EECS 203: Discrete Mathematics
Fall 2023
Homework 10

Due **Tuesday, November 28**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

Reminder: Make sure to leave your answers in combination, permutation, or factorial form and **not** simplified.

1. The Boxer and the Baller [12 points]

How many ways are there to distribute seven balls into five boxes, where each box must have at least one ball in it, if

(a) both the balls and boxes are unlabeled?

(b) the balls are labeled, but the boxes are unlabeled?

(c) both the balls and boxes are labeled?

Solution:

(a) Place 5 balls in 5 boxes, the left 2 balls can either be placed together in one box or be placed separately in two boxes.

$\therefore$ ways # $= 2$

(b) Case 1: $2+2+1+1+1$, $\# = \binom{7}{2}\cdot\binom{5}{2}$.

Case 2: $3+1+1+1+1$, $\# = \binom{7}{3}$

$\therefore \# = \binom{7}{2}\binom{5}{2}+\binom{7}{3} = 245$

(c) Case 1: $2+2+1+1+1$, $\# = \binom{7}{2}\binom{5}{2}\cdot P(5,5)$

Case 2: $3+1+1+1+1$, $\# = \binom{7}{3}\cdot P(5,5)$

$\therefore \# = P(5,5)\left[\binom{7}{2}\binom{5}{2}+\binom{7}{3}\right] = 29400$

2. Sweepstakes Sweep [12 points]

Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

Solution:

$|S| = |\{\text{selections of 3 random winners out of 100 people}\}| = \binom{100}{3}$

$|E| = |\{\text{the 3 guys are winners}\}| = \binom{3}{3}$

$\therefore P(E) = \frac{|E|}{|S|} = \frac{\binom{3}{3}}{\binom{100}{3}} = \frac{3 \cdot 2 \cdot 1}{100 \cdot 99 \cdot 98}$

3. Mississippi Bananas [8 points]

How many different strings can be made by rearranging the letters in the word BANANANANAS?

Solution: String length : 11

① Place the B: $\binom{11}{1}$ choices

② Place the As: $\binom{10}{5}$ choices

③ Place the Ns: $\binom{5}{4}$ choices

④ Place the S: $\binom{1}{1}$ choices

$\therefore$ Answer: $\binom{11}{1}\cdot\binom{10}{5}\cdot\binom{5}{4}\cdot\binom{1}{1}$

4. Probabili-Tee [16 points]

Tom has 30 T-shirts where 10 are blue, 5 are red, and 15 are green. Frank has 20 T-shirts where 13 are blue, 2 are red, and 5 are green. Both Tom and Frank own 1 green EECS 203 T-shirt, but only Tom owns 1 red and 1 blue EECS 203 T-shirt. Assume Frank and Tom pick and wear T-shirts uniformly at random.

(a) What is the probability that Tom and Frank are both wearing their green EECS 203 T-shirts, given that they're both wearing green T-shirts?

(b) What is the probability that Tom and Frank are both wearing a green T-shirt, given that they're both wearing the same type of T-shirt (both EECS 203 T-shirts or both not EECS 203 T-shirts)?

Solution: (a) E_1 = Tom is wearing green EECS 203 T-shirt

F_1 = Tom is wearing green T-shirt.

E_2 = Frank is wearing green EECS 203 T-shirt

F_2 = Frank is wearing green T-shirt.

$$P(E_1|F_1) = \frac{P(E_1 \cap F_1)}{P(F_1)} = \frac{\frac{1}{30}}{\frac{15}{30}} = \frac{1}{15}$$

$$P(E_2|F_2) = \frac{P(E_2 \cap F_2)}{P(F_2)} = \frac{\frac{1}{20}}{\frac{5}{20}} = \frac{1}{5}$$

$\because$ $E_1|F_1$ and $E_2|F_2$ are independent events

$\therefore P(E_1|F_1 \cap E_2|F_2) = P(E_1|F_1) \cdot P(E_2|F_2) = \frac{1}{15} \cdot \frac{1}{5} = \frac{1}{75}$

(b) F = Tom and Frank are bothing wearing a green T-shirt.

E = Tom and Frank are wearing the same type of T-shirt

$\Rightarrow E \cap F$ = Tom and Frank are wearing the same type of green T-shirt

$$P(E \cap F) = \frac{\binom{1}{1}\cdot\binom{1}{1} + \binom{15-1}{1}\cdot\binom{5-1}{1}}{\binom{30}{1}\cdot\binom{20}{1}} = \frac{\binom{1}{1}\binom{1}{1}+\binom{14}{1}\binom{4}{1}}{\binom{30}{1}\cdot\binom{20}{1}}$$

$$P(E) = \frac{\binom{3}{1}\cdot\binom{1}{1} + \binom{30-3}{1}\cdot\binom{20-1}{1}}{\binom{30}{1}\cdot\binom{20}{1}} = \frac{\binom{3}{1}\cdot\binom{1}{1}+\binom{27}{1}\cdot\binom{19}{1}}{\binom{30}{1}\cdot\binom{20}{1}}$$

$$\Rightarrow P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\binom{1}{1}\binom{1}{1}+\binom{14}{1}\binom{4}{1}}{\binom{3}{1}\binom{1}{1}+\binom{27}{1}\binom{19}{1}} \quad \left(= \frac{19}{172}\right)$$

5. Independence Day [10 points]

Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Given that all bitstrings are equally likely to occur, are E and F independent?

Solution:

$S = \{111, 110, 101, 100, 011, 010, 001, 000\}$

$E = \{111, 100, 010, 001\}$

$F = \{101, 100, 110, 111\}$

$E \cap F = \{100, 111\}$

$\therefore P(E|F) = \frac{|E \cap F|}{|F|} = \frac{1}{2}$

$P(E) = \frac{|E|}{|S|} = \frac{4}{8} = \frac{1}{2}$

$\Rightarrow P(E|F) = P(E)$

$\Rightarrow$ E and F are independent.

6. $7 + 5 =$ **[12 points]**

Suppose we roll five fair **seven-sided** dice (there are seven faces, labeled 1 through 7).

(a) What is the probability that exactly four come up even?

(b) What is the probability that exactly two come up even?

Solution: for every dice

E = the dice comes up even

$E = \{2,4,6\}$, $S = \{1,2,3,4,5,6,7\}$, $P(E) = \frac{3}{7}$

$P(\bar{E}) = 1 - P(E) = \frac{4}{7}$

E_n = exactly n dices out of 5 comes up even (the other 5-n dices come up odd)

$\Rightarrow |E_n| = \binom{5}{n} \cdot 3^n \cdot 4^{5-n}$

$\therefore P(n \text{ out of } 5 \text{ is even}) = \frac{|E_n|}{|S|} = \frac{\binom{5}{n} \cdot 3^n \cdot 4^{5-n}}{7^5}$

$\therefore$ (a) $P(4 \text{ out of } 5 \text{ is even}) = \frac{\binom{5}{4} \cdot 3^4 \cdot 4^1}{7^5} = \frac{1620}{16807}$

(b) P(2 out of 5 is even) $= \frac{\binom{5}{2}3^2 \cdot 4^3}{7^5} = \frac{5760}{16807}$

7. Driver's License [20 points]

Suppose we're trying to come up with a new license plate system that must contain exactly 6 characters, each of which can be any of the following: an uppercase letter, lowercase letter, digit, or underscore character. How many possible license plate names are there given the following specifications?

26 26 10 1

(a) License plates cannot have a number character.

(b) License plates must have exactly one underscore character, which cannot be at the beginning or end of the license plate.

(c) License plates must have at least one number.

(d) License plates must have at least one number or at least one underscore character.

Justify your answer for each part.

Solution:

(a) $26+26+1=53$. From 53 chrcters choose 1 every time.

$\therefore$ plate # $= 53^6$ by product rule

(b) 1. place the underscore character in the middle: 4

2. fill the other 5 characters: $(26+26+10)^5 = 62^5$ by product rule.

$\therefore$ plate # $= 4 \cdot 62^5$ by product rule

(c) Consider arrangements where there is not number: 53^6

All plates: 63^6

$\therefore$ plate # $= 63^6 - 53^6$ by difference rule

(d) 1. list all possible plates without restrictions: 63^6 by product rule
2. list all plates which cannot satisfy the requirements, i.e. plates with no number and no underscore character: $(26+26)^6 = 52^6$ by product rule
3. By difference rule, plates $\# = 63^6 - 52^6$

8. Pip Pip Hooray! [10 points]

One pip (small dot on the face of a die) is randomly removed from a standard eight-sided die (where its 8 faces respectively have $\{1, 2, \ldots, 8\}$ pips). **Each pip** has an equal probability of being removed. This means, for example, the face with 8 pips has a greater probability of losing a pip compared to the face with 1 pip.

What is the probability of rolling an even number on this die?

Solution: set E_n = the pip removed is on the face with n pips, $n = 1, 2, \ldots, 8$.

$\Rightarrow |E_n| = n$, $P(E_n) = \frac{|E_n|}{|S|} = \frac{n}{\sum_{n=1}^{8} n} = \frac{n}{\frac{(1+8)\cdot 8}{2}} = \frac{n}{36}$

$\Rightarrow P(E_1) = \frac{1}{36}, P(E_2) = \frac{2}{36}, \ldots, P(E_8) = \frac{8}{36}$

set F = roll an even number on the die.

$\Rightarrow F|E_n$ = when the pip removed is on the face with n pips roll an even number on the die.

$\Rightarrow |F|E_1| = 5$, $|F|E_2| = 3$, $|F|E_3| = 5$, $|F|E_4| = 3$,
$|F|E_5| = 5$, $|F|E_6| = 3$, $|F|E_7| = 5$, $|F|E_8| = 3$.

$\because E = E_1 + E_2 + \ldots + E_8$, $P(E) = 1$

$$\therefore P(F) = P(F|E_1)\cdot P(E_1) + P(F|E_2)\cdot P(E_2) + \ldots + P(F|E_8)\cdot P(E_8) = \sum_{n=1}^{8} P(F|E_n)\cdot P(E_n)$$

$$= \frac{1}{36}\cdot\frac{5}{8} + \frac{2}{36}\cdot\frac{3}{8} + \frac{3}{36}\cdot\frac{5}{8} + \ldots + \frac{8}{36}\cdot\frac{3}{8}$$

$$= \frac{1+3+5+7}{36}\cdot\frac{5}{8} + \frac{2+4+6+8}{36}\cdot\frac{3}{8}$$

$$= \frac{35}{72}$$