EECS 203: Discrete Mathematics
Fall 2023
Homework 5

Due **Thursday, October. 12**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total Points: $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Induction Construction [16 points]

Let $P(n)$ be the statement that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer. In this problem, we will prove this statement via weak induction.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, which is the base case for our inductive step.

(c) In the base case we prove $P(1)$; what do you need to prove in the inductive step?

(d) What is the inductive hypothesis for your proof?

(e) Complete the inductive step, indicating where you used the inductive hypothesis.

(f) Explain why this proof shows $P(n)$ is true for all positive integers n.

Solution:

(a) $P(1)$: $1 \cdot 1! = (1+1)! - 1$.

(b) For $P(1)$, LHS $= 1$
RHS $= 2! - 1 = 2 \cdot 1 - 1 = 1$.
$. \therefore$ LHS $=$ RHS. $\therefore P(1)$ is true.

(c) We need to prove that $P(k) \to P(k+1)$ for any integer n which is ≥ 1.

(d) The inductive hypothesis: Assume $P(k)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.

(e) Let k be an arbitrary positive integer.
Assume $P(k)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$
Want to show: $P(k+1)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! = (n+1+1)! - 1$
Using $P(n)$ we know:

$$\begin{aligned}
1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\
&= (n+1)!(1+n+1) + 1 \\
&= (n+1+1) \cdot (n+1) \cdot n \cdot (n-1) \cdot \cdots \cdot 1 - 1 \\
&= (n+1+1)! - 1
\end{aligned}$$

Thus $P(k) \to P(k+1)$ for any integer n which is ≥ 1.

(f) It is because that: (1) $P(1)$: $1 \cdot 1! = (1+1)! - 1$.
(2)$P(k) \to P(k+1)$ for any integer n which is ≥ 1.
Therefore $P(1) \to P(2) \to P(3) \cdots \to P(n)$
Where n can be any positive integer.
$\therefore$ Since we know (1) is true and (2) is true, $P(n)$ is true for all positive integers n.

2. Base Two Blues [14 points]

Prove using mathematical induction that $\log_2(n) < n$ for every positive integer n. You may assume that the base-2 logarithm function is strictly increasing on its domain.

Fun Fact: $\log_b(n) < n$ is actually true for every positive real number n and arbitrary base $b > 1$, but we're asking you to prove this by induction for the special case where $b = 2$ and n is a positive integer.

Solution:

Let k be an arbitrary positive integer.
Assume $P(k)$: $\log_2 k < k$
Want to show: $P(k+1) : \log_2 k + 1 < k + 1$

Base Case:
$P(0) : \log_2 1 < 1$
Since $\log_2 1 = 0 < 1$, base case is true.

Inductive Step:
Using the property of logarithm we know:

$$\begin{aligned} \log_2{(k+1)} &= \log_2{(\frac{k+1}{k} \cdot k)} \\ &= \log_2{\frac{k+1}{k}} + \log_2{k} \\ &= \log_2{(1+\frac{1}{k})} + \log_2{k} \end{aligned}$$

Since k is a positive integer, $k \geq 1$, $\frac{1}{k} \leq 1$, $\frac{1}{k} + 1 \leq 2$
$\therefore \log_2{(\frac{1}{k}+1)} \leq 1$.
Using $P(n)$ we know: $\log_2 k < k$.
And Since $\log_2{(\frac{1}{k}+1)} \leq 1$ and $\log_2 k < k$
$log_2(k+1) = \log_2{(1+\frac{1}{k})} + \log_2 k < k + 1$.
Then we have proved that $P(K) \to P(k+1)$ for all positive integer k.

Conclusion: $\log_2(n) < n$ for every positive integer n.

3. Inductive Hypothe-six [15 points]

Prove by weak induction that 6 divides $n^3 - n$ where n is a nonnegative integer. Don't include unneeded base cases.

Solution:

Let k be an arbitrary nonnegative integer.
Assume $P(k)$: $6|(k^3 - k)$
Want to show: $P(k+1)$: $6|[(k+1)^3 - (k+1)]$

Base Case:
$P(0)$: $6|0^3 - 0$
Since $0^3 - 0 = 0$, $6|0$, base case is true.

Inductive Step:
Since $P(k)$: $6|(k^3 - k)$, for some integer m, $6m = (k^3 - k)$
Then

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 6m + 3(k^2 + k) \qquad\qquad = 6m + 3k \cdot (k+1)\end{aligned}$$

Since k is an integer, and integers consist of alternating odd numbers and even numbers, one of k and $k+1$ must be even. WLOG assume k is even, then for some integer p, $k = 2p$.
Then $(k+1)^3 - (k+1) = 6m + 3 \cdot 2p \cdot (k+1) = 6m + 6p \cdot (k+1) = 6[m + p(k+1)]$
Since p, m, k are integers, $p(k+1)$ is an integer, $[m + p(k+1)]$ is an integer.
Then $6|[m + p(k+1)]$, i.e. $6|[(k+1)^3 - (k+1)]$.
Therefore we have proved that $P(k) \to P(k+1)$ for any nonnegative integer k.

Conclusion: 6 divides $n^3 - n$ where n is a nonnegative integer.

4. Incorrect Strong Induction [14 points]

For each of the following **incorrect** strong induction proofs, note where the strong induction proof breaks down and is incorrect.

Hint: Consider where the inductive step breaks down.

(a) Proving for every nonnegative integer n, $P(n)\colon 3n = 0$.

Inductive Step:

Assume that $P(j)\colon 3j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. We wish to show $P(k+1)$. We will rewrite $k+1 = a+b$ where a and b are nonnegative integers less than $k+1$. Thus, $3 \cdot (k+1) = 3 \cdot (a+b) = 3a + 3b = 0 + 0 = 0$, therefore $P(k+1)$ is proven.

Base Case: $P(0)\colon 3 \cdot 0 = 0$

Since we have shown the basis step and the inductive step, we have proved for every nonnegative integer n, $P(n) : 3n = 0$.

(b) Proving that every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

Inductive Step:

Assume we can form cent values of j cents for all $3 \leq j \leq k$ using just 3-cent and 4-cent stamps. We wish to show we can form $k+1$ cents using just 3-cent and 4-cent stamps. We can form a $k+1$ cent value by replacing 1 3-cent stamp with 1 4-cent stamp or by replacing 2 4-cent stamps with 3 3-cent stamps.

Base Case:

We can form cent values of 3-cents using one 3-cent stamp and we can form cent values of 4-cents using one 4-cent stamp. This covers our two base cases.

Since we have shown the basis step and the inductive step, we have proved every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

Solution:

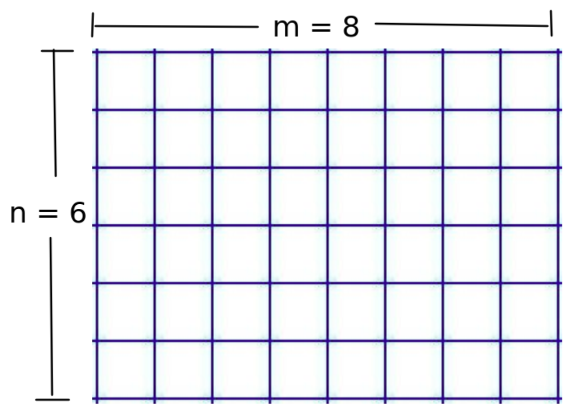
(a) The proof breaks down from the base case when we induce $P(0) \to P(1)$.
At this point, $k+1 = 1$, but we cannot find two integers a and b that are less than 1 while they add up to 1.If a, b are less than 1 and nonnegative, then they can only be both 0. Then they add to to 0 but not 1.
Therefore the proof is incorrect.

(b) The proof breaks down when we induce $(P(3) \wedge P(4)) \to P(5)$.
The inductive step states that we can form a $k+1$ cent value by replacing 1 3-cent

stamp with 1 4-cent or by replacing 2 4-cent stamps with 3 3-cent stam. But at this point, we only have one 4-cent and can not apply the induction.
Therefore the proof is incorrect.

5. Chopping Ice [15 points]

Claire doesn't have an ice tray, so she makes ice by freezing water into a rectangle and then dividing the rectangle into grid-aligned cells. She would like to divide her block of ice into n rows and m columns quickly, before the ice melts! See the image below for an example.



(a) State the number of cuts Claire needs to make to divide her ice block into $n \times m$ cells. One cut means splitting a single rectangle into two rectangles. In other words, you may NOT make a single cut across multiple pieces of ice. You may use n and/or m in your answer.

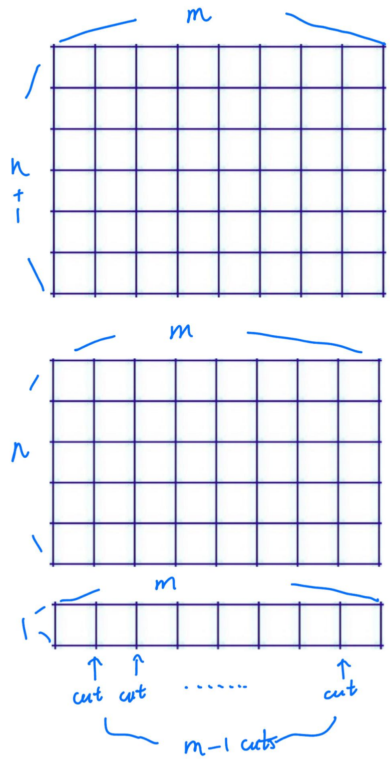
(b) Prove your answer from part (a).

Solution:

(a) $(n-1) + n \cdot (m-1) = mn - 1$
m, n are positive integers.

(b) Let m, n be an arbitrary positive integer. $k(m, n)$ is the number of cuts to make to divide her ice block into $n \times m$ cells.
Assume $P(m, n)$: $k(m, n) = mn - 1$
WLOG (due to symmetry, $k(m+1, n) = k(m, n+1)$), we want to show: $P(m, n+1)$: $k(m, n+1) = m(n+1) - 1$

Inductive Step:
We can first cut the (n+1) row from the block. This requires 1 cut.
Then we get $n \times m$ block and $1 \times m$ block.



From the inductive hypothesis we know, to divide the $n \times m$ block, we need $k(m, n)$ cuts.
And to divide the $1 \times m$ block, since we can not make a single cut across multiple pieces of ice, we can only use $m - 1$ cuts.
$\therefore$ In total, we need:

$$\begin{aligned} 1 + k(m, n) + (m - 1) &= 1 + mn - 1 + m - 1 \\ &= mn + m - 1 \\ &= m(n + 1) - 1 \end{aligned}$$

Base Case: To divide a $P(1, 1) : k(1, 1) = 1 \times 1 - 1 = 0$ is true.

Since $P(1, 1)$, $P(m, n) \to P(m, n{+}1)$, and due to symmetry $P(m, n) \to P(m{+}1, n)$, $P(m, n)$ is true for any positive integer m, n.

Conclusion: We need $mn - 1$ cuts to divide her ice block into $m \times n$ cells.

6. Pastry Recurrence [12 points]

A baker decorates a cookie in 2 minutes, a cupcake in 3 minutes, and a pie in 3 minutes. Let a_n denote the number of distinct ways the baker decorates pastries in exactly n minutes for $n \geq 0$ (where order matters).

(a) Find a recurrence relation for a_n.

(b) What are the initial conditions? Use the fewest initial conditions necessary.

Solution:

(a) Case 1: The last pastry the baker decoratess is cookie.
Then before that, there are a_{n-2} ways.
Case 2: The last pastry the baker decorates is cupcake.
Then before that, there are a_{n-3} ways.
Case 3: The last pastry the baker decorates is pie.
Then before that, there are a_{n-3} ways.
$\therefore a_n = a_{n-2} + 2a_{n-3}$. $(n \geq 3)$

(b) Since a_n is valid only when $n \geq 0$, and we have a_{n-3} in our recurrence relation, we need to know all a_n where $n < 3$.
That is:
$a_0 = 1$ (since the only choice is to do nothing)
$a_1 = 1$ (since the only choice is to do nothing)
$a_2 = 1$ (since the only choice is to decorate a cookie.)

7. Raven's Wrestlers [14 points]

Raven has n weeks to build her wrestling figure collection. Every week, Raven buys one item to add to her collection. There are 4 different types of things she can buy: Figures, T-shirts for her wrestlers to wear, Weapons for them to fight with, or Display Stands to show them off on her shelves.

- Her shelves can fit 2 Stands nicely, so when she buys a Display Stand, she will always buy a second one the next week to finish the shelf. Additionally, the week after buying the second Stand, she will buy something other than a Display Stand (they aren't as exciting to buy)

- When she buys a Figure, she gets very excited about it and wants to buy a new T-shirt for it to wear the following week.

Let a_n represent the number of ways Raven can buy items across the n weeks (where $n \geq 0$)

(a) Find a recurrence relation for a_n.

(b) Which terms would need to be defined with initial conditions (no need to find the value, just which terms)

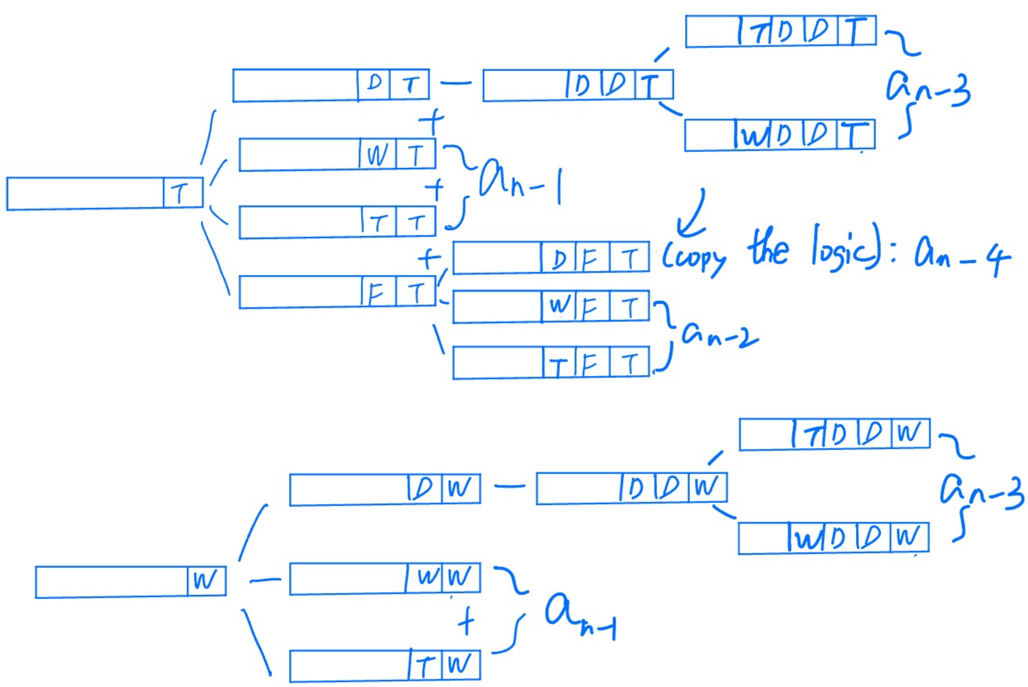
Note 1: Buying the same items in a different order counts as a different way of buying items. We treat all items in a category as identical.

Note 2: on week n, Raven will not buy a Figure (because she knows she will miss buying a T-shirt) or a Stand (what a sad way to end the collection). This information is not needed for the simplest solutions, but some alternate solutions may need to know this.

Solution:

(a) The recurrence relation is:

$$a_n = 2a_{n-1} + a_{n-2} + 2a_{n-3} + a_{n-4}$$



The logic is shown in the picture.
We use T,W,D,F to indicate the four items.

There are two cases for item in week n: T and W.
For the case W in week n, there are three possible choices in week $n-1$: D, W, T. Number of ways ended with W and T are exactly a_{n-1}. And for the way ended with DW, the previous item can only be D, and therefore get TDDW and WDDW. The number of them are exactly a_{n-3}.

For the case T in week n, there are four possible choices in week $n-1$: D, W, T, F. Number of ways ended with WT and TT is exactly a_{n-1}, and for FT, the possible choices in week $n-2$ is D,W,T. Number of ways ended with WFT and TFT are exactly a_{n-2}.
And for the remaining circumstances DT and DFT beginning with D, we can apply the same logic in the DW case, and get a_{n-3} and a_{n-4} respectively.

(b) since for a_n, $n \geq 0$, and there is a_{n-4} in our recurrence relation, $n-4 \geq 0$. So the weeks where $n < 4$ should be set as initial conditions.
Therefore a_1, a_2, a_3, a_4 would need to be defined with initial conditions.