EECS 203: Discrete Mathematics
Fall 2023
Homework 11

Due **Tuesday, December 5**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $9 + 2$ Total Points: $100 + 18$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

Reminder: Make sure to leave your answers in combination, permutation, or factorial form and **not** simplified.

1. Bayes Easy [12 points]

Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.

(a) What is the probability that someone who tests positive has the genetic disease?

(b) What is the probability that someone who tests negative does not have the disease?

Solution: (a) E: the person tests positive

F: the person has the disease

$$\Rightarrow P(F) = \frac{1}{10000}, \; P(\bar{F}) = \frac{10000-1}{10000} = \frac{9999}{10000}$$

$$P(E|\bar{F}) = 0.02\% = \frac{2}{10000} = \frac{1}{5000}$$

$$P(E|F) = 99.9\% = \frac{999}{1000}$$

(a)

$$\Rightarrow P(F|E) = \frac{P(E|F)\cdot P(F)}{P(E)} = \frac{P(E|F)\cdot P(F)}{P(E|F)\cdot P(F) + P(E|\bar{F})\cdot P(\bar{F})}$$

$$= \frac{\frac{999}{1000}\cdot\frac{1}{10000}}{\frac{999}{1000}\cdot\frac{1}{10000} + \frac{1}{5000}\cdot\frac{9999}{10000}} = \underline{\frac{5\cdot 999}{5\cdot 999 + 9999} = \frac{555}{1666}}$$

(b)

$$P(\bar{F}|\bar{E}) = \frac{P(\bar{E}|\bar{F})\cdot P(\bar{F})}{P(\bar{E}|\bar{F})\cdot P(\bar{F}) + P(\bar{E}|F)\cdot P(F)} = \frac{(1-\frac{1}{5000})\cdot\frac{9999}{10000}}{(1-\frac{1}{5000})\frac{9999}{10000} + \frac{1}{1000}\cdot\frac{1}{10000}}$$

$$= \frac{4999\cdot 9999}{4999\cdot 9999 + 5}$$

2. Spaced Out [12 points]

A space probe heading to Mars sends messages back to Earth using bit strings. Suppose that it sends a '1' one-third of the time and a '0' two-thirds of the time. However, the communication channel is noisy—when a 1 is sent, it is possible that noise interferes, causing Earth to receive a 0 and vise versa. Probabilities of different situations are listed:

- When a 0 is sent, the probability that it is received correctly is 0.6.
- When a 0 is sent, the probability that it is received incorrectly (as a 1) is 0.4.
- When a 1 is sent, the probability that it is received correctly is 0.8.
- When a 1 is sent, the probability that it is received incorrectly (as a 0) is 0.2.

(a) Suppose Earth received a '0'. What is the probability that the probe sent '0'?

(b) The space probe then transmits the same bit as part (a) again, and Earth receives '0' a second time. What is the probability the probe sent a 0? You can assume that the event of a bit getting corrupted is independent of any other bit getting corrupted.

Solution:

(a) E = Earth receive '0'.

F = probe sent '0'

$\Rightarrow P(F) = \frac{2}{3},\ P(\bar{F}) = \frac{1}{3}$

$P(E|F) = 0.6,\ P(\bar{E}|F) = 0.4$

$P(\bar{E}|\bar{F}) = 0.8,\ P(E|\bar{F}) = 0.2$

$\therefore P(F|E) = \frac{P(F)\cdot P(E|F)}{P(E)} = \frac{P(F)\cdot P(E|F)}{P(E|F)\cdot P(F) + P(E|\bar{F})\cdot P(\bar{F})}$

$$\frac{P(F)\cdot P(E|F)}{P(E)} = \frac{P(F)\cdot P(E|F)}{P(E|F)\cdot P(F) + P(E|\bar{F})\cdot P(\bar{F})}$$

$$= \frac{\frac{2}{3}\cdot\frac{6}{10}}{\frac{6}{10}\cdot\frac{2}{3} + \frac{2}{10}\cdot\frac{1}{3}}$$

$$= \frac{12}{12+2} = \frac{6}{7}$$

(b) $E =$ Earth receive '0'.

$F =$ probe sent '0'

Now $P(F) = \frac{6}{7}$

$P(E|F) = 0.6$, $P(\bar{E}|F) = 0.4$ as the same

$P(\bar{E}|\bar{F}) = 0.8$, $P(E|\bar{F}) - 0.2$

$$\Rightarrow P(F|E) = \frac{P(F)\cdot P(E|F)}{P(E)} = \frac{P(F)\cdot P(E|F)}{P(E|F)\cdot P(F) + P(E|\bar{F})\cdot P(\bar{F})}$$

$$= \frac{\frac{6}{7}\cdot\frac{6}{10}}{\frac{6}{7}\cdot\frac{6}{10} + \frac{2}{10}\cdot\frac{1}{7}} = \frac{18}{19}$$

3. Aye Aye Esti-matey [8 points]

Give the tightest big-O estimate of the following functions:

(a) $g(n) = (3^n)\cdot(n^2 + \log n)\cdot(2n^4 + n) + (4n + n!)\cdot(1000^{n+1000} + n^n)$

(b) $f(n) = (n^2 + n\log n)\cdot\left(4n + \sum_{i=1}^{10} n^i\right)$

Note: $\sum_{i=1}^{10} n^i = n^1 + n^2 + n^3 + \cdots + n^{10}$

Solution: (a) $\max(n^2, \log n) = n^2$

$\max(2n^4, n) = 2n^4 \Rightarrow n^4$

$\max(4n, n!) = n!$

$\max(1000^{n+1000} + n^n) = n^n$

$3^n \cdot n^2 \cdot n^4 + n! \cdot n^n = n^6 \cdot 3^n + n! n^n,$

$\max(n^6 \cdot 3^n, n! n^n) = n! n^n$

$\therefore$ the tightest big-O estimate is

$\Rightarrow g(n) = O(n! n^n)$

(b) $\max(n^2, n\log n) = n^2$

$4n + \sum_{i=1}^{10} n^i = 4n + n + n^2 + n^3 + n^4 + \ldots + n^{10}$

$= 5n + n^2 + n^3 + n^4 + \ldots + n^{10}$

$\max(5n, n^2, n^3, \ldots, n^{10}) = n^{10}$

$\therefore$ the tightest big-O estimation is $n^2 \cdot n^{10} = n^{12}$

$\Rightarrow g(n) = O(n^{12})$

4. Al Gore, It Him [12 points]

Give a big-O estimate for the number of operations, where an operation is an addition or a multiplication, used in this segment of an algorithm (ignoring comparisons used to test the conditions in the **while** or **for** loop).

Hint: Your estimates may use more than one variable.

(a)
$t \leftarrow 0$
for $i := 1$ to n **do**
  for $j := 1$ to m **do**
    $t \leftarrow t + i + j$
  end for
end for

(b)
$t \leftarrow 0$
for $i := 1$ to n **do**
  $t \leftarrow t \cdot 2$
end for
for $j := 1$ to m **do**
  $t \leftarrow t + j$
end for

(c)
$t \leftarrow 0$
$i \leftarrow 1$
while $i \leq n$ **do**
  $t \leftarrow t - i$
  $i \leftarrow i \cdot 3$
end while

Solution:

(a) operations: $\overbrace{m + m + m + \dots + m}^{n \text{ times}} = mn$

$f(n) = O(nm)$

(b) $\vdots\,]\, n$
$+$
$\vdots\,]\, m$

$f(n) = O(n+m)$

(c) $\overbrace{i \longrightarrow 3i \longrightarrow 9i \longrightarrow \cdots \longrightarrow n}^{\log_3 n \text{ times}}$

$\therefore f(n) = O(\log n)$

(more precisely, $O(\log_3 n)$)

5. Breakout Room [12 points]

In a class with 34 students there are 6 breakout rooms, with 3, 3, 4, 7, 8, and 9 students in each room, respectively.

(a) Suppose we pick a room at random, and consider X to be the random variable defined by the number of people in that room. What is the expected value of X?

(b) Now suppose we pick one of the students at random. Let Y be the random variable defined by the number of people in that student's room. What is the expected value of Y?

Solution:

(a) $S = \{\text{Room1}, \dots \text{Room6}\}$

$X(s)$ = number of students in room s.

$$E(X) = \sum_{s \in S} p(s) \cdot X(s)$$

$$= \frac{1}{6} \times (3 + 3 + 4 + 7 + 8 + 9) = \frac{34}{6} = \frac{17}{3}$$

(b) $S = \{\text{student1}, \dots \text{student34}\}$

$Y(s)$ = the number of students in the room student s is in.

$$E(Y) = \sum_{s \in S} p(s) \cdot Y(s) = \frac{1 \cdot 3}{34} \cdot 3 + \frac{1 \cdot 3}{34} \cdot 3 + \frac{1 \cdot 4}{34} \cdot 4 + \frac{1 \cdot 7}{34} \cdot 7 + \frac{1 \cdot 8}{34} \cdot 8 + \frac{1 \cdot 9}{34} \cdot 9$$

$$= \frac{228}{34} = \frac{114}{17}$$

6. Rolling Dice [12 points]

You roll a fair six-sided die 12 times. Find the probability that:

(a) Exactly two rolls come up as a 6.

(b) Exactly two rolls come up as a 4, given that the first four rolls each came up as 3.

(c) At least two rolls come up as a 6.

Solution:

E = exactly two rolls come up as 6

(a) $p = \frac{|E|}{|S|} = \frac{\binom{12}{2}\cdot(6-1)^{10}}{6^{12}} = \frac{\binom{12}{2}\cdot 5^{10}}{6^{12}}$

(b) E = exactly two rolls come up as 4

F = the first four rolls come up as 3

$\Rightarrow P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{\frac{|E\cap F|}{|S|}}{\frac{|F|}{|S|}} = \frac{|E\cap F|}{|F|}$

$= \frac{\binom{12-4}{2}\cdot(6-1)^{12-4-2}}{6^8} = \frac{\binom{8}{2}\cdot 5^6}{6^8}$

(c) E = at least two rolls come up as 6

$\bar{E}$ = no roll or one roll come up as 6

$P(E) = 1 - P(\bar{E}) = 1 - \frac{|\bar{E}|}{|S|} = 1 - \frac{5^{12}+\binom{12}{1}5^{11}}{6^{12}}$

$= \frac{6^{12}-5^{12}-12\cdot 5^{11}}{6^{12}}$

≈ 0.62

7. More Dice [10 points]

Suppose Emily is rolling a pair of standard dice until the dice roll sums to 8 three times. What is the probability that they will roll more than 4 times?

For example, some sequences of rolls include:

$$(1,2),(4,6),\underline{(4,4)},\underline{(5,3)},(2,5),(2,3),\underline{(6,2)}$$

$$\underline{(4,4)},(2,1),(4,5),(6,6),\underline{(5,3)},(4,3),(5,5),\underline{(4,4)}$$

Solution: $S = \{(1,1),(1,2),\ldots,(6,6)\}, |S| = 36$

$E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}, |E| = 5$

$\therefore P(\text{one time in } E) = \frac{5}{36}$

$\therefore P\left(\begin{matrix}2 \text{ in term } 1^{st}, 2^{nd}, 3^{rd} \text{ in } E, \\ \text{and } 4^{th} \text{ in } E\end{matrix}\right) = \binom{3}{2}\left(\frac{5}{36}\right)^2\left(\frac{31}{36}\right)\cdot\left(\frac{5}{36}\right)$

$P(1^{st}, 2^{nd}, 3^{rd} \text{ term in } E) = \left(\frac{5}{36}\right)^3$

$\therefore P(\text{roll more than 4 times})$

$= 1 - P\left(\begin{matrix}2 \text{ in term } 1^{st}, 2^{nd}, 3^{rd} \text{ in } E \\ \text{and } 4^{th} \text{ in } E\end{matrix}\right)$

$- P(1^{st}, 2^{nd}, 3^{rd} \text{ term in } E)$

$= 1 - 3\cdot\left(\frac{5}{36}\right)^3\left(\frac{31}{36}\right) - \left(\frac{5}{36}\right)^3$

8. Mystery Boba [10 points]

Isabel loves to get bubble tea on campus. On any given day, there is 15% chance she gets taro milk tea, 25% chance she gets a matcha latte, 40% chance she gets passion fruit tea, and 20% chance she doesn't get any bubble tea. Assume Isabel has a maximum of one bubble tea every day.

(a) In a 7-day week, what is the probability that Isabel gets 2 taro milk teas, 1 matcha latte, and 2 passion fruit teas (in any order)?

(b) In a 7-day week, what is the probability that Isabel gets exactly 4 passion fruit teas?

(c) Over 14-days, what is the expected number of taro milk teas Isabel gets?

Solution:

(a) P(Isabel gets 2 taro milk tea, 1 matcha latte, 2 passion fruit teas.)

$$= \binom{7}{2} \cdot \binom{5}{1} \cdot \binom{4}{2} \cdot (0.15)^2 \cdot (0.25)^1 \cdot (0.4)^2 \cdot (0.2)^2$$

(b) P(Isabel gets exactly 4 passion fruit teas.)

$$= \binom{7}{4} \cdot (0.4)^4 \cdot (0.6)^3$$

(c) E(taro milk teas on a day) =

1 · P(get taro milk tea one a day) = 1 · 0.15

= 0.15

∴ E(taro milk teas in 14 days) $= \sum_{i=1}^{14} E_i = 14 \times 0.15$

$= 2.1$

9. The 101 Dalmations Binary Ballet [12 points]

Consider a binary sequence of length 14 selected at random. What is the expected number of times 101 appears in the sequence? For example, it appears 4 times in the string

$$\overline{10\underline{1}}\underline{01}0000\overline{10\underline{1}}\underline{01}.$$

Solution:

$\square\square\ldots\square\square\square$

There are 12 x 3-bit strings together in the sequence

$\because$ Expeccation can be added up without independent

$\therefore$ The expected value of 101 in the 12 substrings

$$E(X) = E(X_1 + X_2 + \ldots + X_{12}) = 12E(X_1)$$

For one 3-bit substring, $P(101) = (\frac{1}{2})^3 = \frac{1}{8}$

$$\therefore E(X) = 12 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$