Lec 5

1. Know technical vocab ① Proof ② Assumption ③ Disproof ④ Counterexample
2. proofs of "for all" propositions
3. proofs of "there exists" propositions
4. proofs of propositions that have mixed quantifiers
   * understand how quantifier order influences the proof structure
5. Disproofs of propositions
   * including propositions with an "if-then" proposition inside of a "for all"
   * including propositions with mixed quantifiers

Today: direct proof

* Pattern of proving "if-then" inside "for all"
1. Arbitrary element
2. Assume
3. End with the if-then statement inside the "for all"

9/12/23

L5 Handout: Direct Proofs

Def: Int x is **"even"** if there exists int k with $x = 2k$.

Prove: For all ~~even~~ integers x, ~~6~~ 4$x + 2$ is even.

Proof:

- Let x be __an arbitrary integer__ ...
- By algebra, $(4x+2) = 2(2x+1)$ 任意的
- Since x is an integer, $(2x+1)$ is also an integer
- So $(4x+2)$ is twice an integer, so $(4x+2)$ is even

List form of a proof

Chains of deductions must end with the statement inside the "for all" proposition

* 2. Prove: For all odd integer x, $(3x-2)$ is odd.
- Let x be an arbitrary odd integer.
- Since x is odd, there exists an integer k with $x = 2k+1$
- $(3x-2) = 3(2k+1) - 2 = 6k+1 = 2(3k)+1$
- Since k is an integer then $3k$ is an integer
- Since $2(3k)$ is twice an integer thus is even, thus $2(3k)+1$ is odd
- Therefore $\forall x \in \mathbb{Z}$, $(3x-2)$ is odd.

① Template: Proving "For all"

Claim: For all x [in the domain], $P(x)$

Proof Template

Let x be an ***arbitrary*** element of the domain

... (make some deductions) ...

Thus, $P(x)$.

Therefore, $P(x)$ holds for all x in the domain.

1. starts with an arbitrary element
2. End with the statement inside the "for all"

② Template: Proving "If-Then"

Claim: If p, then q

Proof Template

Assume p.

... (make some deductions) ...

Therefore, q.

1. Assume
2. Therefore

This pattern often happens inside a **"for all" statement**

eg. prove for all integer, if x is odd, then x^2 is odd.
- Let x be an arbitrary integer
- Assume x is odd, so $x = 2k+1$ for some integer k
- so $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2+2k)+1$
- Since k is an integer, $2k^2+2k$ is also an integer
- So $x^2 = 2(2k^2+2k)+1$ is odd.

③ "If-Then" Proofs

Definition: x is **rational** if there exists integers a and b such that $x = a/b$.

Prove: For all numbers x and y with $y \neq 0$, if x and y are both rational, then x/y is also rational.

Proof:

- Let x and y be __arbitrary numbers__, with $y \neq 0$.
- Assume __x and y are both rational__
- So $x = \frac{a}{b}$, $y = \frac{c}{d}$ for some integers a, b, c, d
- So $\frac{x}{y} = \frac{a/b}{c/d} = \frac{ad}{bc}$

  ad and cb are both integers because a, b, c, d are integers
- So $\frac{x}{y}$ is rational

④ Existence Proofs

The way you prove a "there exists" statement is just by naming a particular example and showing that it works

Prove: There exist integers a, b, c, d such that $a^2 + b^2 + c^2 = d^2$

- **Consider** __$a=0, b=3, c=4, d=5$__
- $a^2+b^2+c^2 = 25$, $d^2 = 25$, then $a^2+b^2+c^2 = d^2$
- Done.

Template: Proving "There exists"

Claim: There exists an x [in the domain] such that $P(x)$

Common Approach

Consider $x =$ __a__

[give a __$P(a)$__ of the domain]

... show that $P(x)$ holds for that value of x.

Mixed Quantifiers

(A) **Proposition:** For all integers x, there exists an integer y such that $x^2 + y = 3$.	(B) **Proposition:** There exists an integer y such that for all integers x, we have $x^2 + y = 3$.
$\forall x\, \exists y\, (x^2+y=3)$	$\exists y\, \forall x\, (x^2+y=3)$

Which one is true? A

Proof (of true one):

- Let x be an arbitrary integer
- consider $y = 3-x^2$
- $x^2+y = x^2+(3-x^2)$
  $= x^2+3-x^2 = 3$ ✓

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Disproofs ("For All")

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$x =$ 2 is called a counterexample to this proposition

1) 6.439

Disprove:
For all integers x, $x < x^2 - 3x + 4$

Disproof:
- We will prove the negation: "There exists an integer x for which $x \geq x^2-3x+4$"

- Consider $x=2$
- We have $x^2-3x+4 = 2^2-3(2)+4 = 2$
- So $x \geq x^2-3x+4$

Disproofs ("There Exists")

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Disprove:
There exists an integer x for which $x > x^2 - 3x + 4$

Disproof:
- We will prove the negation: "For all integer x, we have $x \leq x^2-3x+4$"

- Let x be an arbitrary integer
- We have: $x^2-3x+4 = (x^2-4x+4)+x$
  $= (x-2)^2+x \geq 0+x = x$

Template: Disproofs

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Disproving a proposition is the same as ***proving*** its negation

Claim: There exists x [in the domain] such that $P(x)$	**Claim**: For all x [in the domain], we have $P(x)$
Common Approach	**Common Approach**
We will prove the negation: "For all x [in the domain], $\neg P(x)$"	We will prove the negation: "There exists x [in the domain] such that $\neg P(x)$" counterexample
Let x be arbitrary	Consider $x =$ ____ [called a **counterexample**]
[...deductions...]	[...deductions...]
Therefore, $\neg P(x)$.	Therefore, $\neg P(x)$.

Property of a proof.

1. Consise: not unnecessarily long
2. clear: not ambigious
3. complete: no missing intermediate steps
4. logical: every statement logically follows
5. rigorous: uses mathematical expressions
6. convincing: does not raise questions