EECS 203: Discrete Mathematics
Fall 2023
Discussion 3 Notes

1 Definitions

- **Argument:**
- **Valid Argument/Proof:**
- **Even:**
- **Odd:**
- **Integer:**
- **Rational Numbers:**
- **Divisibility:**
- **Prime Numbers:**
- **Composite Numbers:**
- **Proof:**
- **Types of Proofs:**
  - **Direct Proof:**
  - **Proof by Contraposition:**
- **Disproof:**
- **Without Loss of Generality (WLOG):**
- **Unique:**

Solution:

- **Argument:** An **argument** for a statement S is a sequence of statements ending with S. S is called the **conclusion**. An argument starts with some beginning statements you assume are true, called the **premises**.

- **Valid Argument/Proof:** An argument is **valid** if every statement after the premises is implied ($\rightarrow$) by the some combination of the statements before it. Whenever the premises are true, the conclusion must be true.
- **Even:** An integer x is even if and only if there exists an integer k such that $x = 2k$.
- **Odd:** An integer x is odd if and only if there exists an integer k such that $x = 2k+1$
- **Integer:** a positive or negative whole number (including 0)
- **Rational Numbers:** A number is considered rational iff it can be written as the ratio of two integers: $\frac{p}{q}$.
- **Divisibility:** $n|a$ means "n divides a", or equivalently, "a is divisible by n", also equivalently

  Let n and a be integers.
  $n|a$ iff $\exists k\ [nk = a]$ where k is an integer
- **Prime Numbers** A prime number is a number greater than 1 whose only factors are 1 and itself.
- **Composite Numbers:** A composite number is a number which has at least one factor other than 1 and itself (ie not a prime number). Note that 1 is neither prime nor composite.
- **Proof:** A **proof**/argument for a statement S is a sequence of statements ending with S. S is called the **conclusion**. A proof starts with some beginning statements you assume are true, called the **premises**.
- **Types of Proofs:**
  - **Direct Proof:** Prove that a statement is true without using any more advanced proof techniques (e.g. contrapositive, contradiction, cases).
  - **Direct Proof for $p \rightarrow q$:** Prove that if the proposition p is true, then the other proposition q is true "directly". Start by assuming that p is true, then make some deductions and eventually arrive at the conclusion that q must be true.
  $$p \rightarrow q$$
  - **Proof by Contraposition:** Prove that "if p is true, then q is true" by proving that if q is false, then p is false (since these are logically equivalent).
  $$\neg q \rightarrow \neg p$$

- **Disproof:** To disprove a statement means to prove the negation of that statement.

$$\text{Disprove } P(x) \equiv \text{Prove } \neg P(x)$$

  Note that if the statement you are trying to disprove is a for all statement, all you need to disprove it is a singular counterexample (since $\neg\forall x P(x) \equiv \exists x \neg P(x)$).

- **Without loss of generality (WLOG):** used when the same argument can be made for multiple cases, and there is some symmetry between variables.

- **Unique:** (exactly one) If we say something has a unique solution, we mean that there is a solution and that there is no other solution

2 Exercises

1. Odd Proof

Prove or disprove: The sum of an even and an odd integer is always odd.

Solution: We will prove this statement.
Without loss of generality (WLOG), let x be an **arbitrary** even integer and y be an **arbitrary** odd integer. By definition, then, x and y can be written as $x = 2n$ and $y = 2m + 1$ for some integers n and m. Looking at their sum, we have

$$\begin{aligned} x + y &= 2n + 2m + 1 \\ &= 2(n + m) + 1. \end{aligned}$$

Since $x + y = 2c + 1$, where c is the integer $n + m$, then by definition, $x + y$ is odd. Therefore, this relation holds for all even x and odd y, and we have proved that the sum of an even and an odd integer is odd.

2. Even Proof

Prove (using a direct proof) that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Solution: Using a Direct Proof,

- Let $m + n$ and $n + p$ be **arbitrary** even integers, $m + n = 2a$ and $n + p = 2b$, for some integers a and b.
- $m + p = (m + n) + (n + p) - 2n = 2a + 2b - 2n = 2(a + b - n)$
- Since $a + b - n$ is an integer, let it be k.
- Hence, $m + p = 2k =$ even integer

Therefore, the statement "if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even" holds for all even $m + n$ and $n + p$.

3. Disproofs; Two Sides of the Same Coin

a. **Disprove:** For all real numbers x and y, if they sum to zero, one of them is negative and the other is positive.

b. **Disprove:** For all nonzero rational numbers x and y, if they are multiplicative inverses, $x \neq y$.
Note: Two numbers are multiplicative inverses if their product is 1.

Solution: With 'for all' disproofs, we need to find a counterexample (some values of x and y that make this statement false).

a. Consider $x = 0$ and $y = 0$.
$x + y = 0$, and since 0 is neither negative nor positive, this if-then statement false.

Therefore, it is not true for all real numbers x and y, that if they sum to zero, one of them is negative and the other is positive.

b. Consider $x = 1$ and $y = 1$.
$x \cdot y = 1$, and $x = y$, so this if-then statement is false.

Therefore, it is not true for all nonzero rational numbers x and y that if they are multiplicative inverses, $x \neq y$.

4. Quantifier Proofs

For each part, translate the statement into logical connectives and math symbols. Then prove or disprove it.

a. Each non-zero rational number has a multiplicative inverse (also a rational number) such that their product is 1.

b. Each non-zero integer has a multiplicative inverse that is also an integer.
Note: Two numbers are multiplicative inverses if their product is 1.

Solution:

a. Let x and y come from the domain of all non-zero rational numbers.

$$\forall x \exists y [xy = 1]$$

Prove:

- Take an arbitrary non-zero rational number x.
- By definition of rational numbers, $x = \dfrac{p}{q}$ for some integers p and q where $q \neq 0$.
- Since $x \neq 0$, $p \neq 0$.
- Let $y = \dfrac{q}{p}$. (We can do this since $p \neq 0$)
- y is a rational number by definition
- Since $q \neq 0$, $y \neq 0$.
- $xy = \dfrac{p}{q} \cdot \dfrac{q}{p} = 1$

Thus, for all non-zero rational numbers x there exists an inverse rational number y such that $xy = 1$.

b. Let x and y come from the domain of non-zero integers.

$$\forall x \exists y [xy = 1]$$

Disprove:
Consider $x = 2$ (an element of the nonzero integers). Its (only) multiplicative inverse is $\frac{1}{2}$, but $\frac{1}{2}$ is not an integer. Thus, this for all statement is false (since to be true it would need to be true for every nonzero integer).

5. Proof by Contrapositive I

Prove for all real numbers that if $n^2 + 2$ is irrational, then n is irrational using a proof by contrapositive.

Solution: We will prove the contrapositive, that is: If n is rational, then $n^2 + 2$ is rational.

- Assume n is rational. Then we can write it as $n = \frac{a}{b}$ for arbitrary integers a and b, with $b \neq 0$.
- This means $n^2 + 2 = (\frac{a}{b})^2 + 2$.
  $= \frac{a^2}{b^2} + 2$
  $= \frac{a^2+2b^2}{b^2}$
- Since the integers are closed on addition and multiplication, we can define $c = a^2 + 2b^2$ and $d = b^2$ (and since $b \neq 0$, $b^2 \neq 0$, and therefore $d \neq 0$).
- Therefore, we can say $n^2 + 2 = \frac{c}{d}$, where c and d are integers, and $d \neq 0$.
- Thus from the definition of a rational number, $n^2 + 2$ is rational.

Therefore, by contrapositive we can say that for all real numbers, if $n^2 + 2$ is irrational, then n is irrational.

6. Proof by Contrapositive II

Prove or disprove the following statement: for any two integers a, b, if their product ab is even then either a is even or b is even.

Solution: We will prove this statement.

Proof by Contraposition:
Let a and b be odd.
Then $a = 2k + 1$ and $b = 2c + 1$ for some integers k and c.
$ab = (2k + 1)(2c + 1) = 4kc + 2k + 2c + 1 = 2(2kc + k + c) + 1$
So, ab is odd.
Thus, by contraposition, if ab is even, then a is even or b is even.

Note: this problem can also be solved using a proof by contradiction, which we will talk about next week!