EECS 203: Discrete Mathematics
Fall 2023
Homework 8

Due **Thursday, November 2**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total points: $100 + 42$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Why You Got a 12-Car Garage? [8 points]

Ashu recently acquired three 12-car garages, but he has no cars (yet).

(a) What is the minimum number of cars Ashu has to acquire in order to guarantee that at least one of the garages will have **more** than 6 cars in it? Justify your answer, including an explanation of why it is the minimum number.

(b) If the garages are all adjacent to one another, what is the minimum number of cars Ashu has to acquire in order to guarantee that the middle garage has more than 6 cars in it?

Solution:

(a) Pigeons: Cars, quantity $= n$
Holes: Garages, quantity $= 3$
According to Pigeonhole Principle, we want at least 6 cars in every hole, which means that $\lceil \frac{n}{3} \rceil > 6$.
$\therefore n > 18$.
$\therefore$ the minimum number of cars is 19.

(b) Consider the worst condition: the left and right garages are all full, while the middle garage only has 6 cars. They add up to $12 \times 2 + 6 = 30$ cars.
Now if we add another car, then it can only go to the middle, so there must be 7 cars after that.
If any one of the left and right garage is not full, then that new car can go to the left or right garage. That makes it possible to keep the middle garage with 6 cars.
$\therefore$ if and only if there are more than 30 cars, it is guaranteed that the middle garage has more than 6 cars in it.
$\therefore$ the minimum number of cars is 31.

2. Sum More Counting [14 points]

Consider the set of integers between 1 and 18, inclusive. What is the smallest integer n such that, for any subset $S \subseteq \{1, 2, \ldots, 18\}$ of size $|S| = n$, there are **distinct** integers $x, y \in S$ with $x + y = 18$? Prove that your answer is sufficient to guarantee this, and the minimum necessary number.

Note: To prove that your choice of n is smallest, you must also give an example of a set of size $|S| = n - 1$ that does not contain $x, y \in S$ with $x + y = 18$.

Solution:

The minimum necessary number n is 11.
Consider the partition of the set, shown as elements of a new set:
$\{(1,17),(2,16),(3,15),(4,14),(5,13),(6,12),(7,11),(8,10),9,18\}$
Every element either can form a sum of 18 in its pair or cannot form a sum of 18 with any element (which is 9 and 18).
$\therefore$ There is 10 Holes: partitions of the set, or elements of the new set.
Assume the minimum necessary number is n.
Then according to Pigeonhole Principle, $\lceil \frac{n}{10} \rceil > 1$. (We need at least one full combinition,)
$\therefore n > 10$, the minimum necessary number is 11.

Now we show the choice of 11 is smallest:
Consider the subset $\{9,10,11,12,13,14,15,16,17,18\}$ which has 10 elements, and do not have 2 elements that can sum up to 18

3. Set Sizes [12 points]

Determine which of these sets are finite, countably infinite, or uncountably infinite. Give a short (about 1 line) explanation for each part.

(a) $\{2,3\} \times \mathbb{N}$

(b) $(0,2) - \mathbb{Q}$

(c) $\{x \in \mathbb{R} \mid x^2 - 1 \leq 0\}$

(d) $\{x \in \mathbb{N} \mid x \leq 1000\}$

Solution:

(a) Countably infinite. For any $x \in \mathbb{N}$, We can map $(2,x)$ to an odd positive integer and $(3,x)$ to an even positive integer, alternately. By doing this we can find a 1-to-1 map from $\{2,3\} \times \mathbb{N}$ to $\mathbb{Z}^+$. Then we know $|\{2,3\} \times \mathbb{N}| \leq |\mathbb{Z}^+|$.

(b) Uncountably infinite. Even though there is a set difference from $\mathbb{Q}$, we can still use the table of digit to prove that by diagonalization, that is, to show there does not exist an onto function from $\mathbb{Z}^+$ to $(0,2) - \mathbb{Q}$. Then $|\mathbb{Z}^+| < |(0,2) - \mathbb{Q}|$.

(c) Uncountably infinite. $\{x \in \mathbb{R} \mid x^2 - 1 \leq 0\} = [-1,1]$. We can use the same diagonalization method as in (b) to prove that.

(d) Finite. $|\{x \in \mathbb{N} \mid x \leq 1000\}| = 1001$. It is just integers from 0 to 1000 inclusively.

4. Ready, set, count! [15 points]

Definition: $A \oplus B$ is the symmetric difference of the sets A and B, i.e. the set containing all elements which are in A or in B but not in both.

Provide two **uncountable** sets A and B such that $A \oplus B$ is

(a) finite.

(b) countably infinite.

(c) uncountably infinite.

Include in your justification a desciption of the set $A \oplus B$ without reference to the symmetric difference.

Solution:

(a) $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} = (0, 1]$.
$B = \{x \in \mathbb{R} \mid 0 \leq x < 1\} = [0, 1)$.
$A \oplus B = \{0, 1\}$. It is finite, with only two elements 0 and 1.

(b) $A = \mathbb{R} - \mathbb{Q}$.
$B = \mathbb{R}$.
These two sets are uncountable, which can be proved by table of digit(diagonalization).
Then $A - B = \mathbb{Q}$, which is countably infinite. This can be proved in the same way we prove $|\mathbb{Z}^+ \times \mathbb{Z}^+|$ is the same size as $|\mathbb{Z}^+|$.

(c) $A = \{x \in \mathbb{R} \mid 0 < x < 1\} = (0, 1)$.
$B = \{x \in \mathbb{R} \mid 1 \leq x < 2\} = [1, 2)$.
Then $A + B = \{x \in \mathbb{R} \mid 0 < x < 2\} = (0, 2)$.
We know that they are uncountably infinite in the same logic(diagonalization).

5. Corresponding Counts [18 points]

Prove that $|[0, 2]| = |(3, 6)|$.

For any functions that you name:

- Prove that the function is well-defined, i.e. that for any x in the domain of your function f, $f(x)$ lies in the codomain.

- Prove any function properties that you use (e.g. one-to-one, onto, etc).

Solution:

(a) Proof of $|[0,2]| \leq |(3,6)|$:
We prove it by proving that $\exists$ a one-to-one function $f : [0,2] \to (3,6)$.
Consider $f(x) = \frac{x}{2} + 4$, $x \in [0,2]$.
For this function:

- Proof: It is well defined.
It is to prove that: (1) $\forall x \in [0,2], \exists y = f(x)$,where $3 \leq y \leq 6$.
(2) $\forall y_1, y_2 \in (3,6)$, if $[y_1 \neq y_2] \wedge [y_1 = f(x_1)] \wedge [y_2 = f(x_2)]$, then $x_1 \neq x_2$ (One x cannot map to multiple y).

  Proof of (1):
  Let a be an arbitrary real numbers in $[0,2]$.
  Since $a \geq 0$, $f(a) = \frac{a}{2} + 4 \geq 4 \geq 3$.
  Since $a \leq 2$, $f(a) = \frac{a}{2} + 4 \leq 5 \leq 6$.
  $\therefore \forall x \in [0,2], \exists y = f(x)$, where $3 \leq y \leq 6$.

  Proof of (2):
  Let b_1, b_2 be arbitrary real numbers in $(3,6)$.
  Assume $[b_1 \neq b_2] \wedge [b_1 = f(a_1)] \wedge [b_2 = f(a_2)]$.
  Then $f(a_1) \neq f(a_2)$, $\frac{a_1}{2} + 4 \neq \frac{a_2}{2} + 4$, $\frac{a_1}{2} \neq \frac{a_2}{2}$, $a_1 \neq a_2$.
  $\therefore \forall y_1, y_2 \in (3,6)$, if $[y_1 \neq y_2] \wedge [y_1 = f(x_1)] \wedge [y_2 = f(x_2)]$, then $x_1 \neq x_2$. $\therefore$ $f(x)$ is well-defined.

- Proof: It is one-to-one.
It is to prove that $\forall a_1, a_2 \in [0,2]$, $[f(a_1) = f(a_2)] \to (a_1 = a_2)$.
Let a_1, a_2 be arbitrary real numbers in $[0,2]$.
Assume $f(a_1) = f(a_2)$.
Then $\frac{a_1}{2} + 4 = \frac{a_2}{2} + 4$, $\frac{a_1}{2} = \frac{a_2}{2}$, $a_1 = a_2$.
$\therefore f$ is one-to-one.
$\therefore |[0,2]| \leq |(3,6)|$.

(b) Proof of $|(3,6)| \leq |[0,2]|$:
We prove it by proving that $\exists$ a one-to-one function $f : (3,6) \to [0,2]$.
Consider $g(x) = \frac{x-3}{3}$, $x \in (3,6)$.
For this function:

- Proof: It is well defined.
It is to prove that: (1) $\forall x \in (3,6), \exists y = g(x)$,where $0 \leq y \leq 2$.
(2) $\forall y_1, y_2 \in [0,2]$, if $[y_1 \neq y_2] \wedge [y_1 = g(x_1)] \wedge [y_2 = g(x_2)]$, then $x_1 \neq x_2$ (One x cannot map to multiple y).

Proof of (1):
Let a be an arbitrary real numbers in $(3, 6)$.
Since $a > 3$, $g(a) = \frac{a-3}{3} > 0$, then≥ 0 ($> \rightarrow [> \vee =]$).
Since $a < 6$, $f(a) = \frac{a-3}{3} < 1 \leq 2$.
$\forall x \in (3, 6), \exists y = g(x)$,where $0 \leq y \leq 2$.

Proof of (2):
Let b_1, b_2 be arbitrary real numbers in $[0, 2]$.
Assume $[b_1 \neq b_2] \wedge [b_1 = g(a_1)] \wedge [b_2 = g(a_2)]$.
Then $g(a_1) \neq g(a_2)$, $\frac{a_1-3}{3} \neq \frac{a_2-3}{3}$, $a_1 - 3 \neq a_2 - 3$, $a_1 \neq a_2$.
$\therefore$ $\forall y_1, y_2 \in (3, 6)$, if $[y_1 \neq y_2] \wedge [y_1 = g(x_1)] \wedge [y_2 = g(x_2)]$, then $x_1 \neq x_2$. $\therefore$ $g(x)$ is well-defined.

- Proof: It is one-to-one.
  It is to prove that $\forall a_1, a_2 \in (3, 6)$, $[gf(a_1) = g(a_2)] \rightarrow (a_1 = a_2)$.
  Let a_1, a_2 be arbitrary real numbers in $(3, 6)$.
  Assume $g(a_1) = g(a_2)$.
  Then $\frac{a_1-3}{3} = \frac{a_2-3}{3}$, $a_1 - 3 = a_2 - 3$, $a_1 = a_2$.
  $\therefore$ g is one-to-one.
  $\therefore$ $|(3, 6)| \leq |[0, 2]|$.

(c) Due to Schroeder-Bernstein Theorem, since $|[0, 2]| \leq |(3, 6)|$ and $|(3, 6)| \leq |[0, 2]|$, $|(3, 6)| = |[0, 2]|$.

6. Composition Pr∘∘f [15 points]

Consider functions $g \colon A \rightarrow B$ and $f \colon B \rightarrow C$. **Prove or disprove** that if f and $f \circ g$ are one-to-one, then g is one-to-one.

Solution:

Proof of if f and $f \circ g$ are one-to-one, then g is one-to-one:

Assume f and $f \circ g$ are one-to-one.
According to Definition we know:
(1): $\forall y_1, y_2 \in B$, $[f(y_1) = f(y_2)] \leftrightarrow [y_1 = y_2]$.
(2): $\forall x_1, x_2 \in A$, $[f(g(x_1)) = f(g(x_2))] \leftrightarrow [x_1 = x_2]$.
We want to show that: $\forall x_1, x_2 \in A, [g(x_1) = g(x_2)] \rightarrow [x_1 = x_2]$.

Let a_1, a_2 be arbitrary real numbers in A. $b_1 = g(a_1)$, $b_2 = g(a_2)$.

Assume $b_1 = b_2$, that is, $g(a_1) = g(a_2) \in B$.
Then $f(b_1) = f(b_2)$.
Then $f(g(a_1)) = f(g(a_2))$.
Then from (2) we know: $a_1 = a_2$.
$\therefore$ We have proved that if f and $f \circ g$ are one-to-one, then g is one-to-one.

7. One Hit Wonder [18 points]

For this problem, we will define two new properties. Let S be a set and $f\colon S \to S$ be some function.

We say f is a *one hit wonder* if:

$$\forall x \in S\,[(f \circ f)(x) = f(x)].$$

Some examples of one-hit wonders from $\mathbb{R} \to \mathbb{R}$ are the absolute value function, the ceiling function, and the function which sends every number to 0.

We say f *does nothing* if:

$$\forall x \in S\,[f(x) = x].$$

(a) Prove that if f does nothing, then it is a one-hit wonder.

(b) Prove that if f is a one hit wonder and is one-to-one, then f does nothing.

(c) Prove that if f is a one hit wonder and is onto, then f does nothing.

Solution:

(a) Let a be an arbitrary element of S.
Assume f does nothing, then $f(a) = a \in S$.
Since $f(a) \in S$, then $\exists f(f(a))$, and $f(f(a)) = f(a) = a$.
Then $f \circ f(a) = f(a)$.
$\therefore$ We have proved that if f does nothing, then it is a one-hit wonder.

(b) Let a be an arbitrary element of S.
Assume f is a one hit wonder and is one-to-one.
Since f is a one hit wonder, $[(f \circ f)(a) = f(a)]$, $f(f(a)) = f(a)$.
Since f is one-to-one,$[f(f(a)) = f(a)] \to f(a) = a$.
$\therefore$ for any $x \in S$, $f(x) = x$, i.e. f does nothing.
$\therefore$ If f is a one hit wonder and is one-to-one, then f does nothing.

(c) Let b be an arbitrary element of S.
Assume f is a one hit wonder and is onto.
Since f is onto, $\exists a \in S, f(a) = b$.
Since f is a one hit wonder, $f(f(a)) = f(a) = b$.
Since $f(a) = b$ and $f(f(a)) = b$, $f(b) = b$.
$\therefore$ for any $x \in S$, $f(x) = x$, i.e. f does nothing.
$\therefore$ If f is a one hit wonder and is one-to-one, then f does nothing.

Groupwork

1. Grade Groupwork 7

Using the solutions and Grading Guidelines, grade your Groupwork 7:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/16
Problem 3												/14
Total:												/30

Previous Groupwork 7(1): Raise the Roof [16 points]

Let $f\colon \mathbb{R} \to \mathbb{Z}$, where f is defined as $f(x) = \left\lceil \frac{x+5}{2} \right\rceil + 12$. Is f onto? Prove your answer.

Solution:

f is onto.
i.e. $\forall b \in \mathbb{Z}$, $\exists a \in \mathbb{R}$ such that $b = f(a)$. Proof:
Let b be an arbitrary integer.
Consider $a = 2b - 29$.
Then a is also an integer, $a \in \mathbb{Z} \subseteq \mathbb{R}$, is in the domain.

Then

$$\begin{aligned} f(a) &= \lceil \frac{2b - 29 + 5}{2} \rceil + 12 \\ &= \lceil \frac{2b - 24}{2} \rceil + 12 \\ &= \lceil b - 12 \rceil + 12 \end{aligned}$$

Since b is an integer, $\lceil b - 12 \rceil = b - 12$.
$\therefore f(a) = b - 12 + 12 = b$.
$\therefore \forall b \in \mathbb{Z}, \exists a \in \mathbb{R}$ such that $b = f(a)$.
$\therefore f$ is onto.

Previous Groupwork 7(2): You Mod Bro? [14 points]

Find all solutions of the congruence $12x^2 + 25x \equiv 10 \pmod{11}$.

Solution:

$12x^2 + 25x \equiv 10 \pmod{11}$
$(11 + 1)x^2 + (2 \cdot 11 + 3)x \equiv 10 \pmod{11}$
$x^2 + 3x - 10 \equiv 0 \pmod{11}$
$(x + 5)(x - 2) \equiv 0 \pmod{11}$
$\therefore x \equiv 2 \pmod{11}$ or $x \equiv -5 \pmod{11} \equiv 6 \pmod{11}$.
$\therefore$ The solutions are: all integer x satisfying $x \equiv 2 \pmod{11}$ or $x \equiv 6 \pmod{11}$.

2. Divisibility by Seven [12 points]

In this question we will show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7. You are allowed to cross off zero digits.

For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number $2345 = 7 \cdot 335$.

We will label the digits of an arbitrary 7-digit number as

$$x_6 x_5 x_4 x_3 x_2 x_1 x_0.$$

(a) Prove that there exists some $i < 7$ such that either $x_i x_{i-1} \dots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_j x_{j-1} \dots x_0$ is congruent to it modulo 7.

(b) Use part (a) to prove that if there does not exist some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then there exists $7 > i > j \geq 0$ so that

$$\underbrace{x_i x_{i-1} \dots x_{j+1} 0 \dots 0}_{i+1 \text{ digits total}}$$

is divisible by 7.

(c) Prove the full claim. That is, show that, given a 7-digit number, where all digits except perhaps the last are non-zero, you can cross out some digits at the beginning and at the end such that the remaining number consists of at least one digit and is divisible by 7.

Solution:

(a) There are 7 numbers: $x_6x_5x_4x_3x_2x_1x_0$, $x_5x_4x_3x_2x_1x_0$, $\cdots$, x_0.
Use them as Pigeons.
The remainer of any number mod 7, if the number cannot be divided by 7, has 6 circumstances: 1,2,3,4,5,6.
Use them as Holes.
Assume that all these 7 numbers is not divisible by 7, then there are 7 Pigeons in 6 holes, which means that there are at least 2 Pigeons in one whole.
This means that $i < 7$ and $j < i$ such that $= x_i x_{i-1} \dots x_0$ and $x_j x_{j-1} \dots x_0$ have the same remainer when divided by 7.
Mark the two numbers as a and b.
Then $b \equiv a \pmod 7$.
$(b - a) \equiv 0 \pmod 7$.
Then we have proved the statement: There exists some $i < 7$ such that either $x_i x_{i-1} \dots x_0$ is divisible by 7, or, if it isn't, then there exists some $j < i$ such that $x_j x_{j-1} \dots x_0$ is congruent to it modulo 7.

(b) If there does not exist some $i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7, then from (a) who know: there exists some $j < i$ such that $x_j x_{j-1} \dots x_0$ is congruent to it.
i.e. $(x_i x_{i-1} \dots x_0 - x_j x_{j-1} \dots x_0) \equiv 0 \pmod 7$.
that is, $x_i x_{i-1} \cdots x_{j+1} 0 \cdots 0 \equiv 0 \pmod 7$.
$\therefore$ We have proved the statement.

(c) Given a 7-digit number $x_6x_5x_4x_3x_2x_1x_0$,
Case 1: $\exists i < 7$ such that $x_i x_{i-1} \dots x_0$ is divisible by 7.

Then if $i = 6$, the whole digit is divisible by 7, else we can just cross out all digits from x_6 to x_{i+1}. The remaining digit $x_i x_{i-1} \ldots x_0$ is divisible by 7.

Case 2: $\nexists i < 7$ such that $x_i x_{i-1} \ldots x_0$ is divisible by 7.
Then through (b) we know: there exists some $j < i$ such that $x_i x_{i-1} \ldots x_{j+1} 0 \ldots 0$ is divisible by 7.
We know that $x_i x_{i-1} \ldots x_{j+1} 0 \ldots 0 = x_i x_{i-1} \ldots x_{j+1} \times 100 \cdots$
Then $x_i x_{i-1} \ldots x_{j+1} \times 100 \cdots \equiv 0 \pmod 7$.
Since $100 \cdots \not\equiv 0 \pmod 7$
There must be $x_i x_{i-1} \ldots x_{j+1} \equiv 0 \pmod 7$.
So we can rule out all digits before x_i and after x_j to get a number divisible by 7.

3. A Powerful Proof [30 points]

In this question we will prove that for any set X, $|\mathcal{P}(X)| > |X|$ ($\mathcal{P}(X)$ is the power set of X). Note that while this is simple in the case where X is finite, things get more complicated when we allow X to be infinite. This proof covers all cases.

(a) Show that for all (possibly infinite) sets X, $|\mathcal{P}(X)| \geq |X|$.

(b) Let $g \colon X \to P(X)$ be an arbitrary function. Show that the set $D := \{a \in X \mid a \notin g(a)\}$ is not in the range of g.

(c) Explain why this shows that $|P(X)| \leq |X|$ is false and conclude the proof.

(d) Based on your conclusions above, are there uncountable sets "larger" than $\mathbb{R}$? Explain.

Solution:

(a) We define $f : X \to \mathcal{P}(X)$:
The map of every element in X to a set in $\mathcal{P}(X)$ that only contains that element.
That is: for $X = \{a, b, c, \cdots\}$, $f(a) = \{a\}$, $f(b) = \{b\}$, $\cdots$.
Since for every element in X, we can find a subset of X that only contains that element, and every subset is unique, we know this is a one-to-one function.
$\therefore |X| \leq |\mathcal{P}(X)|$.

(b) The range of g is $\{g(a) | a \in X\}$.
D is a subset of X which contains all the elements that are not an element of its image. We know $D \in \mathcal{P}(X)$.
Assume $D \in range(X)$, then $\exists x \in X$ such that $g(x) = D$, which is an element of

$\mathcal{P}$.
If $x \in D$, then due to the definition of D, $x \notin g(x) = D$.
This causes contradiction.
$\therefore$ D is not in the range of g.

(c) Since every element in D is also in X, it is a subset of X, so D is an element of $\mathcal{P}(X)$, which is in the codomain of g.
However, D is not in the range of g.
$\therefore$ the $range(g) < codom(g)$, there does not exist an onto function g from X to $\mathcal{P}(X)$.
$\therefore$ $|\mathcal{P}(X)| \leq |X|$ is false, $|\mathcal{P}(X)| > |X|$.

(d) From c, we can know that $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. This is a "larger" set than $\mathbb{R}$.