FA23

EECS 203 Discussion 8a

Functions and Pigeonhole Principle

Admin Notes:

Homework:

- HW 7
    - Homework/Groupwork 7 due **Thursday, October 26th**
    - Weekly Check-in 7 due **Thursday, October 26th**
- HW 8
    - Homework/Groupwork 8 due **Thursday, November 2nd**
    - Weekly Check-in 8 due **Thursday, November 2nd**

Functions

Onto and One-to-One Review

- **Function *f: A* → *B*:** associates each element of set A to <u>exactly one</u> element in set B
  - **Domain:** ***A***
  - **Codomain:** ***B***
- **Onto Function *f: A* → *B*:** all elements in B are mapped to by f

$$\forall b \in B \; \exists a \in A \; [\, f(a) = b \,]$$

- **One-to-One Function *f: A* → *B*:** no two elements of A map to the same output in B

$$\forall a,b \in A \; [\, f(a) = f(b) \rightarrow a = b \,]$$

- **Bijective Function:** onto and one-to-one (also called a one-to-one correspondence)

Function Operations Review

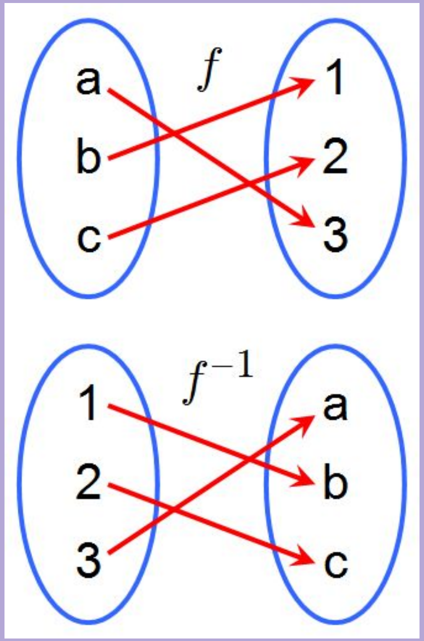
- **Function Inverse f^{-1}:** f must be a <u>bijection</u>

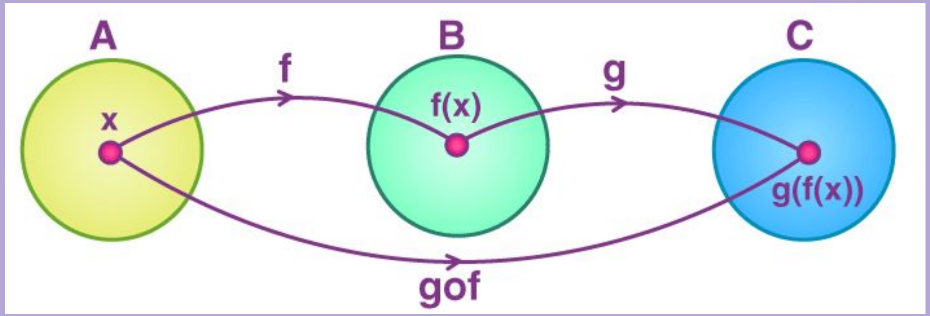
  $f^{-1}(b) = a$ **if and only if** $f(a) = b$

- **Function Composition $f \circ g$:**

  $(f \circ g)(a) = f(g(a))$

- **Adding and Multiplying Functions:**
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
  - $(f_1 f_2)(x) = f_1(x)\, f_2(x)$





Problem

1. Composition and Inverses ⋆

Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Solution

Solution:
Reminder of the definition of f and f^{-1} being inverses: $f^{-1}(b) = a$ if and only if $f(a) = b$. So our goal is to show this property for the pair of functions $f \circ g$ and $g^{-1} \circ f^{-1}$.

Part 1: Proof that $(f \circ g)(a) = b \to (g^{-1} \circ f^{-1})(b) = a$. Let $a \in X, b \in Z$ be arbitrary elements, and assume that $(f \circ g)(a) = b$. Then:

$$\begin{aligned} f(g(a)) &= b \\ f^{-1}(f(g(a))) &= f^{-1}(b) \\ g(a) &= f^{-1}(b) \\ g^{-1}(g(a)) &= g^{-1}(f^{-1}(b)) \\ a &= (g^{-1} \circ f^{-1})(b) \\ (g^{-1} \circ f^{-1})(b) &= a \end{aligned}$$

Part 2: Proof that $(g^{-1} \circ f^{-1})(b) = a \implies (f \circ g)(a) = b$. Let $a \in X, b \in Z$ be arbitrary elements, and assume that $(g^{-1} \circ f^{-1})(b) = a$. Then:

$$\begin{aligned} g^{-1}(f^{-1}(b)) &= a \\ g(g^{-1}(f^{-1}(b))) &= g(a) \\ f^{-1}(b) &= g(a) \\ f(f^{-1}(b)) &= f(g(a)) \\ b &= (f \circ g)(a) \\ (f \circ g)(a) &= b \end{aligned}$$

Thus, $f \circ g$ and $g^{-1} \circ f^{-1}$ are inverses.

Alternate Solution:
We want to show that $(g^{-1} \circ f^{-1}) \circ (f \circ g)(x) = x$ for all $x \in X$ and $(f \circ g) \circ (g^{-1} \circ f^{-1})(z) = z$ for all $z \in Z$. We can apply the definition of the composition function to prove this. So, for every $x \in X$, we have:

$$\begin{aligned} (g^{-1} \circ f^{-1}) \circ (f \circ g)(x) &= (g^{-1} \circ f^{-1})((f \circ g)(x)) \\ &= (g^{-1} \circ f^{-1})(f(g(x))) \\ &= g^{-1}(f^{-1}(f(g(x)))) \\ &= g^{-1}(g(x)) \\ &= x \end{aligned}$$

Similarly for every $z \in Z$, we have:

$$\begin{aligned} (f \circ g) \circ (g^{-1} \circ f^{-1})(z) &= (f \circ g)((g^{-1} \circ f^{-1})(z)) \\ &= (f \circ g)(g^{-1}(f^{-1}(z))) \\ &= f(g(g^{-1}(f^{-1}(z)))) \\ &= f(f^{-1}(z)) \\ &= z \end{aligned}$$

We have shown that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Set Countability

Recall Functions

- **Function *f: A* → *B*:** associates each element of set A to <u>exactly one</u> element in set B
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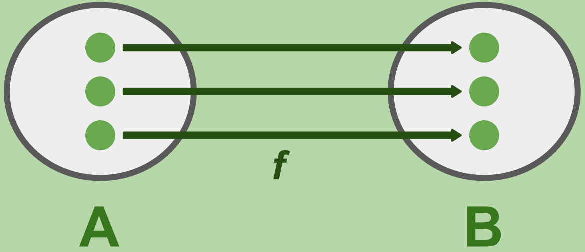
$$\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$$

- **Bijective Function:** onto and one-to-one (also called a one-to-one correspondence)

What do function properties tell us about the set cardinalities?

- **Onto Function f : A → B:** $\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$



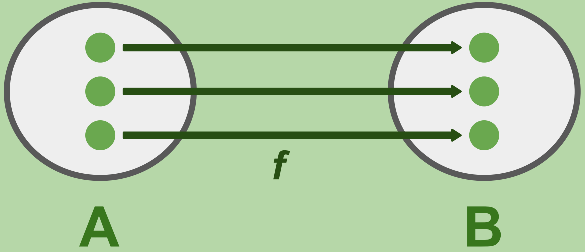
*$|A| = |B|$

- Is it possible that $|A| > |B|$?

- Is it possible that $|B| > |A|$?

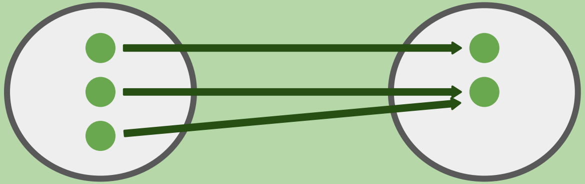
What do these properties tell us about the set cardinalities?

- **Onto Function *f* : A → B:** $\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$



*$|A| = |B|$

  - Is it possible that $|A| > |B|$? **Yes!**



  - Is it possible that $|B| > |A|$? **No**

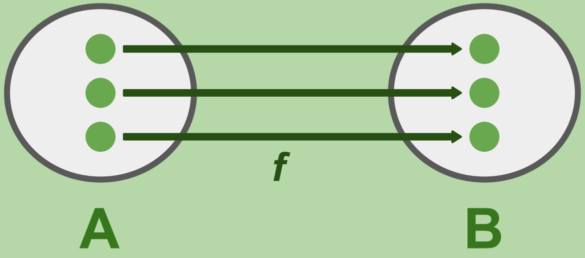
(can't be a function and onto in this case)

*Thus, if we have an onto function from A to B,

$$|A| \geq |B|$$

What do these properties tell us about the set cardinalities?

- **One-to-One Function f : A → B:** $\forall a,b \in A\ [\ f(a) = f(b) \ \rightarrow a = b\ ]$



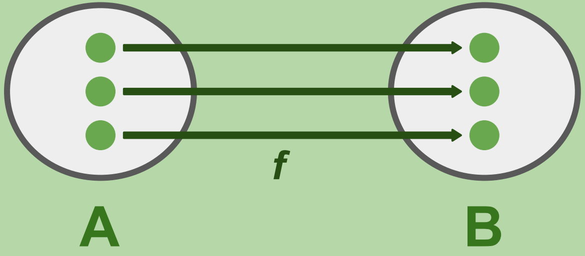
*$|A| = |B|$

  - Is it possible that $|A| > |B|$?

  - Is it possible that $|B| > |A|$?

What do these properties tell us about the set cardinalities?

- **One-to-One Function f : A → B:** $\forall a,b \in A\ [\ f(a) = f(b) \ \rightarrow a = b\ ]$



*$|A| = |B|$

*Thus, if we have an 1-1 function from A to B,

$$|A| \leq |B|$$

- Is it possible that $|A| > |B|$? **No**

(can't be a function and 1-1 in this case)

- Is it possible that $|B| > |A|$? **Yes!**

(still a function, still 1-1)

Countably vs Uncountably Infinite

- **Countably Infinite:** A set is said to be countably infinite if it has the same cardinality as the natural numbers. One way to prove this is by finding a bijection between the set and the natural numbers. Examples:
  - **The natural numbers**
  - **The integers**
  - **The rational numbers**
- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than the cardinality of the natural numbers. Examples:
  - **The real numbers**
  - **The irrational numbers**
  - **(0,1)**

Problem

2. Different Infinities ⋆

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a bijection between the $\mathbb{N}$ and that set. (You do not need to prove that the function you name is indeed a bijection.)

(a) $A = \{x \mid x \in \mathbb{Z} \ \wedge \ x > 10\}$

(b) $B = \{x \mid x \in \mathbb{Z} \ \wedge \ |x| < 1{,}000{,}000\}$

(c) $C = \{x \mid x \in \mathbb{R} \ \wedge \ 0 \leq x \leq 2\}$

(d) $D = \{2, 3\} \times \mathbb{N}$

Solution

Solution:

(a) A is countably infinite. There are many different bijections that are correct. One example of a correct bijection is $f : \mathbb{N} \to A$, $f(x) = x + 11$ (<u>Note</u>: 0 is a natural number in EECS 203).

(b) B is finite. This is because there are a finite number of integers with an absolute value less than $1,000,000$: $-999999, -999998, ..., -1, 0, 1, ..., 999998, 999999$.

(c) C is uncountably infinite. In lecture, we have shown that $[0,1]$ is uncountably infinite. Since $[0,1] \subseteq C$ and $[0,1]$ is uncountably infinite, we know that C must be uncountably infinite.

(d) D is countably infinite. There are many different bijections that are correct. One example of a correct bijection is $f : \mathbb{N} \to D$

$$f(x) = \begin{cases} (2, \frac{x}{2}), & \text{if } x \bmod 2 = 0 \\ (3, \frac{x-1}{2}), & \text{if } x \bmod 2 = 1 \end{cases}$$

Problem

3. Different Infinities with Sets $\star$

Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite

b) countably infinite

c) uncountably infinite

Solution

Solution: There are a lot of possible answers, but here are a few:

a) $A = [0, 1)$ and $B = (-1, 0]$. $A \cap B = \{0\}$

b) $A = \mathbb{R}^+$ and $B = \mathbb{R}^- \cup \mathbb{Z}^+$. $A \cap B = \mathbb{Z}^+$

c) $A = [0, 2]$ and $B = [1, 3]$. $A \cap B = [1, 2]$

Problem

4. Cardinality Proof $\star$

Show that $|(0,1)| \geq |\mathbb{Z}^+|$ by giving a one-to-one function. (You do not need to prove that the function you name is indeed one-to-one.)

Solution

Solution:
We can show that $|(0,1)| \geq |\mathbb{Z}^+|$ is true through the existence of a one-to-one function from $\mathbb{Z}^+ \to (0,1)$.
There are many different correct one-to-one functions. One example of a correct one-to-one example function:
$f : \mathbb{Z}^+ \to (0,1),\ f(x) = \frac{1}{x+1}$

Note: Since $(0,1)$ has exclusive bounds, we cannot write $f : \mathbb{Z}^+ \to (0,1), f(x) = \frac{1}{x}$ because 1 is not in our co-domain.

Schroder-Bernstein & Applications

- **Schroder-Bernstein Theorem:**
  If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

  - So using this theorem, injectivity, and surjectivity, how can we show that $|A| = |B|$ for sets A and B?

    1. Find a **bijection** $f\colon A \to B$ (or a bijection $g\colon B \to A$)
    2. Find **1-1** $f\colon A \to B$ and **1-1** $g\colon B \to A$
    3. Find **onto** $f\colon A \to B$ and **onto** $g\colon B \to A$

$\lvert A\rvert \leq \lvert B\rvert$	$\lvert A\rvert \geq \lvert B\rvert$
$f_1\colon A \to B$ is 1-1	$f_2\colon A \to B$ is onto
$g_1\colon B \to A$ is onto	$g_2\colon B \to A$ is 1-1

Problem

5. Schroder-Bernstein Theorem

Show that $(0, 1)$ and $[0, 1]$ have the same cardinality naming one-to-one functions. (You do not need to prove that the function(s) you name are indeed one-to-one.)

Solution

Solution: By Schroder-Bernstein theorem, it suffices to find two one-to-one functions $f : (0,1) \to [0,1]$ and $g : [0,1] \to (0,1)$. Let $f : (0,1) \to [0,1]$, $f(x) = x$ and $g : [0,1] \to (0,1)$, $g(x) = \frac{(x+1)}{3}$.

Problem

6. REVIEW: Inverses and Sets

Let A, B be sets, let $f : A \to B$ be a function, and let S and T be subsets of B. Prove that

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T).$$

Note: The notation $f^{-1}(S)$ is a shorthand for the set

$$f^{-1}(S) = \{x \in A \mid f(x) \in S\}.$$

Solution

Solution: We need to prove two things in order to show set equality: $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

Part 1: Show that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. First, let x be an arbitrary element such that $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore, $f(x) \in S \vee f(x) \in T$. In the first case $x \in f^{-1}(S)$, and in the second case $x \in f^{-1}(T)$. In both cases, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

Part 2: Show that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$. First, let x be an arbitrary element such that $x \in f^{-1}(S) \cup f^{-1}(T)$. Therefore, $x \in f^{-1}(S) \vee x \in f^{-1}(T)$, so $f(x) \in S \vee f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$ as desired.

Because we have shown both subset relations, we can conclude that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

Problem

7. Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$

(c) Find an explicit one-to-one and onto mapping from the closed interval $(0, 1)$ to the open interval $[0, 1]$.

Solution

Solution:

(a) $\{\frac{1}{n}\}$ where n is an integer greater or equal to 2.

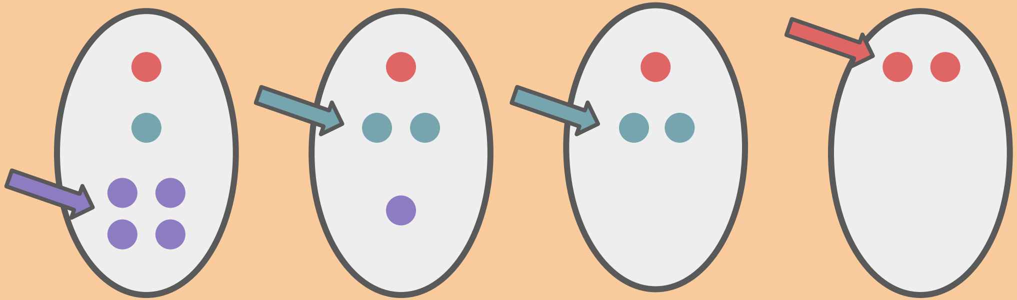
(b) Map $\frac{1}{2}$ to 0, $\frac{1}{3}$ to 1, and for all $n > 3$, map $\frac{1}{n}$ to $\frac{1}{n-2}$

(c) Map every element to itself, except those in A. Map those in A according to the mapping came up with in part b.

Pigeonhole Principle

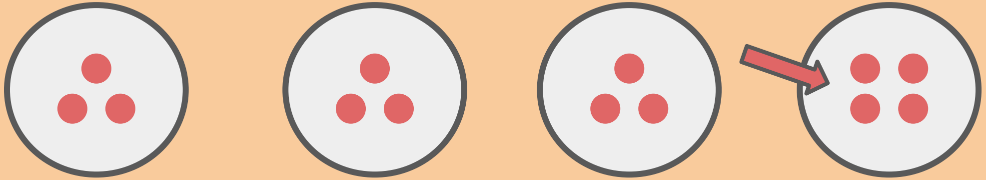
Pigeonhole Principle

- **Pigeonhole Principle:** If we put **k+1 objects** into **k boxes**, then at least one box contains **2 or more** objects.



*__Examples__ of putting 5 objects into 4 bins

- **Generalized Pigeonhole Principle:** If we put **N objects** into **k boxes**, then at least one box contains **ceil(N/k) or more** objects.



*__Example__ of putting 13 objects into 4 bins
ceil(13/4) = ceil(3.25) = 4

Problem

8. Pigeonhole Principle Warm up ⋆

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution

Solution:

(a) $(4 \cdot 21) + 1 = 85$

(b) $(50 \cdot 99) + 1 = 4951$

Problem

9. PHP ⋆

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

Solution

Solution:

(a) 10. Consider colors as boxes, and candies as pigeons. By pigeonhole principle, we have $\lceil \frac{N}{3} \rceil = 4$ where N is the number of pieces we have to grab to make this work. The smallest number N that works is 10.

(b) 33. This is not actually pigeonhole. We specifically need to have 3 orange candies. The only way to make sure this happens is to grab all 15 red, 15 blue, and then the next 3 we grab have to be orange.

Problem

10. Pigeonhole Principle ⋆

How many distinct numbers must be selected from the set

$$\{1, 3, 5, 7, 9, 11, 13, 15\}$$

to guarantee that at least one pair of selected numbers add up to 16?

Solution

Solution: We can group these into pairs that add up to 16 of: $(1, 15), (3, 13), (5, 11), (7, 9)$. Notably, no other pair of numbers sum to 16. Therefore, we must pick at least 5 distinct numbers to guarantee that we pick both numbers from at least one of these pairs.

Problem

11. Pigeonhole Practice

How many integers do we need to select to guarantee that, for two distinct welected integers x, y, the difference $x - y$ is divisible by 10?

Solution

Solution: Let our pigeon holes be (integers equivalent to 0 mod 10), (integers equivalent to 1 mod 10)...(integers equivalent to 9 mod 10). Therefore, we have 10 pigeon holes. After selecting 11 integers, there are more pigeons than pigeon holes, so two of the selected integers are guaranteed to be equivalent mod 10. The difference of these two integers must be divisible by 10.

Problem

12. Sphere

Prove that a sphere with any 5 points along its surface can be split into two equal hemispheres such that one of the hemispheres contains at least 4 points. Points that lie on the dividing line between the two hemispheres may be counted in both hemispheres.

Solution

Solution: Pick any two of the 5 points and connect them. Divide the sphere into hemispheres along the edge of the two connected points (extend the connected edge all the way around the sphere to form a circle). Since points on top of the split can be counted in both hemispheres, we have two hemispheres which both have two points. Now, we are left with the three remaining points and two hemispheres, so by the pigeon hole principle, at least two of the three additional points must be on the same hemisphere. Therefore, one of the hemispheres must have at least 4 of the 5 points (2 from first two picked, and another 2 from the remaining 3).

Problem

13. REVIEW: Pigeonhole Principle

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution

Solution: Let $K(x)$ be the number of other computers that computer x is connected to. The possible values for $K(x)$ are 1, 2, 3, 4, 5. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.