Lecture 13 Handout: Sets -- ANSWERS

(1) • A set is a collection of items

$S = \{ \dots | \quad \}$: set builder notation

- Example:
  - "S is the set of all even numbers between 50 and 100, inclusive"
  - (10) S = { x | Even(x) and 50 ≤ x ≤ 100 }

• Standard Numerical Sets to know:

(2)

Symbol	Elements	Name of Set
∅	{}	Empty set
ℕ	{ 0, 1, 2, 3, … }	Natural numbers
ℤ	{ }	Integers
$\mathbb{Z}^+$	{ 1, 2, 3, 4, … }	Positive integers
$\mathbb{Z}^-$	{ -1, -2, -3, -4, … }	Negative integers
ℚ	{ x | ∃a∈ℤ ∃b∈$\mathbb{Z}^+$ x = a/b }	Rationals
ℝ	*(don't try to list the elements)*	Real numbers
$\mathbb{R}^+$	*(don't try to list the elements)*	Positive reals

(3). A set is unordered.

(4) Each item is either in the set or isn't.

(5) A set can have other sets as elements.

(6) A set can have infinitely many elements!

(7) A set can have zero elements.

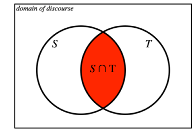
(8). everything not in S | S everything in S

U: domain of discourse 论域

(9). x ∈ S: x is an element of S

x ∉ S: x is not ...

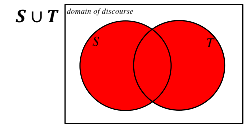
Set Operations



(11) • **S ∩ T** : "the ***intersection*** of S and T" 交集

x ∈ (S ∩ T) ≡ x ∈ S ∧ x ∈ T

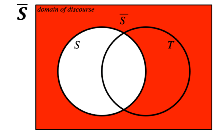
Example: {1,2} ∩ {2,3} = {2}



(12) • **S ∪ T** : "the ***union*** of S and T" 并集

x ∈ (S ∪ T) ≡ x ∈ S ∨ x ∈ T

Example: {1,2} ∪ {2,3} = {1,2,3}



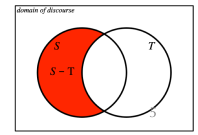
(13) • $\bar{S}$: "the ***complement*** of S" 补集

x ∈ $\bar{S}$ ≡ x ∉ S

Example: $\overline{\{1,2\}}$ = depends on the domain!

if domain is {1,2,3}, then $\bar{S}$ ={3}

if domain is {0,1,2,3,4}, then $\bar{S}$ ={0,3,4}



(14) • **S – T** : "S ***minus*** T" 差集

x ∈ (S – T) ≡ x ∈ S ∧ x ∉ T

Example: {1,2} – {2,3} = {1}

Other Set Operations

The **cardinality** of a set S means the number of elements in S (15) (集和的势/基)

|{a, 22, dog}| = 3

|{a, {e}, {i, o}, u}| = 4

|{x | x ∈ **N** and x² = x}| = 2

|{x | x ∈ **N** and Prime(x)}| = ∞

|**N**| = ∞

|**R**| = ∞

(16) The **Cartesian product** of sets A and B is (笛卡尔积)

A x B = { (a, b) | a∈A ∧ b∈B }

(17) If |A| = x and |B| = y, then |A x B| = xy

A = {soup, salad} and M = {tofu, chicken, steak}

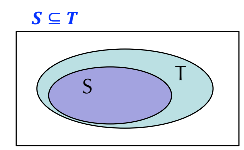
A x M = { (soup, tofu), (soup, chicken), (soup, steak), (salad, tofu), (salad, chicken), (salad, steak) }

|A x M| = 2 · 3 = 6

* X = ℝ, Y = ℝ

X × Y = the set of all coordinates in the Cartesian plane!

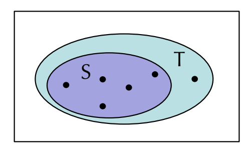
Relationships between Sets



(18) "S is a ***subset*** of T" 子集

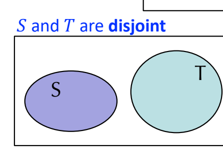
- **S ⊆ T** : ∀x (x ∈ S → x ∈ T)
- "Every element of S is also in T"

T or F	{a, b} ⊆ {a, b, c}	**T** or F	∅ ⊆ {a, b, c}
T or **F**	b ⊆ {a, b, c}	**T** or F	{a, b, c} ⊆ {a, b, c}



(19) "S is a ***proper subset*** of T" 真子集

- **S ⊊ T** : S ⊆ T and S ≠ T
- {1,2} ⊊ {1,2,3}



(20) "S and T are ***disjoint***" 无交集的

- **Means:** S ∩ T = ∅
- {1,2} and {3,4} are disjoint

(21) 幂集

Subsets & Power Sets

For a set S, the set of all subsets is called the **power set** of S, and is denoted as P(S). (22)

Number of subsets, i.e., size of power set: **If |S| = n, then |P(S)| =** 2^n

Examples:

1. P({cat, dog}) = { ∅, {cat}, {dog}, {cat, dog} }
2. P({1, 2, 3}) = {∅, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }
3. P(∅) = { ∅ }
4. P(P(∅)) = { ∅, {∅} }

another proof

$\overline{S \cup T} = \{x \mid x \notin (S \cup T)\}$	definition of complement
$= \{x \mid \neg(x \in (S \cup T))\}$	definition of ∉
$= \{x \mid \neg(x \in S \vee x \in T)\}$	definition of ∪
$= \{x \mid x \notin S \wedge x \notin T\}$	DeMorgan's law
$= \{x \mid x \in \bar{S} \wedge x \in \bar{T}\}$	definition of complement
$= \{x \mid x \in \bar{S} \cap \bar{T}\}$	definition of ∩
$= \bar{S} \cap \bar{T}$	definition of ∈

交的补 = 补的交

Double Subset Equality Proof

Handout

Prove DeMorgan's Law for sets: (23) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

by showing that each side is a subset of the other

Definition: $S \subseteq T$ means that, for all x, *if* $x \in S$, *then* $x \in T$.

Proof: *goal is to prove* $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ *and* $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

For an arb. x:

- Let x be an arbitrary element of the domain.
- Assume $x \in \overline{A \cap B}$.
- So $x \notin A \cap B$
- So $x \notin A$ or $x \notin B$
- So $x \in \bar{A}$ or $x \in \bar{B}$
- Thus $x \in \bar{A} \cup \bar{B}$
- **So** $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

if $x \in S$, *then* $x \in T$

- Assume $x \in \bar{A} \cup \bar{B}$.
- So $x \in \bar{A}$ or $x \in \bar{B}$
- So $x \notin A$ or $x \notin B$
- So $x \notin A \cap B$
- Thus $x \in \overline{A \cap B}$
- **So** $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$
- **Since each side is a subset of the other, we have** $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

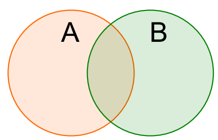
if $x \in T$, *then* $x \in S$

So S= T

(24) $A \subseteq B \wedge B \subseteq A \Rightarrow A = B$!!

Inclusion-Exclusion Principle

(25) |A∪B| = |A| + |B| - |A∩B|



1. add individual sizes

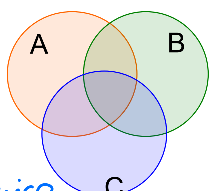
(26) |A∪B∪C| = |A|+|B|+|C|

-|A∩B| - |A∩C| - |B∩C|

+ |A∩B∩C|

2. substract pairwise ∩'s

3. Add 3-wise ∩



(27) Generalization:

$|\bigcup_i A_i| = \{\text{individual sizes}\} - \{\text{pairwise} \cap \text{sizes}\} + \{3\text{-wise} \cap\} - \{4\text{-wise} \cap\} + \{5\text{-wise} \cap\} - \ldots$ (+odd - even ...)

Inclusion-Exclusion Exercise

How many integers between 1 and 300 (inclusive) are divisible by at least one of {2,3,5}?

Let S_m = *Multiples of m from 1 to 300*

- Question is asking: what is $|S_2 \cup S_3 \cup S_5|$?
- Answer:

Inclusion-Exclusion:

$$|S_2 \cup S_3 \cup S_5| = \underbrace{|S_2| + |S_3| + |S_5|}_{\text{Add individual sizes}} \underbrace{- |S_2 \cap S_3| - |S_2 \cap S_5| - |S_3 \cap S_5|}_{\text{Subtract pairwise } \cap\text{'s}} \underbrace{+ |S_2 \cap S_3 \cap S_5|}_{\text{Add 3-wise } \cap}$$

$$= |S_2| + |S_3| + |S_5| - |S_{2\cdot3}| - |S_{2\cdot5}| - |S_{3\cdot5}| + |S_{2\cdot3\cdot5}|$$

$$= |S_2| + |S_3| + |S_5| - |S_6| - |S_{10}| - |S_{15}| + |S_{30}|$$

$$= \frac{300}{2} + \frac{300}{3} + \frac{300}{5} - \frac{300}{6} - \frac{300}{10} - \frac{300}{15} + \frac{300}{30}$$

$$= 150 + 100 + 60 - 50 - 30 - 20 + 10$$

$$|S_2 \cup S_3 \cup S_5| = 310 - 100 + 10 = \mathbf{220}$$

(28) A ∩ (A ∪ B) = A

A ∪ (A ∩ B) = A

Review:

1. (cardinality) $|S|$: the number of elements in S.

2. (Cartesian Product) $A \times B = \{(a,b) \mid a \in A, b \in B\}$

$A \times B \times \ldots \times N = \{(a, b, \ldots, n) \mid a \in A, b \in B, \ldots n \in N\}$

3. $S \cap T$, $S \cup T$, $\bar{S}$, $S - T$

4. subset, proper set, disjoint

5. Power set: $P(S) = \{A \mid A \subseteq S\}$

6. $A \subseteq B \land B \subseteq A \Rightarrow A = B$

7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

8. $|S| = n \Rightarrow |P(S)| = 2^n$

9. $|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cap B| = |A| + |B| - |A \cup B|$

10. $|A + B + C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$|\bigcup_i A_i| = \{\text{individual sizes}\} - \{\text{pairwise } \cap \text{ sizes}\} + \{3\text{-wise } \cap\}$

$- \{4\text{-wise } \cap\} + \{5\text{-wise } \cap\} - \ldots$

Set Identities (FYI)

U = Universal set, i.e., the entire domain

- **Identity** laws

$$A \cap U = A$$
$$A \cup \emptyset = A$$

- **Domination** laws

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

- **Idempotent** laws

$$A \cup A = A$$
$$A \cap A = A$$

- **Complementation** law

$$\overline{(\bar{A})} = A$$

- **Commutative** laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- **Definition of Set Minus** **(not in book)**

$$A - B = A \cap \bar{B}$$

- **Associative** laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive** laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **De Morgan's** laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

- **Absorption** laws

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A$$

- **Complement** laws

$$A \cup \bar{A} = U$$
$$A \cap \bar{A} = \emptyset$$