EECS 203: Discrete Mathematics
F23
Discussion 8 Notes

1 Definitions

- **Function Inverse**
- **Function Composition**
- **Countably Infinite**:
- **Uncountably Infinite**:
- **Schroder-Bernstein Theorem:**
- **Pigeonhole Principle:**
- **Generalized Pigeonhole Principle:**

Solution:

- **Function Inverse** f^{-1}: Let f be a bijection with domain A and codomain B. The inverse function f^{-1} of f is a function with domain B and codomain A. It maps each element $b \in B$ to the unique element $a \in A$ such that $f(a) = b$. Hence, $f^{-1}(b) = a$ if and only if $f(a) = b$.

- **Function Composition** $f \circ g$: Let g be a function with domain A and codomain B, and let f be a function with domain B and codomain C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

- **Countably Infinite:** A set S is countably infinite if it has the same cardinality as the natural numbers $\mathbb{N}$. This can be proven by finding a bijection between S and the natural numbers. Some examples of countably infinite sets include $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

- **Uncountably Infinite:** A set S is uncountably infinite if its cardinality is strictly larger than the set of natural numbers $\mathbb{N}$. An example of a set that is uncountably infinite is $\mathbb{R}$.

- **Schroder-Bernstein Theorem:** For two sets A and B, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ (even when A, B are infinitely large). Note that finding a one-to-one function from A to B shows that $|A| \leq |B|$.
- **Pigeonhole Principle:** If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- **Generalized Pigeonhole Principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

2 Exercises

1. Composition and Inverses $\star$

Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Solution:
Reminder of the definition of f and f^{-1} being inverses: $f^{-1}(b) = a$ if and only if $f(a) = b$. So our goal is to show this property for the pair of functions $f \circ g$ and $g^{-1} \circ f^{-1}$.

Part 1: Proof that $(f \circ g)(a) = b \to (g^{-1} \circ f^{-1})(b) = a$. Let $a \in X, b \in Z$ be arbitrary elements, and assume that $(f \circ g)(a) = b$. Then:

$$\begin{aligned} f(g(a)) &= b \\ f^{-1}(f(g(a))) &= f^{-1}(b) \\ g(a) &= f^{-1}(b) \\ g^{-1}(g(a)) &= g^{-1}(f^{-1}(b)) \\ a &= (g^{-1} \circ f^{-1})(b) \\ (g^{-1} \circ f^{-1})(b) &= a \end{aligned}$$

Part 2: Proof that $(g^{-1} \circ f^{-1})(b) = a \implies (f \circ g)(a) = b$**.** Let $a \in X, b \in Z$ be arbitrary elements, and assume that $(g^{-1} \circ f^{-1})(b) = a$. Then:

$$\begin{aligned}
g^{-1}(f^{-1}(b)) &= a \\
g(g^{-1}(f^{-1}(b))) &= g(a) \\
f^{-1}(b) &= g(a) \\
f(f^{-1}(b)) &= f(g(a)) \\
b &= (f \circ g)(a) \\
(f \circ g)(a) &= b
\end{aligned}$$

Thus, $f \circ g$ and $g^{-1} \circ f^{-1}$ are inverses.

Alternate Solution:
We want to show that $(g^{-1} \circ f^{-1}) \circ (f \circ g)(x) = x$ for all $x \in X$ and $(f \circ g) \circ (g^{-1} \circ f^{-1})(z) = z$ for all $z \in Z$. We can apply the definition of the composition function to prove this. So, for every $x \in X$, we have:

$$\begin{aligned}
(g^{-1} \circ f^{-1}) \circ (f \circ g)(x) &= (g^{-1} \circ f^{-1})((f \circ g)(x)) \\
&= (g^{-1} \circ f^{-1})(f(g(x))) \\
&= g^{-1}(f^{-1}(f(g(x)))) \\
&= g^{-1}(g(x)) \\
&= x
\end{aligned}$$

Similarly for every $z \in Z$, we have:

$$\begin{aligned}
(f \circ g) \circ (g^{-1} \circ f^{-1})(z) &= (f \circ g)((g^{-1} \circ f^{-1})(z)) \\
&= (f \circ g)(g^{-1}(f^{-1}(z))) \\
&= f(g(g^{-1}(f^{-1}(z)))) \\
&= f(f^{-1}(z)) \\
&= z
\end{aligned}$$

We have shown that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

2. Different Infinities $\star$

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a bijection between the $\mathbb{N}$ and that set. (You do not need to prove that the function you name is indeed a bijection.)

(a) $A = \{x \mid x \in \mathbb{Z} \ \wedge \ x > 10\}$

(b) $B = \{x \mid x \in \mathbb{Z} \ \wedge \ |x| < 1,000,000\}$

(c) $C = \{x \mid x \in \mathbb{R} \ \wedge \ 0 \leq x \leq 2\}$

(d) $D = \{2, 3\} \times \mathbb{N}$

Solution:

(a) A is countably infinite. There are many different bijections that are correct. One example of a correct bijection is $f : \mathbb{N} \to A$, $f(x) = x + 11$ (Note: 0 is a natural number in EECS 203).

(b) B is finite. This is because there are a finite number of integers with an absolute value less than $1,000,000$: $-999999, -999998, ..., -1, 0, 1, ..., 999998, 999999$.

(c) C is uncountably infinite. In lecture, we have shown that $[0, 1]$ is uncountably infinite. Since $[0, 1] \subseteq C$ and $[0, 1]$ is uncountably infinite, we know that C must be uncountably infinite.

(d) D is countably infinite. There are many different bijections that are correct. One example of a correct bijection is $f : \mathbb{N} \to D$

$$f(x) = \begin{cases} (2, \frac{x}{2}), & \text{if } x \bmod 2 = 0 \\ (3, \frac{x-1}{2}), & \text{if } x \bmod 2 = 1 \end{cases}$$

3. Different Infinities with Sets ⋆

Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite

b) countably infinite

c) uncountably infinite

Solution: There are a lot of possible answers, but here are a few:

a) $A = [0, 1)$ and $B = (-1, 0]$. $A \cap B = \{0\}$

b) $A = \mathbb{R}^+$ and $B = \mathbb{R}^- \cup \mathbb{Z}^+$. $A \cap B = \mathbb{Z}^+$

c) $A = [0, 2]$ and $B = [1, 3]$. $A \cap B = [1, 2]$

4. Cardinality Proof ⋆

Show that $|(0,1)| \geq |\mathbb{Z}^+|$ by giving a one-to-one function. (You do not need to prove that the function you name is indeed one-to-one.)

Solution:
We can show that $|(0,1)| \geq |\mathbb{Z}^+|$ is true through the existence of a one-to-one function from $\mathbb{Z}^+ \to (0,1)$.
There are many different correct one-to-one functions. One example of a correct one-to-one example function:
$f : \mathbb{Z}^+ \to (0,1),\ f(x) = \frac{1}{x+1}$

Note: Since $(0,1)$ has exclusive bounds, we cannot write $f : \mathbb{Z}^+ \to (0,1), f(x) = \frac{1}{x}$ because 1 is not in our co-domain.

5. Schroder-Bernstein Theorem

Show that $(0,1)$ and $[0,1]$ have the same cardinality naming one-to-one functions. (You do not need to prove that the function(s) you name are indeed one-to-one.)

Solution: By Schroder-Bernstein theorem, it suffices to find two one-to-one functions $f : (0,1) \to [0,1]$ and $g : [0,1] \to (0,1)$. Let $f : (0,1) \to [0,1],\ f(x) = x$ and $g : [0,1] \to (0,1),\ g(x) = \frac{(x+1)}{3}$.

6. REVIEW: Inverses and Sets

Let A, B be sets, let $f : A \to B$ be a function, and let S and T be subsets of B. Prove that

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T).$$

Note: The notation $f^{-1}(S)$ is a shorthand for the set

$$f^{-1}(S) = \{x \in A \mid f(x) \in S\}.$$

Solution: We need to prove two things in order to show set equality: $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

Part 1: Show that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. First, let x be an arbitrary element such that $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore, $f(x) \in S \vee f(x) \in T$. In the first case $x \in f^{-1}(S)$, and in the second case $x \in f^{-1}(T)$. In both cases, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

Part 2: Show that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$. First, let x be an arbitrary element such that $x \in f^{-1}(S) \cup f^{-1}(T)$. Therefore, $x \in f^{-1}(S) \vee x \in f^{-1}(T)$, so $f(x) \in S \vee f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$ as desired.

Because we have shown both subset relations, we can conclude that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

7. Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$

(c) Find an explicit one-to-one and onto mapping from the closed interval $(0, 1)$ to the open interval $[0, 1]$.

Solution:

(a) $\{\frac{1}{n}\}$ where n is an integer greater or equal to 2.

(b) Map $\frac{1}{2}$ to 0, $\frac{1}{3}$ to 1, and for all $n > 3$, map $\frac{1}{n}$ to $\frac{1}{n-2}$

(c) Map every element to itself, except those in A. Map those in A according to the mapping came up with in part b.

8. Pigeonhole Principle Warm up ⋆

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution:

(a) $(4 \cdot 21) + 1 = 85$

(b) $(50 \cdot 99) + 1 = 4951$

9. PHP $\star$

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

Solution:

(a) 10. Consider colors as boxes, and candies as pigeons. By pigeonhole principle, we have $\lceil \frac{N}{3} \rceil = 4$ where N is the number of pieces we have to grab to make this work. The smallest number N that works is 10.

(b) 33. This is not actually pigeonhole. We specifically need to have 3 orange candies. The only way to make sure this happens is to grab all 15 red, 15 blue, and then the next 3 we grab have to be orange.

10. Pigeonhole Principle $\star$

How many distinct numbers must be selected from the set

$$\{1, 3, 5, 7, 9, 11, 13, 15\}$$

to guarantee that at least one pair of selected numbers add up to 16?

Solution: We can group these into pairs that add up to 16 of: $(1, 15), (3, 13), (5, 11), (7, 9)$. Notably, no other pair of numbers sum to 16. Therefore, we must pick at least 5 distinct numbers to guarantee that we pick both numbers from at least one of these pairs.

11. Pigeonhole Practice

How many integers do we need to select to guarantee that, for two distinct welected integers x, y, the difference $x - y$ is divisible by 10?

Solution: Let our pigeon holes be (integers equivalent to 0 mod 10), (integers equivalent to 1 mod 10)...(integers equivalent to 9 mod 10). Therefore, we have 10 pigeon holes. After selecting 11 integers, there are more pigeons than pigeon holes, so two of the selected integers are guaranteed to be equivalent mod 10. The difference of these two integers must be divisible by 10.

12. Sphere

Prove that a sphere with any 5 points along its surface can be split into two equal hemispheres such that one of the hemispheres contains at least 4 points. Points that lie on the dividing line between the two hemispheres may be counted in both hemispheres.

Solution: Pick any two of the 5 points and connect them. Divide the sphere into hemispheres along the edge of the two connected points (extend the connected edge all the way around the sphere to form a circle). Since points on top of the split can be counted in both hemispheres, we have two hemispheres which both have two points. Now, we are left with the three remaining points and two hemispheres, so by the pigeon hole principle, at least two of the three additional points must be on the same hemisphere. Therefore, one of the hemispheres must have at least 4 of the 5 points (2 from first two picked, and another 2 from the remaining 3).

13. REVIEW: Pigeonhole Principle

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution: Let $K(x)$ be the number of other computers that computer x is connected to. The possible values for $K(x)$ are 1, 2, 3, 4, 5. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.