terminology

graph ; node/vertex ; edge ; degree ;
图 节点 边 度

cycle ; complete; hypercube; isomorphism ; invariant;
圈图 完全图 立方体图 同构 不变量

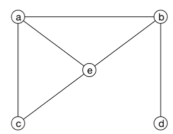
subgraph ; path ; connected (node pair); connected (graph) ; connected component ; tree
子图 通路 点连通 图连通 连通分支 (极大连通子图) 树

10/31/23

Lecture 18: Graphs 1 (Structure and Relations)-- ANSWERS

- ① A **graph** G is a pair of sets (V , E).
- The elements of the set V are called vertices or nodes. ② 节点
- The elements of the set E are called edges ③ 边
  - Every edge is a set of 2 vertices.

two vertices of an edge : its endpoints (端点)

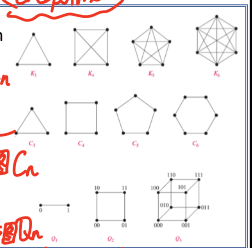


$V = \{a, b, c, d, e\}$
$E = \{\{a,b\},\{a,c\},\{a,e\},\{b,d\},\{b,e\},\{c,e\}\}$

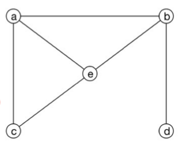
K_n : the **complete** graph on n vertices ④ 完全图 K_n every

C_n : the **cycle** on n vertices ⑤ 圈图 C_n

Q_n : "Hypercubes" 立方体图 Q_n

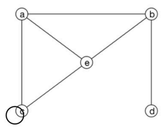


cube (1-d) cube (2-d) cube (3-d)

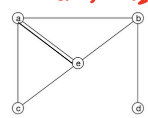


Graph

① What is not a graph? (But in a broader sense is)



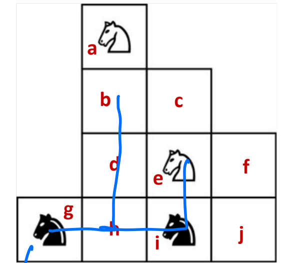
Not a graph due to "edge" $\{c, c\}$
(but it is a "graph with self-loops," a broader object that we won't talk about)



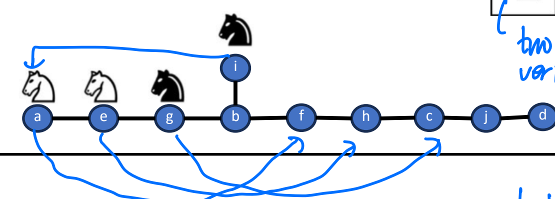
Not a graph due to two edges $\{a, e\}$
(but it is a "multigraph," a broader object that we won't talk about)

Knight Swap Puzzle

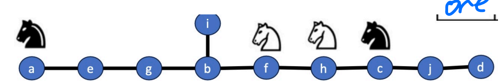
- **Goal:** Move the knights around the restricted chessboard to swap the positions of the black and white knights
  - Knights move in an L-shape
  - Must end move inside the board
  - No two knights may occupy the same square
- Draw the **graph** for this problem below:



two neighbouring vertices of g: b, e



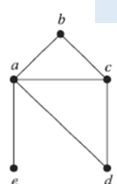
⟹



one knight done, then the others

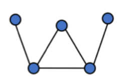
Node degrees

⑧ The **degree** of a node v in a graph $G = (V, E)$ is the number of edges that contain v.



eg
$\deg(a) = 4$
$\deg(b) = 2$
$\deg(c) = 3$
$\deg(d) = 2$
$\deg(e) = 1$

Draw a graph with 5 vertices whose degrees are 1,1,2,3,3.



(or other possibilities)

Draw a graph with 5 vertices whose degrees are 1,1,2,3,4.

Impossible: Sum of degrees is odd, which violates handshake theorem

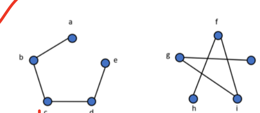
Handshake Theorem: In a graph $G = (V, E)$,
$$\sum_{v \in V} \deg(v) = 2|E|$$
★ ⑨

one edge exist ⟹ two vertices +1 degree
every edge $\{u,v\} \in E$ contributes 1 to deg(u) and 1 to deg(v).

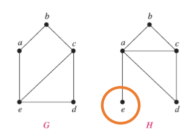
How to Prove/Disprove Isomorphism

Two graphs G, H are ⑩ **isomorphic** if we can **relabel** the vertices of G to make it **equal** to H.

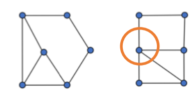
Which of these pairs of graphs are isomorphic? How do you know?



Yes
$a \to h, b \to f, c \to i, d \to g, e \to j$



No
H has a degree 1 node, G does not



No
H has a degree 4 node, G does not

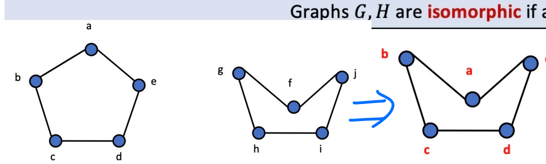
rigorously

Prove two graphs are isomorphic by giving the bijection between vertices (i.e., give the relabeling)

Disprove two graphs are isomorphic by naming an ***invariant*** for one of the graphs that is violated in the other graph.

⑪ 不变量
Invariant = a property that is preserved under isomorphism (e.g., has a degree 1 node, or has a 5-cycle)

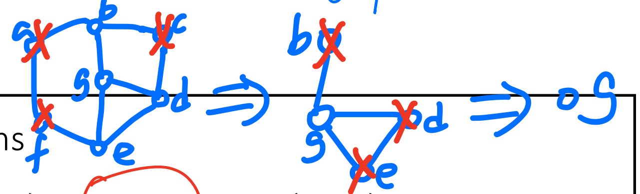
Formally:
An **isomorphism** from $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a bijection $f: V_G \to V_H$, where $\{u, v\} \in E_G \leftrightarrow \{f(u), f(v)\} \in E_H$
Graphs G, H are **isomorphic** if an isomorphism exists.



• All vertices in a K_4 has degree 3,
• ∴ Any vertex with degree ≤ 2 cannot participate in a K_4 subgraph

how to prove?

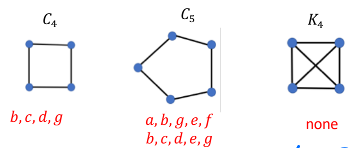
• ∴

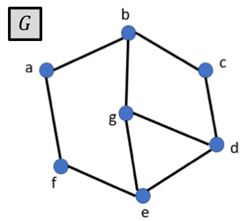


Subgraphs

(12) $H = (V_H, E_H)$ is a **subgraph** of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

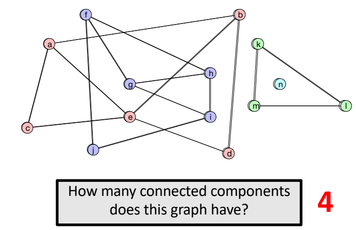
In graph G on the right, find all instances (if any) of each as a subgraph:





Paths and Connectivity

(13)
- A **path** in a graph is a contiguous sequence of nodes and edges.
- (14) A path is **simple** if it does not repeat nodes.
- The **endpoints** of a path are its first and last node (as an ordered pair). (15)
- (16) Two nodes s, t in a graph are **connected** if there exists a path with endpoints (s, t).
- (17) A graph is **connected** if all pairs of nodes are connected.
- (18) The **connected components** of a graph G are its maximal connected subgraphs.



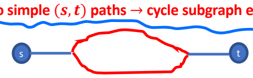
* (16): one-node path is valid! ex. • f
(f, f) are connected.

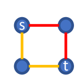
Trees

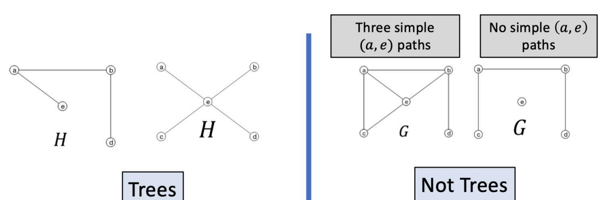
(19)
- A **tree** is a graph that is 1. connected and that 2. does not have a **cycle subgraph**.
- (17) iff
- A **tree** is a graph that has **exactly one** simple path between each pair of vertices.

(20) **Theorem:** These two definitions of trees are **equivalent**.

Assume that G is connected and has no cycle subgraph. Let (s, t) be an arbitrary node pair. G has **at least one** simple (s, t) path because: **G is connected** G has **at most one** simple (s, t) path because: **Two simple (s, t) paths → cycle subgraph exists**	**(Contrapositive):** Assume that G is **not** connected or has a cycle subgraph. **Case 1: G is not connected** There is a pair (s, t) with **zero** simple (s, t) paths because: **Otherwise, G would be connected.** **Case 2: G has a cycle subgraph** There is a pair (s, t) with **two (or more)** simple (s, t) paths because: **Let s, t be nodes on the cycle.** **Either direction around the cycle gives a simple (s, t) path.**

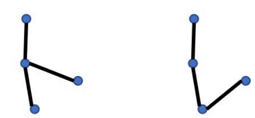






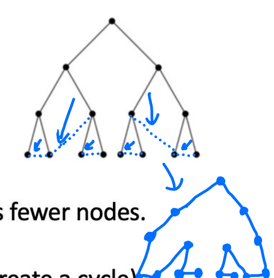
Edges in a Tree

(21)
Theorem: Any tree T on n nodes has exactly n-1 edges.



Proof: We will use induction on the number of nodes.
Base Case: n=1. A tree with 1 node has 0 edges.

Inductive Step (Sketch):
(22)
Claim: T should have some node v of degree 1 (this called a "leaf")
Hint: If all nodes had degree ≥ 2, there would be a cycle. Why?



Consider the graph T' = T-{v} (i.e. v and incident edge removed). T' has fewer nodes.
1) Is it still connected? Why? (recall v is a leaf)
2) Can it have cycles? Why? (deleting vertices and edges cannot create a cycle)

By inductive hypothesis, T' has n-2 edges (as it has n-1 nodes). So T has n-1 edges.

(8) degree : ex

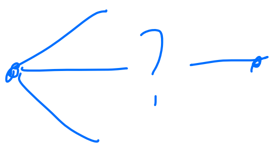
Claim: There is no graph with 5 vertices of degrees 1,1,2,3,4.

Proof:
- By the handshake theorem, the sum of degrees in any graph $G = (V, E)$ is $2|E|$, which is **even**.
- But the sum of these degrees is $1 + 1 + 2 + 3 + 4 = 11$, which is **odd**.
- So no graph with these degrees may exist.

The condition "Sum of all degrees is even" is necessary for a graph to exist, but it is not sufficient.

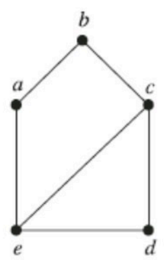
E.g. graphs with these degree sequence cannot exist (even though sum of degrees is even)

1. (3,1)
2. (3,2,1,0)
3. (4,3,1,1,1)

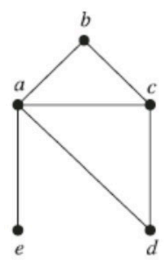


(10) isomorphism : ex

1°



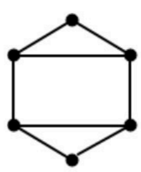
G



H

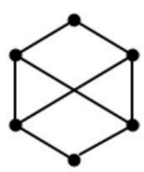
- H has a node of degree 1, but G does not.
- Therefore G, H are not isomorphic.

2°



E

Has 1 4-cycle

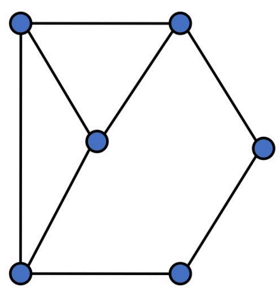


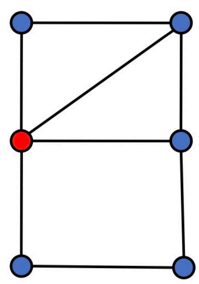
F

Has more than 1 4-cycle

3°

All vertices have degree ≤ 3

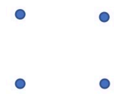




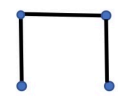
Has a degree 4 node

(19) Tree Supplement

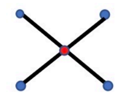
Bonus: Steiner Tree Puzzle



Given 4 cities at corners of a square (say distance 1)
Find the shortest way to connect them all

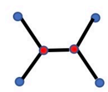


Direct Edges. Best we can do is 3.



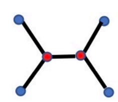
Better Solution: Put a node at center
Length = $2\sqrt{2}$ = 2.828… (Pythagoras)

Can we do better?
Perhaps have two intermediate nodes?
Yes!
Cost = $1 + \sqrt{3}$ = 2.732…
(trigonometry)



All internal angles 120 degrees

Can we do even better?
Perhaps with three or more intermediate nodes?



All internal angles 120 degrees
Cost = 2.732…

Can we do even better with ≥ 3 intermediate (red) nodes?
These red nodes are called Steiner nodes.

Theorem: Three or more Steiner nodes do not help!
Proof: Some starting observations.
1) Can assume graph is a tree. Why?
2) Each Steiner node has degree >= 3. Why?

Suppose the optimal solution uses **k Steiner nodes.**
By 1) k+4 nodes and it's a tree, we have $|E|$ = k+3 edges.
By 2) Sum of degrees $\geq$ 4 + 3k
So, By handshake thm, $2|E| = 2(k+3) \geq 4 + 3k$. Rearranging gives $k \leq 2$.