EECS 203: Discrete Mathematics
Fall 2023
Homework 7

Due **Thursday, October 26**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Growing your Growth Mindset [5 points]

(a) Watch the linked video about developing a growth mindset. This is a different video than the one you saw in lecture.

(b) Rewrite the last two fixed mindset statements as growth mindset statements.

(c) Write down one of your recurring fixed mindset thoughts, then write a thought you can replace it with that reflects a growth mindset.

Video: Developing a Growth Mindset (tinyurl.com/eecs203growthMindset)

What to submit: Your three pairs of fixed and growth mindset statements (the two from the table, and one that you came up with on your own).

Fixed Mindset Statement	**Growth Mindset Statement**
When I have to ask for help or get called on in lecture, I get anxious and feel like people will think I'm not smart.	The question I have is likely the same question someone else in lecture may have. It's important for me to ask so I can better understand what I am learning.
I'm jealous of other people's success.	I am inspired and encouraged by other people's success. They show me what is possible.
I didn't score as high on the exam as I expected. I'm not going to do well in this class and should drop it.	I learned from my mistakes on exam 1, and exam 2 will be a new opportunity for me to practice what I've learned.
This class is hard for me, so I am not fit for this major.	[FILL IN YOUR OWN]
Either I'm good at Discrete Math, or I'm not.	[FILL IN YOUR OWN]
[FILL IN YOUR OWN]	[FILL IN YOUR OWN]

Solution:

Fixed Mindset Statement	Growth Mindset Statement
When I have to ask for help or get called on in lecture, I get anxious and feel like people will think I'm not smart.	The question I have is likely the same question someone else in lecture may have. It's important for me to ask so I can better understand what I am learning.
I'm jealous of other people's success.	I am inspired and encouraged by other people's success. They show me what is possible.
I didn't score as high on the exam as I expected. I'm not going to do well in this class and should drop it.	I learned from my mistakes on exam 1, and exam 2 will be a new opportunity for me to practice what I've learned.
This class is hard for me, so I am not fit for this major.	This class is hard for me and also for others. Every time I am stuck and finally solve a hard question, I am one step closer to the mastery of my major.
Either I'm good at Discrete Math, or I'm not.	I could be bad at Discrete Math, but if I work hard, I would be good at it.
I am afraid of the upcoming EECS 281 next term since it can be even more challenging.	EECS 281 could be more challenging, but I do not need to be anxious about it. As long as I lay a solid basis in this course, I will be well prepared.

2. Home on the Range [15 points]

Find the integer a such that

(a) $a \equiv 74 \pmod{15}$ and $-5 \leq a \leq 9$

(b) $a \equiv 144 \pmod{27}$ and $5 \leq a \leq 31$

(c) $a \equiv -85 \pmod{31}$ and $120 \leq a \leq 150$

Solution:

(a) For some integer m, $a = 74 + 15m$
Since $-5 \leq a \leq 9$
$-5 \leq 15m + 74 \leq 9$

$-5.8 \leq m \leq -4.3$
Since m is an integer, m can only be -5.
$\therefore a = 74 - 15 \times 5 = -1$

(b) For some integer m, $a = 144 + 27m$
Since $5 \leq a \leq 31$
$5 \leq 144 + 27m \leq 31$
$-5.5 \leq m \leq -4.1$
Since m is an integer, m can only be -5.
$\therefore a = 144 - 27 \times 5 = 9$

(c) For some integer m, $a = -85 + 31m$
Since $120 \leq a \leq 150$
$120 \leq 31m - 85 \leq 150$
$6.61 \leq m \leq 7.58$
Since m is an integer, m can only be 7.
$\therefore a = -85 + 31 \times 7 = 132$

3. How low can you go? [15 points]

Suppose $a \equiv 6 \pmod 7$ and $b \equiv 5 \pmod 7$. In each part, find c such that $0 \leq c \leq 6$ and

(a) $c \equiv 2a^2 + b^3 \pmod 7$

(b) $c \equiv b^{24} + 1 \pmod 7$

(c) $c \equiv a^{99} \pmod 7$

Show your work! You should be doing the arithmetic/making substitutions **without using a calculator**. Your work must not include numbers above 50.

Solution:

(a)

$$\begin{aligned}
c &\equiv 2a^2 + b^3 \pmod 7 \\
&\equiv 2 \times 6 \times 6 + 5 \times 5 \times 5 \pmod 7 \\
&\equiv 2 \times (5 \times 7 + 1) + 5 \times (3 \times 7 + 4) \pmod 7 \\
&\equiv 2 + 21 \pmod 7 \\
&\equiv 2 + 3 \times 7 \pmod 7 \\
&\equiv 2 \pmod 7
\end{aligned}$$

$\because 0 \leq c \leq 6$
$\therefore c = 2$

(b)

$$\begin{aligned} c &\equiv b^{24} + 1 \pmod 7 \\ &\equiv 5^{24} + 1 \pmod 7 \\ &\equiv (3 \times 7 + 4)^{12} + 1 \pmod 7 \\ &\equiv 4^{12} + 1 \pmod 7 \\ &\equiv (2 \times 7 + 2)^{6} + 1 \pmod 7 \\ &\equiv 64 + 1 \pmod 7 \\ &\equiv 7 \times 9 + 1 + 1 \pmod 7 \\ &\equiv 2 \pmod 7 \end{aligned}$$

$\because 0 \leq c \leq 6$
$\therefore c = 2$

(c)

$$\begin{aligned} c &\equiv a^{99} \pmod 7 \\ &\equiv 6^{99} \pmod 7 \\ &\equiv (7-1)^{99} \pmod 7 \\ &\equiv (-1)^{99} \pmod 7 \\ &\equiv -1 \pmod 7 \end{aligned}$$

$\because 0 \leq c \leq 6$
$\therefore c = 7 - 1 = 6$

4. Mod-tastic Mixing and Modding [15 points]

Let x and y be integers with $x \equiv 2 \pmod{14}$ and $y \equiv 5 \pmod{21}$. For each of the following expressions, either compute the value, or explain why there is not enough information to determine the value.

(a) $(y - 4x) \bmod 7$

(b) $(x + y) \bmod 14$

(c) $(xy^2 + 12) \bmod 7$

Solution:

Since $x \equiv 2 \pmod{14}$ and $y \equiv 5 \pmod{21}$, there are some integer p and q such that $x = 14p + 2$, $y = 21q + 5$.

(a) $y - 4x = 21q + 5 - 56p - 8 = 7(3q - 8p) - 3$
Since p and q are integers, $3p - 8q$ is an integer, so $7|7(3q - 8q)$.
$\therefore (y - 4x) \equiv -3 \pmod 7 \equiv 4 \pmod 7$
$\therefore (y - 4x) \bmod 7 = 4$

(b) $x + y = 14p + 2 + 21q + 5 = 7(2p + 3q + 1)$
Since p, q are integers, $2p + 3q + 1$ is an integer.
$\therefore 7|7(2p + 3q + 1)$
Then if $2p + 3q + 1$ is even, i.e. $2|7(2p + 3q + 1)$, $14|7(2p + 3q + 1)$;
And if $2p + 3q + 1$ is odd, $14 \not| 2p + 3q + 1$.
However, we do not know whether $2p + 3q + 1$ is even or odd.
$\therefore$ there is not enough information to determine the value.

(c) Since $x = 2 \cdot 7p + 2$, $x \equiv 2 \pmod 7$.
Since $y = 21q + 5$, $y^2 = 21 \cdot 21q^2 + 10 \times 21q + 25 = 7(21 \cdot 3q^2 + 30q + 3) + 4$.
Since p, q are integers, $(21 \cdot 3q^2 + 30q + 3)$ is an interger.
$\therefore 7(21 \cdot 3q^2 + 30q + 3) + 4 \equiv 4 \pmod 7 = 4$, $y^2 \equiv 4 \pmod 7$
$\therefore (xy^2 + 12) \equiv (2 \times 4) \pmod 7 \equiv 1 \pmod 7$.
$\therefore (xy^2 + 12) \bmod 7 = 1$.

5. Sample Functions [15 points]

Determine if each of the examples below are functions or not. If a given construction is not a function, prove it by showing that a single input can have multiple outputs (not well defined) or that some input doesn't have an output (not total). If it is a function, explain why you think each input has exactly one output.

(a) $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = y$ iff $3y = \frac{1}{x-3}$

(b) $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = y$ iff $y \leq x$

(c) $f\colon$ Compound Propositions $\to \{T, F\}$ such that $f(x) = T$ iff x is satisfiable, and $f(x) = F$ otherwise.

Example: $f(p \land \neg p) = F$.

Solution:

(a) Not a function.
Consider $x = 3$, $3y = \frac{1}{0}$ does not exist. That means for $x = 3$, there is no output.
$\therefore$ not a function.

(b) Not a function.
Consider $x = 2$, $y = 1$ and $y = 0$ all satisfy $y \leq 1$, so $f(x)$ can be both 0 and 1.
$\therefore$ have multiple outputs, not a function.

(c) It is a function.
For any Compound Proposition, it must has a truth value which can only be either T or F.
$\therefore$ for any satisfiable input x, there is exactly an output which is either T or F.
$\therefore$ is a function.

6. Functions are Fun(ctions) [20 points]

For each of the following functions, prove or disprove that it is onto and that it is one-to-one. Conclude whether it is a bijection or not, and why.

(a) $f\colon [2,3] \to [7,9]$, with $f(x) = 2x + 3$
Reminder: $[a,b]$ is the set of all real numbers between a and b, including both a and b.
Reminder: Always make sure to reference whether things are elements of the domain and codomain when needed. This is true for all such proofs, but this part has more unusual sets, so it is extra important.

(b) $f\colon \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$, with $f(x,y) = \frac{x}{y}$

(c) $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = \lfloor x \rfloor + x$

(d) $f\colon \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Z}$ where $f(x,y) = x^{2y}$

Solution:

(a) We prove it is onto and one-to-one.

i. $\forall a_1, a_2 \in [2,3]$, $[f(a_1) = f(a_2)] \to (a_1 = a_2)$
Proof:
Let a_1, a_2 be arbitrary real numbers in $[2,3]$.
Assume $f(a_1) = f(a_2)$,
Then $2a_1 + 3 = 2a_2 + 3$, $2a_1 = 2a_2$, $a_1 = a_2$.
$\therefore f$ is one-to-one.

ii. $\forall b \in [7,9], \exists a \in [2,3]$ such that $f(a) = b$.
Proof:
Let b be an arbitrary real numbers in $[7,9]$.
Consider $a = \frac{b-3}{2}$
Then $\frac{7-3}{2} \leq a \leq \frac{9-3}{2}$, $2 \leq a \leq 3$, a is in the domain.
And $2a + 3 = b$.
$\therefore f$ is onto.

$\therefore f$ is onto and one-to-one.
$\therefore f$ is a bijection.

(b) We prove it is onto but not one-to-one.

i. $\exists (a_1, b_1), (a_2, b_2)$ where $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{Z}^+$, $[f(a_1, b_1) = f(a_2, b_2)] \wedge [(a_1, b_1) \neq (a_2, b_2)]$.
Proof:
Consider $x_1 = 1$, $y_1 = 2$, $x_2 = 2$, $y_2 = 4$.
Then $f(x_1, y_1) = \frac{x_1}{y_1} = \frac{1}{2}$, $f(x_2, y_2) = \frac{x_2}{y_2} = \frac{2}{4} = \frac{1}{2}$
$\therefore f$ is not one-to-one.

ii. $\forall c \in \mathbb{Q}$, $\exists (a, b)$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, such that $c = \frac{a}{b}$.
Proof:
Let c be an arbitrary rational numbers.
Then for some integers p, q, $c = \frac{p}{q}$. If q is positive, consider $a = p$ and $b = q$.
Then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, and $c = \frac{a}{b}$.
If q is negative, consider $a = -p$ and $b = -q$.
Then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, and $c = \frac{a}{b}$.
$\therefore f$ is onto.

$\therefore f$ is onto but not one-to-one.
$\therefore f$ is not a bijection.

(c) We prove it is one-to-one but not onto.

i. $\forall a_1, a_2 \in \mathbb{R}$, $[f(a_1) = f(a_2)] \rightarrow (a_1 = a_2)$.
Proof:
Let a_1 and a_2 be arbitrary real numbers.
Assume $f(a_1) \neq f(a_2)$, seeking contradiction.
Case 1: $a_2 > a_1$
Then $\lfloor a_2 \rfloor \geq \lfloor a_1 \rfloor$, $\lfloor a_2 \rfloor + a_2 < \lfloor a_1 \rfloor + a_1$, $f(a_2) > f(a_1)$.
Case 2: $a_2 < a_1$
Then $\lfloor a_2 \rfloor \leq \lfloor a_1 \rfloor$, $\lfloor a_2 \rfloor + a_2 < \lfloor a_1 \rfloor + a_1$, $f(a_2) < f(a_1)$.
$\therefore$ contradiction.
$\therefore a_1$ can only equal to a_2.
$\therefore f$ is one-to-one.

ii. $\exists\, b \in \mathbb{R}$ such that $\forall\, a \in \mathbb{R}$, $f(a) \neq b$.
Proof:
Consider $b = 1$.
When $a < 0$, $f(a) = \lfloor a \rfloor + a < 0$.
When $0 < a < 1$, $\lfloor a \rfloor$ is always 0, and $f(a) = \lfloor a \rfloor + a = 0 + a < 1$.
And at the point where a reaches 1, $\lfloor a \rfloor$ immediately becomes 1, and $f(a) = \lfloor a \rfloor + a = 1 + 1 = 2$.
After that when $a > 1$, $\lfloor a \rfloor \geq 1$ and $f(a) = \lfloor a \rfloor + a > 1$.
$\therefore$ for $b = 1$, there is no such a that $f(a) = b$. $\therefore f$ is not onto.

$\therefore f$ is one-to-one but not onto.
$\therefore f$ is not a bijection.

(d) We prove it is not onto and not one-to-one.

i. $\exists (a_1, b_1), (a_2, b_2)$ where $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{Z}^+$, $[f(a_1, b_1) = f(a_2, b_2)] \wedge [(a_1, b_1) \neq (a_2, b_2)]$.
Proof:
Consider $a_1 = 4, b_1 = 1,\ a_2 = 2, b_2 = 2$,
Then $f(a_1, b_1) = f(a_2, b_2) = 16$.
$\therefore f$ is not one-to-one.

ii. $\exists c \in Z$, $\forall a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, $f(a, b) \neq c$.
Proof:
Consider c $= 2$.
Since $b \in \mathbb{Z}+$, $2b \geq 2$.
We know that $2^2 = 4$ and for a power a^n where $|a| > 1$, when $n > 1$, the lager n is, the larger $|a^n|$ is,
$\therefore$ for all a $\in \mathbb{Z}$ where $|a| \geq 2$ and $b \in \mathbb{Z}^+$, $f(a, b) \geq 4 > c$.
And when $|a| = 0$ or 1, no matter what b is, $f(a, b)$ is 0 or $\pm 1 \neq 2$.
$\therefore$ for $c = 2$, there is no correspoding a, b such that $f(a, b) = c$.
$\therefore f$ is not onto.

$\therefore f$ is not onto and not one-to-one.
$\therefore f$ is not a bijection.

7. Comp∘sition(Functions) [15 points]

For each of the following pairs of functions f and g, find $f \circ g$ and $g \circ f$, and name their domains and codomains. If either can't be computed, explain why.

(a) $f\colon \mathbb{N} \to \mathbb{Z}^+,\ f(x) = x^2 + 1$

$g\colon \mathbb{Z}^+ \to \mathbb{N},\ g(x) = x + 2$

(b) $f\colon \mathbb{Z} \to \mathbb{R},\ f(x) = \left(4x + \frac{3}{7}\right)^3$

$g\colon \mathbb{R} \to \mathbb{R}_{\geq 0},\ g(x) = |x|$

Note: $\mathbb{R}_{\geq 0}$ is the set of real numbers greater than or equal to 0.

Solution:

(a) Since $codom(g) \subseteq dom(f)$, $f \circ g$ exists.
$f \circ g(x) = f(g(x)) = (x+2)^2 + 1 = x^2 + 4x + 5.$
The domain of $f \circ g(x)$ which equals to the domain of $g(x)$ is $\mathbb{Z}^+$.
So $f \circ g(x) \geq f \circ g(1) = 10.$
$\therefore$The codomain of $f \circ g(x)$ is $\mathbb{Z}_{\geq 10}$.

Since $codom(f) \subseteq dom(g)$, $g \circ f$ exists.
$g \circ f(x) = g(f(x)) = (x^2 + 1) + 2 = x^2 + 3.$
The domain of $g \circ f(x)$ whith equals to the domain of $f(x)$ is $\mathbb{N}$.
Since $x^2 + 3 \pm 3$, and at $x = 0$, $x^2 + 3 = 3$,
The codomain of $g \circ f(x)$ is $\mathbb{Z}_{\geq 3}$.

(b) Since $codom(g) \not\subseteq dom(f)$, $f \circ g$ does not exist.

Since $codom(f) \subseteq dom(g)$, $g \circ f$ exists.
$g \circ f(x) = g(f(x)) = |(4|x| + \frac{3}{7})^3|.$
The domain of $g \circ f(x)$ which equals to the domain of $f(x)$ is $\mathbb{Z}$.
Since $4|x| \geq 0$, $g \circ f(x) \geq (\frac{3}{7})^3$. At x = 0 its value is $\frac{27}{343}$.
The codomain of $g \circ f(x)$ is $\mathbb{R}_{\geq \frac{27}{343}}$.