EECS 203: Discrete Mathematics
Fall 2023
Homework 4

Due **Thursday, Sept. 28**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $6 + 2$ Total Points: $100 + 50$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Let's be rational (numbers) [16 points]

(a) **Prove or disprove:** for all real numbers x and y, if xy is irrational, then x is irrational or y is irrational.

(b) **Prove or disprove:** for all real numbers x and y, if x is irrational or y is irrational, then xy is irrational.

Solution:

(a) Prove.
We can prove it by proving its contrapositive: "For all real numbers x and y, if x is rational and y is rational, then xy is rational." If x is rational and y is rational, for some integers p, q, k, m, $x = \frac{p}{q}$ and $y = \frac{k}{m}$.
Since p, q, k, m are integers, pk and qm are integers. Then $xy = \frac{pk}{qm}$, therefore xy is rational.
Therefore we have proved the original proposition by proving its contrapositive.

(b) Disprove.
Consider $x = \sqrt{2}$ and $y = \frac{\sqrt{2}}{2}$, $xy = 1$ which is rational.

2. Irrational Pr00f [14 points]

Prove or disprove that for all nonzero rational numbers x and irrational numbers y, xy is irrational.

Solution:

Proof.
Let x be an arbitrary rational number that is not 0, and y be an arbitrary irrational number. For some integer p and q, $x = \frac{p}{q}$.

Assume xy is rational, then for some integers m, n, $xy = \frac{m}{n}$. Then

$$\frac{p}{q}y = \frac{m}{n}$$
$$y = \frac{qm}{pn}$$

Since p, q, m, n are integers, qm and pn are integers, so y is rational, which causes contradiction.
Then we have proved the proposition by contradiction.

3. That's Really Odd... (and Even) [17 points]

In this problem, you may use the following statement without proof: "for all integers n, n^2 is odd if and only if n is odd."

(a) Prove that for all integers n, $n^2 - n + 1$ is odd.

(b) Prove that for all integers x and y, if $x^2 + y^2$ is even, then $x + y$ is even.

Solution:

(a) Let n be an arbitrary integer and n can fall into 2 cases.

  (i) Case 1: Assume that n is odd.
  Since n^2 is odd if and only if n is odd, n^2 is odd.
  Then For some integer p, q, $n = 2p + 1$, $n^2 = 2q + 1$.
  Then $n^2 - n + 1 = 2(q - p) + 1$, is odd.

  (ii) Case 2: Assume that n is even.
  Since n^2 is odd if and only if n is odd, n^2 is even.
  Then for some integer p, q, $n = 2p$, $n^2 = 2q$.
  Then $n^2 - n + 1 = 2(q - p) + 1$, is odd.

  Since $n^2 - n + 1$ is odd holds in all the cases, $n^2 - n + 1$ is true.
  Therefore We have proved the proposition.

(b) We prove it by proving its contrapositive: "For all integers x and y, if $x + y$ is odd, $x^2 + y^2$ is odd."
  Since n^2 is odd if and only if n is odd, $(x + y)^2 = x^2 + 2xy + y^2$ is odd if $x + y$ is odd.
  Since $x^2 + 2xy + y^2$ is odd, it equals to $2k + 1$ for some integer k. Then $x^2 + y^2 = 2k + 1 - 2xy = 2(k - xy) + 1$. Since k, x, y are integers, $k - xy$ is an integer, so $x^2 + y^2$ is odd.
  Therefore we have proved the original proposition by proving its contraposotive.

4. All that remains [21 points]

Definition: For integers n, r, d, we say that r is the textbfremainder of n when divided by d if and only if $0 \leq r < d$, and there exists integer q such that $n = dq + r$. For example, the remainder of 14 when divided by 4 is 2 since $14 = 4 \cdot 3 + 2$.

(a) Prove that for all integers n, the remainder of n^2 when divided by 4 is either 0 or 1.

(b) Prove that for all prime numbers p greater than 3, the remainder of p^2 when divided by 3 is 1. **Hint:** Consider the possible remainders when dividing p by 3.

Solution:

(a) Let n be an arbitrary integer, and n can fall into 2 cases.

   (i) Case 1: n is even.
   That means for some integer k, $n = 2k$.
   Then $n^2 = 4k^2$, $\frac{n^2}{4} = k^2$. Its remainder when divided by 4 is 0.

   (ii) Case 2: n is odd.
   That means for some integer k, $n = 2k + 1$.
   Then $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Its remainder when divided by 4 is 1.

   Since the remainder of n^2 when divided by 4 is 0 or 1 for all cases, we have proved the proposition.

(b) Let p be an arbitrary prime number which is greater than 3. Since p is a primer number, it is not divisible by 3, so p equals to $3k + 1$ or $3k + 2$ for some integer k.

   (i) Case 1: $p = 3k + 1$
   Then $p^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Its remainder when divided by 3 is 1.

   (ii) Case 2: $p = 3k + 2$
   Then $p^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Its remainder when divided by 3 is 1.

   Since the remainder of p^2 when divided by 3 is 1 for all cases, we have proved the proposition.

5. FOtter's Day [17 points]

Every year on Father's Day, each otter pup at the Ann Arbor Zoo gives a rock to each adult otter at the zoo. We will prove that if there are an even number of otters at the Ann Arbor

Zoo, and an even number of rocks were gifted this year, then there are an even number of otter pups and an even number of adult otters.

(a) Let x be the number of otter pups and y be the number of adult otters. Rewrite the above statement in terms of x and y.

(b) Prove the statement you wrote in (a).

Solution:

(a) Let x be the number of otter pups and y be the number of adult otters, then the total number of otters is $x + y$. Each otter pup at the Ann Arbor Zoo gives a rock to each adult otter at the zoo. Therefore the number of Rocks given should be xy.

Therefore we can rewrite the statement as: "For all(in the case should be positive, but the condition is unnecessary) integers x, y, if $x + y$ is even and xy is even, then x and y are all even."

(b) Since $x + y$ is even, for some integers p, $x + y = 2p$.
If x is even, then for some integer k, $x = 2k$, then $y = 2(p - k)$ is even.
If x is odd, then for some integer k, $x = 2k + 1$, then $y = 2(p - k) - 1$ is odd.
Therefore we know, x and y can only be both even or both odd.
Then we will prove that x and y can only be both even by contradiction.

(i) Case 1: x and y are both even.
Then for some integers p, q, $x = 2p$ and $y = 2q$.
Then $xy = 4pq = 2(2pq)$ is even.

(ii) Case 2: x and y are both odd.
Then for some integers p, q, $x = 2p + 1$ and $y = 2q + 1$.
Then $xy = (2p + 1)(2q + 1) = 4pq + 2p + 2q + 1 = 2(2pq + p + q) + 1$ is odd, which causes a contradiction with the condition that xy is even. Therefore the case does not exist.

Then we have proved the proposition.

6. False Inequality [15 points]

In this problem, you may use the following axiom: "for all real numbers a, b, c, if $a < b$ and c is positive, then $ac < bc$." We will disprove the following proposition p:

"There exists a real number x such that $x^2 < x < x^3$."

(a) Prove that for all real numbers x satisfying $x^2 < x$, x is positive.

(b) Using part (a), disprove p.

Solution:

(a) let x be an arbitrary real number, then $x^2 \geq 0$.
If and only if $x^2 = 0$, $x = 0$, x is not positive. However, in this case, $x^2 = x$, it is not in the domain.
Therefore for all real numbers satisfying $x^2 < x$, x is positive.
Then we have proved the proposition.

(b) We disprove p by contradiction.
Assume for real number x, $x^2 < x < x^3$.
Since $x^2 < x$, we know that x is positive through part (a).
Since x is positive and $x^3 > x^2$, $\frac{x^3}{x} > \frac{x^2}{x}$, that is, $x^2 > x$.
$x^2 > x$ and $x^2 < x$ causes a contradiction. Therefore there does not exist such real number x.