Exam 3 Solutions
EECS 203, Fall 2023

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

8-Digit UMID: ______________________________

Instructions

- When you receive this packet, fill in your name, Uniqname, and UMID above.
- Once the exam begins, make sure you have problems 1-16 in this booklet.
- Write your UMID in the blank at the top of every other page.
- No one may leave within the last 10 minutes of the exam.
- After you complete the exam, sign the Honor Code below. If you finish when time is called, your proctor will give you time to sign the Honor Code.
- Do not detach the scratch paper at the end of the packet.
- Do not discuss the exam until solutions have been released!

Materials

- No electronics allowed, including calculators.
- You may use one 8.5″ by 11″ note sheet, front and back, created by you.
- You may not use any other sources of information.

Honor Code

This exam is administered under the College of Engineering Honor Code. Your signature endorses the pledge below. We will not grade your exam without your signature.

I have neither given nor received unauthorized aid on this examination, nor have I concealed any violations of the Honor Code. I further agree not to discuss any aspect of this examination in any way, shape, or form until the solutions have been published.

Signature: ______________________________

This is a space for scratch work. DO NOT DETACH THIS PAPER FROM YOUR EXAM.

Part A: Single Answer Multiple Choice

Instructions

- There are 6 questions in this section.
- Shade the bubble you believe to be correct.
- There is only one correct answer option for each question in this section. If you shade more than one bubble, your answer will be marked as incorrect.

Example.	ⓐ ⓑ ● ⓓ ⓔ

Make sure to SHADE A BUBBLE next to the question title, as shown above.

Problem 1. (4 points)	ⓐ ⓑ ⓒ ⓓ ⓔ

Emily is throwing three distinguishable balls into three distinguishable buckets. How many ways can she put all the balls in the buckets such that at least one bucket is empty?

(a) 3

(b) 6

(c) 12

(d) 21

(e) 27

Solution

(d)

There are two cases: you can either put all 3 balls in the same bucket, or put 2 balls in one bucket and the remaining ball in another bucket.
Case 1: put all 3 balls in the same bucket
You just need to choose which of the 3 distinguishable buckets to put the balls in, so there are $\binom{3}{1}$ ways to put all 3 balls in the same bucket.
Case 2: put 2 balls in one bucket and the remaining ball in another bucket
First, choose the 2 balls that will be put together in the same bucket: $\binom{3}{2}$.
Next, choose the 2 buckets that will contain at least one ball: $\binom{3}{2}$.
Finally, pick the bucket out of these 2 that contains the 2 balls you chose earlier.
There are $\binom{3}{2}\binom{3}{2} \cdot 2$ ways to put 2 balls in one bucket and the remaining ball in another bucket.

In total, there are $21 = \binom{3}{1} + \binom{3}{2}\binom{3}{2} \cdot 2$ ways to put all the balls in the buckets such that at least one bucket is empty.

Alternate Solution:
Start by counting all ways to place them regardless of whether a bucket is empty. Each ball has 3 options, so we have $3^3 = 3 \cdot 3 \cdot 3 = 27$ ways to place them. This overcounts by all the ways we put them in buckets such that they are all full. This can only be done with each bucket having 1 ball in it. There are $3! = 6$ ways to rearrange which buckets they go in. This makes $27 - 6 = 21$ ways that are valid.

Problem 2. (4 points)

a b c d e

Suppose we color each of the vertices of C_4 red or blue at random (equally likely). What is the probability that no edge connects two vertices of the same color?

(a) 1/16

(b) 1/8

(c) 1/4

(d) 1/3

(e) 1/2

Solution

(b),
With C_4, there are only two ways we can color the vertices such that no two adjacent vertices are the same color. Starting from the top-left, colors alternating like Red-Blue-Red-Blue or the other way around, with Blue-Red-Blue-Red. Thus the size of the event space is 2.

The size of the sample space is the total number of ways one can color the graph using 2 colors. Each vertex will have 2 options - Red or Blue. Thus, the total number of combinations would be

$$2 \times 2 \times 2 \times 2 = 2^4 = 16$$

Putting this together, we have that the probability $p = \frac{|E|}{|S|} = \frac{2}{16} = \frac{1}{8}$

Alternate solution:
Fix one vertex as the starter vertex. In any coloring, it must have a color. For that color of that vertex, there are $2^3 = 8$ ways we could color the other vertices, only 1 of which has no two adjacent vertices the same color. Again, this gives $p = \frac{|E|}{|S|} = \frac{1}{8}$.

Problem 3. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

What is the Big-Θ bound on the runtime of this algorithm?

```
procedure POKER(n)
    for i := 1 to n do
        j ← 1
        while j < n do
            j ← j * 2
            print 'All in'
        end while
    end for
    for i := 1 to n do
        print 'Doubled up'
    end for
end procedure
```

(a) $\Theta(n^3)$

(b) $\Theta(n^2 \log(n))$

(c) $\Theta(n^2)$

(d) $\Theta(n \log(n))$

(e) $\Theta(n)$

Solution

There are two main loops to examine in this algorithm. Looking at the inside of the first one first: we see that j starts at 1 and doubles until it is no longer less than n, at which point this terminates. This is repeated n times overall. The runtime of the inner loop is thus $\log n$ for an overall runtime of $n \log n$ of the first nested loop. As the second loop is only n runtime, we can absorb it into the first loop and the runtime of the algorithm is $\Theta(n \log n + n) = \Theta(n \log n)$

Problem 4. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

Mariko likes to study at a local cafe. She noticed the following about her caffeine and study habits:

- When Mariko drinks coffee, the probability that she finishes her homework is 3/4.
- When she finishes her homework, the probability that she drank coffee is 3/7.
- She drinks coffee 2/7 of the time.

How often does she finish her homework?

(a) 1/4

(b) 3/7

(c) 1/2

(d) 3/4

(e) 3/14

Solution

(c)
Let H = Mariko finishes her homework, and let C = Mariko drinks coffee.
$P(H|C) = 3/4, P(C|H) = 3/7, P(C) = 2/7$.

Solution 1:
These terms are 3 of the 4 terms in Bayes Theorem when finding $P(H|C)$:

$$P(H|C) = \frac{P(C|H)P(H)}{P(C)}$$
$$\frac{3}{4} = \frac{3/7 \cdot P(H)}{2/7}$$

Solving for $P(H)$, we get $P(H) = 1/2$.

Solution 2:
We can use conditional probabilities to compute their intersectional probability. For example, we have $P(H|C) = 3/4$ and $P(C) = 2/7$, so we know $P(H \cap C) = P(H|C)P(C) = 3/4 \cdot 2/7 = 3/14$.
We can also compute this using $P(C|H)$. $P(H \cap C) = P(C|H)P(H)$. We know $P(C|H) = 3/7$, and from above, we know that $P(H \cap C) = 3/14$, so apparently $3/14 = 3/7 \cdot P(H)$. Dividing through by 3/7 yields $P(H) = 1/2$.

Problem 5. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

You're setting up groups for students to work on a homework assignment. How many ways are there to split 7 students in two groups of two and one group of three? You may assume that the students are distinguishable.

(a) $\binom{7}{2}\binom{5}{2}$

(b) $P(7,2) + P(5,2) + P(3,3)$

(c) $\binom{7}{2}\binom{5}{2} \cdot \frac{1}{2}$

(d) $P(7,2)P(5,2)P(3,3)$

(e) $3!2!2!$

Solution

(c) You choose 2 out of the 7 students for your first group of two, and then choose the next 2 out of the remaining 5 students for your second group of two. The 3 students left, form the group of three. However, this way of counting introduces order between the first and the second group (not the third group since its different size makes it distinguishable compared to the groups of 2). Since the two groups are indistinguishable, the order in which the groups are picked shouldn't matter (ie, there should be no difference between the "first" and the "second" group of 2). So we divide by 2!.

Incorrect Options:

(a) This introduces order between the two groups of 2. Example: if you choose students A and B for your first group of 2 followed by students C and D. This is considered different from choosing C and D first followed by A and B. To remove this ordering, the correct solution divides by 2!

(b) This is the same as option d, however, the terms are added instead of multiplied. Adding the terms is further incorrect because adding implies that selecting the three groups are three different cases which are mutually exclusive. Since choosing the three groups is a sequential action, the correct option multiplies the terms.

(d) This option considers the groups distinguishable like in option a, and takes into account the order in which students are selected in each group (since it uses permutation instead of combination).

(e) This option has the students for each group picked already and counts the number of ways to order the students in each group. 3! for the students in the group of 3, 2! for one of the groups of 2 and 2! for the other group of 2.

Problem 6. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

What is the Big-Θ bound on the runtime of this algorithm?

```
procedure AdvancedAlgorithm(n)

    if n ≤ 1 then
        return
    end if

    for i := 1 to 10 do
        AdvancedAlgorithm(n/2)
    end for

    for i := 1 to n do
        for j := 1 to n do
            for k := 1 to n do
                print 'final stretch!'
            end for
        end for
    end for
end procedure
```

(a) $\Theta(n^3)$

(b) $\Theta(\log n)$

(c) $\Theta(n^4)$

(d) $\Theta(n^3 \log n)$

(e) $\Theta(n^{\log_2 10})$

Solution

(e) $\Theta(n^{\log_2 10})$

This algorithm calls itself recursively 10 times, each with size $n/2$. It then prints "final stretch" a bunch of times. We have 3 nested loops each running n cycles, so we have $n \cdot n \cdot n = n^3$ runtime for the loops. This gives a recurrence of $T(n) = 10T(n/2) + n^3$. This is in the form that the Master Theorem applies to. $a = 10$, $b = 2$, and $d = 3$, so $a/b^d = 10/2^3 = 10/8 > 1$. This means the algorithm runs in time $O(n^{\log_b a}) = O(n^{\log_2 10})$

Part B: Multiple Answer Multiple Choice

Instructions

- There are 3 questions in this section.
- Shade whichever boxes you believe are correct. **This could be all answers, no answers, or anything in between.**
- If there are no correct answers, leave all the boxes blank.

Example	a b c d e

Make sure to SHADE 0 OR MORE BOXES next to the question title, as shown above.

Problem 7. (4 points)	a b c d e

Which of the following can you always find if you only know $P(E|F)$ and $P(F)$?

(a) $P(E \cap F)$

(b) $P(\overline{E} \cap F)$

(c) $P(E \cap \overline{F})$

(d) $P(E|\overline{F})$

(e) $P(\overline{E}|F)$

Solution

(a), (b), (e)

(a) $P(E \cap F) = P(E|F)P(F)$

(b) $P(\overline{E} \cap F) = P(\overline{E}|F)P(F)$ (by part e, we can find $P(\overline{E}|F)$)
Alternately: $P(\overline{E} \cap F) = P(F) - P(E \cap F)$ (by part a, we can find $P(E \cap F)$

(c) $P(E \cap \overline{F}) = P(E|\overline{F})P(\overline{F})$. While we can compute $P(\overline{F})$, any value for $P(E|\overline{F})$ would still be consistent with our fixed values. Knowing $P(E)$ would also be sufficient to determine the value

(d) see part c

(e) $P(\overline{E}|F) = 1 - P(E|F)$

Problem 8. (4 points)

a b c d e

A graph G has 10 edges. Each vertex of G has either degree 2 or 3. Which of the following choices are possible values for the total number of vertices in G?

(a) 4

(b) 5

(c) 6

(d) 7

(e) 8

Solution

(d), (e)
By the handshake theorem, with 10 edges in G, we know that the total number of degrees would be 20.

Given that each vertex has degree 2 or 3, with n vertices, the maximum number of degrees this graph can have is $3n$. Thus we can eliminate answer options (a), (b) and (c) as the maximum number of degrees these graphs can have are 12, 15 and 18 respectively.

Looking at the remaining two options, we see that both of these are possible by the following breakdown of degrees,
d) $6 \cdot 3 + 1 \cdot 2$
e) $4 \cdot 3 + 4 \cdot 2$
You can further confirm by drawing out graphs (a 7-cycle with 3 nonadjacent diagonals, and an 8-cycle with 2 nonadjacent diagonals)

Problem 9. (4 points)

a b c d e

Consider positive-valued functions f and g, where:

- f is $O(n^3)$ and $\Omega(n)$
- g is $O(n^2)$ and $\Omega(\log n)$

Which of the following **could** be true?

(a) $f \cdot g$ is $\Theta(n^2)$

(b) $f \cdot g$ is $\Theta(n^5 \log n)$

(c) $f + g$ is $\Theta(n)$

(d) $f + g$ is $\Theta(n^2 \log n)$

(e) $1000f$ is $\Theta(n^3)$

Solution

(a), (c), (d), (e)

A tight-bound estimate must grow no faster than the upper-bound estimate and no slower than the lower-bound estimate. To get the lower-bound and upper-bound estimates for $f \cdot g$, we can multiply their respective Big-Ω and Big-O functions:

- $f \cdot g$ is $\Omega(n \log n)$
- $f \cdot g$ is $O(n^5)$

To get the lower-bound and upper-bound estimates for $f + g$, we can take the maximum their respective Big-Ω and Big-O functions:

- $f + g$ is $\Omega(n)$
- $f + g$ is $O(n^3)$

(a) $f \cdot g$ is $\Theta(n^2)$ could be true since n^2 grows faster than $n \log n$ and slower than n^5.

(b) $f \cdot g$ is $\Theta(n^5 \log n)$ could NOT be true since that would mean that it grows faster than our upper-bound estimate for $f \cdot g$.

(c) $f + g$ is $\Theta(n)$ could be true since n grows at the same rate as n and slower than n^3.

(d) $f + g$ is $\Theta(n^2 \log n)$ could be true since $n^2 \log n$ grows at the same rate as n and slower than n^3.

(e) $1000f$ is $\Theta(f)$, so we have the same bounds on $1000f$ as we did on f. This means $1000f$ could be $\Theta(n^3)$.

Part C: Short Answer

Instructions

- There are 2 questions in this section
- Write your solution in the space provided below the question
- Don't simplify your answers
- Show your work and include justification

Problem 10: Bipartite Distances (7 points)

Let G be an **unweighted bipartite** graph and let s and t be nodes with a distance of 10 between them. Suppose we delete one edge from G. What are all possible new values for the distance between s and t?
Briefly justify your answer, but you do not need to give a proof.

Final Answer:

The distance between s and t could be ______________________________

Justification:

Solution

Any even integer ≥ 10, or ∞.

The new distance between s and t:

could still be 10

If an edge that was not on the shortest path from s to t was deleted (or there are multiple shortest paths and the deleted edge is not on one of them), then the distance would remain 10.

could be infinity

If an edge on the shortest path from s to t was deleted and there was only one path between s and t, then this deletion would disconnect s and t and the distance between them would be ∞. Note that we accepted all answers that indicated in some way that s and t could be disconnected.

must be greater than or equal to 10

If an edge on the shortest path from s to t was deleted and there was more than one path between s and t, then this deletion would make the distance be based on a different path. By definition, the distance between two nodes is the length of their shortest path(s). Because we know that 10 was the length of the (or one of the) shortest path(s) between s and t, the new distance must be greater than or equal to 10.

must be even

As a continuation of the previous case where an edge on the shortest path from s to t was deleted and there was more than one path between s and t, because the graph G is bipartite, the new distance must be even. This can be justified in either of the two following ways:

Justification using partitions:

Because the distance between s and t was even, the two nodes are in the same partition of a bipartite graph and any new distance between them must also be even. (If you're not fully convinced, feel free to read the paragraph under the section "Further explanation on justification using partitions".)

justifications using even cycles: Because G is bipartite, it cannot contain any odd cycles. Since the inital distance was 10, the alternative path must be even as well.

Further justifications using even cycles: In the case where an edge is removed from the original path and there still exists a path from s to t, we know that G must contain a cycle since there was an alternative way to reach the destination. Because G is bipartite, we know that this cycle must be an even length as bipartite graphs cannot have odd cycles.

WLOG. we can break down the initial path into two parts, the parts that are part of a cycle and the parts that isn't. Denote the total distance in the path not part of the cycle as x, then the cycle portion would be $10-x$. Because all cycles in a bipartite graph must be even cycles, denote the total length of the cycles as $2k$. Then We know that the path taken in the new route woukd have distance $x+(2k-(10-x) = 2k+2x-10 = 2(k+x+5)$, thus even.

Correct justifications following either lines of thought were given full credit.

Further explanation on justification using partitions

If you'd like further convincing for the justification using partitions, read on but we did **not** require an explanation as extensive as the following paragraph to receive full credit:

By the definition of bipartite, we can partition the vertices of G into 2 sets, call them A and B, such that all edges go between the sets. Because s and t have an even distance of 10 between them, they must be in the same partition. (To see this, we can consider WLOG that s is in set A. Then, every two edges along the length 10 path to t would first visit a vertex in set B and then return to a vertex in set A. Because 10 is a multiple of 2, the final edge taken to reach t would end at set A.) Because s and t are in the same partition of a bipartite graph, the length of any path between s and t must be even, and the resulting distance must also be even.

Grading Guidelines [7 points]

- +1 Includes 10 as a possibility
- +1 Recognizes distances must be ≥ 10
- +0.5 Correctly justifies that distances must be ≥ 10
- +2 Recognizes distances must be even
- +1 Correctly justifies that distances must be even
- +1 Recognizes that ∞ is possible and/or indicates that s and t can be disconnected
- +0.5 Correctly justifies that ∞ is a possible distance
- -0.2 Incorrectly states that the distance between s and t must be greater than 10 when an edge on the shortest path between the nodes is removed

Common Mistakes:

- Stating that in any graph G, if you remove an edge on a path of length 10 from s to t, the resulting distance between s and t will be ∞. This is incorrect because even if one path no longer exists, there could still be other paths connecting s and t. This often came up in justifying that ∞ is a possible distance, and such submissions did not receive the 0.5 points for correctly justifying that ∞ is a possible distance.
- Stating that if an edge on the shortest path between s and t is deleted, the next shortest path must have a distance greater than 10. This is not true because if there was another path of the same length 10, the resulting distance between the two nodes could still be 10. The correct statement is that if an edge on the shortest path between s and t is deleted, the resulting distance between the two nodes must be greater than or equal to 10. Such submissions were penalized by -0.2 points.
- Providing insufficient justification when showing that the new distance must be even. It was not sufficient to state that the new distance must be even because graph G is bipartite. Without any explanation following the partition or cycle arguments as detailed above, a submission only received +0.5/1 points for the rubric item for "Correctly justifies that distances must be even".
- Showing that even distances are possible but not showing that odd distances aren't possible. Some submissions showed that the new distance could be even but did not explain why odd distances were not possible and/or why only even distances were possible. This was insufficient justification to show that the new distance **must** be even, and such submissions did not receive the rubric item for "Correctly justifies that distances must be even".
- Using the following notation to indicate distances from 10 to infinity: $[10, \infty)$. Using this notation indicates that infinity is not included in the range of values that is being described. This error was not penalized as long as it was made clear

in the justification that it was meant to indicate the case in which s and t were disconnected. If this was not made clear, the submission didn't receive the 1.5 points relating to ∞. Students who instead wrote $[10, \infty]$ were given credit.

Problem 11: King Arthur's Round Tables (7 points)

After experiencing financial trouble and moving to a smaller castle, King Arthur sold the original round table and found two smaller identical round tables that could each seat 50 knights. Find the number of unique seating arrangements for 150 knights if a seating arrangement is considered the same as long as each knight has the same knight seated to his left and the same knight to his right.

Note: Since there are 100 seats, King Arthur won't be able to seat 50 of the knights.

Final Answer: ______________________________

Justification:

Solution

$\frac{P(150,50)\cdot P(100,50)}{2!(50^2)}$

First, order 50 out of the 150 knights in a line and place them in order at one of the tables. There are $P(150, 50)$ possible orderings of these 50 knights.
Next, order 50 out of the 100 remaining knights in a line and place them at the other table. There are $P(100, 50)$ possible orderings of these 50 knights.
It doesn't matter which table the knights are seated at, as long as they have the same knights to their left and right. Divide by 2! to account for indistinguishable tables.
Rotations around each table are also considered the same seating arrangement, so divide by 50^2 (one for each table) to account for circular tables.

Alternate Solutions:
Any solution that is numerically equivalent to the solution above merits full credit. Some examples include:

- $\binom{150}{100}\binom{100}{50}\left(\frac{50!}{50}\right)^2\left(\frac{1}{2!}\right)$

- $\binom{150}{100}\binom{100}{50}\left(\frac{50!}{50}\right)^2\left(\frac{1}{2!}\right)$
- $P(150, 100)/(50^2 \cdot 2)$

Grading Guidelines [7 points]

+1.5 Selects 100 out of the 150 knights
+1 Assigns 50 knights per table
+1.5 Permutes the knights
+1.5 Correctly accounts for circular tables
+1.5 Correctly accounts for indistinguishable tables
-0.5 Extraneous terms in an otherwise correct solution
-0.5 Extraneous terms that are unjustified from above rubric item

Common Mistakes:

- Dividing by 3! to account for the circular tables instead of 50^2. This question is different than the problem from the homework, where in the homework problem, we divided by 3! 4 times to account for each side's 3 people's order not mattering. The reason we're dividing by 50 in this problem is because we have a circular table, and when we say an arrangement is the same when each knight has the same left and right neighbor, there are 50 ways we can rotate all the knights and still have them in the same arrangement, and thus we're overcounting by a factor of 50 for each table, so 50^2 for 2 tables.
- Using sum rule when ordering the knights. Notice that the decisions we're making in this problem are all being done sequentially in the same case, and thus we should be using product rule. Sum rule applies when we have different non-overlapping cases.
- Multiplying by 50! only once. We have two tables, and we need to order our knights on *both* tables.
- Dividing by 50 only once. Again, we have two tables, and need to account for the overcounting for *both* tables.
- Dividing by 50! or 100! to account for circular tables. Consider how we are overcounting. We are overcounting due to the circular nature of the table and how having the same left and right neighbor equates to an equivalent arrangement, thus we overcount by a factor of 50 per table. Overcounting by 50! would mean that the ordering of these knights on the table doesn't matter, which is incorrect.
- Using combinations, but not multiplying by 50!. Knights are distinguishable, so the order in which they sit matters too. After choosing the 50 knights for a table, we must also multiply that by 50! since there are 50! possible orderings of these 50 distinct knights.
- Using the combination formula for permutations, or vice versa. Review the notation for each of these, as they are different and must be used in different situations.

Part D: Free Response

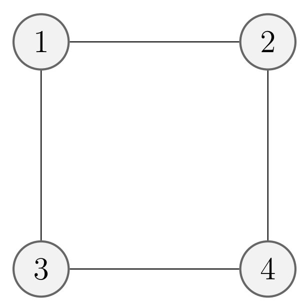
Instructions

- There are 5 questions in this section
- Write your solution in the space provided
- Write down your answer with care: **answers that are unreadable (such as too faint or too messy) will not be graded**
- If you have multiple answers, you must indicate which one you want graded. Otherwise, we will grade your least favorable answer.
- **Show your work** and include justification

Problem 12: A Colorful Question (10 points)

Ian wants to color the vertices of the graph C_4 (depicted below) with one restriction: if two vertices are adjacent, then they must have different colors. Ian has 3 colors to choose from: red, yellow, and blue. How many ways can he color the graph?

Note: The vertices are distinguishable, and Ian does not have to use all three colors.



Remember to justify your answer.

Final Answer: ____________________________

Solution

18

There are many valid solutions, here are a few of them:

Cases by Number of Colors:

Ian must use at least 2 colors, so he can use either 2 or 3 colors.

Case 1: He uses 2 colors. First we choose which of the two colors he uses ($\binom{3}{2}$ options). Then once he chooses a color for vertex 1 (2 options), the colors for the rest of the vertices are determined, so there are 2 possibilities.

Case 2: He uses 3 colors. Then there is one way to choose the three colors: $\binom{3}{3} = 1$.

Case 2a: 2 and 3 have different colors. Choose a color for 1 (3 choices), then for 2 (2 choices), then for 3 (1 choice). Then vertex 4 must have the same color as 1 (1 choice). So we have $3 \cdot 2$ choices.

Case 2b: 2 and 3 have the same color. Choose a color for 1 (3 choices), then for 2 and 3 (2 choices). At this point we've only used 2 colors, so 4 must use the third color so that we use exactly 3 colors (1 choice). So we have $3 \cdot 2$ choices again.

Alternatively we can directly count this case by choosing which color to use twice (3 ways), then there are two choices for which pair of vertices receive this color (1 and 4, or 2 and 3), and then two ways to use the remaining two colors, yielding $3 \cdot 2 \cdot 2$ ways.

So overall our count is $\binom{3}{2} \cdot 2 + \binom{3}{3} \cdot (3 \cdot 2 + 3 \cdot 2) = \binom{3}{2} \cdot 2 + \binom{3}{3} \cdot (3 \cdot 2 \cdot 2) = 18$.

Cases on Vertices 2 and 3:

We can directly compute the number of ways Ian can color the graph using at most 3 colors.

Case 1: 2 and 3 have different colors. First choose a color for vertex 1 (3 choices), then for 2 (2 choices), then 3 (1 choice). Then since vertex 4 can't be the same colors as 2 or 3, we have 1 choice, yielding $3 \cdot 2 \cdot 1^2 = 3 \cdot 2$ choices overall.

Case 2: 2 and 3 have the same color. Then first choose a color for vertex 1 (3 choices), then for 2 and 3 (2 choices). Then there are 2 choices for vertex 4, yielding $3 \cdot 2^2$ choices.

So overall we have $3 \cdot 2 + 3 \cdot 2^2$ choices.

Subtractive Alternate Solution:

If we do not account for the restriction, we can color each vertex 3 ways accounting for the 3 different colors, resulting in 3^4 total colorings. Next, we must subtract the total number of colorings which violate this constraint. We can break these into cases by how many colors are used.

Case 1: We only use one color. All colorings using just one color violate the constraint trivially, and we see there are 3 ways to color the graph if we only use one color.

Case 2: We can color the graph using 2 colors in a way that violates the constraint in two ways:

Case 2a: Two adjacent vertices share the same color, and the remaining two share a different color. There are 2 ways we can group the vertices: either $\{1, 2\}$ are one color and $\{3, 4\}$ are both a different color, or $\{1, 3\}$ are one color and $\{2, 4\}$ are both a different color. We multiply this by the number of ways to choose a color for the first set of vertices, which is 3, and then by the number of ways to choose a color for the second set of vertices, which is 2 because we cannot choose the color of the first set. This totals $2 \cdot 3 \cdot 2$ ways.

Case 2b: Three vertices share the same color, and the remaining vertex is a different color. The number of ways to pick a vertex that is a different color from the rest is 4. The number of ways to pick the color for this vertex is 3, and the number of ways to pick the color of the remaining vertices is 2. This results in $4 \cdot 3 \cdot 2$ ways.

Case 3: Lastly, we must subtract ways to color the graph with 3 different colors in a way that violates the constraint. There are 4 ways to pick a pair of adjacent vertices that will share a color: $\{1, 2\}$, $\{2, 4\}$, $\{4, 3\}$, and $\{3, 1\}$. We multiply this by the number of ways to choose the color of these two vertices, which is 3, and then by the number of ways to color the remaining two vertices, which is 2, for 2 possible permutations of the remaining two colors. This results in $4 \cdot 3 \cdot 2$.

When we subtract the number of ways to color the graph in a way that violates the constraint from the total number of colorings, we get $3^4 - (3 + 2 \cdot 3 \cdot 2 + 4 \cdot 3 \cdot 2 + 4 \cdot 3 \cdot 2)$

<u>Subtractive Alternate Solution 2:</u> We can start at vertex 1 and move around the graph clockwise then subtract the number of invalid colorings counted. There are 3 colors to choose from for vertex 1. We multiply this be the 2 ways we can color vertex 2, because it cannot be the same color as vertex 1. Similarly, vertex 4 can be either color other than what vertex 2 is, so it has 2 options. If we ignore the restriction between vertices 1 and 3, there are also 2 ways to color vertex 3, resulting in $3 \cdot 2 \cdot 2 \cdot 2$ total colorings.

Now we must subtract the number of colorings that violate the restriction between 1 and 3 but satisfy all others. In this case 1 and 3 are the same color but 2 and 4 are each a different color. There are 3 ways to choose a color for 1 and 3. We multiply this by the 2 ways to permute the remaining 2 colors across vertices 2 and 3, resulting in $3 \cdot 2$. This yields a final answer of $3 \cdot 2 \cdot 2 \cdot 2 - 3 \cdot 2$

<u>Cases on which pairs of vertices share a color:</u>
Because there are only 3 colors and 4 vertices, there must always be at least one pair of diagonal vertices which share a color. We can break the problem down into three non-overlapping cases based on which vertices share a color and add them together.

Case 1: Vertices 1 and 4 share a color, and the remaining two are different colors. There are 3 ways to pick a color for 1 and 4, and $2 \cdot 1$ ways to permute the remaining two colors over the remaining two vertices, yielding $3 \cdot 2 \cdot 1$.

Case 2: Vertices 2 and 3 share a color, and the remaining two are different colors. Same as in the last case, there are 3 ways to pick a color for 2 and 3, and $2 \cdot 1$ ways to permute the remaining two colors over the remaining two vertices, yielding $3 \cdot 2 \cdot 1$.

Case 3: Vertices 1 and 4 share a color *and* vertices 2 and 3 share a color. There are 3 ways to pick a color for the first pair, and 2 ways to pick a color for the second pair so that it is different from the color of the first pair. This results in $3 \cdot 2$ ways.

Once we add all of the cases together, there are $3 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1 + 3 \cdot 2$ ways to color it.

Direct Solution:
First we compute the number of ways to assign colors to vertices 1 and 2. There are 3 ways to assign a color to vertex 1, which is multiplied by 2 ways to choose a color for vertex 2, as it cannot be the same color as vertex 1. For the remaining two vertices there are 3 ways to color them:

- Vertex 3 is the same color as vertex 2, and vertex 4 is the same color as vertex 1. In this case, only two colors are used.
- Vertex 3 is the same color as vertex 2, and vertex 4 is the third, previously unused, color. In this case, all three colors are used.
- Vertex 4 is the same color as vertex 1, and vertex 3 is the third, previously unused, color. In this case, all three colors are used.

We multiply by 3 to account for these distinct cases, resulting in $3 \cdot 2 \cdot 3$.

Grading Guidelines [10 points]

Primary rubric (cases)
+1.5 At least 1 correct case
+2 Case breakdown is fully correct
+1.5 At least 1 case is computed correctly
+1.5 At most one case is incorrect
+2 Counts each case/term correctly with justification
+1.5 Correctly combines terms/cases

Partial credit rubric (no cases)
+3 Uses the adjacency restriction to argue that a particular vertex has fewer than 3 possible colors
+3 Mostly correct solution but fails to account for one adjacency restriction

Alternate rubric (brute-force)
+1.5 Mostly correct list (may be missing only a few possibilities)
+1.5 Fully correct list
+3.5 Some identification of cases within the list
+3.5 Complete justification of exhaustiveness

Common Mistakes

- A common mistake was failing to account for the last edge restriction, and simply walking around the cycle (e.g. clockwise) assigning colors. For example, this might result in (3 colors for Node #1) $\times$ (2 colors for Node #2) $\times$ (2 colors for Node #4) $\times$ (2 colors for Node #3) = 24. This correctly accounts for the edges (1, 2), (2, 4), and (4, 3), but does not consider the edge (1, 3), thus overcounting.

  Such solutions generally received 6/10 (second rubric).

- Some solutions correctly counted the cases but then applied the division rule to "account for" different orientations of the cycle. This might be the correct strategy if the vertices were *indistinguishable*, but in this problem we actually do consider different orientations to be distinct, because the vertices are labeled.

  Such solutions received a maximum 8.5/10 (incorrectly combining terms).

- Some students attempted to solve the problem by brute-force enumeration of the possible colorings. These solutions, even when they correctly listed the 18 valid outcomes, generally did not fully justify why this was an exhaustive list.

  Maximum 3/10 for a correct list of outcomes with no justification.

- Many solutions by cases included an extra factor of 2. These correctly counted 3 choices for vertex 1 and 2 choices each for vertices 2 and 3, and then noted that the number of options for vertex 4 depends on the outcomes of the previous choices (whether vertex 2 matches vertex 3). However, students overlooked that once we split in cases by the color of vertex 3, we have to take another look at the previous steps: to arrive at the correct answer, we should no longer multiply by 2 to count the possibilities for vertex 3, because that's already baked into the case breakdown.

  Maximum typically 7/10.

- Some solutions attempted to compensate for overcounting at the end by multiplying their result by 4, 4!, etc. For example, to account for rotations of the cycle or choice of the first node. Since the nodes are distinguishable, this is not necessary.

  Maximum 8.5/10 (incorrectly combines terms).

- By similar reasoning to the previous item, some solutions included the choice of the node to look at first, so multiplied by $\binom{4}{1}$ when counting. This typically leads to overcounting, because the temporal order of the coloring doesn't actually appear in the final outcome.

  Maximum typically 7/10 (incorrect computation of cases).

- It was common to misuse "WLOG," for example to introduce cases. We didn't automatically deduct points for this if the rest of the reasoning was sound.

Problem 13: Parking Tickets (10 points)

Ava drives to North Campus every day. She parks in the Orange Lot if there is space, and in the Blue Lot otherwise. The probability that the Orange Lot is full is always $\frac{1}{4}$. If she parks in the Orange Lot, the probability that she gets a parking ticket is $\frac{1}{9}$, but if she parks in the Blue Lot, the probability that she gets a parking ticket is $\frac{2}{3}$.

Remember to justify your answers.

(a) On any given day, what is the probability that she gets a parking ticket?

Final Answer: _________

(b) Suppose she starts driving to North Campus on day 1. On which day does she expect to get her first parking ticket?

Final Answer: _________

(c) The Fall 2023 semester is 84 days long. What is the expected number of parking tickets that Ava will get throughout the whole semester?

Final Answer: _________

Solution

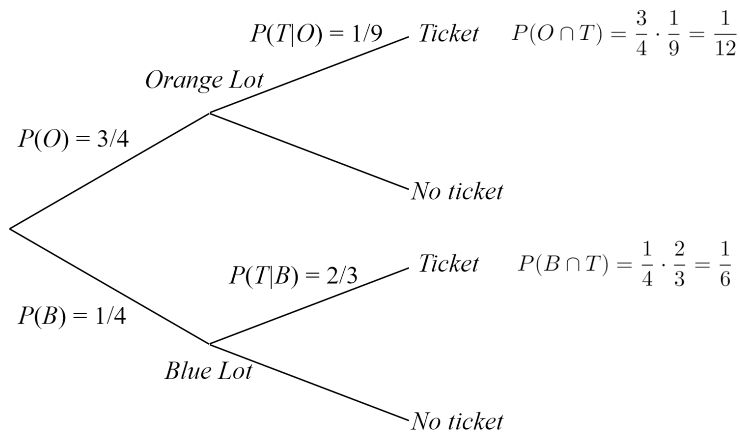
Let B denote that Ava parks in the Blue lot and O denote that Ava parks in the Orange lot. Note that $O = \overline{B}$. Let T denote that Ava gets a parking ticket.
From the problem, we have that:
$P(O) = 1 - \frac{1}{4} = \frac{3}{4}$
$P(B) = \frac{1}{4}$
$P(T|O) = \frac{1}{9}$
$P(T|B) = \frac{2}{3}$

(a) Law of Total Probability
Since parking in the blue and orange lots are disjoint events, we use the law of total probability to find the probability that Ava gets a ticket:
$P(T) = P(T|O)P(O) + P(T|B)P(B) = \frac{1}{9} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{4}$

(b) Geometric Distribution
Based on the probability from (a), we see that the act of Ava getting her first ticket follows a geometric distribution with $p = \frac{1}{4}$. The expected value can then be found by using the geometric distribution formula:
$E(X) = \frac{1}{P(T)} = \frac{1}{\frac{1}{4}} = 4$
Thus, Ava expects to get her first parking ticket on the 4th day.

(c) Binomial Distribution
Now that we have a fixed number of days, the number of tickets Ava receives follows a binomial distribution with $p = \frac{1}{4}$ and $n = 84$. We can then use this distribution to find the expected number of tickets across this time period:
E(number tickets) $= n \cdot P(T) = 84 \cdot \frac{1}{4} = 21$ tickets

Tree method for part (a)
Part (a) can also be solved with the assistance of the tree method. We can interpret the given probabilities with the following tree. Then $P(T) = P(O \cap T) + P(B \cap T) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$.



Grading Guidelines [10 points]

Part (a) : 4 points
+1 Attempts to use law of total probability
+1.5 Correct expression of the law of total probability
+1.5 Plugs in correct values for probabilities
Part (b) : 3 points
+1 Correct answer based on their probability from (a)
+1 Justification: Recognizes geometric distribution
+1 Justification: Applies expected value formula for geometric distribution
Part (c) : 3 points
+1 Correct answer based on their probability from (a)
+1 Justification: Recognizes binomial distribution
+1 Justification: Applies expected value formula for binomial distribution

-0.25 Mistakenly uses $P(O) = \frac{1}{4}$ rather than $P(O) = \frac{3}{4}$
-0.25 Simplification error (any part)

Common Mistakes

- Justifying the answer to part (b) using a binomial distribution. For example, saying that the expected number of parking tickets over 4 days is $4 \cdot P(T) = 4 \cdot \frac{1}{4} = 1$. This is not the same as saying that she expects to get her first parking ticket on the 4th day.
- Misinterpreting "the probability that the Orange Lot is full is always $\frac{1}{4}$" as that the probability of Ava parking in the Orange Lot is $\frac{1}{4}$. If the Orange Lot is full, then Ava does *not* park in the Orange Lot. Thus, the correct probability of Ava parking in the Orange Lot is $1 - \frac{1}{4} = \frac{3}{4}$.

Problem 14: Happy Meals (10 points)

Congratulations! McDonald's is giving you 10 free Happy Meals. In each Happy Meal, one of the six following toys appear with equal probability: Mickey Mouse, Minnie Mouse, Donald Duck, Daisy Duck, Goofy, and Pluto.

Remember to justify your answers.

Note: If you can't solve an earlier part, you can still get credit for later parts by phrasing your answer in terms of the earlier part. For example, in Part (c), you could say "let a be the answer to Part (a) and let b be the answer to Part (b)," and then write your answer to Part (c) in terms of a and/or b.

(a) What is the probability that you get at least one Mickey Mouse toy?

Final Answer: ________________________

(b) Eric, a big Disney fan, wants a Mickey Mouse toy. If you get one, he will pay you $3 for it (he will only buy 1, though). How much do you expect to earn?

Final Answer: ________________________

(c) Now suppose instead that Eric is willing to buy one of each type of toy you have. He will pay you $3 for each distinct toy that appears in your Happy Meals. For example, if you get 4 Donald Ducks, 2 Goofys, and 4 Plutos, he will pay you $9, buying 3 total toys from you. What is the expected value of your earnings?

Final Answer: ________________________

Solution

(a) Let M be a random variable that represents how many Mickey Mouse toys I get. Note that for any given Happy Meal, there is a 5/6 probability of not getting a Mickey Mouse toy. Using the Complement and Product Rules,
$P(M \geq 1) = 1 - P(M = 0) = 1 - (5/6)^{10}$.

(b) Let D be the number of dollars I earn. Therefore, $E(D) = \Sigma_{s \in S}\ p(s) \cdot D(s) = 0 \cdot (5/6)^{10} + 3 \cdot (1 - (5/6)^{10}) = 3(1 - (5/6)^{10})$. This can also be written as $3a$, where a is the answer to Part (a).

(c) Let D be the number of dollars I earn. Number each type of toy 1 through 6. Let D_i be the random variable that represents the number of dollars I earn from toy i. Using Linearity of Expectation and the fact that each toy has an equal probability of being received by me, $E(D) = E(D_1 + D_2 + D_3 + D_4 + D_5 + D_6) = E(D_1) + E(D_2) + E(D_3) + E(D_4) + E(D_5) + E(D_6) = 6E(D_1) = 6(3(1 - (5/6)^{10}))$. This can also be written as $18a$, $6 \cdot 3a$, or $6b$, where a is the answer to Part (a) and b is the answer to Part (b).

Alternate Solution:

We also accepted the $|E|/|S|$ solution for part A, which should have evaluated to $(6^{10} - 5^{10})/6^{10}$. The sample space is 6^{10}, which represents selecting which of the 6 toys you get for each of the 10 toys you are receiving. The event E removes all instances in which there are no Mickeys, or 5^{10} options from the sample space.

Common Mistakes

- Not justifying the use of linearity of expectation in part c.
- Making an argument such as "It is more likely than not that you will get one Mickey Mouse toy. Thus you expect to get a toy, and expect to get three dollars". This possibly intuitive notion of expectation is not the mathematical meaning of expectation. (Students making this mistake often chose 3 as the answer to part b and/or 18 as the answer to part c).
- Part c. Attempting to calculate the expectation directly without using linearity of expectation. This is too unwieldy, because the events of getting at least one of each type of toy are not independent. Fortunately, linearity of expectation holds even for dependent events.
- Attempting to use formulas for binomial or geometric random variables.
- Part a. Computing the probability of getting exactly 1 Mickey Mouse toy, or similarly, no Mickey Mouse toys.

Problem 15: Preeti's Coins (10 points)

Preeti has 5 coins, two of which are double-headed, two are double-tailed, and one is a standard coin (one side is heads, one side is tails).

Notes:
For each part, write your answer as a single, fully-simplified number.
Each part of this question builds on the previous parts.
Remember to justify your answers.

(a) She shuts her eyes, picks a coin at random, and tosses it.
What is the probability that the lower face of the coin is a head?

Final Answer: ________

(b) She opens her eyes and sees that the coin is showing heads.
What is the probability that the lower face is a head, i.e., that the coin is double-headed?

Final Answer: ________

(c) She shuts her eyes again, and tosses the same coin again.
What is the probability that the lower face is a head?

Final Answer: ________

(d) She opens her eyes again and sees that the coin is showing heads.
What is the probability that the lower face is a head, i.e., that the coin is double-headed?

Final Answer: ________

Solution

Let DH refer to double-headed coin, DT refer to double-tailed coin, and S refer to standard coin. Further, let H refer to getting a Heads on a flip (by symmetry, this will be the same whether seeking heads face down or heads face up, so we will actually use the two interchangeably)
P(DH) = 2/5
P(DT) = 2/5
P(S) = 1/5

a) Trying to find P(H). Using the law of total probability,
$P(H) = P(H|DH) \cdot P(DH) + P(H|DT) \cdot P(DT) + P(H|S) \cdot P(S)$.
This gives us $P(H) = 1 \cdot 2/5 + 0 \cdot 2/5 + 1/2 \cdot 1/5 = 2/5 + 0 + 1/10 = 1/2$
b) Trying to find P(DH|H). Using Bayes Theorem, we get
$P(DH|H) = \frac{P(H|DH) \cdot P(DH)}{P(H)} = \frac{1 \cdot 2/5}{1/2} = 4/5$.

For the next two parts, we are using the same coin, so the probabilities that we flip a head or tails are changed based on what we know from part (b). These probabilities are updated to:
P(DH) = 4/5
P(DT) = 0 (It's impossible to flip a head using a double-tailed coin)
P(S) = 1−4/5 = 1/5
We will focus on using these probabilities to solve the remaining parts and ignore any probabilites that we found in parts (a) and (b)

c) Trying to find the new P(H). Similar to part (a),
$P(H) = P(H|DH) \cdot P(DH) + P(H|DT) \cdot P(DT) + P(H|S) \cdot P(S)$.
This gives us $P(H) = 1 \cdot 4/5 + 0 \cdot 0 + 1/2 \cdot 1/5 = 4/5 + 1/10 = 9/10$.
d) Trying to find P(DH|H). Similar to part (b), we will use Bayes Theorem to solve this.
$P(DH|H) = \frac{P(H|DH) \cdot P(DH)}{P(H)} = \frac{1 \cdot 4/5}{9/10} = 8/9$

Alternate Solution
a) There are 5 coin sides that are heads and 5 coin sides that are tails. Heads and tails are equally likely (by symmetry), and the probability that the lower face is a head is 1/2.
b) Since Preeti sees that the coin is showing heads, she is either seeing one of the 4 sides from the double-headed coins or the 1 heads side from the normal coin. In the first 4 cases, the bottom side will be a heads, while in the 5th, it will be a tails. Thus, the probability that the lower face is a head is 4/5.
c) Since Preeti is repeating the experiment, the 5 possible outcomes become 10 (5 that stay the same way and 5 that result in the coin being on the other side). 8 of these outcomes are from double-headed coins, and 2 of these are from standard coins. All 8 outcomes with double-headed coins are guaranteed to have heads on the bottom, while

a standard coin only has one side with a head. Thus, the probability that the lower face is a head is 9/10.
d) Since she sees that the coin is showing heads, there are now 9 total outcomes (one outcome is eliminated). As stated in part c, 8 of these outcomes come from double-headed coins. Thus, the probability that the lower face is a head is 8/9.

Grading Guidelines [10 points]

Part (a) : 2.5 points
+1 Correct solution
+1.5 Correct justification
Part (b) : 2.5 points
+1 Correct solution in terms of previous parts
+1.5 Correct justification
Part (c) : 2.5 points
+1 Correct solution in terms of previous parts
+1.5 Correct justification
Part (d) : 2.5 points
+1 Correct solution in terms of previous parts
+1.5 Correct justification (must be different from part b for partial credit)

Common Mistakes

- For part b, counting the number of coins with heads rather than the number of heads out of the total number of faces possible. If you only count coins, you don't consider that it is more likely for a double-headed coin to show a head than a standard coin. This approach gives you 3 total coins to choose from, 2 of which are double-headed, resulting in a probability of 2/3.

- For part(s) c and/or d not viewing b as providing new information i.e. not updating the probability of the coin being double heads or falling heads down.

- In part c, approaching the problem as a series of sequential events rather than bayes using the updated probability from part b

Problem 16: Ham-iltonian Sandwich (10 points)

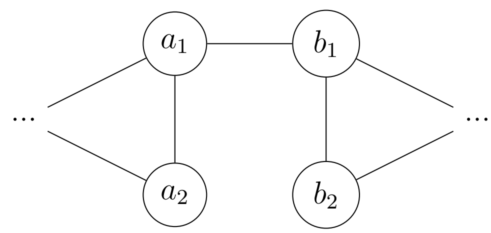
Let $G = (V, E)$ be a **connected** graph that contains two cycles C_A, C_B such that for every vertex $v \in V$, we have $v \in C_A$ or $v \in C_B$ but not both. Prove that it is possible to add at most one new edge to G so that it then contains a Hamiltonian cycle.

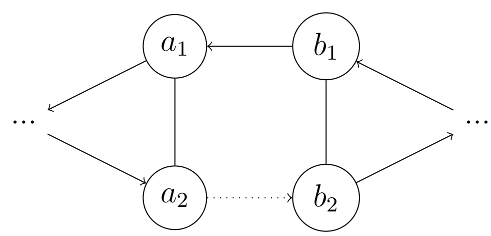
Notes:

- A Hamiltonian cycle is a cycle that contains every vertex in G exactly once.
- It may be helpful to draw a diagram and then explain in words where your new edge is added and how the Hamiltonian cycle is formed. A more formal description than that is not required for this proof.

Solution

G is connected, so there must be an edge e_1 between some a_1 and b_1, where $a_1 \in C_A$ and $b_1 \in C_B$. If it does not already exist, you can add an edge e_2 between a_2 and b_2 where $a_2 \in C_A$ and a_2 is adjacent to a_1, and $b_2 \in C_B$ and b_2 is adjacent to b_1. This creates a Hamiltonian cycle since you can start at a_1, walk C_A to get to a_2, take edge e_2 to get to b_2, walk C_B to get to b_1, and take edge e_1 to get back to a_1.





Alternate Solution:
Proof by Contradiction: Assume for the sake of contradiction that it is not possible to add at most one new edge to G so that it then contains a Hamiltonian cycle...(rest of the proof is almost identical to main solution, except we must end with a contradiction because we were able to form a Hamiltonian cycle)

Common Mistakes:

- Considering only graphs with a single edge "bridging" the two cycles. While it is true the graph must contain at least two cycles (C_A and C_B), there is no restriction that it cannot contain more. This was the most common reason for receiving the partial credit item "Arguments are general, but for some smaller class of graphs."

It also means that a graph G may already contain a Hamiltonian cycle (thus the wording of "at most one new edge").

- Describing only a specific graph and its cycles, without describing a process by which this can be generalized.
- Considering two cycles who share a single vertex, rather than two cycles connected by one (or more) edges.
- Claiming a "bridge" must exist because the two cycles are disjoint – mentioning that the graph is connected is a key step in this proof.
- Attempting to use the handshake theorem (the degree of vertices has no inherent bearing on whether a Hamiltonian cycle is possible).
- Adding an intermediary vertex on the "bridge" between C_A and C_B (this is not possible due to the way we defined G).
- Not inherently a mistake, but drawing a diagram with two C_3 can lead to missing the realization that the added edge's vertices must be adjacent to the original "bridging" edge's vertices (since in such a case any pair of vertices will work).
- No points were taken off for this, but the term "connected" was frequently used in place of "neighboring" or "adjacent." "Connected" refers either to a graph in which all vertices have a path to any other vertex, or to two vertices in particular who have a path between them, while "neighboring" or "adjacent" (synonyms) refer to two vertices whose distance to each other is 1. Adjacent vertices are by definition connected, but connected vertices are not inherently adjacent.

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