EECS 203: Discrete Mathematics
Fall 2023
Discussion 4 Notes

1 Definitions

- **Proof:**
- **Types of Proofs:**
  - **Direct Proof:**
  - **Proof by Contraposition:**
  - **Proof by Contradiction:**
  - **Proof by Cases:**
- **Disproof:**
- **Without Loss of Generality (WLOG):**
- **Unique:**

Solution:

- **Types of Proofs:**
  - **Direct Proof:** Prove that a statement is true without using any more advanced proof techniques (e.g. contrapositive, contradiction, cases).
  - **Direct Proof for $p \to q$:** Prove that if the proposition p is true, then the other proposition q is true "directly". Start by assuming that p is true, then make some deductions and eventually arrive at the conclusion that q must be true.

$$p \to q$$

  - **Proof by Contraposition:** Prove that "if p is true, then q is true" by proving that if q is false, then p is false (since these are logically equivalent).

$$\neg q \to \neg p$$

- **Proof by Contradiction:** Prove p is true by assuming $\neg p$, and arriving at a contradiction, i.e. a conclusion that we know is false.
  When using a proof by contradiction to prove "if p is true then q is true", we assume that p is true and that q is false, and derive a contradiction. This shows us that if p is true, then q is true.

  $$\neg(p \to q) \equiv (p \wedge \neg q) \to F \to \neg(p \wedge \neg q) \equiv (p \to q)$$

  A simpler way to view this: Assume p is true and show that

  $$\neg q \to \mathrm{F} \to q$$

- **Proof by Cases:** Prove by considering all possibilities, or all categories of possibilities (i.e., cases), and showing that in each of those cases, the proposition you're trying to prove is true.

- **Disproof:** To disprove a statement means to prove the negation of that statement.

  $$\text{Disprove } P(x) \equiv \text{Prove } \neg P(x)$$

  Note that if the statement you are trying to disprove is a for all statement, all you need to disprove it is a singular counterexample (since $\neg\forall x P(x) \equiv \exists x \neg P(x)$).

- **Without loss of generality (WLOG):** used when the same argument can be made for multiple cases, and there is some symmetry between variables.

- **Unique:** (exactly one) If we say something has a unique solution, we mean that there is a solution and that there is no other solution

2 Exercises

1. Contraposition vs Contradiction $\star$

Show that for all integers n, if $n^3 + 5$ is odd, then n is even, using

a) a proof by contraposition.

b) a proof by contradiction.

Note: The algebra in either case is the same. You don't need to rewrite the algebra for part (b), just reformat your proof from (a) into a proof by contradiction.

Solution:

a) We will prove the contrapositive of the proposition, which is: "if n is odd, then n^3+5 is even".
Since n is odd, n can be written as $2k+1$, where k is some integer. Then,

$$\begin{aligned} n^3+5 &= (2k+1)^3+5 \\ &= (8k^3+12k^2+6k+1)+5 \\ &= 8k^3+12k^2+6k+6 \\ &= 2(4k^3+6k^2+3k+3) \end{aligned}$$

So $n^3+5=2m$, where m is the integer $4k^3+6k^2+3k+3$. Because n^3+5 is two times some integer, we can say that n^3+5 is even.

b) We will use a proof by contradiction. Let n^3+5 be odd. *Seeking a contradiction*, assume that n is odd. Since n is odd, it can be written as $2k+1$, where k is some integer. So

$$\begin{aligned} n^3+5 &= (2k+1)^3+5 \\ &= (8k^3+12k^2+6k+1)+5 \\ &= 8k^3+12k^2+6k+6 \\ &= 2(4k^3+6k^2+3k+3) \end{aligned}$$

Since $n^3+5=2m$, for an integer m ($m=4k^3+6k^2+3k+3$), then n^3+5 is even. Since the premise was that n^3+5 is odd, this completes the contradiction. Therefore, our assumption that n is odd must be false, leading to the conclusion that n is even.

Note:
You can also start this proof by contradiction by assuming the negation of the entire "if ... then" statement. Here, this would entail starting with "Seeking contradiction, assume that n^3+5 is odd and n is odd." From here, the logic of finding a contradiction by showing that n^3+5 is even is almost identical.

2. Odd Proof III

Prove that for all integers a and b, if a divides b and $a+b$ is odd, then a is odd.

Solution: Proof by Contradiction

- We are supposed to prove: $[(a \text{ divides } b) \wedge (a+b \text{ is odd})] \rightarrow a$ is odd

- Seeking contradiction, assume the negation of the above statement: $\neg$ [[a divides b $\wedge$ $a + b$ is odd] $\rightarrow$ a is odd], which is (a divides b) $\wedge$ ($a + b$ is odd) $\wedge$ (a is even).

- Since a is even, $a = 2k$ for some integer k.

- Since a divides b we have $b = m \cdot a$.

- So, $a + b$ becomes $2k + m(a) = 2k + m(2k) = 2(k + km) = 2p$, where p is an integer equal to $k + km$

- Thus $a + b = 2p$ and is even. However, we had originally assumed that $a + b$ is odd. This leads to our **contradiction**.

- Hence the assumption in the second bullet point is false, and [(a divides b) $\wedge$ ($a + b$ is odd)] $\rightarrow$ a is odd

3. Proof Practice $\star$

Prove or disprove that for all irrational numbers x and rational numbers y, $2x - y$ is irrational.

Solution: Proof by Contradiction

We prove the statement via proof by contradiction. Let x be an arbitrary irrational number. Let y be an arbitrary rational number such that $y = \frac{a}{b}$ with a and b as integers and $b \neq 0$. For the sake of contradiction, we assume that $2x - y$ is rational, which means that $2x - \frac{a}{b}$ is rational. Then we can write $2x - \frac{a}{b} = \frac{p}{q}$ for some integers p and q with $q \neq 0$. This gives $2x = \frac{p}{q} + \frac{a}{b} = \frac{pb+aq}{bq}$, so $x = \frac{pb+aq}{2bq}$. Note that both the numerator and the denominator are integers, and that $2bq \neq 0$ since b and q were both nonzero. Therefore, x is, by definition, a rational number, which is a contradiction since x was assumed to be irrational. Hence, it must be that the sum of a rational number and an irrational number is irrational.

4. Polynomial Proof $\star$

Prove that there does not exist a rational number x satisfying the equation $x^3 + x + 1 = 0$.
Hint: Use the fact that 0 is an even number.
You can use the following lemmas without proving:

- Odd $\times$ Even = Even

- Odd $\times$ Odd = Odd

- Even $\times$ Even = Even

- Odd + Even = Odd
- Odd + Odd = Even
- Even + Even = Even

Solution:
Suppose there is. Let a solution be $\frac{a}{b}$, with a, b in reduced form.
Then we know that $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0 \iff a^3 + ab^2 + b^3 = 0$.
Since the RHS is even, LHS should be even as well.

Case 1: a, b both odd.
Then we have LHS = odd + odd $\times$ odd + odd = odd.
Case 2: a is odd, b is even.
Then we have LHS = odd + even + even = odd.
Case 3: a is even, b is odd.
(note that WLOG does not apply here since a, b are not symmetric; there is a term ab^2).
Then we have LHS = even + even + odd = odd.
Case 4: a, b are both even.
This cannot occur since a, b is in reduced form.

Each case results in LHS being odd which is a contradiction if LHS = 0. Thus we have proved by contradiction that the equation $x^3 + x + 1$ has no solution in $\mathbb{Q}$.

5. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a positive integer ≥ 2 such that p and 1 are the only positive integers that divide p.

Solution: We can consider two cases: either p is even, or it is odd.

- Case 1: Consider the even primes, which is just $p = 2$. $p^2 + 11 = 15$, and $15 = 5 \cdot 3$ is composite.
- Case 2: Now we consider the odd primes, or any prime greater than 2. Since p is odd, we have $p = 2k + 1$ for some integer $k > 1$. Then

$$p^2 + 11 = (2k+1)^2 + 11 = 4k^2 + 4k + 12 = 2(2k^2 + 2k + 6).$$

Hence, p^2+11 can be factored into 2 and $2k^2+2k+6$, therefore p^2+11 is composite.

We have exhausted all non-overlapping cases and proved that for all primes p, $p^2 + 11$ is composite.

6. Rational Proof $\star$

1. Prove or disprove: For all nonzero rational numbers x and y, x^y is rational
2. Prove or disprove: For all nonzero integers x and y, x^y is rational

Solution:

1. This is false. Let $x = 2$ and $y = \frac{1}{2}$. Then $x^y = \sqrt{2}$ which is irrational.
2. This is true. We prove this by cases. Case 1: $y > 0$ Then x^y is x multiplied by itself y times - and thus x^y is an integer. As we know all integers are rational, x^y must be rational. Case 2: $y < 0$ Then $x^y = \frac{1}{x^{-y}}$. As $y < 0$, $-y > 0$ so x^{-y} is an integer. As both 1 is an integer, and x^{-y} is an integer, we know $\frac{1}{x^{-y}}$ is rational.

7. Proving the Triangle Inequality

Prove the triangle inequality, which states that for all real numbers x and y, we have $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

Solution: This is a proof by cases. There are 4 cases to consider:

- x and y are both nonnegative
- x and y are both negative
- $x \geq 0$, $y < 0$, $x \geq -y$
- $x \geq 0$, $y < 0$, $x < -y$

Since x and y play symmetric roles (you can switch the values of x and y without impacting the validity of the triangle inequality), we can assume without loss of generality (WLOG) for the last two cases that $x \geq 0$ and $y < 0$.

- Case 1: If x and y are both nonnegative, then $|x| + |y| = x + y = |x + y|$.
- Case 2: If x and y are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.
- Case 3: If $x \geq 0$ and $y < 0$ and $x + y \geq 0$, then $|x| + |y| = x + (-y)$ is some number greater than x. $|x + y|$ is some positive number less than x since y is negative. Thus, $|x| + |y| \geq x \geq |x + y|$.

- Case 4: If $x \geq 0$ and $y < 0$ and $x + y < 0$, then $|x| + |y| = x + (-y)$ is some number greater than $-y$. $|x + y| = -(x + y) = (-x) + (-y)$ which is some positive number less than or equal to $-y$, since $-x$ is nonpositive. Thus, we have $|x| + |y| \geq -y \geq |x + y|$.

We have now proved for all cases that the triangle inequality is valid. This example is purposely lengthy to show in full detail a proof by cases.