# EECS 203: Discrete Mathematics
# Fall 2023
# Homework 8

## Due **Thursday, November 2**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total points: $100 + 42$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

# Individual Portion

## 1. Why You Got a 12-Car Garage? [8 points]

Ashu recently acquired three 12-car garages, but he has no cars (yet).

(a) What is the minimum number of cars Ashu has to acquire in order to guarantee that at least one of the garages will have **more** than 6 cars in it? Justify your answer, including an explanation of why it is the minimum number.

(b) If the garages are all adjacent to one another, what is the minimum number of cars Ashu has to acquire in order to guarantee that the middle garage has more than 6 cars in it?

**Solution:**

We can use pigeonhole to solve this problem.

(a) There are three holes and we need to find how many pigeons are needed to guarantee at least one garage will have 7 cars in it. So we need to find the smallest number of cars $x$ such that $\lceil \frac{x}{3} \rceil = 7$. The smallest possible $x$ would be 1 more than $\frac{x_0}{3} = 6$. Since $x_0 = 18$, Ashu would need 19 cars.

**Note:** The capacity of the garage was not needed for this part.

(b) In order to guarantee that the middle garage must have more than 6 cars, we must have that the other two garages are full. Therefore, Ashu must have a total of $12 + 12 + 7 = 31$ cars.

**Draft Grading Guidelines [8 points]**

**Part a:**

+1.5 correct numerical answer
+1.5 valid use of pigeonhole principle
+1.5 correctly justifies that answer is the minimum by discussing solutions to the equation $\lceil \frac{x}{3} \rceil = 7$

**Part b:**

+1.5 recognizes that to guarantee the middle garage having at least six cars, the other garages must be full
+2 correct numerical answer

## 2. Sum More Counting [14 points]

Consider the set of integers between 1 and 18, inclusive. What is the smallest integer $n$ such that, for any subset $S \subseteq \{1, 2, \dots, 18\}$ of size $|S| = n$, there are **distinct** integers $x, y \in S$ with $x + y = 18$? Prove that your answer is sufficient to guarantee this, and the minimum necessary number.

**Note:** To prove that your choice of $n$ is smallest, you must also give an example of a set of size $|S| = n - 1$ that does not contain $x, y \in S$ with $x + y = 18$.

**Solution:**

The minimum number we must select is 11.

**Sufficient:** We can prove this with the Pigeonhole Principle. Our strategy is to make the pigeons the numbers we select, and the holes a partition of the set we're choosing from. Ideally, if two pigeons end up in the same hole, their sum will be 18, since that would line up with the Pigeonhole Principle and our goal for the proof.

We'll start making our holes by grouping the numbers into subsets of size 2, which is the same as pairs of unique integers, that add up to 18.

$$\{1, 17\}, \{2, 16\}, \{3, 15\}, \{4, 14\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{8, 10\}$$

We don't include $\{9, 9\}$ because $|\{9, 9\}| = 1$. Alternatively, since we're picking *unique* integers, we can't pick 9 twice, so no two pigeons can ever end up in that hole. In addition to 9, there's no match for 18. However, we want every pigeon to go into exactly one hole, so both need to go into some set. We want them in their own sets so that no two pigeons can be in those holes. Thus, if two pigeons do end up in the same hole, they must add up to 18. So now, we construct our holes as follows:

$$\{1, 17\}, \{2, 16\}, \{3, 15\}, \{4, 14\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{8, 10\}, \{9\}, \{18\}$$

In total, there are 10 holes. So, if we select 11 numbers, by the Pigeonhole Principle, one of the holes must have at least two pigeons. We'll call them $a$ and $b$. Since only the first 8 holes can possibly have more than one pigeon, $a$ and $b$ must both be distinct elements of one of those sets. We chose those sets because any two distinct elements add up to 18. Hence, $a + b = 18$, so it is sufficient to select 11 numbers.

**Minimum necessary:** Seeking contradiction, suppose we can guarantee that a pair adds up to 18 by selecting fewer than 11 numbers. In particular, that means we can do it when we select 10. Consider the following selection of 10 distinct integers from our set:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 18$$

None of the numbers below 9 can add up to 18, since they would need to be added to a number between 10 and 17 inclusive, but there are none. We can't use 9 to get to 18, since we would need another 9. We can't use 18 to get to 18, since adding anything to it will make it too big. Thus, we could not actually make our guarantee, a contradiction. Thus, we need to select at least 11 distinct integers.

**Draft Grading Guidelines [14 points]**

+4 identifies 8 pairs which add to 18
*+2 partial credit: missing a pair (if explicitly listed)*
*+2 partial credit: extra pair for* $9 + 9$
+3 identifies that we can pick 9 and 18 without using a "hole"
+4 correctly applies pigeonhole principle to get 11
+3 correctly reasons about minimal requirement

## 3. Set Sizes [12 points]

Determine which of these sets are finite, countably infinite, or uncountably infinite. Give a short (about 1 line) explanation for each part.

(a) $\{2, 3\} \times \mathbb{N}$

(b) $(0, 2) - \mathbb{Q}$

(c) $\{x \in \mathbb{R} \mid x^2 - 1 \leq 0\}$

(d) $\{x \in \mathbb{N} \mid x \leq 1000\}$

**Solution:**

(a) Countably infinite. We can list the elements in order $(2, 0)$, $(3, 0)$, $(2, 1)$, $(3, 1)$, $(2, 2)$, $(2, 3)$, etc., and map those to the natural numbers (0, 1, 2, 3, etc.).

(b) Uncountable. This is the set of irrationals between 0 and 2. To see why this is uncountable, assume it is countable. Then this set, unioned with $\mathbb{Q} \cap (0, 2)$ would give the set (0, 2). Since the union of two countable sets is countable, this implies that $(0, 2)$ is countable, but we know that $(0, 2)$ is uncountable, oops.

(c) Uncountable. The polynomial has roots $\pm 1$, and is concave up, so the set is $[-1, 1]$, which is uncountable.

(d) Finite. This is the set $\{0, 1, \ldots, 1000\}$ in which there are 1001 numbers.

**Draft Grading Guidelines [12 points]**

**For each part:**

+1.5 correct cardinality
+1.5 correct justification

## 4. Ready, set, count! [15 points]

**Definition:** $A \oplus B$ is the symmetric difference of the sets $A$ and $B$, i.e. the set containing all elements which are in $A$ or in $B$ but not in both.

Provide two **uncountable** sets $A$ and $B$ such that $A \oplus B$ is

(a) finite.

(b) countably infinite.

(c) uncountably infinite.

Include in your justification a description of the set $A \oplus B$ without reference to the symmetric difference.

**Solution:**

**Note:** There are many possible sets that satisfy the given requirements! We give (at least) one example for each.

(a) Let $A = B = \mathbb{R}$. Then $A \oplus B = \emptyset$, which is finite with cardinality 0.

As another example, we can let $A = (0, 1)$ and $B = [0, 1]$. Then $A \oplus B = \{0, 1\}$ which is finite.

(b) We can let $A = \mathbb{R}$ and $B$ be the set of real numbers that are not integers, i.e. $B = \mathbb{R} - \mathbb{Z}$. Then $A \oplus B = \mathbb{Z}$, which is countably infinite.

(c) We can let $A = \mathbb{R}^+$ and $B = \mathbb{R}^-$. Then $A \oplus B = \mathbb{R} - \{0\}$, which is uncountable.

**Draft Grading Guidelines [15 points]**

**For each part:**

+2 $A$ and $B$ are uncountable
+1 correctly identifies the set $A \oplus B$
+2 $A \oplus B$ has the correct cardinality

## 5. Corresponding Counts [18 points]

Prove that $|[0,2]| = |(3,6)|$.

For any functions that you name:

- Prove that the function is well-defined, i.e. that for any $x$ in the domain of your function $f$, $f(x)$ lies in the codomain.
- Prove any function properties that you use (e.g. one-to-one, onto, etc).

**Solution:**

This can be proved with the Schroeder-Bernstein Theorem. We need to show that $|[0,2]| \leq |(3,6)|$ and $|(3,6)| \leq |[0,2]|$.

To prove the former, we need to find a one-to-one function $f\colon [0,2] \to (3,6)$. One such function is $f(x) = x + \frac{7}{2}$.

**Example proof that $f(x)$ is well-defined:**

Since $x \in [0,2]$, $x \in \mathbb{R}$, so $f(x) = x + \frac{7}{2}$ will be in $\mathbb{R}$ as well. $0 \leq x \leq 2$, so $\frac{7}{2} \leq x + \frac{7}{2} \leq \frac{11}{2}$. Since $3 < \frac{7}{2}$ and $6 > \frac{11}{2}$, we have that $3 < f(x) < 6$.

**Example proof that $f(x)$ is one-to-one:**

Let $x, y \in [0,2]$ and assume $f(x) = f(y)$. Then

$$x + \frac{7}{2} = y + \frac{7}{2}$$
$$x = y$$

We have shown that $f(x)$ satisfies the definition of one to one, that is for all $a, b \in \operatorname{dom}(f)$, $f(a) = f(b) \to a = b$, and therefore $|[0,2]| \leq |(3,6)|$.

To prove the latter inequality, we need to find a one-to-one function $g\colon (3,6) \to [0,2]$. One such function is $g(x) = \frac{x}{3}$.

**Example proof that $g(x)$ is well-defined:**

Since $x \in (3,6)$, $x \in \mathbb{R}$, so $g(x) = \frac{x}{3}$ will be in $\mathbb{R}$ as well. $3 < x < 6$, so $1 < \frac{x}{3} < 2$, thus $0 \leq g(x) \leq 2$.

**Example proof that $g(x)$ is one-to-one:**

Let arbitrary $x, y \in (3,6)$ and assume $g(x) = g(y)$. Then

$$\frac{x}{3} = \frac{y}{3}$$
$$x = y$$

We have shown that $g(x)$ satisfies the definition of one to one, so $|(3,6)| \leq |[0,2]|$.

**Alternate method, construct $g(x)$ that is onto from $[0,2]$ to $(3,6)$:**

Let $g\colon [0,2] \to (3,6)$. $g$ is defined as follows:

$$g(x) = \begin{cases} 3x & 1 < x < 2 \\ 5 & x \leq 1 \text{ or } x = 2 \end{cases}$$

First we must show $g$ is well-defined. Let $x \in [0,2]$. If $x \leq 1$ or $x = 2$ then $g(x) = 5 \in (3,6)$. Otherwise we have $1 < x < 2$, so $3 < 3x = g(x) < 6$, so $g(x) \in (3,6)$. We prove $g$ is onto by definition. Let $y \in (3,6)$. Then $\frac{y}{3} \in (1,2) \subseteq [0,2]$, and $g(y) = 3 \cdot \frac{y}{3} = y$. So $g$ is onto.

By the above we have that $|[0,2]| \leq |(3,6)|$ and $|(3,6)| \leq |[0,2]|$. Thus, by the Schroeder-Bernstein Theorem, $|[0,2]| = |(3,6)|$.

**Draft Grading Guidelines [18 points]**

+0.5 concludes that by showing $|[0,1]| \leq |(3,6)|$ and $|[0,1]| \geq |(3,6)|$ that we can conclude that $|[0,1]| = |(3,6)|$

**For proving** $|[0,2]| \leq |(3,6)|$

+2 attempts to write a one-to-one function from $[0,2]$ to $(3,6)$ or onto function from $(3,6)$ to $[0,2]$
+1 finds a valid function meeting requirements above
+2 correctly justifies that the function is well-defined
+2 correct assumption and attempted conclusion for proving stated property
+1.75 correct algebra to prove stated property

**For proving** $|[0,2]| \geq |(3,6)|$

+2 attempts to write an onto function from $[0,2]$ to $(3,6)$ or one-to-one function from $(3,6)$ to $[0,2]$
+1 finds a valid function meeting requirements above
+2 correctly justifies that the function is well-defined
+2 correct assumption and attempted conclusion for proving stated property
+1.75 correct algebra to prove stated property

## 6. Composition Pr$\circ\circ$f [15 points]

Consider functions $g\colon A \to B$ and $f\colon B \to C$. **Prove or disprove** that if $f$ and $f \circ g$ are one-to-one, then $g$ is one-to-one.

**Solution:**

This statement is true. First, without loss of generality, let $g$ be a function from $X$ to $Y$ and $f$ a function from $Y$ to $Z$. So $f \circ g$ is a function from $X$ to $Z$. Symbolically $g\colon X \to Y$, $f\colon Y \to Z$, and $f \circ g\colon X \to Z$.

**Solution 1:**

Let's consider a proof by contradiction and assume that $f$ and $f \circ g$ are one-to-one but $g$ is not one-to-one. If $g$ is not one-to-one, there must exist some $x_1, x_2 \in X$, such that $x_1 \neq x_2$ but $g(x_1) = g(x_2)$. Both $g(x_1)$ and $g(x_2)$ exist in $Y$, so, we have that $f(g(x_1)) = f(g(x_2))$. This contradicts our original assumption that $f \circ g$ is one-to-one. Therefore, $g$ must also be one-to-one.

**Solution 2:**

We will prove that $g$ is one-to-one directly. Let $x_1, x_2 \in X$, and suppose $g(x_1) = g(x_2)$. So, $f(g(x_1)) = f(g(x_2))$, which is the same as saying $(f \circ g)(x_1) = (f \circ g)(x_2)$. Since $f \circ g$ is one-to-one, $x_1 = x_2$. Thus, $g$ is one-to-one.

Note that neither solution requires that $f$ is one-to-one.

**Draft Grading Guidelines [15 points]**

+3 chooses to prove

**Proof by Contradiction:**

+3 correct assumption for proof by contradiction
+3 uses the definition of one-to-one to find elements in the domain of $g$ such that $x_1 \neq x_2$ but $g(x_1) = g(x_2)$
+3 uses definition of one-to-one and applies it to the composed functions
+3 arrives at a contradiction and concludes that $g$ is one-to-one

**Direct Proof:**

+3 takes arbitrary $x_1$ and $x_2$ in the domain of $g$ and assumes $g(x_1) = g(x_2)$
+3 plugs both sides into $f$
+3 notes $f(g(x)) = (f \circ g)(x)$
+3 applies the definition of one-to-one to conclude $x_1 = x_2$

## 7. One Hit Wonder [18 points]

For this problem, we will define two new properties. Let $S$ be a set and $f\colon S \to S$ be some function.

We say $f$ is a *one hit wonder* if:

$$\forall x \in S\,[(f \circ f)(x) = f(x)].$$

Some examples of one-hit wonders from $\mathbb{R} \to \mathbb{R}$ are the absolute value function, the ceiling function, and the function which sends every number to 0.

We say $f$ *does nothing* if:

$$\forall x \in S\,[f(x) = x].$$

(a) Prove that if $f$ does nothing, then it is a one-hit wonder.

(b) Prove that if $f$ is a one hit wonder and is one-to-one, then $f$ does nothing.

(c) Prove that if $f$ is a one hit wonder and is onto, then $f$ does nothing.

**Solution:**

(a) Let $x \in S$ be arbitrary. Then we have:

$$(f \circ f)(x) = f(f(x)) = f(x)$$

by using the fact that $f$ does nothing on the inner $f(x)$

(b) Let $x \in S$ be arbitrary. Because $f$ is a one hit wonder, we know $f(f(x)) = f(x)$. Then because $f$ is one-to-one, we have $f(x) = x$.

(c) Let $y \in S$ be arbitrary. Because $f$ is onto, we know there is some $x \in S$ such that $f(x) = y$. Then we have:

$$\begin{aligned}
f(x) &= y \\
f(f(x)) &= f(y) && \text{(apply } f\text{)} \\
f(x) &= f(y) && (f \text{ is a one hit wonder}) \\
y &= f(y) && \text{(definition of } y\text{)}
\end{aligned}$$

**Draft Grading Guidelines [18 points]**

**Part a**:

+1 attempts to prove correct property
+2 correctly applies definition of does nothing

**Part b**:

+2 correctly applies definition of one hit wonder
+3 correctly applies definition of one-to-one

+2 reaches correct conclusion

**Part c**:

+2 correctly applies definition of onto
+2 applies $f$ again to apply definition of one hit wonder
+2 correctly applies definition of one hit wonder
+2 reaches correct conclusion

# Groupwork

## 1. Grade Groupwork 7

Using the solutions and Grading Guidelines, grade your Groupwork 7:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) | (viii) | (ix) | (x) | (xi) | Total: |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Problem 2 | | | | | | | | | | | | /16 |
| Problem 3 | | | | | | | | | | | | /14 |
| Total: | | | | | | | | | | | | /30 |

## 2. Divisibility by Seven [12 points]

In this question we will show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7. You are allowed to cross off zero digits.

For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number $2345 = 7 \cdot 335$.

We will label the digits of an arbitrary 7-digit number as

$$x_6x_5x_4x_3x_2x_1x_0.$$

(a) Prove that there exists some $i < 7$ such that either $x_ix_{i-1}\ldots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_jx_{j-1}\ldots x_0$ is congruent to it modulo 7.

(b) Use part (a) to prove that if there does not exist some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then there exists $7 > i > j \geq 0$ so that

$$\underbrace{x_i x_{i-1} \dots x_{j+1} 0 \dots 0}_{i+1 \text{ digits total}}$$

is divisible by 7.

(c) Prove the full claim. That is, show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7.

**Solution:**

(a) If we cross off digits starting on the left, we have the numbers $x_6x_5x_4x_3x_2x_1x_0$, $x_5x_4x_3x_2x_1x_0$, $\dots$, $x_1x_0$, $x_0$. This list has 7 numbers. If any are a multiple of 7, then we're done. If not, then each number will be congruent to 1,2,3,4,5, or 6 mod 7. By the pigeonhole principle, two of them will be equivalent modulo 7. Notice that the difference of two numbers that are equivalent modulo 7 is a multiple of 7.

(b) By part (a), if there does not exist some $i < 7$ such that either $x_i x_{i-1} \dots x_0$ is divisible by 7, then there exists some $j < i$ such that $x_i x_{i-1} \dots x_0$ and $x_j x_{j-1} \dots x_0$ are congruent mod 7. Let's take the difference of these two integers. This results in

$$\underbrace{x_i x_{i-1} \dots x_{j+1} 0 \dots 0}_{i+1 \text{ digits total}}$$

The difference of two numbers that are equivalent mod 7 is a multiple of 7. Therefore this difference is a multiple of 7.

(c) If there exists some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then we have found a number formed by crossing off digits at the end that is divisible by 7. Otherwise, by part (b), there exists a number formed by crossing off digits from the left and replacing $n$ of the digits on the right with 0 such that this number is divisible by 7. You can take the $n$ zeros off the right by dividing by $10^n$, leaving us with $x_i x_{i-1} \dots x_{j+1} 0 \dots 0$. Remember $x_i x_{i-1} \dots x_{j+1} 0 \dots 0$ is divisible by 7; $10^n$ can't be a multiple of 7, so the subsequence must be a multiple of 7. Formalizing this fact is beyond the scope of this problem, but this argument can be made rigorous by citing the fundamental theorem of arithmetic. Done!

**Grading Guidelines [12 points]**

**Part a:**

(i) +2 considers a 7 digit number as a sequence of variables
(ii) +2 generates a list of 7 numbers by crossing off digits starting on the left
(iii) +3 applies pigeonhole principle to conclude that two numbers in the list are equivalent modulo 7

**Part b:**

(iv) +3 states that the difference of 2 numbers that are equivalent modulo 7 from part (b) is a multiple of 7

**Part c:**

(v) +2 concludes argument by identifying a sub-sequence divisible by 7

## 3. A Powerful Proof [30 points]

In this question we will prove that for any set $X$, $|\mathcal{P}(X)| > |X|$ ($\mathcal{P}(X)$ is the power set of $X$). Note that while this is simple in the case where $X$ is finite, things get more complicated when we allow $X$ to be infinite. This proof covers all cases.

(a) Show that for all (possibly infinite) sets $X$, $|\mathcal{P}(X)| \geq |X|$.

(b) Let $g \colon X \to P(X)$ be an arbitrary function. Show that the set $D := \{a \in X \mid a \notin g(a)\}$ is not in the range of $g$.

(c) Explain why this shows that $|P(X)| \leq |X|$ is false and conclude the proof.

(d) Based on your conclusions above, are there uncountable sets "larger" than $\mathbb{R}$? Explain.

**Solution:**

(a) One way to do this is to construct a one-to-one function from $X$ to $\mathcal{P}(X)$. Consider the function $f \colon X \to \mathcal{P}(X)$, $f(x) = \{x\}$. Then for distinct $a, b \in X$, we have $f(a) = \{a\} \neq \{b\} = f(b)$, so $f$ is one-to-one. Thus $|\mathcal{P}(X)| \geq |X|$.

(b) Seeking contradiction assume $D$ is in the range of $g$. Then there exists $d \in X$ such that $g(d) = D$. Now we consider two cases:

(i) If $d \in D$, then this means $d \in g(d)$ as $g(d) = D$. But this is a contradiction, because by the definition of $D$ if $d \in g(d)$ then $d \notin D$.

(ii) If $d \notin D$, then $d \notin g(d)$. But again by the definition of $D$ this implies $d \in D$.

Since in each case we reach a contradiction, this completes the proof.

(c) This shows that there does not exist an onto function from $X$ to $\mathcal{P}(X)$. Thus by definition $|\mathcal{P}(X)| \not\leq |X|$ which implies $|\mathcal{P}(X)| > |X|$ by (a).

(d) Yes. The above means that $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \cdots$ so we have infinitely many uncountable sets with a greater cardinality than $\mathbb{R}$.

**Grading Guidelines [30 points]**

**Part a:**

(i) +5 attempts to construct a suitable function
(ii) +5 correct proof

**Part b:**

(iii) +3 assumes there exists $d \in X$ such that $g(d) = D$
(iv) +3 breaks into cases based on whether or not $d \in D$
(v) +4 correct case work

**Part c:**

(vi) +5 correct conclusion

**Part d:**

(vii) +5 correct answer with explanation