Practice Exam 2
QUESTIONS PACKET
EECS 203
Winter 2023

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

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*****MAKE SURE YOU HAVE PROBLEMS 1 - 18 IN THIS BOOKLET.*****

General Instructions

You have 120 minutes to complete this exam. You should have two exam packets.

- **Questions Packet:** Contains ALL the questions for this exam, worth 100 points total:
  - 2 Single Answer Multiple Choice questions (4 points each),
  - 10 Multiple Answer Multiple Choice questions (4 points each),
  - 1 Short Answer question (6 points), and
  - 5 Free Response questions (9 or 10 points each)

  Questions Packet is for scratch work only. Work in this packet will not be graded.

- **Answers Packet:** Write all of your answers in the Answers Packet, including your answers to multiple choice questions.

  For free response questions, you must show your work! Answers alone will receive little or no credit.

- You may bring **one** 8.5" by 11" note sheet, front and back, created by you.

- You may **NOT** use any other sources of information, including but not limited to electronic devices (including calculators), textbooks, or notes.

- After you complete the exam, sign the Honor Code on the front of the Answers Packet.

- You must turn in both parts of this exam.
- **You are not to discuss the exam until the solutions are published.**

Part A1: Single Answer Multiple Choice

Problem 1. (4 points)

Consider an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. Which of the following is the correct **assumption** to begin a one-to-one proof?

(a) For arbitrary real number b assume there exists a real number a such that $f(a) = b$

(b) For arbitrary real numbers a, b assume $f(a) = f(b)$

(c) For arbitrary real numbers a, b assume $(f(a) = f(b)) \to (a = b)$

(d) For arbitrary real numbers a, b assume that $f(a) = b$

(e) For arbitrary real number a assume there exists a real number b such that $f(a) = b$

Solution: (b)

In order to prove a function is one-to-one, we need to deduce that if the outputs $f(a)$ and $f(b)$ are equal, then it must be the case that the inputs a and b are also equal! Otherwise, there would be two different inputs matching to the same output. Therefore, we must start by assuming $f(a) = f(b)$, and then prove that $a = b$.

Problem 2. (4 points)

Consider the recurrence relation

$$f(n) = 2f(n-1) + 4f(n-2) + f(n-5).$$

What is the **minimum** number of initial conditions required for this recurrence?

(a) 2

(b) 3

(c) 4

(d) 5

(e) Not enough information

Solution: (d)
(a) - This would not allow us to calculate $f(n-5)$, as we need to be able to look back at least 5 away. Say we knew $f(0)$ and $f(1)$, when we try to calculate $f(3)$ we would get $f(n-5) = f(3-5) = f(-2)$, which is not valid. If we did not have $f(n-5)$ this would be correct.
(b) - The $f(n-5)$ requires us to have at least 5 base cases.
(c) - Same as b
(d) - The $f(n-5)$ requires a minimum of 5 initial conditions.
(e) - If we are given the recurrence relation we can figure out the minimum number of base cases by finding the function that requires us to look back the furthest - in this case, f(n-5) requires us to have at least five base cases.

Part A2: Multiple Answer Multiple Choice

Problem 3. (4 points)

For each Inductive Step, determine whether it's possible to complete an inductive proof that $P(n)$ is true for all $n \geq n_0$ using the specified number of base cases.

(a) Inductive Step: $P(k) \to P(k+1)$. Number of Base Case(s): 1

(b) Inductive Step: $P(k-2) \to P(k)$. Number of Base Case(s): 1

(c) Inductive Step: $P(k-4) \to P(k)$. Number of Base Case(s): 4

(d) Inductive Step: $P(k-1) \to P(k+1)$. Number of Base Case(s): 2

(e) Inductive Step: $P(k-1) \to P(k+3)$. Number of Base Case(s): 4

Solution: (a), (c), (d), (e)

(a) - Our inductive step tells us that if $P(k)$ is true, then $P(k+1)$ is also true. To establish this, we need a single base case: $P(n_0)$. Thus, one base case is sufficient.
(b) - Our inductive step tells us that if $P(k-2)$ is true, then $P(k)$ is also true. However, to start the chain of inductions, we need both $P(n_0)$ and $P(n_0+1)$. Hence, one base case is not sufficient.
(c) - For this step, knowing $P(k-4)$ lets us conclude $P(k)$. To cover all propositions, we need $P(n_0)$, $P(n_0+1)$, $P(n_0+2)$, and $P(n_0+3)$. With four base cases provided, this is sufficient.
(d) - Here, if $P(k-1)$ is true, then $P(k+1)$ is also true. This skips over $P(k)$. To cover all propositions, we need $P(n_0)$ and $P(n_0+1)$. Two base cases are sufficient for this step.
(e) - In this scenario, knowing $P(k-1)$ lets us determine $P(k+3)$, skipping over $P(k)$, $P(k+1)$, and $P(k+2)$. To ensure all propositions are established, we'd need $P(n_0)$, $P(n_0+1)$, $P(n_0+2)$, and $P(n_0+3)$. Four base cases are therefore sufficient.

Problem 4. (4 points)

Let $n \equiv 2 \pmod 6$. Which of the following statements are **guaranteed** to be true?

(a) $n \equiv 0 \pmod 2$

(b) $n \equiv 5 \pmod 2$

(c) $n \equiv 8 \pmod{12}$

(d) $n \equiv 2 \pmod{12}$

(e) $n \equiv 2 \pmod{3}$

Solution: (a), (e)

(a) $n = 2 + 6k$ for some integer k by the integer definition of mod.
So $n \equiv 2 + 6k \equiv 2 \equiv 0 \pmod{2}$

(b) This is guaranteed to be false, since $5 \equiv 1 \not\equiv 0 \pmod{2}$ and we know $n \equiv 0 \pmod{2}$ from answer choice (a).

(c) We know that $n = 2 + 6k$ for some integer k. If $k = 0$, then $n = 2$ and $n \equiv 2 \not\equiv 8 \pmod{12}$. So n is not guaranteed to be equivalent to $8 \pmod{12}$.

(d) By similar reasoning to answer choice (c), we know that $n = 2+6k$ for some integer k. Then, if $k = 1$, $n \equiv 8 \not\equiv 2 \pmod{12}$.
So, n is not guaranteed to be equivalent to $2 \pmod{12}$.

(e) $n = 2 + 6k$ for some integer k by the integer definition of mod.
So $n \equiv 2 + 6k \equiv 2 \pmod{3}$

Problem 5. (4 points)

Which of these sets are uncountable?

(a) $[10, 17)$

(b) $[5, 6] - (5, 6)$

(c) $\mathbb{R} - \mathbb{Q}$

(d) $\{0, 1\} \times \mathbb{R}$

(e) $\mathbb{Q} \times \mathbb{Q}$

Solution: (a), (c), (d)

(a) any interval of real numbers with nonzero length is uncountable.

(b) this is just $\{5, 6\}$ which is finite and countable

(c) irrational numbers are uncountable

(d) R is uncountably infinite and thus a Cartesian product involving R and a nonempty set is uncountably infinite

(e) rationals are countably infinite, and a Cartesian product of countable sets is still countable.

Problem 6. (4 points)

$$f(x) = \sqrt{x + 203}$$

Reminder: $\sqrt{a}$ returns the positive square root of a.

For which of the following domain-codomain pairs is $f(x)$ a function?

(a) $\mathbb{R}^+ \to \mathbb{R}^-$

(b) $\{4\} \to \{\sqrt{207}\}$

(c) $\mathbb{Q} \to \mathbb{R}^+$

(d) $\mathbb{Q}^+ \to \mathbb{R}$

(e) $\mathbb{R} \to \mathbb{R}$

Solution: (b), (d)

(a) $\sqrt{a}$ will only return the positive square root, so a codomain of only negative values will not work.

(b) the one element of the domain successfully maps to the one element of the codomain.

(c) Some elements of the domain do not map anywhere. For example, $f(-204)$ is undefined, as that would map to $\sqrt{-1}$, which is not an element of $\mathbb{R}^+$.

(d) All positive rationals when added to 203 can successfully be square rooted to produce a real number.

(e) Like (c), there are inputs that do not map, such as $f(-204)$.

Problem 7. (4 points)

If $x \equiv 3 \pmod 5$ and $y \equiv 1 \pmod 6$, compute $6x + 5y \bmod 15$.

(a) 1

(b) 5

(c) 8

(d) 12

(e) Not enough information

Solution: (c)
We may rewrite $x = 5a + 3$ and $y = 6b + 1$. Substituting, we have $6x + 5y = 6(5a + 3) + 5(6b + 1) = 30a + 18 + 30b + 5$. Applying $\pmod{15}$ simplifies $30a$ and $30b$ to 0, and we are left with $23 \pmod{15} \equiv 8 \pmod{15}$

Problem 8. (4 points)

Which of the following definitions of $f(x)$ satisfy both of the following:

- are valid functions $f : [2, 4] \to (8, 16)$, and
- prove that $|[2, 4]| \leq |(8, 16)|$

(a) $f(x) = 7$

(b) $f(x) = \sqrt{x} + 8$

(c) $f(x) = 4x$

(d) $f(x) = \begin{cases} 10, & x \in \{2, 4\} \\ 4x, & 2 < x < 4 \end{cases}$

(e) $f(x) = -2x + 18$

Solution: (b), (e)

(a) False. This function is not one-to-one and therefore is insufficient.

(b) True. When operated on the domain of $[2, 4]$, the range of this function is $[\sqrt{2} + 8, 10]$, which is a subset of the target codomain of $(8, 16)$. This function is also one-to-one when operated on the domain of $[2, 4]$. Therefore, this function is sufficient.

(c) False. Note that $f(2) = 4 * 2 = 8$, and $f(4) = 4 * 4 = 16$. This function's range is $[8, 16]$, which includes points not in the target codomain of $(8, 16)$. Therefore, this is not a function.

(d) False. Although this function is onto on the domain of $[2, 4]$ and codomain of $(8, 16)$, it is not one-to-one. For example, $f(2) = f(2.5) = 10$, but $2 \neq 2.5$. Therefore, this function is insufficient.

(e) True. When operated on the domain of $[2, 4]$, the range of this function is $[10, 14]$, which is a subset of the target codomain of $(8, 16)$. This function is also one-to-one when operated on the domain of $[2, 4]$. Therefore, this function is sufficient.

Problem 9. (4 points)

What is the least number of distinct integers you would need to draw from the set $\{1, 2, \ldots, 20\}$ that would guarantee you have numbers x and y such that $y - x = 10$.

(a) 10

(b) 20

(c) 5

(d) 11

(e) 2

Solution: (d)
We can split our 20 numbers into 10 pairs, where each pair of numbers has a difference of 10: $\{1, 11\}, \ldots, \{9, 19\}, \{10, 20\}$. If we select 11 distinct integers from the set $\{1, 2, \ldots, 20\}$ there are only 10 pairs, so, by the Pigeonhole Principle, at least two of the selected integers must belong to the same pair. Those two have a difference of 10.

Problem 10. (4 points)

Which of the following functions are **bijective**?

(a) $f\colon \mathbb{Z} \to \mathbb{Z},\ f(x) = x^3$

(b) $g\colon \mathbb{Z}^+ \to \mathbb{Z},\ g(x) = x^3$

(c) $p\colon \mathbb{R} \to \mathbb{R}^+,\ p(x) = |x|$

(d) $q\colon \mathbb{R}^+ \to \mathbb{R}^+,\ q(x) = |x|$

(e) $r\colon \mathbb{R} \to \mathbb{R}^+,\ r(x) = 2^x$

Solution: (d), (e)

(a) Because this only works with integers, it is not onto. For example, there is no x such that $f(x) = 2$, because $\sqrt[3]{2}$ is not an integer.

(b) This has the same problem as (a), except worse, as now it also misses all negative numbers.

(c) This is not one-to-one. For example, $f(-2) = |-2| = 2 = |2| = f(2)$, but $-2 \neq 2$. In fact, this technically is not even a function, as $f(0)$ should map to 0, but that is not an element of the codomain.

(d) When working only with positive numbers, $|x| = x$, so we will never have collisions violating one-to-one, and every real number can be reached by inputting itself to the function.

(e) Exponential functions like 2^x will only ever output positive values. By plugging in negative values of x, we can get as close to 0 as we want. The function is continuous and always increasing, so it must be one-to-one.

Problem 11. (4 points)

You're making friendship bracelets for Tako, your octopus friend. What is the minimum number of friendship bracelets you have to make to guarantee that Tako will have 5 or more bracelets on at least one tentacle? (Tako has 8 tentacles.)

(a) 5

(b) 6

(c) 32

(d) 33

(e) 41

Solution: (d) That's a lot of bracelets. Better get to work!

(a) With 5 bracelets, Tako could have 1 bracelet on 5 tentacles, which would not guarantee one tentacle with at least 5 bracelets.

(b) With 6 bracelets, Tako could have 1 bracelet on 6 tentacles, which would not guarantee one tentacle with at least 5 bracelets.

(c) If you make 32 bracelets, Tako could have 4 bracelets on each tentacle with no tentacles that have 5 or more bracelets. So, 32 does not guarantee that Tako will have 5 or more bracelets on at least one tentacle.

(d) Generalized pigeonhole principle: if you make 32 bracelets, then worst case each tentacle has 4 bracelets, and if you add one more with 33 bracelets, then we have 5 on one tentacle.

(e) 41 bracelets guarantees that Tako will have 5 or more bracelets on at least one tentacle, but this is not the minimum bracelets required.

Problem 12. (4 points)

Consider two arbitrary functions $g : A \to B$ and $f : B \to C$. Which of the following must be true for $f \circ g$ to be a bijection?

(a) $|A| = |B|$

(b) $|A| = |C|$

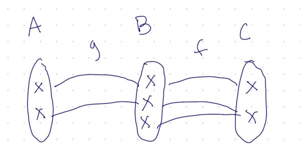
(c) f is a bijection

(d) g is one-to-one

(e) g is onto

Solution: (b), (d)

(a) This is not required, as seen in this example of function definitions:



(b) $f \circ g$ is a function from A to C, so if it is a bijection, then $|A| = |C|$

(c) The counterexample from (a) will also work for this, as f is not a bijection, as it is not one-to-one.

(d) This is necessary. One way to see this is to note that if g were not one-to-one, then there would be two elements of A, $a_1 \neq a_2$, such that $g(a_1) = g(a_2)$. This means $f(g(a_1)) = f(g(a_2))$, which together with $a_1 \neq a_2$ contradicts the requirement that $f \circ g$ be a bijection (and thus one-to-one).

(e) The counterexample from (a) will also work for this, as g is not onto.

Part B: Short Answer

Problem 13. (6 points) Modular Computation

Let $x \equiv 5 \pmod 6$ and $y \equiv 7 \pmod 9$. Compute the following values. If a value cannot be computed, write "N/A" as your answer.

(a) $4x + y \bmod 3$.

(b) $4x + y \bmod 18$.

Solution:

(a) The answer is 0. Possible work: let j, k be integers with $x = 5 + 6j, y = 7 + 9k$. Then

$$\begin{aligned} 4x + y \bmod 3 &= 20 + 24j + 7 + 9k \text{ mod} 3 \\ &= 27 \bmod 3 \\ &= 0. \end{aligned}$$

Alternate Solution:
Since we know that 6 and 9 are multiples of 3, then we know that $x \equiv 5 \pmod 3$ and $y \equiv 7 \pmod 3$. Because of this, we can plug in x and y directly:

$$4x + y \bmod 3 = 4(5) + 7 \bmod 3 = 27 \bmod 3 = 0$$

Common Mistakes

- Not simplifying all the way to 0, after applying the operation mod 3, the only possible remainders are 0,1, or 2.
- Using the same variable for both x and y. i.e. $x = 5 + 6k, y = 7 + 9k$. Then

$$\begin{aligned} 4x + y \bmod 3 &= 20 + 24k + 7 + 9k \text{ mod} 3 \\ &= 27 + 33k \bmod 3 \\ &= 0. \end{aligned}$$

- Answering N/A because 6 and 9 are greater than 3: since 6 and 9 are multiples of 3, we are able to convert the mods.
- Algebra mistakes: 27 mod 3 = 1, 20+7 = 26, 4(5+6j) = 18 + 24j etc

Grading Guidelines

- +3 points for correct approach with correct answer

- -0.5 points for minor algebra mistake (i.e. +2.5 points for correct approach but incorrect solution due to algebra error)
- -1 points for insufficient work (i.e. +1.0 points for correct answer but insufficient work)
- -1 point for reusing the same variable name when expressing both x and y in the same equation
- +1 point partial credit for demonstrating some basic understanding of modular arithmetic (e.g., writing $x = 5 + 6k$ or "x has a remainder of 5 when divided by 6)
- -0.5 point for final answer not being simplified (i.e 27 or 0)

(b) The answer is N/A. For example, $x = 5, y = 7$ leads to an answer of 9, but $x = -1, y = -2$ leads to an answer of 12.

Common mistakes:

- Final answer of 9 with work like:

  Solution:
  $x \equiv 5 \pmod 6 \rightarrow x \equiv 5 \pmod{18}$
  $y \equiv 7 \pmod 9 \rightarrow y \equiv 7 \pmod{18}$
  So $4x + y \equiv 4(5) + 7 \equiv 20 + 7 \equiv 27 \equiv 9 \pmod{18}$

  However, we don't actually know what x and y are congruent to mod 18. For example, $5 \equiv 11 \pmod 6$, but $5 \not\equiv 11 \pmod{18}$.

  The idea of changing mod like this works when going from mod n to one of its factors p, not the other way around. To demonstrate, since $p \mid n$, there is an integer q such that $n = pq$. So for all integers x and y, if $x \equiv y \pmod n$, there is some integer k such that $x = y + nk$, so $x = y + pqk$, so $x \equiv y \pmod p$.

- Final answer of 2 with work like:

  Solution:
  $x \equiv 5 \pmod 6 \rightarrow \exists k_1[x = 5 + 6k_1] \rightarrow \exists k_1[x = 15 + 18k_1]$
  So $x \equiv 15 \pmod{18}$.
  $y \equiv 7 \pmod 9 \rightarrow \exists k_2[y = 7 + 9k_2] \rightarrow \exists k_2[y = 14 + 18k_2]$
  So $x \equiv 14 \pmod{18}$.
  Thus, $4x + y \equiv 4(15) + 14 \equiv 60 + 14 \equiv 74 \equiv 2 \pmod{18}$

However, going from $5 + 6k_1$ to $15 + 18k_1$ and $7 + 9k_2$ to $14 + 18k_2$ requires multiplying by 3 and 2 respectively. If we multiply one side of an equation by a value, we also have to multiply the other side by the same value, so we really end up learning that $3x \equiv 15 \pmod{18}$ and $2y \equiv 14 \pmod{18}$. Modular arithmetic does not give us a way out of this rule.

Grading Guidelines

- +3 points for giving correct answer N/A by any means
- +2.5 points for correct approach but incorrect answer due to algebra error
- +2 points for showing values of $4x + y \mod 18$ for all possible x and y, but not concluding that the value cannot be computed
- -1 point for reusing the same variable name when expressing x and y
- +1 point for demonstrating some basic understanding of modular arithmetic (including an answer of 0, with work shown)
- +0 points for incorrect answer with no work shown or work shown that does not sufficiently demonstrate understanding of modular arithmetic

Part C: Free Response

Problem 14. (10 points)

Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = \left|\dfrac{2}{x}\right|$

(a) Prove or disprove f is one-to-one.

(b) Prove or disprove f is onto.

Solution:

(a) Let $a_1, a_2 \in \mathbb{R}^+$ be arbitrary. Assume $f(a_1) = f(a_2)$.

$$\begin{aligned}
f(a_1) &= f(a_2) \\
\left|\frac{2}{a_1}\right| &= \left|\frac{2}{a_2}\right| \\
\frac{2}{a_1} &= \frac{2}{a_2} && \frac{2}{a_1}, \frac{2}{a_2} > 0 \text{ since } a_1, a_2 \in \mathbb{R}^+ \\
2a_2 &= 2a_1 \\
a_1 &= a_2
\end{aligned}$$

(b) Let $b \in \mathbb{R}^+$ be arbitrary. Let $a = \frac{2}{b}$. Note $b > 0$ so $a \in \mathbb{R}^+$.

$$\begin{aligned}
f(a) &= f(2/b) \\
&= \left|\frac{2}{2/b}\right| \\
&= |b| \\
&= b && b > 0
\end{aligned}$$

Grading Guidelines:
Part (a):
+1 Choose prove
+1 Takes two arbitrary elements a_1, a_2 such that $f(a_1) = f(a_2)$
+1 Attempts to show $a_1 = a_2$
+1 Correct arithmetic (despite possible absolute value mistakes)
+1 Addresses that the quantities inside the absolute values are positive when removing the absolute values

Part (b):
+1 Choose prove
+1 Pick arbitrary b in the codomain
+1 Gives a value of a in the domain
+1 Correctly shows that $f(a) = b$ (despite possible absolute value mistakes)
+1 Addresses that the quantities inside the absolute values are positive when removing the absolute values

Common Mistakes:
The most common mistake revolved around correctly addressing the absolute value operation in each parts. Acceptable ways to address it include:

- as our solution does, explicitly stating that the quantities inside the absolute value are positive, then removing them in the next step. This doesn't necessarily need to be done in the same step that the solution does it in.
- for part (a), we also accepted solutions which stated $a_1, a_2 > 0$ rather than $\frac{2}{a_1} > 0$
- noting in one or both parts that since the codomain is positive, $f(x) = \frac{2}{x}$ for all x.
- breaking into positive and negative cases, then explaining why the ones which don't work are impossible (note that for part (a), you would need four cases).

Insufficient ways to address it include:

- noting in the assumption of part (a) that $a_1, a_2 \in \mathbb{R}^+$ (or similarly for b in part (b)), but not restating it when removing absolute value: this doesn't explain that this information is relevant when removing the absolute value bars
- addressing the absolute value in a step in one part but not the other: since these are different quantities, you should be addressing them each separately
- including a step with the absolute value bars then another without them (with no justification): you need to explain how you can make that jump, since it isn't true in general
- for part (b), making your element $a = |2/b|$: although this is a fine definition for a, absolute value bars don't cancel each other out, so this doesn't help you to remove them

Although this may seem pedantic, not needing to address absolute value would somewhat trivialize the proof, so it's important that it was done properly.

Another mistake we noticed was "undoing" f to get $a = 2/b$, but not then applying f to show $f(a) = b$. This would only work if we already knew f has an inverse, which would assume the proof.

Problem 15. (9 points)

Prove using weak induction that for all positive integers n:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

Solution:
Let $P(n)$ be the statement that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

Base case:

$$\frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = \frac{6}{3} = 2 = 1 \cdot 2 = 1 \cdot (1+1)$$

Inductive step:
Let $P(k)$ be true for some $k \geq 1$. Thus,

$$1 \cdot 2 + 2 \cdot 3 + ... + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$$

Now we show that $P(k+1)$ is true.

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + ... + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

Thus, $\forall k \geq 1 \quad P(k) \to P(k+1)$.
Therefore, by induction, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Alternate Inductive Step:
Let $P(k)$ be true for some $k \geq 1$. Thus,

$$1 \cdot 2 + 2 \cdot 3 + ... + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$$

Start with our inductive hypothesis and add $(k+1)(k+2)$ to both sides, then simplify the right hand side:

$$\begin{aligned}
1 \cdot 2 + 2 \cdot 3 + ... + k(k+1) &= \frac{k(k+1)(k+2)}{3} \\
1 \cdot 2 + 2 \cdot 3 + ... + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3}
\end{aligned}$$

Thus, $\forall k \geq 1 \quad P(k) \rightarrow P(k+1)$.
Therefore, by induction, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Other Combinations of Base Case and Inductive Hypothesis:
Alternate inductive step for P(1) base case:

- For arbitrary $k \geq 2$, show $P(k-1) \rightarrow P(k)$

Base case P(0):

$$\frac{0(0+1)(0+2)}{3} = 0 = \text{empty sum}$$

Possible accompanying inductive steps:

- For arbitrary $k \geq 0$, show $P(k) \rightarrow P(k+1)$
- For arbitrary $k \geq 1$, show $P(k-1) \rightarrow P(k)$

Grading Guidelines:
Base case:
+1 Correct input for base case (i.e., P(1))
+1 Correct work for base case
Inductive Step:
+2 Correct Inductive Hypothesis
+1.5 Goes from LHS to RHS of the equation, or vice versa
+1.5 Correct application of IH
+1 Correct algebra

+1 Correct final expression or "want to show"

Common Mistakes:
Base cases:

- Having a base case of P(0) is not technically incorrect, as long as your inductive step matches up with your base case, so your inductive hypothesis in this case must be "assume P(k) for an arbitrary integer $k \geq 0$."
- If your (smallest) base case is an integer greater than 1, then you have not proven that the predicate is true for all positive integers n.
- Always work from the LHS to RHS, even for base cases. Setting LHS = RHS and evaluating both sides is still incorrect. We did not deduct points for this, but just as a note.

Inductive Step:

- We found many students factoring out the last term of the summation, doing something along the lines of $(k+1)(k+2) = k^2 + 3k + 2 = k(k+1) + 2k + 1$. This is technically valid computation, but remember that $1 \cdot 2 + 2 \cdot 3 + ... + (k+1)(k+2)$ is a summation and already includes the term $k(k+1)$, immediately before the $(k+1)(k+2)$ term. This means that performing that first factorization gives us the value $k(k+1)$ twice, and we only use it once in the inductive hypothesis, leading to many issues if this wasn't accounted for.
- Stating that $P(k) = \frac{k(k+1)(k+2)}{3}$. $P(k)$ is a predicate - not a numeric function. Saying that $P(k)$ is equal to any number doesn't quite make sense for a proof.
- Not specifying the domain for k in the inductive hypothesis. It's important to give the domain for our value of k otherwise we wouldn't be able to use the inductive hypothesis as a valid proof for "all positive integers", like in this question.
- Assuming $P(k)$ to be true **for all** k as opposed to **for some** (single value). Assuming $P(k)$ is true for all uses circular reasoning.
- Not working from LHS to RHS. Similar to the base cases, we must work from the LHS to the RHS as opposed to using equality.
- Misinterpreting the LHS of the predicate P(n) to be $n(n+1)$ instead of $1 \cdot 2 + 2 \cdot 3 + ... + n(n+1)$. This led to issues in proving $P(k+1)$ in the inductive step.

Problem 16. (9 points)

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* when, for all $x, y \in \mathbb{R}$,

$$f(x) < f(y) \text{ if and only if } x < y.$$

(a) Using the definition of one-to-one, prove that if f is strictly increasing, then f is one-to-one.

(b) Using the definition of strictly increasing, prove that if f and g are strictly increasing, then $f \circ g$ is strictly increasing.

Solution:

(a) Let $x, y \in \mathbb{R}$, and suppose $x \neq y$. Then either $x < y$ or $x > y$. WoLoG assume the former. Then we have $f(x) < f(y)$, so $f(x) \neq f(y)$. Therefore f is one-to-one.

Alternate (Prove one-to-one directly):
Let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Assume seeking a contradiction that $x \neq y$. Then either $x < y$ or $x > y$. WoLoG assume the former. Then $f(x) < f(y)$, contradicting the fact that $f(x) = f(y)$. So $x = y$, and thus f is one-to-one.

Alternate (Contraposition of strictly increasing):
By taking the contrapositive of the definition of strictly increasing, we have that $f(x) \geq f(y)$ if and only if $x \geq y$. Now assume $f(x) = f(y)$. Then $f(x) \geq f(y)$ and $f(x) \leq f(y)$. So $x \geq y$ and $x \leq y$. These two statements imply $x = y$, thus f is one-to-one.

Alternate (Proof by contradiction):
Assume seeking a contradiction that f is strictly increasing and not one-to-one. Then there exist $x, y \in \mathbb{R}$ such that $x \neq y$ and $f(x) = f(y)$. Since $x \neq y$, either $x < y$ or $x > y$. In the first case, $f(x) < f(y)$ because f is strictly increasing, contradicting that $f(x) = f(y)$. In the second case, we have $f(y) < f(x)$, once again yielding a contradiction. Since in either case we reach a contradiction, we conclude that f must be one-to-one.

(b) Let $x, y \in \mathbb{R}$ and assume $x < y$. Then $g(x) < g(y)$ because g is strictly increasing, and thus $f(g(x)) < f(g(y))$ because f is strictly increasing. Therefore $(f \circ g)(x) < (f \circ g)(y)$, so $f \circ g$ is strictly increasing.

Note: the original statement of strictly increasing in this question (and the definition most commonly used) did not include "if and only if," but instead just said "$f(x) < f(y)$ if $x < y$." This was changed late in the exam editing process, but we only ever intended to grade one direction of this proof; the reverse direction is almost identical to the forward direction:

$$\begin{aligned}
&(f \circ g)(x) < (f \circ g)(y) \\
\Rightarrow\ &f(g(x)) < f(g(y)) \\
\Rightarrow\ &g(x) < g(y) \\
\Rightarrow\ &x < y.
\end{aligned}$$

Grading Guidelines:
Part A:
+1 Writes definition of one-to-one or properly outlines it in proof
+1 Assumes $x \neq y$
+1 Splits into cases $x < y$, $x > y$, or equivalent
+0.5 Uses definition of strictly increasing productively
+1 Shows that $f(x) > f(y)$ or $f(x) < f(y)$, possibly using WoLoG
+1 Justifies the conclusion that $f(x) \neq f(y)$ (or appropriate conclusion for proof style) with prior supporting work
Part B:
+1 Assumes $x < y$
+1 Shows $g(x) < g(y)$
+1 Shows $f(g(x)) < f(g(y))$
+0.5 Justifies work by citing f and g are strictly increasing

Common Mistakes:
Part A:

- Assuming that $f(x) < f(y)$ or $f(x) > f(y)$, and showing $x \neq y$ in both cases. To prove one-to-one, we must prove $f(x) = f(y) \to x = y$. This common mistake attempts to prove one-to-one by showing $f(x) \neq f(y) \to x \neq y$, which is not logically equivalent. In fact, this must be true for all functions, not just those which are one to one, by definition of a function.

- Taking a specific strictly increasing function (e.g. $f(x) = 2x$), and proving that this function is one-to-one. We cannot use a proof by example here because we must show that for **all** functions, if the function is strictly increasing, then it is one-to-one.

- Assuming $x \neq y$, such that x and y have a specific relationship (e.g. "Let $y = x + 1$"). When we define variables in this way, we limit the proof to only be valid when this assumption holds true, so the proof cannot show that the function is one-to-one. We must show $x \neq y \to f(x) = f(y)$ for **all** x, y.

Part B:

- Showing that $f \circ g$ is one-to-one (i.e. $(f \circ g)(x) = (f \circ g)(y) \rightarrow x = y$) and making the claim that this implies $f \circ g$ is strictly increasing. Though in part a, we showed that if a function is strictly increasing, it is also one-to-one, the converse is not always true, so it cannot be used here.

Problem 17. (9 points)

Michelle the triathlete will do one of four things on any given day: Swim, Run, Bike, or Nap. To maximize training efficiency and prevent injuries, Michelle follows the following rules when constructing a training plan.

- If Michelle Runs on a given day, she must have Napped on the previous day.
- If Michelle Swims on a given day, she must have Run or Biked on the previous day.

Note that the above rules mean that she cannot Swim or Run on the first day.

a) Find a recurrence relation for $f(n)$, the number of ways Michelle can train in n days.

b) Find the initial conditions for your recurrence relation in part (a). For full credit, you must provide the fewest initial conditions possible to satisfy your recurrence.

Solution:
(a) Working backwards, on day n, Michelle has the following 4 options:

- Case 1: Naps on day n
  Since there are no restrictions on what she can do on the $(n-1)$th day, she has 4 options again - the same options we started out on day n. Hence there are $f(n-1)$ ways for the $(n-1)$ days.
- Case 2: Bikes on day n
  There are no restricitions for the $(n-1)$th day, hence similar to case 1, there are $f(n-1)$ ways to train.
- Case 3: Runs on day n
  If she runs on the nth day, she can only nap on the $(n-1)$th day, but she is free to do anything on the $(n-2)$nd day, hence there are $f(n-2)$ ways.
- Case 4: Swims on day n
  There are 2 options for what she can do on day $(n-1)$. Either bike or run. If she bikes, she is free to do anything on day $(n-2)$, hence $f(n-2)$ ways for the rest of the $(n-2)$ days. If she runs on day $(n-1)$, she has to nap on day $(n-2)$. There are no restrictions for the (n-3)rd day, hence $f(n-3)$ to count this scenario.

Adding up all the cases gives us $f(n) = 2f(n-1) + 2f(n-2) + f(n-3)$

(b) Since $f(n)$ requires knowing $f(n-3)$, a minimum of 3 base cases are required. Either $f(0)$, $f(1)$, $f(2)$ or $f(1)$, $f(2)$, $f(3)$ can be provided.

- $f(0) = 1$, if we have 0 days, there is nothing to select.

- $f(1) = 2$, she can either bike or nap on day 1.
- $f(2) = 6$. If she naps on day 1, she can either rest, nap or bike on day 2. If she bikes on day 1, she can either bike, swim or nap on day 2. Thus a total of 6 ways.
- $f(3) = 17$. The possible options include {NNN, NNB, NNR, NBB, NBN, NBS, NRN, NRB, NRS, BBB, BBN, BBS, BNB, BNR, BNN, BSN, BSB}

Part (a) Alternate Solution (Working forwards)
Working forwards, we start with the restriction that Michelle can only Nap or Bike on the first day (since she cannot Run or Swim on the first day). As a result, we want to work forwards until we reach the **same** restrictions that we begin with. If we apply $f(x)$, that only accounts for ways that would be valid across x days, so if we are allowed more than just Nap or Bike, those extras are not accounted for by $f(x)$. So, we have 2 initial cases:

1. Case 1: Naps on the first day.
   She could Nap, Bike, or Run on the second day:
   (a) If she Naps or Bikes on the second day, this is the same restriction we started with, so we've found the recursive component for this case. The first day is fixed [as Napping], and the restrictions apply to the following $n-1$ days: $f(n-1)$
   (b) If she Runs on the second day:
   She could Nap, Bike, or Swim on the third day:
      i. Subcase: Nap or Bike on the third day. This is the same restrictions as on day 1, but with the first and second day fixed, the restrictions apply to the following $n-2$ days: $f(n-2)$
      ii. Subcase: Swim on the third day. This means that on the fourth day, the options are Nap or Bike. The first, second, and third day are fixed, so the restrictions apply to the following $n-3$ days: $f(n-3)$
2. Case 2: Bikes on the first day.
   She could Nap, Bike or Swim on the second day:
   (a) Nap or Bike on the second day. This is the same restriction as the we started with. The the first day fixed [as Biking], and the restriction applies to the following $n-1$ days: $f(n-1)$
   (b) Swims on the second day. On the third day, she can only Nap or Bike. This is the same restriction as the first day, with the first and second day fixed [as Biking then Swimming]: $f(n-2)$

Adding all of these cases together, we have $f(n) = 2f(n-1) + 2f(n-2) + f(n-3)$

Grading Guidelines [9 points]

Part a: [2 points total] (backward method)
+0.5 attempt to split into cases
+1 correct term for Biking case ($f(n-1)$)
+1 correct term for Napping case ($f(n-1)$)
+1.5 correct term for Running case ($f(n-2)$)
+1 correct term for Swimming/Running subcase ($f(n-3)$)
+1 correct term for Swimming/Biking subcase ($f(n-2)$)
+1 Adds cases together to get recurrence relation

Part a: [2 points total] (forward method):
+0.5 attempt to split into cases
+1 correct term for Biking case ($f(n-1)$)
+1 correct term for Napping case ($f(n-1)$)
+1.5 correct term for Napping-Running case ($f(n-2)$)
+1 correct term for Napping-Running-Swimming subcase ($f(n-3)$)
+1 correct term for Biking-Swimming subcase ($f(n-2)$)
+1 Adds cases together to get recurrence relation

Part b: [2 points total]
+1 correct number of initial conditions (based on final recurrence relation)
+1 correct values for initial conditions (+0.5 for partially correct, ie at least one base case, but not all, is correct)

Common Mistakes:

1. Incorrect notation: Ex. writing $(n-1)$ instead of $f(n-1)$. Note that we are not taking points off for this.

2. If using the forwards method (starting with bike and nap cases), students are unable to enumerate the remaining options until they get back bike and nap cases, where recurrence is then allowed. (ie you have to loop back around to how you started)

3. In the swim-bike-nap case, assuming it's $f(n-2)$ instead of $f(n-3)$. Essentially, students forget that napping must happen the day before biking happens.

4. Incorrectly evaluating initial condition $f(0) = 0$, when it should be $f(0) = 1$ as there is 1 way to do nothing.
5. Incorrectly evaluating initial condition $f(2) = 5$ or 7 when it should be 6. Or claim that $f(3) = 16$ or 18 when it should be 17.
6. Using recurrence relation when evaluating $f(2) = 2f(1) + 2f(0) + f(-1)$, which is invalid, since $f(-1)$ is not defined.
7. Listing the constraints (eg. Must have napped before biking etc.) as the initial conditions instead of actually calculating them.
8. Having the wrong number of initial conditions based on the recurrence relation. Eg, having only 2 initial conditions while recurrence relation in part (a) shows f(n-3), which requires 3 initial conditions.

Problem 18. (9 points)

The Martian monetary system uses colored beads instead of coins. In particular,

- A red bead is worth 3 Martian credits
- A green bead is worth 7 Martian credits
- A blue bead is worth 8 Martian credits.

Let $P(n)$ be the predicate "There is some combination of red, green and blue beads that is worth exactly n Martian credits." Prove that $\forall n \geq 6,\ P(n)$.

Solution:
The intended solution for this question was using Strong Induction, but there are other valid proofs as well.

Solution 1:

Inductive step:
Let k be an arbitrary integer such that $k \geq 9$, and assume that for all integers j such that $6 \leq j \leq k-1$, there is some combination of red, green and blue beads worth j credits. That is, assume $P(j)$ holds for all j where $6 \leq j \leq k-1$. We now need to show there is some combination of red, green and blue beads worth k credits, i.e., we need to show that $P(k)$ holds.

We can amass k credits by adding one red bead (worth 3 credits) to some combination of beads that is worth $k-3$ credits, i.e. $k = (k-3)+3$. Because $k \geq 9$, it is true that $6 \leq k-3 \leq k-1$, so by our inductive hypothesis, $P(k-3)$ holds, i.e., there is some combination of red, green and blue beads worth $k-3$ credits. We can make a new combination of beads worth k credits by adding one red bead. Thus, $P(k)$ holds.

Base cases:

$$\begin{aligned} &P(6) : 6 = 3+3 && \text{two red beads} \\ &P(7) : 7 = 7 && \text{one green bead} \\ &P(8) : 8 = 8 && \text{one blue bead} \end{aligned}$$

Thus, by the principle of induction, $P(n)$ holds for all $n \geq 6$.

Solution 2:

Inductive step: Let k be an arbitrary integer such that $k \geq 9$, and assume $P(k-3)$ is true. We now need to show there is some combination of red, green and blue beads worth k credits. That is, we need to show $P(k)$ holds.

We can amass k credits by adding one red bead (worth 3 credits) to some combination of beads that is worth $k-3$ credits, i.e. $k = (k-3)+3$. By our inductive hypothesis there is some combination of red, green and blue beads worth $k-3$ credits. We can make a new combination of beads worth k credits by adding one red bead. Thus, $P(k)$ holds. This completes the inductive step.

Base cases:

$$\begin{aligned} &P(6): 6 = 3+3 && \text{two red beads} \\ &P(7): 7 = 7 && \text{one green bead} \\ &P(8): 8 = 8 && \text{one blue bead} \end{aligned}$$

Thus, by the principle of induction, $P(n)$ holds for all $n \geq 6$.

Solution 3, using Weak Induction:
Base case: $P(6)$ holds because $6 = 3+3$ is two red beads.

Inductive step: Let $k \in \mathbb{N}$ be arbitrary such that $k \geq 6$. Assume $P(k)$. WTS $P(k+1)$.

By the Inductive Hypothesis, we know we can create k credits. Let Let r, g, b be the number of red, green, and blue beads (respectively) used to create k credits. Then, $k = 3r + 7g + 8b$.

Note that because $k \geq 6$, we are guaranteed to have at least two red beads, or at least one green bead, or at least one blue bead among our k credits (more details below the cases). We consider these 3 cases:

- Case $r \geq 2$: Create $k+1$ credits by replacing two reds with one green bead, giving $k - 2(3) + 1(7) = k+1$ credits.

- Case $g \geq 1$: Create $k+1$ credits by replacing one green bead with one blue bead, giving $k - 7 + 8 = k+1$ credits.

- Case $b \geq 1$: Create $k+1$ credits by replacing one blue bead with three red beads, giving $k - 8 + 3(3) = k+1$ credits.

Since $k \geq 6$, these cases are exhaustive (to be excluded from all cases, it must be that $r \in \{0,1\}$ and $g = b = 0$, but then $k \in \{0,3\}$, a contradiction). Since, in each case, we can create $k+1$ credits, then we can conclude that $P(k+1)$ holds.

Thus, by the principle of mathematical induction, $P(n)$ holds for all $n \geq 6$.

Solution 4, using a mod-like argument with cases.
Let n be an arbitrary integer with $n \geq 6$. Consider the remainder r when n is divided by 3. We know that for any n, $r \in \{0, 1, 2\}$ and we can write n as $n = 3k + r$, where $k \in \mathbb{Z}$. Further, since $n \geq 6$ we know $k \geq 2$.

Consider the three cases, where $r = 0, 1, 2$:

- Case $r = 0$: Then $n = 3k$. So n Martian credits are formed from k red beads (worth 3 credits each).
- Case $r = 1$: Then $n = 3k+1$. We can rewrite this as $n = 3k+1 = 3k-6+6+1 = 3(k-2)+7$. Since $k \geq 2$, we know $k-2$ is non-negative. So n Martian credits are formed from $k-2$ red beads (3 credits each) and 1 green bead (7 credits each).
- Case $r = 2$: Then $n = 3k+2$. We can rewrite this as $n = 3k+2 = 3k-6+6+2 = 3(k-2)+8$. Since $k \geq 2$, we know $k-2$ is non-negative. So n Martian credits are formed from $k-2$ red beads (3 credits each) and 1 blue bead (8 credits each).

These cases are exhaustive, i.e., they cover every value of $n \geq 6$. In every case, we could create n credits using a combination of red, green, and blue beads. Thus, we have proven that $P(n)$ holds $\forall n \geq 6$.

Grading guidelines:
Inductive step [7 points total]

a) +1 Introduce arbitrary variable $k > 8$ (or the largest base case).

   Note: To receive this rubric item, you must introduce k at the beginning of the inductive step (i.e. *before* the variable is used) *and* use the strictest possible bound.

b) +2 Correct inductive hypothesis: $P(j)\ \ \forall\ 6 \leq j \leq k-1$.

   Note: $P(k-3)$ is also valid, but the bounds $(6 \leq k-3 \leq k-1)$ must be justified.

c) +2 Cite inductive hypothesis to show $P(k-3)$.

d) +2 Conclude $P(k)$ by adding 3 Martian credits to $k-3$.

Base cases [2 points total]

e) +2 Sufficient and minimal base cases, given the inductive step.

f) +1 (Partial credit) Sufficient base cases, given the inductive step.

Note: Correct solutions may use different indexing than what we use here (e.g. proving $k + 1$). Grading is not determined by the choice of indexing.

Common mistakes:

1. Off-by-one indexing error connecting the base cases to the inductive step. Usually, this means that the case $k = 9$ is not covered by either the base cases or the inductive step, so the proof is not complete.

   These solutions do not receive rubric item (a): Introduce arbitrary variable k.

2. Using variables before they're defined. The most common example of this is to state the inductive hypothesis without having defined k. For example, consider the following part of a proof:

   Suppose $P(j)$ is true for all j such that $6 \leq j < k$. Let $k > 8$ be arbitrary.

   It uses k in the inductive hypothesis, but we haven't yet said what k is. The sentence defining it needs to come first.

   These solutions do not receive rubric item (a): Introduce arbitrary variable k.

3. Conflating propositions with algebraic expressions, i.e. adding, equating, or comparing propositions or predicates with numbers. For example, saying that $P(6) = 3 + 3$. Each $P(i)$ is a logical statement that says some combination adding up to i exists, not an actual combination. That's why we can assume $P(i)$; we can assume logical statements, but it doesn't make sense to assume a number.

4. Incorrect quantifiers, often with the use of keywords for all, for some, etc. For example, saying that $P(j)$ is the IH without quantifying it or quantifying incorrectly by assuming it is true for only one value of j.