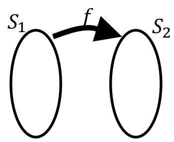
Countable
Uncountable
countably infinite
uncountably infinite
diagonalization: 对角化

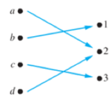
Lec 17: Countability-- ANSWERS

more exact:
① $|A| \leq |B|$ iff $\exists$ one-to-one $f: A \to B$

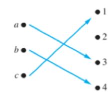
② $|A| = |B|$ iff $\exists$ bijection $f: A \to B$

③ $|A| < |B|$ iff $\nexists$ onto $f: A \to B$





Function is "Onto" means that: all codomain elements have *at least* one incoming arrow



Function is "1-to-1" means that: all codomain elements have *at most* one incoming arrow

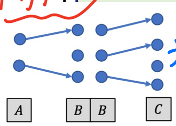
For *infinite* sets S_1, S_2:

$|S_1| = |S_2|$ means that there exists a function $f: S_1 \to S_2$ that is a bijection

$|S_1| \leq |S_2|$ means that there exists a function $f: S_1 \to S_2$ that is one-to-one

↓ reverse, same

$|S_1| \geq |S_2|$ means that there exists a function $g: S_2 \to S_1$ that is one-to-one



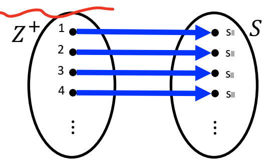
∗ $\exists f: A \to B$ one to one $\wedge$ $\exists g: B \to C$ one to one
$\Rightarrow g \circ f: A \to C$ one to one

Set Comparisons

Definition: A set S is **countable** iff $|S| \leq |\mathbb{Z}^+|$
(otherwise it's *uncountable*)

$|\mathbb{Z}^+| = \aleph_0$ = "aleph null"

One way to think about countability:
A set is countable if you can list its elements, one after the other, and any given element would be listed within finitely many steps

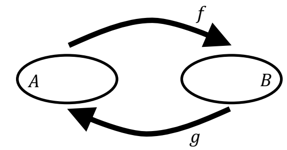


Some Countable Sets	**Some Uncountable Sets**
$\mathbb{Z}^+$	$\mathbb{R}$
$\mathbb{N}, \mathbb{Z}$	$[0, 1]$
$\mathbb{Q}$	$\mathbb{R} - \mathbb{Q}$
$\mathbb{Z} \times \mathbb{Z}$	$P(\mathbb{Z})$
and many more!	and many more!

One-to-one, Onto, and Cardinality

If $f: A \to B$ is one-to-one,
then $|A| \leq |B|$

If $g: B \to A$ is one-to-one,
then $|A| \geq |B|$



Schroeder-Bernstein Theorem:
④ If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

Ways to show $\lvert A\rvert \leq \lvert B\rvert$	Ways to show $\lvert B\rvert \leq \lvert A\rvert$
$f: A \to B$ is one-to-one	$g: B \to A$ is one-to-one

To prove $|A| = |B|$:
- Give a bijection $A \to B$, or a bijection $B \to A$, OR
- **** Easiest **** Give a **one-to-one** function $A \to B$ **and** a **one-to-one** function $B \to A$

Three ways to prove $|A| = |B|$:
- Give a bijection $A \to B$, OR
- Give a bijection $B \to A$, OR
- Give a one-to-one function $A \to B$ **and** a one-to-one function $B \to A$ (this is sometimes much easier)

$\mathbb{Z}^+$ vs $\mathbb{N}$

Are these sets the same size? How do you know?

$$\mathbb{Z}^+ = \{1,2,3,4, \dots\}$$

$$\mathbb{N} = \{0,1,2,3,4, \dots\}$$

Same size because there is a **bijection** $f: \mathbb{N} \to \mathbb{Z}^+$
given by $f(n) = n + 1$

Equivalently, we could use the **bijection** $f: \mathbb{Z}^+ \to \mathbb{N}$ given by $f(z) = z - 1$

Idea for a bijection:

- Use the **even integers** in $\mathbb{N}$ to cover the **nonnegative integers** in $\mathbb{Z}$
- Use the **odd integers** in $\mathbb{N}$ to cover the **negative integers** in $\mathbb{Z}$

$\mathbb{Z}$ vs $\mathbb{N}$

Are these sets the same size? How do you know?

$$\mathbb{N} = \{0,1,2,3,4,5,6,7 \dots\}$$

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Same size because there is a **bijection** $f: \mathbb{Z} \to \mathbb{N}$ given by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

Equivalently, we could use the **bijection** $f: \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$|\mathbb{Z}^+ \times \mathbb{Z}^+|$ vs $|\mathbb{Z}^+|$ $\Rightarrow$ same size

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

$\mathbb{Z}^+ \times \mathbb{Z}^+$:

f(1) (1, 1)	f(3) (1, 2)	f(6) (1, 3)	(1, 4)	...
f(2) (2, 1)	f(5) (2, 2)	(2, 3)	(2, 4)	...
f(4) (3, 1)	(3, 2)	(3, 3)	(3, 4)	...

f(1) = (1, 1)
f(2) = (2, 1)
f(3) = (1, 2)
f(4) = (3, 1)
f(5) = (2, 2)
f(6) = (1, 3)
...

A bijection $f: \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ is like **putting these elements in an order.**

- Assign one element $f(1)$
- Assign one element $f(2)$
- ...

12

Yes, same size

Proof: [0, 1] is Uncountable

$z \rightarrow \boxed{f} \rightarrow f(z)$

- Let $f: \mathbb{Z}^+ \to [0,1]$ be arbitrary. The goal is to show f is not onto.
- Make a table of the digits of $f(z)$. For example:

$z \in \mathbb{Z}^+$	$f(z) \in [0, 1]$		digit 1 of f(z)	digit 2 of f(z)	digit 3 of f(z)	digit 4 of f(z)	...
1	0	0.	0	0	0	0	...
2	$\pi/10$	0.	3	1	4	1	...
3	0.149	0.	1	4	9	0	...
4	$e/10$	0.	2	7	1	8	...
⋮	⋮						

- How can we generate a number in [0,1], but **not** in any row of the table (so not in $range(f)$)?

Take the diagonal elements, and add 5 (mod 10): 0.5643 ...

or any int as long as different from original one

Additional Practice: Countability

Determine whether each set is countable or uncountable.

a)	{red, green, blue}	countable	uncountable
b)	**N**	countable	uncountable
c)	**N** × **N**	countable	uncountable
d)	**R** – **Q** (the irrationals)	countable	uncountable
e)	$\{x \in R \mid \lfloor x \rfloor = x\}$	countable	uncountable
f)	$\{x \in R \mid \sqrt{x} \text{ is rational}\}$	countable	uncountable
g)	$\mathcal{P}(\mathbf{Z})$	countable	uncountable

5

$|\mathbb{N}|$ vs $|\mathbb{Z}^+|$

A new guest, Mx. Zero, arrives. Can they be accommodated?
- (A) Yes!
- (B) No!

36

6

Q2: Yes!

$|\mathbb{Z}|$ vs $|\mathbb{Z}^+|$

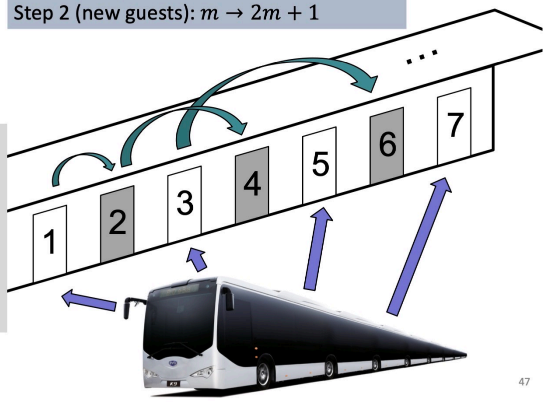
Step 1 (old guests): $n \rightarrow 2n$
Step 2 (new guests): $m \rightarrow 2m+1$

Hilbert's Hotel is full again!

But now **infinitely many new guests arrive**.

Ask each guest to move from room n to room $2n$

Put new guests in the gaps!



47

7

Q3: Yes!

Re-assign guests to rooms as indicated here.

$|\mathbb{Z}^+ \times \mathbb{Z}^+|$ vs $|\mathbb{Z}^+|$

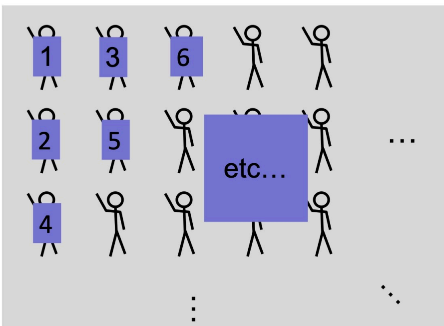
Passenger

1 2 3 4 5

Hotel Rooms

Bus # 1

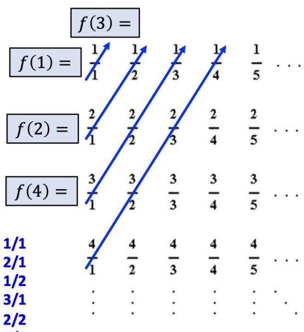
2



8

Similar Trick:
$|\mathbb{Q}^+| = |\mathbb{Z}^+|$

$|\mathbb{Q}^+|$ vs $|\mathbb{Z}^+|$



f(1) = 1/1
f(2) = 2/1
f(3) = 1/2
f(4) = 3/1
f(5) = 2/2
f(6) = 1/3
...

$$|\mathbb{Q}^+| \leq |\mathbb{Z}^+ \times \mathbb{Z}^+| = |\mathbb{Z}^+|$$

Consider f: $\boldsymbol{Q}^+ \rightarrow \boldsymbol{Z}^+ \times \boldsymbol{Z}^+$
f(r) = (a,b) where r = a/b
(reduced so that a,b have no common factor)

f is one-to-one (different rationals → different fractions)

But we also have $|\mathbb{Z}^+| \leq |\mathbb{Q}^+|$, from $f(z) = z$
identity function is one-to-one

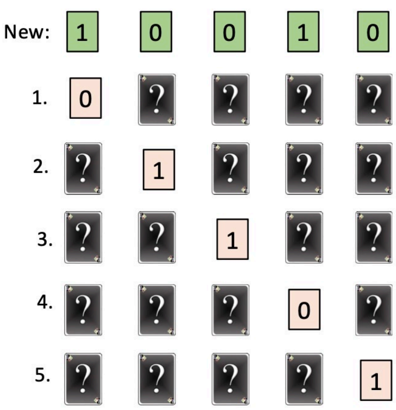
By Schröder-Bernstein, $|\mathbb{Z}^+| = |\mathbb{Q}^+|$.

9. Diagonalization (and proof of $|\mathbb{Z}| < |[0,1]|$)

- Each card has either a **0 or a 1** on the other side.
- You may flip over any number of cards for **$1 per flip**
- If you can name a 5-bit string that you are **certain** is different from all 5 listed here, you win $10.

Can you make money on this game? How much? What's your strategy?

A Game



Best strategy:
- Pay $5 to flip over the cards **along the main diagonal.**
- Name the **flipped bitstring** as your guess:
  01101 → **10010**
- Make $5 overall.

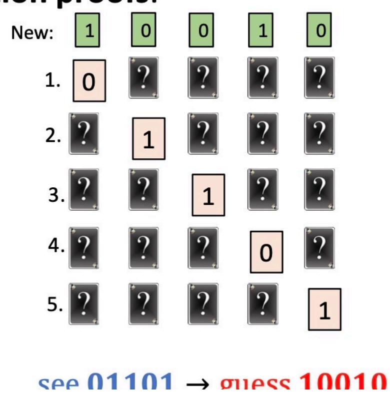
Diagonalization

- The strategy we used in the game is an example of a **diagonalization** argument
  - A powerful proof style, for showing impossibility of things!

Diagonalization

General format for **diagonalization proofs**:

- Arrange items in a table
- Look at the elements along the main diagonal
- Manipulate each diagonal element and put them together to create a *new row* that is guaranteed **not** to be in the table
  - Is the new row the same as row i of the table?
  - **No**: its i^{th} bit is different from the string in row i.



Proof: [0, 1] is Uncountable

$[0,1]$ vs $|\mathbb{Z}^+|$

Theorem:

$|\mathbb{Z}^+| < |[0,1]|$ equivalently There is no **onto** function $f: \mathbb{Z}^+ \to [0,1]$

- Let $f: \mathbb{Z}^+ \to [0,1]$ be an arbitrary function.
- Imagine an infinite table with the **digit expansion** of $f(1)$ in the first row, of $f(2)$ in the second row…
- Let d be the **"diagonal number"** formed by taking digits from the diagonal of the table
- Let s be the **"shifted diagonal number"** formed by adding 5 to each digit (rollover 9 → 0)
- s cannot be in the table! For all i, s is different from the number in row i of the table, because its i^{th} digit is different
- So f is not **onto**.

$z \in \mathbb{Z}^+$	$f(z) \in$ [0, 1]		digit 1 of f(z)	digit 2 of f(z)	digit 3 of f(z)	digit 4 of f(z)	...
1	0	0.	0	0	0	0	...
2	$\pi/10$	0.	3	1	4	1	...
3	0.149	0.	1	4	9	0	...
4	$e/10$	0.	2	7	1	8	...
⋮	⋮						

d = 0.0198…
s = 0.5643…

THERE ARE DIFFERENT INFINITIES

Punchline: The number ∞ is not just one number!!
There are lots of different sets of size ∞, with different sizes than each other

(10) $\aleph_0$: aleph null

Instead:

- $|\mathbb{Z}^+| = |\mathbb{N}| = \aleph_0$ = "aleph null"
- $|[0,1]| = |\mathbb{R}| = \aleph_1$ = "continuum" or "aleph one"
  - Actually, [Paul Cohen' 63] showed that it's **unprovable** whether $|\mathbb{R}|$ is the second smallest infinity, $\aleph_1$, but it's standard to assume that it is.
- There is no largest infinity. There's always a bigger one. $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$
  - Fun fact: taking a power set always makes it larger. $|P([0,1])| > |[0,1]|$

"S is countably infinite" means $|S| = \aleph_0$

Possible set cardinalities: