EECS 203 Exam 2 Review

Day 2

Today’s Review Topics

- Functions
  - Properties
  - Compositions
- Countability
  - Schroder-Bernstein
- Recurrence Relations
- Pigeonhole Principle

Functions

(Functions) Definitions

- **Function** $f : A \to B$: A function f is a relation between two sets, say A and B that associates each element of set A to exactly one element from the set B. The set A and set B are respectively called the domain and codomain of f. The range of f is the set of all elements in the codomain which are mapped to by an element in the domain.

1. Each "x"-value is mapped to something
2. Each "x"-value maps to exactly one thing

- **Onto**: A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

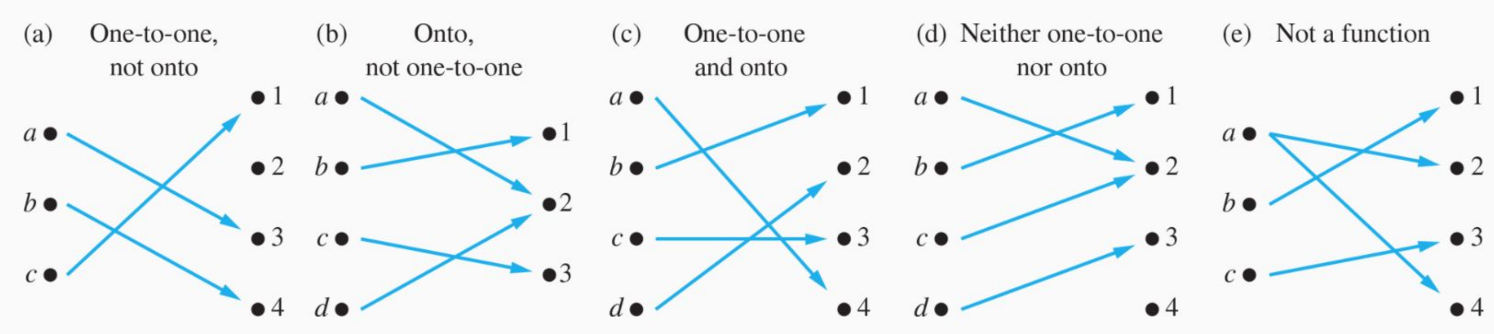
$$\forall b \in B, \exists a \in A\ [f(a) = b]$$

- **One-to-One**: A function f is said to be one-to-one, or an injection, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

$$\forall a_1, a_2\ [(f(a_1) = f(a_2)) \to (a_1 = a_2)]$$

- **Bijection**: A function f is called a bijection (or one-to-one correspondence) if it is both one-to-one and onto.

Some Pictures



Function Property Proof Templates

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

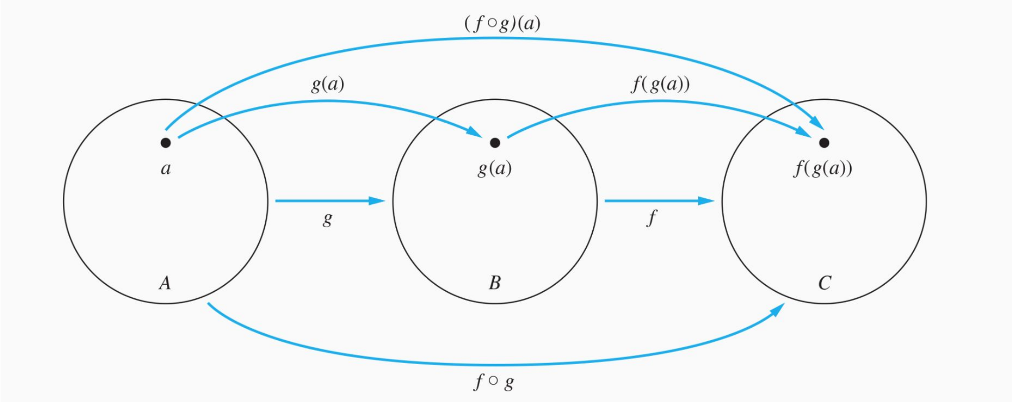
To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

More (Functions) Definitions

- f^{-1}: Let f be a bijection from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Bijective == Invertible

- $f \circ g$: Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Inverse

What is the inverse of function f?

$$f : \mathbb{Z} \to \mathbb{Z} \quad \text{where} \quad f(x) = \begin{cases} x - 2 & : x \geq 5 \\ x + 1 & : x \leq 4 \end{cases}$$

(A)

$$f^{-1}(x) = \begin{cases} x - 2 & : x \geq 4 \\ x + 1 & : x \leq 5 \end{cases}$$

(B)

$$f^{-1}(x) = \begin{cases} x + 2 & : x \geq 5 \\ x - 1 & : x \leq 4 \end{cases}$$

(C)

$$f^{-1}(x) = \begin{cases} x + 2 & : x \geq 4 \\ x - 1 & : x \leq 5 \end{cases}$$

(D) f does not have an inverse

Solution

D, because the function is not a bijection (both 4 and 7 map to 5).

Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$: $f(x) = \log_{10}(x)$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^+$: $g(x) = 3^x$. Which of the following can we conclude?

(a) f is one-to-one

(b) $f \circ g$ is a bijective function

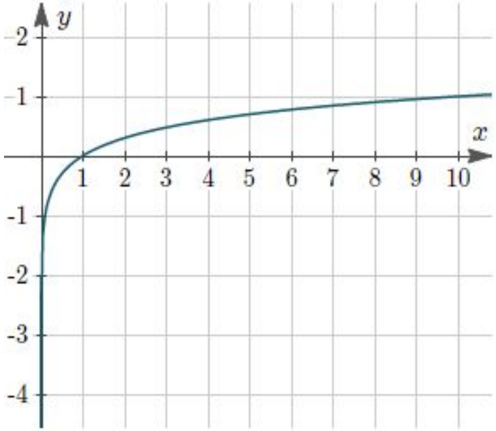
(c) $g \circ f$ is a bijective function

(d) f is onto

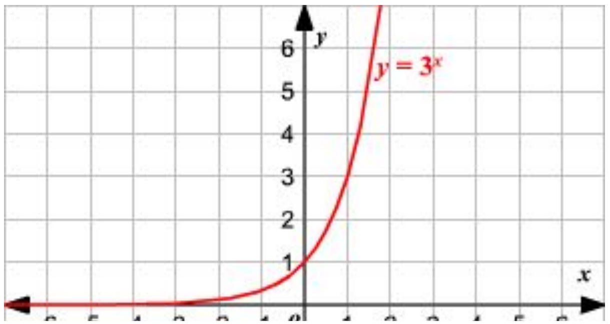
(e) g is bijective

Solution: All of the above

(a), (d), and (e) are true, because the domain and codomain are limited to the ones that every item is mapped to. (b) and (c) are both true because the composition of two bijections is a bijection.



$f : \mathbb{R}^+ \longrightarrow \mathbb{R}$



$g : \mathbb{R} \longrightarrow \mathbb{R}^+$

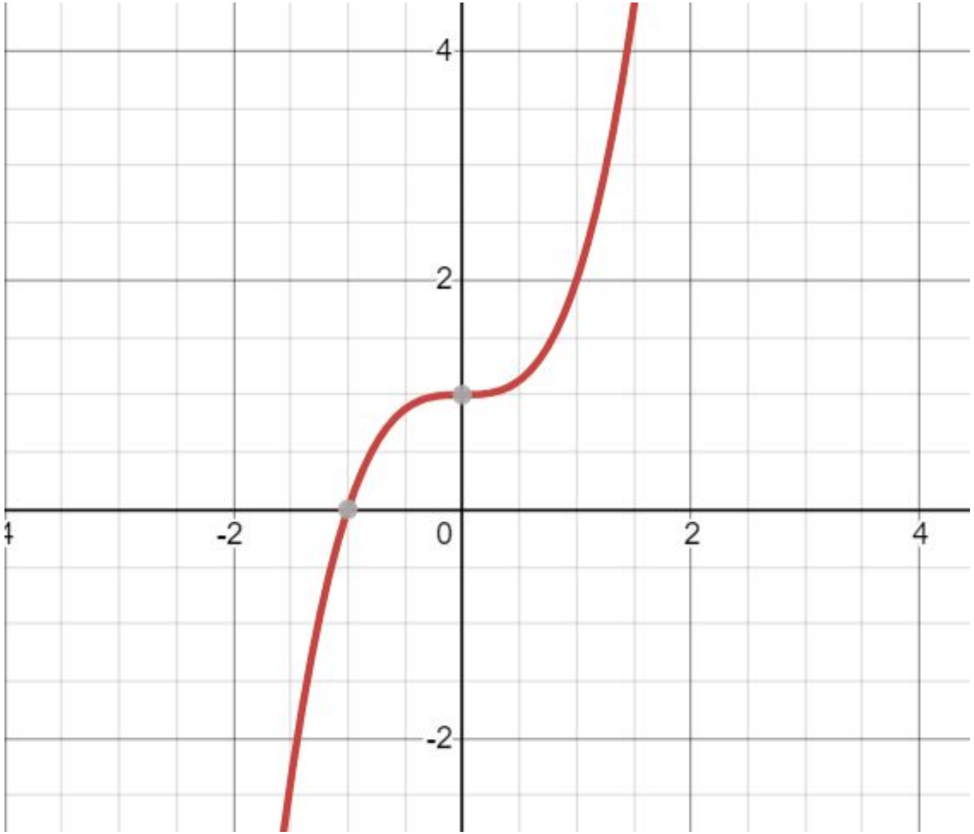
Practice

For each of the functions below determine whether it is i) one-to-one, ii) onto, iii) bijective. Prove your answers.

a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 + 1$

b) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -3x^2 + 7$

a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 + 1$



x^3 graphs are generally onto and one-to-one

a) onto and one-to-one; bijective

i) f is one-to-one. To prove this we need to show that for any $x_1, x_2 \in \mathbb{R}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Let $x_1, x_2 \in \mathbb{R}$, and let $f(x_1) = f(x_2)$. Then

$$\begin{aligned} f(x_1) &= f(x_2) && \text{Assumption} \\ (x_1)^3 + 1 &= (x_2)^3 + 1 && \text{defn of } f(x) \\ (x_1)^3 &= (x_2)^3 && \text{Subtract 1 from both sides} \\ x_1 &= x_2. && \text{Cube-root of both sides} \end{aligned}$$

Therefore, f is one-to-one, by the definition of one-to-one.

ii) f is onto. To prove this, we need to show that for any $b \in \mathbb{R}$ (the codomain), there is some $a \in \mathbb{R}$ (the domain) such that $f(a) = b$.

Given an arbitrary $b \in \mathbb{R}$, let $a = (b-1)^{1/3}$. Then,

$$\begin{aligned} f(a) &= a^3 + 1 \\ &= ((b-1)^{1/3})^3 + 1 \\ &= (b-1) + 1 \\ &= b. \end{aligned}$$

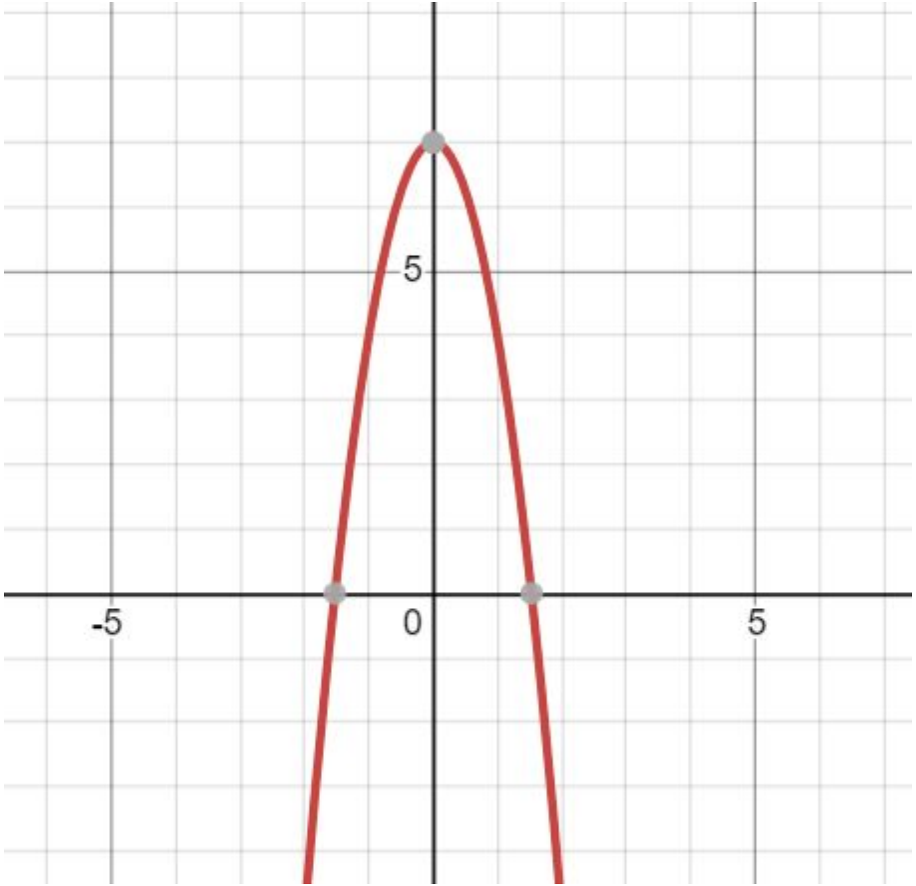
Given an arbitrary $b \in \mathbb{R}$, we provide an $a \in \mathbb{R}$ such that $f(a) = b$. This is precisely the definition of onto.

Note: We didn't pull the value of a that we use in the proof above out of thin air. We did the following calculations 'on the side' first, to find an a that would work: We need $f(a) = b$, so

$$\begin{aligned} f(a) &= b \\ a^3 + 1 &= b \\ a^3 &= b - 1 \\ a &= (b-1)^{1/3}. \end{aligned}$$

iii) Because f is both one-to-one and onto, it is bijective (by definition).

b) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -3x^2 + 7$



Parabolas are generally not one-to-one (fails horizontal line test) or onto (here, nothing larger than 7 in the codomain is hit)

b) neither one-to-one nor onto; not bijective

i) f is not one-to-one. For example, let $x = -1$ and $y = 1$. Note $x, y \in \mathbf{R}$ (the domain).
$f(x) = f(-1) = -3(-1)^2 + 7 = -3 + 7 = 4.$
$f(y) = f(1) = -3(1)^2 + 7 = -3 + 7 = 4.$
We have $x \neq y$, but $f(x) = f(y)$, which violates the definition of one-to-one.

ii) f is not onto because there are elements of the codomain that are not mapped to by f. As an example, $b = 10$ is in the codomain but is not mapped to by any number in the domain. To find $a \in \mathbb{R}$ that maps to $b = 10$, we need $f(a) = -3a^2 + 7 = 10$. Solving for a gives us $a = \sqrt{-1} = i$, which is not a real number, i.e., not in the domain of f. So there is no real number that maps to $b = 10$ under f, which means f is not onto.

Also, since the range of f is the set $(-\infty, 7]$, which is not the same as its codomain ($\mathbb{R}$), we know that the function is not onto.

iii) To be bijective, a function needs to be both one-to-one and onto. This function is neither, so it is not bijective.

Prove

Suppose that g is a function from A to B and f is a function from B to C.

c) If $f \circ g$ is a bijection, then g is onto if and only if f is one-to-one.

Proof

Given: $f \circ g$ is a bijection.

Want to show: g is onto if and only if f is one-to-one

Equivalent: g is onto ↔ f is one-to-one

Split into two implications: g is onto → f is one-to-one

f is one-to-one → g is onto

Sufficient to prove both of them separately.

First side: g is onto → f is one-to-one

Let g be onto. We will show that f is one-to-one.

Consider $y_1, y_2 \in B$ such that $f(y_1) = f(y_2)$.

Because g is onto, there exist $x_1, x_2 \in A$ such that $g(x_1) = y_1$, $g(x_2) = y_2$.

This implies that $f(g(x_1)) = f(g(x_2))$.

We also know that $f \circ g$ is bijective, so this implies $x_1 = x_2$.

Since $x_1 = x_2$, then $g(x_1) = g(x_2)$, which implies that $y_1 = y_2$.

Thus, f is one-to-one.

Second side: f is one-to-one $\rightarrow$ g is onto

Seeking contradiction, assume that f is one-to-one, but g is not onto.

Because g is not onto, there exists some $b \in B$ that is not in the image of g.

Consider $f(b) = c \in C$.

Since $f \circ g$ is bijective, there exists some $a \in A$ such that $f(g(a)) = c$.

Since f is one-to-one, $f(g(a)) = c = f(b) \longrightarrow g(a) = b$.

Thus, we arrive at a contradiction, because b was not supposed to be in the image of g. Thus, if f is one-to-one, then g is onto.

Countability

Countability definitions

- **Cardinality**: The number of elements in a set. Cardinality of a set S is denoted by $|S|$

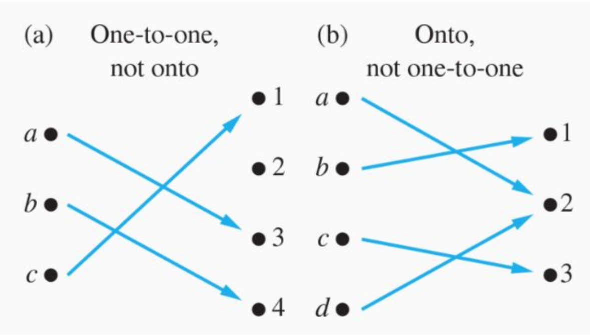
- **Countably Infinite:** A set is countably infinite if it has the same cardinality as the natural numbers. This can be proven for a set by finding a one-to-one correspondence between it and the natural numbers. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are common examples of sets that are countably infinite.

- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than that of the set of all natural numbers. $\mathbb{R}$ is one example of a set that is uncountably infinite.

- **Schroder-Bernstein Theorem:** For two sets A and B, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. Note that finding an onto function from A to B shows that $|A| \geq |B|$, and finding a one-to-one function from A to B shows that $|A| \leq |B|$.

Function Properties & Cardinality

	$f : A \to B$	$f : B \to A$
f is **one-to-one**	$\lvert A\rvert \leq \lvert B\rvert$	$\lvert B\rvert \leq \lvert A\rvert$
f is **onto**	$\lvert A\rvert \geq \lvert B\rvert$	$\lvert B\rvert \geq \lvert A\rvert$



Let $A = \mathbb{R}$. For which of the following will $A - B$ be countably infinite?

(a) $B = (\mathbb{R} - \mathbb{Q}) \cup (\mathbb{Z} \times \mathbb{Z})$

(b) $B = \mathbb{Q}$

(c) $B = \mathbb{Q} \oplus \mathbb{R}^{+}$

(d) $B = \{x | x \in \mathbb{R} \land x \neq -x\}$

(e) $B = \{x | x \in \mathbb{R} \land \exists y (y \in \mathbb{Z} \land |x - y| < \frac{1}{2})\}$

Solution: A, E

A: This removes all irrational numbers, so $A - B$ is then $\mathbb{Q}$. Since none of $\mathbb{Z} \times \mathbb{Z}$ is in R, the union with that does not affect the difference.

B: $\mathbb{R} - \mathbb{Q}$ is the irrational numbers, which is uncountable.

C: $\mathbb{Q} \oplus \mathbb{R}^+$ is the nonpositive rationals unioned with positive irrationals. Thus the difference in question is positive rationals unioned with negative irrationals, which is still uncountable.

D: The only real number not in this set is 0, so the difference in question is $\{0\}$, which is finite.

E: B is all numbers who are less than $\frac{1}{2}$ from some integer, which is all numbers except those of the form X.5, so numbers of those form make up the difference. We can easily map this to $\mathbb{Z}$, so this is countably infinite.

Given the following, which of the statements must be **FALSE**?

$$|A| = 5,\ |B| = 8,\ |C| = 7,\ |D| = 4$$
$$f : A \to B,\ g : B \to C,\ h : C \to D$$

Scantron 10

(A) f can be one to one

(B) g can be onto

(C) h can be onto

(D) $h \circ g$ can be onto

(E) $g \circ f$ can be bijective

Solution: E. $g \circ f$ is a function from A to C. Since $|A| \neq |C|$, $g \circ f$ cannot be bijective.

	$f : A \to B$	$f : B \to A$
f is **one-to-one**	$\|A\| \leq \|B\|$	$\|B\| \leq \|A\|$
f is **onto**	$\|A\| \geq \|B\|$	$\|B\| \geq \|A\|$

Problem 13. (4 points)

Which of the following functions can be used to show that $|\mathbb{Z}| = |\mathbb{N}|$? (Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

(a) $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod 2 \\ -\frac{n+1}{2}, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

(b) $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = n$$

(c) $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0 \end{cases}$$

(d) $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n, & \text{if } n \geq 0 \\ -2n-1, & \text{if } n < 0 \end{cases}$$

(e) $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 4n, & \text{if } n \geq 0 \\ -4n-1, & \text{if } n < 0 \end{cases}$$

Solution: a,d

a) Since it maps all even numbers to positive numbers and odd numbers to negative integers, it is one-to-one and onto.
b) Not onto
c) Not one-to-one
d) Bijection
e) Not onto

Since a and d are valid bijective functions, they are your answers.

Prove that $\mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}$

(Hint: use Schroder-Bernstein)

Solution:

$|\mathbb{Z}^+| \leq |\mathbb{Z}|$: One-to-one function from $\mathbb{Z}^+$ to $\mathbb{Z}$:

$f(x) = x$

$|\mathbb{Z}^+| \geq |\mathbb{Z}|$: One-to-one function from $\mathbb{Z}$ to $\mathbb{Z}^+$:

$$f(x) = \begin{cases} -2x & \text{when } x \text{ is negative} \\ 2x + 1 & \text{when } x \text{ is non-negative} \end{cases}$$

By Schroder-Bernstein, $|\mathbb{Z}^+| = |\mathbb{Z}|$

Which of the following sets have the same cardinality?

(a) $\mathbb{Z}$ and $\mathbb{Q}^+$

(b) $\mathbb{Q}^- \cap \mathbb{R}^-$ and $[0, 203)$

(c) $\mathbb{Z}^- \cap \mathbb{R}+$ and $\mathbb{Z}^-$

(d) $\mathbb{N} \cup (4, 20)$ and the set of even integers

(e) $\mathbb{Z} \times \mathbb{Q}$ and $\mathbb{N}$

Which of the following sets have the same cardinality?

(a) $\mathbb{Z}$ and $\mathbb{Q}^+$

both are countably infinite

(b) $\mathbb{Q}^- \cap \mathbb{R}^-$ and $[0, 203)$

First is the same as negative rationals (countably infinite), second is an interval of the reals (uncountably infinite)

(c) $\mathbb{Z}^- \cap \mathbb{R}+$ and $\mathbb{Z}^-$

First is empty set (negative integers and positive reals have no intersection), which is finite, and second is countably infinite

(d) $\mathbb{N} \cup (4, 20)$ and the set of even integers

First set includes an interval of reals (uncountable), second set is an infinite subset of the countably infinite integers (countably infinite)

(e) $\mathbb{Z} \times \mathbb{Q}$ and $\mathbb{N}$

Cartesian product of two countably infinite sets is countably infinite, naturals are countably infinite

Solution: a, e

Break

Recurrence

Recurrence Recap

- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses $\{a_n\}$ in terms of one or more of the previous terms of the sequence.

$$a_n = 8a_{n-1} - 16a_{n-2}$$

- Includes associated base cases
- Will be relevant when solving recursive problems
- For example, finding Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

with the initial values

$$F_1 \text{ and } F_2 = 1.$$

Warm up

You are trying to climb up a total of n rungs up a ladder, and you can climb up either 2 or 3 rungs at a time. Find the recurrence relation for the number of ways to climb up n rungs.

Solution

$a_n = a_{n-2} + a_{n-3}$ because in order to get to nth rung, you can start from n-2 rungs and go up 2, or start from n-3 rungs and go up 3

Base cases $a_0 = 1$, $a_1 = 0$, $a_2 = 1$

Q: Why is n=3 not included as a base case?

A: With n=1,2, you can calculate a_3. Provide minimal base cases.

Finding Recurrence Relations

- Either think about the “first steps” or the “last steps”
  - For example, the stairs problem solution thought about the "last steps".
- Try to go forward from "first steps" or backwards from "last steps" as little as possible; use the recursive leap of faith whenever possible
- Base cases are usually the first cases that aren't covered by the equation (they're like edge cases)
- Every term after base cases should be able to be calculated from the recurrence

More practice with recurrence relations.

Suppose that Bob has 3 different hats, a red one, a blue one, and a yellow one, where he wears one hat per day. He has a couple of rules on how he distributes wearing them.

1) If he wears a red hat, he must have worn a blue hat on the day before.
2) If he wears a yellow hat, he cannot wear the yellow hat again the next day, unless he also wore a blue hat on the day before he wore the first yellow hat.

Find a recurrence relation to describe how many ways there are for Bob to wear his hats in a period of n days. For example, for n = 5, a valid way for Bob to wear his hats is BRBYY.

Let a_n represent the number of valid sequences of hat-wearing for n days.
We can solve this by looking at the last hat that Bob wore. There are 3 cases.

1) **Last hat was blue**: There are no restrictions on when Bob can wear a blue hat, so there are a_{n-1} ways to have a sequence ending in Bob wearing blue.
2) **Last hat was red**: If Bob ended on a red hat, he must have worn a blue hat the day before, so not every sequence of length n-1 is compatible with ending on a red hat. Thus, the only ways we can have this happen are when a sequence of n-2 days is followed by a blue hat, so there are a_{n-2} ways to have a sequence ending with Bob wearing red.
3) **Last hat was yellow**: We should split this up into 2 cases, one where Bob doesn't end on 2 yellows, and one where Bob ends on 2 yellows.
    a) Case 1: Much like the case where the last hat was blue, this means that the sequence of n-2 was followed by a red hat or blue hat. If it was followed by a red hat, then the hat before it was blue, so the sequence would end with BRY. There are a_{n-3} ways to do this. If it was followed by a blue hat, there are a_{n-2} ways to do it.
    b) Case 2: If Bob ends on 2 yellows, he must have had a blue hat before, so the sequence of hats must have ended on BYY. There are a_{n-3} ways to do this.

Thus, the final relation is $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3}$
Base cases: $a_0 = 1$ (1 way to wear hats over no days), $a_1 = 2$ (for one day, you can only wear blue or yellow, since red requires blue the day before), $a_2 = 4$ (enumerating these out, it can be BB, BR, BY, YB)

At a local candy shop, Milky Way, Twix, and Kit-Kat are each \$2 per bar and Reese's and Hershey's are each \$3 per bar.

a) Let a_n represent the number of ways to buy exactly $\$n$ worth of candy bars where the order in which candy bars are bought matters. Find a recurrence relation for a_n. Include an explanation for why this is the correct recurrence.

b) What are the initial conditions for the recurrence relation from part (a)? For full credit, you must use the fewest number of initial conditions necessary for your recurrence and must start at a_0. Also, please provide brief justification for each initial condition.

c) Compute a_6. Express your answer as a single number.

a) finding the recurrence relation

Thinking backwards about the last candy bar being bought with \$n, there are two cases: the last candy bar was 2-cost, or the last candy bar was 3-cost.

If the last candy bar bought was 2-cost (Milky Way, Twix, Kit Kat) then the sequence of candy bars that are bought before the final candy bar was \$(n - 2). Using the recursive leap of faith, there are a_{n-2} sequences of candy bars for \$(n - 2).

To get the \$n sequences from this case, we take each of the \$(n - 2) sequences and tack on one of the three 2-costs at the end. This gives us $3a_{n-2}$.

If the last candy bar bought was 3-cost (Reese's, Hershey) then the sequence of candy bars that are bought before the final candy bar was \$(n - 3). Using the recursive leap of faith, there are a_{n-3} sequences of candy bars for \$(n - 3).

To get the \$n sequences from this case, we take each of the \$(n - 3) sequences and tack on one of the two 3-costs at the end. This gives us $2a_{n-3}$.

Combining these two cases together, $a_n = 3a_{n-2} + 2a_{n-3}$.

Solution:

a) $a_n = 3a_{n-2} + 2a_{n-3}$
Since there are 3 options of candy bars for \$2 dollars each, we add $3a_{n-2}$ to our recurrence because for any ordering of bars we could have had with the first $n-2$ dollars, we can now add any one of the 3 \$2 bars to the end of the sequence of purchases in order to spend exactly n dollars. Becuase there were a_{n-2} sequences of purchases for exactly $n-2$ dollars spent, then there will be $3a_{n-2}$ sequences of this form of length n. Furthermore, since there are 2 options of candy bars for \$3 dollars each, we add $2a_{n-3}$ to our recurrence because any ordering of bars we could have had with the first $n-3$ dollars, we can now add any one of the 2 \$3 bars to the end of that sequence of purchases in order to spend exactly n dollars.

b) $a_0 = 1, a_1 = 0, a_2 = 3$
$a_0 = 1$ because there is exactly one way to spend \$0 dollars: purchase no candy bars at all.
$a_1 = 0$ because there is no way to spend exactly \$1 dollar if the candy bars are \$2 and \$3 dollars each.
$a_2 = 3$ because in order to spend exactly \$2 dollars, one would need to buy exactly one \$2 bar and there are 3 options of this type.
Note, we need 3 initial conditions because our recurrence relation references as far back as a_{n-3}, so once we have 3 initial conditions, we can calculate all future a_ns using our recurrence relation and these initial conditions.

c) $a_3 = 3 \cdot 0 + 2 \cdot 1 = 2$
$a_4 = 3 \cdot 3 + 2 \cdot 0 = 9$
$a_5 = 3 \cdot 2 + 2 \cdot 3 = 12$
$a_6 = 3 \cdot 9 + 2 \cdot 2 = \mathbf{31}$

Pigeonhole Principle

Pigeonhole Principle Recap

- If there are *k+1* objects and *k* boxes, at least one box must contain 2 or more objects
- Generalized Version: If *N* objects are placed into *k* boxes, then there is at least one box containing at least *ceiling(N/k)* objects
- This does NOT guarantee which box will have 2 or more objects
- This does NOT guarantee the number of objects in the special box

Warm-Up

What's the minimum number of phone numbers in order to ensure that at least three people have the same area code where X can be any number from 0 to 9 (XXX-213-435)?

Warm-Up Solution

What's the minimum number of phone numbers in order to ensure that at least three people have the same area code where X can be any number from 0 to 9 (XXX-213-435)?

There are $10^3 = 1000$ possible area codes since there are 10 possibilities for each of the 3 digits. If we select N phone numbers we can guarantee $\lceil \frac{N}{1000} \rceil$ have the same phone number by generalized pigeonhole principle. So we want to find minimum N that satisfies $\lceil \frac{N}{1000} \rceil = 3$. If $N = 2000$ then $\frac{N}{1000}$ exactly equals 2, so we need at least $N = 2000 + 1 = 2001$ so that its ceiling equals 3.

Therefore 2001 is the minimum number of phone numbers required in order to ensure that at least three people have the same area code.

Pigeonhole Principle

- Think of the “worst case scenarios”
- If the question asks “what is the least number required to get x number of objects per box ?” consider “how many can I have that will allow for x - 1 per box?”

Practice Problem

If there are n people who can shake hands with one another (where $n > 1$), show that there is always a pair of people who will shake hands with the same number of people.

Solution

At minimum, someone can shake hands with 0 people. At maximum, someone can shake hands with everyone else (with $n-1$ people). This gives n different numbers of people that one can shake hands with, which is too many to apply the pigeonhole principle.

We thus consider two cases.

Assume there exists someone who shakes hands with 0 people. Then there is nobody who shakes hands with everyone else. So the maximum number of hands someone can shake becomes $n-2$. This gives $n-1$ different numbers of people that one can shake hands with.

Assume nobody shakes hands with 0 people. So the minimum number of hands someone can shake becomes 1. This gives $n-1$ different numbers of people that one can shake hands with.

In both cases, by the pigeonhole principle we can conclude that if we have n people, it guaranteed that there is a pair of people who will shake hands with the same number of people.

Have a great weekend!