EECS 203: Discrete Mathematics
Fall 2023
Homework 5

Due **Thursday, October. 12**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total Points: $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Induction Construction [16 points]

Let $P(n)$ be the statement that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer. In this problem, we will prove this statement via weak induction.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, which is the base case for our inductive step.

(c) In the base case we prove $P(1)$; what do you need to prove in the inductive step?

(d) What is the inductive hypothesis for your proof?

(e) Complete the inductive step, indicating where you used the inductive hypothesis.

(f) Explain why this proof shows $P(n)$ is true for all positive integers n.

Solution:

(a) $P(1)$ is the statement that $1 \cdot 1! = (1+1)! - 1$

(b) $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1 = 1 \cdot 1 = 1 \cdot 1!$. Therefore $P(1)$ is true.

(c) We want to show that for all $k \geq 1$, $P(k) \rightarrow P(k+1)$.

(d) The inductive hypothesis is $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$ for some positive integer k.

(e) Assume the inductive hypothesis. That is, let $k \geq 1$ be an integer and assume $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$. Then,

$$\begin{aligned}
&1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! \\
&\quad = (k+1)! - 1 + (k+1) \cdot (k+1)! && \text{(by IH)} \\
&\quad = (k+1)! + (k+1) \cdot (k+1)! - 1 \\
&\quad = (k+1)!\big[1 + (k+1)\big] - 1 \\
&\quad = (k+1)!(k+2) - 1 \\
&\quad = (k+2)! - 1
\end{aligned}$$

(f) Since we have shown $P(1)$ is true, and $P(k) \rightarrow P(k+1)$ for all positive integers k, we have shown $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ is true for all positive integers n by weak induction.

Grading Guidelines [16 points]

Parts a through d:

+2 correct $P(1)$
+2 correctly shows $P(1)$ is true
+3 correctly states we want to show that for all $k \geq 1$, $P(k) \rightarrow P(k+1)$
+3 correct inductive hypothesis

Part e:

+2 correct application of IH
+2 simplification after application of IH

Part f:

+2 conclusion

2. Base Two Blues [14 points]

Prove using mathematical induction that $\log_2(n) < n$ for every positive integer n. You may assume that the base-2 logarithm function is strictly increasing on its domain.

Fun Fact: $\log_b(n) < n$ is actually true for every positive real number n and arbitrary base $b > 1$, but we're asking you to prove this by induction for the special case where $b = 2$ and n is a positive integer.

Solution:

Inductive Step:

Assume $\log_2(k) < k$.

$$\begin{aligned}
\log_2(k+1) &\leq \log_2(2k) && \text{(since } k \geq 1,\ k+1 \leq 2k\text{)} \\
&= \log_2(2) + \log_2(k) \\
&= 1 + \log_2(k) \\
&< k + 1. && \text{(by IH)}
\end{aligned}$$

Base case:

Let $n = 1$. $\log_2(1) = 0 < 1$. Therefore the base case holds.

By induction, we have proven that for every positive integer n, $\log_2(n) < n$.

Grading Guidelines [14 points]

+3 assumes proper inequality $\log_2(k) < k$
+3 substituted $2k$ for $k+1$
+3 applies log properties to break up $\log_2(2k)$ into $\log_2(2) + \log_2(k)$
+3 applies IH
+2 correct base case

3. Inductive Hypothe-six [15 points]

Prove by weak induction that 6 divides $n^3 - n$ where n is a nonnegative integer. Don't include unneeded base cases.

Solution:

Let $P(n)$ be the predicate that 6 divides $n^3 - n$.

Base case: $n = 0$

$P(0)$ means 6 divides $0^3 - 0$. Because $0^3 - 0 = 0$, and $6 \mid 0$ since $6(0) = 0$, $P(0)$ is true.

Inductive step:

For our inductive hypothesis, assume that $P(k)$ is true for some $k \geq 0$. Then 6 divides $k^3 - k$, so there exists some integer a such that $k^3 - k = 6a$.

We want to show that $P(k+1)$ is true, meaning we want to prove that 6 divides $(k+1)^3 - (k+1)$ by demonstrating that there exists some integer b such that $(k+1)^3 - (k+1) = 6b$.

Our solution will take $(k+1)^3 - (k+1)$, work on it so we can apply the inductive hypothesis, then do more work so we end up writing it as 6 times some integer.

$$\begin{aligned}
(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) + (-k - 1) \\
&= (k^3 - k) + (3k^2 + 3k) + (1 - 1) \\
&= (k^3 - k) + 3k(k+1) \\
&= 6a + 3k(k+1) && \text{by IH} \\
&= 6a + 3(2c) && c \in \mathbb{Z} \text{ see } \bigstar \\
&= 6a + 6c \\
&= 6(a + c)
\end{aligned}$$

$\bigstar$ Since either k or $k+1$ is even, and an even integer times any integer is even, $k(k+1)$ is even.

So, $(k+1)^3 - (k+1) = 6(a+c)$. Because $a + c$ is an integer, that means 6 divides $(k+1)^3 - (k+1)$, so $P(k+1)$ is true.

Hence, by induction, 6 divides $n^3 - n$ where n is a nonnegative integer.

Grading Guidelines [15 points]

+2 correct base case
+3 correct inductive hypothesis (assuming $P(n)$ or $P(n-1)$)
+4 correct application of inductive hypothesis
+3 correct algebra up to substitution of inductive hypothesis
+3 correct algebra to show $P(n+1)$ if assuming $P(n)$ or $P(n)$ if assuming $P(n-1)$

4. Incorrect Strong Induction [14 points]

For each of the following **incorrect** strong induction proofs, note where the strong induction proof breaks down and is incorrect.

Hint: Consider where the inductive step breaks down.

(a) Proving for every nonnegative integer n, $P(n)\colon 3n = 0$.

Inductive Step:

Assume that $P(j)\colon 3j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. We wish to show $P(k+1)$. We will rewrite $k+1 = a+b$ where a and b are nonnegative integers less than $k+1$. Thus, $3 \cdot (k+1) = 3 \cdot (a+b) = 3a + 3b = 0 + 0 = 0$, therefore $P(k+1)$ is proven.

Base Case: $P(0)\colon 3 \cdot 0 = 0$

Since we have shown the basis step and the inductive step, we have proved for every nonnegative integer n, $P(n) : 3n = 0$.

(b) Proving that every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

Inductive Step:

Assume we can form cent values of j cents for all $3 \leq j \leq k$ using just 3-cent and 4-cent stamps. We wish to show we can form $k+1$ cents using just 3-cent and 4-cent stamps. We can form a $k+1$ cent value by replacing 1 3-cent stamp with 1 4-cent stamp or by replacing 2 4-cent stamps with 3 3-cent stamps.

Base Case:

We can form cent values of 3-cents using one 3-cent stamp and we can form cent values of 4-cents using one 4-cent stamp. This covers our two base cases.

Since we have shown the basis step and the inductive step, we have proved every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

Solution:

(a) This inductive step is not valid for $j = 1$, since the only two nonnegative integers a and b that add up to 1 are 0 and 1, however you can not use $P(1)$ in a proof to prove $P(1)$.

(b) This inductive step is not valid for 5-cent values, since for the 4-cent value composed of 1 4-cent stamp, there are no 3-cent stamps to turn to 4-cent stamps and no 2 4-cent stamps to turn into 3 3-cent stamps.

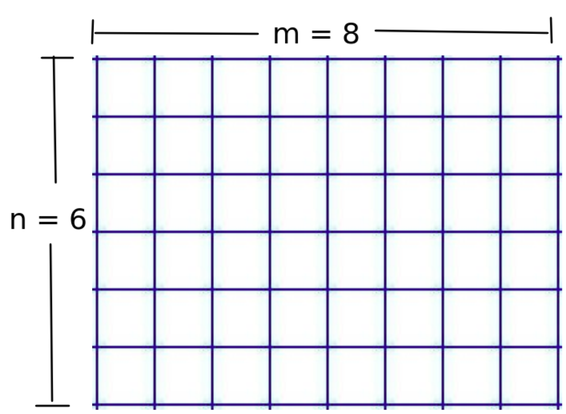
Grading Guidelines [14 points]

Parts a and b:

+3 correct identification of incorrect step
+4 explanation of incorrect step

5. Chopping Ice [15 points]

Claire doesn't have an ice tray, so she makes ice by freezing water into a rectangle and then dividing the rectangle into grid-aligned cells. She would like to divide her block of ice into n rows and m columns quickly, before the ice melts! See the image below for an example.



(a) State the number of cuts Claire needs to make to divide her ice block into $n \times m$ cells. One cut means splitting a single rectangle into two rectangles. In other words, you may NOT make a single cut across multiple pieces of ice. You may use n and/or m in your answer.

(b) Prove your answer from part (a).

Solution:

(a) $nm - 1$

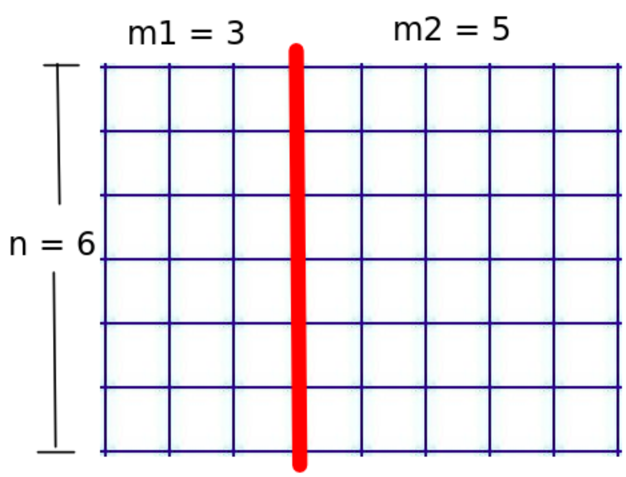
(b) Let $P(A)$ mean "the minimum number of cuts to split a rectangle with area $A = n \times m$ is $nm - 1$." Use strong induction.

Inductive Hypothesis:

Assume for all A' s.t. $1 \leq A' < A$, $P(A')$ is true (meaning for any rectangle of area $A' = n'm'$ for some n' and m', $n'm' - 1$ cuts are the number of cuts necessary to split this block of ice).

Inductive step: We want to show $P(A)$ given the inductive hypothesis.

Suppose $A = n \times m$ for some integers n and m. Now, enumerate the first cut we made. Suppose we made a cut to divide our $n \times m$ rectangle into two rectangles of area A_1 and A_2 rectangle where $1 \leq A_1 < A$, and $1 \leq A_2 < A$. Note that $A_1 + A_2 = A$. Suppose $A_1 = n_1 \times m_1$ and $A_2 = n_2 \times m_2$ for some integers n_1, n_2, m_1, m_2. By our inductive hypothesis, it would take $(n_1 m_1 - 1) + (n_2 m_2 - 1)$ cuts to completely cut up both smaller rectangles. In total, it would take $(n_1 m_1 - 1) + (n_2 m_2 - 1) + 1 = A_1 + A_2 - 2 + 1 = A - 1 = nm - 1$ cuts. Thus $P(A)$ holds.



Base Case:

$P(1)$: 0 cuts are necessary to split a 1 by 1 rectangle since one cube does not need to be split anymore.

Alternate Base Cases:

$P(2)$: 1 cut is necessary to split an area 2 rectangle (1 by 2 or 2 by 1 rectangle) into two 1 by 1 pieces.

Grading Guidelines [15 points]

+3 states $nm - 1$ (or equivalently area $- 1$) in part (a)
+1 states the correct base case: $A = 1$, or equivalently, $m = n = 1$. $A = 2$ (i.e. either $m = 1$, $n = 2$ or $m = 2$, $n = 1$) is acceptable if **both** combinations of m, n are explicitly specified values are correctly included.
+2 correctly proves the base case(s) from the previous item
+2 correct inductive hypothesis
+2 correct "want to show" (can be implicit) for the inductive step
+5 correctly shows $P(A)$ using the inductive hypothesis and algebra

6. Pastry Recurrence [12 points]

A baker decorates a cookie in 2 minutes, a cupcake in 3 minutes, and a pie in 3 minutes. Let a_n denote the number of distinct ways the baker decorates pastries in exactly n minutes for $n \geq 0$ (where order matters).

(a) Find a recurrence relation for a_n.

(b) What are the initial conditions? Use the fewest initial conditions necessary.

Solution:

(a) **Solution: "Walking Backwords"**

If the last pastry decorated was a cookie, then there are a_{n-2} possible ways to decorate. If the last pastry decorated was a cupcake, there are a_{n-3} ways. If the last pastry decorated was a pie, there are a_{n-3} ways. Therefore, our recurrence is $a_n = a_{n-2} + a_{n-3} + a_{n-3} = a_{n-2} + 2a_{n-3}$.

Solution: "Walking Forwards"

If the first decorated pastry was a cookie, then the first 2 minutes is used to decorate a cookie, and there are a_{n-2} ways to use the rest of the minutes. If the first decorated pastry was a cupcake or pie, then there are 2 options for the first 3 minutes, and there are a_{n-3} ways to use the rest of the minutes. This would mean there are $2a_{n-3}$ ways in this case. Therefore, our recurrence is $a_n = a_{n-2} + a_{n-3} + a_{n-3} = a_{n-2} + 2a_{n-3}$.

(b) For 0 minutes, there is one possibility (no baked goods decorated) so $a_0 = 1$. For 1 minute, there are zero possibilities (you can't use exactly 1 minute to decorate) so $a_1 = 0$. For 2 minutes, there is one possibility (a cookie) so $a_2 = 1$.

Grading Guidelines [12 points]

Part a:

+3 correctly handles the case when the last pastry decorated was a cookie
+3 correctly handles the case when the last pastry decorated was a cupcake
+3 correctly handles the case when the last pastry decorated was a pie

Part b:

+1 correct number of initial conditions
+2 correct value of initial conditions

7. Raven's Wrestlers [14 points]

Raven has n weeks to build her wrestling figure collection. Every week, Raven buys one item to add to her collection. There are 4 different types of things she can buy: Figures, T-shirts for her wrestlers to wear, Weapons for them to fight with, or Display Stands to show them off on her shelves.

- Her shelves can fit 2 Stands nicely, so when she buys a Display Stand, she will always buy a second one the next week to finish the shelf. Additionally, the week after buying the second Stand, she will buy something other than a Display Stand (they aren't as exciting to buy)
- When she buys a Figure, she gets very excited about it and wants to buy a new T-shirt for it to wear the following week.

Let a_n represent the number of ways Raven can buy items across the n weeks (where $n \geq 0$)

(a) Find a recurrence relation for a_n.

(b) Which terms would need to be defined with initial conditions (no need to find the value, just which terms)

Note 1: Buying the same items in a different order counts as a different way of buying items. We treat all items in a category as identical.

Note 2: on week n, Raven will not buy a Figure (because she knows she will miss buying a T-shirt) or a Stand (what a sad way to end the collection). This information is not needed for the simplest solutions, but some alternate solutions may need to know this.

Solution:

(a) **Forward solution:**

On week 1, Raven can purchase any of the 4 items.

Case 1: T-shirt

There are no further requirements, so on week 2, she can again buy anything. This means the number of ways to select items to buy for the remaining $n-1$ weeks is a_{n-1}, as you can just ignore week 1.

Case 2: Weapon

Similarly, there are no further requirements, so there are a_{n-1} ways to select the remaining weeks.

Case 3: Figure

In this case, Raven will buy a T-shirt on week 2. After that, though, we are back to being able to buy anything, so there are a_{n-2} ways to select for the remaining $n-2$ weeks.

Case 4: Stand

In this case, Raven will have to buy a stand on week 2, and for week 3, we know she will not buy another stand, so we can't just use a_{n-2}. Instead, we need subcases for the other options:

Case 4a: Stand, Stand, T-shirt. a_{n-3}

Case 4b: Stand, Stand, Weapon. a_{n-3}

Case 4c: Stand, Stand, Figure

In this case, she will buy a T-shirt on week 4, leaving only $n-4$ weeks to decide with no requirements. This means there are a_{n-4} for this subcase.

Putting all this together, we get $a_n = 2a_{n-1} + a_{n-2} + 2a_{n-3} + a_{n-4}$

Backward solution:

On week n, Raven can only purchase T-shirts or Weapons. This means that if we ever find ourselves with more options than just these 2, we will need to branch those off as subcases.

Case 1: Weapon

If Raven buys a Weapon on week n, she still cannot buy a Figure on week $n-1$ (otherwise, she'd have to buy a T-shirt on week n). She could, however, buy a Stand on week $n-1$.

Case 1a: Weapon, T-shirt or Weapon

If we know she bought a T-shirt or Weapon on week $n-1$, that actually means that the first $n-1$ weeks have the same constraints as if there were no week n (because she can't buy Stands or Figures on the last week). This means there are a_{n-1} ways to buy items with a Weapon on week n and either a T-shirt or Weapon on week $n-1$.

Case 1b: Weapon, Stand

If Raven buys a Weapon on week n and a Stand on week $n-1$, then she must have also bought a stand on week $n-2$. Before that, however, we are back to the default of only being able to purchase a Weapon or T-shirt on the last week. That means there are a_{n-3} ways to end with a Weapon on week n and a Stand on week $n-1$.

Case 2: T-shirt

If Raven buys a T-shirt on week n, she could actually buy anything at all the previous week. As with case 1, a recursive term would only deal with the T-shirt or Weapon options, but we will need other subcases for Stand and Figure

Case 2a: T-shirt, T-shirt or Weapon

As with case 1a, this is a perfect setup to use a_{n-1}, as the requirements on week $n-1$ match the original week n.

Case 2b: T-shirt, Stand

As with case 1b, this means Raven must buy a Stand on week $n-2$ as well, and week $n-3$ could not be a third Stand in a row, nor a Figure, so the remaining $n-3$ weeks can be decided by a_{n-3}

Case 2c: T-shirt, Figure

If Raven buys a T-shirt on week n and a Figure on week $n-1$ (which is valid because she did buy a T-shirt the next week), then the options for week $n-2$ are T-shirt, Weapon, or Stand. Obnoxiously enough, once again, a recursive call can only deal with the T-shirt and Weapon options, so we will need sub sub cases:

Case 2ci: T-shirt, Figure, Weapon or T-shirt

Similar to cases 1a and 2a, we have a_{n-2} ways to select what to buy.

Case 2cii: T-shirt, Figure, Stand

Similar to cases 1b and 2b, we know that she must buy a second Stand on week $n-3$, leaving the remaining $n-4$ weeks free to be anything (with the requirement that week $n-4$ be something that can come before a stand, which is only T-shirts or Weapons, just like we wane). This means there are a_{n-4} options here.

Adding all this up, we get exactly the same recurrence as the other solution: $a_n = 2a_{n-1} + a_{n-2} + 2a_{n-3} + a_{n-4}$

(b) Regardless of which solution we use, the furthest back recurrence term is a_{n-4}, so we need 4 initial conditions: a_0, a_1, a_2, a_3.

Grading Guidelines [14 points]

Part a (forward solution):

+3 correctly handles T-shirt case
+3 correctly handles Weapon case

+3 correctly handles Figure case
+3 correctly handles Stand case

Part a (backward solution):

+3 identifies correct cases (Weapon and T-shirt)
+3 identifies correct sub-cases (1a, 1b, 2a, 2b, and 2c)
+3 obtains correct terms from Case 1
+3 obtains correct terms from Case 2

Part b:

+2 correct initial conditions (does not need to provide values)

Groupwork

1. Grade Groupwork 4

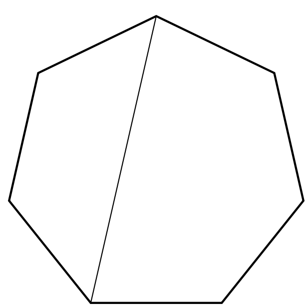
Using the solutions and Grading Guidelines, grade your Groupwork 4:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/20
Problem 3												/30
Total:												/50

2. Polly Gone [12 points]

A convex polygon is a 2D shape with all straight edges such that any line segment between any two non-adjacent vertices passes entirely through its interior (see example picture below). Show via induction that the sum of the interior angles of a convex polygon with n sides is $(n-2) \cdot 180°$. Don't include unneeded base cases.



Hint 1: It is helpful to know that a triangle's interior angles always sum to $180°$. You may assume this is true for the problem.

Hint 2: In order to apply your inductive hypothesis to a convex polygon, you'll need to think of it in terms of smaller convex polygons. How can you make smaller polygons out of a big one?

Solution:

Base case: A triangle's interior angles sum to $(3-2)180° = 180°$.

Inductive step:

We claim that for $k < n$, the sum of the interior angles is $(k-2)180°$. Now, we can note that any convex n-sided polygon can be divided into a triangle and a $n-1$-sided polygon i.e. for an n-sided polygon, we can take any three consecutive vertices and draw a line between the first and the third of those. The sum of the interior angles in the triangle and $n-1$-gon equal the sum the sum of the angles in the n-gon. From our inductive hypothesis, we know the triangle's interior angles sum to $180°$ and the $n-1$-gon's interior angles sum to $((n-1)-2)180°$. Their sum is

$$\begin{aligned} 180° + ((n-1)-2)180° &= 180° + (n-3)180° \\ &= (1 + (n-3))180° \\ &= (n-2)180°. \end{aligned}$$

Therefore, by induction, we've shown that the sum of the interior angles of a convex n-gon is $(n-2)180°$.

Alternatively, we can solve this with strong induction, where we divide the polygon along a line between any two non-consecutive vertices. For the inductive step, we divide the polygon into a $(n-i+1)$-gon and an $(i+1)$-gon for some $2 \leq i \leq n-2$. Since $n-i+1$ and $i+1$ are both less than n, by our inductive hypothesis the sum of the interior angles is

$$\begin{aligned} (n-i+1-2)180° + (i+1-2)180° &= ((n-i+1-2) + (i+1-2))180° \\ &= (n-i+i+1+1-2-2)180° \\ &= (n-2)180°. \end{aligned}$$

Draft Grading Guidelines [12 points]

(i) +4 specifies triangle as the only correct base case
(ii) +4 correct inductive hypothesis
(iii) +4 correctly subdivides polygons based on induction type

3. Running Recurrence [8 points]

An EECS 203 student goes to lecture everyday. On each day, she always chooses exactly one method of transportation, and always chooses to walk, bike, or take a bus. She also follows additional rules:

- She never walks two days in a row.
- If she takes the bus, she must have biked two days ago and walked a day ago.

Let a_n denote the number of ways she can go to EECS 203 lecture across n days for $n \geq 0$.

(a) Find a recurrence relation for a_n.

(b) What are the initial conditions? Use the fewest initial conditions necessary.

Solution:

(a) **Solution: "Walking Backwards"**

We consider the different cases for what method of transportation was chosen on the n-th day.

Case 1 Bike: If the student biked on the n-th day, there are no restrictions on the previous day. Because the last day is set and there are no restrictions on the $(n-1)$-th day, there are a_{n-1} ways for this case.

Case 2 Bus: If the student took a bus on the n-th day, she must have biked on the $(n-2)$-th day and walked on the $(n-1)$-th day. Because the last three days are set, and there are no restrictions for the $(n-3)$-th day, there are a_{n-3} ways for this case.

Case 3 Walk: If the student walked on the n-th day, she must have not walked the day before, and there are two cases for the $(n-1)$-th day.

- If she biked on the $(n-1)$-th day, there are no restrictions for the $(n-2)$-th day, and there are a_{n-2} ways for this case.
- If she took a bus on the $(n-1)$-th day, she must have biked on the $(n-3)$-th day and walked on the $(n-2)$-th day. Because the last 4 days are set, there are a_{n-4} ways for this case.

Therefore, if the student walked on the n-th day, there are $a_{n-2} + a_{n-4}$ ways.

The recurrence is $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$.

(b) There is 1 way to go to lecture 0 times so $a_0 = 1$. There are two ways to go to lecture on the first day (bike, walk) so $a_1 = 2$. There are 3 ways to go to lecture by the second day (bike bike, bike walk, walk bike) so $a_2 = 3$. There are 6 ways to go to lecture by the third day (bike bike bike, bike bike walk, bike walk bus, bike walk bike, walk bike walk, walk bike bike) so $a_3 = 6$.

Draft Grading Guidelines [8 points]

Part a:

(i) +2 correctly handles Case 1
(ii) +2 correctly handles Case 2
(iii) +2 correctly handles Case 3

Part b:

(iv) +1 correct number of initial conditions
(v) +1 correct value of initial conditions