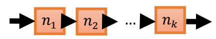
① ex. string (8char) each from 0-9, a-z, A-Z: $(10+26+26)^6 = 62^6$ (sum+product Rule)

Lec 21 Handout: Intro to Counting

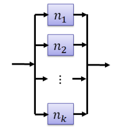
- **Product Rule:** k sequential **tasks/stages**, with ***exactly*** n_i ***possible choices*** for task i, means:

  $\prod_{i=1}^{k} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$ possible choices



- **Sum Rule:** k parrallel **tasks**, with ***exactly*** n_i ***possible choices*** for task i, means:

  $\sum_{i=1}^{k} n_i = n_1 + n_2 + n_3 + \ldots + n_k$ possible choices



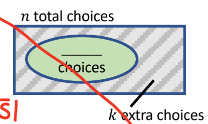
- **Division Rule:** A process with n **total choices**, and each choice represented **exactly k times**, means:

  there are $\frac{n}{k}$ possible choices

- **Difference Rule:** A process with n **total choices**, which has k **extra choices** that shouldn't have counted, means:

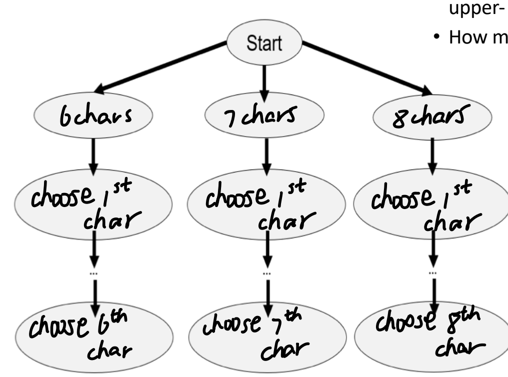
  $n-k$ possible choices

  $|S| = |U| - |\bar{S}|$



Password Counting

- A password must be 6, 7, or 8 characters
- Each character must be a digit (0-9) or an upper- or lower-case letter (a-z, A-Z)
- How many different passwords?



\# choices in first branch: 62^6

\# choices in second branch: 62^7

\# choices in third branch: 62^8

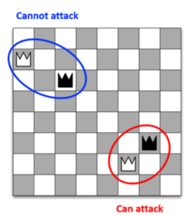
\# choices **total**: $62^6 + 62^7 + 62^8$

Review: $|P(S)| = 2^{|S|}$

$|A \cdot B| = |A| \cdot |B|$

Chessboard Arrangements

- A king can attack any adjacent square, including diagonally.
- How many ways can a white and black king be arranged so that neither is attacking the other?



place black → place white (product rule)

First Case: Place black King on a corner

\# possibilities in first case:

$4 \cdot (64-4) = 4 \cdot 60$

Second Case: Place black King on a side

\# possibilities in second case:

$(28-4) \cdot (64-6) = 24 \cdot 58$

Third Case: Place black King on the interior

\# possibilities in third case:

$6 \cdot 6 \cdot (64-9) = 36 \cdot 55$

Total # of ways?

$4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55$

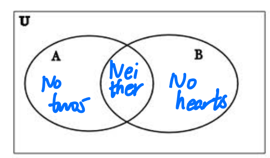
Total # of ways if **both** kings are black?

$\frac{4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55}{2}$

Inclusion-Exclusion

How many ordered 5-card poker hands don't contain a two, or don't contain a heart?

52 cards in a deck, 4 twos, 13 hearts, 1 two of hearts



The Inclusion/Exclusion Principle:

$|A \cup B| = |A| + |B| - |A \cap B|$

\# that don't contain a two:

start: $52-4=48$ $|A| = 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$

\# that don't contain a heart:

start: $52-13=39$ $|B| = 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35$

\# that don't contain a two **and** don't contain a heart:

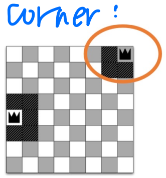
start: $39-3=36$ $|A \cap B| = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32$

\# that don't contain a two **or** don't contain a heart:

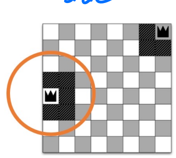
$|A \cup B| = |A| + |B| - |A \cap B| =$

$48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 + 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 - 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32$

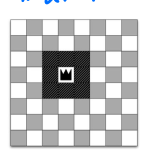
corner:



side:

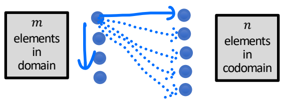


interior:



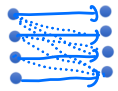
Counting Functions

How many functions from a set of size m to a set of size n?



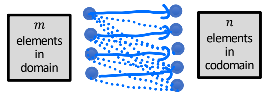
$\underbrace{n\cdot n\cdot\ldots\cdot n}_{m \text{ times}} = n^m$

How many **bijections** $f:\{1,\ldots,n\}\to\{1,\ldots,n\}$?



$n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1 = n!$

How many **one-to-one** $f:\{1,\ldots,m\}\to\{1,\ldots,n\}$? $(m \le n)$



$\underbrace{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-m+1)}_{m\,\#}$

Counting Blitz: Strings of English Letters

- How many strings of 8 English letters are there
  1. That contain no vowels, if letters can be repeated? 元音(a e i o u)
  2. That contain no vowels, if letters cannot be repeated?
  3. That start with a vowel, if letters can be repeated?
  4. That contain at least one vowel, if letters can be repeated?
  5. That contain exactly one vowel, if letters can be repeated?
  6. That contain exactly 2 vowels, letters can be repeated?
  7. That contain exactly 2 vowels, not consecutive, letters can be repeated?

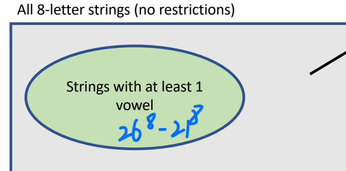
Counting Blitz: Strings of English Letters (work space)

1) $(26-5)^8 = 21^8$
2) $21\cdot 20\cdot\ldots\cdot 15\cdot 14 = \frac{21!}{13!}$
3) $5\cdot 26^{8-1} = 5\cdot 26^7$
4) $26^8 - (26-5)^8 = 26^8 - 21^8$
5) $8\cdot 5\cdot 21^7$
6) $\frac{8\cdot 7}{2}\cdot 5^2\cdot 21^6$
7) $\frac{2\cdot 6+6\cdot 5}{2}\cdot 5^2\cdot 21^6$

Strings with at least one vowel

How many strings of 8 lowercase English letters have at least one vowel? (26 letters, possibly with repeats, vowel=$\{a, e, i, o, u\}$)

All 8-letter strings (no restrictions)



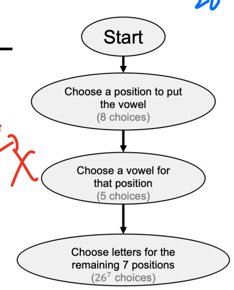
21^8

26^8

8-letter strings: 26^8

8-letter strings, no vowels: 21^8

8-letter strings, ≥ 1 vowels: $26^8 - 21^8$



cannot do this

Wrong!
Multiple ways to generate many strings

Even Worse:
The division rule only works when there are **the same number of ways to count each item.**

Here, the # of ways to count a string *depends on its # of vowels*, so the division rule won't help.