Groupwork

1. Grade Groupwork 9

Using the solutions and Grading Guidelines, grade your Groupwork 9:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2	-0	-0	-2	-1	-1							11 /15
Problem 3	-0	-2	-3	-0	-0							10 /15
Total:												21 /30

Previous Groupwork 9(1): Square the Cycle [15 points]

Prove that every n-node graph ($n \geq 3$) in which all nodes have degree at least $\lceil \sqrt{n} \rceil$ has a 3-cycle subgraph or a 4-cycle subgraph.

Hint: One useful concept is the neighborhood of a vertex; the neighborhood of $v \in V$ is the set $N(v) = \{u \in V : u \text{ is adjacent to } v\}$. We can also define the neighborhood of a set $A \subseteq V$:

$$N(A) = \{u \in V : u \text{ is adjacent to some } v \in A\}.$$

We recommend using a proof by contradiction, although this can also be done with a clever direct proof. Suppose a graph satisfying the above condition does not have a 3-cycle or 4-cycle. Fix a vertex $v \in V$. What can we say about the size of $N(v)$? What about $N(N(v))$?

We are good, but we need to consider more accurately and carefully!

Solution:

Seeking contradiction, assume:

$\exists$ n-node graph ($n \geq 3$) s.t. $\forall v \in V$, $deg(v) \geq \lceil \sqrt{n} \rceil$, but it does not contain a 3-cycle or 4-cycle subgraph. +3

Let v be an arbitrary node of it, v has $\geq \lceil \sqrt{n} \rceil$ neighbours.

$\Rightarrow$ each node in $N(v)$ also has $\geq \lceil \sqrt{n} \rceil$ deg. +4

$\Rightarrow |N(N(v))| \geq |N(v)| \cdot \lceil \sqrt{n} \rceil \geq \lceil \sqrt{n} \rceil \cdot \lceil \sqrt{n} \rceil \geq n$ +2 (-1) -2

Proof: $|N(N(v))| - \{v\}$ and $N(v)$ are disjoint $\Rightarrow |N(N(v)) - \{v\}| \geq n - \sqrt{n}$

This implies there are at least n distinct nodes in the graph, which contradicts the fact the graph has only n nodes.

i.e. Claim 3 but lacks the proof and deduction: $|\{v\} \cup N(v) \cup N(N(v)) - \{v\}| \geq 1 + \sqrt{n} + n - \sqrt{n} = n + 1 > n = |V|$ $\not\downarrow$

$\therefore$ The graph must contain at least one cycle

$\therefore$ contradicts the assumption that it does not contain a 3-cycle or 4-cycle subgraph.

+2 (-1)

QED

Previous Groupwork 9(2): The Office Allocation [15 points]

Consider a new office building with n floors and k offices per floor in which you must assign $2nk$ people to work, each sharing an office with exactly one other person. Find a closed form solution for the number of ways there are to assign offices if from floor to floor the offices are distinguishable, but any two offices on a given floor are not.

We used mathematica to test our result (by $k=2$) and found that it's the same as the answer $\frac{(2k n)!}{2^{kn}(k!)^n}$. It might not be the best solution, but we did our best!

Solution:

(1) To assign $2k \cdot n$ people to n floors, $2k$ people per floor

$$C(2nk, 2k) \cdot C(2(n-1)k, 2k) \cdot \ldots \cdot \underbrace{C(2k, 2k)}_{=1}$$

(2) To assign the $2k$ people per floor to k offices (not distinguishable)

(2a) $$C(2k,2) \cdot C(2(k-1),2) \cdot \ldots \cdot \underbrace{C(2,2)}_{=1}$$

(2b) Since k offices in a floor is not distinguishable,

$$\Rightarrow \frac{C(2k,2) \cdot C(2(k-1),2) \cdot \ldots \cdot C(2,2)}{P(k,k)}$$

(for it's too complex an answer, deduct 5 points from 2 and 3 which is a smarter way)

(2c) Since there are n floors

$$\Rightarrow \left(\frac{C(2k,2) \cdot C(2(k-1),2) \cdot \ldots \cdot C(2,2)}{P(k,k)}\right)^n$$

(3) So in total:

$$\prod_{i=1}^{n} C(2ik, 2k) \cdot \left(\frac{\prod_{j=1}^{k} C(2j,2)}{k!}\right)^n$$

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Values

n	1	2	3	4	5
$2^{-3n}(4n)!$	3	630	935550	5108103000	74246277105000

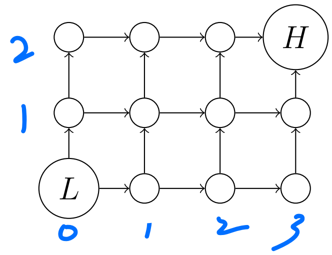
Values

n	1	2	3	4	5
$8^{-n}\Gamma(4n+1)$	3	630	935550	5108103000	74246277105000

2. Lily's Lily Pads [15 points]

Lily the Frog is on a lily pad and wants to get to her home! She can jump from lily pad to lily pad to help reach this goal. The lily pads are arranged in a grid. Lily starts on the **bottom-left** lily pad, and her home is at the **top-right** lily pad. Lily can only move one lily pad **upward** or one lily pad **rightward** at a time.

Each lily pad has coordinates of the form $(x, y) \in \mathbb{N} \times \mathbb{N}$, where x represents how far rightward a point is from the left of the grid, and y represents how far upward a point is from the bottom of the grid. Lily starts at location $(0, 0)$, and her home is at location $(x_H, y_H) \in \mathbb{N}\times\mathbb{N}$.



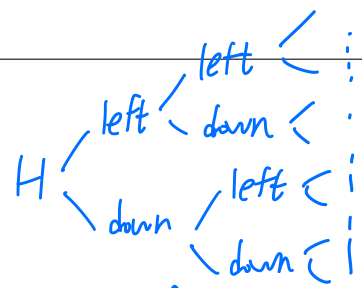
In the above example, $(x_H, y_H) = (3, 2)$. In the general case, though, (x_H, y_H) could be any ordered pair of natural numbers.

(a) How many different paths can Lily take to get home?

(b) Lily's frog friend, Francine, is also on the grid at coordinates $(x_F, y_F) \in \mathbb{N} \times \mathbb{N}$ such that $0 \le x_F \le x_H$ and $0 \le y_F \le y_H$. What is the probability that Lily meets Francine on her path home? You may assume that any two paths home are equally likely for Lily to take.

Solution:

(a)



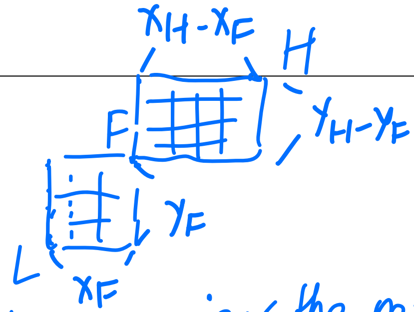
i.e. Starting from home, tracking back. Every last move is "left" or "down", s.t. "left" $\# \le x_H$, "down" $\# \le y_H$. Once "left" $\# = x_H$ $\wedge$ "down" $\# = y_H$, a path is formed.

Then we know: (1) for one path, total move is $x_H + y_H$, which contains "down" $\# = y_H$, "left" $\# = x_H$.

$\therefore$ This problem is equivalent to: from $x_H + y_H$ steps, choose x_H steps to be "left" (and the rest one "down".

$\therefore$ total $\# = \binom{x_H + y_H}{x_H} = \binom{x_H + y_H}{y_H}$

(b)



We can view the paths where Lily meets Francine as two stages: (1) Lily starts from (0,0), ends at Francine. (2) Lily starts from Francine (x_F, y_F), ends at Home (x_H, y_H)

Similar to question a:

$\#$paths from $(0,0)$ to $(x_F, y_F) = \binom{x_F + y_F}{x_F} = \binom{x_F + y_F}{y_F}$

$\#$paths from (x_F, y_F) to $(x_H, y_H) = \binom{x_H + y_H - x_F - y_F}{x_H - x_F} = \binom{x_H + y_H - x_F - y_F}{y_H - y_F}$

$$\therefore p = \frac{|E|}{|S|} = \frac{\binom{x_F + y_F}{x_F} \cdot \binom{x_H + y_H - x_F - y_F}{x_H - x_F}}{\binom{x_H + y_H}{x_H}}$$

3. Random Connections [15 points]

We say that a *random graph* is an undirected graph where, for each pair of vertices, there is an independent $\frac{1}{3}$ chance that they are adjacent. It's a bit like Lily's pond, except that the vertices aren't in a grid, and you can move in any direction.

We want to learn about the connectedness of random graphs.

Let G be a finite random graph. Let's split the vertices into two nonempty sets, $A, B \subseteq V$.

(a) Let $a \in A$. What is the probability that no element of B is adjacent to a?

(b) What is the probability that there is some $a \in A$ and $b \in B$ such that a is adjacent to b?

(c) Let's imagine doing this with larger and larger graphs. Define $f(a, b)$ be your answer to the previous problem when $|A| = a$ and $|B| = b$. What is

$$\lim_{a+b\to\infty} f(a, b)?$$

(d) This isn't quite a proof, but your answer to (c) might lead you to some ideas. What might you conjecture about the connectedness of infinite random graphs?

Solution: (a) P_i: vertex i in B is of adjacent to a

$\Rightarrow \forall i \in B,\ P_i = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ P(no element of B is adjacent to a)

$= \prod_{i \in B} P_i = \left(\frac{2}{3}\right)^{|B|}$

(b) P(some $a \in A$ and $b \in B$ are adjacent)

$= 1 - P(\forall a \in A \forall b \in B$, a, b are not adjacent)

$P(\forall a \in A \forall b \in B$, a, b are not adjacent)

$= \prod_{a \in A}$ P(no element of B is adjacent to a)

$= \prod_{a \in A} \left(\frac{2}{3}\right)^{|B|} = \left(\frac{2}{3}\right)^{|B| \cdot |A|}$

$\therefore P$(some $a \in A$ and $b \in B$ are adjacent) $= 1 - \left(\frac{2}{3}\right)^{|A| \cdot |B|}$

(c) $f(a,b) = 1 - \left(\frac{2}{3}\right)^{ab}$

$\Rightarrow \lim_{a,b \to \infty} f(a,b) = \lim_{n \to \infty} 1 - \left(\frac{2}{3}\right)^{\infty} = 1$

(d) Conjecture: if for an infinite random graph, the chance that two vertices are adjacent $\neq 0$, then some must exist some adjacent vertices in the graph.