TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x\forall y P(x, y)$ $\forall y\forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x\exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x\forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x\exists y P(x, y)$ $\exists y\exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Proposition - declarative statement that is either true or false
Tautology - compound proposition that is always true
Contradiction - compound proposition that is always false
Satisfiable/Consistent - some assignment of truth values that make the compound proposition true

Proofs

Always announce your proof style

To prove a true **for-all statement:**
- Let x be an **arbitrary** integer
- (Prove that x has the named property)

To prove a true **there-exists statement:**
- Name a **specific** integer:
  - "Consider $x = 2$..."
- (Prove that this x has the named property)

To **disprove** a false for-all statement:
- Name a **specific** integer
  - "Consider $x = -1$..."
- (Prove that x does not have the named property)

To **disprove** a false there exists statement:
- Let x be an **arbitrary** integer
- (Prove that x does not have the named property)

Original statement: "If *[prop 1]*, then *[prop 2]*"
Contrapositive: "If *[not prop 2]*, then *[not prop 1]*"

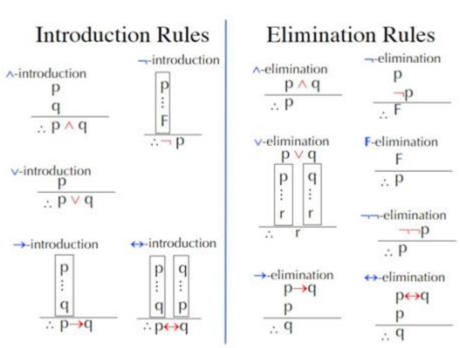
Contradiction: Assume negation and prove false
Common Mistake: Negation of "p implies q" is "p and not q" (you can use logical equivalences to prove it! See table 7 on slide 7)

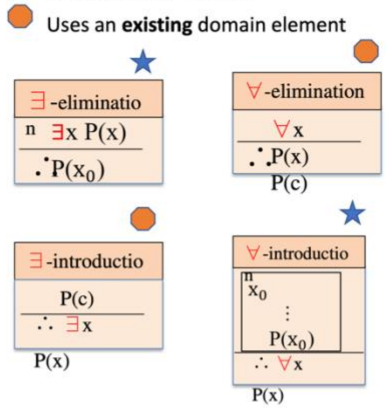
Cases: split into **cases**, where you assume various things about **x**.
Make sure **at least one** of our cases is true for any x!
Prove the proposition in **every** possible case.

And always have a closing statement concluding the proof

Natural deduction rules

★ Creates a **new** variable
⬢ Uses an **existing** domain element





- **Standard Numerical Sets**
  - **N** = {0,1,2,3,…} *natural numbers*
  - **Z** = {…,-2,-1,0,1,2,3,…} *integers*
  - **Z⁺** = {1,2,3,…} *positive integers*
  - **Q** = {x | ∃a∈**Z** ∃b∈**Z⁺** x = a/b} *rationals*
  - **R** *reals*
  - **R⁺** *positive reals*
  - Intervals: [a,b], (a,b], [a,b), (a,b)
  - e.g. [a,b) = {x∈R | a ≤ x < b}

"The set of all real x" "such that" "x is at least a and less than b"

Recall: a ≡ b (mod m) means a = b + km for some integer k (and assuming m is a positive integer)
Suppose a ≡ b (mod m) and c ≡ d (mod m).
Claim: a+c ≡ b+d (mod m) ***(Addition works!)***
Claim: a–c ≡ b–d (mod m) ***(Subtraction works!)***
Claim: ac ≡ bd (mod m) ***(Multiplication works!)***
Some proofs. Let a = b + km and c = d + jm.
So a+c = b+km+d+jm = (b+d) + (k+j)m ≡ b+d (mod m)
So ac = (b+km)(d+jm) = bd + (bj+dk+kjm)m ≡ bd (mod m)

If p is prime, then for any positive int a<p, there exists a unique positive a^{-1}<p s.t. $aa^{-1} \equiv 1 \ (mod\ p)$

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Prove that A ⊆ B

a. assume an arbitrary element x in A
b. show that this element must also be in B

Compound Proposition	Expression in English
$\neg p$	"It is not the case that p"
$p \wedge q$	"Both p and q"
$p \vee q$	"p or q (or both)"
$p \oplus q$	"p or q (but not both)"
$p \rightarrow q$	"if p then q" "p implies q"
$p \leftrightarrow q$	"p if and only if q"

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$