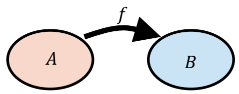
Technical verbs: ① mapping, ② function, ③ domain, ④ codomain, ⑤ range, ⑥ onto, ⑦ one-to-one, ⑧ bijection, ⑨ inverse function, ⑩ function composition

10/19/23

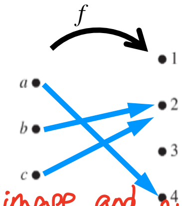
Lec 15: Functions & Properties



② ① mapping

- A **function** $f : A \to B$ is a machine where you input an element from a given set (the "③ domain 定义域") and it outputs an element from another given set (the "④ codomain 陪域").
- **Key Requirements** (to be a function):
  - For every input, the function produces exactly one output
  - A mapping not necessarily satisfying these requirements is a **relation**.

原像 map 像
preimage ⟶ image



Domain = $\{a, b, c\}$

Codomain = $\{1, 2, 3, 4\}$

Range = $\{2, 4\}$

值域
⑤ range: the set of elements in the codomain that do get mapped at least once.

* image and preimage! 像

given $f: A \to B$, for $x \in A$, the image of x is value of $f(x)$. (** only one);
for $y \in B$, the preimage 原像 of y is a value of ☆☆ x where $f(x) = y$. (** may not exist or have ≥ 2 !!)

eg: $f: \underset{\text{domain}}{\mathbb{Z}} \to \underset{\text{codomain}}{\mathbb{N}}$, $f(x) = x^2 + 2$
range: $[2, +\infty)$

Also, image and preimage can be for a set:
eg: $f(\{a, b, c\}) = \{f(a), f(b), f(c)\} = \{2, 1, 2\} = \{1, 2\}$
then $\{1, 2\}$ is the image of $\{a, b, c\}$, and $\{a, b, c\}$ is the preimage of $\{1, 2\}$.

Image of domain given f is its range;
Preimage of codomain and range is domain!

Examples (or not!) of Functions

1. Is there a function getChild, with domain/codomain both the set of all people, and
   getChild(x) = y iff y is the child of x?
   A. Yes
   B. No
   ① Some x has no child
   ② Some x has ≥2 children
2. Is there a function getMother, with domain/codomain both the set of all people, and
   getMother(x) = y iff y is the mother of x?
   A. Yes
   B. No
   Not everyone has exactly one mother.
3. Is there a function f: ℝ → ℝ with
   $f(x) = y$ iff $x = y^2$?
   A. Yes
   B. No
   ① Some $x (<0)$ has no square root
   ② Some $x (>0)$ has 2 square roots.
4. Is there a function f: ℝ → ℝ with
   f(x) = y iff y = 1/x²
   A. Yes
   B. No
   $x = 0$ has no mapping

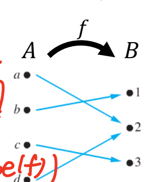
Onto, One-to-one, Bijections, Inverse Functions

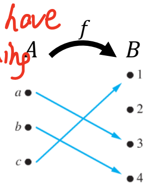
Given a function $f: A \to B$

- ⑥ f is **onto** iff (or surjection 满射) all codomain points ☆ have at least one incoming arrow. (i.e. codomain(f) = range(f)) (陪域 = 值域)
- ⑦ f is **one-to-one** iff (or injection 单射) all codomain points have at most one ☆ incoming arrow. (无多x对应同y)

Given $f: A \to B$, $\neg\exists a_1, a_2 \in A, (a_1 \neq a_2) \wedge (f(a_1) = f(a_2))$
$\equiv$
$\forall a_1, a_2 \in A, [(f(a_1) = f(a_2)) \to (a_1 = a_2)]$

- ⑧ f is a **bijection** iff onto $\wedge$ 1→1
- ⑨ f has an inverse f^{-1} iff f is bijection ☆
  - $f^{-1}(b) = a \quad \leftrightarrow \quad f(a) = b$

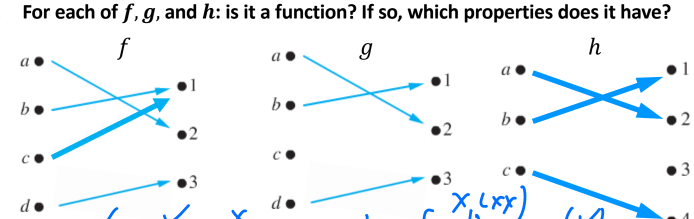




Special Properties:
(1) Bijections have inverse
(2) For bijection $A \to B$, $|A| = |B|$

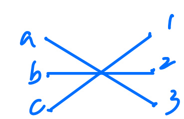
Exercise: One-to-one and Onto

1. **For each of f, g, and h: is it a function? If so, which properties does it have?**

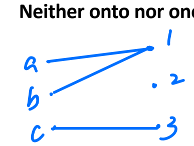


f: function ✓; onto ✓; ¬1→1 ✗
g: not a function ✗ (✗✗)
h: function (✓); ¬onto (✗); 1→1 (✓)

2. **Draw a function that is**
   a) **Onto and one-to-one**



   b) **Neither onto nor one-to-one**



- The properties of **onto**, **one-to-one**, and **invertibility** are important in:
  - Counting (later this term)
  - Hashing, Cryptography, Error-correcting codes, Computational Geometry, …

Proof: $f(x)$ is one-to-one

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$.
Prove: f is one-to-one.

Goal: Prove logical expression

$\forall a_1, a_2 \in \mathbb{R}, [f(a_1) = f(a_2)] \rightarrow (a_1 = a_2)$

Let a_1, a_2 be arbitrary real numbers
Assume $f(a_1) = f(a_2)$

$2a_1 + 4 = 2a_2 + 4$
$2a_1 = 2a_2$
$a_1 = a_2$

So $a_1 = a_2$
Thus f is one-to-one

Proof: $f(x)$ is onto

Let $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2x + 4$.
Prove: f is onto.

Goal: Prove expression

$\forall b \in \mathbb{R}, \exists a \in \mathbb{R}$ such that $f(a) = b$

Let b be an arbitrary real number.
Consider $a = \frac{b}{2} - 2$
Note that a is a real number, and thus in the domain of f.
So $2a = b - 4$
$2a + 4 = b$
So $f(a) = b$

Side work:
$f(a) = b$
$2a + 4 = b$
$2a = b - 4$
$a = \frac{b}{2} - 2$

(10) Composition of functions 复合函数
Def: $(g \circ f)(x) = g(f(x))$. We can only compose $g \circ f$ when $\text{codom}(f) \subseteq \text{dom}(g)$

Prove or Disprove: $f(x)$ is onto

Let $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3 + 1$.
Prove or Disprove (circle one): f is onto.

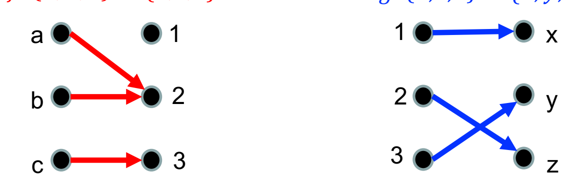
We will try to prove the expression:

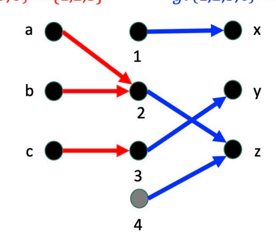
$\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, we have $f(x) \neq y$

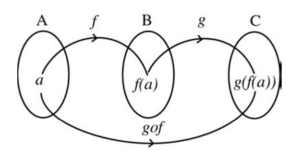
Consider $b = 3$
Let a be an arbitrary integer
Seeking contradiction, assume that $f(a) = b$
So $a^3 + 1 = 3$
$a^3 = 2$
$a = 2^{\frac{1}{3}} \notin \mathbb{Z}$
This contradicts that $a \in \mathbb{Z}$.
So $f(a) \neq b$
In other words that only a that solves $f(a) = 3$ is not in the domain of f.
Therefore f is not onto.

Inverses and Composition

$f: \{a, b, c\} \to \{1,2,3\}$ $\qquad$ $g: \{1,2,3\} \to \{x, y, z\}$







Which of these exist?

1. f^{-1} ✗ f not a bijection
2. g^{-1} ✓ g is a bijection
3. $f \circ g$ ✗ $\text{codom}(g) \nsubseteq \text{dom}(f)$
4. $g \circ f$ ✓ $\text{codom}(f) \subseteq \text{dom}(g)$

Caution:
Order matters! $(f \circ g)(x)$ is not the same as $(g \circ f)(x)$
Here, $(g \circ f)(x) = g(2x + 3) = 3(2x + 3) + 1 = 6x + 7$

- First 1/3 of the course: **Logic & Proofs**
- Second 1/3: **Discrete Structures (Sets, Functions, Graphs)**
- Last 1/3: **Counting & Probability**