1. Recurrence
2. Recurrence Relations
3. Initial Conditions

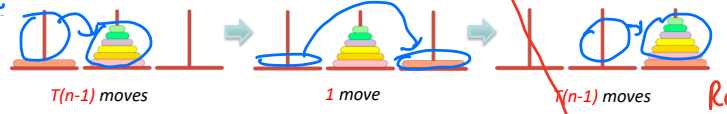
10/5/23

L12: Induction & Recurrence Relations -- ANSWERS

- **Recurrence Relations**
  - $T(n)$ is defined in terms of previous values, i.e., $T(k), k < n$
  - To fully define the recurrence, need initial value(s) of T
  - Recursion = induction

***Example*: Towers of Hanoi**

How many moves are required to solve the Tower of Hanoi puzzle for n disks?



Define $T(n)$ = number of moves when there are n disks.

$T(n) = T(n-1) + 1 + T(n-1)$
$= 2T(n-1) + 1$

- Initial condition/base case: $T(0) = 0$

$T(n-1) = 2T(n-2)+1$
$T(n-2) = 2T(n-3)+1$
$\vdots$
$T(0) = 0$

$T(1)=1,\ T(2)=3,\ T(3)=7,\ T(4)=15$
$T(n) = 2^n - 1$

Explanation
实际上整个过程即3步：
1. 把n-1个移到II
2. 把1个移到III
3. 把n-1个移到III
1和3作为子步骤，有同逻辑的下级子步骤，在子步骤中重复算出即可.

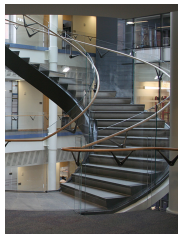
$T(n-2) \xrightarrow{+1} T(n-1) \xrightarrow{+1} T(n)$

Recursive Algorithm for ToH in Pseudocode
ToH(n, A, B, C):
if n=0, stop.
ToH(n-1, A, C, B) → T(n-1) moves
Move disk n from A to C → 1 move
ToH(n-1, B, A, C) → T(n-1) moves

(1 A 1 B 1 C)

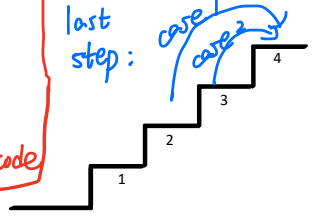
Example: Climbing Stairs

How many ways are there to climb n stairs if you can take one stair or two stairs at a time?



Example: $n = 4$

last step: case 1, case 2



Ways to climb 4 stairs:
if you can take one stair or two stairs at a time

- 1,2,3,4
- 2,4
- 1,2,4
- 2,3,4
- 1,3,4

5 ways

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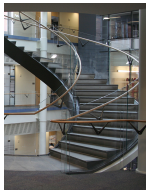
Example: Climbing Stairs

How many ways are there to climb n stairs if you can take one stair or two stairs at a time?



Approach #2: Consider the different ways you could make your first step from the bottom

Let s_n be the number of ways to climb n stairs

- Two disjoint cases, based on what you do for your *first step*
- Case 1: Start by climbing one stair
  - #ways to do this = #ways to climb the remaining $n-1$ stairs — s_{n-1} ways
- Case 2: Start by climbing two stairs at once
  - #ways to do this = #ways to climb the remaining $n-2$ stairs — s_{n-2} ways
- total # of ways = #ways with case 1 + #ways with case 2

$s_n = s_{n-1} + s_{n-2}$

With initial conditions $s_0 = 1$ and $s_1 = 1$

Need 2 initial conditions since recurrence goes back to $n-2$:
- 1 way to climb 0 stairs
- 1 way to climb 1 stair

Another approach (solution in lecture slides): Consider the different ways you could **arrive at the n-th stair**

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$S_n = \begin{cases} S_{n-1} = \begin{cases} S_{n-2} \\ + \\ S_{n-3} \\ \vdots \end{cases} \\ + \\ S_{n-2} = \begin{cases} S_{n-3} \\ + \\ S_{n-4} \\ \vdots \end{cases} \end{cases}$

$S_2 = S_1 + S_0$

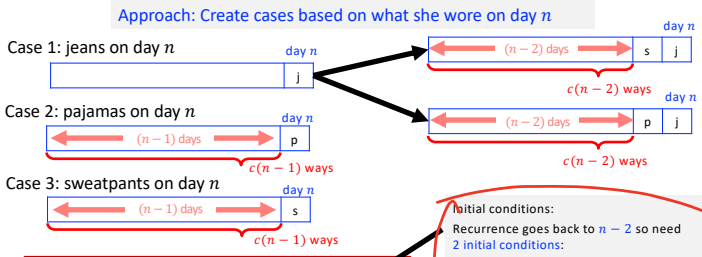
$S_1 = 1,\ S_0 = 0$

Example: Katie's Pandemic Outfits

Katie has three outfit choices while social distancing. Each day she wears either pajamas, sweatpants, or jeans. Her only rule is that she never wears jeans on two or more days in a row.

Let $c(n)$ represent the number of ways Katie can choose outfits across n days, $n \geq 0$. Find a recurrence relation for $c(n)$ including the initial conditions

Approach: Create cases based on what she wore on day n



Sum counts across all cases to get:
$c(n) = 2c(n-1) + 2c(n-2)$
With initial conditions:
$c(0) = 1, \quad c(1) = 3$

Initial conditions:
Recurrence goes back to $n-2$ so need 2 initial conditions:
- Easiest to do 0 days and 1 day (but could do 1 day and 2 days)
- $c(0) = 1$ (1 way to choose over 0 days, i.e., **1 way to do nothing**)
- $c(1) = 3$ (3 ways to choose over 1 day)

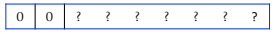
$C(n) = \begin{cases} \text{Case 1: } C(n-2) + C(n-2) = 2C(n-2) \\ \text{Case 2: } C(n-1) = C(n-1) \\ \text{Case 3: } C(n-1) = C(n-1) \end{cases}$ ……

Example: Bitstrings with 00 (Solution 1)

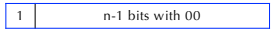
- How many bitstrings (strings of 0s and 1s) of length n contain "00" somewhere? Find a recurrence $b(n)$ for this value.

Approach: Consider how the bitstring starts

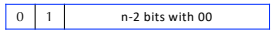
- Case 1: start with 00
  - We have our 00! We can just fill the rest with any bits
    - 2^{n-2}



- Case 2: has 00 later (BUT not at the start)
  - Case 2a: starts with 1 (thus cannot start with 00)
    - $b(n-1)$



  - Case 2b: starts with 0
    - In order to not start with 00, the second bit must be 1
    - $b(n-2)$



- These 3 cases count everything and don't overlap!
  - $b(n) = 2^{n-2} + b(n-1) + b(n-2)$
  - $b(0) = 0,\ b(1) = 0$
    - (Could instead use $b(1) = 0, b(2) = 1$ as initial conditions, if you prefer)
  - If you looked at solution 2 (in the lecture slides), this looks very different, but in fact, it generates the same sequence!

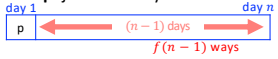
$b(n) = \begin{cases} (\text{Case 1: } 00\ldots) & 2^{n-2} \\ (\text{Case 2: } 01\ldots) & + b(n-2) \\ (\text{Case 3: } 1\ldots) & + b(n-1) \end{cases}$

Additional Exercises - Solutions

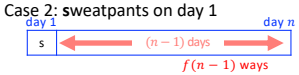
2. Suppose Katie chooses each day between **4 pant types**: pajamas, sweatpants, jeans, and dress pants. She **never wears jeans 3 days in a row**. Find a recurrence for $f(n)$, the number of ways she can choose outfits over n days.

Approach 1: Create cases based on what she wore on **day n**
Approach 2: Create cases based on what she wore on **day 1**
Either approach will work here; we'll use the "day 1" approach to get some practice with it.

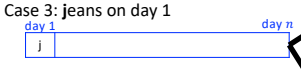
Case 1: **p**ajamas on day 1



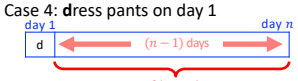
Case 2: **s**weatpants on day 1



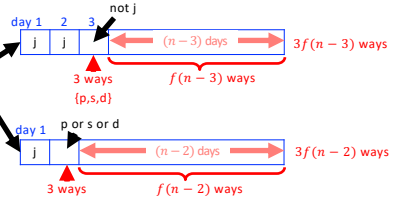
Case 3: **j**eans on day 1



Case 4: **d**ress pants on day 1



Sum across all cases to get:
$f(n) = 3f(n-1) + 3f(n-2) + 3f(n-3)$
With initial conditions:
$f(0) = 1,\ f(1) = 4,\ f(2) = 16$



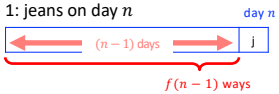
All same logic

Additional Exercises - Solutions

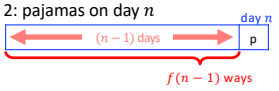
1. Suppose Katie chooses each day between **3 pant types**: pajamas, sweatpants, and jeans. Her only restriction is that **she can only wear sweatpants if she wore pajamas the day before.** Find a recurrence for $f(n)$, the number of ways she can choose outfits over n days.

Approach: Create cases based on what she wore on day n

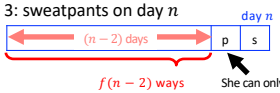
Case 1: jeans on day n



Case 2: pajamas on day n



Case 3: sweatpants on day n



She can only wear sweatpants on day n if she wore pajamas on day $n-1$

Sum counts across all cases to get:
$f(n) = 2f(n-1) + f(n-2)$
With initial conditions:
$f(0) = 1, \quad f(1) = 2$

Initial conditions:
Recurrence goes back to $n-2$ so need 2 initial conditions:
- $f(0) = 1$ (1 way to choose over 0 days, i.e., **1 way to do nothing**)
- $f(1) = 2$ noting that she cannot wear sweatpants on day 1 because it was the first day so there is no way she could wear pajamas on "the day before".

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