The Master Theorem: If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } (a/b^d) < 1 \\ \Theta(n^d \log n) & \text{if } (a/b^d) = 1 \\ \Theta(n^{\log_b a}) & \text{if } (a/b^d) > 1 \end{cases}$$

better — worse

$\Theta(1),\ \log n,\ n,\ n\log n,\ n^2,\ n^3,\ (\text{maybe } n^4, \dots),\ 2^n,\ n!$

$= n^d \cdot \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$

$T\left(\frac{n}{b^a}\right) = 1 \Rightarrow a = \log_b n$

$a^{\log_b n} = n^{\log_b a}$

Updates cut remaining list in half each time:
j-i≈n, then n/2, then n/4, …
↪ loop iterates $\log n$ times. $\Rightarrow \Theta(\log n)$

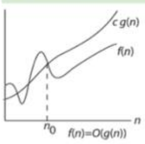
```
procedure bar(n: integer)
    a := (n * n - 7) / 2
    for i := 1 to n
        j := n
        while j > 1
            print "hi"
            j := j / 2
    print "bye"

    for i := 1 to 500n
        print "203 is fun!"
```

$\log n$; n $\Rightarrow \Theta(n\log n)$

$500n \Rightarrow \Theta(n)$

Asymptotic upper bound	Asymptotic lower bound	Asymptotic tight bound
Big-O is like $\le$:	Big-Ω ("Big-Omega") the opposite, like $\ge$.	Big-Θ is like $=$:
$x = O(x)$ $x = O(x^2)$ $x = O(2^x)$ $\log x = O(x)$ $2^x = O(2^x)$ $2^x = O(3^x)$	$f(x) = O(g(x))$ is the same thing as $g(x) = \Omega(f(x))$ $x = \Omega(x)$ $x = \Omega(\log x)$ $2^x = \Omega(2^x)$ $2^x = \Omega(x)$	$f(x) = \Theta(g(x))$ means $f(x) = O(g(x))$ AND $f(x) = \Omega(g(x))$ $2^x = \Theta(2^x)$ $3x^2 + 2 = \Theta(x^2)$



- Let $f, g: \mathbb{R}^+ \to \mathbb{R}^+$ read: f is the big O of
- **Big-O:** "f is $O(g)$"
  1. ▪ Means f grows no faster than as g
     ▪ $\exists k, c$ such that for all $n \ge k$: $f(n) \le cg(n)$
- **Big-Omega** "f is $\Omega(g)$"
  2. ▪ Means f grows as least as fast as g
     ▪ $\exists k, c$ such that for all $n \ge k$: $f(n) \ge cg(n)$
- ***Big-Theta*:** "f is $\Theta(g)$"
  ▪ Means f grows at the same rate as g
  3. ▪ f is $\Theta(g)$ iff $f = O(g)$ and $f = \Omega(g)$
     ▪ $\exists k, c_1, c_2$ such that for all $n > k$: $c_1 g(n) \le f(n) \le c_2 g(n)$

Consider positive-valued functions $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$.

- **Addition** $(f_1 + f_2)(n) = \Theta(\max(g_1(n), g_2(n)))$
- **Scalar multiplication** $af(n) = \Theta(f(n))$
- **Product** $(f_1 \cdot f_2)(n) = \Theta(g_1(n) \cdot g_2(n))$

$f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$ means there exists constants c_1, c_2, k_1 and c_3, c_4, k_2 such that

$$c_1 g_1(n) \le f_1(n) \le c_2 g_1(n) \text{ for all } n \ge k_1$$
$$c_3 g_2(n) \le f_2(n) \le c_4 g_2(n) \text{ for all } n \ge k_2$$

Adding these inequalities gives

$c_1 g_1(n) + c_3 g_2(n) \le (f_1 + f_2)(n) \le c_2 g_1(n) + c_4 g_2(n)$ for all $n \ge \max(k_1, k_2)$

Now $c_2 g_1(n) + c_4 g_2(n) \le (c_2 + c_4)\max(g_1(n) + g_2(n))$

and $c_1 g_1(n) + c_3 g_2(n) \ge c \max(g_1(n), g_2(n))$ for c = min(c_1,c_2)

This gives $(f_1 + f_2)(n) = \Theta(\max(g_1(n), g_2(n)))$

- **Product Rule:** k sequential tasks/stages, with *exactly* n_i ***possible choices*** for task i, means:
  $\prod_{i=1}^{k} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$ possible choices
- **Sum Rule:** k parrallel tasks, with *exactly* n_i ***possible choices*** for task i, means:
  $\sum_{i=1}^{k} n_i = n_1 + n_2 + n_3 + \ldots + n_k$ possible choices
- **Division Rule:** A process with n **total choices**, and each choice represented **exactly k times**, means:
  there are $\frac{n}{k}$ possible choices
- **Difference Rule:** A process with n **total choices**, which has k **extra choices** that shouldn't have counted, means:
  $n-k$ possible choices $|S| = |U| - |\bar{S}|$

$P(n,k) = \frac{n!}{(n-k)!}$

- # ways to select a sequence of k things from a set of size n

Combinations

$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

- # ways to select a set of k things from a set of size n

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

- $0 \le p(E) \le 1$ (3)
- $p(E) + p(\bar{E}) = 1$ (4)
- $p(\bar{E}) = 1 - p(E)$ (5)
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

If S is a sample space of **equally likely** outcomes, the **probability** of an event E is

(1) $p(E) = \frac{|E|}{|S|}$

Events E and F are **independent** if and only if any/all of the following equivalent conditions hold:

$\Pr(E|F) = \Pr(E)$

$\Pr(F|E) = \Pr(F)$

$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$

one is true ⇒ the other twos are true.

If S consists of **unequally likely** outcomes, then

(2) $p(E) = \sum_{s \in E} p(s)$

Suppose that E and F are events from a sample space S both with non-zero probabilities. Then

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}$$

Alternative form of Bayes', with expanded denominator:

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

Denominator is the total probability of E

Two ways to find E(X):

- $E(X) = \sum_{s \in S} p(s) \cdot X(s)$ (weighted sum over outcomes)
- $E(X) = \sum_{r \in range(X)} p(X = r) \cdot r$ (weighted sum over range of X)

Example: Find the expected winnings for the dice game with the *unfair* die

$E(Y) = \sum_{r \in range(Y)} p(Y = r) \cdot r$

$= p(Y = 5)(5) + p(Y = 10)(10) + p(Y = -3)(-3)$

$= (\frac{1}{10})(5) + (\frac{1}{10})(10) + (\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{2})(-3)$

$= \frac{1}{2} + 1 - \frac{24}{10} = -\frac{9}{10}$

The **conditional probability** of event E given event F is

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{|E \cap F|}{|F|}$$

(3) $E(X + Y) = E(X) + E(Y)$

$E(aX + b) = aE(X) + b$ for any constants a, b

- (4) Does linearity require that X and Y be *independent*? NO, linearity *always* works
- (5) Does $E(XY) = E(X)E(Y)$? Only when X and Y are independent iff!!

Geometric Distribution:

The probability of requiring exactly k trials to achieve the first success in a sequence of Bernoulli trials is

$$p(X = k) = (1-p)^{k-1}p$$

And $E(X) = 1/p$

Binomial Distribution:

The probability of exactly k successes in n independent (and identically distributed) Bernoulli trials is

$$p(X = k) = \binom{n}{k} p^k q^{n-k}$$

And $E(X) = np$

- Expected value of a roll of the unfair die from dice game:

S = {1,2,3,4,5,6},
p(1)=p(2)= p(3)=p(4)=p(5) = 1/10 and p(6) = ½
X(1) = 1, X(2) = 2, X(3) = 3, X(4) = 4, X(5) = 5, X(6) = 6

$E(X) = \sum_{s \in S} p(s) \cdot X(s)$

$= p(1)X(1) + p(2)X(2) + p(3)X(3) + p(4)X(4) + p(5)X(5) + p(6)X(6)$

$= (1/10)\cdot 1 + (1/10)\cdot 2 + (1/10)\cdot 3 + (1/10)\cdot 4 + (1/10)\cdot 5 + (1/2)\cdot 6$

$= \frac{15}{10} + 3 = \mathbf{4.5}$

- An *indicator variable* I_F indicates whether an event F happened or not:
  $I_F = \begin{cases} 1 & \text{, if } F \text{ happened} \\ 0 & \text{, if not} \end{cases}$
- Let $p(F) = p$. Then $E(I_F) = 1 \cdot p + 0 \cdot (1-p) = p$ (7)

Recall: a ≡ b (mod m) means a = b + km for some integer k (and assuming m is a positive integer)

Suppose a ≡ b (mod m) and c ≡ d (mod m).

Claim: a+c ≡ b+d (mod m) ***(Addition works!)***

Claim: a–c ≡ b–d (mod m) ***(Subtraction works!)***

Claim: ac ≡ bd (mod m) ***(Multiplication works!)***

Some proofs. Let a = b + km and c = d + jm.

So a+c = b+km+d+jm = (b+d) + (k+j)m ≡ b+d (mod m)

So ac = (b+km)(d+jm) = bd + (bj+dk+kjm)m ≡ bd (mod m)

If p is prime, then for any positive int a<p, there exists a unique positive a^{-1}<p s.t. $aa^{-1} \equiv 1\ (mod\ p)$

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

One-to-one correspondence.
- If $f: A \to B$ is 1-1, then $|A| \le |B|$.
- If $f: A \to B$ is 1-1 and onto, then $|A| = |B|$.

The Schroeder-Bernstein Theorem: If $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$.

(We will use this result w/o proof.)

Definition: A set S is **countable** iff $|S| \le |\mathbb{Z}^+|$
(otherwise it's *uncountable*)

f, g onto $\to g \circ f$ onto

$g, g \circ f$ onto $\not\to$ f onto

K_n : the **complete** graph on n vertices

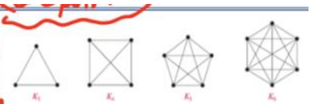
完全图K_n

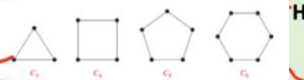
C_n : the **cycle** on n vertices

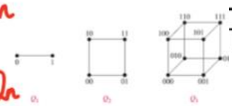
圈图C_n

Q_n : "Hypercubes"

立方体图Q_n





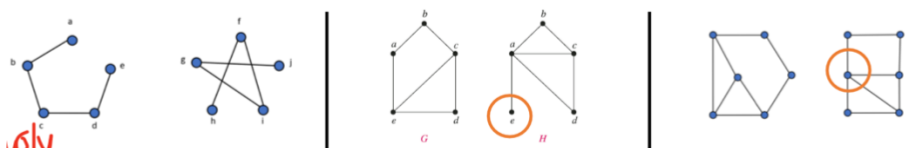


The **degree** of a node v in a graph $G = (V, E)$ is the number of edges that contain v.

Invariant = a property that is preserved under isomorphism (e.g., has a degree 1 node, or has a 5-cycle)

Two graphs G, H are **isomorphic** if we can **relabel** the vertices of G to make it **equal** to H.

Which of these pairs of graphs are isomorphic? How do you know?



Handshake Theorem: In a graph $G = (V, E)$,
$$\sum_{v \in V} \deg(v) = 2|E|$$

Draw a graph with 5 vertices whose degrees are 1,1,2,3,4.

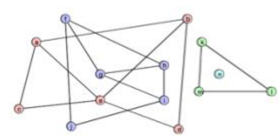
Impossible: Sum of degrees is odd, which violates handshake theorem

Trees

- A **tree** is a graph that is **connected** and that does not have a **cycle subgraph**.
- A **tree** is a graph that has **exactly one** simple path between each pair of vertices.

Paths and Connectivity

- A **path** in a graph is a **contiguous sequence** of nodes and edges.
- A path is **simple** if it does not repeat nodes.
- The **endpoints** of a path are its **first** and **last** node (as an ordered pair).
- Two nodes s, t in a graph are **connected** if there exists a path with endpoints (s, t).
- A graph is **connected** if all pairs of nodes are connected.
- The **connected components** of a graph G are its maximal connected subgraphs.



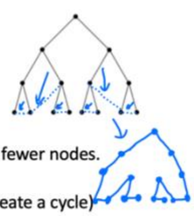
How many connected components does this graph have? **4**

Edges in a Tree

Theorem: Any tree T on n nodes has exactly n-1 edges.

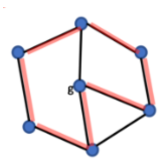
Proof: We will use induction on the number of nodes.
Base Case: n=1. A tree with 1 node has 0 edges.

Inductive Step (Sketch):
Claim: T should have some node v of degree 1 (this called a "leaf")
Hint: If all nodes had degree ≥ 2, there would be a cycle. Why?



Consider the graph T' = T-{v} (i.e. v and incident edge removed). T' has fewer nodes.
1) Is it still connected? Why? (recall v is a leaf)
2) Can it have cycles? Why? (deleting vertices and edges cannot create a cycle)

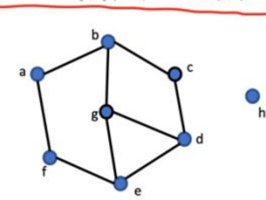
By inductive hypothesis, T' has n-2 edges (as it has n-1 nodes). So T has n-1 edges.



- The **length** of a path p is the number of **edges** in p.
- An (s, t) path p is a **shortest path** if there is no other (s, t) path of smaller length.
- The **distance** between two nodes is the length of **their shortest path(s).**
  - Note: multiple shortest paths all have the same length
  - If no shortest (s, t) path, then $dist(s, t) = \infty$.

A **Hamiltonian Cycle** in a graph G is a path that that contains every **vertex** exactly once, and returns where it started.

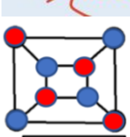
Equivalently: in an n-node graph, a **Hamiltonian Cycle** is a **C_n subgraph**.



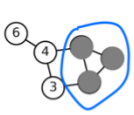
$dist(a, e) = 2$
$dist(b, d) = 2$
$dist(f, c) = 3$
$dist(e, e) = 1$
$dist(h, c) = \infty$
$dist(c, c) = 0$

A graph $G = (V, E)$ is ***bipartite*** if there is a partition $V = A \cup B$ with $E \subseteq A \times B$.

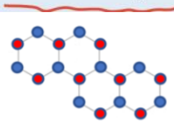
Theorem: For any graph G, the following are equivalent:
- G is **bipartite**: we can partition nodes into sets A, B so that all edges go between A and B
- G is **2-colorable**: we can assign colors "red" and "blue" to its vertices so that no edge has same-color endpoints
- G has **no odd cycles** : for any odd integer k, G does not have C_k as a subgraph



Yes: A = red nodes, B = blue nodes



No (has a C_3 subgraph)

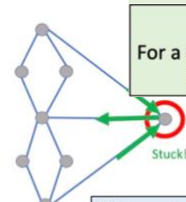


Yes: A = red nodes, B = blue nodes

An **Euler Circuit** is a path in a graph that contains every **edge** exactly once, and that starts and ends at the **same node** node.

Euler's Theorem
For a connected graph G, G has an Euler circuit if and only if all nodes in G have **even degree**.

Euler's Theorem #2
For a connected graph G, G has an Euler path if and only if at most two nodes in G have odd degree (equivalently at least |V|-2 nodes have even degree)



This node has degree 3. Each time an Euler circuit enters the node, it needs to leave it along a new edge.

If G has a node of odd degree, then it does not have an Euler circuit.

Is that the "only reason?" If G has all even-degree nodes, must it have an Euler circuit?