"Logical Equivalence" proof styles:
- Proof by contradiction $(\neg p \to F) \to p$
- Proof by contrapositive $(p \to q) \equiv (\neg q \to \neg p)$
- Proof by cases $((a \vee b) \wedge (a \to p) \wedge (b \to p)) \to p$

"Counting" proof styles:
- Proof by induction
- pigeonhole principle
- diagonalization

Some sets to know:
- $\mathbb{Z}$: integers
- $\mathbb{Z}^+$: positive integers
- $\mathbb{N}$: the natural numbers

$\in$: "in" / "is an element of"

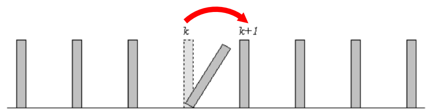
9/26/23

Lec 9: Proof by Induction - ANSWERS

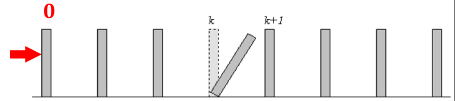
Let $P(n)$ be a predicate. $P(n)$: $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Goal: Prove that $P(n)$ is true for all $n \in \mathbb{N}$

Step 1: "Inductive Step"
If you can knock down one domino, then you can knock down the next one.
For any $k \in \mathbb{N}$, $P(k) \to P(k+1)$



Step 2: "Base Case"
You can knock down the first domino
$P(0)$



Therefore, you can knock down all dominos.

Example 1

Prove, $\forall n \in \mathbb{N}$, $P(n)$: $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Handout

2. **Inductive Step**: (1) Let k be an arbitrary natural number.
- (2) Assume $P(k)$: $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ ← Inductive Hypothesis
- (3) Want to show $P(k+1)$: $0 + 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

(4) Algebra, show $P(k) \to P(k+1)$

$$\begin{aligned} 0 + 1 + 2 + \cdots + k + (k+1) &= (0 + 1 + 2 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{Inductive Hypothesis} \\ &= \frac{(k+1)(k+2)}{2} && \text{Algebra} \end{aligned}$$

- Thus $P(k) \to P(k+1)$ for any natural number k

3. **Base case**: $P(0)$: $0 = \frac{0(0+1)}{2}$.

4. **Conclusion:** By induction, $P(n)$ is true for all natural numbers n.

Guide for Induction Proofs

- Restate the claim you are trying to prove
- **Base case**: Prove the claim holds for the "first" value of n
  - Prove $P(n_0)$ is true
- **Inductive Step**: Prove that $P(k) \to P(k+1)$ for an arbitrary integer k in the desired range.
  - Let k be an arbitrary integer with $k \geq n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

  Equivalently: Show $P(k-1) \to P(k)$
- Conclusion: explain that you've proven the desired claim.

Induction Proof: Another Equality

- Claim: $\underbrace{\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}}_{P(n)}$ $\quad \forall n \geq 1$

Inductive Step: Consider an arbitrary integer $k \geq 1$.

Assume **P(k)**: $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} = 1 - \frac{1}{3^k}$

Want to show **P(k+1)**: $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^{k+1}}$

$$\begin{aligned} \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} + \frac{2}{3^{k+1}} &= 1 - \frac{1}{3^k} + \frac{2}{3^{k+1}} && \text{(by I.H., i.e., apply } \mathbf{P(k)}) \\ &= 1 - \frac{3}{3} \cdot \frac{1}{3^k} + \frac{2}{3^{k+1}} \\ &= 1 - \frac{3}{3^{k+1}} + \frac{2}{3^{k+1}} \\ &= 1 - \frac{1}{3^{k+1}} \end{aligned}$$

Base Case: **P(1)**: $\frac{2}{3^1} = 1 - \frac{1}{3^1}$

$$\frac{2}{3^1} = \frac{2}{3} = 1 - \frac{1}{3} = 1 - \frac{1}{3^1}$$

By mathematical induction, the claim holds for all $n \geq 1$.

Warning: Always work in **One Direction**

Summary: To prove LHS = RHS (or LHS < RHS, etc.)

Correct approach:

1. Start with one side
2. Work your way in one direction until you get the other side

In your own words, why *shouldn't* we start with LHS = RHS (our desired conclusion) and work both sides of that equation until we get the same expression on both sides?

Answers will vary. Some possible answers:

- Logically, do this is equivalent to saying "if q, then true", which does not prove that q is true.
- Not every mathematical operation is reversible
- Doing so can lead to incorrect "proofs", e.g., 1024 = -57 because I can multiply both sides by 0 to get 0 = 0.

Induction Proof: An Inequality

- Claim: $2n + 3 \leq 2^n \quad \forall n \geq 4$

Inductive Step: Assume **P(k)** for some $k \geq 4$. That is, assume: $2k + 3 \leq 2^k$

Want to show **P(k+1)**: $2(k+1) + 3 \leq 2^{k+1}$

$$\begin{aligned} 2(k+1)+3 &= 2k+2+3 \\ &= 2k+3+2 \\ &\leq 2^k + 2 && \text{(by I.H., i.e., apply } \mathbf{P(k)}) \\ &< 2^k + 2^k && (\text{because } 2 < 2^k \quad \forall k > 1) \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Base Case: **P(4)**: $\mathbf{2(4) + 3 \leq 2^4}$

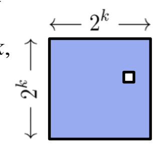
$$2(4) + 3 = 11 \leq 16 = 2^4$$

By mathematical induction, $2n + 3 \leq 2^n$ holds for all $n \geq 4$.

14

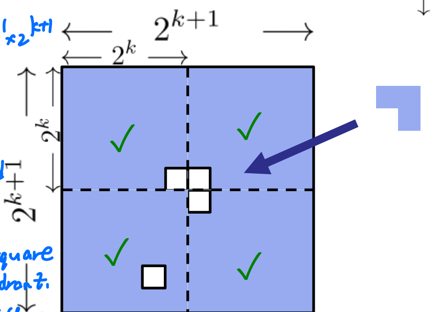
Tiling a Checkerboard

– Inductive Hypothesis: For an arbitrary positive integer k, a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominos.



– Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with any one square removed. We can tile it as follows: Didide checkerboard into four $2^k \times 2^k$ quadrants.

– The missing square is in one quadrant. From the other three quadrants, remove the square closest to the center of the checkerboard



– Assuming IH (Inductive Hypothesis), each of the four quadrants missing one square can be tiled

16

– Thus, ... (conclusion)

$k=1$