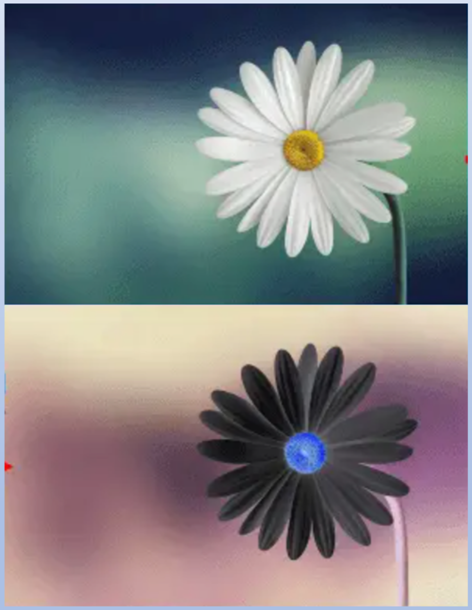
Lecture 6

Proofs 2:
Proofs by Contrapositive



Special Section

Special Discussion Sections

Focus on Fundamentals (FoF) and CSP Discussions

- 2 hours
- Smaller class size, focus on group-work
- Need to change your section enrollment!

- **021: Friday 2-4 in 1230 USB**
  - If interested, fill out the Admin form and we'll get you an override
  - Must happen before Monday's drop/add deadline

- **023: Monday 5:30-7:30 in 2150 DOW**
  - Self-enroll on Wolverine Access

Extended Discussion Sections

- 1.5 hours
- Same format as standard discussions, just meet for longer
- Open to anyone: **no need** to change your section enrollment

  - 017: F 12-1:30 in 1005 DOW
  - 018: F 3-4:30 in 185 EWRE
  - 019: M 12-1:30 in 185 EWRE
  - 020: T 3-4:30. in 2150 DOW

Reminder: Surveys

- **2/3 surveys due today!**
- Last survey (exam date confirmation) due **next Tuesday, Sep 19**
  - But you could do it today anyways, just for fun

Learning Objectives

After today's lecture (and the associated readings, discussion, & homework), you should be able to:

- **Know Technical Vocab:** Proof by contrapositive, "without loss of generality" (WLOG)
- Read and write proofs by contrapositive
- Recognize propositions for which proof by contrapositive might be helpful
- Understand when it is and isn't valid to use "without loss of generality" in a proof

Outline

- **Disproofs**
- Proofs by Contrapositive
- “Without Loss of Generality”

Disproofs

- A **proof** is a logical argument showing that a given proposition is **true**.
- A **disproof** is a logical argument showing that a given proposition is **false**.

- A disproof is **not** the same as a failed proof.
  - Attempting a proof but then getting stuck somewhere does **not** show that the statement is false! Maybe another proof would work.

To Disprove $\boldsymbol{p}$:

First state the negation $\neg p$

Then **prove** $\neg p$.

(Showing that $\neg p$ is true is the same as showing that p is false.)

Disproofs

Def: An integer x is **a multiple of m** if there is an integer $\boldsymbol{k}$ with $\boldsymbol{x = km}$.

Disprove: For all integers x,
if x is a multiple of 3, then x is a multiple of 9.

Disproof:

Disproofs

Def: An integer x is **a multiple of m** if there is an integer $\boldsymbol{k}$ with $\boldsymbol{x = km}$.

Disprove: For all integers x,
if x is a multiple of 3, then x is a multiple of 9.

Disproof:

- We will prove the negation:

  "There exists an integer x such that x is a multiple of 3 and x is not a multiple of 9."

Reminder:
The negation of "If p, then q" is "p and not q"

- **Consider** $x = 3$.
- $x = 3 \cdot 1$, so x is a multiple of 3.
- By algebra, when $x = 3$ the only number with $9k = x$ is $k = \frac{1}{3}$.
- Since $\frac{1}{3}$ is not an integer, this means that x is not a multiple of 9.
- (optional concluding sentence) We have now proved that x is a multiple of 3 and not a multiple of 9, so the negation is proved.

Disproofs on Mixed Quantifiers

(A) **Proposition:**
For all integers x, there exists an integer y such that $x^2 + y = 3$.

True

(B) **Proposition:**
There exists an integer y such that for all integers x, we have $x^2 + y = 3$.

False

Disproof:

…try it – can you disprove this?

Disproofs

Disprove:
There exists an integer y such that for all integers x, we have $x^2 + y = 3$.

Disproof:

Disproofs

Disprove:
There exists an integer y such that for all integers x, we have $x^2 + y = 3$.

Disproof:

- We will prove the negation:
  "For all integers y, there exists an integer x such that $x^2 + y \neq 3$."

- Let y be an **arbitrary** integer.
- **Consider** $x = \begin{cases} 1 & \text{if } y = 3 \\ 0 & \text{if } y \neq 3 \end{cases}$
- So $x^2 + y = \begin{cases} 1 + y = 4 \neq 3 & \text{if } y = 3 \\ 0 + y = y \neq 3 & \text{if } y \neq 3 \end{cases}$
- So $x^2 + y \neq 3$.
- (Optional concluding sentence) We proved the negation, so we disproved the original proposition.

Many different choices for x would work here.
Piecewise definitions are often helpful!

A quick look-ahead

Disprove:
There exists an integer y such that for all integers x, we have $x^2 + y = 3$.

Disproof:

- We will prove the negation:
  "For all integers y, there exists an integer x such that $x^2 + y \neq 3$."

- Let y be an **arbitrary** integer.
- **We will consider two cases**: either $y = 3$, or $y \neq 3$.

- **Case 1: Assume that $y = 3$.**
  - Then consider $x = 1$…

- **Case 2: Assume that $y \neq 3$.**
  - Then consider $x = 0$…

- In either case, $x^2 + y \neq 3$, so the proposition is proved.

This is an example of a **"proof by cases"**

We'll talk about this a lot next week!

Outline

- Disproofs
- **Proofs by Contrapositive**
- “Without Loss of Generality”

Handout

Some Proofs

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Prove: For all integers x,
if x is even, then x^2 is even.

Prove: For all integers x,
if x^2 is even, then x is even.

Proof 1

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Prove: For all integers x,
if x is even, then x^2 is even.

Proof 1

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Prove: For all integers x,
if x is even, then x^2 is even.

Proof:

- Let x be an **arbitrary** integer.
- **Assume** that x is even.

- So there is an integer k with $x = 2k$.
- So $\boldsymbol{x^2} = (2k)^2$
  $= 4k^2$
  $= \boldsymbol{2(2k^2)}$
- Since k is an integer, $2k^2$ is also an integer
- So x^2 is even.

Proof 2

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Prove: For all integers x,
if x^2 is even, then x is even.

Proof 2

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

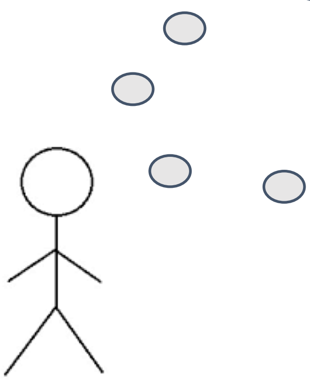
Prove: For all integers x,
if x^2 is even, then x is even.

Proof Attempt:

- Let x be an **arbitrary** integer.
- **Assume** that x^2 is even.
- So there exists an integer k with $x^2 = 2k$
- So $\boldsymbol{x} = \sqrt{2k}$
- ...
- *...then what?*
- ...
- So x is even

Contrapositives

“If I bring an umbrella, then I will stay dry.”



“If I do **not** stay dry, then I did **not** bring an umbrella.”

$$p \rightarrow q \;\equiv\; \neg q \rightarrow \neg p$$

Recall: these statements are equivalent!

They are **contrapositives** of each other.

This gives us a powerful new tool for word proofs

Proof by Contrapositive

Prove: For all integers x, if x^2 is even, then x is even.

contrapositive

Prove the contrapositive: For all integers x, if x is odd, then x^2 is odd.

This statement has the **same logical meaning** as the original, so we can prove it instead!

Axiom: an integer is not even if and only if it is odd

- **"Proof By Contrapositive:"**
  - Any proof that starts out by modifying the given proposition, replacing all or part of it with its contrapositive.
- **"Direct Proof:"**
  - A proof that does **not** use this contrapositive strategy, nor any of the other new proof styles we'll see soon.
  - Before this, all proofs we'd seen were direct proofs.

Proof by Contrapositive

Handout

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x,
if x^2 is even, then x is even.

Proof:

- Let x be ____.
- We will prove the **contrapositive:** "____."
- **Assume** ______________________________

Proof by Contrapositive

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x,
if x^2 is even, then x is even.

Proof:

- Let x be an **arbitrary** integer.
- We will prove the **contrapositive**:

  "If x is odd, then x^2 is odd."

  We actually proved this last lecture, but we'll recap:

- **Assume** that x is odd.
- So there is an integer k with $x = 2k + 1$.
- So $\boldsymbol{x^2} = (2k+1)^2$
  $= 4k^2 + 4k + 1$
  $= \boldsymbol{2(2k^2 + 2k) + 1}$
- Since k is an integer, $2k^2 + 2k$ is an integer.
- So x^2 is odd.

Whenever you use a **"proof style"**
(contrapositive, or others that we will talk about soon)
you should always:

1. Announce it, and
2. Write down any logically modified statements that you will consider.

We're using this box marks the "main" part of the proof, where we show the if-then.
(it's just a visual guide – totally optional in your proofs)

Handout

Template: Proof by Contrapositive

Claim: If p, then q

Proof Template

We will prove the contrapositive: [state the contrapositive]
Assume not(q).

… (make some deductions) …

Therefore, not(p).

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Another Contrapositive

Prove: For all integers x, if $x^2 + 6x + 5$ is even, then x is odd.

Things that should make you think *"maybe a proof by contrapositive will be helpful…"*

- An if-then statement
- In which the "if" part feels more complicated or harder to work with than the "then" part.

Handout

Another Contrapositive

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, if $x^2 + 6x + 5$ is even, then x is odd.

Proof:

- Let x be ____
- We will prove the **contrapositive:** "____."
- **Assume** ______________________________

Another Contrapositive

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, if $x^2 + 6x + 5$ is even, then x is odd.

Proof:

- Let x be an **arbitrary** integer.
- We will prove the **contrapositive**:

  "If x is even, then $x^2 + 6x + 5$ is odd."

- **Assume** that x is even
- So there is an integer k with $x = 2k$
- So $\boldsymbol{x^2 + 6x + 5} = (2k)^2 + 6(2k) + 5$
  $= 4k^2 + 12k + 5$
  $= \boldsymbol{2(2k^2 + 6k + 2) + 1}$
- Since k is an integer, $2k^2 + 6k + 2$ is an integer
- So $x^2 + 6x + 5$ is odd.

Handout

You Try It

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, if $5x^2 + 4$ is even, then x is even.

Prove: For all integers x, y, if xy is even, then x is even or y is even.

You Try It

Handout

Prove: For all integers x,
if $5x^2 + 4$ is even, then x is even.

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

You Try It

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x,
if $5x^2 + 4$ is even, then x is even.

You Try It

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x,
if $5x^2 + 4$ is even, then x is even.

Proof:

- Let x be an **arbitrary** integer.
- We prove the **contrapositive**:
  "If x is odd, then $5x^2 + 4$ is odd."

- **Assume** that x is odd.
- So there is an integer k with $x = 2k + 1$.
- So $\mathbf{5x^2 + 4} = 5(2k + 1)^2 + 4$
  $= 5(4k^2 + 4k + 1) + 4$
  $= 20k^2 + 20k + 9$
  $\mathbf{= 2(10k^2 + 10k + 4) + 1}$
- Since k is an integer, $10k^2 + 10k + 4$ is an integer.
- Therefore $5x^2 + 4$ is odd.

You Try It

Handout

Prove: For all integers x, y,
if xy is even, then x is even or y is even.

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

You Try It

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, y, if xy is even, then x is even or y is even.

You Try It

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, y, if xy is even, then x is even or y is even.

Proof:

- Let x, y be **arbitrary** integers.
- We prove the **contrapositive**:

  "If x is odd and y is odd, then xy is odd."

- **Assume** that x is odd and y is odd.
- So there are integers j, k with $x = 2j + 1$ and $y = 2k + 1$.
- So $\boldsymbol{xy} = (2j + 1)(2k + 1)$

  $= 4jk + 2j + 2k + 1$

  $= \boldsymbol{2(2jk + j + k) + 1}$
- Since j, k are integers, $2jk + j + k$ is an integer.
- Therefore xy is odd.

Outline

- Disproofs
- Proofs by Contrapositive
- **“Without Loss of Generality”**

A Fork in the Road

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, y, if $x + y$ is even, then x, y have the same parity
(meaning both are even or both are odd)

A Fork in the Road

Handout

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

Prove: For all integers x, y, if $x + y$ is even, then x, y have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be ___________________________________
- We will prove the **contrapositive**: ____
- **Assume** that ____
- **Assume without loss of generality (WLOG)** that ___________________________________

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

A Fork in the Road

Prove: For all integers x, y, if $x + y$ is even, then x, y have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be **arbitrary** integers.
- We will prove the **contrapositive**:

"If x, y have different parities (one is even and the other is odd), **then $x + y$ is odd."**

- **Assume** that x, y have different parities
- *… then what?*
  - *Next step is to apply even/odd defs to x, y*
  - *But we don't know **which** of x, y is even and which is odd.*
  - *Does it really matter which is which? It's essentially the same proof either way …*

"Without Loss of Generality"

In proofs, you can use **"Assume without loss of generality…"** or **"Assume WLOG…"** when:

- There are several possibilities about the state of the world
  - E.g. (x is even and y is odd) or (x is odd and y is even)
- But these possibilities are completely **symmetric**, and the proof would look essentially the same under one possibility as the other.
  - E.g. x, y are both arbitrary integers and we have assumed nothing else about them
  - So we might as well say x is the even one and y is the odd one.
- So we can write "**Assume WLOG** that [one of the two possibilities holds]."

Be careful with WLOG! Don't assume things unless you are sure that there really is symmetry.

You will never **have** to use WLOG – it's just a time-saving tool. The alternative is to consider each possibility separately, and repeat the proof in each case.
(We'll talk more about this alternative next week.)

Def: Int x is **"even"** if there exists an int k such that $\boldsymbol{x = 2k}$.

Def: Int x is **"odd"** if there exists an int k such that $\boldsymbol{x = 2k + 1}$.

A Fork in the Road

Prove: For all integers x, y, if $x + y$ is even, then x, y have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be **arbitrary** integers.
- We will prove the **contrapositive**:

"If x, y have different parities (one is even and the other is odd), **then $x + y$ is odd."**

- **Assume** that x, y have different parities.
- **Assume without loss of generality (WLOG)** that x is even and y is odd.
- So there are integers j, k with $x = 2j$ and $y = 2k + 1$.
- So $\boldsymbol{x + y} = 2j + 2k + 1$
  $= \boldsymbol{2(j + k) + 1}$
- Since j, k are integers, $j + k$ is an integer
- So $x + y$ is odd.

A Fork in the Road

Prove: For all integers x, y, if $x + y$ is even, then $x, 3y$ have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be **arbitrary** integers.
- We will prove the **contrapositive**:

"If $x, 3y$ have different parities (one is even and the other is odd), **then $x + y$ is odd."**

- **Assume** that $x, 3y$ have different parities.

We **can't** continue using WLOG here: $x, 3y$ are not symmetric!
(One is any integer, the other is any multiple of 3)

A more involved proof strategy would be needed.

Def: We say that **5 divides** an int x if there exists an int k such that $\boldsymbol{x = 5k}$.

You Try It

Prove: For all integers x, y, if 5 does not divide xy, then 5 does not divide x and 5 does not divide y.

Def: We say that **5 divides** an int x if there exists an int k such that $\boldsymbol{x = 5k}$.

You Try It

Prove: For all integers x, y, if 5 does not divide xy, then 5 does not divide x and 5 does not divide y.

Proofs:

- Let x, y be **arbitrary** integers.
- We will prove the **contrapositive:**

"If 5 divides x or 5 divides y, then 5 divides xy."

- **Assume** that 5 divides x or 5 divides y.
- **Assume without loss of generality** that 5 divides x.
  - (x, y are completely symmetric: might as well call x the one that's divisible by 5.)
- So there is an integer k with $x = 5k$.
- So $\boldsymbol{xy = 5ky}$.
- Since k, y are integers, ky is an integer.
- So 5 divides xy.

Wrapup

- **Proofs by Contrapositive** are a connection between proofs and logical equivalences:
    - Modify the proposition using logical equivalences to make it easier to prove
    - Then prove it

- This strategy is much broader than just contrapositive!
    - Apply implication breakout to an **“if-then”** or **“or”** proposition before proving it
    - Next week: **proofs by contradiction** and **proofs by cases**