EECS 203: Discrete Mathematics
Fall 2023
Homework 10

Due **Tuesday, November 28**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

Reminder: Make sure to leave your answers in combination, permutation, or factorial form and **not** simplified.

1. The Boxer and the Baller [12 points]

How many ways are there to distribute seven balls into five boxes, where each box must have at least one ball in it, if

(a) both the balls and boxes are unlabeled?

(b) the balls are labeled, but the boxes are unlabeled?

(c) both the balls and boxes are labeled?

Solution:

(a) 2

Since both the balls and boxes are unlabeled, there are only 2 ways to distribute the balls, i.e. the number of balls in each box is 1/1/1/1/3 or 1/1/1/2/2.

(b) $\binom{7}{3} + \binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}$

We split into cases based on the number of balls in each box.

- If the number of balls in the boxes are 1/1/1/1/3, we only need to pick the three balls that are in the same box. There are $\binom{7}{3}$ ways.
- If the number of balls in the boxes are 1/1/1/2/2, we need to pick the two groups of two balls. We might try to initially count this as $\binom{7}{2}\binom{5}{2}$, however we overcount by a factor of 2, because deciding to pair balls 1 and 2 then balls 3 and 4 is the same as making the pairings in reverse order. So, there are $\binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}$ ways.

In total, there are $\binom{7}{3} + \binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}$ ways.

(c) $\left[\binom{7}{3} + \binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}\right] \cdot 5!$

If we solve this problem from scratch, we'd end up redoing most of the work in part (c). We can simply multiply our answer from (c) by 5!, accounting for the number of ways to assign labels to the boxes after we place the balls in. So in total there are $\left[\binom{7}{3} + \binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}\right] \cdot 5!$.

Draft Grading Guidelines [12 points]

Part a:

+3 correct answer with justification

Part b:

+2 splits into the cases identified in (a)
+1 obtains $\binom{7}{3}$ for the 1/1/1/1/3 case
+1 obtains $\binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}$ for the 1/1/1/2/2 case
+1 correct answer with justification

Part c:

+2 splits into correct answer or attempts to use answer from (b)
+2 correct answer with justification

2. Sweepstakes Sweep [12 points]

Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

Solution:

The number of ways for the drawing to turn out is $100 \cdot 99 \cdot 98$. The number of ways for the drawing to cause Kumar, Janice, and Pedro each to win a prize is $3 \cdot 2 \cdot 1$ (three ways for one of these to be picked to win first prize, two ways for one of the others to win second prize, one way for the third to win third prize). Therefore, the probability we seek is

$$\frac{3 \cdot 2 \cdot 1}{100 \cdot 99 \cdot 98} = \frac{1}{161700}$$

Alternate Solution:

We define the sample space as all ways of choosing the top 3 winners out of 100 people. Since the top 3 is the same no matter who comes in first, second, or third, it's unordered. So, we can count this with $\binom{100}{3}$.

Now, out of these possibilities, we're only interested in just 1: the event that Kumar, Justice, and Pedro are the top 3.

So, our overall probability is

$$\frac{1}{\binom{100}{3}}$$

Draft Grading Guidelines [12 points]

+3 correct size of sample space
+3 correct size of event
+3 correct final answer
+3 correct justification

3. Mississippi Bananas [8 points]

How many different strings can be made by rearranging the letters in the word BANANANANAS?

Solution:

In the word, BANANANANAS, B occurs once, A occurs 5 times, N occurs 4 times and S occurs once. Hence, we are counting the number of permutations of $n = 11$ letters with $n_B = 1$ occurrences of letter B, $n_A = 5$ occurrences of letter A, $n_N = 4$ occurrences of letter N, and $n_S = 1$ occurrences of letter S. Thus the answer is

$$\frac{n!}{n_B! \cdot n_A! \cdot n_N! \cdot n_S!} = \frac{11!}{1! \cdot 5! \cdot 4! \cdot 1!}$$

Draft Grading Guidelines [**8 points**]

+2 identifies the situation follows a multinomial coefficient problem
+3 correct answer
+3 correct justification

4. Probabili-Tee [16 points]

Tom has 30 T-shirts where 10 are blue, 5 are red, and 15 are green. Frank has 20 T-shirts where 13 are blue, 2 are red, and 5 are green. Both Tom and Frank own 1 green EECS 203 T-shirt, but only Tom owns 1 red and 1 blue EECS 203 T-shirt. Assume Frank and Tom pick and wear T-shirts uniformly at random.

(a) What is the probability that Tom and Frank are both wearing their green EECS 203 T-shirts, given that they're both wearing green T-shirts?

(b) What is the probability that Tom and Frank are both wearing a green T-shirt, given that they're both wearing the same type of T-shirt (both EECS 203 T-shirts or both not EECS 203 T-shirts)?

Solution:

Since Frank and Tom pick T-shirts uniformly at random, their decisions are independent of one another.

(a) Let A be the event that Tom and Frank are both wearing green T-shirts and let B be the event that Tom and Frank are both wearing their green EECS 203 T-shirts. We want $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$. $P(A \cap B) = P(B) = \frac{1}{30} \cdot \frac{1}{20} = \frac{1}{600}$. $P(A) = \frac{15}{30} \cdot \frac{5}{20} = \frac{75}{600}$. Therefore, $\frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{600}}{\frac{75}{600}} = \frac{1}{75}$. Therefore, $P(B \mid A) = \frac{1}{75}$.

Alternate Solution: Using the same definitions for A and B, we could assume $P(A)$ to be true, and then find $P(B)$ under this assumption. Given that Frank and Tom are both wearing green T-shirts, $P(B) = \frac{1}{5} \cdot \frac{1}{15} = \frac{1}{75}$

(b) Let A be the event that Frank and Tom are both wearing EECS 203 T-shirts, and B the event that they are both not wearing EECS 203 T-shirts. Let C be the event that both are wearing green T-shirts. So in notation, we're looking for

$$P(C \mid A \cup B) = \frac{P((A \cup B) \cap C)}{P(A \cup B)}.$$

First, we find $P(A \cup B)$. Since these events are mutually exclusive, we have that

$$P(A \cup B) = P(A) + P(B) = \frac{1}{20} \cdot \frac{3}{30} + \frac{19}{20} \cdot \frac{27}{30} = \frac{516}{600}.$$

Next, we find $P((A \cup B) \cap C)$. By set identities, and because A and B are mutually exclusive, we have

$$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C).$$

The probability that Tom and Frank are both wearing EECS 203 T-shirts and both wearing green T-shirts is

$$P(A \cap C) = \frac{1}{20} \cdot \frac{1}{30} = \frac{1}{600}.$$

The probability that Tom and Frank are both not wearing EECS 203 T-shirts and both wearing green T-shirts is

$$P(B \cap C) = \frac{4}{20} \cdot \frac{14}{30} = \frac{56}{600}.$$

So, we get that

$$P((A \cup B) \cap C) = \frac{1}{20} \cdot \frac{1}{30} + \frac{4}{20} \cdot \frac{14}{30} = \frac{57}{600}.$$

Therefore,

$$P(C \mid (A \cup B)) = \frac{\frac{1}{20} \cdot \frac{1}{30} + \frac{4}{20} \cdot \frac{14}{30}}{\frac{1}{20} \cdot \frac{3}{30} + \frac{19}{20} \cdot \frac{27}{30}} = \frac{\frac{57}{600}}{\frac{516}{600}} = \frac{57}{516}.$$

Draft Grading Guidelines [16 points]

Part a:

+2 correct value for $P(A)$ or $|A|$ or $\frac{1}{15}$ for Tom wearing green EECS 203 shirt given wearing a green T-shirt
+2 correct value for $P(A \cap B)$ or $|A \cap B|$ or $\frac{1}{5}$ for Frank wearing green EECS 203 shirt given wearing a green T-shirt
+3 Correctly combines terms to get 1/75

Part b:

+3 correct value for $P(A)$
+3 correct value for $P(A \cap B)$
+3 Correctly combines terms to get $\frac{57}{516}$ or more simplified

5. Independence Day [10 points]

Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Given that all bitstrings are equally likely to occur, are E and F independent?

Solution:

First, we find $P(E)$. There are four bit strings of length three that have an odd number of 1s: (001, 010, 100, 111). There are $2^3 = 8$ bitstrings in total, since we have 2 choices for each of the 3 spots. So, $P(E) = \frac{4}{8} = \frac{1}{2}$.

Now we find $P(E \mid F)$ and compare it to $P(E)$. Out of the four bit strings that start with a 1, two have an odd number of ones: 100 and 111. So $P(E \mid F) = \frac{1}{2} = P(E)$, and so E and F are independent.

Alternate solution:

We can show that $P(E \cap F) = P(E) \cdot P(F)$. Since there are four bit strings of length 3 that start with 1, $P(F) = \frac{4}{8} = \frac{1}{2}$. As enumerated above, there are 2 bit strings that both start with 1 and have an odd number of ones, so $P(E \cap F) = \frac{2}{8} = \frac{1}{4}$. Since $P(E) = P(F) = \frac{1}{2}$ it is indeed the case that $P(E \cap F) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(E) \cdot P(F)$.

Draft Grading Guidelines [10 points]

+3 correct $P(E)$
+3 (Correct $P(E \mid F)$) or (correct $P(E \cap F)$ and $P(F)$)
+4 correct justification

6. $7 + 5 =$ [12 points]

Suppose we roll five fair **seven-sided** dice (there are seven faces, labeled 1 through 7).

(a) What is the probability that exactly four come up even?

(b) What is the probability that exactly two come up even?

Solution:

(a) The best way to solve this problem is to treat the dice as distinguishable. The sample space is possibililties for all five dice, which is 7^5. The event is that exactly four of the five come up even. We need to choose which four dice come up even, then pick which even number each one comes up as. Finally, we pick which odd number the remaining dice come up as. That means $|E| = \binom{5}{4}3^4 \cdot 4^1$. Then the probability is:

$$\frac{5 \cdot 3^4 \cdot 4}{7^5}$$

Alternate: Suppose we want the probability that the first four come up even and the last one comes up odd. Then the probability is $\left(\frac{3}{7}\right)^4\left(\frac{4}{7}\right)^1$. There are $\binom{5}{4} = 5$ distinct cases for which four come up even, so the total probability is:

$$5 \cdot \left(\frac{3}{7}\right)^4\left(\frac{4}{7}\right)$$

Alternate: We can recognize that this is equivalent to the probability that exactly one die comes up odd, then use either of the two previous methods to get the same answer, possibly in a different order.

(b) We solve this in the same way as the previous part. We have the same sample space, so $|S| = 7^5$. The event is the sequences with two even faces and three odd faces. This gives us $|E| = \binom{5}{2} \cdot 3^2 \cdot 4^3$. Then our final probability is:

$$\frac{\binom{5}{2} \cdot 3^2 \cdot 4^3}{7^5}$$

There are similar alternates to the previous part.

Draft Grading Guidelines [12 points]

For each part, where k is the number that come up even:

+2 7^5 in the denominator
+2 $\binom{5}{k}$ (or equivalent) in numerator
+1 3^k (or equivalent) in the numerator
+1 4^{5-k} (or equivalent) in the numerator

7. Driver's License [20 points]

Suppose we're trying to come up with a new license plate system that must contain exactly 6 characters, each of which can be any of the following: an uppercase letter, lowercase letter, digit, or underscore character. How many possible license plate names are there given the following specifications?

(a) License plates cannot have a number character.

(b) License plates must have exactly one underscore character, which cannot be at the beginning or end of the license plate.

(c) License plates must have at least one number.

(d) License plates must have at least one number or at least one underscore character.

Justify your answer for each part.

Solution:

There are 26 uppercase letters, 26 lowercase letters, 10 number characters, and 1 underscore character.

(a) If license plates cannot have a number, then there are $(26+26+1)^6$ possible license plates.

(b) There are 4 choices for where the underscore must go, and then $(26 + 26 + 10)^5$ choices for the remaining characters, so the final count is $4 \cdot (26 + 26 + 10)^5$.

(c) If license plates must have at least one number, we can use the complement rule to subtract all the license plates with no numbers. There are $(26 + 26 + 10 + 1)^6$ total license plates and $(26 + 26 + 1)^6$ with no numbers, so our final answer is $(26 + 26 + 10 + 1)^6 - (26 + 26 + 1)^6$.

(d) Denote by S the set of all license plates without restrictions, N to be "at least one number" and U to be "at least one underscore character." Similar to part (c), we can use the complement rule here. $|S| - |\overline{N \cup U}| = |S| - |\overline{N} \cap \overline{U}|$ by De Morgan's law. $|\overline{N} \cap \overline{U}|$ is when there is no number and no underscore character, so the total number of ways is $(26 + 26 + 10 + 1)^6 - (26 + 26)^6$.

Alternatively, we can just reason about this in English. License plates that don't have at least one number or at least one underscore character are license plates that have no numbers and have no underscore characters. This means that license plates that violate this specification are those that only contain letters. Thus, we can count the license plates that fit the given specification by subtracting the number of license plates with only letters from the total combinations: $(26+26+10+1)^6-(26+26)^6$.

Draft Grading Guidelines [20 points]

For part a, b and c:

+4 correct answer with justification

For part d:

+8 correct answer with justification

8. Pip Pip Hooray! [10 points]

One pip (small dot on the face of a die) is randomly removed from a standard eight-sided die (where its 8 faces respectively have $\{1, 2, \ldots, 8\}$ pips). **Each pip** has an equal probability of being removed. This means, for example, the face with 8 pips has a greater probability of losing a pip compared to the face with 1 pip.

What is the probability of rolling an even number on this die?

Solution:

There are a total of 36 dots on a standard eight-sided die. We can then split our analysis into two separate cases, when the pip is removed from an even face and when it is removed from an odd face.

Let E be the pip is removed from a face that was originally even and F be an even number is rolled. We are then finding $P(F) = P(F \cap E) + P(F \cap \overline{E})$.

The probability of a pip being removed from an even face is

$$P(E) = \frac{2 + 4 + 6 + 8}{36} = \frac{20}{36}$$

This will then turn that face into an odd face, thus we will have 3 even faces resulting in $P(F \mid E) = \frac{3}{8}$. From here, we get that

$$P(F \cap E) = P(F \mid E)P(E) = \frac{3}{8} \cdot \frac{20}{36}.$$

The probability that the dot is removed from an odd face is

$$P(\overline{E}) = 1 - P(E) = \frac{16}{36}.$$

This will then turn that face into an even face, and thus we will have 5 even faces resulting in $P(F \mid \overline{E}) = \frac{5}{8}$. From here, we get that

$$P(F \cap \overline{E}) = P(F|\overline{E})P(\overline{E}) = \frac{5}{8} \cdot \frac{16}{36}.$$

Thus, we can conclude that

$$P(F) = P(F \cap E) + P(F \cap \overline{E}) = \frac{20}{36} \cdot \frac{3}{8} + \frac{16}{36} \cdot \frac{5}{8} = \frac{35}{72}.$$

Draft Grading Guidelines [10 points]

+2 attempts to apply the law of total probability
+2 correct value for $P(E)$
+2 correct value for $P(F \mid E)$
+2 correct value for $P(F \mid \overline{E})$
+2 correct final answer

Groupwork

1. Grade Groupwork 9

Using the solutions and Grading Guidelines, grade your Groupwork 9:

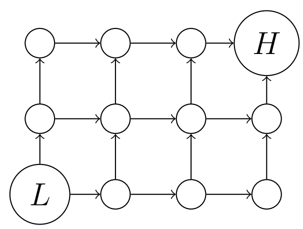
- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/15
Problem 3												/15
Total:												/30

2. Lily's Lily Pads [15 points]

Lily the Frog is on a lily pad and wants to get to her home! She can jump from lily pad to lily pad to help reach this goal. The lily pads are arranged in a grid. Lily starts on the **bottom-left** lily pad, and her home is at the **top-right** lily pad. Lily can only move one lily pad **upward** or one lily pad **rightward** at a time.

Each lily pad has coordinates of the form $(x, y) \in \mathbb{N} \times \mathbb{N}$, where x represents how far rightward a point is from the left of the grid, and y represents how far upward a point is from the bottom of the grid. Lily starts at location $(0, 0)$, and her home is at location $(x_H, y_H) \in \mathbb{N}\times\mathbb{N}$.



In the above example, $(x_H, y_H) = (3, 2)$. In the general case, though, (x_H, y_H) could be any ordered pair of natural numbers.

(a) How many different paths can Lily take to get home?

(b) Lily's frog friend, Francine, is also on the grid at coordinates $(x_F, y_F) \in \mathbb{N} \times \mathbb{N}$ such that $0 \le x_F \le x_H$ and $0 \le y_F \le y_H$. What is the probability that Lily meets Francine on her path home? You may assume that any two paths home are equally likely for Lily to take.

Solution:

(a) There are $\binom{x_H+y_H}{x_H} = \binom{x_H+y_H}{y_H}$ ways. (Either answer is acceptable.)

This is because Lily must make $x_H + y_H$ total moves, and the only difference between each valid sequence of moves is there order that they occur in. Out of the $x_H + y_H$ moves, we can choose x_H of them to be the movements to the right.

(b) The probability is $\frac{\binom{x_F+y_F}{x_F}\binom{x_H-x_F+y_H-y_F}{x_H-x_F}}{\binom{x_H+y_H}{x_H}}$. (Each x in the lower part of each binomial coefficient can be replaced with y, and the answer would still be acceptable.)

This can be found by letting the probability be $\frac{|E|}{|S|}$, where E is the event that Lily meets her friend Francine on her path home, and S is the event that Lily makes it home. There are $\binom{x_F+y_F}{x_F}$ ways for Lily to get to Francine and $\binom{x_H-x_F+y_H-y_F}{x_H-x_F}$ ways for Lily to get from Francine to home, so $|E| = \binom{x_F+y_F}{x_F}\binom{x_H-x_F+y_H-y_F}{x_H-x_F}$. As shown in part (a), $|S| = \binom{x_H+y_H}{x_H}$.

Draft Grading Guidelines [15 points]

(i) +2 Part A: Answer includes x_H and y_H
(ii) +2 Part A: Correct answer
(iii) +3 Part A: Correct justification
(iv) +2 Part B: Answer includes x_H, y_H, x_F, and y_F
(v) +1 Part B: Correct cardinality of event
(vi) +1 Part B: Correct cardinality of sample space
(vii) +1 Part B: Correct answer
(viii) +3 Part B: Correct justification

3. Random Connections [15 points]

We say that a *random graph* is an undirected graph where, for each pair of vertices, there is an independent $\frac{1}{3}$ chance that they are adjacent. It's a bit like Lily's pond, except that the vertices aren't in a grid, and you can move in any direction.

We want to learn about the connectedness of random graphs.

Let G be a finite random graph. Let's split the vertices into two nonempty sets, $A, B \subseteq V$.

(a) Let $a \in A$. What is the probability that no element of B is adjacent to a?

(b) What is the probability that there is some $a \in A$ and $b \in B$ such that a is adjacent to b?

(c) Let's imagine doing this with larger and larger graphs. Define $f(a, b)$ be your answer to the previous problem when $|A| = a$ and $|B| = b$. What is

$$\lim_{a+b\to\infty} f(a, b)?$$

(d) This isn't quite a proof, but your answer to (c) might lead you to some ideas. What might you conjecture about the connectedness of infinite random graphs?

Solution:

(a) Since B is finite, let's say that its elements are $b_1, b_2, \ldots, b_{|B|}$. Then this probability is the same as the probability that a is not adjacent to b_1, and not adjacent to b_2, and so on.

These probabilities are all independent, so we can calculate the overall odds through multiplication, and the probability that a is not adjacent to some vertex is $1-\frac{1}{3}=\frac{2}{3}$. So, the probability is

$$\frac{2}{3}\cdot\frac{2}{3}\cdots=\left(\frac{2}{3}\right)^{|B|}$$

overall.

(b) This is the complement of the probability that no elements of A and B are adjacent.

Like before, since A is finite, let's say its elements are $a_1, a_2, \ldots, a_{|A|}$. Then the complement is the same as the probability that a_1 is not adjacent to anything in B, and a_2 is not adjacent to anything in B, and so on.

These are questions that we answered in the previous question, and the probabilities are independent, so the probability is

$$1-\left(\frac{2}{3}\right)^{|B|}\cdot\left(\frac{2}{3}\right)^{|B|}\cdots=1-\left(\left(\frac{2}{3}\right)^{|B|}\right)^{|A|}=1-\left(\frac{2}{3}\right)^{|A|\cdot|B|}$$

overall.

(c) Since A and B are nonempty, we restrict $a, b > 0$. Then $\lim_{a+b\to\infty} a\cdot b=\infty$, so

$$\lim_{a+b\to\infty}\left(1-\left(\frac{2}{3}\right)^{a\cdot b}\right)=1-\lim_{a\cdot b\to\infty}\left(\frac{2}{3}\right)^{a\cdot b}=1-0=1$$

(d) There is no "right" conjecture, and we haven't proven this, but (c) may lead us to guess that infinite random graphs are usually connected.

This turns out to be true. Infinite random graphs are (almost always) connected.

In fact, it turns out that there's (basically) only one infinite random graph! It (almost) doesn't matter which particular vertices end up adjacent, and it even doesn't matter if we change $\frac{1}{3}$ to any other probability in $(0,1)$; it's possible (just about all the time) to build a graph isomorphism.

The caveat is that these are true *with probability 1.* There are counterexamples – for example, we might get super unlucky, and the graph ends up without any edges – but the probability of failure is incredibly low. It's like randomly choosing a real number and ending up with an integer.

(i) +2 part a: considers the probabilities of a being adjacent to some elements of B

(ii) +3 part a: correct answer
(iii) +2 part b: attempts to find the probability via the complement
(iv) +3 part b: correct answer
(v) +2 part c: correct answer
(vi) +3 part d: some reasonable conjecture about the probability of adjacency, connectedness, etc