use proof by cases inside a for-all statement or at top level
use proof by cases when one or more cases are impossible
decide which proof style to use, when several might work

9/21/23

Lec 8 Handout – Proof by Cases – ANSWERS

Prove:
For all integers x, x^2+1 is not divisible by 3.

Proof:

- Let x be an **arbitrary** integer.

Case 1: **Assume** that there is an integer k with $x = 3k$.

- So $\mathbf{x^2+1} = (3k)^2+1$
  $= 9k^2+1$
  $= \mathbf{3(3k^2)+1.}$
- Since k is an integer, $3k^2$ is an integer.
- So **remainder** of x^2+1 divided by 3 **is 1**
- So x^2+1 is not divisible by 3. (by axiom)

Case 2: **Assume** that there is an integer k with $x = 3k+1$.

- So $\mathbf{x^2+1} = (3k+1)^2+1$
  $= 9k^2+6k+2$
  $= \mathbf{3(3k^2+2k)+2.}$
- Since k is an integer, $3k^2+2k$ is an integer.
- So **remainder** of x^2+1 divided by 3 **is 2**
- So x^2+1 is not divisible by 3.

Case 3: **Assume** that there is an integer k with $x = 3k+2$

- So $\mathbf{x^2+1} = (3k+2)^2+2$
  $= 9k^2+12k+6$
  $= \mathbf{3(3k^2+4k)+2}$
- Since k is an integer, $3k^2+4k$ is an integer.
- So **remainder** of x^2+1 divided by 3 is **2**
- So x^2+1 is not divisible by 3.

2

Template: Proof by Cases (within a "for all")

1

Claim: for all x P(x)

Set of cases must cover the **entire domain**

Proof Template

- Let x be an arbitrary element in the domain. Suppose x can fall into n cases.

3

Case 1: Assume that x is ____
… (make some deductions) …
P(x) is true

.
.

Case n: Assume that x is ____
… (make some deductions) …
P(x) is true

2

Each case must lead to the **same conclusion**

- Since P(x) holds in all the cases, P(x) is true

Therefore P(x) holds for all elements in the domain

Why does proof by cases work?
For any a, b, p, we have: $((a \vee b) \vee (a \to p) \vee (b \to p)) \to p$

Even Squares, Revisited

Proposition: For all integers x, if x^2 is even, then x is even.

Proof:

- Let x be an **arbitrary** integer.
- **Assume** that x^2 is even

Case 1: Assume that x is even

- So x is even.

Case 2: Assume that x is odd

- So there is an integer k with $x = 2k+1$
- So $\mathbf{x^2} = (2k+1)^2$
  $= 4k^2+4k+1$
  $= \mathbf{2(2k^2+2k)+1}$
- Since k is an integer, $2k^2+2k$ is an integer
- So x^2 is odd.
- But x^2 is even (from line 2), so we have a contradiction.
- So this case can't happen.

What do we conclude about Case 2? It doesn't happen (since we proved that x^2 is both even and odd)

WLOG Woes

Prove: For all integers x, y, if $x+y$ is even, then $x, 3y$ have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be **arbitrary** integers
- **Assume** that $x+y$ is even
- So there is an integer k with $x+y = 2k$
- So $\mathbf{y = 2k - x}$
- Now consider two cases, on whether x is even or odd:

Case 1: Assume that x is even

- So there is an integer j with $x = 2j$
- So $\mathbf{3y} = 3(2k-x)$
  $= 3(2k-2j)$
  $= 6k-6j$
  $= \mathbf{2(3k-3j)}$
- Since j, k are integers, $3k-3j$ is an integer
- So $3y$ is even, and so $x, 3y$ have the same parity

Case 2: Assume that x is odd

- So there is an integer $2j$ with $x = 2j+1$
- So $\mathbf{3y} = 3(2k-x)$
  $= 3(2k-(2j+1))$
  $= 6k-6j-3$
  $= \mathbf{2(3k-3j-2)+1}$
- Since j, k are integers, $3k-3j-2$ is an integer
- So $3y$ is odd, and so $x, 3y$ have the same parity

(optional concluding sentence)
In either case, we proved that $x, 3y$ have the same parity

What if:

Case 1: $(\sqrt{2})^{\sqrt{2}}$ is rational $\Rightarrow x=\sqrt{2}, y=\sqrt{2}$, x^y is rational

Case 2: $(\sqrt{2})^{\sqrt{2}}$ is irrational $\Rightarrow ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = 2$ is rational

$\therefore$ Proved

Rational Powers

Prove:
There exist irrational numbers x, y such that x^y is rational.

Proof Attempt 1:

- **Consider** $x = \sqrt{2}$ and $y = \sqrt{2}$.
- So $x^y = \sqrt{2}^{\sqrt{2}}$.

Do we know if x is irrational? Yes (by lemma)

Do we know if y is irrational? Yes (by lemma)

Do we know if x^y is rational? No

Proof Attempt 2:

- **Consider** $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$.
- So $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$
$= (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}}$
$= (\sqrt{2})^2$
$= 2.$

Do we know if x is irrational? No

Do we know if y is irrational? Yes (by lemma)

Do we know if x^y is rational? Yes $(2 = 2/1)$

10

Additional Arguments

Prove: For all integers x, y, if $3x^2y^2$ is odd, then x, y are both odd.

I. Prove it by contrapositive

Proof 1:

- Let x, y be **arbitrary** integers
- **We will prove the contrapositive:**
"If x is even or y is even, then $3x^2y^2$ is even."
- **Assume** that x is even or y is even.
- **Assume WLOG** that x is even (note: x, y are symmetric)
- So there exists an integer k with $x = 2k$
- So we have $3x^2y^2 = 3(2k)^2y^2$
$= 12k^2y^2$
$= 2(6k^2y^2)$
- Since k, y are integers, $6k^2y^2$ is an integer
- So $3x^2y^2$ is even.

Additional Arguments

Prove: For all integers x, y, if $3x^2y^2$ is odd, then x, y are both odd.

II. Disprove its negation by contradiction

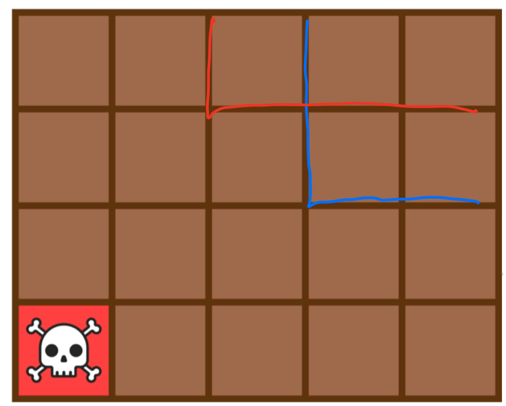
Proof 2:

- **Seeking a contradiction, assume the negation:**
"There exist integers x, y for which $3x^2y^2$ is odd and (x is even or y is even)."
- **Assume** WLOG that x is even.
- So there exists an integer k for which $x = 2k$
- So we have $3x^2y^2 = 3(2k)^2y^2$
$= 12k^2y^2$
$= 2(6k^2y^2)$
- So $3x^2y^2$ is even
- This completes the contradiction: we have proved that $3x^2y^2$ is both even and odd.

The algebra in the middle of this proof is very similar to the previous proof!

This is not an accident: it's the **same argument**, phrased in two **different proof styles**.

Play Chomp



In the game, the first player has a winning strategy

If 1x1 chomp is a winning first move, then the first player has a winning strategy.

If not, then the first player can steal the second player's winning response strategy. (Starting with an $a \times b$ chomp)