

EECS 203: Discrete Mathematics
Fall 2023
Homework 6

Due **Friday, October 20**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $5 + 2$

Total Points: $100 + 33$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Set Me Free [18 points]

- (a) For each of the following, determine if the statement is true or false. Justify your answers.
- (i) $\emptyset \in \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
 - (ii) $\emptyset \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
 - (iii) $\emptyset \times \{0, 1\} \subseteq \emptyset$
 - (iv) $\{\{1, 2\}\} \subsetneq \{\{1, 2\}, \{2\}\}$
- (b) Find the cardinality of $\{\emptyset, \emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$.
- (c) (i) Find the cardinality of $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$.
(ii) Find $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$. Make sure it has the cardinality you wrote in (i).

Solution:

- (a) (i) False.
 $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ has only two elements: $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$, \emptyset is not an element.
- (ii) True.
 \emptyset is a subset of every set. So it is also a subset of $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.
- (iii) True.
The Cartesian product of sets A and B is: $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.
When A is \emptyset , $a \in A$ is false for any a . $\therefore a \in A \wedge b \in B$ is also false.
 \therefore there is no such (a, b) that can be an elements of $\emptyset \times B$.
 $\therefore \emptyset \times \{0, 1\} = \emptyset$
 $\therefore \emptyset \subseteq \emptyset$
 $\therefore \emptyset \times \{0, 1\} \subseteq \emptyset$.
- (iv) True.
The only element in $\{\{1, 2\}\}$ is $\{1, 2\}$.
And this element is also in $\{\{1, 2\}, \{2\}\}$.
 $\therefore \{\{1, 2\}\} \subseteq \{\{1, 2\}, \{2\}\}$.
Notice the element $\{2\}$, $\{2\} \in \{\{1, 2\}, \{2\}\}$, $\{2\} \notin \{\{1, 2\}\}$
 $\therefore \{\{1, 2\}\} \subsetneq \{\{1, 2\}, \{2\}\}$.
- (b) Since the two \emptyset in $\{\emptyset, \emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$ are identical, we should remove one to simplify it.
And since the two \emptyset in the element $\{\emptyset, \emptyset\}$ in \emptyset in $\{\emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$ are identical, we

must remove one to simplify it.

After doing that, we have two identical elements $\{\emptyset\}$ in $\{\emptyset, \{\emptyset\}, \{\emptyset\}\}$, so we must remove one $\{\emptyset\}$ to simplify it.

\therefore the simplest form of the original set is: $\{\emptyset, \{\emptyset\}\}$.

So the cardinality is 2 since there are two elements in the set.

- (c) (i) There are 3 elements in $\{\emptyset, \{a\}, \{b, c\}\}$.
 So there are $2^3 = 8$ elements in $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$.
 \therefore the cardinality of $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$ is 8.
- (ii) $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\}) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b, c\}\}, \{\{a\}, \{b, c\}\}, \{\emptyset, \{b, c\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{a\}, \{b, c\}\}\}$.

2. Is This Your Card(inality)? [18 points]

Suppose we define $A = \{m, a, r, c, h\}$, $B = \{a, r, t, i, c, h, o, k, e\}$, and $C = \{a, e, i, o, u\}$. Determine the cardinality of each of the following sets. You may leave large exponents unsimplified.

- (a) $A \times B$
- (b) $\mathcal{P}(A \cup B)$
- (c) $\mathcal{P}((A \cap B) \times (B - C))$

Solution:

- (a) $\because |A| = 5, |B| = 9,$
 $\therefore |A \times B| = 5 \times 9 = 45.$
- (b) $|A \cup B| = |A| + |B| - |A \cap B|$
 $A \cap B = \{a, r, c, h\}, |A \cap B| = 4$
 $\therefore |A \cup B| = 5 + 9 - 4 = 10$
 $\therefore \mathcal{P}(|A \cup B|) = 2^{10} = 1024$
- (c) $A \cap B = \{a, r, c, h\},$
 $B - C = \{r, t, c, h, k\},$
 $\therefore |(A \cap B) \times (B - C)| = 4 \times 5 = 20.$
 $\therefore |\mathcal{P}((A \cap B) \times (B - C))| = 2^{20} = 1024 \times 1024 = 1048576$

3. Ice Cream Truck! [20 points]

There are 40 IAs for EECS 203 and they are all waiting in line for the ice cream truck! At the ice cream truck, every IA orders at least one flavor of ice cream, but they have the option to order more.

- 22 of them ordered chocolate ice cream
- 25 of them ordered strawberry ice cream
- 16 of them ordered vanilla ice cream

These numbers add up to more than 40, so we know that some IAs must have ordered multiple flavors. We also know:

- 12 IAs ordered chocolate ice cream and strawberry ice cream,
- 8 ordered strawberry ice cream and vanilla ice cream,
- 10 ordered vanilla cream and chocolate ice cream,
- and of these, some IAs ordered all three.

How many IAs ordered all three flavors? Show all your calculations.

Note: A Venn diagram is not acceptable justification for this question.

Solution:

We use A to represent the set of IAs who ordered chocolate ice cream;

and B : the set of IAs who ordered strawberry ice cream;

and C : the set of IAs who ordered vanilla ice cream;

Then the given information is that:

Above all, $|A \cup B \cup C| = 40$.

And $|A| = 22$, $|B| = 25$, $|C| = 16$.

And $|A \cap B| = 12$, $|B \cap C| = 8$, $|A \cap C| = 10$.

Since $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$,

$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| = 40 - 22 - 25 - 16 + 12 + 8 + 10 = 7$

4. Subset Fun [20 points]

Let A , B , and C be sets. Prove that

$$(B - A) \cup (C - A) = (B \cup C) - A$$

by showing that each is a subset of the other.

Solution:

- (a) We prove that $(B - A) \cup (C - A) \subseteq ((B \cup C) - A)$, that is, any element in $(B - A) \cup (C - A)$ is also in $(B \cup C) - A$.

Let x be an arbitrary element in the domain.

If $x \in (B - A) \cup (C - A)$, $x \in (B - A) \vee x \in (C - A)$ due to definition of set union.

$\therefore (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A)$ due to definition of set subtraction.

$\therefore (x \in B \wedge \neg x \in A) \vee (x \in C \wedge \neg x \in A)$

$\therefore \neg x \in A \wedge (x \in B \vee x \in C)$ due to logic operator's distributive law.

$\therefore (x \in B \cup C) \wedge (x \notin A)$

$\therefore x \in ((B \cup C) - A)$ due to definition of set union.

$\therefore (B - A) \cup (C - A) \subseteq ((B \cup C) - A)$ due to definition of subset.

- (b) We prove that $((B \cup C) - A) \subseteq (B - A) \cup (C - A)$, that is, any element in $(B \cup C) - A$ is also in $(B - A) \cup (C - A)$.

Let x be an arbitrary element in the domain.

If $x \in ((B \cup C) - A)$, $(x \in (B \cup C)) \wedge (x \notin A)$ due to definition of set subtraction.

$\therefore ((x \in B) \vee (x \in C)) \wedge (\neg x \in A)$ due to definition of set union.

$\therefore (x \in B \wedge \neg x \in A) \vee (x \in C \wedge \neg x \in A)$ due to logic operator's distributive law.

$\therefore (x \in (B - A)) \vee (x \in (C - A))$ due to definition of set subtraction.

$\therefore ((B \cup C) - A) \subseteq (B - A) \cup (C - A)$ due to definition of subset.

Since $(B - A) \cup (C - A) \subseteq ((B \cup C) - A)$ and $((B \cup C) - A) \subseteq (B - A) \cup (C - A)$,
 $(B - A) \cup (C - A) = (B \cup C) - A$.

5. Subset Size Question [24 points]

Given that A, B, C are sets with $|A| = 13$, $|B| = 8$, $|C| = 10$, find the maximum *and* the minimum possible cardinalities of the following sets in the domain of integers.

- (a) $\overline{A} \cap B$
- (b) $A \cap (B \cup C)$
- (c) $\mathcal{P}(B \times C)$
- (d) $(A - B) \cap C$

Solution:

- (a) Due to intersection, $|\overline{A} \cap B| \leq |B| = 8$
 Since a set cannot have fewer than 0 element, $|\overline{A} \cap B| \geq 0$.
 When $A \cap B = \emptyset$, then $\overline{A} \cap B = B$, then $|\overline{A} \cap B| = |B| = 8$.
 When $B \subseteq A$, $A \cap B = B$, then $\overline{A} \cap B = \emptyset$, then $|\overline{A} \cap B| = |\emptyset| = 0$.
 \therefore These two situations are all possible,
 \therefore the maximum possible cardinality is 8, and the minimum cardinality of $\overline{A} \cap B$ is 0.
- (b) Due to union, $|B \cup C| \geq \max(|B|, |C|) = 10$ and $|B \cup C| \leq |B| + |C| = 18$.
 Due to intersection, $|A \cap (B \cup C)| \geq 0$ and $|A \cap (B \cup C)| \leq \min(|A|, |B \cup C|)$.
 $\therefore |A| = 13, 10 \leq |B \cup C| \leq 18$,
 When $10 \leq |B \cup C| < 13$, $\min(|A|, |B \cup C|) = |B \cup C| < 13$;
 When $13 \leq |B \cup C| \leq 18$, $\min(|A|, |B \cup C|) = |A| = 13$.
 \therefore the maximum value of $|A \cap (B \cup C)|$ is 13.
 \therefore The minimum possible cardinality is 0 and maximum possible cardinality is 13.
- (c) Since $|B| = 8, |C| = 10$, B, C cannot be \emptyset .
 $\therefore |B \times C|$ can only be $|B| \times |C| = 80$.
 $\therefore \mathcal{P}(B \times C)$ can only be $2^{|B \times C|} = 2^{80}$
 \therefore The maximum possible cardinality and the minimum possible cardinality are both 2^{80} .
- (d) Due to subtraction, $|A| - |B| \leq |A - B| \leq |A|$.
 $\therefore 5 \leq |A - B| \leq 13$.
 Due to intersection, $0 \leq |(A - B) \cap C| \leq \min(|A - B|, |C|)$
 When $|A - B| < 10$, $\min(|A - B|, |C|) = |A - B| < 10$. When $|A - B| \geq 10$, $\min(|A - B|, |C|) = |C| = 10$.
 \therefore the overall range is $0 \leq |(A - B) \cap C| \leq 10$.
 \therefore the maximum possible cardinality is 10 and the minimum possible cardinality is 0.