Outline
- more Weak Induction
  - divisibility
  - Towers of Hanoi
- Strong Induction
  - stamps
  - piles of stones
- Questionable Inductive Proof: Horses

(iff: if and only if)

※ $a|b$: a divides b
- $a|b$ iff b is a multiple of a
- $a|b$ iff b is divisible by a
- $a|b$ iff $am=b$ for some $m \in \mathbb{Z}$

Example: $3|(5^3-5)$ because $5^3-5=120=3(40)$

9/27/23

Lec 10 Handout: Strong Induction - ANSWERS

- How are you feeling about induction overall?
  - Answers will vary
- Which proof from last lecture did you understand the most? The least?
  - Answers will vary
- Write down a few questions you have about induction.
  - Answers will vary

※ An introduction to well-ordering: A set A with ordering "<" is well-ordered iff every non-empty subset $S \subseteq A$ has a least element. ($a, b \in S$, a is "less than" b iff $a<b$)

Z^+ is well-ordered

(So far, we have induction over N or Z^+ but can we do induction over R^+? No, no "next step")

$R^+ \cup \{0\}$ is not well-ordered under "<". There are some subsets that do not have a least element. eg: $(0,1]$, every x has $\frac{x}{2}$ which is smaller

Divisibility Example

Prove $\forall n \in \mathbb{N}\ \ 3 \mid n^3 - n$

Claim: $\forall n \in \mathbb{N}\ \ 3|n^3-n$

Base Case: $n=0$: $3 \mid 0^3-0$ because $0^3-0=3m$ for $m=0$, so $P(0)$ holds.

Inductive Step:
- Consider an arbitrary integer k, with $k \geq 0$.
- Assume $3|k^3-k$
- Want to show $3|(k+1)^3-(k+1)$

From our inductive hypothesis, $k^3-k=3a$ for some integer a.

We want to show that $(k+1)^3-(k+1)=3b$ for some integer b.

$$\begin{aligned}(k+1)^3-(k+1) &= k^3+3k^2+3k+1-k-1\\ &= k^3-k+3k^2+3k\\ &= 3a+3k^2+3k \quad \text{by IH}\\ &= 3(a+k^2+k)\end{aligned}$$

Since a and k are integers, $a+k^2+k$ is an integer.

So $(k+1)^3-(k+1)$ is 3 times an integer, and thus $3|(k+1)^3-(k+1)$

Therefore, by induction, $3|n^3-n$ for all $n \in \mathbb{N}$.

6

An Inductive Logic Puzzle

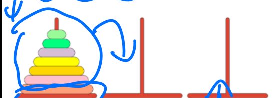
- **For all $n \in \mathbb{N}$, $P(n)$:** The Tower of Hanoi puzzle has a solution with n discs initially on peg 1.
- **Proof: Base Case : P(_ 0 _)**

When n = 0, start configuration = end configuration

Inductive Step:

Assume P(k-1): the puzzle has a solution for $k-1$ discs.

Want to show P(k): the puzzle has a solution for k discs.



then repeat..

1. Use **inductive hypothesis** to move top k-1 disks over one.
2. Move biggest disc to peg 3.
3. Use **inductive hypothesis** to move top k-1 disks to peg 3.

∴ If you can solve 0, you can solve 1, 2, 3, ⋮, n discs

$[P(1) \wedge P(2) \wedge \dots \wedge P(k)] \rightarrow P(k+1)$

Strong Induction as Dominos

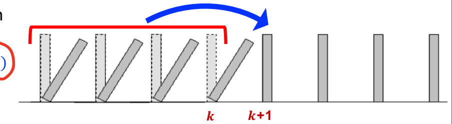
Let $P(n)$ be a predicate.

Goal: Prove that $P(n)$ is true for all $n \in Z^+$

Step 1: [Strong] Inductive Step

If you can knock down *all the previous* dominos, then you can knock down the k+1st one.

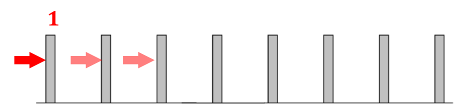
$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$



Step 2: Base Case(s)

You can knock down the first domino(s) $P(1)$

Possibly also: $P(2)$ and $P(3)$ and more



Therefore, you can knock down all dominos.

10

	Weak induction	.VS	Strong Induction
Basic Step	$P(1)$	Same	$P(1)$
Inductive Step	$\forall k \in Z^+ [P(k) \rightarrow P(k+1)]$		$\forall k \in Z^+ [P(1) \wedge \dots \wedge P(k) \rightarrow P(k+1)]$
Conclusion	$\forall n \in Z^+ P(n)$	Same	$\forall n \in Z^+ P(n)$

(1) Strong Induction 有更充分的条件

(2) 能weak Induction的一定能strong Induction, 反之不然

Stamps Proof

P(n): "n cents can be made using 3- and 5-cent stamps"
Claim: P(n) for all $n \geq 8$

(Strong) Inductive step.

- Let k be an integer with $k \geq 11$
- Assume $P(j)$ is true for all $8 \leq j \leq k-1$.
  - $P(j)$: "j cents can be made using …"
- Want to show: $P(k)$
  - $P(k)$: "k cents can be made using …"

Base cases:
P(8): $8 = 3 + 5$
P(9): $9 = 3 + 3 + 3$
P(10): $10 = 5 + 5$

- $P(k-3)$ is true, by IH (because $8 \leq k-3 \leq k-1$)
  - i.e., we can make $k-3$ cents
- To make k cents: first make $k-3$ cents, then add an additional 3-cent stamp.
- Therefore, $P(k)$ is true.

12

Guide for Strong Induction Proofs

1. Restate the claim you are trying to prove

Equivalently: Show
$[P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k})] \rightarrow P(\boldsymbol{k+1})$

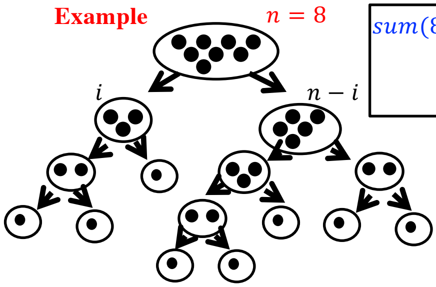
2. **Inductive Step**: Prove that for an arbitrary integer k in the desired range,
$$[P(\boldsymbol{n_0}) \wedge P(n_0+1) \wedge \cdots \wedge P(\boldsymbol{k-1})] \rightarrow P(\boldsymbol{k})$$
   - (1) Let k be an arbitrary integer with $k \geq$ ____ (value depends on the proof)
   - (2) Assume $P(j)$ is true for all $\boldsymbol{n_0} \leq j \leq \boldsymbol{k-1}$ ☆
   - (3) Show that $P(\boldsymbol{k})$ holds
3. **Base case(s)**: Prove the claim holds for the "first" **value(s)** of n
   - (1) Prove $P(n_0)$ is true
   - * (2) May also need to prove $P(n_0+1)$ and more, depending on the inductive step
4. Conclusion: explain that you've proven the desired claim.

Piles of stones

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones, $sum(n) = \frac{n(n-1)}{2}$

- Suppose you begin with a pile of n stones.
  - Split the pile into two smaller piles of size i and n-i
  - Multiply together the number of stones in each of the two smaller piles, $i(n-i)$, and add that product to your running sum
  - Repeat until you get n piles of one stone each.
- **What sum do you get for $n = 8$?** $sum = 28$

Example $n = 8$



$sum(8) = 15 + 2 + 6 + 1 + 2 + 1 + 1$
$= 28$
$= \frac{8(8-1)}{2}$

$(3 \cdot 5)$
$+(2 \cdot 1) + (3 \cdot 2)$
$+(1 \cdot 1) + (2 \cdot 1) + (1 \cdot 1)$
$+(1 \cdot 1)$

16

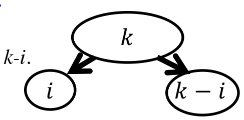
Proof.. Piles of stones

Handout

Claim: $\forall n \geq 1\ P(n)$.
$P(n)$: with n stones, $sum(n) = \frac{n(n-1)}{2}$

Inductive step: Assume $P(j)\ \forall j\ \boxed{1 \leq j \leq n-1}$

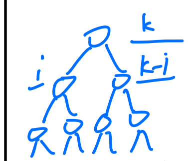
- Want to show $P(k)$, i.e., $sum(k) = \frac{k(k-1)}{2}$.
  - Take a pile of size k.
  - Divide the pile into two smaller piles of size i and k-i.



$$sum(k) = i(k-i) + sum(i) + sum(k-i)$$
$$= \frac{2}{2}\, i(k-i) + \frac{i(i-1)}{2} + \frac{(k-i)(k-i-1)}{2} \quad \text{(Inductive Hypothesis)}$$
$$= \frac{1}{2}\,[\,2i(k-i) + i(i-1) + (k-i)(k-i-1)\,]$$
$$= \frac{1}{2}\,[\,2ik - 2i^2 + i^2 - i + k^2 - ki - k - ik + i^2 + i\,]$$
$$= \frac{1}{2}\,[\,i^2(\underbrace{\cancel{-2+1}+1}_{0}) + k^2 + ik(\underbrace{\cancel{2-1-1}}_{0}) \cancel{-i} + i - k\,]$$
$$= \frac{1}{2}\,[k^2 - k]$$
$$= \frac{k(k-1)}{2}$$

Base case: $n =$ 1
- $sum(n) = sum(1) =$ 0 ✓
- $\frac{n(n-1)}{2} = \frac{0(0-1)}{2} = 0$ ✓

By strong induction, $P(n)$ holds $\forall n \geq 1$.



* sum(k) = sum(i) + sum(k-i) + i(k-i)
and that's enough
(sum(k-i) = sum(i) + sum(k-2i) + i(k-2i))
......
Recursion