

EECS 203: Discrete Mathematics
Fall 2023
Homework 9

Due **Thursday, November 16**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $9 + 2$

Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

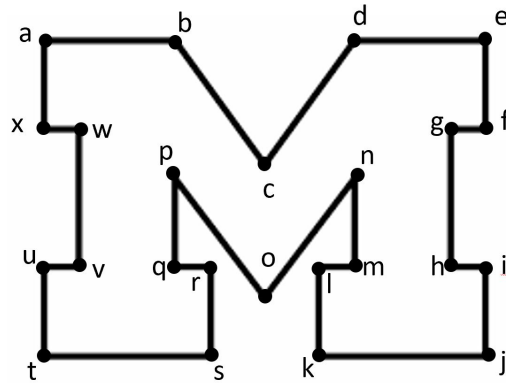
Individual Portion

1. Shortest Paths [12 points]

Provide the shortest path distances between the following point pairings in their respective graphs. Justify your answer by providing a shortest path, or by stating that there is no such path.

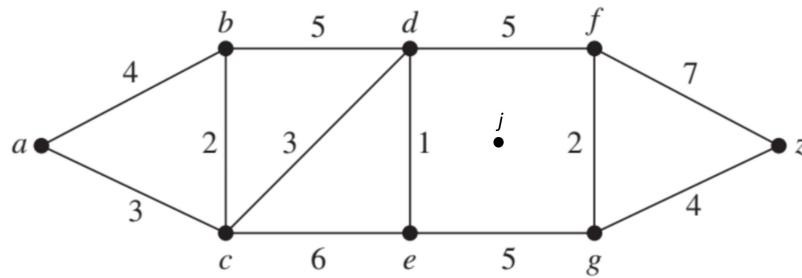
- (a) (i) (x, p)

- $$(ii) \quad (c, o)$$

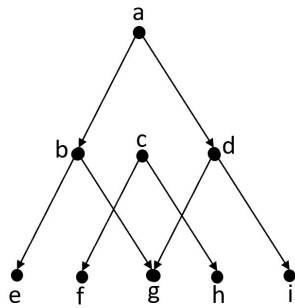


- (b) (i) (a, z)

- (ii) (b, j)

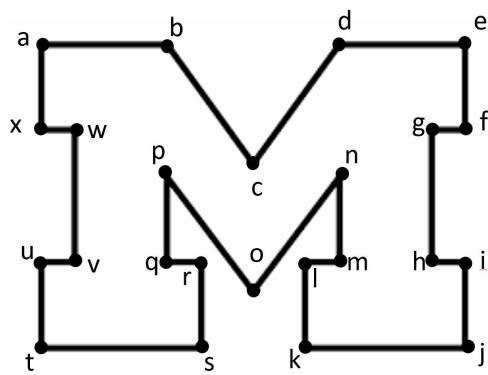


- (c) (i) (a, g)
(ii) (i, b)

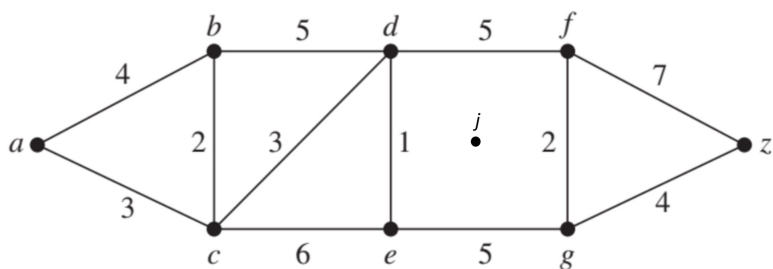


Solution:

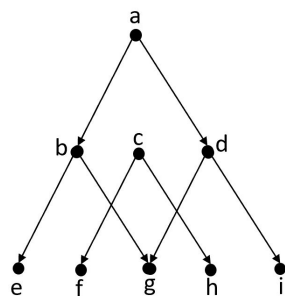
(a)



(b)



(c)



2. Alexander Hamiltonian [12 points]

For each of the following parts, state whether a graph with $n \geq 3$ vertices and the given properties **always**, **sometimes**, or **never** contains a Hamiltonian cycle. Justify your response for each part.

- (a) The complete graph K_n .

(b) A tree.

(c) A bipartite graph.

(d) A graph that contains a vertex v where $\deg(v) = 1$.

Solution:

(a)

(b)

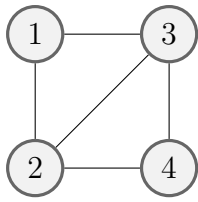
(c)

(d)

3. Bipartite? [12 points]

For each of the following parts, determine if the graph is bipartite and justify your answer.

- (a) The cycle C_5
- (b) The hypercube Q_2
- (c) The complete graph K_5
- (d)



Solution:

(a)

(b)

(c)

(d)

4. Melman the Graph [12 points]

In a simple, undirected graph with 5 vertices, what are all possible values for the number of vertices with odd degree? Justify your answer.

For each possible value k , construct a graph with 5 vertices that has k vertices of odd degree.

Solution:

5. Keeping things Merry! [12 points]

Mary has three dogs, Sarge, Duke, and Brady. However, when left alone, Duke will pick a fight with either of the other two dogs. Mary wants to walk all 3 dogs from her house across

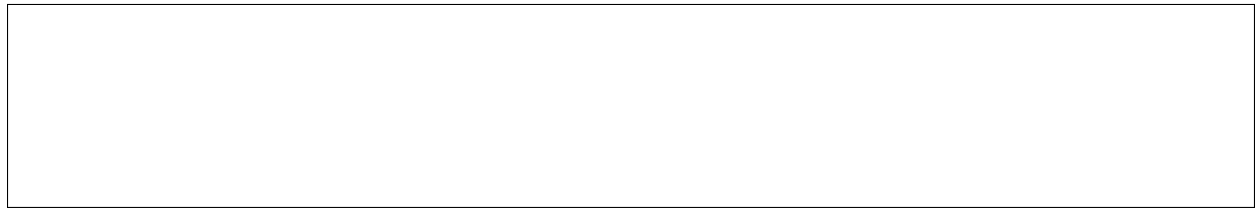
the road to the dog park, can only walk 1 dog at a time, and wants to avoid a fight between any of the dogs. She can make multiple trips across the road, and can leave some of her dogs on one side of the road when walking another across.

- (a) Draw a graph where the nodes are the legal configurations of the puzzle and the edges represent possible transitions from one state to another. Each node should contain M, S, D, B, and || representing Mary, Sarge, Duke, Brady, and the road. For instance SB||MD would represent the Sarge and Brady being the the left side of the road at home with Mary and Duke on the other side at the park. Start with MSDB|| as your initial state.
- (b) Identify the nodes that represent the start and end configurations of the puzzle. Is the puzzle solvable? Explain in terms of your graph.

Solution:

(a)

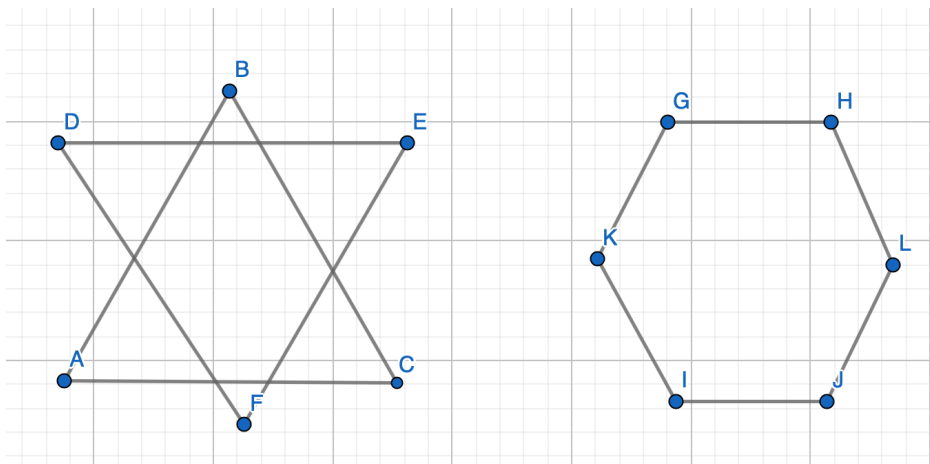
(b)



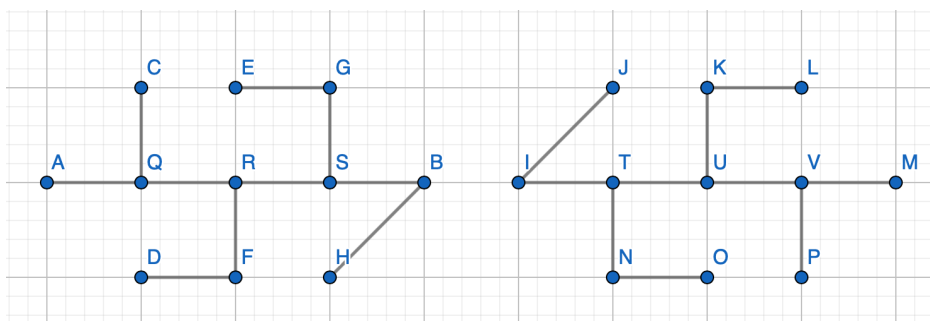
6. It's Iso-Morphin' Time [12 points]

Determine whether or not each of the following pairs of graphs are isomorphic. If yes, provide an isomorphism. If not, explain why.

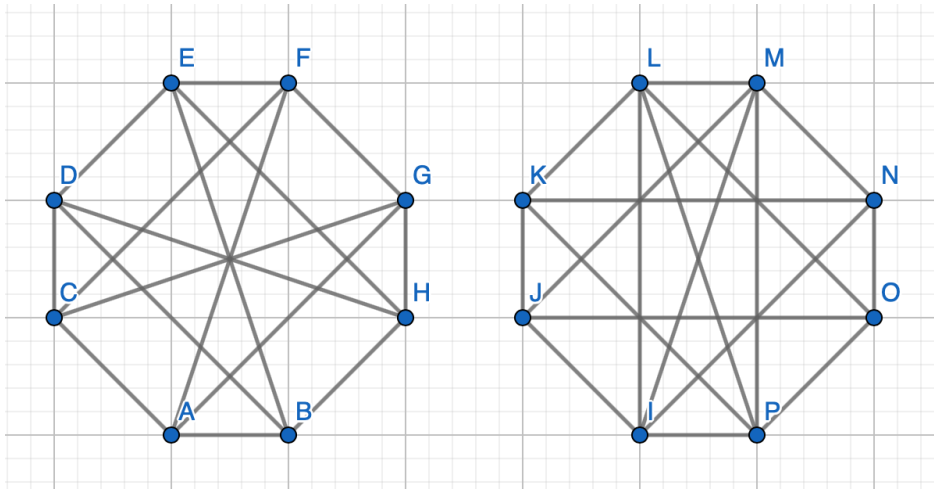
(a)



(b)

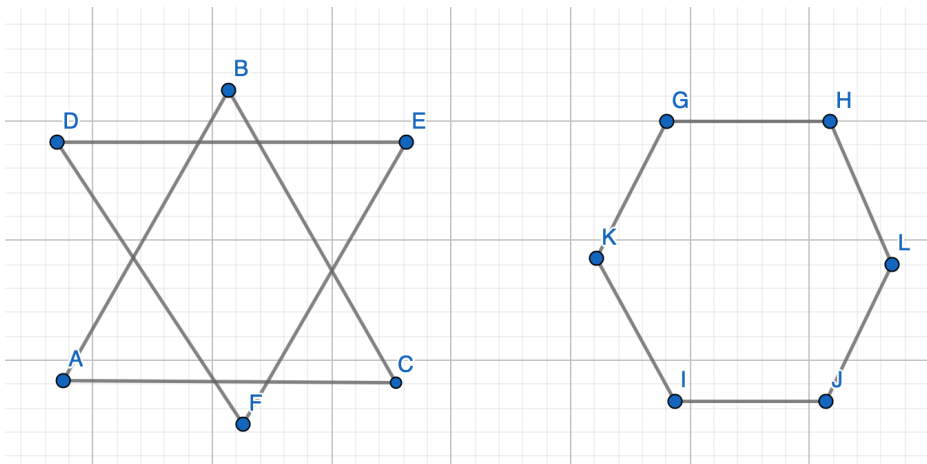


(c)

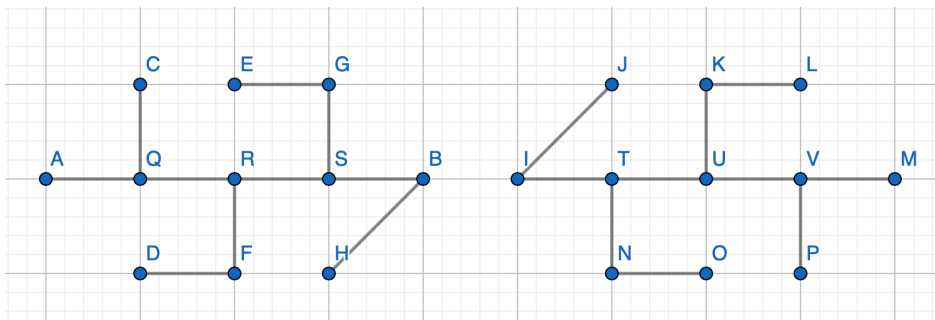


Solution:

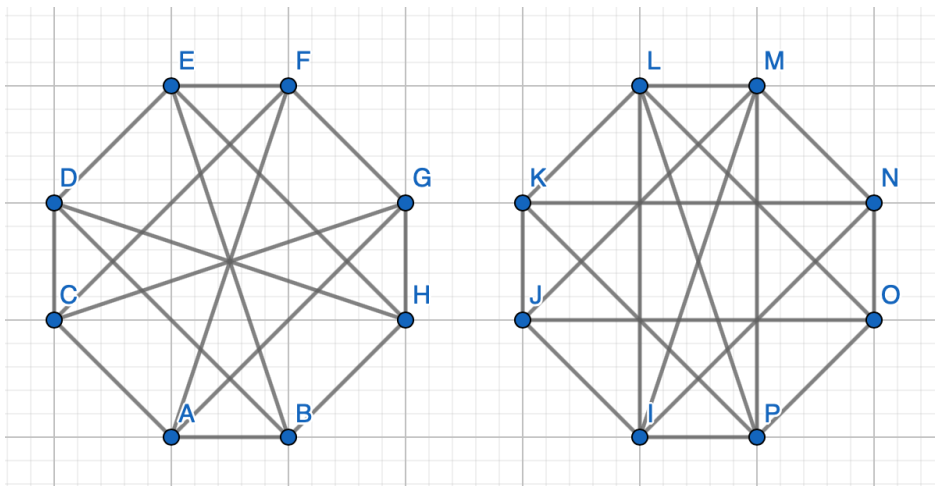
(a)

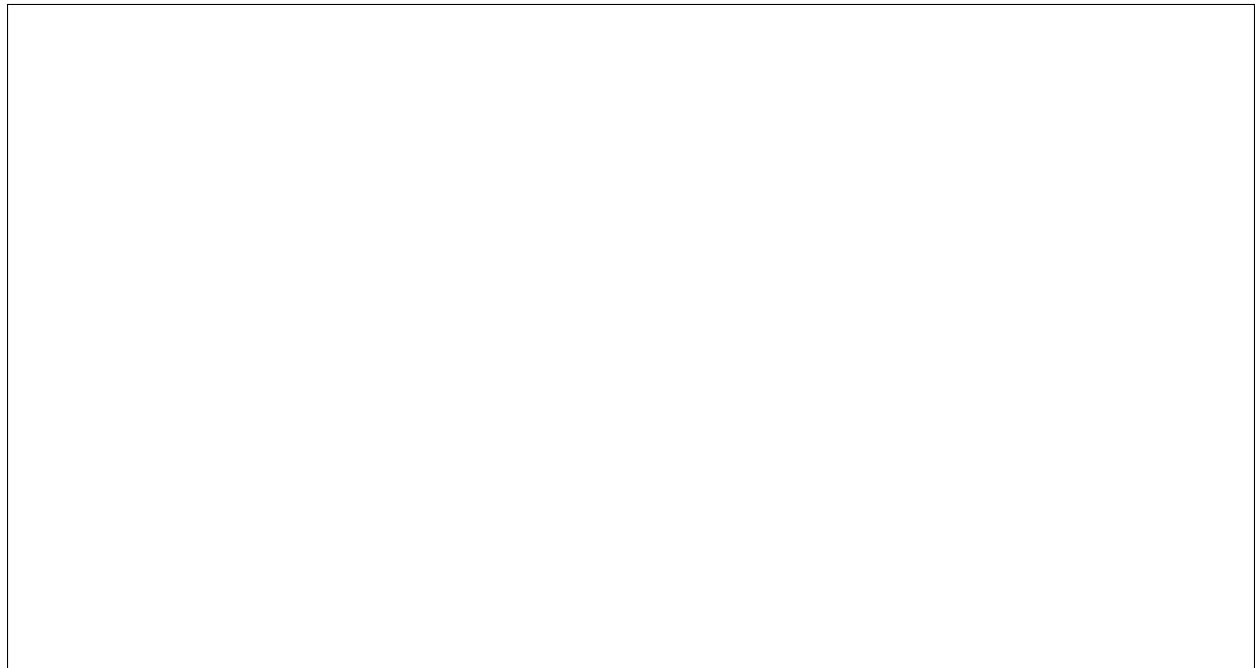


(b)



(c)





7. Reduce, Reuse, Recycle [12 points]

For which values of $n \geq 4$ do these graphs have an Euler cycle?

- (a) The complete graph K_n
- (b) The cycle C_n
- (c) The wheel W_n
- (d) The hypercube Q_n

Solution:

- (a)

(b)

(c)

(d)

8. Counting [8 points]

How many positive integers less than 1000

- (a) are divisible by 7 but not by 11?
- (b) have distinct digits?
- (c) have distinct digits and are even?

You do **not** need to simplify your answer.

Solution:

(a)

(b)

(c)

9. (Not So) Round and Round [8 points]

Suppose we have a square-shaped table which seats 3 people on each side. How many ways are there to seat 12 people at the table, where seatings are considered the same if everyone is in the same group of 3 on a side?

Solution:

Groupwork

1. Grade Groupwork 8

Using the solutions and Grading Guidelines, grade your Groupwork 8:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
 - If your current group submitted the same groupwork last time, grade it together.
 - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) | (viii) | (ix) | (x) | (xi) | Total: |
|-----------|-----|------|-------|------|-----|------|-------|--------|------|-----|------|--------|
| Problem 2 | | | | | | | | | | | | /12 |
| Problem 3 | | | | | | | | | | | | /30 |
| Total: | | | | | | | | | | | | /42 |

Previous Groupwork 8(1): Divisibility by Seven [12 points]

In this question we will show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7. You are allowed to cross off zero digits.

For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number $2345 = 7 \cdot 335$.

We will label the digits of an arbitrary 7-digit number as

$$x_6x_5x_4x_3x_2x_1x_0.$$

- (a) Prove that there exists some $i < 7$ such that either $x_ix_{i-1}\dots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_jx_{j-1}\dots x_0$ is congruent to it modulo 7.

- (b) Use part (a) to prove that if there does not exist some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then there exists $7 > i > j \geq 0$ so that

$$\underbrace{x_i x_{i-1} \dots x_{j+1} 0 \dots 0}_{i+1 \text{ digits total}}$$

is divisible by 7.

- (c) Prove the full claim. That is, show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7.

Solution:

- (a) There are 7 numbers: $x_6 x_5 x_4 x_3 x_2 x_1 x_0$, $x_5 x_4 x_3 x_2 x_1 x_0$, \dots , x_0 .

Use them as Pigeons.

The remainder of any number mod 7, if the number cannot be divided by 7, has 6 circumstances: 1,2,3,4,5,6.

Use them as Holes.

Assume that all these 7 numbers is not divisible by 7, then there are 7 Pigeons in 6 holes, which means that there are at least 2 Pigeons in one whole.

This means that $i < 7$ and $j < i$ such that $x_i x_{i-1} \dots x_0$ and $x_j x_{j-1} \dots x_0$ have the same remainder when divided by 7.

Mark the two numbers as a and b .

Then $b \equiv a \pmod{7}$.

$(b - a) \equiv 0 \pmod{7}$.

Then we have proved the statement: There exists some $i < 7$ such that either $x_i x_{i-1} \dots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_j x_{j-1} \dots x_0$ is congruent to it modulo 7.

- (b) If there does not exist some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then from (a) we know: there exists some $j < i$ such that $x_j x_{j-1} \dots x_0$ is congruent to it. i.e. $(x_i x_{i-1} \dots x_0 - x_j x_{j-1} \dots x_0) \equiv 0 \pmod{7}$. that is, $x_i x_{i-1} \dots x_{j+1} 0 \dots 0 \equiv 0 \pmod{7}$.
 \therefore We have proved the statement.

- (c) Given a 7-digit number $x_6 x_5 x_4 x_3 x_2 x_1 x_0$,

Case 1: $\exists i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7.

Then if $i = 6$, the whole digit is divisible by 7, else we can just cross out all digits from x_6 to x_{i+1} . The remaining digit $x_i x_{i-1} \dots x_0$ is divisible by 7.

Case 2: $\nexists i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7.

Then through (b) we know: there exists some $j < i$ such that $x_i x_{i-1} \dots x_{j+1} 0 \dots 0$ is divisible by 7.

We know that $x_i x_{i-1} \dots x_{j+1} 0 \dots 0 = x_i x_{i-1} \dots x_{j+1} \times 100 \dots$

Then $x_i x_{i-1} \dots x_{j+1} \times 100 \dots \equiv 0 \pmod{7}$.

Since $100 \dots \not\equiv 0 \pmod{7}$

There must be $x_i x_{i-1} \dots x_{j+1} \equiv 0 \pmod{7}$.

So we can rule out all digits before x_i and after x_j to get a number divisible by 7.

Previous Groupwork 8(2): A Powerful Proof [30 points]

In this question we will prove that for any set X , $|\mathcal{P}(X)| > |X|$ ($\mathcal{P}(X)$ is the power set of X). Note that while this is simple in the case where X is finite, things get more complicated when we allow X to be infinite. This proof covers all cases.

- Show that for all (possibly infinite) sets X , $|\mathcal{P}(X)| \geq |X|$.
- Let $g: X \rightarrow \mathcal{P}(X)$ be an arbitrary function. Show that the set $D := \{a \in X \mid a \notin g(a)\}$ is not in the range of g .
- Explain why this shows that $|\mathcal{P}(X)| \leq |X|$ is false and conclude the proof.
- Based on your conclusions above, are there uncountable sets “larger” than \mathbb{R} ? Explain.

Solution:

- We define $f: X \rightarrow \mathcal{P}(X)$:

The map of every element in X to a set in $\mathcal{P}(X)$ that only contains that element. That is: for $X = \{a, b, c, \dots\}$, $f(a) = \{a\}$, $f(b) = \{b\}$, \dots .

Since for every element in X , we can find a subset of X that only contains that element, and every subset is unique, we know this is a one-to-one function.

$\therefore |X| \leq |\mathcal{P}(X)|$.

- The range of g is $\{g(a) \mid a \in X\}$.

D is a subset of X which contains all the elements that are not an element of its image. We know $D \in \mathcal{P}(X)$.

Assume $D \in \text{range}(g)$, then $\exists x \in X$ such that $g(x) = D$, which is an element of \mathcal{P} .

If $x \in D$, then due to the definition of D , $x \notin g(x) = D$.

This causes contradiction.

$\therefore D$ is not in the range of g .

- (c) Since every element in D is also in X , it is a subset of X , so D is an element of $\mathcal{P}(X)$, which is in the codomain of g .
 However, D is not in the range of g .
 \therefore the $\text{range}(g) < \text{codom}(g)$, there does not exist an onto function g from X to $\mathcal{P}(X)$.
 $\therefore |\mathcal{P}(X)| \leq |X|$ is false, $|\mathcal{P}(X)| > |X|$.
- (d) From c, we can know that $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. This is a “larger” set than \mathbb{R} .

2. Square the Cycle [15 points]

Prove that every n -node graph ($n \geq 3$) in which all nodes have degree at least $\lceil \sqrt{n} \rceil$ has a 3-cycle subgraph or a 4-cycle subgraph.

Hint: One useful concept is the neighborhood of a vertex; the neighborhood of $v \in V$ is the set $N(v) = \{u \in V : u \text{ is adjacent to } v\}$. We can also define the neighborhood of a set $A \subseteq V$:

$$N(A) = \{u \in V : u \text{ is adjacent to some } v \in A\}.$$

We recommend using a proof by contradiction, although this can also be done with a clever direct proof. Suppose a graph satisfying the above condition does not have a 3-cycle or 4-cycle. Fix a vertex $v \in V$. What can we say about the size of $N(v)$? What about $N(N(v))$?

Solution:

3. The Office Allocation [15 points]

Consider a new office building with n floors and k offices per floor in which you must assign $2nk$ people to work, each sharing an office with exactly one other person. Find a closed form solution for the number of ways there are to assign offices if from floor to floor the offices are distinguishable, but any two offices on a given floor are not.

Solution: