# EECS 203: Discrete Mathematics
# Fall 2023
# Discussion 2 Notes

## 1 Definitions

- **Logical Equivalence**:
- **DeMorgan's Laws**:
- **Contrapositive**:
- **Implication Breakout**:
- **Identity Laws:**
- **Distributive Laws:**
- **Commutative Laws:**
- **Associative Laws:**
- **Tautology**:
- **Contradiction**:
- **Satisfiable:**
- **Predicate**:
- **Quantifiers:**
  - **Universal quantifier:**
  - **Existential quantifier:**
- **Nested Quantifiers:**

**Solution:**

- **Logical Equivalence**: When two compound propositions have the same truth values under any inputs (i.e., they have the same truth table).

- **DeMorgan's Laws**:
  - **Negating And/Or Statements:**
    $\neg(p \wedge q) \equiv \neg p \vee \neg q$
    $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - **Negating Quantifiers:**
    $\neg\forall\text{x}P(x) \equiv \exists\text{x}\neg P(x)$
    $\neg\exists\text{x}P(x) \equiv \forall\text{x}\neg P(x)$

- **Contrapositive:** When given an "if...then" statement, the contrapositive of the statement can be found by switching the hypothesis and the conclusion, and negating both terms. So the contrapositive of $(p \to q)$ is $(\neg q \to \neg p)$. An implies statement and its contrapositive are logically equivalent.
  - **Example:** The statement "If I am teaching, then my materials are prepared" is logically equivalent to the contrapositive statement: "If my materials are not prepared, then I am not teaching."

  $p \to q \equiv \neg p \to \neg q$

- **Implication Breakout**:
  $p \to q \equiv \neg p \vee q$

- **Identity Laws:**
  $p \wedge T \equiv p$
  $p \vee F \equiv p$

- **Distributive Laws:** (You can distribute in from either side. Distributing in from the left is shown below.)
  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- **Commutative Laws:**
  $p \wedge q \equiv q \wedge p$
  $p \vee q \equiv q \vee p$

- **Associative Laws:**
  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
  $p \vee (q \vee r) \equiv (p \vee q) \vee r$

- **Tautology:** A compound proposition that is always true, regardless of its input values

- **Example:** $p \vee \neg p$
  $p$'s value can either be true or false. If $p$ is true, then the statement must be true. Alternatively, if $p$ is false, then $\neg p$ is true, showing that the proposition is true for all possible values of $p$.

- **Contradiction:** A compound proposition that is always false, regardless of its input values

  - **Example:** $p \wedge \neg p$
    $p$ and $\neg p$ can never both be true, as $\neg p$ is a negation of $p$. Because of this, the proposition will always be false.

- **Satisfiable:** A compound proposition that can take on a value of true. All tautologies are satisfiable, but a satisfiable proposition can also have false values.

  - **Example:** $p \wedge q$
    If $p$ and $q$ are both true, then the proposition is true, making it satisfiable. Note that this proposition can also be false, for example if $p$ is true and $q$ is false.

- **Predicate:** A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified. For example, "$x > 10$" is a predicate that depends on the parameter $x$.

- **Quantifiers:**

  - **Universal quantifier:** Denoted by $\forall$ and translated as "for all", it specifies that the following propositional function is true for all possible parameters in the domain. Note that $\forall$ represents a chain of ands. For example, if the domain is $\{0, 1, 2, 3\}$, then $\forall x P(x)$ has the same truth value as $P(0) \wedge P(1) \wedge P(2) \wedge P(3)$.
  - **Existential quantifier:** Denoted by $\exists$ and translated as "there exists", it specifies that the following propositional function is true for at least one parameter in the domain. Note that $\exists$ represents a chain of ors. For example, if the domain is $\{a, b, c\}$, then $\exists x P(x)$ has the same truth value as $P(a) \vee P(b) \vee P(c)$.

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers ($\forall$ or $\exists$) to quantify a compound proposition. In nested quantifiers, order matters, e.g. $\forall x \exists y P(x, y)$ is different from $\exists y \forall x P(x, y)$.

## 1. Finding Truth Values of Compound Propositions

For each compound proposition, find its truth value when $p = T$, $q = F$, $r = F$, $s = F$, $t = T$, $u = F$, and $v = F$

a) $(q \to \neg p) \vee (\neg p \to \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

**Solution:**

a) $(q \to \neg p) \vee (\neg p \to \neg q)$
$\equiv (F \to \neg T) \vee (\neg T \to \neg F)$
$\equiv (F \to F) \vee (F \to T)$
$\equiv T \vee T$
$\equiv T$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$
$\equiv (T \vee \neg T) \wedge (T \vee \neg F)$
$\equiv (T \vee F) \wedge (T \vee T)$
$\equiv T \wedge T$
$\equiv T$

c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$
$\equiv (T \to F) \vee (\neg F \to \neg T) \vee (\neg F \to F)$
$\equiv (T \to F) \vee (T \to F) \vee (T \to F)$
$\equiv F \vee F \vee F$
$\equiv F$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$
$\equiv (T \wedge F \wedge F) \vee (F \wedge T) \vee (F \wedge \neg T)$
$\equiv (T \wedge F \wedge F) \vee (F \wedge T) \vee (F \wedge F)$
$\equiv F \vee F \vee F$
$\equiv F$

## 2. English to Logic Translation I

Let $p$, $q$, and $r$ be the propositions defined as follows.

- $p$: Grizzly bears have been seen in the area.

- $q$: Hiking is safe on the trail.
- $r$: Berries are ripe along the trail.

Write these propositions in logic using $p$, $q$, $r$, logical connectives (including negations), and parentheses.
***Reminder:** $\wedge$ denotes "and", $\vee$ denotes "or", $\leftrightarrow$ denotes"if and only if", and $\neg$ denotes "not".

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe

**Solution:**

a) $r \wedge \neg p$

b) $\neg p \wedge q \wedge r$

c) $r \rightarrow (q \leftrightarrow \neg p)$

d) $\neg q \wedge \neg p \wedge r$

## 3. Logic to English Translation

Consider the following propositions:

- $g$: you can graduate
- $m$: you owe money to the university
- $r$: you have completed the requirements of your major
- $b$: you have an overdue library book

Translate the following statement to English: $g \rightarrow (r \wedge \neg m \wedge \neg b)$

**Solution:**
You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book.
Equivalently, if you can graduate, then you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book.

## 4. Tautologies

a) Determine whether $[\neg p \wedge (p \rightarrow q)] \rightarrow \neg q$ is a tautology.

b) Show that this conditional statement is a tautology by using truth tables.

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

**Solution:**

a) This is not a tautology. We can show this by finding values for $p$ and $q$ which make the proposition false. In order to make the proposition false, we can set $p = F$ and $q = T$.

$[\neg p \wedge (p \rightarrow q)] \rightarrow \neg q$
$\equiv [\neg F \wedge (F \rightarrow T)] \rightarrow \neg T$
$\equiv [T \wedge T] \rightarrow F$
$\equiv T \rightarrow F$
$\equiv F$

b) As seen in the truth table, all combinations of boolean assignments result in the statements being true. Therefore, it is a tautology.

| $p$ | $q$ | $p \rightarrow q$ | $p \wedge (p \rightarrow q)$ | $[p \wedge (p \rightarrow q)] \rightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

Of note, we can also show this statement is equivalent to true (a tautology) by our logical equivalences.
$[p \wedge (p \rightarrow q)] \rightarrow q$
$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$ Implication Breakout
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$ Distributive Law
$\equiv [F \vee (p \wedge q)] \rightarrow q$ ($p \wedge \neg p \equiv F$, contradiction)
$\equiv [(p \wedge q) \rightarrow q]$ Identity Law
$\equiv \neg(p \wedge q) \vee q$ Implication Breakout
$\equiv (\neg p \vee \neg q) \vee q$ De Morgan's
$\equiv \neg p \vee (\neg q \vee q)$ Associative
$\equiv \neg p \vee T$ ($\neg q \vee q \equiv T$, tautology)
$\equiv T$ Domination Law

**Note:** We will not ask you to do lengthy line-by-line logical equivalence calculations like this one on homework or exams, and you only need to know the names of the logical equivalence laws explicitly mentioned in lecture. In general, truth tables always work for logical equivalence proofs.

## 5. Truth Values of Predicates

Let $Q(x)$ be the predicate "$x + 1 > 2x$". Determine the truth values of the following propositions. The domain of quantifiers is all integers.

a) $Q(-1)$

b) $Q(1)$

c) $\exists x Q(x)$

d) $\forall x Q(x)$

e) $\exists x \neg Q(x)$

f) $\forall x \neg Q(x)$

**Solution:**

a) True. Substitute $x = -1$ into the predicate, and check if the the inequality holds.

b) False. Substitute $x = 1$ into the predicate, and check if the the inequality holds. In this case, the inequality does not hold.

c) True. To prove an exists statement is true, we just need to find one element of the domain for which the statement holds. Let's consider $x = -10$. $Q(-10)$ is the statement "$-10 + 1 > 2(-10)$". This is true since $-10 + 1 = -11 > -20 = 2(-10)$. There are many other values of $x$ for which the statement holds as well (any integer less than 1 will work).

d) False. To prove a for all statement is true, we need to show $Q(x)$ is true for all $x$ in the domain. Or we could prove it is false by finding a counterexample. In this case, we can provide a counterexample $x = 1$. Now we have proved this statement is false.

e) True. To prove this statement is true, we only need to find one example where $Q(x)$ is false. We have found this example in part (b).

f) False. To prove this to be false, we only need one example to show $Q(x)$ is true, and we can use examples such as $x = 0$ or $x = -1$.

## 6. Quantifiers and Negations

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x(-4 < x \leq 1)$

b) $\forall z \exists x \exists y(x^3 + y^3 = z^3)$

**Solution:**

a) $\neg(\exists x(-4 < x \leq 1))$
$\equiv \forall x(\neg(-4 < x \leq 1))$
$\equiv \forall x((x \leq -4) \vee (x > 1))$

b) $\neg(\forall z \exists x \exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \neg(\exists x \exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \forall x \neg(\exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \forall x \forall y(\neg(x^3 + y^3 = z^3))$
$\exists z \forall x \forall y(x^3 + y^3 \neq z^3)$

## 7. Quantified Statement Counterexamples

Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y(x = 1/y)$

b) $\forall x \exists y(y^2 - x < 100)$

c) $\forall x \forall y(x^2 \neq y^3)$

**Solution:**

a) Consider $x = 2$, then there is no y among the integers such that $2 = 1/y$, since the only solution of this equation is $y = 1/2$.

b) Consider $x = -200$. The statement claims there exists a $y$ such that $y^2 + 200 < 100$. This would require our $y^2$ to be negative, which is not possible in the domain of integers.

c) Consider $x = y = 0$. $x^2 = 0$ and $y^3 = 0$, so $x^2 = y^3$.

## 8. Quantifier Translations I

Let $P(x)$ be "$x$ is perfect"; let $F(x)$ be "$x$ is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

**Solution:** Let $P(x)$ be "$x$ is perfect"; let $F(x)$ be "$x$ is your friend"; and let the domain (universe of discourse) be all people.

a) This means that everyone has the property of being not perfect: $\forall x \neg P(x)$. Alternatively, we can write this as $\neg \exists x P(x)$, which says that there does not exist a person who is perfect.

b) This is just the negation of "Everyone is perfect": $\neg \forall x P(x)$. Alternatively we could write $\exists x \neg P(x)$ (i.e. there exists someone who is not perfect).

c) If someone is your friend, then that person is perfect: $\forall x(F(x) \implies P(x))$. Note the use of conditional statement with universal quantifiers.

d) $\exists x(F(x) \land P(x))$. Note the use of conjunction ($\land$) with existential quantifiers to restrict the domain. (Note this allows the possibility that more than one of your friends are perfect.)

e) The expression is $\forall x(F(x) \land P(x))$. Note that here we did use a conjunction with the universal quantifier because $\forall$ distributes over $\land$. We can also split up the expression into two quantified statements and write $(\forall x F(x)) \land (\forall x P(x))$.

f) This is a disjunction. The expression is $(\neg \forall x F(x)) \lor (\exists x \neg P(x))$.