EECS 203: Discrete Mathematics
Fall 2023
Homework 2

Due **Thursday, Sept. 14**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $6 + 2$ Total Points: $100 + 25$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. N∃gations and Qu∀ntifiers [18 points]

Negate the following statements. Simplify your answers so that all negation symbols immediately precede predicates. Make sure to show all intermediate steps.

(a) $(x \lor y) \land ((a \land b) \lor z)$

(b) $[\forall x P(x)] \lor [\exists y Q(y)]$

(c) $\forall x \exists y \forall z [L(x, y) \to [R(y, z) \to T(z, x)]]$

Solution:

(a)

$$\begin{aligned}
&\neg[(x \lor y) \land ((a \land b) \lor z)] \\
&\quad \equiv \neg(x \lor y) \lor \neg((a \land b) \lor z)] \\
&\quad \equiv (\neg x \land \neg y) \lor (\neg(a \land b) \land \neg z)] \\
&\quad \equiv (\neg x \land \neg y) \lor ((\neg a \lor \neg b) \land \neg z)]
\end{aligned}$$

(b)

$$\begin{aligned}
&\neg(\forall x P(x) \lor \exists y Q(y)) \\
&\quad \equiv \neg \forall x P(x) \land \neg \exists y Q(y) \\
&\quad \equiv \exists x \neg P(x) \land \forall y \neg Q(y)
\end{aligned}$$

(c)

$$\begin{aligned}
&\neg \forall x \exists y \forall z [L(x, y) \to [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \neg \exists y \forall z [L(x, y) \to [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \forall y \neg \forall z [L(x, y) \to [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \forall y \exists z \neg [L(x, y) \to [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \forall y \exists z \neg [\neg L(x, y) \lor [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \forall y \exists z [L(x, y) \land \neg [R(y, z) \to T(z, x)]] \\
&\quad \equiv \exists x \forall y \exists z [L(x, y) \land \neg [\neg R(y, z) \lor T(z, x)]] \\
&\quad \equiv \exists x \forall y \exists z [L(x, y) \land [R(y, z) \land \neg T(z, x)]]
\end{aligned}$$

Grading Guidelines [18 points]

For each part:

+2 correct answer (all negations immediately precede predicates in the final expression and it is logically equivalent to the correct answer)
+2 correct use of De Morgans or applies negation to implies statement properly
+2 correct justification (some intermediate steps shown)

2. To Tell the Truth [18 points]

Prove or disprove whether each of the following compound propositions is a tautology. **Justify your answers.**

(a) $(p \wedge q) \to (p \vee q)$

(b) $((p \wedge q) \vee s) \to (s \to (p \vee q))$

Solution:

(a) This is a tautology. We will prove this in two ways: using a truth table and using logical equivalences. Only one method is needed to receive full credits (combining both is also fully correct, i.e. doing some logical equivalences to simplify the statement and then constructing a truth table for that compound proposition).

Solution 1, Truth Table:

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Since the values in the last column of the truth table (the desired compound proposition) are all true, this compound proposition is a tautology.

Solution 2, Logical Equivalences:

$$\begin{aligned}
&(p \wedge q) \to (p \vee q) && \\
&\equiv \neg(p \wedge q) \vee (p \vee q) && \text{Implication Breakout} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{De Morgan's Law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{Commutative Law} \\
&\equiv T \vee (\neg q \vee q) && \text{Negation Law} \\
&\equiv T && \text{Domination Law}
\end{aligned}$$

(b) This is not a Tautology. We can disprove this by providing an example where the statement does not result in True.

Consider the truth values $p = F, q = F$, and $s = T$. The compound proposition would be:

$$((F \wedge F) \vee T) \to (T \to (F \vee F)) \equiv (F \vee T) \to (T \to F) \equiv T \to F \equiv F.$$

Therefore this is not a Tautology.

Note: Simply drawing a truth table would not suffice. If a truth table was used, some statement either pointing out that not all values in the last column are true or highlighting a row that results in false is required for full credit.

Draft Grading Guidelines [18 points]

Part a, truth table:

+2 correct number of rows
+2 correct intermediate steps (columns)
+3 correct final column values
+2 correct result (prove that it is a tautology) with some sentence describing why

Part a, logic equivalences:

+3 correctly shows intermediate steps (citations not necessary)
+3 correctly identifies that it is a Tautology
+3 correct ends with the proof with a row for T

Part b:

+2 states the proposition is not a tautology
+4 provides a variable assignment for which it is false (this can be done through a truth table as well)
+3 shows the assignment results in the proposition being false (this can also be done with a truth table), and explains this means it is not a tautology

3. Not it! [12 points]

Find the negations of the following statements. You must simplify your answers using De Morgan's Laws or other logical equivalences for full credit

(a) I like to add both ham and pineapple to my pizza.

(b) If Bob studies computer science at the University of Michigan, they will take EECS 203.

(c) Every student has at least one friend who lives in Baits.

Solution:

(a) I don't like to add ham or I don't like to add pineapple to my pizza

(b) Bob studies computer science at the University of Michigan and does not take EECS 203

(c) There is a student who does not have a friend who lives in Baits

Grading Guidelines [12 points]

For each part:

+4 correct answer
+1 partial credit: correctly negates part of the expression, for example, when a student forgets to flip the logical connective when applying De Morgan's Laws

4. Quantifier Quandary [18 points]

Determine the truth value of each of these statements, where the domain of each quantified variable is all real numbers. **Briefly justify your answers.**

(a) $\exists x(x^3 = -1)$

(b) $\exists x(x^2 = -1)$

(c) $\exists x(x^4 < x^2)$

(d) $\forall x(2x > x)$

(e) $\forall x \exists y(x^2 = y)$

(f) $\forall x \exists y(y^2 = x)$

Solution:

(a) True. $x = -1$.

(b) False. The square of any real number is non-negative.

(c) True. $x = 0.5$.

(d) False. $x = -1$.

(e) True. Simply let $y = x^2$.

(f) False. Let $x = -1$, like in part b. Then there is no real value of y where $y^2 = -1$.

Grading Guidelines [18 points]

For each part:

+1 correct answer (true/false)
+2 correct justification (does not need to exactly match provided witnesses for existential statements)

5. Internet Connections [16 points]

A strange Internet outage has struck campus. Some people have internet, but others don't.

- Let $I(x)$ mean "x has internet access"
- Let $F(x, y)$ mean "x is friends with y"

Using the given predicates, logical operators ($\wedge$, $\vee$, $\neg$, $\rightarrow$), and quantifiers ($\forall$, $\exists$), express the following statements. The domain of every quantifier you use must be "students on campus." For purposes of this question, we'll say that $F(x, y)$ always has the same truth value as $F(y, x)$, and so these may be used interchangeably.

(a) Someone does not have internet access.

(b) Nobody is friends with everybody.

(c) Everyone with internet access has a friend without internet access.

(d) Everyone with internet access has *exactly one* friend without internet access.

Solution:

There are many equivalent ways to write every proposition. We wrote the solutions in the most common way(s), but anything else equivalent to those is also correct.

(a) $\exists x\, \neg I(x)$, or equivalently, $\neg \forall x\, I(x)$

(b) $\neg \exists x \forall y\, F(x, y)$, or equivalently $\forall x\, \neg \forall y\, F(x, y)$ or $\forall x \exists y\, \neg F(x, y)$

(c) $\forall x\, [I(x) \rightarrow \exists y\, [F(x, y) \wedge \neg I(y)]]$

(d) $\forall x[I(x) \rightarrow \exists y\,[F(x,y) \wedge \neg I(y) \wedge \forall z\,[(F(x,z) \wedge \neg I(z)) \rightarrow z = y]]]$

Equivalently,

- $\forall x[I(x) \rightarrow \exists y\,[F(x,y) \wedge \neg I(y) \wedge \forall z\,[z \neq y \rightarrow (\neg F(x,z) \vee I(z))]]]$
- $\forall x[I(x) \rightarrow \exists y\,[F(x,y) \wedge \neg I(y) \wedge \neg \exists z\,[z \neq y \wedge F(x,z) \wedge \neg I(z)]]]$

Grading Guidelines [16 points]

For each part:

+2 correctly uses quantifiers and variables to represent people in the statements
+2 correct expression

Note to graders: If the solution to (c) is used in (d), treat it as correct for purposes of checking (d).

6. Flip the Switch [18 points]

Determine whether or not each of the following implications is true, regardless of the definition of the predicate $P(x, y)$. Give a brief explanation for your answer.

(a) $\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)$

(b) $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$

Solution:

(a) This is not a correct implication. One can see this because even though all x have some y for which $P(x, y)$ is true, this does not necessarily mean that the y is the same for all x. A real-world example would be considering some number of shops that are open some days of the week. We could say that all shops (all x) have a day (y) for which they are open. However, this does not mean that there is a day for which all shops are open.

(b) This is a correct implication. One can see this because if there is a single y such that for all x $P(x, y)$ then for any x one may take the single y that works for any given x, and it will work for this x. We can consider our real-world example with this. If we guaranteed that there was a single day of the week, say Tuesday, on which every store is open, can you conclude that, for each store, there is a day of the week on which it is open? Yes, it is open on Tuesday.

Grading Guidelines [18 points]

For each part:

+4 correct answer (is/is not a correct implication)
+5 correct justification

Groupwork

1. Grade Groupwork 1

Using the solutions and Grading Guidelines, grade your Groupwork 1:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1												/14
Problem 2												/10
Total:												/24

2. Implication Inception [13 points]

Consider two propositions, A and B.

(a) Prove via a truth table that $A \equiv [(A \to B) \to A]$.

(b) Consider the compound proposition

$$\underbrace{((A \to B) \to A) \to B \ldots}_{\text{203 letters}}$$

which has 203 total letters in it, alternating between A and B. Fill in the rest of the following truth table. You don't need to manually make all 203 columns of the truth table to solve this, so try to find a pattern and think of a shortcut.

A	B	$\underbrace{((A \to B) \to A) \to B \dots}_{\text{203 letters}}$
T	T	
T	F	
F	T	
F	F	

(c) Consider the following truth table. This is similar to the truth table in part (b), but it contains each iteration of the compound proposition from 1 to 203 letters.

A	B	$\underbrace{A}_{\text{1 letter}}$	$\underbrace{A \to B}_{\text{2 letters}}$	$\underbrace{(A \to B) \to A}_{\text{3 letters}}$	...	$\underbrace{((A \to B) \to A) \to B \dots}_{\text{203 letters}}$
T	T	T				
T	F	T				
F	T	F				
F	F	F				

What is the total number of cells that will be T among all 4 rows and 203 columns? (Note that the two initial A and B columns do *not* count as columns that should be counted. However, the $\underbrace{A}_{\text{1 letter}}$ column and all following columns do count.)

Solution:

(a)

A	B	$A \to B$	$[(A \to B) \to A]$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

For the first two rows of the truth table, both A and $[(A \to B) \to A]$ are T. For the other two rows of the truth table, both A and $[(A \to B) \to A]$ are F. In all cases, A and $[(A \to B) \to A]$ have the same truth value, so they are logically equivalent to each other. In other words, $A \equiv [(A \to B) \to A]$.

(b)

A	B	$\underbrace{((A \to B) \to A) \to B \dots}_{\text{203 letters}}$
T	T	T
T	F	T
F	T	F
F	F	F

As shown in part (a), $A \equiv [(A \to B) \to A]$. Therefore, in the sequence of $((A \to B) \to A) \to B \dots$, whenever the last proposition is A, its overall truth table will be the same as that of A. This can be shown by repeatedly substituting A in place of $(A \to B) \to A$. For example, $(A \to B) \to A$ will have the same truth table as A, so will $(((A \to B) \to A) \to B) \to A$, and so on. Note that A is the first proposition, and it alternates with B. Therefore, all of the odd-indexed predicates will be A, so the 203rd proposition will be A. Since the sequence ends with A, it will have the same truth table as A, so the truth table is filled in as such.

(c) As explained in part (b), the columns with an odd number of letters end with A and each have 2 T cells. Also, the columns with an even number of letters end with B and each have 3 T cells. There are 102 odd numbers and 101 even integers from 1 through 203, inclusive. Therefore, there are $102 \cdot 2 + 101 \cdot 3 = 204 + 303 = 507$ total T cells among the 203 columns.

Draft Grading Guidelines [13 points]

Part a:

(i) +2 $(A \to B) \to A$ column is correct (+1 partial credit if $(A \to B) \to A$ column would be correct except it has exactly 1 mistake)

(ii) +1 explains that the A and $(A \to B) \to A$ columns have all of the same truth values, so $A \equiv [(A \to B) \to A]$

Part b:

(iii) +2 notices that there is a repeating cycle

(iv) +1 notices that the cycle alternates every 2 columns

(v) +2 correct truth table

Part c:

(vi) +1 considers how some cells have two T cells and others have three T cells

(vii) +2 notes that there are 102 columns with the same amount of T cells as each other and 101 with the same amount of T cells as each other

(viii) +2 correct answer

3. Functionally Complete [12 points]

A logical operator (or a set of logical operators) is considered to be *functionally complete* if it can be used to make any truth table.

(a) One set of functionally complete logical operators is $\{\vee, \neg\}$. In other words, we can use the $\vee$ and $\neg$ operators to make any truth table. Let's test this out with an example! Consider two propositions p and q. Write a compound proposition that is logically equivalent to $p \wedge q$ by only using p, q, $\vee$, $\neg$, and parentheses.

(b) Now, let's consider a new logical operator: NAND. The symbol for NAND is $\overline{\wedge}$. Below is the truth table for NAND. *(If you take EECS 370, you will get to use NAND even more!)*

p	q	$p \,\overline{\wedge}\, q$
T	T	F
T	F	T
F	T	T
F	F	T

Let's start trying to figure out whether $\overline{\wedge}$ is functionally complete. Is it possible to write a proposition that is logically equivalent to $\neg p$ by only using p and $\overline{\wedge}$? If so, write the proposition. If not, explain why it is impossible to do so.

(c) Is it possible to write a compound proposition that is logically equivalent to $p \vee q$ by only using p, q, $\overline{\wedge}$, and parentheses? If so, write the compound proposition. If not, explain why it is impossible to do so.

(d) Based on parts (a), (b), and (c), is $\overline{\wedge}$ functionally complete? Why or why not?

Solution:

(a) $\neg(\neg p \vee \neg q)$. This can be derived using De Morgan's Law.

(b) Yes. $p \,\overline{\wedge}\, p \equiv \neg p$.

(c) Yes. $(p \,\overline{\wedge}\, p) \,\overline{\wedge}\, (q \,\overline{\wedge}\, q) \equiv \neg p \,\overline{\wedge}\, \neg q \equiv p \vee q$.

(d) Yes. It was given in part (a) that $\{\vee, \neg\}$ is functionally complete, so $\{\vee, \neg\}$ can be used to make any truth table. In part (b), we showed that it is possible to make $\neg$ out of $\overline{\wedge}$. In part (c), we showed that it is possible to make $\vee$ out of $\overline{\wedge}$. Because we can make $\vee$ and $\neg$ from $\overline{\wedge}$, and $\{\vee, \neg\}$ can be used to make any truth table, then $\overline{\wedge}$ can be used to make any truth table. Therefore, $\overline{\wedge}$ is functionally complete!

Note: In this question, we were able to boil a problem down to something we already have been given the answer to. This proof style is called Proof by Reduction. If you take EECS 376, you will learn more about this proof style!

Grading Guidelines [12 points]

Part a:

(i) +3 correct compound proposition (note that there may be multiple correct compound propositions besides the one that we gave)

Part b:

(ii) +1 states that the answer is yes (or attempts to provide a proposition)
(iii) +2 correct compound proposition (again, there may be multiple correct answers)

Part c:

(iv) +1 states that the answer is yes (or attempts to provide a proposition)
(v) correct compound proposition (again, there may be multiple correct answers)

Part d:

(vi) +1 correct answer (yes)
(vii) +2 correct explanation