Groupwork

1. Grade Groupwork 10

Using the solutions and Grading Guidelines, grade your Groupwork 10:

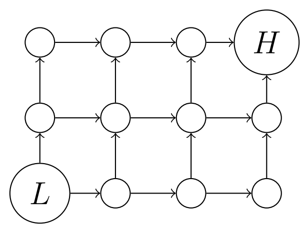
- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2	+2	+2	+3	+2	+1	+1	+1	+3				15 /15
Problem 3	+2	+3	+2	+3	+2	+3						15 /15
Total:												30 /30

Previous Groupwork 10(1): Lily's Lily Pads [15 points]

Lily the Frog is on a lily pad and wants to get to her home! She can jump from lily pad to lily pad to help reach this goal. The lily pads are arranged in a grid. Lily starts on the **bottom-left** lily pad, and her home is at the **top-right** lily pad. Lily can only move one lily pad **upward** or one lily pad **rightward** at a time.

Each lily pad has coordinates of the form $(x, y) \in \mathbb{N} \times \mathbb{N}$, where x represents how far rightward a point is from the left of the grid, and y represents how far upward a point is from the bottom of the grid. Lily starts at location $(0, 0)$, and her home is at location $(x_H, y_H) \in \mathbb{N}\times\mathbb{N}$.



In the above example, $(x_H, y_H) = (3, 2)$. In the general case, though, (x_H, y_H) could be any ordered pair of natural numbers.

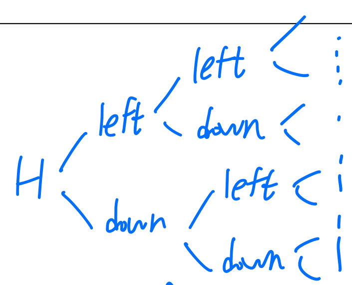
(a) How many different paths can Lily take to get home?

(b) Lily's frog friend, Francine, is also on the grid at coordinates $(x_F, y_F) \in \mathbb{N} \times \mathbb{N}$ such that $0 \leq x_F \leq x_H$ and $0 \leq y_F \leq y_H$. What is the probability that Lily meets Francine on her path home? You may assume that any two paths home are equally likely for Lily to take.

Solution:

Perfect work of us! There is nothing wrong or insufficient!

(a)



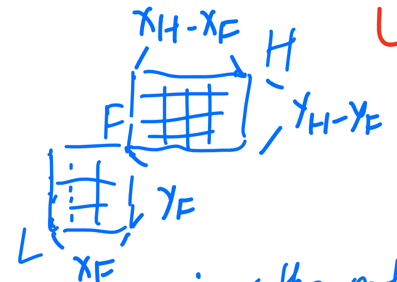
i.e. Starting from home, tracking back. Every last move is "left" or "down", s.t. "left"# $\leq x_H$, "down"# $\leq y_H$. Once "left"# $= x_H$ $\wedge$ "down"# $= y_H$, a path is formed.

Then we know: (1) for one path, total move is $x_H + y_H$, which contains "down" # = y_H, "left" # = x_H +3

∴ This problem is equivalent to: from $x_H + y_H$ steps, choose x_H steps to be "left" (and the rest one "down".

∴ total # = $\binom{x_H+y_H}{x_H} = \binom{x_H+y_H}{y_H}$ +2 +2

(b)



We can view the paths where Lily meets Francine as two stages: (1) Lily starts from (0,0), ends at Francine. (2) Lily starts from Francine (x_F, y_F), ends at Home (x_H, y_H) +3

Similar to question a:

#paths from (0,0) to $(x_F, y_F) = \binom{x_F+y_F}{x_F} = \binom{x_F+y_F}{y_F}$

#paths from (x_F, y_F) to $(x_H, y_H) = \binom{x_H+y_H-x_F-y_F}{x_H-x_F} = \binom{x_H+y_H-x_F-y_F}{y_H-y_F}$

$$\therefore p = \frac{|E|}{|S|} = \frac{\binom{x_F+y_F}{x_F}\cdot\binom{x_H+y_H-x_F-y_F}{x_H-x_F}}{\binom{x_H+y_H}{x_H}}$$

+1 +1 +2+1

Previous Groupwork 10(2) Random Connections [15 points]

We say that a *random graph* is an undirected graph where, for each pair of vertices, there is an independent $\frac{1}{3}$ chance that they are adjacent. It's a bit like Lily's pond, except that the vertices aren't in a grid, and you can move in any direction.

We want to learn about the connectedness of random graphs.

Let G be a finite random graph. Let's split the vertices into two nonempty sets, $A, B \subseteq V$.

(a) Let $a \in A$. What is the probability that no element of B is adjacent to a?

(b) What is the probability that there is some $a \in A$ and $b \in B$ such that a is adjacent to b?

(c) Let's imagine doing this with larger and larger graphs. Define $f(a, b)$ be your answer to the previous problem when $|A| = a$ and $|B| = b$. What is

$$\lim_{a+b\to\infty} f(a, b)?$$

(d) This isn't quite a proof, but your answer to (c) might lead you to some ideas. What might you conjecture about the connectedness of infinite random graphs?

We did it very elegantly and rigorously!

Solution: (a) P_i: vertex i in B is of adjacent to a

$\Rightarrow \forall i \in B,\ P_i = 1 - \frac{1}{3} = \frac{2}{3}$ +2

$\therefore P(\text{no element of } B \text{ is adjacent to } a)$

$= \prod_{i \in B} P_i = \left(\frac{2}{3}\right)^{|B|}$ +3

(b) $P(\text{some } a \in A \text{ and } b \in B \text{ are adjacent})$

$= 1 - P(\forall a \in A \forall b \in B,\ a, b \text{ are not adjacent})$ +2

$P(\forall a \in A \forall b \in B,\ a, b \text{ are not adjacent})$

$= \prod_{a \in A} P(\text{no element of } B \text{ is adjacent to } a)$

$= \prod_{a \in A} \left(\frac{2}{3}\right)^{|B|} = \left(\frac{2}{3}\right)^{|B| \cdot |A|}$ +3

$\therefore P(\text{some } a \in A \text{ and } b \in B \text{ are adjacent}) = 1 - \left(\frac{2}{3}\right)^{|A| \cdot |B|}$

(c) $f(a, b) = 1 - \left(\frac{2}{3}\right)^{ab}$

$\Rightarrow \lim_{a,b\to\infty} f(a,b) = \lim_{n\to\infty} 1-\left(\frac{2}{3}\right)^{\infty} = 1$

+2

(d) Conjecture: if for an infinite random graph, the chance that two vertices are adjacent $\neq 0$, then some must exist some adjacent vertices in the graph.

+3

(almost always connected no matter how much in $(0,1)$ we change $\frac{1}{3}$ to)

2. I Am Speed [10 points]

Suppose we have two algorithms, $\mathcal{A}$ and $\mathcal{B}$. Suppose that on inputs of size n, algorithm $\mathcal{A}$ runs in time $\Theta(n^2)$, while algorithm $\mathcal{B}$ runs in time $\Theta(n^3)$. Show that there exists some n_0 such that for any $n > n_0$, algorithm $\mathcal{A}$ will take less time to run than algorithm $\mathcal{B}$ on inputs of size n.

Note: you may find it useful for your notation to let the runtime of $\mathcal{A}$ on inputs of size n be denoted as $f_{\mathcal{A}}(n)$, and similarly for algorithm $\mathcal{B}$ as $f_{\mathcal{B}}(n)$.

Solution:

Set the runtime of A on inputs of size n as $f_A(n)$,
the runtime of B on inputs of size n as $f_B(n)$

$\because f_A(n) = \Theta(n^2)$

$\therefore \exists c_1$ s.t. $f_A(n) \leq c_1 n^2$

$\because f_A(n) = \Theta(n^3)$

$\therefore \exists c_2$ s.t. $f_B(n) \geq c_2 n^3$

$C_2 n^3 - C_1 n^2 = n^2 (C_2 n - C_1)$

Consider $n_0 = \frac{C_1}{C_2}$,

$\forall n > n_0$, $C_2 n^3 - C_1 n^2 = n^2 \underbrace{(C_2 n - C_1)}_{>0} > 0$

$C_2 n^3 > c_1 n^2$

$\Rightarrow f_B(n) \geq C_2 n^3 > c_1 n^2 \geq f_A(n)$

qed.

3. GameStop or GameRoll? [8 points]

In a game of repeated die rolls, a player is allowed to roll a standard die up to n times, where n is determined prior to the start of the game. On any roll except the last, the player may choose to either keep that roll as their final score, or continue rolling in hopes of a higher roll later on. If the player rolls all n times, then on the n-th roll the player must keep that roll as their final score. A player always acts to maximize their expected final score. Finally, let V_n denote the final score in a game with a max of n rolls allowed.

(a) Compute $E(V_2)$ with justification.

(b) Compute $E(V_3)$ with justification.

(c) Find the smallest n such that $E(V_n) \geq 5$.

Solution:

(a) $E(V_1) = \frac{1+2+\cdots+6}{6} = \frac{(1+6)\cdot 3}{6} = \frac{7}{2}$

$\Rightarrow$ If a player rolls $n \in \{1,2,3\}$, we should expect them to roll again ($p = \frac{3}{6} = \frac{1}{2}$)

$$\therefore E(V_2) = \frac{1}{2}(3.5) + \frac{1}{2}\left(\frac{4+5+6}{3}\right) = 4.25$$

(b) $\because E(V_2) = 4.25$

$\therefore$ if a player rolls $n \in \{1,2,3,4\}$, reroll

expected

$$\Rightarrow E(V_3) = \frac{2}{3}(4.25) + \frac{1}{3}\left(\frac{5+6}{2}\right) = 4.67$$

(c) Similarly

$$E(V_4) = \frac{3}{4}(4.67) + \frac{1}{4}\left(\frac{5+6}{2}\right) = 4.8775$$

$$E(V_5) = \frac{4}{5}\cdot(4.875) + \frac{1}{5}\cdot\left(\frac{5+6}{2}\right) = 5$$

$\therefore$ It will take 5 rolls