EECS 203: Discrete Mathematics
Fall 2023
Homework 4

Due **Thursday, Sept. 28**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $6 + 2$ Total Points: $100 + 50$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Let's be rational (numbers) [16 points]

(a) **Prove or disprove:** for all real numbers x and y, if xy is irrational, then x is irrational or y is irrational.

(b) **Prove or disprove:** for all real numbers x and y, if x is irrational or y is irrational, then xy is irrational.

Solution:

(a) **Prove.** We will prove the contrapositive: if x and y are both rational then xy is rational. Then let x and y be arbitrary rational numbers. Then there are integers a and b where $b \neq 0$ such that $x = \frac{a}{b}$ and integers c and d where $d \neq 0$ such that $y = \frac{c}{d}$. Then $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Since a, c, b and d are all integers and b and d are both non-zero then ac and bd will be integers and bd will be non-zero. Therefore xy is rational. Therefore we have proven the contrapositive.

(b) **Disprove.** Let $x = \sqrt{2}$ and $y = \sqrt{2}$. Then $xy = \sqrt{2} \cdot \sqrt{2} = 2$. Therefore we have shown there exists two irrational numbers that when multiplied have a rational product.

Alternate Disproof: Let $x = \sqrt{2}$ and $y = 0$. Then $xy = \sqrt{2} \cdot 0 = 0$, which is rational.

Part a:

+1 chooses to prove
+2 chooses to use proof by contrapositive
+3 correct contrapositive
+3 correctly defines the rational numbers as fractions of integers
+3 correct proof and conclusion

Note to graders: Award full points for a correct and equivalent proof by contradiction (supposing that the premise is true and the conclusion is false, and then showing the premise is false).

Part b:

+1 chooses to disprove
+3 gives a counterexample and correctly disproves

2. Irrational Pr00f [14 points]

Prove or disprove that for all nonzero rational numbers x and irrational numbers y, xy is irrational.

Solution:

Proof by Contradiction:

Let x be an arbitrary nonzero rational number and let y be an arbitrary irrational number. Since x is rational, there exists integers a and b such that $x = \frac{a}{b}$ and $b \neq 0$.

Assume $\frac{a}{b}y$ is not an irrational number. Therefore, $\frac{a}{b}y$ is rational. Since $\frac{a}{b}y$ is rational, $\frac{a}{b}y = \frac{c}{d}$, for some integers c, d, where $d \neq 0$. Since $\frac{a}{b} \neq 0$, $a \neq 0$. Therefore, $y = \frac{c}{d} \cdot \frac{b}{a} = \frac{cb}{da}$. Since a, b, c, and d are all integers, cb and da are also integers with da not zero. Therefore, $y = \frac{cb}{da}$ is a rational number.

However, this contradicts the fact that y is an irrational number. Therefore the assumption that $\frac{a}{b}y$ is not an irrational number is false. This implies that $\frac{a}{b}y$ is an irrational number, completing the proof.

Grading Guidelines [14 points]

+3 assumes the product of some rational number and some irrational number is rational
+3 correctly defines the rational numbers as fractions of integers
+3 shows the irrational number is rational
+3 states this is a contradiction
+2 concludes the product of a rational and irrational is irrational

3. That's Really Odd... (and Even) [17 points]

In this problem, you may use the following statement without proof: "for all integers n, n^2 is odd if and only if n is odd."

(a) Prove that for all integers n, $n^2 - n + 1$ is odd.

(b) Prove that for all integers x and y, if $x^2 + y^2$ is even, then $x + y$ is even.

Solution:

(a) Let n be an arbitrary integer. Consider two possible cases for n: n is odd and n is even.

Case 1: n is odd. Assume n is an odd number. Then by definition of odd, we can write $n = 2k + 1$ where k is an integer, moreover we know n^2 is odd so $n^2 = 2j + 1$

for some integer j. Then $n^2-n+1 = 2j+1-2k-1+1 = 2(j-k)+1$. Since $j-k$ is an integer, by the definition of odd we have shown n^2-n+1 is odd in this case.

Case 2: n is even. Assume n is even. Then by definition of even we can write $n = 2k$ where k is an integer. We also have that n^2 is even, so $n^2 = 2j$ for some integer j. Then $n^2 - n + 1 = 2j - 2k + 1 = 2(j-k) + 1$. Since $j-k$ is an integer, by the definition of odd $n^2 - n + 1$ is odd.

Since $n^2 - n + 1$ is odd in all cases, $n^2 - n + 1$ is odd. Therefore for all integers n, $n^2 - n + 1$ is odd.

Alternate Solution:

Let n be an arbitrary integer. Notice $n^2 - n + 1 = n(n-1) + 1$. Since n is an integer, n and $n-1$ will have opposite parity. Then this implies $n(n-1)$ will be an even number, and then $n(n-1)+1$ will be an even number plus one, and therefore $n^2 - n + 1$ will be odd.

(b) Let x and y be arbitrary integers.

Assume that x^2+y^2 is even. Then $x^2+y^2 = 2k$ for some integer k. Adding $2xy$ on both sides gives $x^2+2xy+y^2 = 2k+2xy$ which implies $(x+y)^2 = 2(k+xy)$. Since k, x and y are all integers, so is $k+xy$. Hence, $(x+y)^2 = 2p$ with $p = k+xy$. Therefore $(x+y)^2$ is an even integer. Now from the statement given in the problem, we can conclude that $x+y$ is also even.

Alternate solution:

We can prove this by contraposition: if $x+y$ is odd, then x^2+y^2 is odd. We let $x+y$ be odd. From the statement given in the problem, $(x+y)^2$ must also be odd, so $(x+y)^2 = 2k+1$ for some integer k. We can expand the binomial and rearrange this to say:

$$\begin{aligned} 2k+1 &= (x+y)^2 \\ &= x^2+2xy+y^2 \\ x^2+y^2 &= 2(k-xy)+1 \end{aligned}$$

Since k, x and y are integers, so is $k-xy$. Hence $x^2+y^2 = 2p+1$, where $p = k-xy$, so we can say x^2+y^2 is odd.

Grading Guidelines [17 points]

Note to graders: As usual, fully correct alternate solutions should receive full points.

Part a:

+2 identifies correct cases (n is odd or n is even)
+4 correctly shows if n is even then $n^2 - n + 1$ is odd
+4 correctly shows if n is odd then $n^2 - n + 1$ is odd

Part b:

+2 correct assumption for chosen proof method
+2 correctly applies the definition of even/odd
+3 correctly proves the conclusion for chosen proof method, optionally using the statement given in the problem

4. All that remains [21 points]

Definition: For integers n, r, d, we say that r is the **remainder** of n when divided by d if and only if $0 \le r < d$, and there exists integer q such that $n = dq + r$. For example, the remainder of 14 when divided by 4 is 2 since $14 = 4 \cdot 3 + 2$.

(a) Prove that for all integers n, the remainder of n^2 when divided by 4 is either 0 or 1.

(b) Prove that for all prime numbers p greater than 3, the remainder of p^2 when divided by 3 is 1.

Hint: Consider the possible remainders when dividing p by 3.

Solution:

(a) Let n be an arbitrary integer. n can fall into two cases: n is even and n is odd.

Case 1: n is even. Assume n is even. Then $n = 2k$ for some integer k, so $n^2 = 4k^2$. Then letting $q = k^2$, $n^2 = 4q + 0$. Therefore the remainder of n^2 when divided by 4 is 0. Therefore the remainder of n^2 when divided by 4 is 0 or 1.

Case 2: n is odd. Assume n is odd. Then $n = 2k + 1$ for some integer k, so $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Then letting $q = k^2 + k$, $n^2 = 4q + 1$. Therefore the remainder of n^2 when divided by 4 is 1. Therefore the remainder of n^2 when divided by 4 is 0 or 1.

Since the remainder of n^2 when divided by 4 is 0 or 1 in all cases, the remainder of n^2 when divided by 4 is 0 or 1.

Therefore for all integers n, the remainder of n^2 when divided by 4 is either 0 or 1.

(b) Let p be an arbitrary prime number greater than 3. Let r be the remainder when dividing p by 3. Then there exists some integer k such that $p = 3k + r$. r can fall into three cases: $r = 0$, $r = 1$, and $r = 2$.

Case 1: $r = 0$. Assume $r = 0$. Then $p^2 = (3k + 0)^2 = 9k^2 = 3(3k^2)$. Then p is divisible by 3. But p is greater than 3, therefore p is not prime. Since p is prime and p is not prime, we have a contradiction. Assuming $r = 0$ led to a contradiction. Therefore $r \neq 0$.

Case 2: $r = 1$. Assume $r = 1$. Then $p^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$. Then letting $q = 3k^2 + 2k$, $p^2 = 3q + 1$. Therefore the remainder when dividing p^2 by 3 is 1.

Case 3: $r = 2$. Assume $r = 2$. Then $p^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1$. Then letting $q = 3k^2 + 4k + 1$, $p^2 = 3q + 1$. Therefore the remainder when dividing p^2 by 3 is 1.

In all cases, the remainder when dividing p^2 by 3 is 1. Therefore the remainder when dividing p^2 by 3 is 1.

Therefore for all prime numbers p greater than 3, the remainder of p^2 when divided by 3 is 1.

Grading Guidelines [21 points]

Part a:

+2 identifies correct cases (n is odd vs. even)
+3 correctly handles case where n is even
+3 correctly handles case where n is odd
+2 correct conclusion
Also award full points for a solution that instead correctly uses 4 cases on the remainder of n when divided by 4 (remainder equals 0, 1, 2, 3).

Part b:

+1 correctly applies the definition of remainder to p
+2 identifies correct cases (possible remainders r are $0, 1, 2$ when dividing p by 3)
+2 correctly handles case where $r = 0$
+2 correctly handles case where $r = 1$
+2 correctly handles case where $r = 2$
+2 correct conclusion

5. FOtter's Day [17 points]

Every year on Father's Day, each otter pup at the Ann Arbor Zoo gives a rock to each adult otter at the zoo. We will prove that if there are an even number of otters at the Ann Arbor Zoo, and an even number of rocks were gifted this year, then there are an even number of otter pups and an even number of adult otters.

(a) Let x be the number of otter pups and y be the number of adult otters. Rewrite the above statement in terms of x and y.

(b) Prove the statement you wrote in (a).

Solution:

(a) We can rewrite the statement as "if $x + y$ and xy are even, then x and y are both even."

(b) The contrapositive of the statement is "if x or y is odd, then $x + y$ or xy is odd." Then we have the following cases:

Case 1: x is odd, y is even. Then $x = 2j + 1$ for some integer j and $y = 2k$ for some integer k. Then $x + y = (2j + 1) + (2k) = 2(j + k) + 1$. Since j and k are integers, so is $j + k$. As such, $x + y = 2l + 1$ with $l = j + k$, so $x + y$ is odd.

Case 2: x is even, y is odd. Then $x = 2j$ for some integer j and $y = 2k + 1$ for some integer k. Then $x + y = (2j) + (2k + 1) = 2(j + k) + 1$. Since j and k are integers, so is $j + k$. As such, $x + y = 2l + 1$ with $l = j + k$, so $x + y$ is odd.

Note: We can combine cases 1 and 2 using WoLoG.

Case 3: x and y are both odd. Then $x = 2j+1$ for some integer j and $y = 2k+1$ for some integer k. Then $xy = (2j+1)(2k+1) = 4jk+2j+2k+1 = 2(2jk+j+k)+1$. Since j and k are integers, so is $2jk+j+k$. As such, $xy = 2l+1$ with $l = 2jk+j+k$, so xy is odd.

Therefore, if $x + y$ and xy are even, then x and y are even.

Grading Guidelines [17 points]

Part a:

+2 does not use otters nor rocks in the rewritten statement
+2 correctly rewrites as an equivalent statement in terms of x and y

Part b:

+3 takes the correct contrapositive of the statement
+3 correctly handles case where x is odd and y is even
+3 correctly handles case where x is even and y is odd (or correctly applies without loss of generality)
+3 correctly handles case where x is odd and y is odd
+1 correct conclusion

6. False Inequality [15 points]

In this problem, you may use the following axiom: "for all real numbers a, b, c, if $a < b$ and c is positive, then $ac < bc$."

We will disprove the following proposition p:

"There exists a real number x such that $x^2 < x < x^3$."

(a) Prove that for all real numbers x satisfying $x^2 < x$, x is positive.

(b) Using part (a), disprove p.

Solution:

(a) Let x be an arbitrary real number and suppose $x^2 < x$. For any real number x, $x^2 \geq 0$. Then $0 \leq x^2 < x$. Therefore $0 < x$. Therefore x is positive. Therefore for all real numbers x satisfying $x^2 < x$, x is positive.

(b) Assume that there exists a real number x such that $x^2 < x < x^3$. Then $x^2 < x^3$ and $x^2 < x$. Since $x^2 < x$, x is positive by part (a). Also since $x^2 < x$, $x^2 \cdot x < x \cdot x$, meaning $x^3 < x^2$. This contradicts that $x^2 < x^3$. Therefore our original assumption is false, so there does not exist a real number x such that $x^2 < x < x^3$.

Grading Guidelines [15 points]

Part a:

+2 correctly assumes $x^2 < x$ for arbitrary real number x
+3 identifies that $x^2 \geq 0$
+2 concludes that since $x^2 < x$, x is positive

Part b:

+1 correctly assumes p is true
+3 shows that since $x^2 < x$, $x^3 < x^2$. **Note to graders:** Still award this point if any alternate fact is shown that is used to arrive at a valid contradiction
+2 arrives at a valid contradiction
+2 concludes that p is false

Groupwork

1. Grade Groupwork 3

Using the solutions and Grading Guidelines, grade your Groupwork 3:

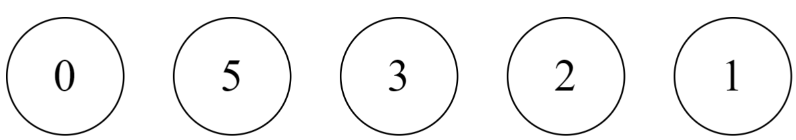
- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1												/10
Problem 2												/20
Total:												/30

2. Diag Squirrels [20 points]

Sammy and Sapphire the Diag squirrels are playing a game. There is a row of n holes, each starting with 203 acorns in it. They also have a large, unlimited pile of extra acorns. Sammy and Sapphire take turns, starting with Sammy; when all the holes are empty on one of the squirrel's turn, that squirrel loses. On each turn, a squirrel picks a hole, eats exactly one acorn from it, then places any number of extra acorns they wish into each hole to the right of that hole. They may place a different number of acorns into each other hole.

For example, suppose they are playing with $n = 5$ holes and it is Sammy's turn. Suppose the number of acorns in each hole at the start of Sammy's turn are as follows.



On their turn, Sammy must pick a hole and eat exactly one acorn from it. Suppose they pick the third hole. Then the counts become the following:

0 5 2 2 1

Sammy may then place any number of extra acorns into each hole to the right of that hole. In this case, they can place into the fourth and fifth holes. Suppose they choose to place 3 acorns in the fourth hole and 1 in the fifth. Then the at the end of Sammy's turn the acorn counts are the following:

0 5 2 5 2

A winning strategy for a player is a sequence of moves which guarantees that they will win regardless of what moves their opponent makes. We will construct a winning strategy for Sammy. We will need an important but non-obvious fact about this game: the game must reach a state where every hole is empty except for the right-most hole.

(a) Prove that, once we reach the state where all but the right-most hole is empty, Sammy has a winning strategy if and only if there are an odd number of acorns in the hole at the start of their turn.

(b) Prove that if a squirrel starts their turn with all holes having an even number of acorns (and the game is not over), then at the end of their turn, at least one hole will have an odd number of acorns.

(c) Prove that if a squirrel starts their turn with at least one hole having an odd number of acorns, they can end their turn with all holes having an even number of acorns

(d) Using the previous parts, prove that Sammy has a winning strategy.

Solution:

(a) Assume that there are an odd number of acorns in the right-most hole at the start of Sammy's turn. Then we can express the number of acorns as $2k+1$ for some natural number k (k must be natural because we can't have a negative number of acorns). Then after Sammy eats an acorn, there are $2k$ acorns left. Now on each subsequent turn, Sapphire will eat an acorn, then Sammy will eat an acorn, meaning two acorns were eaten. This leaves us with $2(k-1)$ acorns. This process will continue for k turns until there are 0 acorns at the beginning of Sapphire's turn.

We will prove the reverse direction by contrapositive. Assume there are an even number of acorns at the start of Sammy's turn. Then by the above argument, the game will end after k turns when there are 0 acorns left at the start of Sammy's turn. Additionally, notice that the only possible move is for Sammy to eat a single acorn, since there are no holes to the right. Thus, Sammy does not have a winning strategy.

(b) Assume a squirrel starts their turn with all holes having an even number of acorns. Since the game is not over, there is at least one hole that is not empty. Thus, the squirrel is forced to eat an acorn from one of the non-empty holes, leaving it with an odd number. Since they can only add acorns to holes to the right, then the hole they ate out of will still have an odd number at the end.

(c) Assume a squirrel starts their turn such that at least one hole has an odd number of acorns. Then there must be a left-most hole with an odd number. The squirrel is able to eat out of this hole, leaving it with an even number. Consider each hole to the left of that hole. By assumption, they all have an even number of acorns. Next, consider each hole to the right of it. If it has an even number, then the squirrel can leave it. If it has an odd number, then the squirrel can add one acorn to make it an even number.

(d) At the start of Sammy's first turn, all holes have 203 acorns, which is an odd number. Thus, Sammy is able to end their turn with an even number in each hole by part (c). After this turn, Sapphire is forced to end her turn with an odd number in at least one hole by part (b). Then this process continues until, by the lemma given in the question, we end up with all holes empty except for the rightmost hole. Since Sammy will start each of their turns with at least one hole having an odd number, and every hole has zero except for the rightmost hole, this hole has an odd number in it. Then by part (a), Sammy has a winning strategy.

Grading Guidelines [20 points]

Part a:

(i) +2 proves that if there are an odd number of acorns, Sammy has a winning strategy
(ii) +3 proves that if Sammy has a winning strategy, there are an odd number of acorns

Part b:

(iii) +5 correct proof

Part c:

(iv) +5 correct proof

Part d:

(v) +5 correct proof

3. The Third Dimension [30 Points]

In lecture, we discussed the problem of tiling a chessboard with dominoes of dimension 2×1. We also saw that this question can be made more interesting by changing the shape of the board.

A related idea to tiling is packing. In a packing question, we no longer care that the board gets completely covered, instead it is enough to show that a certain number of dominoes can fit on the board. For example, 32 or fewer 1×2 dominoes can be packed into a 8×8 chess board, but 33 or more cannot.

In this problem, we will investigate packing dominoes into a three dimensional "chess board". In particular, we will prove that it is impossible to pack 53 $1 \times 1 \times 4$ dominoes into a $6 \times 6 \times 6$ board.

(a) As a warm-up, first show that you *can* pack 54 dominoes into the board provided that you're allowed to break the dominoes in half.

(b) We can divide our board evenly into $2 \times 2 \times 2$ regions. Consider coloring these regions red and blue in an alternating fashion. We say that each $1 \times 1 \times 1$ cell of a domino is colored red if it lies in a red region and colored blue if it lies in a blue region.

For any domino, list all possible colorings of its 4 cells. Conclude that exactly half of each domino must lie in a red region.

(c) Prove that it is impossible to pack 53 dominoes into a $6 \times 6 \times 6$ board.

Hint: Figure out how many cells of each color there are, and apply part (b).

Solution:

(a) We will construct a packing by breaking each domino in half, leaving us with 108 $1 \times 1 \times 2$ tiles. We can then pack 18 of these tiles into one layer of our board by lining them in 6 1×6 rows. Finally we can repeat this process for each layer using up all $18 \times 6 = 108$ tiles.

Alternatively, we can pack the tiles upright, placing 36 in one layer, 36 more in a second layer, and the remaining $108 - 2 \cdot 36 = 36$ in the final layer.

(b) There are 4 possible colorings for each domino: RRBB, BRRB, BBRR, RBBR. Hence, no matter where we place our domino exactly half of it will be red.

(c) Using the coloring from part (b), note that there are 27 total regions. Provided that the corner regions are colored blue, there are 13 red regions and 14 blue regions. Since each region contains 8 cells, there are $8 * 13 = 104$ red cells in total. However, from part (b) we know that each of our 53 tiles need to sit in 2 red cells. This means that we need 106 red cells in order to pack our dominoes into the board, so no such packing exists.

Grading Guidelines [30 points]

Part a:

+6 produces a valid packing

Part b:

+3 correctly identifies all possible colorings of a domino
+3 concludes that exactly half of the domino sits in red (or blue) cells

Part c:

+6 identifies that there are 13 regions of one color
+3 notes that there are exactly 104 red (or blue) cells in the board
+6 applies part (b) to say 106 red (or blue) cells are needed for a packing to exist
+3 concludes that no valid packing exists