EECS 203: Discrete Mathematics
Fall 2023
Homework 9

Due **Thursday, November 16**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $9 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

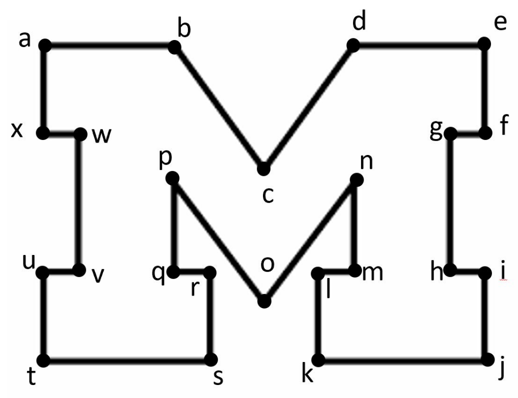
Individual Portion

1. Shortest Paths [12 points]

Provide the shortest path distances between the following point pairings in their respective graphs. Justify your answer by providing a shortest path, or by stating that there is no such path.

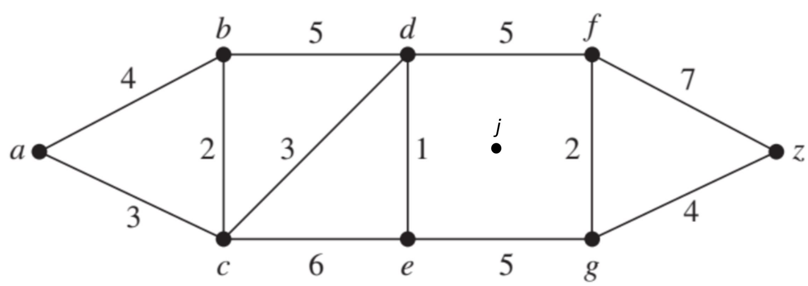
(a) (i) (x, p)

(ii) (c, o)



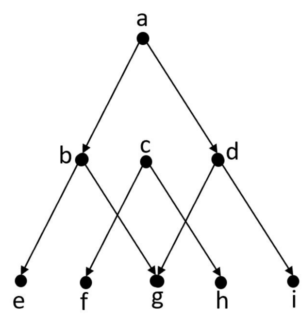
(b) (i) (a, z)

(ii) (b, j)



(c) (i) (a, g)

(ii) (i, b)



Solution:

(a) (i) 8. Shortest path: $(x, w, v, u, t, s, r, q, p)$.

(ii) 12. Shortest path: $(c, d, e, f, g, h, i, j, k, l, m, n, o)$. Alternate shortest path: $(c, b, a, x, w, v, u, t, s, r, q, p, o)$.

(b) (i) 16. Shortest path: (a, c, d, e, g, z).

(ii) ∞ or undefined; there is no bj path.

(c) (i) 2. Shortest path: (a, b, g). Alternate shortest path: (a, d, g).

(ii) ∞ or undefined; no ib path exists.

Draft Grading Guidelines [12 points]

For each pair of vertices:

+1 correct value
+1 correct path

2. Alexander Hamiltonian [12 points]

For each of the following parts, state whether a graph with $n \geq 3$ vertices and the given properties **always**, **sometimes**, or **never** contains a Hamiltonian cycle. Justify your response for each part.

(a) The complete graph K_n.

(b) A tree.

(c) A bipartite graph.

(d) A graph that contains a vertex v where $\deg(v) = 1$.

Solution:

(a) Always. C_n is a subgraph of K_n for all $n \geq 3$, so K_n contains a C_n subgraph and therefore contains a Hamiltonian cycle.

(b) Never. By definition, a tree contains no cycles. So, since a graph containing a Hamiltonian cycle must contain a C_n subgraph, a tree with $n \geq 3$ vertices will not contain a Hamiltonian cycle.

(c) Sometimes. C_4 is a bipartite graph that contains a Hamiltonian cycle. A tree with $n = 3$ vertices is a bipartite graph that does not contain a Hamiltonian cycle.

(d) Never. If there is a vertex v in G with degree 1, then that vertex must not be a part of a cyclic subgraph of G (since all vertices in a cycle have degree 2). Thus, G must not contain a Hamiltonian cycle.

Draft Grading Guidelines [12 points]

For each part:

+1 correct always/sometimes/never
+2 correct justification

3. Bipartite? [12 points]

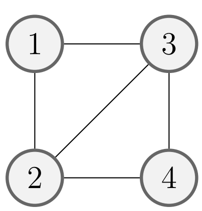
For each of the following parts, determine if the graph is bipartite and justify your answer.

(a) The cycle C_5

(b) The hypercube Q_2

(c) The complete graph K_5

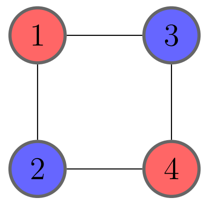
(d)



Solution:

(a) C_5 is not bipartite because there is a cycle with an odd number of nodes.

(b) Q_2 is bipartite because there are no cycles with an odd number of nodes. The following is one possible coloring:



(c) K_5 is not bipartite because there are cycles with an odd number of nodes.

(d) This graph is not bipartite because it contains odd cycles, for example $(1, 2, 3)$.

Draft Grading Guidelines [12 points]

For each part:

+1 correct answer
+2 valid justification

4. Melman the Graph [12 points]

In a simple, undirected graph with 5 vertices, what are all possible values for the number of vertices with odd degree? Justify your answer.

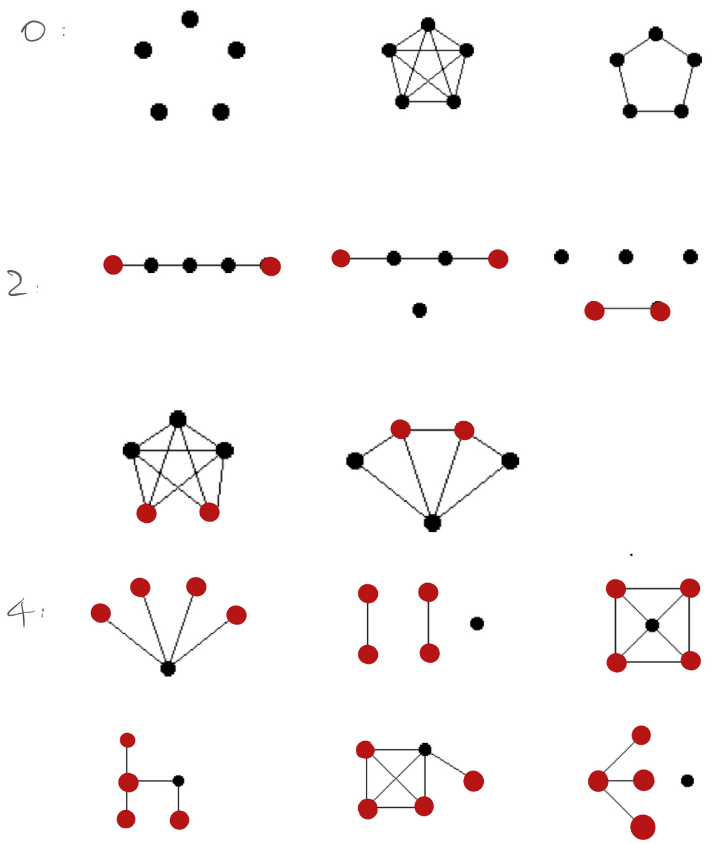
For each possible value k, construct a graph with 5 vertices that has k vertices of odd degree.

Solution:

By the Handshake Theorem, the sum of the degrees in a graph is twice the number of edges, and so is even. Therefore, a graph must have an even number of vertices with odd degree, since otherwise the sum of the degrees of the vertices would be odd.

Alternatively, we have a corollary of the Handshake Theorem from lecture directly stating that all graphs have an even number of vertices with odd degree.

So at most, the possible numbers of vertices of odd degree is 0, 2, or 4. Because we can find examples of graphs with all of these, the possibilities are exactly 0, 2, and 4 vertices of odd degree. Below are some (not all) possible graphs.



Draft Grading Guidelines [12 points]

+2 correct set of possible vertex counts: 0, 2, 4
+0.5 partial credit for giving 2/3 correct vertex counts and no more
+4 correctly justifies why there cannot be 1, 3, or 5 vertices of odd degree without claiming any of 0, 2, or 4 are impossible; for example, using the Handshake Theorem or citing its corollary to rule out odd numbers of odd degree vertices
+1 partial credit for correct justification for why there cannot be 2/3 of 1, 3, and 5 vertices of odd degree, without claiming any of 0, 2, or 4 are impossible

For each correct number of vertices:

+2 at least one correct graph (and no incorrect graphs if some were given) for that value

5. Keeping things Merry! [12 points]

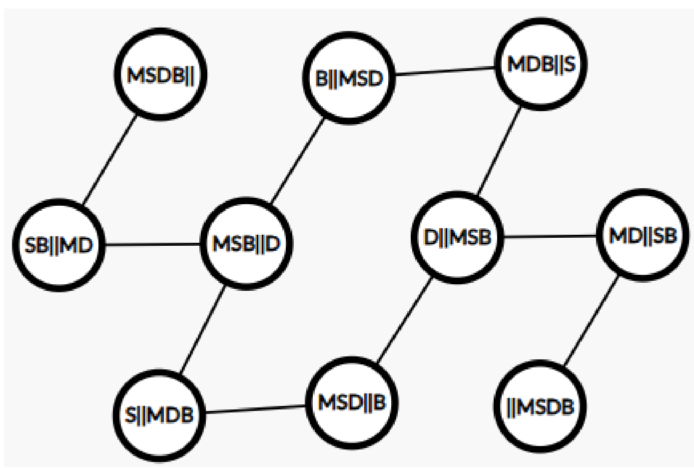
Mary has three dogs, Sarge, Duke, and Brady. However, when left alone, Duke will pick a fight with either of the other two dogs. Mary wants to walk all 3 dogs from her house across the road to the dog park, can only walk 1 dog at a time, and wants to avoid a fight between any of the dogs. She can make multiple trips across the road, and can leave some of her dogs on one side of the road when walking another across.

(a) Draw a graph where the nodes are the legal configurations of the puzzle and the edges represent possible transitions from one state to another. Each node should contain M, S, D, B, and || representing Mary, Sarge, Duke, Brady, and the road. For instance SB||MD would represent the Sarge and Brady being the the left side of the road at home with Mary and Duke on the other side at the park. Start with MSDB|| as your initial state.

(b) Identify the nodes that represent the start and end configurations of the puzzle. Is the puzzle solvable? Explain in terms of your graph.

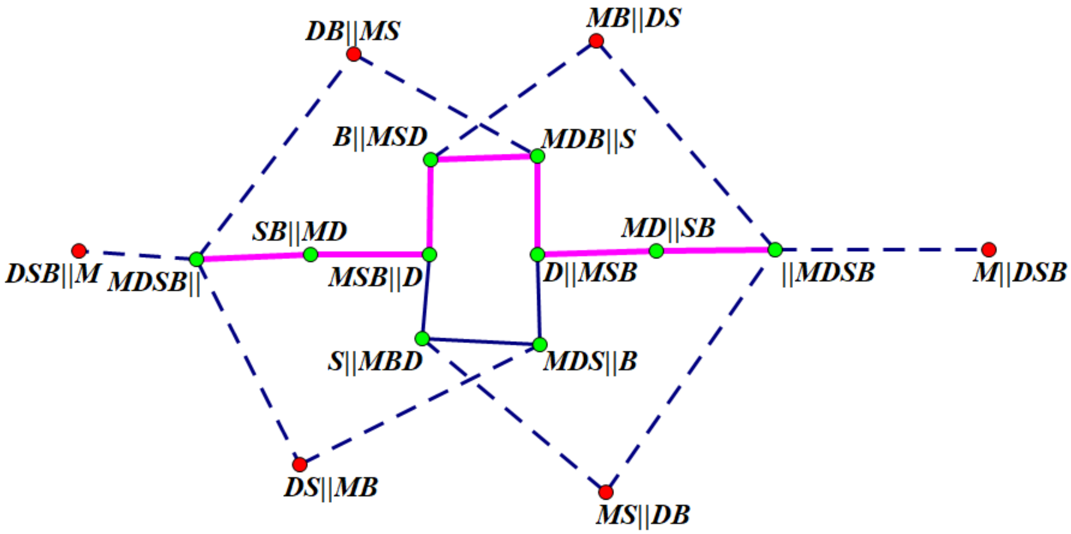
Solution:

(a)



Alternate Solution:

If you wanted to include the nodes that would be reached by invalid moves, you would need to make it clear that they are invalid, such as by color coding and labeling. For example, the graph below uses red nodes to mark the bad states, green nodes to mark the good ones, and dashed lines to mark moves that cannot be made. A solution path is marked in magenta.



(b) Start: MSDB||, End: ||MSDB. Seeing that these states are connected by a valid set of transitions between states, there is a valid path from MSDB|| to ||MSDB and therefore the puzzle is solvable.

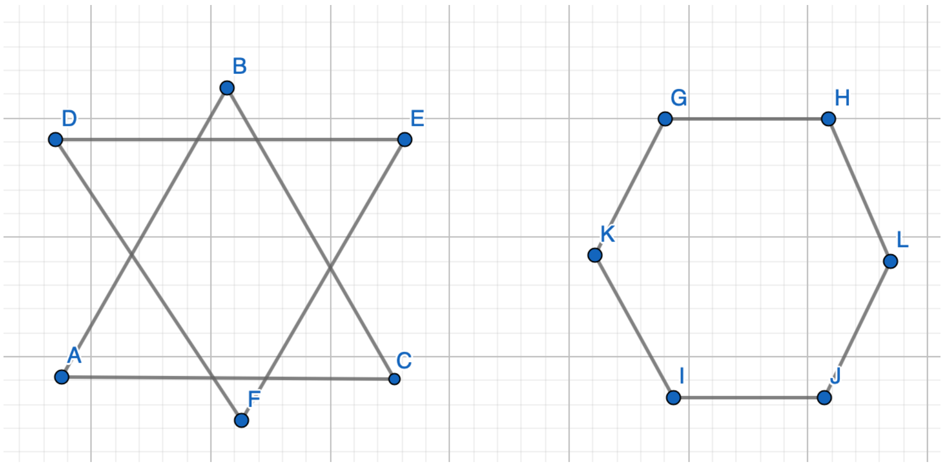
Draft Grading Guidelines [12 points]

+1 correctly identifies MSDB|| to SB||MD as the only viable first step
+5 correctly completes the rest of the graph
partial credit for item above: -1 *per extraneous/missing edge/vertex up to* -5 *points*
+2 identifies MSDB|| and ||MSDB as the start and end nodes respectively
+4 states puzzle is solvable and uses graph as justification

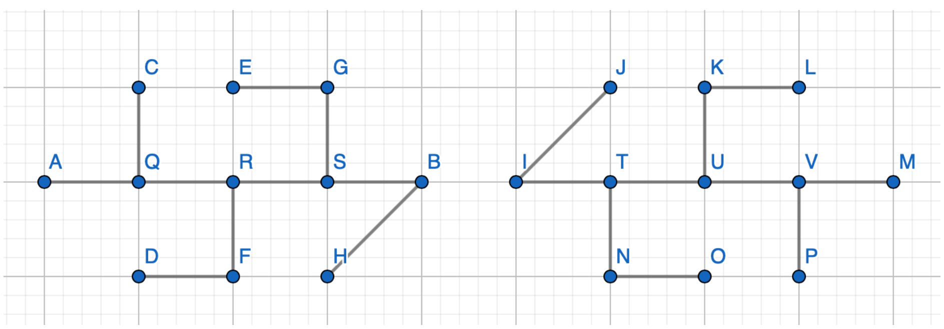
6. It's Iso-Morphin' Time [12 points]

Determine whether or not each of the following pairs of graphs are isomorphic. If yes, provide an isomorphism. If not, explain why.

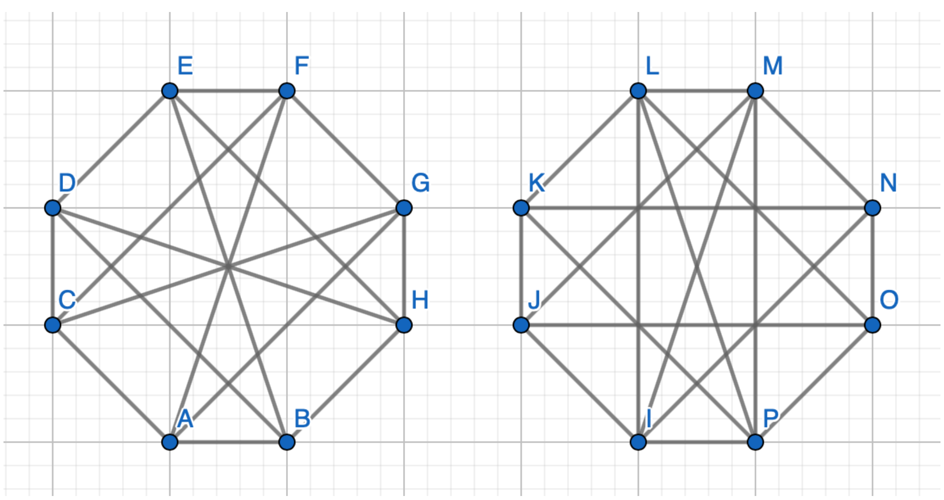
(a)



(b)



(c)



Solution:

(a) **Not isomorphic.** Some possible reasons:

- first graph has 2 connected components while second graph has 1
- (equivalent to the above) graph 1 is not connected, graph 2 is connected
- graph 2 has a path of length 6 while graph 1 does not
- graph 1 is not bipartite, graph 2 is bipartite

(b) **Isomorphic.** We can actually imagine flipping either of these graphs horizontally and vertically and getting the other one. Below is one isomorphism between the two graphs (there may be others).

$$f(C) = P$$
$$f(E) = O$$
$$f(G) = N$$
$$f(A) = M$$
$$f(Q) = V$$
$$f(R) = U$$
$$f(S) = T$$
$$f(B) = I$$
$$f(H) = J$$
$$f(F) = K$$
$$f(D) = L$$

(c) **Not isomorphic.** Some possible reasons:

- different degree sequences
- different number of edges

Draft Grading Guidelines [12 points]

For each part:

+2 correctly states isomorphic or not isomorphic
+2 if answered correctly, then provides an isomorphism if the graphs are isomorphic, or identifies a broken invariant if they are not

7. Reduce, Reuse, Recycle [12 points]

For which values of $n \geq 4$ do these graphs have an Euler cycle?

(a) The complete graph K_n

(b) The cycle C_n

(c) The wheel W_n

(d) The hypercube Q_n

Solution:

(a) The degrees of the vertices are all $(n-1)$. So, the degrees are even if and only if n is odd. Therefore, there is an Euler circuit if and only if n is odd.

(b) For all $n \geq 4$, C_n has an Euler circuit, namely itself.

(c) Since the degrees of the vertices around the rim are all odd, no wheel has an Euler circuit.

(d) The degrees of the vertices are all n. Therefore, there is an Euler circuit if and only if n is even.

Draft Grading Guidelines [12 points]

For each part:

+1 correct answer
+2 correct reasoning

8. Counting [8 points]

How many positive integers less than 1000

(a) are divisible by 7 but not by 11?

(b) have distinct digits?

(c) have distinct digits and are even?

You do **not** need to simplify your answer.

Solution:

Let the universe be $U = \{1, 2, \ldots, 999\}$. Define $S_k = \{x \in U : k \mid x\}$ to be the multiples of k in U.

(a) 130

The question asks for $|S_7 - S_{11}|$.

$$\begin{aligned} |S_7 - S_{11}| &= |S_7| - |S_7 \cap S_{11}| && (1)\\ &= |S_7| - |S_{77}| && (2)\\ &= \left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{999}{77} \right\rfloor && (3)\\ &= 142 - 12 && (4)\\ &= 130. && (5) \end{aligned}$$

(1) follows from the fact that we must only subtract the multiples of 11 that are also in S_7, i.e. the numbers which are multiplies of both 7 and 11. This leads in to (2), as the multiples of both 7 and 11 are exactly the multiples of $7 \cdot 11 = 77$.

(b) $9 + 9 \cdot 9 + 9 \cdot 9 \cdot 8 = 9 + 81 + 648 = 738$

We break the problem down into three cases: one-digit numbers, two-digit numbers and three-digit numbers.

There are 9 one-digit numbers, and each of them has distinct digits.

There are 90 two-digit numbers (10 through 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. An alternative way to compute this is to note that the first digit must be 1 through 9 (9 choices), and the second digit must be something different from the first digit (9 choices out of the 10 possible digits), so by the product rule, we get $9 \cdot 9 = 81$ choices in all.

This approach also tells us that there are $9 \cdot 9 \cdot 8 = 648$ three-digit numbers with distinct digits (again, work from left to right; in the ones place only 8 digits are left to choose from).

So the final answer is $9 + 81 + 648 = 738$.

(c) $738 - (5 + 5 \cdot 8 + 5 \cdot 8 \cdot 8) = 738 - (5 + 40 + 320) = 738 - 365 = 373$

It turns out to be easier to count the odd numbers with distinct digits and subtract from our answer to part (b), so let us proceed that way.

There are 5 odd one-digit numbers. For two-digit numbers, first choose the ones digit (5 choices), then choose the tens digit (8 choices, since neither the ones digit value nor 0 is available); therefore there are 40 such two-digit numbers. (Note that this is not exactly half of 81.)

If we had tried to compute the number of even numbers with distinct digits directly, then we would have to deal with the fact that if the last digit was 0, then the first could be one of 9 digits (1-9), but if it was any other even number then it could only be one of 8 (1-9 without the chosen digit). This is why it's easier to count odd numbers with distinct digits.

For the three-digit numbers, first choose the ones digit (5 choices), then the hundreds digit (8 choices), then the tens digit (8 choices), giving us 320 in all. So there are $5 + 40 + 320 = 365$ odd numbers with distinct digits.

Thus the final answer is $738 - 365 = 373$.

Draft Grading Guidelines [8 points]

Part a:

+1 correct application of difference rule
+1 correct final answer

Parts b and c:

+1 at least one of the three cases (one digit number, two digit number, and three digit number) is correct
+2 correct final answer

9. (Not So) Round and Round [8 points]

Suppose we have a square-shaped table which seats 3 people on each side. How many ways are there to seat 12 people at the table, where seatings are considered the same if everyone is in the same group of 3 on a side?

Solution:

There are 12! ways to order 12 people around the table. Then, we need to figure out how this is overcounting the number of orderings according to which seatings are considered the same. We are going to divide out the number of arrangements in which seatings are considered to be the same to account for this overcounting. If we rearrange 3 people on any given side, the seating will remain the same. Thus, we need to divide by 3! for each side of the table, which gives us $(3!)^4$. The positioning of groups relative to each other also does not affect the seating. Since there are 4 sides, we must also divide by 4!. So, one final answer is:

$$\frac{12!}{(3!)^4 \cdot 4!}.$$

Alternate Solution 1:

We only care about which people are sitting together on a side, so we can choose the 4 groups of 3 with $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}$. However, this implicitly assigns an order to the groups, which we don't want, because we don't care which side the groups sit on. Therefore, we need to divide this by 4!. This gives us another final answer of:

$$\frac{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!}.$$

Alternate Solution 2:

Let us start by numbering the people 1 through 12. Person 1 must be sitting at the table somewhere. There are 2 people sitting with them on the same side. Select which two (unordered) with $\binom{11}{2}$. Of the remaining 9 people, select the smallest numbered person and call them x (that might be person 2, but if 2 is on the same side as 1, it might be a higher number). x is by definition not sitting on the same side as 1, but is sitting at the table on some other side. x has 2 people sitting with them, that we must select from the remaining 8 people with $\binom{8}{2}$. Now we have 6 people left, and once again, we will pick the smallest numbered person and call them y. Once again, by definition, y is not on the same side as 1 or x, so they must be sitting on yet a third side. They have 2 people sitting with them, which we can select with $\binom{5}{2}$. Finally, the remaining 3 people must be sitting together on whatever side of the table is left. Putting this together, we get:

$$\binom{11}{2}\binom{8}{2}\binom{5}{2}$$

Draft Grading Guidelines [8 points]

+2 has factor of 12!
+3 divides by $(3!)^4$ (can be accomplished by explicitly dividing or selecting groups via combinations like in the alternate solution)
+3 divides by 4!

Groupwork

1. Grade Groupwork 8

Using the solutions and Grading Guidelines, grade your Groupwork 8:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/12
Problem 3												/30
Total:												/42

2. Square the Cycle [15 points]

Prove that every n-node graph ($n \geq 3$) in which all nodes have degree at least $\lceil \sqrt{n} \rceil$ has a 3-cycle subgraph or a 4-cycle subgraph.

Hint: One useful concept is the neighborhood of a vertex; the neighborhood of $v \in V$ is the set $N(v) = \{u \in V : u \text{ is adjacent to } v\}$. We can also define the neighborhood of a set $A \subseteq V$:

$$N(A) = \{u \in V : u \text{ is adjacent to some } v \in A\}.$$

We recommend using a proof by contradiction, although this can also be done with a clever direct proof. Suppose a graph satisfying the above condition does not have a 3-cycle or 4-cycle. Fix a vertex $v \in V$. What can we say about the size of $N(v)$? What about $N(N(v))$?

Solution:

We use a proof by contradiction. We consider a graph with n nodes without 3-cycles or 4-cycles satisfying the problem's criteria, and then reach a contradiction by showing it must have too many vertices. We do this by demonstrating, for some arbitrary vertex v, that $|N(v)| \geq \sqrt{n}$, and $|N(N(v)) - \{v\}| \geq n - \sqrt{n}$. We show that these two sets are disjoint, thus implying that G has $n+1$ vertices, completing the contradiction.

Assume seeking a contradiction that there exists a graph G with $n \geq 3$ vertices, satisfying the above criteria except that it has no 3-cycle or 4-cycle. Pick an arbitrary vertex $v \in V$.

Claim 1: $|N(v)| \geq \sqrt{n}$. By our assumption, every vertex in G has at least degree $\lceil \sqrt{n} \rceil$. This means that $|N(v)| \geq \lceil \sqrt{n} \rceil \geq \sqrt{n}$.

Claim 2: $N(N(v)) - \{v\}$ and $N(v)$ are disjoint. Assume seeking a contradiction that there exists some vertex u in $N(v)$ and $N(N(v)) - \{v\}$. Because $u \in N(v)$, v is adjacent to u. Because $u \in N(N(v)) - \{v\}$, u is adjacent to some other neighbor of v $u' \in N(v)$. However the sub-graph with vertices $\{v, u, u'\}$ forms a 3-cycle, contradicting our assumption, therefore $N(N(v)) - \{v\}$ and $N(v)$ are disjoint.

Claim 3: $|N(N(v)) - \{v\}| \geq n - \sqrt{n}$. Since each vertex in G has degree at least $\lceil \sqrt{n} \rceil$, every neighbor of v has at least this degree. Therefore every neighbor of v connects to at least $\lceil \sqrt{n} \rceil - 1$ other vertices besides v. Now assume seeking a contradiction that two distinct vertices $u_1, u_2 \in N(v)$ share an adjacent vertex other than v; call this vertex u_3. Then the sub-graph containing v, u_1, u_2, and u_3 forms a 4-cycle in G, thus all neighbors of v have distinct neighbors other than v. Therefore $|N(N(v)) - \{v\}| \geq \lceil \sqrt{n} \rceil \big(\lceil \sqrt{n} \rceil - 1 \big) \geq \sqrt{n}(\sqrt{n} - 1) = n - \sqrt{n}$.

By definition we know $\{v\}$ shares no elements with $N(v)$ or $N(N(v)) - \{v\}$, so $\{v\}$, $N(v)$, and $N(N(v)) - \{v\}$ are all disjoint. Note that $\{v\} \cup N(v) \cup (N(N(v)) - \{v\}) \subseteq V$, but

$$\begin{aligned}
&|\{v\} \cup N(v) \cup (N(N(v)) - \{v\})| \\
&\quad = |\{v\}| + |N(v)| + |N(N(v)) - \{v\}| && \text{(claim 2)} \\
&\quad \geq 1 + \sqrt{n} + n - \sqrt{n} && \text{(claims 1 and 3)} \\
&\quad = n + 1 \\
&\quad > n \\
&\quad = |V|.
\end{aligned}$$

A subset of V cannot contain more vertices than V, so we have reached a contradiction, and thus G must contain a 3-cycle or 4-cycle.

Alternate Solution (Direct Proof):

Let V be the set of all vertices in the graph. Since $|V| = n \geq 3$, there must exist some $v \in V$. We consider the neighbors of the vertex $N(v)$. Furthermore we define $N_v = N(v) \cup \{v\}$, and $\overline{N_v} = V - N_v$.

Case 1: there are two adjacent vertices $u, w \in N(v)$. So, the subgraph with just the vertices $\{u, v, w\}$ is a 3-cycle subgraph.

Case 2: no two vertices in $N(v)$ are adjacent. Each neighbor of v must have at least $\lceil\sqrt{n}\rceil$ incident edges, so excluding the edge connecting them to v, there are at least $\lceil\sqrt{n}\rceil - 1$ incident edges to each neighbor that connect to vertices not in N_v. So in total, there are at least

$$|N(v)| \cdot (\lceil\sqrt{n}\rceil - 1) \geq \sqrt{n} \cdot (\sqrt{n} - 1) = n - \sqrt{n}$$

edges connecting to other vertices. No two neighbors are adjacent, so all such edges connect to vertices in $\overline{N_v}$. Since

$$|\overline{N_v}| = n - (|N(v)| + 1) \leq n - \sqrt{n} - 1,$$

we know that $|\overline{N_v}| \leq |N(v)| \cdot (\lceil\sqrt{n}\rceil - 1)$. We have essentially shown that there are more edges connected to the vertices in N_v than the number of vertices not in that set. We now show that this implies there exists a 4-cycle.

Conclusion via averages: The average number of edges incident to elements of $N(v)$ for vertices in $\overline{N_v}$ is at least

$$\frac{|N(v)| \cdot (\lceil\sqrt{n}\rceil - 1)}{|\overline{N_v}|} \geq 1.$$

At least one vertex $v_1 \in \overline{N_v}$ must be equal to or above average. So, there are at least two distinct vertices $u_1, u_2 \in N(v)$ adjacent to v_1. Thus, the subgraph with just the vertices $\{u_1, v, v_1, u_2\}$ is a 4-cycle.

Conclusion via Pigeonhole Principle: By the Pigeonhole Principle, there must be a vertex $v_1 \in \overline{N_v}$ with an edge to two distinct vertices $u_1, u_2 \in N(v)$. Thus, the subgraph with just the vertices $\{u_1, v, v_1, u_2\}$ is a 4-cycle.

In either case the graph has a 3-cycle or 4-cycle subgraph.

Grading Guidelines [15 points]

Proof by contradiction:

(i) +3 correct assumption for proof by contradiction
(ii) +4 considers some vertex and counts its neighbors
(iii) +2 shows that neighbors of neighbors must be disjoint from neighbors because of 3-cycles
(iv) +3 counts the neighbors of neighbors using the above fact
(v) +3 finds a contradiction by adding up distinct vertices

Direct proof:

(i) +3 considers some vertex and its neighbors
(ii) +4 considers neighbors of neighbors (including splitting into cases based on neighbors being adjacent to each other)
(iii) +2 correctly finds a 3-cycle subgraph in the adjacent neighbors case
(iv) +3 makes an argument based around degree, number of edges, or number of non-neighbor vertices in the non-adjacent neighbors case
(v) +3 correctly finds a 4-cycle subgraph in the non-adjacent neighbors case, using either averages or the Pigeonhole Principle

3. The Office Allocation [15 points]

Consider a new office building with n floors and k offices per floor in which you must assign $2nk$ people to work, each sharing an office with exactly one other person. Find a closed form solution for the number of ways there are to assign offices if from floor to floor the offices are distinguishable, but any two offices on a given floor are not.

Solution:

First, assign everyone a number $1, \ldots, 2kn$; there are $(2kn)!$ ways to do this. Then for odd i assign $i+1$ and i to be office partners. There is over counting by a factor of $2^{kn}(kn)!$: there are $(kn)!$ ways to order the kn pairs and then there is a factor of 2^{kn} to account for swapping i and $i+1$ in each pair. So there are

$$\frac{(2kn)!}{2^{kn}(kn)!}$$

ways to form pairs.

Now since offices on any given floor are indistinguishable, we need only assign the pairs to floors. Assign each pair a unique number $1, \ldots, kn$; there are $(kn)!$ ways to do this. Then numbers 1 through k go on floor 1, numbers k through $2k$ go on floor 2, etc. Using this construction, we over-count the number of possibilities for each floor by a factor of $k!$, so since there are n floors we over-count by $(k!)^n$ in total. So by the product rule there are

$$\frac{(2kn)!}{2^{kn}(kn)!} \cdot \frac{(kn)!}{(k!)^n} = \frac{(2kn)!}{2^{kn}(k!)^n}$$

ways to assign people to offices.

Grading Guidelines [15 points]

(i) +3 solution contains the factor $(2kn)!$
(ii) +3 divides by 2^{kn} to account for swapping the office mates

(iii) +2 divides by $(kn)!$ to account for order not mattering when selecting office mates
(iv) +3 computes factor of $(kn)!$ to select floors for the pairs
(v) +4 divides by $(k!)^n$ to account for indistinguishable offices