EECS 203: Discrete Mathematics
Fall 2023
Discussion 5 Notes

1 Definitions

- **Mathematical Induction**:
- **Induction Steps**:
  - **Base Case**:
  - **Inductive Hypothesis**:
  - **Inductive Step**:

Solution:

- **Mathematical Induction**: Mathematical Induction is a proof method used to prove a predicate $P(n)$ holds for "all" $n \geq n_0$. Often "all" n is $\mathbb{N}$ or $\mathbb{Z}^+$, but the desired domain of n varies by problem. Mathematical induction consists of a base case and an inductive step, which proves: $[P(n_0) \wedge \forall k \geq n_0(P(k) \implies P(k+1))] \implies \forall n \geq n_0, P(n)$
- **Base Case**: The part of the inductive proof which directly proves the predicate for the *first* value in the domain (generally n_0). The base case does not rely on $P(k)$ for any other value of k. Often this will be $P(0)$ or $P(1)$
- **Inductive Hypothesis**: The assumption we make at the beginning of the inductive step. The inductive hypothesis assumes that the predicate holds for some *arbitrary* member of the domain
- **Inductive Step**: The proof which shows that the predicate holds for the "next" value in the domain. The inductive step should make use of the inductive hypothesis.

2 Exercises

1. Mathematical Induction ⋆

Prove by mathematical induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Solution:

Inductive step:
Let k be an arbitrary positive integer. Assume $P(k)$: 3 divides $k^3 + 2k$. We want to show $P(k+1)$: 3 divides $(k+1)^3 + 2(k+1)$.

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3k^2 + 3k + 3 \\ &= (k^3 + 2k) + 3(k^2 + k + 1)\end{aligned}$$

3 divides $3(k^2 + k + 1)$. By the inductive hypothesis, 3 divides $k^3 + 2k$. Thus, 3 divides $(k+1)^3 + 2(k+1)$, so $P(k+1)$ is true.

Base case:
Prove $P(1)$: 3 divides $1^3 + 2 \cdot 1$. $1 + 2 = 3$. Since 3 is divisible by 3, $P(1)$ is true.

By mathematical induction, we have proven that for every positive integer n, 3 divides $n^3 + 2n$.

Alternate Solution for Inductive Step: Let k be an arbitrary positive integer. Assume $P(k)$: 3 divides $k^3 + 2k$. We want to show $P(k+1)$: 3 divides $(k+1)^3 + 2(k+1)$.

$$\begin{aligned}3|k^3 + 2k &\to 3|k^3 + 2k + 3(k^2 + k + 1) \\ &\to 3|k^3 + 2k + 3k^2 + 3k + 3 \\ &\to 3|k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &\to 3|(k+1)^3 + 2(k+1)\end{aligned}$$

2. Bandar's Blunder ⋆

Bandar writes a proof for the following statement:

$$n! > n^2 \text{ for all } n \geq 4.$$

His proof is incorrect, and it's your task to help him identify his mistake!

Proof:

Inductive step:
Let k be arbitrary. Assume $P(k) : k! > k^2$. We need to show $P(k+1) : (k+1)! > (k+1)^2$

$$
\begin{aligned}
(k+1)! &= (k+1)\cdot k! \\
&> (k+1)\cdot k^2 && \text{(By the Inductive Hypothesis)} \\
&= (k+1)(k\cdot k) \\
&\geq (k+1)(2\cdot k) && \text{(Because } k \geq 2\text{)} \\
&= (k+1)(k+k) \\
&\geq (k+1)(k+1) && \text{(Because } k \geq 1\text{)} \\
&= (k+1)^2
\end{aligned}
$$

This proves $(k+1)! > (k+1)^2$.

Base Case:
Prove $P(0) : 0! > 0^2$, $0! = 1 > 0^2 = 0$

Thus by mathematical induction, $n! > n^2$ for all $n \geq 0$.

What is wrong with Bandar's proof?

Solution: The key idea here is that although we have a valid base case, and a valid inductive step, they don't work together. In particular, the inductive step requires $k \geq 4$, but our base case only shows that $k = 0$ is valid (and in fact, $k = 1, k = 2,$ and $k = 3$ are false). A valid proof could have used the same inductive step with a base case of $n = 4$.

Some possible explanations:

- The base case and inductive step are individually valid, but the base case can't be used with the inductive step.
- The base case doesn't help prove the statement is true for $n = 4$, and this case can't be proved with the inductive step.
- The inductive step doesn't work with the given base case.

3. Sum Mathematical Induction

Using induction, prove that for all integers $n \geq 1$:

$$\sum_{r=1}^{n}(r+1)\cdot 2^{r-1} = n\cdot 2^n$$

Solution:

Inductive Step:
Let k be an arbitrary integer that is greater or equal to 1.

Assume $P(k) : \sum_{r=1}^{k}(r+1)\cdot 2^{r-1} = k\cdot 2^k$.

We want to show $P(k+1) : \sum_{r=1}^{k+1}(r+1)\cdot 2^{r-1} = (k+1)\cdot 2^{k+1}$

$$\begin{aligned}
&\sum_{r=1}^{k+1}(r+1)\cdot 2^{r-1} \\
&= [\sum_{r=1}^{k}(r+1)\cdot 2^{r-1}] + (k+1+1)\cdot 2^{k+1-1} \\
&= [\sum_{r=1}^{k}(r+1)\cdot 2^{r-1}] + (k+2)\cdot 2^{k} \\
&= [k\cdot 2^k] + (k+2)\cdot 2^k \text{ (by Inductive Hypothesis)} \\
&= k\cdot 2^k + k2^k + 2^{k+1} \\
&= 2k\cdot 2^k + 2^{k+1} \\
&= k\cdot 2^{k+1} + (1)\cdot 2^{k+1} \\
&= (k+1)\cdot 2^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true.

Base Case:
Prove $P(1) : \sum_{r=1}^{1}(r+1)\cdot 2^{r-1} = 1\cdot 2^1$. $LHS = (1+1)\cdot(2)^0 = 2, RHS = (1)\cdot(2)^1 = 2$, so $LHS = RHS$. Therefore, $P(1)$ is true.

Therefore we have shown by mathematical induction that for all integers $n \geq 1$, $\sum_{r=1}^{n}(r+1)\cdot 2^{r-1} = n\cdot 2^n$

4. REVIEW: Satisfiability $\star$

Determine whether each of these compound propositions is satisfiable.

(a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

(b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$

Solution:

(a) Satisfiable. The expression is satisfied when p is False and q is False. You could draw up a truth table to help you think through the possible combinations of truth values for p and q.

(b) Unsatisfiable (ie a contradiction)

p	q	$p \rightarrow q$	$p \rightarrow \neg q$	$\neg p \rightarrow q$	$\neg p \rightarrow \neg q$	$(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$
T	T	T	F	T	T	F
T	F	F	T	T	T	F
F	T	T	T	T	F	F
F	F	T	T	F	T	F

Since all boolean assignments of p and q result in the expression being False, this is compound proposition is unsatisfiable.

Alternate Solutions:

- Using Equivalence Laws:

$$\begin{aligned}
&(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q) \\
&\qquad \equiv (\neg p \vee q) \wedge (\neg p \vee \neg q) \wedge (p \vee q) \wedge (p \vee \neg q) \\
&\qquad \equiv (\neg p \vee (q \wedge \neg q)) \wedge (p \vee q) \wedge (p \vee \neg q) \\
&\qquad \equiv \neg p \wedge (p \vee q) \wedge (p \vee \neg q) \\
&\qquad \equiv \neg p \wedge (p \vee (q \wedge \neg q)) \\
&\qquad = \neg p \wedge p \\
&\qquad = F
\end{aligned}$$

- Verbal Argument: In order to show that this statement is not satisfiable, we will consider every possible assignment of p and q and show that in every case, the statement is false. When p is true and q is true, $p \rightarrow \neg q$ is false so the whole statement is false. When p is true and q is false, $p \rightarrow q$ is false, so the whole statement is false. When p is false and q is true, $\neg p \rightarrow \neg q$ is false, so the whole statement is false. When p is false and q is false, $\neg p \rightarrow q$ is false, so the whole statement is false. Therefore, in every possible assignment of p and q, the statement is false, which means that the statement is not satisfiable.

5. REVIEW: Nested Quantifier Translations

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

Solution:

a) There is a student in your class who has taken a computer science course [at your school].

b) There is a student in your class who has taken every computer science course.

c) Every student in your class has taken at least one computer science course.

d) There is a computer science course that every student in your class has taken.

e) Every computer science course has been taken by at least one student in your class.

f) Every student in your class has taken every computer science course.

6. REVIEW: Direct Proof

Use a direct proof to show that the product of two odd numbers is odd.

Solution: Using a Direct Proof,
Let a and b be arbitrary odd integers. Then, a and b can be written as $a = 2m + 1$ and $b = 2n + 1$ for some integers n and m. Looking at their product, we have

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

Since $ab = 2k + 1$, where k is the integer $2mn + m + n$, then by definition ab is odd.

7. REVIEW: Proof by Contradiction ⋆

Prove that for all integers n, if $n^2 + 2$ is even, then n is even using a proof by contradiction.

Solution: Let n is an arbitrary integer. For the sake of contradiction, assume $n^2 + 2$ is even and n is odd.
(Note that we could have also assumed the negation of the entire statement: "Assume that there exists some n such that $n^2 + 2$ is even and n is odd".)

- Since n is odd, we can say $n = 2k + 1$ for some integer k.
- This means $n^2 + 2 = (2k + 1)^2 + 2$.
$= 4k^2 + 4k + 1 + 2$
$= 2(2k^2 + 2k + 1) + 1$
$= 2j + 1$, where j is an integer equal to $2k^2 + 2k + 1$
- Thus from the definition of an odd number, $n^2 + 2$ is odd. This contradicts our earlier assumption that $n^2 + 2$ is even.

Therefore, using proof by contradiction, we have showed that for all integers n, if n is odd, then $n^2 + 2$ is odd.

8. REVIEW: Proof by Contrapositive ⋆

Prove that for all integers x and y, if xy^2 is even, then x is even or y is even.

Solution:
We will prove the statement via proof by contrapositive. Let x and y be arbitary integers. Because we are using proof by contrapositive, we want to assume x is odd and y is odd and eventually conclude that xy^2 is odd. First, we will assume x is odd and y is odd. Since x and y are odd, $x = 2k+1$ and $y = 2n+1$ where k and n are integers. Therefore, $xy^2 = (2k+1)(2n+1)^2 = (2k+1)(4n^2+4n+1) = 8kn^2+8kn+2k+4n^2+4n+1 = 2(4kn^2+4kn+k+2n^2+2n)+1 = 2j+1$ where j is an integer and $j = 4kn^2+4kn+k+2n^2+2n$. Therefore, xy^2 is odd. Thus, we have shown via proof by contrapositive that for all integers x and y, if xy^2 is even, then x is even or y is even.

9. REVIEW: Proof by Cases/Disproofs ⋆

a) Prove or Disprove that for all integers n, $n^2 + n$ is even

b) Prove or Disprove that for all integers a and b, $\frac{a}{b}$ is a rational number.

Solution:

a) We prove the statement via proof by cases. Let x be an arbitrary integer.

- **Case 1:** x is even
  Since x is even, $x = 2k$ where k is an integer. Therefore, $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.
- **Case 2:** x is odd
  Since x is odd, $x = 2k+1$ where k is an integer. Therefore, $x^2 + x = (2k+1)^2 + (2k+1) = (4k^2+4k+1) + (2k+1) = 4k^2+6k+2 = 2(2k^2+3k+1) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.

For all cases of x, we have shown that $x^2 + x$ is even. Therefore, we have shown that for all integers n, $n^2 + n$ is even.

b) We will disprove this statement. Consider the case, $a = 1$ and $b = 0$. In this case, $\frac{a}{b}$ is not a rational number because $b = 0$.