Groupwork

1. Grade Groupwork 6

Using the solutions and Grading Guidelines, grade your Groupwork 6:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2	+3	+4	+1	+2	0	+1	0	+1	+2	+1		15 /17
Problem 3	+4	+2	+4	0	0							10 /16
Total:												25 /33

Previous Groupwork 6(1): (Set)ting up a (Power)ful Proof [17 points]

(a) Suppose we want to prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$. What is the Inductive Hypothesis for your proof? *Hint: Consider what variable you should do induction on.*

(b) Prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$.

Solution:

(a) for $S(k)$ be an arbitrary set s.t. $|S(k)| = k$. Assume that $\mathcal{P}(S(k)) = 2^k$.

(b) Let k be an arbitrary nonnegative integer.
Let $S(k)$ be an arbitrary set s.t. $|S(k)| = k$, i.e. $S(k)$ has k elements.
Assume: $\mathcal{P}(S(k)) = 2^k$. +1
+3 Want to show: $\mathcal{P}(S(k+1)) = 2^{k+1}$.

We can do even better!

Base Case:
$k = 0. S(k) = \emptyset = \{\}$ +4
The only subset of $S(k)$ is $\emptyset$.
$\therefore \mathcal{P}(S(k)) = 1 = 2^0$ is true.

Inductive Step: i.e. remove an element. $P(S') = 2^{|S'|}$
Mark the elements in $S(k)$ as $a_1, a_2, ..., a_k$, i.e., $S(k) = \{a_1, a_2, ..., a_k\}$.
And mark the new element in $S(k+1)$ as a_{k+1}, i.e., $S(k) = \{a_1, a_2, ..., a_{k+1}\}$.
Then all the new subsets created by a_{k+1} is exactly a subset of $S(k)$ plus the element a_{k+1}.
(−1−1 for not clear) +1+1
And every subset plus an element of a_{k+1} is a included in $\mathcal{P}(S(k+1))$.
$\therefore$ The number of elements in $\mathcal{P}(S(k+1))$ is $|\mathcal{P}(S(k))| + |\mathcal{P}(S(k))| = 2|\mathcal{P}(S(k))| = 2 \times 2^k = 2^{k+1}$
+2 $\therefore |\mathcal{P}(S(k+1))| = 2^{k+1}$ +2
$\therefore$ we have proved that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$. +1

Previous Groupwork 6(2): Out of the ordinary [16 points]

The (von Neumann) ordinals are a special kind of number. Each one is represented just in terms of sets. We can think of every natural number as an ordinal. We won't deal with it in this question, but there are also infinite ordinals that "keep going" after the natural numbers, which works because there are infinite sets.

The smallest ordinal, 0, is represented as $\emptyset$. Each ordinal after is represented as the set of all smaller ordinals. For example,

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

(a) What is the ordinal representation of 4, in terms of just sets?

(b) If the sets X and Y are ordinals representing the natural numbers x and y respectively, how can we tell if $x \leq y$ in terms of X and Y? Why does this work?

(c) If X is the ordinal representing the natural number x, what is the ordinal representation of $x + 1$? Why does this work?

Solution:

Mistake is acceptable. We will do better next time!

(a) $4 = \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$ +4

(b) We can tell $x \leq y$ by comparing the cardinality of X and Y.
+2 i.e., if $|X| \leq |Y|$, then $x \leq y$. (iff, but does not affect the conclusion)
+4 This works because every ordinal is represented as the set of all smaller ordinals than it, so it has more elements than ordinals smaller than it.

(c) It is $\mathcal{P}(X)$. ✗ $X \cup \{X\}$
Since every ordinal is represented as the set of all smaller ordinals than it,
$x+1$ is represented as $\{0, 1, 2, 3, ..., x\}$, in which $0, 1, 2, 3, ...x - 1$ are all the subsets of X but X itself, and plus X itself, they form a set of all the subsets of X, which is also defined as the Power Set of X, i.e. $\mathcal{P}(X)$.
This can be proved by induction.
Assume $x = \{0, 1, ..., x - 1\} = \mathcal{P}(X - 1)$
Base Case:
$0 = \emptyset, 1 = \{0\} = \{\emptyset\} = \mathcal{P}(0)$
Inductive Step:
$x + 1 = \{0, 1, ..., x - 1, x\}$, $0, 1, ...x - 1$ is all the elements in $\mathcal{P}(X - 1)$ which is all the subsets of the ordinal of $x - 1$, and by plusing x, we have all the subsets of x in the set.
$\therefore$ it is $\mathcal{P}(X)$.

it's only one element not number of all elements in one set, so it's wrong

2. Raise the Roof [16 points]

Let $f : \mathbb{R} \to \mathbb{Z}$, where f is defined as $f(x) = \left\lceil \frac{x+5}{2} \right\rceil + 12$. Is f onto? Prove your answer.

Solution:

f is onto.
i.e. $\forall b \in \mathbb{Z}, \exists a \in \mathbb{R}$ such that $b = f(a)$. Proof:
Let b be an arbitrary integer.
Consider $a = 2b - 29$.
Then a is also an integer, $a \in \mathbb{Z} \subseteq \mathbb{R}$, is in the domain.

Then

$$\begin{aligned} f(a) &= \lceil \frac{2b-29+5}{2} \rceil + 12 \\ &= \lceil \frac{2b-24}{2} \rceil + 12 \\ &= \lceil b-12 \rceil + 12 \end{aligned}$$

Since b is an integer, $\lceil b-12 \rceil = b-12$.
$\therefore f(a) = b - 12 + 12 = b$.
$\therefore \forall b \in \mathbb{Z}, \exists a \in \mathbb{R}$ such that $b = f(a)$.
$\therefore f$ is onto.

3. You Mod Bro? [14 points]

Find all solutions of the congruence $12x^2 + 25x \equiv 10 \pmod{11}$.

Solution:

$12x^2 + 25x \equiv 10 \pmod{11}$
$(11+1)x^2 + (2 \cdot 11 + 3)x \equiv 10 \pmod{11}$
$x^2 + 3x - 10 \equiv 0 \pmod{11}$
$(x+5)(x-2) \equiv 0 \pmod{11}$
$\therefore x \equiv 2 \pmod{11}$ or $x \equiv -5 \pmod{11} \equiv 6 \pmod{11}$.
$\therefore$ The solutions are: all integer x satisfying $x \equiv 2 \pmod{11}$ or $x \equiv 6 \pmod{11}$.