In Exercises 1–14, to establish a big-O relationship, find witnesses C and k such that $|f(x)| \leq C|g(x)|$ whenever $x > k$.

1. Determine whether each of these functions is $O(x)$.

a) $f(x) = 10$
b) $f(x) = 3x + 7$
c) $f(x) = x^2 + x + 1$
d) $f(x) = 5 \log x$
e) $f(x) = \lfloor x \rfloor$
f) $f(x) = \lceil x/2 \rceil$

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$.

5. Show that $(x^2 + 1)/(x + 1)$ is $O(x)$.

7. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a) $f(x) = 2x^3 + x^2 \log x$
b) $f(x) = 3x^3 + (\log x)^4$
c) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
d) $f(x) = (x^4 + 5 \log x)/(x^4 + 1)$

9. Show that $x^2 + 4x + 17$ is $O(x^3)$ but that x^3 is not $O(x^2 + 4x + 17)$.

11. Show that $3x^4 + 1$ is $O(x^4/2)$ and $x^4/2$ is $O(3x^4 + 1)$.

13. Show that 2^n is $O(3^n)$ but that 3^n is not $O(2^n)$.

15. Explain what it means for a function to be $O(1)$.

17. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$. Show that $f(x)$ is $O(h(x))$.

19. Determine whether each of the functions 2^{n+1} and 2^{2n} is $O(2^n)$.

21. Arrange the functions $\sqrt{n}$, $1000 \log n$, $n \log n$, $2n!$, 2^n, 3^n, and $n^2/1{,}000{,}000$ in a list so that each function is big-O of the next function.

23. Suppose that you have two different algorithms for solving a problem. To solve a problem of size n, the first algorithm uses exactly $n(\log n)$ operations and the second algorithm uses exactly $n^{3/2}$ operations. As n grows, which algorithm uses fewer operations?

25. Give as good a big-O estimate as possible for each of these functions.

a) $(n^2 + 8)(n + 1)$ **b)** $(n \log n + n^2)(n^3 + 2)$
c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

27. Give a big-O estimate for each of these functions. For the function g in your estimate that $f(x)$ is $O(g(x))$, use a simple function g of the smallest order.

a) $n \log(n^2 + 1) + n^2 \log n$
b) $(n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$
c) $n^{2^n} + n^{n^2}$

29. For each function in Exercise 2, determine whether that function is $\Omega(x^2)$ and whether it is $\Theta(x^2)$.

2. Determine whether each of these functions is $O(x^2)$.

a) $f(x) = 17x + 11$ **b)** $f(x) = x^2 + 1000$
c) $f(x) = x \log x$ **d)** $f(x) = x^4/2$
e) $f(x) = 2^x$ **f)** $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

35. Express the relationship $f(x)$ is $\Theta(g(x))$ using a picture. Show the graphs of the functions $f(x)$, $C_1|g(x)|$, and $C_2|g(x)|$, as well as the constant k on the x-axis.

37. Explain what it means for a function to be $\Theta(1)$.

41. Suppose that $f(x)$ is $O(g(x))$, where f and g are increasing and unbounded functions. Show that $\log |f(x)|$ is $O(\log |g(x)|)$.