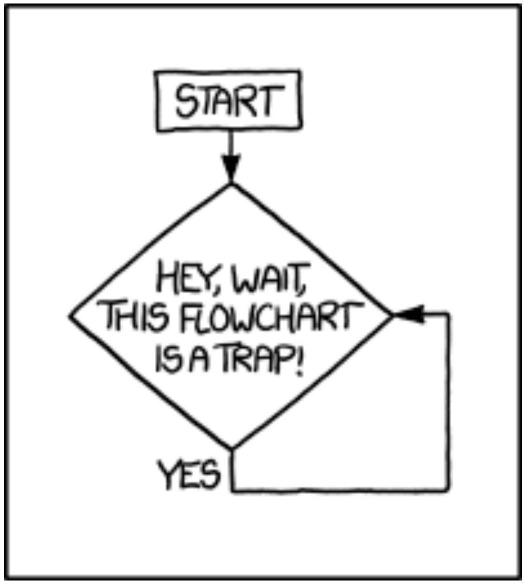
Lecture 4
Predicates & Quantifiers



Today!

M | COMPUTER SCIENCE & ENGINEERING

INTRO COURSES STUDENT MIXER

Thursday, September 7, 4-6PM
Beyster Building Foyer

Students in EECS 110, 183, 203, 280

Don't know anyone in your class? Looking to meet your instructor? Stop by our Intro Courses Student Mixer!

Event will include wellness stations, light refreshments, and an informal atmosphere to meet your classmates and instructors

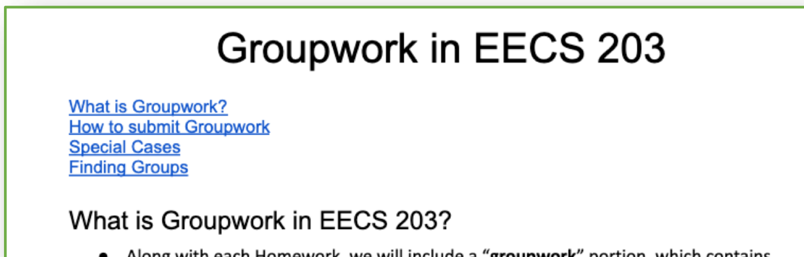
SCAN ME

Announcements

- Reminder: HW1 is due **TONIGHT** at 10 pm
  - Make sure to **MATCH PAGES** when you upload to Gradescope
  - Groupwork: Make sure that tag your groupmates / have been tagged by your groupmate
- Do the 3 surveys – they're part of your grade ("free" points):
  - FCI Beginning of the Term Survey Due Thu 9/14
  - BBCS Entry Survey Due Thu 9/14
  - Exam Date Confirmation Form Due Tue 9/19
  - These are for a (completion) grade, so make sure to fill them out

Groupwork

- "All About Groupwork" doc on Canvas
    - Should answer many of your questions



Groupwork in EECS 203

What is Groupwork?
How to submit Groupwork
Special Cases
Finding Groups

What is Groupwork in EECS 203?

- Along with each Homework, we will include a "**groupwork**" portion, which contains

- **If you change groups**
    - For the first groupwork with your new group, you will need to submit individually for yourself
        - Grade *your own* previous Groupwork (from your old group)
        - Submit your new group's solutions to the new Groupwork questions
    - After the first Groupwork with your new group
        - You can submit a single assignment with the rest of your new group, since you'll be grading the same "previous work"

Sample Groupwork Grading

- Starting with Homework 2, part of your assignment is for your group to grade your previous groupwork
- The groupwork grading "question" will ask you to fill out a table summarizing **your grading of yourself**
- Here's an example grading hw0:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	Total:
Exponents	+0.5	+0	+0.5	+0	+0.5	+0.5	+0.5		2.5/3.5
Logarithms	+0.5	+0.5	+0	+0	+0.5	+0.5	+0.5	+0.5	3/4
Equations 1	+1	+0							1/2
Equations 2	+1	+1							2/2
Total:									8.5/11.5

Groupwork scores

We grade *your grading* for effort. If you've done an honest job grading your previous work, you should get 100%.

- You can get +1% extra credit on the assignment by writing positive message(s) to yourself. E.g.:
  - "Good effort!"
  - "I can do this"
  - "I learned a lot"
  - "It's ok that I scored low, I just need more practice"

I put in effort to this question, and it paid off, I have to review some exponent rules, but that doesn't discount the ones I got correct.

Learning Objectives

After today’s lecture (and the associated readings, discussion, & homework), you should be able to:

- **Know Technical Vocab:** predicate, quantifier, “there exists", “for all", nested quantifiers, domain [of discourse], variable scope
- Know logic symbols for "there exists" and "for all"
- Translate predicates from English to logic and logic to English
  - Correctly interpret variable scope
  - Correctly interpret domain restrictions in for-all/there-exists statements
- Determine the truth value of quantified statements
- Understand when you can and cannot switch the order of nested quantifiers while preserving meaning

Outline

- **Predicates**
  - **Recap of Quantifiers: For all, There exists**
  - Nested Quantifiers
  - Translation Practice
- DeMorgan's for Quantifiers
- Domain Restrictions
- Quantifier Scoping

Predicates & Quantifiers Recap

<u>**Universal Quantifier** (aka “for all”)</u>

- $\forall$ means “for all”, “for each”, “for every”
- Like a big chain of ANDs
  $\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots$

Read as “For all x, $P(x)$”

<u>**Existential Quantifier** (aka “there exists”)</u>

- $\exists$ means “there exists”
- Like a big chain of ORs
  $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots$

Read as “There exists an x such that $P(x)$”

Specify the domain of x.
Domain = set of possible values

Ex 1: $\boldsymbol{P(x)}$ = **“$\boldsymbol{x}$ likes Bubble Tea”**
- $\boldsymbol{\forall x\, P(x)}$ [domain = people]
- means “Everyone likes Bubble Tea”

Ex 2: $\boldsymbol{Q(x)}$ = **“$\boldsymbol{x^2 > 2}$”**
- $\boldsymbol{\forall x\, Q(x)}$ [domain = integers]
- means “For all integers x, $\boldsymbol{x^2 > 2}$”
  (false proposition)

Ex 1: $\boldsymbol{P(x)}$ = **“$\boldsymbol{x}$ likes Bubble Tea”**
- $\boldsymbol{\exists x\, P(x)}$ [domain = people]
- means “Someone likes Bubble Tea”

Ex 2: $\boldsymbol{Q(x)}$ = **“$\boldsymbol{x^2 > 2}$”**
- $\boldsymbol{\exists x\, Q(x)}$ [domain = integers]
- means “There exists an integer x, such that $\boldsymbol{x^2 > 2}$” (true proposition)

Truth values of Quantified statements

Example:
P(x) = "x likes broccoli"

- When is $\forall x\, P(x)$ true? When is it false?
- What about $\exists x\, P(x)$?

	True iff …	**False** iff …
$\forall x\, P(x)$		
$\exists x\, P(x)$		

Truth values of Quantified statements

Example:
P(x) = “x likes broccoli”

- When is $\forall x\, P(x)$ true? When is it false?
- What about $\exists x\, P(x)$?

	True iff …	**False** iff …
$\forall x\, P(x)$	P(x) is true for *every* x in the domain	P(x) is false for *at least one* x in the domain
$\exists x\, P(x)$	P(x) is true for *at least one* x in the domain	P(x) is false for *every* x in the domain

Handout

Predicates & Quantifiers

A **predicate** is a statement with ______________________________

Ex: $P(x)$ = "x likes Bubble Tea", $Q(x)$ = "$2x = 7$"

Quantifiers:

$\forall$ means ____________________

(like a big chain of ___________)

$\exists$ means ____________________

(like a big chain of ___________)

	True iff …	**False** iff …
$\forall x\ P(x)$	P(x) is __________ for ____________________ in the domain	P(x) is __________ for ____________________ in the domain
$\exists x\ P(x)$	P(x) is __________ for ____________________ in the domain	P(x) is __________ for ____________________ in the domain

Outline

- Predicates
  - Recap of Quantifiers: For all, There exists
  - **Nested Quantifiers**
  - Translation Practice
- DeMorgan's for Quantifiers
- Domain Restrictions
- Quantifier Scoping

Multivariable Predicates

- Predicates can also have **several variables**.
  - $P(x, y) = "x + y = 2"$
  - $C(x, y) = "city\ x\ is\ the\ capital\ of\ state\ y"$
  - $T(a, b, c) = "a^2 + b^2 = c^2"$

- You can apply **different quantifiers** to these variables if you want.

Nested Quantifiers

Domain of x: people in this room
Domain of y: all countries in the world

$B(x, y)$ = person x has been to country y

Translate each quantified statement:

- $\exists x\ B(x, \text{Japan})$ =
- $\exists x \forall y\ \ B(x, y)$ =
- $\exists y \forall x\ \ B(x, y)$ =

Nested Quantifiers

Domain of x: people in this room
Domain of y: all countries in the world

$B(x, y)$ = person x has been to country y

Translate each quantified statement:

- $\exists x\ B(x, \text{Japan})$ = Someone has been to Japan.
- $\exists x \forall y\ \ B(x, y)$ = Someone has been to every country, i.e., there is a person who has been to all the countries.
- $\exists y \forall x\ \ B(x, y)$ = There is a country that we've all been to.

Question: Does the order of the quantifiers matter?

Is $\exists \boldsymbol{x} \forall \boldsymbol{y}\ \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})$ equivalent to $\forall \boldsymbol{y} \exists \boldsymbol{x}\ \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})$?

Let's take a closer look through the next example

Nested Quantifiers – additional exercises

Handout

Let **P(x,y) = "4x – y = 0",** domain = integers

Write each proposition using quantifiers, then determine whether it is true or false:

Quantifiers

__________ 1. There exists **x** such that there exists **y** such that **P(x, y).** True / False

__________ 2. There exists **x** such that for all **y**, **P(x, y).** True / False

__________ 3. For all **y**, there exists **x** such that **P(x, y).** True / False

__________ 4. For all **x**, there exists **y** such that **P(x, y).** True / False

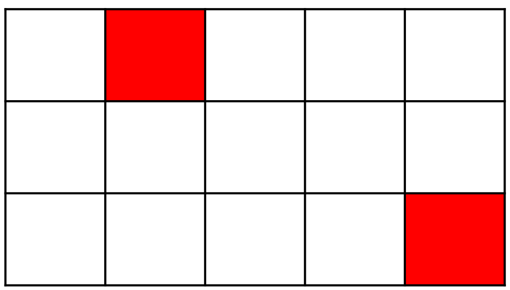
__________ 5. There exists **y** such that for all **x**, **P(x, y).** True / False

__________ 6. For all **x**, for all **y**, **P(x, y)**. True / False

Nested Quantifiers – additional exercises

Handout

P(x,y) = “the square at row **x**, column **y** is colored”



(a) P(0, 0) is false

P(0, 1) is true

P(1, 0) is ________

P(2, 4) is ________

- *Note*: we’ll start our indexing at 0, so P(0, 0) refers to the square at the top left corner of the grid

(b) What is the **domain of x**? ______________

What is the **domain of y**? ______________

(c) If $\exists \boldsymbol{x} \exists \boldsymbol{y}\, \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$ is **true**, the minimum number of shaded squares is ______, and the maximum number is _________.

Nested Quantifiers – additional exercises

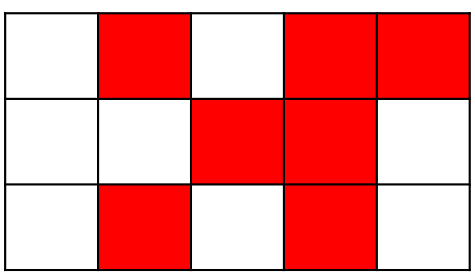
Handout

P(x,y) = “the square at row **x**, column **y** is colored”

Select the logical expression that **matches the English statement**.

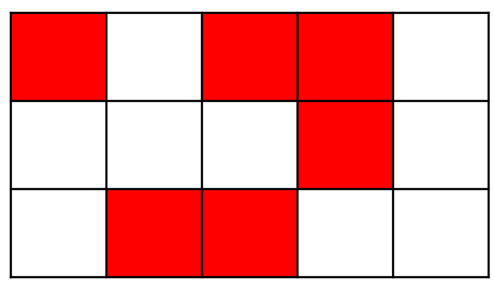
(d) *English*: **The grid has an entire column that is shaded.**

Logic: ________________



(e) *English*: **Every row has at least one shaded square.**

Logic: ________________



Answer options:

$A.\ \exists x\, \forall y\ \ P(x, y)$

$B.\ \forall y\, \exists x\ \ P(x, y)$

$C.\ \forall x\, \exists y\ \ P(x, y)$

$D.\ \exists y\, \forall x\ \ P(x, y)$

Outline

- Predicates
  - Recap of Quantifiers: For all, There exists
  - Nested Quantifiers
  - **Translation Practice**
- DeMorgan's for Quantifiers
- Domain Restrictions
- Quantifier Scoping

Translating Logic to English #1

Handout

Domain of:
m: all movies
x, y: people in this room

$V(x, m)$: "person x has seen movie m"

a) $\exists x\, \exists y\, [x \neq y\ \wedge\ \exists m\, \big(V(x,m) \wedge V(y,m)\big)]$

b) $\exists x\, \exists y\, [x \neq y\ \wedge \forall m\, \big(V(x,m) \leftrightarrow V(y,m)\big)]$

Symbol	"Main" Words
¬p	"not p"
p∧q	"p and q"
p∨q	"p or q"
p→q	"if p, then q"
∀x P(x)	"for all x, P(x)"
∃x P(x)	"there exists x such that P(x)"

Translating Logic to English #1

Handout

Domain of:
m: all movies
x, y: people in this room

$V(x,m)$: "person x has seen movie m"

a) $\exists x\, \exists y\, [x \neq y \wedge \exists m\, (V(x,m) \wedge V(y,m))]$

Two different people have seen the same movie.

b) $\exists x\, \exists y\, [x \neq y \wedge \forall m\, (V(x,m) \leftrightarrow V(y,m))]$

There are two different people who have seen the exact same set of movies.

Symbol	"Main" Words
¬p	"not p"
p∧q	"p and q"
p∨q	"p or q"
p→q	"if p, then q"
∀x P(x)	"for all x, P(x)"
∃x P(x)	"there exists x such that P(x)"

Outline

- Predicates
  - Recap of Quantifiers: For all, There exists
  - Nested Quantifiers
  - Translation Practice
- **DeMorgan's for Quantifiers**
- Domain Restrictions
- Quantifier Scoping

DeMorgan's over Quantifiers

P(x) = x goes to the park
Domain = everyone in this room

Negate the quantified statements:

Proposition 1: We will all go to the park. $\forall x\ P(x)$

Proposition 2: Someone will go to the park. $\exists x\ P(x)$

DeMorgan's over Quantifiers

P(x) = x goes to the park
Domain = everyone in this room

Negate the quantified statement:

Proposition 1: We will all go to the park. $\forall x\, P(x)$

Negation: It's not true that (we will all go to the park). $\neg \forall x\, P(x)$

simplify: At least one of us will not go to the park. $\exists x\, \neg P(x)$

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Proposition 2: Someone will go to the park. $\exists x\, P(x)$

Negation: It's not true that (someone will go to the park). $\neg \exists x\, P(x)$

simplify: None of us will go to the park. $\forall x\, \neg P(x)$

DeMorgan's over Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Connection between DeMorgan's for Quantifiers and 'regular' DeMorgan's Law:

- Recall: **"for all x"** can be interpreted as a **chain of "and"s**

P(x) = x goes to the park
Domain = everyone in this room

"It is not the case that (we will *all* go to the park)."

$$\neg \forall x\, P(x)$$

$$\equiv \neg[P(April) \wedge P(Austin) \wedge P(Karthik) \wedge \cdots \wedge P(Yongling)]$$

apply DeMorgan's law for AND...

$$\equiv \neg P(April) \vee \neg P(Austin) \vee \neg P(Karthik) \vee \cdots \vee \neg P(Yongling)$$

$$\equiv \exists x\, \neg P(x)$$

"*There exists* someone who won't go to the park."

DeMorgan's over Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Connection between DeMorgan's for Quantifiers and 'regular' DeMorgan's Law:

- Recall: **"for all x"** can be interpreted as a **chain of "and"s**
- Recall: **"there exists an x**" can be interpreted as a **chain of "or"s**

P(x) = x goes to the park
Domain = everyone in this room

"It is not the case that (*there exists* someone who will go to the park)."

$$\neg\exists x\, P(x)$$

$$\equiv \neg[P(April) \vee P(Austin) \vee P(Karthik) \vee \cdots \vee P(Yongling)]$$

apply DeMorgan's law for AND…

$$\equiv \neg P(April) \wedge \neg P(Austin) \wedge \neg P(Karthik) \wedge \cdots \wedge \neg P(Yongling)$$

$$\equiv \forall x\, \neg P(x)$$

"*Everyone* won't go to the park."

DeMorgan’s Laws (recap)

Negation on the outside		Negation on the inside
$\neg(p \wedge q)$	$\equiv$	$\neg p \vee \neg q$
$\neg(p \vee q)$	$\equiv$	$\neg p \wedge \neg q$
$\neg \forall x\, P(x)$	$\equiv$	$\exists x\, \neg P(x)$
$\neg \exists x\, P(x)$	$\equiv$	$\forall x\, \neg P(x)$

Negate both parts and flip the operator (or to and; and to or)

Push the negation inside and switch the quantifier

DeMorgan's Laws for Quantifiers

Handout

$$\neg \exists x\, P(x) \equiv \underline{\hspace{6cm}}$$

$$\neg \forall x\, P(x) \equiv \underline{\hspace{6cm}}$$

Exercise: Simplify each statement (so that negation appears only directly before a predicate):

a) $\neg\ \exists x\ \forall y\ \exists z\ [\neg P(x, y, z) \vee \neg Q(x, y, z)]$

b) $\neg\ \exists x\ [P(x) \rightarrow \neg Q(x)]$

An Exercise

P(x, y, z), Q(x, y, z) are unknown predicates

Simplify:

$\neg\, \exists x\, \forall y\, \exists z\, [\neg P(x, y, z) \vee \neg Q(x, y, z)]$

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$$\neg \exists x \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$$

$$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$$

DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x\, \forall y\, \exists z\, [\neg P(x, y, z) \vee \neg Q(x, y, z)]$

$\equiv \forall x\, \neg \forall y\, \exists z\, [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x\, \exists y\, \neg \exists z\, [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \neg \exists z [\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

$\equiv \forall x \exists y \forall z \neg[\neg P(x, y, z) \vee \neg Q(x, y, z)]$ DeMorgan (quantifier)

P(x, y, z), Q(x, y, z) are unknown predicates

An Exercise

Simplify:

$\neg \exists x \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$

$\equiv \forall x \neg \forall y \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$	DeMorgan (quantifier)
$\equiv \forall x \exists y \neg \exists z [\neg P(x, y, z) \lor \neg Q(x, y, z)]$	DeMorgan (quantifier)
$\equiv \forall x \exists y \forall z \neg[\neg P(x, y, z) \lor \neg Q(x, y, z)]$	DeMorgan (quantifier)
$\equiv \forall x \exists y \forall z [P(x, y, z) \land Q(x, y, z)]$	DeMorgan (normal)

Handout

An Exercise

Simplify: $\neg \exists x \, [P(x) \longrightarrow \neg Q(x)]$

An Exercise

Simplify: $\neg \exists x\, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg\, [P(x) \longrightarrow \neg Q(x)]$

DeMorgan (quantifier)

An Exercise

Simplify: $\neg \exists x \, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \, \neg \, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \, \neg \, [\neg P(x) \vee \neg Q(x)]$

DeMorgan (quantifier)

Implication Breakout

An Exercise

Simplify: $\neg \exists x\, [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [P(x) \longrightarrow \neg Q(x)]$

$\equiv \forall x \neg [\neg P(x) \vee \neg Q(x)]$

$\equiv \forall x\, [P(x) \wedge Q(x)]$

DeMorgan (quantifier)

Implication Breakout

DeMorgan (normal)

Outline

- Predicates
  - Recap of Quantifiers: For all, There exists
  - Nested Quantifiers
  - Translation Practice
- DeMorgan's for Quantifiers
- **Domain Restrictions**
- Quantifier Scoping

Domain Restrictions

S(x) = "x is a student in this class"

C(x) = "x has studied calculus"

"Every student in this class has studied calculus."

Is this translation correct? $\forall x\ [\ S(x) \land C(x)\ ]$ ***(domain = all people)***

Domain Restrictions

S(x) = "x is a student in this class"

C(x) = "x has studied calculus"

"Every student in this class has studied calculus."

Is this translation correct? $\forall x\,[\,S(x) \wedge C(x)\,]$ ***(domain = all people)***

NO! This says: "Everyone is a student in this class who has studied Calculus."

Use **if/then** with **universal quantifiers** to make a proposition about only **some** of the domain.

$$\forall x\,[\,S(x) \rightarrow C(x)\,]$$

Recall Lec 2: If-Then can be Counterintuitive

p	q	p→q
T	T	T
T	F	**F**
F	T	T
F	F	T

- It takes practice to get the right intuitions about p→q.
- One useful perspective:
  - If p is true, p→q says something about q.
  - If p is false, p→q says nothing about q. (Or, if p is false, all bets are off about q, at least based on p→q.)
- Another one: The *only way* for p→q to be false, is for p to be true, and q to be false.
- There will be another great help when we have quantifiers

"Every student in this class has studied calculus."

$$\forall x\ \underbrace{[\ S(x) \rightarrow C(x)\ ]}$$ ***(domain = all people)***

Need this portion to be **true for *each* person x**

- Including when x is not in this class, i.e., when $S(x) \equiv F$
- Therefore, the truth table for "implies" must have

$$F \rightarrow anything = T$$

Domain Restrictions

S(x) = "x is a student in this class"

G(x) = "x plays Golf"

"**Some** student in this class plays Golf."

Is this translation correct? $\exists x\ [\ S(x) \rightarrow G(x)\ ]$ ***(domain = all people)***

Domain Restrictions

NO! Remember: if-then with existential quantifiers is unintuitive
(This is satisfied by any x with ¬S(x), because F → anything is true)

S(x) = "x is a student in this class"
G(x) = "x plays Golf"

"**Some** student in this class plays Golf."

Is this translation correct? ∃x [S(x) → G(x)] ***(domain = all people)***

There exists a person, such that
[if they are a student in this class, then they play golf]

Can we find a person, x, to satisfy this "exists" statement?

- Someone who makes $S(x) \rightarrow G(x) \equiv \boldsymbol{T}$
- Could be $T \rightarrow T \equiv \boldsymbol{T}$
- Could also be $F \rightarrow anything \equiv \boldsymbol{T}$
  - Let $x =$ your grandmother
  - $S(x) \equiv F, G(x) \equiv ?$
  - $S(x) \rightarrow G(x) \equiv F \rightarrow ? \equiv \boldsymbol{T}$

p	q	p → q **If** p **then** q
T	T	T
T	F	F
F	T	T
F	F	T

Domain Restrictions

NO! Remember: if-then with existential quantifiers is unintuitive
(This is satisfied by any x with ¬S(x), because F → anything is true)

S(x) = "x is a student in this class"
G(x) = "x plays Golf"

"**Some** student in this class plays Golf."

Is this translation correct? $\exists x \, [\, S(x) \rightarrow G(x) \,]$ ***(domain = all people)***

There exists a person, such that
[if they are a student in this class, then they play golf]

Correct translation:

$$\exists x \, [\, S(x) \wedge G(x) \,]$$

Use **and** with **existential quantifiers** to make a proposition about only **some** of the domain.

p	q	p → q **If** p **then** q
T	T	T
T	F	F
F	T	T
F	F	T

How would you translate this?

Handout

"There is a person in this class all of whose friends in this class will get As"

C(x) : "x is in this class" F(x,y) : "x and y are friends"

A(x) : "x will get an A

Select all correct translations.

(A) $\exists x\ [C(x) \rightarrow \forall y\ (F(x,y) \wedge C(y) \wedge A(y))\ ]$

(B) $\exists x \forall y\ [(C(x) \wedge F(x,y)) \rightarrow (C(y) \wedge A(y))\ ]$

(C) $\exists x\ [C(x) \wedge \forall y\ [\ (F(x,y) \wedge C(y)) \rightarrow A(y)\ ]\ ]$

(D) $\exists x\ [C(x) \wedge \forall y\ [F(x,y) \rightarrow (C(y) \rightarrow A(y))\ ]\ ]$

(E) $\exists x\ [C(x) \wedge \forall y\ [F(x,y) \wedge (C(y) \rightarrow A(y))\ ]\ ]$

How would you translate this?

Handout

"There is a person in this class all of whose friends in this class will get As"

C(x) : "x is in this class" F(x,y) : "x and y are friends"

A(x) : "x will get an A

Select all correct translations.

(A) $\exists x\, [C(x) \rightarrow \forall y\, (F(x,y) \land C(y) \land A(y))\,]$

(B) $\exists x \forall y\, [(C(x) \land F(x,y)) \rightarrow (C(y) \land A(y))\,]$

(C) $\exists x\, [C(x) \land \forall y\, [\, (F(x,y) \land C(y)) \rightarrow A(y)\,]\,]$

(D) $\exists x\, [C(x) \land \forall y\, [F(x,y) \rightarrow (C(y) \rightarrow A(y))\,]\,]$

Correct answers: C and D

(E) $\exists x\, [C(x) \land \forall y\, [F(x,y) \land (C(y) \rightarrow A(y))\,]\,]$

How would you translate this?

"There is a person in this class all of whose friends in this class will get As"

$C(x)$: "x is in this class" $F(x,y)$: "x and y are friends"

$A(x)$: "x will get an A"

Here are two possible translations:

There is someone who is in this class (x) and, for each person (y):

if (x and y are friends AND y is in the class), then y will get an A

(C) $\exists x\,[C(x) \wedge \forall y\,[\,(F(x,y) \wedge C(y)) \rightarrow A(y)]\,]$

(D) $\exists x\,[C(x) \wedge \forall y\,\,[\,F(x,y) \rightarrow (C(y) \rightarrow A(y))\,]\,]$

There is someone who is in this class (x) and, for each person (y):

if (x and y are friends), then (if y is in the class, they'll get an A)

Outline

- Predicates
  - Recap of Quantifiers: For all, There exists
  - Nested Quantifiers
  - Translation Practice
- DeMorgan's for Quantifiers
- Domain Restrictions
- **Quantifier Scoping**

Scoping

Handout

Translate the following logical statements into English.

Do the two statements have the same meaning? ____________

B(x, y) = "x buys a y"

Logic	**English**
$\forall x\ [\ B(x, \text{umbrella}) \lor B(x, \text{raincoat})\ ]$	
$[\forall x\ B(x, \text{umbrella})] \lor [\forall x\ B(x, \text{raincoat})]$	

Scoping

$B(x, y)$ = "x buys a y"

Translate: ***(domain = all people)***

$\forall x\ [B(x,\text{umbrella}) \lor B(x,\text{raincoat})]$

"Everyone buys an umbrella or a raincoat"

(meaning that each person could buy either; they **don't** have to all buy the same thing)

$[\forall x\ B(x,\text{umbrella})] \lor [\forall y\ B(y,\text{raincoat})]$

"Everyone buys an umbrella or everyone buys a raincoat"

(meaning that everybody has to buy the same thing)

"Variable scope:" the part of the expression in which a variable created by a for-all/there-exists exists is still active.

This matters! It causes these two propositions to have different meanings.

Scoping

S(x,y) = "x is **S**iblings with y"

B(x)= "x plays **B**oard games"

"Everyone who has a sibling plays board games."

(A) $\forall x\ [\ (\exists y\ S(x,y)) \rightarrow B(x)\ ]$

(B) $\forall x\ \exists y\ [\ S(x,y) \rightarrow B(x)\ ]$

Scoping

S(x,y) = "x is **S**iblings with y"

B(x)= "x plays **B**oard games"

"Everyone who has a sibling plays board games."

(A) $\forall x\, [\, (\exists y\, S(x,y)) \rightarrow B(x)\,]$

(B) $\forall x\, \exists y\, [\, S(x,y) \rightarrow B(x)\,]$

For each expression, consider:

can the expression still be true,
even if
there is someone **with a sibling** who
does **not** play board games?

"there exists a person y, such that …"

i.e., Consider the case when x has a sibling.
Can you find a y that will make the statement true
even when $B(x) \equiv F$?

Scoping

S(x,y) = "x is **S**iblings with y"
B(x)= "x plays **B**oard games"
"Everyone who has a sibling plays board games."

Scope of *y*:

(A) $\forall x \,[\, (\exists y\, \boxed{S(x,y)}) \rightarrow B(x)\,]$

A: Scope of y is limited to *inside the premise* of the implication

(B) $\forall x\, \exists y\, \boxed{[\, S(x,y) \rightarrow B(x)\,]}$

B: Scope of y is *the entire* implication

i.e., Consider the case when x has a sibling.
Can you find a y that will make the statement true
even when $B(x) \equiv F$?

Scoping

S(x,y) = "x is **S**iblings with y"
B(x)= "x plays **B**oard games"
"Everyone who has a sibling plays board games."

Scope of *y*:

(A) $\forall x\ [\ (\exists y\ \boxed{S(x,y)})\rightarrow B(x)\ ]$

A: Scope of y is limited to *inside the premise* of the implication

(B) $\forall x\ \exists y\ \boxed{[\ S(x,y)\rightarrow B(x)\ ]}$

B: Scope of y is *the entire* implication

Recall: $F\rightarrow F=T$
So for (B) to be true, we just need to find a person y that makes $S(x,y)\equiv F$

i.e., Consider the case when x has a sibling.
Can you find a y that will make the statement true even when $B(x)\equiv F$?

Scoping

S(x,y) = "x is **S**iblings with y"
B(x)= "x plays **B**oard games"
"Everyone who has a sibling plays board games."

"For all x:
if (there is someone, y, who is x's Sibling), then (x plays Board games)."

(A) $\forall x\ [\ (\exists y\ \boxed{S(x,y)})\rightarrow B(x)\ ]$

A: Scope of y is limited to *inside the premise* of the implication

(B) $\forall x\ \exists y\ \boxed{[\ S(x,y) \rightarrow B(x)\ ]}$

B: Scope of y is *the entire* implication

"For all x, there is someone, y, such that:
if (x is Siblings with y), then (x plays Board games)."

Scoping

S(x,y) = "x is **S**iblings with y"
B(x)= "x plays **B**oard games"
"Everyone who has a sibling plays board games."

"For all x:
if (there is someone, y, who is x's Sibling), then (x plays Board games)."

(A) $\forall x \, [\, (\exists y \, \boxed{S(x,y)}) \rightarrow B(x) \,]$

A: Scope of y is limited to *inside the premise* of the implication

For any person x, there are tons of people who are not x's sibling, so we can always find a person y that makes the premise false

(B) $\forall x \, \exists y \, \boxed{[\, S(x,y) \rightarrow B(x) \,]}$

B: Sc

"For all x, there is someone, y, such that:
if (x is Siblings with y), then (x plays Board games)."

Scoping

S(x,y) = "x is **S**iblings with y"

B(x)= "x plays **B**oard games"

"Everyone who has a sibling plays board games."

Scope of *y*:

(A) $\forall x\ [\ (\exists y\ \underline{\boxed{S(x,y)}}) \rightarrow B(x)\ ]$

(B) $\forall x\ \exists y\ \underline{\boxed{[\ S(x,y) \rightarrow B(x)\ ]}}$

If/Then directly inside an existential quantifier is a red flag! Probably non-meaningful behavior

Wrap Up

- We’ve studied logic, translation, negation, logical equivalences, quantified statements, all of which are preparation for….

- PROOFS!
  - Next 4 lectures (and beyond)