F23

EECS 203 Discussion 6

Sets

Admin Notes:

Homework:

- HW + GW 5 was due **Thursday, October 12th**
- Homework/Groupwork 6 due **Thursday, October 19th**
- Weekly Check-in 6 due **Thursday, October 19th**

Surveys:

- Wellness Check-in Survey
  - Open **Oct. 10 - Oct. 19**
  - Scan QR Code to take the survey! →

Intro to Sets

Set Terminology

- **Set:** A set is an unordered collection of distinct objects
- **Universe:** In set theory, a universe is a collection that contains all the entities one wishes to consider in a given situation.
- **Union $S \cup T$:** The set containing the elements that are in S or T

  $$S \cup T = \{x \mid x \in S \vee x \in T\}$$

- **Intersection $S \cap T$:** The set containing the elements that are in S and T

  $$S \cap T = \{x \mid x \in S \wedge x \in T\}$$

- **Complement $\bar{A}$ of A:** The set containing the elements that are in the universe U but not in A.

  $$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$$

- **Minus $S - T$:** The set containing the elements that are in S but not in T

  $$S - T = \{x \mid x \in S \wedge x \notin T\}$$

Set Terminology

- **Subset:** The set A is a subset of B if and only if every element of A is also an element of B. Denoted $A \subseteq B$. Note that A and B may be the same set.

  $$\boldsymbol{A \subseteq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B]}$$

- **Proper Subset:** The set A is a proper subset of B if and only if A is a subset of B and $A \neq B$. That is, A is a subset of B and there is at least one element of B that is not in A.

  $$\boldsymbol{A \subsetneq B.\ A \subsetneq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B] \wedge (A \neq B)}$$

- **Disjoint:** The sets A and B are disjoint if and only if they do not share any elements

- **Empty Set:** The empty set, denoted $\varnothing$ or **{}**, is the unique set having no elements.

Set Terminology

- **Inclusion−Exclusion Principle:** The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Power Set:** The power set of a set S is the set of all subsets of S. $P(S)$ denotes the power set of S.

  $$P(S) = \{T \mid T \subseteq S\}$$

- **Cardinality:** The number of elements in a set. The cardinality of a set S is denoted by $|S|$.

- **Cartesian Product:** $A \times B$ is the set of all ordered pairs of elements (a, b) where $a \in A$ and $b \in B$.

  $$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Standard Numerical Sets

- **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- **Integers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$
- **Positive Integers:** $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$
- **Negative Integers:** $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$
- **Rational Numbers:** $\mathbb{Q}$

    Ex: **1, -2/3, 4/7**

- **Real Numbers:** $\mathbb{R}$

    Ex: **1, -2/3, π, e**

- **Positive Real Numbers:** $\mathbb{R}^+$

    Ex: **1, π, e**

Note: We consider 0 to be a natural number in EECS 203.

Set Identities – Logical Equivalences for Sets

- Identity Laws

  $$A \cap U = A \quad A \cup \emptyset = A$$

- Domination Laws

  $$A \cup U = U \quad A \cap \emptyset = \emptyset$$

- Idempotent Laws

  $$A \cup A = A \quad A \cap A = A$$

- Complementation Law

  $$\overline{(\overline{A})} = A$$

- Commutative Laws

  $$A \cup B = B \cup A$$
  $$A \cap B = B \cap A$$

- Associative Laws

  $$A \cup (B \cup C) = (A \cup B) \cup C$$
  $$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive Laws

  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
  $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- De Morgan's Laws

  $$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption Laws

  $$A \cap (A \cup B) = A \quad A \cup (A \cap B) = A$$

- Complement Laws

  $$A \cup \overline{A} = U$$
  $$A \cap \overline{A} = \emptyset$$

Practice Problems

Problem:

1. Set Exploration $\star$

a) What is $|\emptyset|$?

b) Let $A = \{1, 2, 3\}$, $B = \{\emptyset\}$, $C = \{\{\emptyset\}\}$, $D = \{4, 5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?

ii. Is $\emptyset \subseteq A$?

iii. Is $\emptyset \in B$?

iv. Is $\emptyset \subseteq B$?

v. Is $\emptyset \in C$?

vi. Is $\emptyset \subseteq C$?

vii. What is $A \cap D$?

viii. What is $B \cap C$?

ix. What is $B \cap E$?

x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?

ii. What is $P(C)$?

iii. Generalize what $|P(S)|$ is for any set S.

iv. What is $C \times A$?

v. What is A^2? ($A^2 = A \times A$)

vi. Generalize what $|A \times B|$ is for any sets A and B.

Solution:

Solution:

a) $|\emptyset| = 0$

b) i. No, $\emptyset$ is not an element of A, you would see it in A if it was.

ii. Yes, $\emptyset$ is a subset of all sets. All elements of $\emptyset$ (none) are elements of A. $\{\} \subseteq \{1, 2, 3\}$

iii. Yes, $\emptyset \in \{\emptyset\}$

iv. Yes, $\emptyset$ is a subset of all sets.

v. No, $\emptyset \notin \{\{\emptyset\}\}$

vi. Yes, $\emptyset$ is a subset of all sets.

vii. $A \cap D = \emptyset$

viii. $B \cap C = \emptyset$

ix. $B \cap E = \{\emptyset\}$

x. $|B| = 1$, $|C| = 1$, $|E| = 2$

c) i. $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

ii. $\mathcal{P}(C) = \Big\{\ \emptyset,\ \{\{\emptyset\}\}\ \Big\}$

iii. $|\mathcal{P}(S)| = 2^{|S|}$

iv. $C \times A = \Big\{\Big(\{\emptyset\}, 1\Big), \Big(\{\emptyset\}, 2\Big), \Big(\{\emptyset\}, 3\Big)\Big\}$

v. $A^2 = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

vi. $|A \times B| = |A| \times |B|$

Problem:

2. Double Subset Equality

Prove the following set equality by showing that each side is a subset of the other.
$A - (B \cap C) = (A - B) \cup (A - C)$

Solution:

Solution:
First, let's show $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
Let x be an arbitrary element of the domain.
Assume $x \in A - (B \cap C)$

- $x \in A \land x \in \overline{(B \cap C)}$
- $x \in A \land (x \notin B \lor x \notin C)$ (using DeMorgan's Law)
- $(x \in A \land x \notin B) \lor (x \in A \land x \notin C)$ (using the distributive property)
- $(x \in A - B) \lor (x \in A - C)$
- $x \in (A - B) \lor (A - C)$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Now we will show $(A - B) \cup (A - C) \subseteq A - (B \cap C)$
Let x be an arbitrary element of the domain.

Assume $x \in (A - B) \cup (A - C)$
So $x \in A - B$ or $x \in A - C$

Case 1: $x \in A - B$

- $x \in A \land x \notin B$
- $x \in A$
- $x \notin B$
- $x \notin B \lor x \notin C$
  (**Note:** we can add whatever we want with an or statement, since we know the first half is always true!)
- $x \in A \land (x \notin B \lor x \notin C)$

Case 2: $x \in A - C$

- $x \in A \land x \notin C$
- $x \in A$
- $x \notin C$
- $x \notin B \lor x \notin C$
- $x \in A \land (x \notin B \lor x \notin C)$

Now we need to use the conclusions of our cases:

- In both cases, we have $x \in A \land (x \notin B \lor x \notin C)$
- $x \in A \cap (\overline{B} \cup \overline{C})$
- $x \in A \cap \overline{(B \cap C)}$ (using DeMorgan's Law)
- $x \in A - (B \cap C)$

Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Since each side is a subset of the other, we can say $(A - B) \cup (A - C) = A - (B \cap C)$

Problem:

3. Subset Proofs $\star$

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

Solution:

Solution:

a) Let $x \in (A \cap B \cap C)$ be arbitrary
- By the definition of intersection, we have $(x \in A) \wedge (x \in B) \wedge (x \in C)$
- So we have $(x \in A) \wedge (x \in B)$
- Thus we have, $x \in (A \cap B)$

Therefore, $(A \cap B \cap C) \subseteq (A \cap B)$ by definition.

b) Let $x \in (A - B) - C$ be arbitrary
- By definition of set difference, we know that $(x \in A - B) \wedge (x \notin C)$
- Since $x \in A - B$, we know that $(x \in A) \wedge (x \notin B)$
- Thus, $(x \in A) \wedge (x \notin B) \wedge (x \notin C)$
- Then we have $(x \in A) \wedge (x \notin C)$
- Finally, by definition of set minus, we have $x \in A - C$

Therefore, we have shown that $(A - B) - C \subseteq A - C$

Problem:

4. Power Set of a Cartesian Product

Prove or disprove that for all sets A, B, we have $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Solution:

Solution: (Disprove) For a counterexample, let $A = B = \emptyset$. Then $A \times B = \emptyset$ and $\mathcal{P}(A \times B) = \{\emptyset\}$, whereas $\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset\}$ and $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$. Thus, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$.

Another counterexample: let $A = \{a\}$ and $B = \{b\}$. Then $A \times B = \{(a, b)\}$ whereas $\mathcal{P}(A) = \{\emptyset, a\}$ and $\mathcal{P}(B) = \{\emptyset, b\}$, and $\mathcal{P}(A) = \{(\emptyset, \emptyset), (\emptyset, b), (a, \emptyset), (a, b)\}$. Thus, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$.

Problem:

5. Inclusion-Exclusion

Suppose there is a group of 120 U of M students. Here's what you know:

- There are 31 in Engineering.
- There are 65 in LSA.
- There are 44 in Ross.
- There are 20 that are not in any of these 3 schools.
- There are 15 in Engineering and Ross.
- There are 17 in Engineering and LSA.
- There are 18 in LSA and Ross.

How many are in all 3 schools?

Solution:

Solution: We see that the 20 people that are not in any of the schools do not contribute to the count of any other number, and so we can simply look at the $120 - 20 = 100$ students that are in the schools that we are looking at.

By the inclusion-exclusion principle, we have to add the number of people in the 3 individual categories independently, then subtract the pairwise totals, and add back the number of those in all 3 categories. This should yield our original number of students.

$$31 + 65 + 44 - 15 - 17 - 18 + x = 100$$

Doing some algebra, we see that $x = 10$. Thus there are 10 people in all three schools.

Problem:

6. Set Operations

Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

Solution:

Solution: This can be done easily by drawing the Venn diagram. This is a good way to think about.

Here's a formal solution:

We need to find A and B such that $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$ and $A \cap B = \{3, 6, 9\}$.

Since $A \cap B = \{3, 6, 9\}$, this means that $3, 6, 9 \in A$ and $3, 6, 9 \in B$.

$A - B = \{1, 5, 7, 8\}$ means that the elements $1, 5, 7, 8 \in A$ but $1, 5, 7, 8 \notin B$.

$B - A = \{2, 10\}$ means that the elements $2, 10 \in B$ but $2, 10 \notin A$.

Thus, $A = \{1, 3, 5, 6, 7, 8, 9\}$ and $B = \{2, 3, 6, 9, 10\}$.

Problem:

7. Power Sets

Prove or Disprove: if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

Solution:

Solution:

Direct Proof: The union of all the elements in the power set of a set X must be exactly X. In other words, we can recover X from its power set, uniquely. Since a power set has exactly one originating set associated with it, A and B must be the same set.

Proof by Contrapositive: Consider the contrapositive "if $A \neq B$, then $\mathcal{P}(A) \neq \mathcal{P}(B)$" Let A and B be sets and assume $A \neq B$. Because A and B are not equal, WLOG there exists an element $x \in A$ such that $x \notin B$. Therefore we conclude that $\{x\} \in \mathcal{P}(A)$, and that $\{x\} \notin \mathcal{P}(B)$, therefore $\mathcal{P}(A) \neq \mathcal{P}(B)$. Thus by the contrapositive, the original statement is true.

Problem:

8. More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution:

Solution:

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) We know that the power set a set of size n has 2^n elements, but this set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set. Set cardinality aside, the set $\{\emptyset, a\}$ may come to mind, but $P(\{\emptyset, a\}) = \{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\}$.

d) This is the power set of $\{a, b\}$.

Problem:

9. Sets Warm Up

Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution:

Solution: Since this is an iff statement we must prove each direction.

Proof that $\mathcal{P}(\mathbf{A}) \subseteq \mathcal{P}(\mathbf{B})$ implies $\mathbf{A} \subseteq \mathbf{B}$:
Let $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ for sets A and B.
Take an arbitrary element $a \in A$
Then $\{a\} \subseteq A$, so $\{a\} \in \mathcal{P}(A)$
Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$,
$\{a\} \in \mathcal{P}(B) \implies \{a\} \subseteq B \implies a \in B$
Thus we've shown that $A \subseteq B$.

Proof that $\mathbf{A} \subseteq \mathbf{B}$ implies $\mathcal{P}(\mathbf{A}) \subseteq \mathcal{P}(\mathbf{B})$:
Let $A \subseteq B$ for sets A and B.
Take an arbitrary element $a \in \mathcal{P}(A)$
By definition of the power set of A, $a \subseteq A$
And since $A \subseteq B$, $a \subseteq B$
Thus by def of power set of B, $a \in \mathcal{P}(B)$
Thus we've shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$