

Part 2: G connected and all nodes have even degree → G has an Euler circuit

We will use strong induction on the number of edges.
Base case: |E|=0, then if G is connected then |V|=1. Clearly true.

Inductive Step.
- Find ***some*** cycle C. Let E_C denote edges of C.
- Let $G' = (V, E - E_C)$?
- Each vertex of G' has even degree (but not necessarily connected)
- By strong induction, each connected comp. $G_1, \ldots, G_k$ of G' has an Eulerian Circuit C_i (for i=1,…k).
- As G is connected, C and each C_i share some vertex, say v_i.
- Form an Euler cycle for G by taking C and "splicing" C_i into it at v_i.



⑬ Eular path

An ***Euler path*** in a graph is a path that contains every edge exactly once,
(it may or may not start and end at the same node)

Using a similar argument as for Euler circuit one can also show the following

Euler's Theorem #2
For a connected graph G, G has an Euler path if and only if at most two nodes in G have odd degree (equivalently at least |V|-2 nodes have even degree)