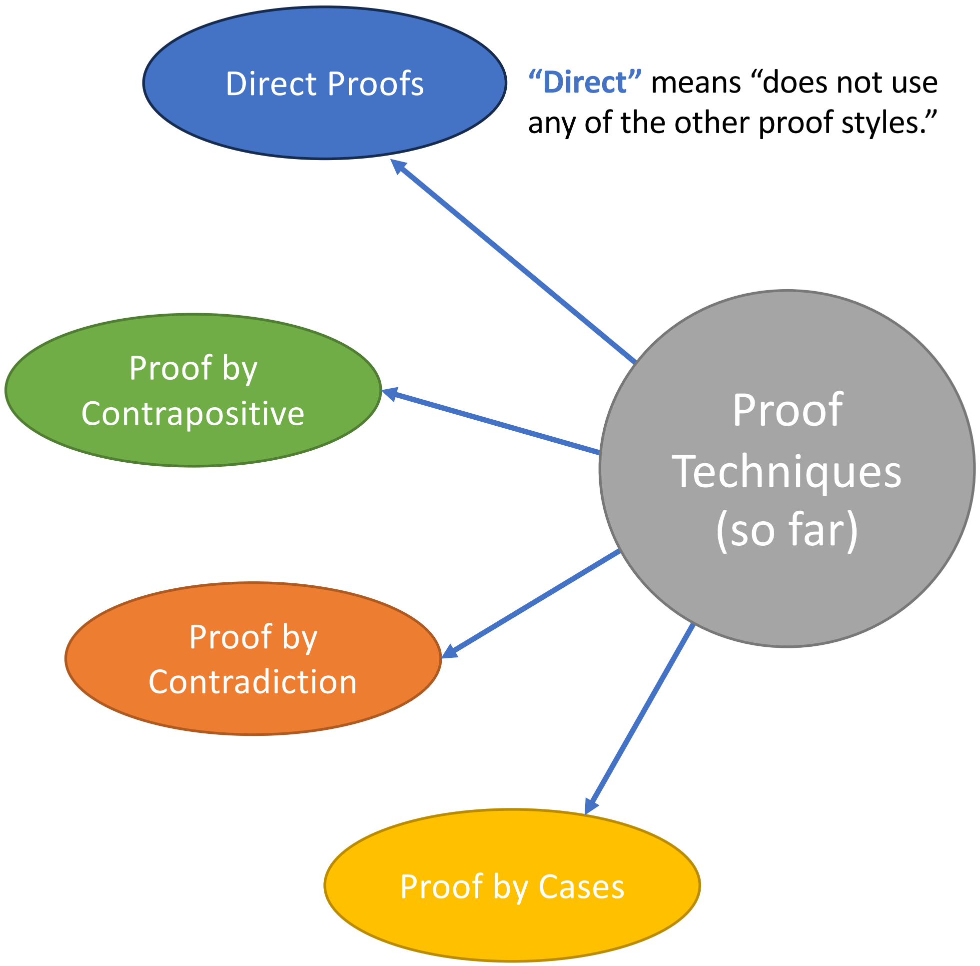
Proof Templates

EECS 203

Properties of a proof

Properties

- Concise (not unnecessarily long)
- Clear (not ambiguous)
- Complete (no missing intermediate steps)
- Logical (every statement logically follows)
- Rigorous (uses mathematical expressions)
- Convincing (does not raise questions)
- The way a proof is presented might be different from the way the proof is discovered.



Formatting your proofs

- **List format (RECOMMENDED):**
  - List each step of your proof on a new line (often with bullets).
  - Can be helpful for you in organizing your ideas
  - Helpful for the reader in understanding your ideas
- **Paragraph format:** write out the steps of the proof as a paragraph(s). Often takes up less space, but can be harder to follow. Textbooks often use paragraph format.

Template: Proving a “For all” statement

Claim: For all x [in the domain], $P(x)$

Proof Template

Let x be an ***arbitrary*** element of the domain (e.g., integer, student, etc).

… (make some deductions, probably involving the arbitrary x) …

Thus, $P(x)$.

Therefore, $P(x)$ holds for all x in the domain.

Template: Proving an "Implies" Statement (direct proof)

Claim: If p, then q

Proof Template

Assume p.

… (make some deductions) …

Therefore, q.

This pattern often happens **inside a "for all" statement**

How To Prove an "Exists" statement

Claim: There exists an x [in the domain] such that $P(x)$

Common Approach

Consider $x =$ ________ [give a specific element of the domain]

… show that $P(x)$ holds for that value of x.

- No need to classify all elements of the domain that satisfy the claim, just name ONE valid choice.

Template: Disproofs

Claim: There exists x [in the domain] such that $P(x)$

Common Approach

We will prove the negation:

"For all x [in the domain], $\neg P(x)$"

Let x be arbitrary

[...deductions...]

Therefore, $\neg P(x)$.

Claim: For all x [in the domain], we have $P(x)$

Common Approach

We will prove the negation:

"There exists x [in the domain] such that $\neg P(x)$"

Consider $x =$ _____ [called a **counterexample**]

[...deductions...]

Therefore, $\neg P(x)$.

Template: Proof by Contrapositive

Claim: If p, then q

Proof Template

We will prove the contrapositive: [state the contrapositive]
Assume not(q).

… (make some deductions) …

Therefore, not(p).

Template: Proof by Contradiction

Special case: when the claim is an "if-then" statement

Claim: $a \to b$

Remember: the negation of $a \to b$ is
a and $\neg b$

Claim: p

Proof Template

Seeking a contradiction, assume: [state the negation of p]

… (make some deductions, eventually leading to a contradiction) …

Common contradictions: a number is an **integer and is not an integer**; a number is both **even and odd**; a number is both **rational and irrational**.

Since *[restate contradictory statements]*, we have a contradiction.
Assuming $\neg p$ led to a contradiction. Therefore, p must be true. (optional concluding sentence)

Proof by cases (at top level)

Given: p_1 or p_2 or …

Claim: q

Often this isn't explicitly given, but rather something we know (e.g., a number is either even or odd; positive, negative, or zero; etc.)

Proof

- Proof by cases:
  - **case 1:** Assume p_1.
  - *…(deductions)…*
  - q.

  - **case 2:** Assume p_2.
  - *…(deductions)…*
  - q

  - …
- Thus, q.

Template: Proof by Cases (within a “for all”)

Claim: for all x P(x)

Proof Template

- Let x be an arbitrary element in the domain. Suppose x can fall into n cases.

Case 1: Assume that x is ____
… (make some deductions) …
P(x) is true

.
.
.

Case n: Assume that x is ____
… (make some deductions) …
P(x) is true

- Since P(x) holds in all the cases, P(x) is true

Therefore P(x) holds for all elements in the domain

Optional: “Without Loss of Generality”

In proofs, you can use **“Assume without loss of generality…”** or **“Assume WLOG…”** when:

- There are several possibilities about the state of the world
  - E.g. (x is even and y is odd) or (x is odd and y is even)
- But these possibilities are completely **symmetric**, and the proof would look essentially the same under one possibility as the other.
  - E.g. x, y are both arbitrary integers and we have assumed nothing else about them
  - So we might as well say x is the even one and y is the odd one.
- So we can write “**Assume WLOG** that [one of the two possibilities holds].”

Be careful with WLOG! Don’t assume things unless you are sure that there really is symmetry.

You will never **have** to use WLOG – it’s just a time-saving tool. The alternative is to consider each possibility separately, and repeat the proof in each case.
(We’ll talk more about this alternative next week.)