EECS 203: Discrete Mathematics
Fall 2023
Homework 9

Due **Thursday, November 16**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $9 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

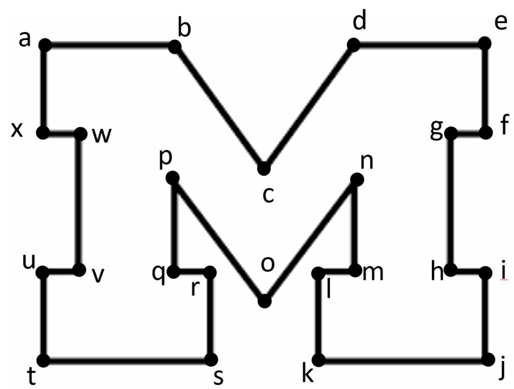
Individual Portion

1. Shortest Paths [12 points]

Provide the shortest path distances between the following point pairings in their respective graphs. Justify your answer by providing a shortest path, or by stating that there is no such path.

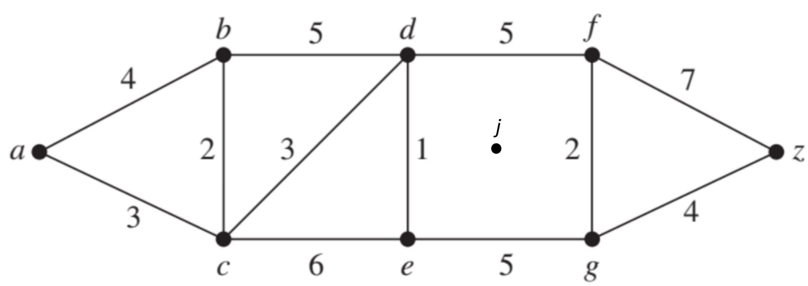
(a) (i) (x, p)

(ii) (c, o)



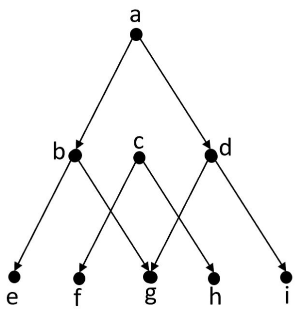
(b) (i) (a, z)

(ii) (b, j)



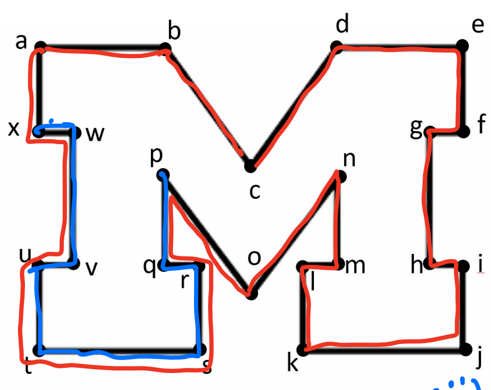
(c) (i) (a, g)

(ii) (i, b)



Solution:

(a)



(ii)

(i) The shortest path distance between (x, p) is 8. Because there are two paths from x to p, one' distance is 8, another is 24 − 8 = 16.

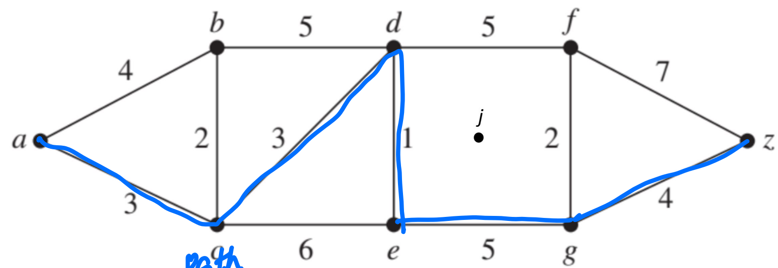
the shortest path between (x, p) is x → w → v → u → t → s → r → q → p

(ii) The shortest path distance between (c, o) is 12, because due to symmetry, there are 2 shortest paths between (c, o), each is 24/2 = 12 in distance

one is: c → d → e → f → g → h → i → j → k → l → m → n → o

another is: c → b → a → x → w → v → u → t → s → r → q → p → o

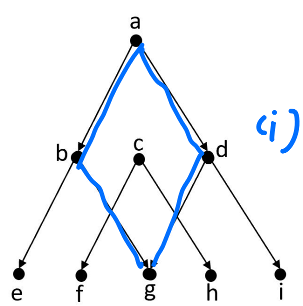
(b)



(i) The shortest path distance between (a,z) is $3+3+1+5+4=16$. The shortest path is $a \xrightarrow{3} c \xrightarrow{3} d \xrightarrow{1} e \xrightarrow{5} g \xrightarrow{4} z$.

(ii) The shortest path distance between (b,j) is ∞ because they are not connected (no such path)

(c)



(i) The shortest path distance between (a,g) is 2 because both paths $a \to b \to g$ and $a \to d \to g$ are shortest, of the distance of 2.

(ii) The shortest path distance between (i,b) is ∞ because they are not connected (no such path)

2. Alexander Hamiltonian [12 points]

For each of the following parts, state whether a graph with $n \geq 3$ vertices and the given properties **always**, **sometimes**, or **never** contains a Hamiltonian cycle. Justify your response for each part.

(a) The complete graph K_n.

(b) A tree.

(c) A bipartite graph.

(d) A graph that contains a vertex v where $\deg(v) = 1$.

Solution:

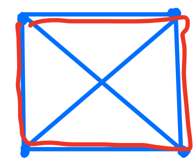
(a) K_n always contain a Hamiltonian cycle

Proof Let $V = \{a_1, a_2, a_3, \dots, a_n\}$

$K_n \Rightarrow \forall i \in \{1, 2, \dots, n\}$, a_i is connected with $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n$

$\therefore \{a_1, a_2\}, \{a_2, a_3\}, \dots \{a_{n-1}, a_n\}, \{a_n, a_1\}$ are edges.

$\therefore$ Hamiltonian path $a_1 \to a_2 \to \dots \to a_n \to a_1$ exists

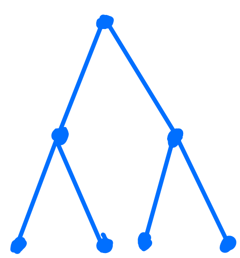


(b) A tree never contains a Hamiltonian cycle.

Proof. $\because$ A tree does not contain a cycle subgraph

And a Hamiltonian cycle is a cycle subgraph where $V_H = V_G$, $E_H = E_G$

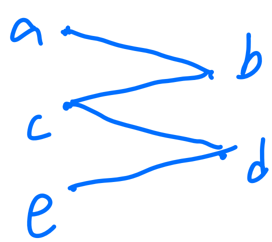
$\therefore$ A tree does not contain a Hamiltonian cycle.



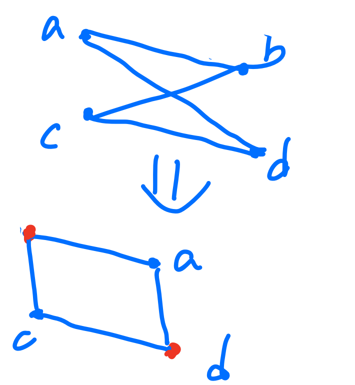
(c)

(d) A bipartite graph sometimes has a Hamiltonian cycle.

Example 1. No Hamiltonian cycle.



Example 2 has Hamiltonian $a \to b \to c \to d \to a$



$\therefore$ Both case exists

A graph that contains a vertex of $deg(v) = 1$ never has a Hamiltonian cycle.

Proof: If a graph has a Hamiltonian cycle, then every vertex has to has an edge with at least 2 other vertices.

$\therefore \forall v,\ deg(v) \geq 2$

$\therefore$ If $\exists v$ s.t. $deg(v) = 1$, it cannot has a Hamiltonian cycle.

3. Bipartite? [12 points]

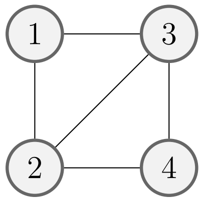
For each of the following parts, determine if the graph is bipartite and justify your answer.

(a) The cycle C_5

(b) The hypercube Q_2

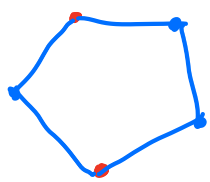
(c) The complete graph K_5

(d)



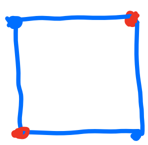
Solution:

(a)



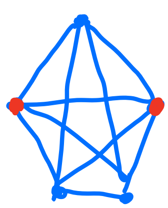
C_5 is not a bipartite because it is not 2-colorable since C_5 itself is it's odd cycle subgraph

(b)



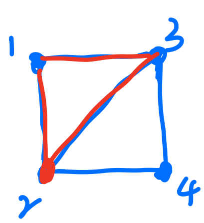
Q_2 is a bipartite because it is 2-colorable, shown in the sketch.

(c)



K_5 is not a bipartite because it is not 2-colorable since it has a C_5 odd cycle subgraph

(d)



The graph is not a bipartite because vertices 1, 2, 3 form an odd cycle subgraph

4. Melman the Graph [12 points]

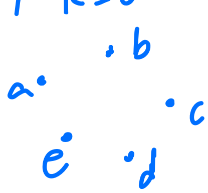
In a simple, undirected graph with 5 vertices, what are all possible values for the number of vertices with odd degree? Justify your answer.

For each possible value k, construct a graph with 5 vertices that has k vertices of odd degree.

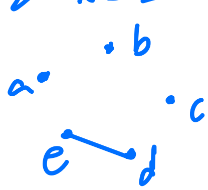
Solution:

According to the handshake theorem, the number of vertices with odd degree must be even, so the only possible circumstances are $k=0$, $k=2$, $k=4$

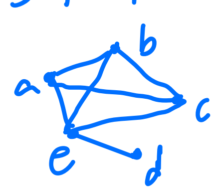
1° $k=0$



2° $k=2$



3° $k=4$



$\deg(a)=3$
$\deg(b)=3$
$\deg(c)=3$
$\deg(d)=1$
$\deg(e)=4$

$\therefore$ $k=0$, $k=2$, $k=4$ are all possible.

5. Keeping things Merry! [12 points]

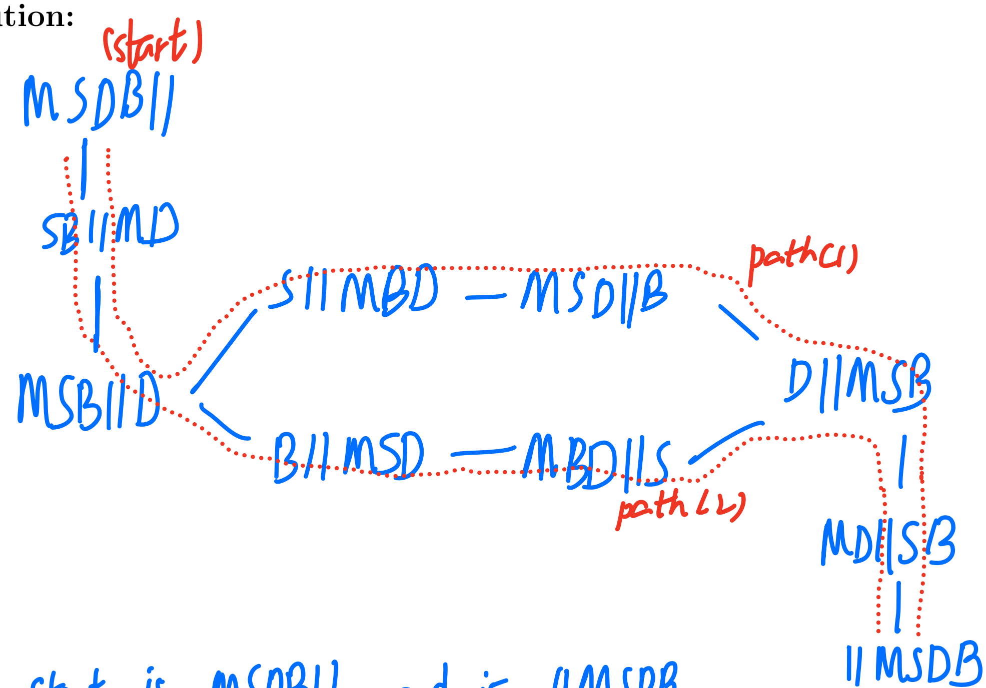
Mary has three dogs, Sarge, Duke, and Brady. However, when left alone, Duke will pick a fight with either of the other two dogs. Mary wants to walk all 3 dogs from her house across the road to the dog park, can only walk 1 dog at a time, and wants to avoid a fight between any of the dogs. She can make multiple trips across the road, and can leave some of her dogs on one side of the road when walking another across.

(a) Draw a graph where the nodes are the legal configurations of the puzzle and the edges represent possible transitions from one state to another. Each node should contain M, S, D, B, and || representing Mary, Sarge, Duke, Brady, and the road. For instance SB||MD would represent the Sarge and Brady being the the left side of the road at home with Mary and Duke on the other side at the park. Start with MSDB|| as your initial state.

(b) Identify the nodes that represent the start and end configurations of the puzzle. Is the puzzle solvable? Explain in terms of your graph.

Solution:

(a)



(b) start is MSDB||, end is ||MSDB.
The question is: to find a path between node MSDB|| and ||MSDB. So looking at our graph, the puzzle is solvable. There are two such paths:

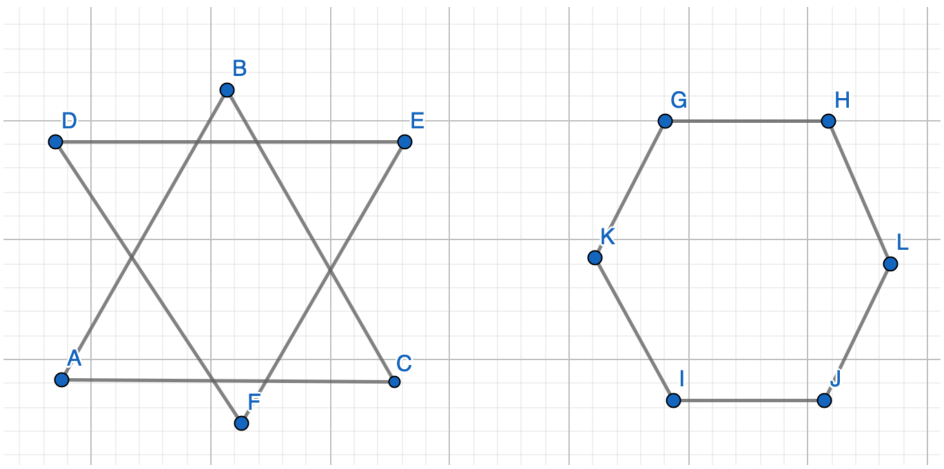
(1) MSDB|| — SB||MD — MSB||D — S||MBD — MSD||B — D||MSB — MD||SB — ||MSDB

(2) MSDB|| — SB||MD — MSB||D — B||MSD — MBD||S — D||MSB — MD||SB — ||MSDB

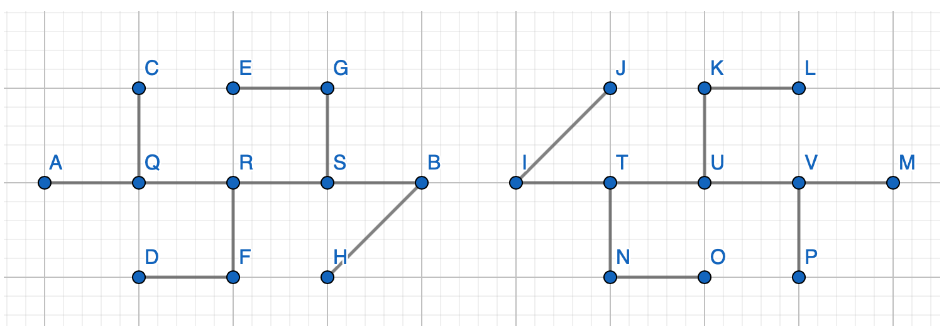
6. It's Iso-Morphin' Time [12 points]

Determine whether or not each of the following pairs of graphs are isomorphic. If yes, provide an isomorphism. If not, explain why.

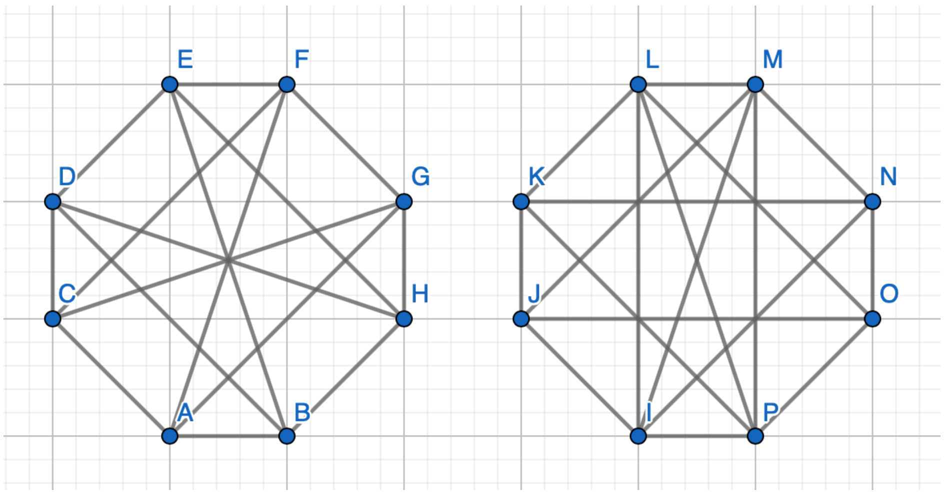
(a)



(b)

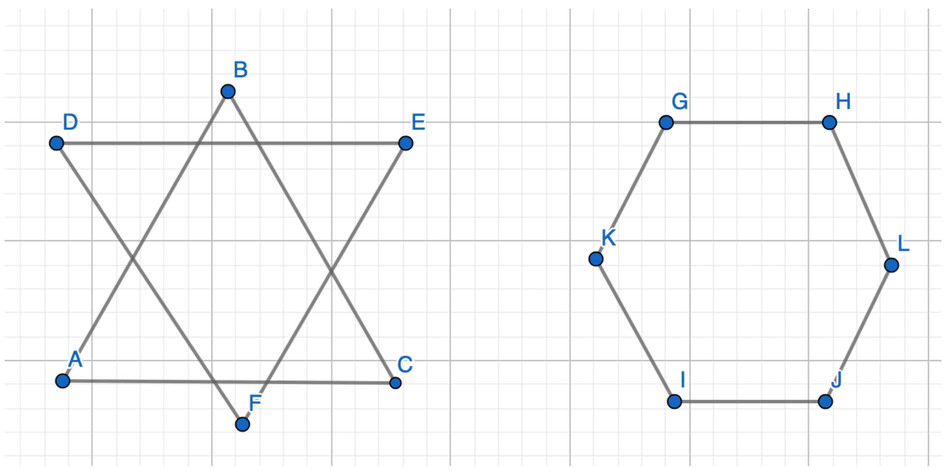


(c)



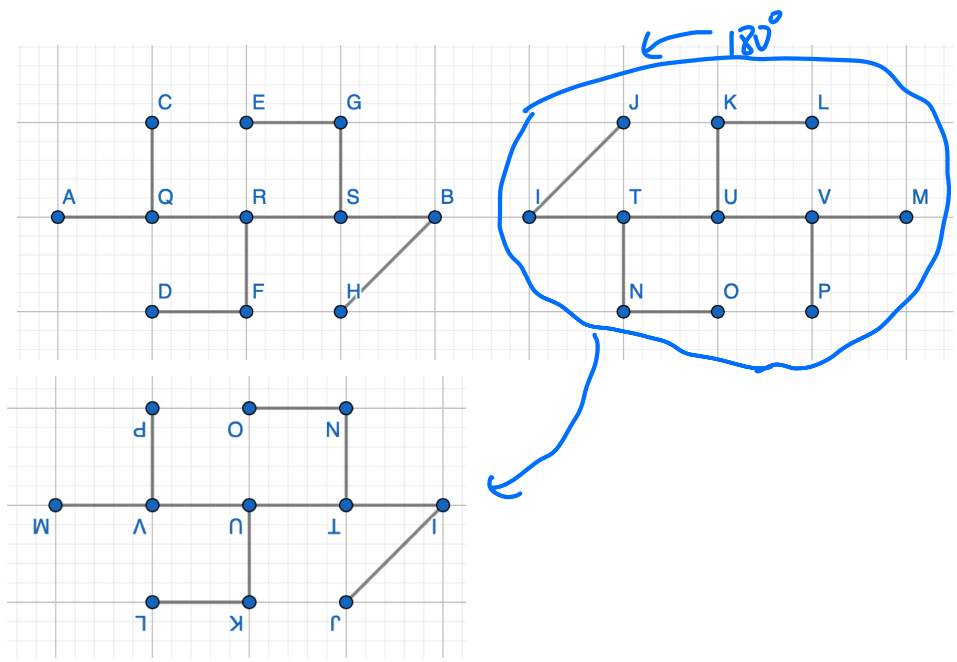
Solution:

(a)



They are not isomorphic. The graph on the left has 2 cycle subgraphs but the graph on the right only has 1 cycle subgraph.

(b)

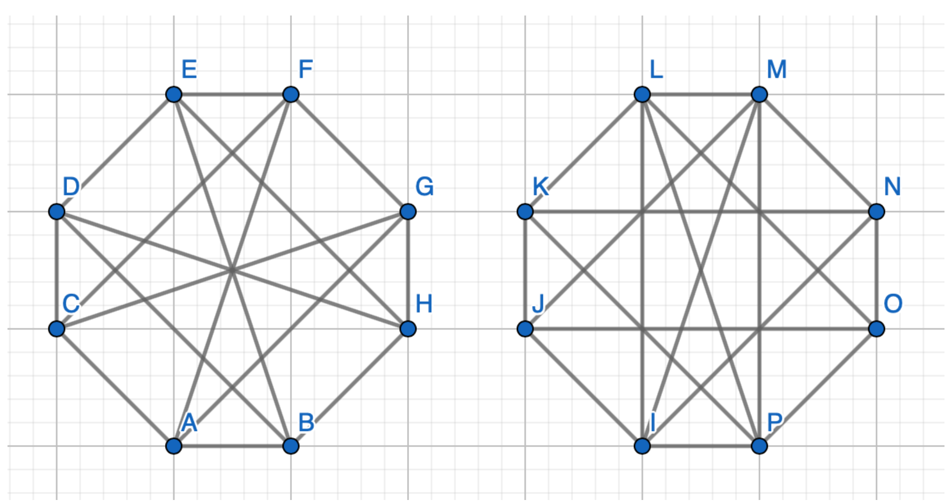


Isomorphic.

bijection: $f(A)=M$, $f(C)=P$, $f(Q)=V$, $f(R)=U$, $f(F)=K$, $f(D)=L$,

$f(S)=T$, $f(G)=N$, $f(E)=O$, $f(B)=I$, $f(H)=J$

(c)



They are not isomorphic.

Consider L, $\deg(L) = 5$

They graph on the left has no vertex that has a degree of 5.

7. Reduce, Reuse, Recycle [12 points]

For which values of $n \geq 4$ do these graphs have an Euler cycle?

(a) The complete graph K_n

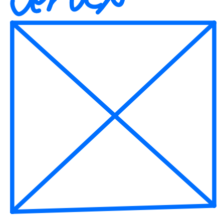
(b) The cycle C_n

(c) The wheel W_n

(d) The hypercube Q_n

Solution:

(a) We know for K_n, every vertex has an edge with any other vertex, so $\forall v \in V$, $\deg(v) = n-1$

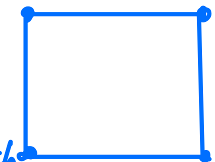


By Euler's Theorem, a graph has an Euler cycle iff. all vertices have even degree, so K_n has an Euler cycle iff $n-1$ is even $\Rightarrow$

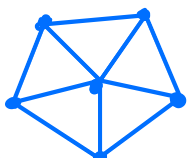
n is odd

$\therefore$ for all $n = 2k+1$ $(k \geq 2, k \in \mathbb{Z})$, K_n has an Euler cycle.

(b) For C_n, every vertex has an edge with exactly 2 other vertices, so $\forall v \in V$, $\deg(v) = 2$ which is even, so by Euler's Theorem, C_n has an Euler Cycle for all integers $n \geq 4$.

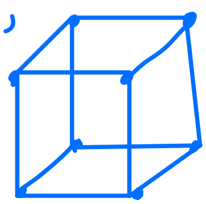


(c) W_n can never has an Euler Cycle in W_n for any n.



Because $\forall v \in V$ that isn't the central vertex, its degree is 3. (one edge with the central vertex, two with its adjacent vertices) which is odd, so by Euler's Theorem, it can never has an Euler Cycle.

(d) We know that for Q_n, every vertex has degree of n, so by Euler's Theorem, it has an Euler Cycle when n is even.



i.e. when $n = 2k$ $(k \in \mathbb{Z}, k \geq 2)$

8. Counting [8 points]

How many positive integers less than 1000

(a) are divisible by 7 but not by 11?

(b) have distinct digits?

(c) have distinct digits and are even?

You do **not** need to simplify your answer.

Solution:

(a) $A = \{ n \mid 7 \mid n \wedge n \in \mathbb{Z}^+ \wedge n < 1000 \}$

$B = \{ n \mid 7 \mid n \wedge 11 \mid n \wedge n \in \mathbb{Z}^+ \wedge n < 1000 \}$

$= \{ n \mid 77 \mid n \wedge n \in \mathbb{Z}^+ \wedge n < 1000 \}$

$\therefore$ the answer is $|A| - |B| = \lfloor \frac{1000}{7} \rfloor - \lfloor \frac{1000}{77} \rfloor$

$= 142 - 12 = 130$

(b) $\underbrace{9}_{1 \sim 9} + \underbrace{9 \times (10-1)}_{10 \sim 99} + \underbrace{9 \times (10-1) \times (10-1-1)}_{100 \sim 999}$

$= 738$

(c) $\underbrace{4}_{1\text{st}} + \underbrace{1\times 9 + 4\times 8}_{2^{nd}=0,\ 2^{nd}\neq 0} + \underbrace{9\times 8}_{3^{rd}=0} + \underbrace{4\times 8}_{2^{nd}=0} + \underbrace{4\times 8\times 7}_{\text{no } 0}$

$= 4 + 41 + 328$

$= 373$

9. (Not So) Round and Round [8 points]

Suppose we have a square-shaped table which seats 3 people on each side. How many ways are there to seat 12 people at the table, where seatings are considered the same if everyone is in the same group of 3 on a side?

Solution: Step 1. Pick out members of the 4 groups:

$$\underbrace{C(12,3)}_{1^{st}\text{ group}} \cdot \underbrace{C(9,3)}_{2^{nd}\text{ group}} \cdot \underbrace{C(6,3)}_{3^{rd}\text{ group}} \cdot \underbrace{1}_{4^{th}\text{ group}}$$

Step 2.: Since order of groups does not matter, and we have permutations of group A·B·C·D here, we need to divide the result by P(4,4)

$\therefore$ result $= \dfrac{C(12,3)\cdot C(9,3)\cdot C(6,3)\cdot 1}{4!} = 15400$