Final Exam Review

Day 1

Topics Covered

- Graphs
- Counting
  - Sum and Product Rule
  - Combinations and Permutations
  - Generalized Combinations and Permutations

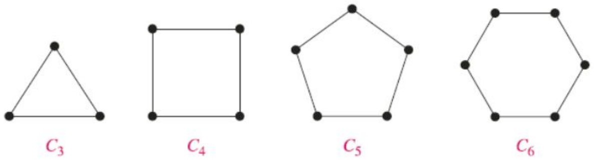
Graphs

Graphs Recap

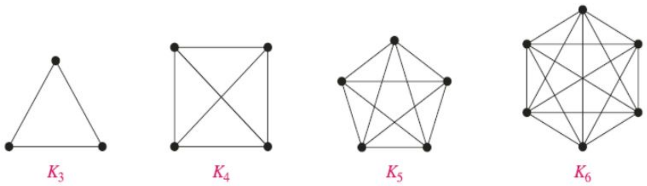
- A graph G = (V, E) is made of
  - V: the set of **vertices** (nodes)
  - E: set of **edges**, where an edge $e = (v_1, v_2)$
- Properties of graphs
  - **Directed/undirected** edges
  - **Weighted/unweighted** edges
- Simple graph: undirected, unweighted, at most 1 edge between two vertices

Special Graphs

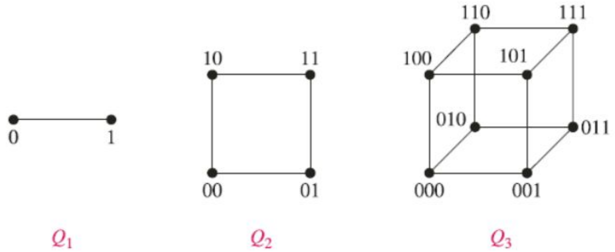
- Cycle $\mathbf{C_n}$ - each vertex has exactly two edges attached to it
- Complete graph $\mathbf{K_n}$ - each pair of vertices connected by an edge
- Hypercube $\mathbf{Q_n}$ - useful to think of it inductively i.e. Q_n is two copies of Q_{n-1} with edges between corresponding vertices



Cycle



Complete Graph



Hypercube

Subgraphs, Cycles, and Trees

- **Subgraph: H = (V_H, E_H)** is a subgraph of G = (V_G, E_G) iff $\mathbf{V_H \subseteq V_G}$ and $\mathbf{E_H \subseteq E_G}$
- **Cyclic Graph:** a graph containing at least one cycle
- **Acyclic Graph:** a graph having no cyclic subgraphs
- **Tree:** a connected, acyclic graph
- **Tree Theorems:**
  - If T = (V, E) and $u,v \in V$, there is a unique simple path from u to v
  - Every tree on n vertices contains n-1 edges

Paths and Connectivity

- **Connected Vertices:** Two vertices *u* and *v* are connected iff there exists a path from *u* to *v*
- **Connected Component:** A nonempty <u>set of vertices</u> in which every pair of vertices in the set is connected.
  - 1 connected component ↔ connected graph

Degrees

- Degree of a vertex deg(v): the number of “edge-ends” incident to it
- Handshake Theorem:
  - An undirected graph has an even number of vertices of odd degree

$$2|E| = \sum_{v \in V} \deg(v)$$

Graphs 1

Prove that an undirected graph with $n \geq 2$ vertices is connected if the degree of each vertex is at least $\frac{n-1}{2}$.

Graphs 1 Solution

Solution: Suppose G is not connected. It must be split into at minimum two components that are not connected to each other at all (c_1, c_2). However, this means that c_1 has a vertex that borders at least $\frac{n-1}{2}$ vertices, and so c_1 must have at least that many vertices plus 1. Same argument arises from examining c_2. This gives a total of $2(\frac{n-1}{2}+1) = n+1$ vertices, a contradiction.

Graphs 2

Suppose G is connected, and each vertex in G is of degree 40. Use the Handshake Theorem to prove that the graph is still connected if we remove any one edge.

Graphs 2 Solution

Solution: Suppose that G is no longer connected. This means we end up with a graph with two vertices of degree 39 (the ends of the removed edge). Since the graph separated because the edge between these two vertices got removed, it must separate the graph into 2 parts with one vertex of degree 39 in each. We may treat each part as an independent graph, in which case it must obey the Handshake Theorem. This means that there are a bunch of vertices of degree 40, and one of degree 39 in this graph, giving an odd total degree, which is a contradiction. G is connected.

Graphs 3

Trees are connected graphs without any cycles. They always have $n - 1$ edges.

Prove that any tree must have a vertex that has degree 1.

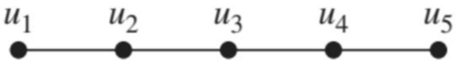
Graphs 3 Solution

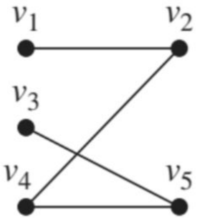
Solution: Suppose no vertex has degree 1. By definition, there cannot be any nodes of degree 0. Thus every vertex has degree at minimum 2. This means that the total degree of the vertices is at least $2n$. This means that there are at minimum n edges in the tree. This contradicts the fact that trees do not have any more than $n-1$ edges. Thus there exists a vertex with degree 1.

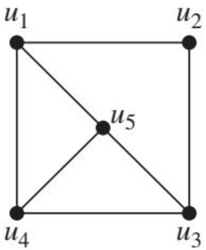
Graph Isomorphism

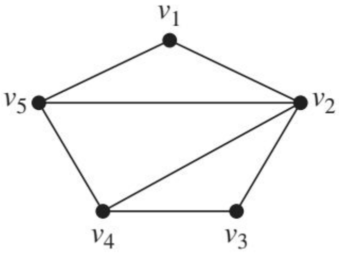
- Graphs G and H are isomorphic if a mapping can be made from G's vertices to make it equal to H
- There exists a bijective mapping $f: V(G) \rightarrow V(H)$ such that if $a, b \in V(G)$ are adjacent, $f(a)$ and $f(b) \in V(H)$ are adjacent
- Proving isomorphism
  - Show a bijective mapping (i.e. $f(a) = b$, $f(x) = y$, …)
- Disproving isomorphism
  - Find a graph invariant that differs between G and H
    - number of vertices
    - number of edges
    - degree sequence
    - existence of subgraphs, path properties
    - cyclic/acyclic
    - having paths of certain length
    - etc.

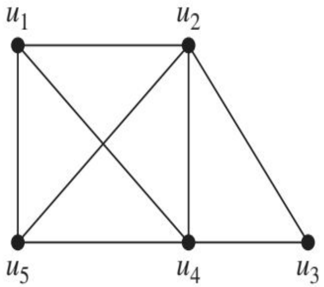
Are They Isomorphic❗❓

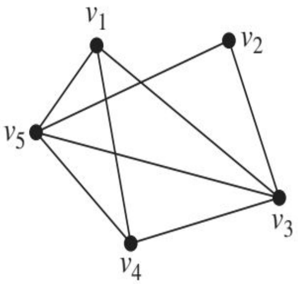




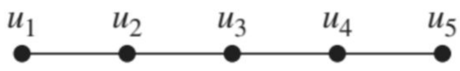


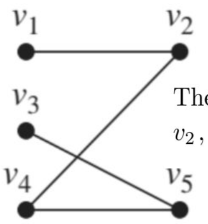




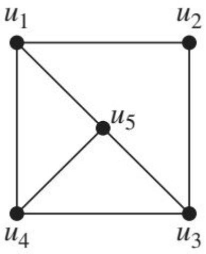


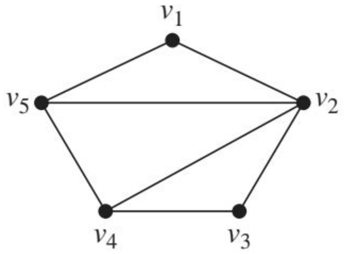
Are They Isomorphic❗❓



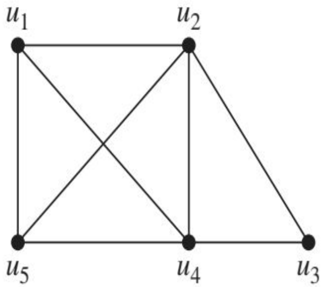


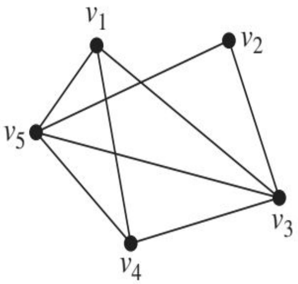
These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$.





These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.





These two graphs are isomorphic. Each consists of a K_4 with a fifth vertex adjacent to two of the vertices in the K_4. Many isomorphisms are possible. One is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

Graphs 4

Let $G = (V, E)$ be an undirected connected graph. We say an edge $e \in E$ is a *bridge* if $G - e$ is disconnected. Show that G is a tree iff every edge in G is a bridge.

Graphs 4 Solution

Solution: First, we'll show that if G is a tree, then every edge in G is a bridge. We proceed via proof by contraposition. Suppose there exists an edge $(u, v) \in E$ that is not a bridge. Specifically, if (u, v) is not a bridge, then there exists a path P from u to v in $G - (u, v)$. $P \cup (u, v)$ is a cycle, and so G is not a tree.

To show the converse, we suppose G is connected and every edge is a bridge. Seeking contradiction, suppose that G is not a tree. Then, G must contain a cycle. If we remove some edge (u, v) in the cycle, we can still find a path from u to v by traversing the other way around the cycle. This means that the graph is still connected. Thus, there exists an edge that is not a bridge, a contradiction.

Eulerian paths, Hamiltonian paths

An **Eulerian path** in a graph is a path that goes through every single *edge* and doesn't double back on itself.

Eulerian circuit: Eulerian path that starts and ends at the same vertex.

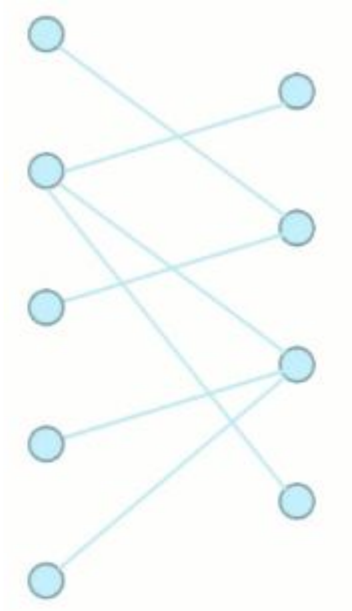
Euler's Theorem is the statement that a Eulerian path exists if and only if the number of vertices with odd degree is at most 2.

A **Hamiltonian path** in a graph is a path between two vertices that visits every *vertex* exactly once

If a Hamiltonian path exists whose endpoints are adjacent, then the resulting graph cycle is called a Hamiltonian cycle.

Bipartite Graphs

- A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2 such that every edge in the graph connects a vertex in V1 and a vertex in V2.
- (V1, V2) is a bipartition of the vertex set V
- A simple graph is bipartite iff it is two-colorable
- A graph is 2-colorable iff all cycles have even length



Graphs 5

Suppose you're a mouse with a 3x3x3 cube of cheese. You eat the block of cheese via the following procedure: pick any of the 27 subcubes to start at; then you're only allowed to eat adjacent subcubes (that is, you can't travel diagonally). You want to eat the entire block of cheese, and can't move to an empty space. Show that you can't eat the middle cube last.

Graphs 5 Solution

We begin with the observation that the block of cheese is two-colorable. WLOG, let’s say we color the middle cube blue. Following this coloring, we have 13 blue vertices; we’ll say the remaining 14 vertices are red. Since the graph is bipartite, any path through it will alternate colors. We want a path that ends with blue and uses all 27 vertices; however, there are fewer blue than red vertices, so it is impossible to find such a path.

Counting

Cheat Sheet !!!

Permutations of n objects, with n_i repetitions of type i:

$$\frac{n!}{n_1!n_2!\ldots n_k!} \; where \sum_{i=1}^{k} n_i = n$$

Ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects go in box i :

$$\frac{n!}{n_1!n_2!\ldots n_k!} \; where \sum_{i=1}^{k} n_i = n$$

Complement Rule: |E| = |S| - |Ē|

- useful when the complement of the event results in a small number of cases
- "at least" is often a signal to use this

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r

$P(n,\ k) =$ *number of ways to choose k things out of n things when order matters, sequence*

$$P(n,\ k) = \frac{n!}{(n-k)!}$$

$C(n,\ k) =$ *number of ways to choose a set of k things out of n things when order doesn't matter*

$$C(n,\ k) = \frac{P(n,\ k)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

Permutation

- n! = $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Permutations
  - P(n,k) = Number of ways to choose *k* things *(order counts!)* out of *n* things
  - P(n,k) = $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \dfrac{n!}{(n-k)!}$

n choices for first thing

n-1 choices for second thing

n-*k*+1 choices for k^{th} thing

Combinations

- C(n,k) = Number of ways to choose a set of *k* things (order doesn't matter) out of *n* things
- C(n,k) = $\dfrac{P(n,k)}{k!} = \dfrac{n!}{(n-k)!\,k!} = \dbinom{n}{k}$ ← read *"n choose k"*

This is P(*k*,*k*)

This is an instance of the **Division Rule**

If each of the effectively different ways of doing things has *d* variations in the total set of ways *n*, then the number of effectively different ways of doing things is *n/d*.

Counting Question 1

How many strings of eight uppercase English letters are there that start or end with the letters AB (in that order), if letters can be repeated?

(a) $26^6 - 26^4$

(b) 26^4

(c) $26^6 + 26^6 - 26^4$

(d) $26^8 + 26^8 - 26^4$

(e) $26^8 - 26^6 - 26^6 + 26^4$

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Solution:

C, there are 26^6 ways to have AB at the beginning and there are 26^6 ways to have AB at the end of the string, but we have overcounted. Thus, we have to subtract off the overlap which is 26^4. In total, there are then $26^6 + 26^6 - 26^4$ strings.

Counting Question 2

1. How many tuples of digits, (x_1, x_2, x_3, x_4), are there such that $0 \leq x_1 < x_2 < x_3 < x_4$?

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Solution:

This is the same as the number of sets of 4 distinct numbers $\{x_1, x_2, x_3, x_4\}$ from between 0 and 9. The number of tuples is $\binom{10}{4}$.

Counting Question 3

How many ways can the letters from the word TREES be ordered such that each anagram starts with a consonant and end with a vowel?

`Scantron 8`

(A) 9

(B) 18

(C) 24

(D) 27

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Scantron 8

(A) 9

(B) 18

(C) 24

(D) 27

Solution:

B, because the last letter must be a vowel, thus we must have an E at the end. There are then 4 letters left to arrange. The first position can have any of the consonants, which is 3 letters, then after that we have 3 letters, 2 letters and 1 letter. Thus, there are 3*3*2*1=18 ways to arrange letters.

Counting Question 4

In how many ways can 2 cats and 4 dogs sit in a circle so that no two cats sit next to each other?

Solution

Solution 1:
We can think about this as first placing the cats and then placing the dogs. There is only 1 way to place the first cat in the circle. Then there are 3 places we could place the second cat (because the second cat cannot be to the immediate right or left of the first). Then there are 4! ways to arrange the dogs in the other spots.
Final answer: $1 \cdot 3 \cdot 4!$

Solution 2:
We can think about placing the dogs and then placing the cats. There are $\frac{4!}{4}$ ways we could arrange the dogs in circle of size 4 (4! ways to arrange the dogs in a straight line, which overcounts the circle rotations by 4). Then we have 4 possible gaps between the dogs for the cats. Each cat needs to be in a different spot, so we have 4 options for the first cat and 3 options for the second.
Final answer: $\frac{4!}{4}(4 \cdot 3)$

Counting Question 5

A team of science bowl contestants consists of 4 members. These members are selected from a group of 16 students, where 4 of these students are majoring in microbiology. If there must be at least 2 student on the team majoring in microbiology, how many different teams can be formed? Please leave the answer unsimplified.

Counting Question 5

A team of science bowl contestants consists of 4 members. These members are selected from a group of 16 students, where 4 of these students are majoring in microbiology. If there must be at least 2 student on the team majoring in microbiology, how many different teams can be formed? Please leave the answer unsimplified.

Solution: $\binom{4}{2}\binom{12}{2} + \binom{4}{3}\binom{12}{1} + \binom{4}{4}$
We can choose either 2, 3, or 4 people majoring in microbiology then choose the rest of the people.

Distinguishable and Indistinguishable problems

- Distinguishable boxes and distinguishable objects → permutation with repetition, either C(n, n1)C(n - n1, n2)... or below

THEOREM 4 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

- Indistinguishable boxes and distinguishable objects→ depends on situations
- Indistinguishable boxes and Indistinguishable objects → cases of number of objects in boxes
- Distinguishable boxes and indistinguishable objects → cases of number of objects in boxes, then number of ways to order those sized groups per case (be careful about overcounting)

Counting Question 6

How many ways are there to distribute seven balls into five boxes if each box must have at least one ball in it if

a) both the balls and boxes are unlabeled?

b) the balls are unlabeled, but the boxes are labeled?

c) the balls are labeled, but the boxes are unlabeled?

d) both the balls and boxes are labeled?

Counting Solution 6

a) Since both the balls and boxes are unlabeled, there are only 2 ways to distribute the balls, i.e. the number of balls in each box is 1,1,1,1,3 or 1,1,1,2,2.

b) Take the cases from a). The boxes are now labelled, so we care which box has each distinct number of balls. In the case with 1,1,1,1,3 balls in each box, there are $C(5,1)$ ways to choose the box that has 3 balls. In the case with 1,1,1,2,2 balls in each box, there are $C(5,2)$ ways to choose the boxes that have 2 balls. So total there are $C(5,1)+C(5,2)=15$ ways.

c) (1) If the number of balls in the boxes are 1,1,1,1,3, we only need to pick the three balls that are in the same box. There are $C(7,3)=35$ ways. (2) If the number of balls in the boxes are 1,1,1,2,2, we need to pick the two groups of two balls. there are $C(7,2)\times C(5,2)/2=105$ ways. In total, there are 140 ways.

d) (1) If the number of balls in the boxes are 1,1,1,1,3, we need to first select the box that has 3 balls and then select the balls within each box. There are $C(5,1)\times C(7,1)\times C(6,1)\times C(5,1)\times C(4,1)=4200$ ways. It will also be equivalent to $C(5,1)\times 7!/3!$. (2) If the number of balls in the boxes are 1,1,1,2,2, we need to pick the two boxes with two balls, then select the balls within each box. There are $C(5,2)\times C(7,1)\times C(6,1)\times C(5,1)\times C(4,2)=12600$ ways. In total, there are 16800 ways.

Counting Question 7

How many unique anagrams are there of the string EECSEECSEECS?

Counting Question 7

How many unique anagrams are there of the string EECSEECSEECS?

Solution:
Using multinomial coefficients, we can count the number of unique anagrams by dividing the number of permutations of all 12 letter strings by the factor over overcounting resulting from permuting Es, Cs and Ss. Specifically, the number of unique anagrams equals 12! / (6! * 3! * 3!)

Have a great rest of the weekend!