EECS 203: Discrete Mathematics
Fall 2023
Homework 2

Due **Thursday, Sept. 14**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $6 + 2$ Total Points: $100 + 25$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. N∃gations and Qu∀ntifiers [18 points]

Negate the following statements. Simplify your answers so that all negation symbols immediately precede predicates. Make sure to show all intermediate steps.

(a) $(x \vee y) \wedge ((a \wedge b) \vee z)$

(b) $[\forall x P(x)] \vee [\exists y Q(y)]$

(c) $\forall x \exists y \forall z [L(x, y) \rightarrow [R(y, z) \rightarrow T(z, x)]]$

Solution:

(a) $\neg[(x \vee y) \wedge [(a \wedge b) \vee z]]$
$\equiv \neg(x \vee y) \vee \neg[(a \wedge b) \vee z]$
$\equiv (\neg x \wedge \neg y) \vee [\neg(a \wedge b) \wedge \neg z]$
$\equiv (\neg x \wedge \neg y) \vee [(\neg a \vee \neg b) \wedge \neg z]$
$\equiv (\neg x \wedge \neg y) \vee (\neg a \wedge \neg z) \vee (\neg b \wedge \neg z)$

(b) $\neg[[\forall x P(x)] \vee [\exists y Q(y)]]$
$\equiv [\neg \forall x P(x)] \wedge [\neg \exists y Q(y)]$
$\equiv [\exists x \neg P(x)] \wedge [\forall y \neg Q(y)]$

(c) $\neg[\forall x \exists y \forall z [L(x, y) \rightarrow [R(y, z) \rightarrow T(z, x)]]]$
$\equiv \exists x \neg[\exists y \forall z [L(x, y) \rightarrow [R(y, z) \rightarrow T(z, x)]]]$
$\equiv \exists x \forall y \neg[\forall z [L(x, y) \rightarrow [R(y, z) \rightarrow T(z, x)]]]$
$\equiv \exists x \forall y \exists z \neg[L(x, y) \rightarrow [R(y, z) \rightarrow T(z, x)]]$
$\equiv \exists x \forall y \exists z \neg[\neg L(x, y) \vee [R(y, z) \rightarrow T(z, x)]]$
$\equiv \exists x \forall y \exists z [L(x, y) \wedge \neg[R(y, z) \rightarrow T(z, x)]]$
$\equiv \exists x \forall y \exists z [L(x, y) \wedge \neg[\neg R(y, z) \vee T(z, x)]]$
$\equiv \exists x \forall y \exists z [L(x, y) \wedge R(y, z) \wedge \neg T(z, x)]$

2. To Tell the Truth [18 points]

Prove or disprove whether each of the following compound propositions is a tautology. **Justify your answers.**

(a) $(p \wedge q) \rightarrow (p \vee q)$

(b) $((p \wedge q) \vee s) \rightarrow (s \rightarrow (p \vee q))$

Solution:

(a) $(p \wedge q) \to (p \vee q)$
$\equiv \neg(p \wedge q) \vee (p \vee q)$
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$
$\equiv T \vee T$
$\equiv T$
$\therefore$ tautology.

(b) $((p \wedge q) \vee s) \to (s \to (p \vee q))$
mark $p \wedge q$ as m, $p \vee q$ as n. (That is, $(p \wedge q) \equiv m$, $(p \vee q) \equiv n$.)
then the proposition is: $[(m \vee s) \to (s \to n)] \equiv [(m \vee s) \to (\neg s \vee n)]$.
Since for "if-then" proposition, there is only one circumstance where its truth value is F, suppose $(m \vee s) \to (\neg s \vee n) \equiv F$, then $(m \vee s) \equiv T$ and $(\neg s \vee n) \equiv F$
Since for $(\neg s \vee n)$ to be false, $\neg s$ and n should all be false, and if $\neg s \equiv F$, then $s \equiv T$, so $(m \vee s) \equiv T$.
$\therefore$ if whatever the truth value of m, as long as there is a case where $n \equiv F$, then $(m \vee s) \to (\neg s \vee n) \equiv F$.
$\therefore$ Consider $p \equiv F$, $q \equiv F$, $s \equiv T$, the proposition's truth value is F.
$\therefore$ not a tautology.

3. Not it! [12 points]

Find the negations of the following statements. You must simplify your answers using De Morgan's Laws or other logical equivalences for full credit

(a) I like to add both ham and pineapple to my pizza.

(b) If Bob studies computer science at the University of Michigan, they will take EECS 203.

(c) Every student has at least one friend who lives in Baits.

Solution:

(a) $p(x)$: "I like to add ham to my pizza";
$q(x)$: "I like to add pineapple to my pizza."
then the original proposition is $p(x) \wedge q(x)$, and the negation is $\neg[p(x) \wedge q(x)] \equiv [\neg p(x) \vee \neg q(x)]$.
$\therefore$ the negation is: I like to add no ham or no pineapple or neither to my pizza.

(b) $p(x)$: x studies computer science at the Univerisity of Michigan;
$q(y)$: y will take EECS 203.
The domain of all variables are all students.
then the original proposition is: $p(Bob) \to q(Bob)$, and the negation is $\neg[p(Bob) \to q(Bob)], \equiv [p(Bob) \wedge \neg q(Bob)]$
$\therefore$ the negation is: Bob studies computer science at the University of Michigan, and they will not take EECS 203.

(c) $p(x, y)$: "x has a friend y."
$q(x)$: "x lives in Baits." .
The domains of all variables are all students.
Then the proposition becomes: $\forall x \exists y[p(x, y) \wedge q(y)]$, and the negation is $\neg\forall x \exists y[p(x, y) \wedge q(y)]$
$\equiv \exists x \neg\exists y[p(x, y) \wedge q(y)]$
$\equiv \exists x \forall y \neg[p(x, y) \wedge q(y)$
$\equiv \exists x \forall y[\neg p(x, y) \vee \neg q(y)]$
$\therefore$ the negation is: There exists a student who has no friend that lives in Baits. (Any other student either is not that one's friend or does not live in Baits.)

4. Quantifier Quandary [18 points]

Determine the truth value of each of these statements, where the domain of each quantified variable is all real numbers. **Briefly justify your answers.**

(a) $\exists x(x^3 = -1)$

(b) $\exists x(x^2 = -1)$

(c) $\exists x(x^4 < x^2)$

(d) $\forall x(2x > x)$

(e) $\forall x \exists y(x^2 = y)$

(f) $\forall x \exists y(y^2 = x)$

Solution:

(a) $\exists x(x^3 = -1) \equiv T$.
Consider when $x = -1$, $x^3 = -1$.

(b) $\exists x(x^2 = -1) \equiv F$.
If and only if $x = \pm i$, $x^2 = -1$, but $\pm i$ is not in the domain (all real numbers.) Therefore F.

(c) $\exists x(x^4 < x^2) \equiv T$.
Consider $x = 0.5$, $x^4 = 0.0625$, $x^2 = 0.25$, $x^4 < x^2$,

(d) $\forall x(2x > x) \equiv F$.
Consider $x = -1$, then $2x = -2 < x$
Therefore F.

(e) $\forall x \exists y(x^2 = y) \equiv T$
Let x be an arbitrary real number since real number has a square, there exists a $y = x^2$ that is the square of x.

(f) $\forall x \exists y(y^2 = x) \equiv F$
Consider $x = -1$, since in the domain of all real numbers there is no y that can have a square which is less than 0, no y can match it.

5. Internet Connections [16 points]

A strange Internet outage has struck campus. Some people have internet, but others don't.

- Let $I(x)$ mean "x has internet access"
- Let $F(x, y)$ mean "x is friends with y"

Using the given predicates, logical operators ($\wedge$, $\vee$, $\neg$, $\rightarrow$), and quantifiers ($\forall$, $\exists$), express the following statements. The domain of every quantifier you use must be "students on campus." For purposes of this question, we'll say that $F(x, y)$ always has the same truth value as $F(y, x)$, and so these may be used interchangeably.

(a) Someone does not have internet access.

(b) Nobody is friends with everybody.

(c) Everyone with internet access has a friend without internet access.

(d) Everyone with internet access has *exactly one* friend without internet access.

Solution:

(a) $\exists x \neg I(x)$

(b) $\neg\exists x \forall y F(x, y)$
$(\equiv \forall x \exists y \neg F(x, y))$

(c) $\forall x[I(x) \rightarrow \exists y[\neg I(y) \wedge F(x, y)]]$
Logic: For all x, if $I(x)$, then there exists an y, such that $\neg I(y)$ and $F(x, y)$.

(d) $\forall x[I(x) \rightarrow \exists y[\neg I(y) \wedge F(x, y) \wedge \forall z[[\neg I(z) \wedge F(x, z)] \rightarrow (z = y)]]]$
Logic: For all x, if $I(x)$, then there exists an y, such that $\neg I(y)$ and $F(x, y)$ and searching the whole domain of x, the y must be unique. That means inside the "there exists y" which is inside "for all x", we need to ensure the only result is that y. Therefore we can use a different variable z to search the domain of x to assure the uniqueness of y and put it in the restrictions of "there exists a y".

6. Flip the Switch [18 points]

Determine whether or not each of the following implications is true, regardless of the definition of the predicate $P(x, y)$. Give a brief explanation for your answer.

(a) $\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)$

(b) $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$

Solution:

(a) The truth value of $[\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)]$ depends on truth value of $P(x, y)$.

Proof:
Consider this example: Suppose for $P(x, y)$, when $x < 0$, $y = a$; and when $x \geq 0$, $y = a + 1$. Then $\forall x \exists y P(x, y) \equiv T$.
And in this example, for $y = a$, when $x \geq 0$, $\neg P(x, y)$; for $y = a + 1$, when $x < 0$, $\neg P(x, y)$; for other values of y, $\neg P(x, y)$. Therefore $\exists y \forall x P(x, y) \equiv F$.
$\therefore$ In this circumstance $[\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)] \equiv F$
$\therefore$ the truth value of $[\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)]$ depends on truth value of $P(x, y)$.

(b) $[\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)] \equiv T$ regardless of the definition of the predicate $P(x, y)$.

Proof:
The only circumstance that $[\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)] \equiv F$ is that $\exists y \forall x P(x, y) \equiv T$ and $\forall x \exists y P(x, y) \equiv F$. So let's suppose $\exists y \forall x P(x, y) \equiv T$.

Without Loss Of Generality, assume when $y = a$, $\forall x P(x, y) \equiv T$;
Then $\forall x$, at least when $y = a$, $P(x, y) \equiv T$.
$\therefore$ regardless of $P(x, y)$, $\exists y \forall x P(x, y) \equiv T$.