Exam 3
QUESTIONS PACKET
EECS 203
Practice Exam 2

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

8-Digit UMID: ______________________________

*****MAKE SURE YOU HAVE PROBLEMS 1 - 19 IN THIS BOOKLET.*****

General Instructions

You have 120 minutes to complete this exam. You should have two exam packets.

- **Questions Packet:** Contains ALL the questions for this exam, worth 100 points total:
  - 10 Single Answer Multiple Choice questions (4 points each),
  - 2 Multiple Answer Multiple Choice questions (4 points each),
  - 5 Short Answer questions (6-8 points each), and
  - 2 Free Response questions (10 points each)

  Questions Packet is for scratch work only. Work in this packet will not be graded.

- **Answers Packet:** Write all of your answers in the Answers Packet, including your answers to multiple choice questions.

  For free response questions, you must show your work! Answers alone will receive little or no credit.

- You may bring **one** 8.5" by 11" note sheet, front and back, created by you.
- You may **NOT** use any other sources of information, including but not limited to electronic devices (including calculators), textbooks, or notes.
- After you complete the exam, sign the Honor Code on the front of the Answers Packet.

Part A1: Single Answer Multiple Choice

Problem 1. (4 points)

Suppose two fair six-sided dice are rolled, one yellow and the other blue. Given their sum is less than or equal to 5, what is the probability that the value of the blue die is 2?

(a) $\frac{3}{10}$

(b) $\frac{1}{3}$

(c) $\frac{4}{10}$

(d) $\frac{1}{2}$

(e) $\frac{2}{3}$

Solution: (a)
Let $E = \{\text{sum is} \leq 5\}$ and $F = \{\text{blue is } 2\}$. The problem asks for $P(F|E)$. The outcomes, as (yellow, blue) pairs, are all equally likely, so by definition of conditional probability:

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{|E \cap F|}{|E|}.$$

- We have 1 way to sum to 2, 2 ways to sum to 3, 3 ways to sum to 4, and 4 ways to sum to 5; thus $|E| = 10$ and $P(E) = 10/36 = 5/18$.
- Of these 10 outcomes, the (yellow, blue) values such that the blue die comes up 2 are (1, 2), (2, 2), and (3, 2). So $|E \cap F| = 3$ and $P(E \cap F) = 3/36$.

Therefore $P(F|E) = 3/10$. Alternatively, by Bayes' Rule:

- Given that the blue die comes up 2, the conditional probability that the sum is ≤ 5 is the same as the conditional probability that the yellow die comes up ≤ 3. Thus $P(E|F) = 3/6 = 1/2$.
- $P(F) = 1/6$ because the dice are fair.
- From above, $P(E) = 5/18$.

$$P(F|E) = \frac{P(E|F) \cdot P(F)}{P(E)} = \frac{1/2 \cdot 1/6}{5/18} = \frac{1/2}{5/3} = \frac{3}{10}.$$

Problem 2. (4 points)

Nolan wears his lucky EECS 203 staff T-shirt to all of his job interviews. When Nolan wears the lucky shirt, he has a 70% chance of receiving the job offer. Suppose Nolan has 5 upcoming interviews this week (yikes!). What is the probability that he will receive exactly 3 job offers?

(a) $3!\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$

(b) $\binom{5}{3}\left(\frac{7}{10}\right)^2\left(\frac{3}{10}\right)^3$

(c) $\binom{5}{3}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$

(d) $P(5,3)\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$

(e) $\frac{7^3 \cdot 3^2}{10^5}$

Solution: (c) Bernoulli trials/binomial distribution
The interviews are independent, identically distributed Bernoulli trials, where getting a job offer is a "success". The number of job offers received is a random variable that follows the binomial distribution, made up of 5 of these trials. So, we can calculate the probability of getting exactly 3 successes in 5 Bernoulli trials, each with probability of success of 0.7.

- P(gets offer) = 7/10
- P(does not get offer) = 3/10

P(exactly 3 successes in 5 trials) = $\binom{5}{3} \cdot (7/10)^3 \cdot (3/10)^2$

Problem 3. (4 points)

A bucket contains 20 indistinguishable Red Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples. How many ways are there to choose 3 apples from the bucket?

(a) 9

(b) 10

(c) 11

(d) 12

(e) 13

Solution: (b) - We can choose three apples in the following ways:

- Choose 3 apples of one kind (apples of the same kind are indistinguishable, so we simply need to choose one of the three types).
- Choose 2 apples of one kind, and 1 apple of a different kind (which can be done by selecting the type we will take 2 of and then the type we will take one of). This can be done $3 \cdot 2 = 6$ ways by the product rule.
- Choose 1 apple of each kind (which can be only done in one way).

By the sum rule, there are $3 + 6 + 1 = 10$ ways to choose these three apples.

Problem 4. (4 points)

What is the tightest big-O bound of the function: $f(n) = (n^2 \log n)(5n^3 + 4)(n^4 + 3^n)$?

(a) $O(n^5 \log n)$

(b) $O(3^n)$

(c) $O(3^n n^5 \log n)$

(d) $O(3^n n^3 \log n)$

(e) $O(n^5 \log n + 3^n)$

Solution: (c)
Begin by applying the sum rule to each expression to see that the 3 terms in the product are $O(n^2 \log n)$, $O(5n^3)$, and $O(3^n)$, respectively. Then, by the product rule, $f(n)$ is $O(3^n 5n^3 \log n)$. Dropping scalar multiples, we see that $f(n)$ is $O(3^n n^5 \log n)$.

Problem 5. (4 points)

What is the runtime complexity of the following algorithm?

function GOOFYLOOP($n \in \mathbb{N}$)
  for $i \leftarrow 1$ to 10 **do**
    GOOFYLOOP($\lfloor \frac{n}{5} \rfloor$)
  end for
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to 20 **do**
      print "I'm stuck in a loop"
    end for
  end for
  print "I'm freeee"
end function

(a) $\Theta(n)$

(b) $\Theta(n \log n)$

(c) $\Theta(n^{10})$

(d) $\Theta(n^{10} \log n)$

(e) $\Theta(n^{\log_5 10})$

Solution: (e)
The first loop executes 10 recursive calls on inputs of size $\lfloor n/5 \rfloor$. For the second set of (nested) loops: the outer loop executes n times, and the inner loop executes 20 times for each iteration of the outer loop, for a total of $20n$ times that "I'm stuck in a loop" gets printed. Finally "I'm freeee" is printed only once. So, the work within the non-recursive portion of the `GOOFYLOOP` function is $\Theta(20n) = \Theta(n)$, giving the recurrence relation:

$$T(n) = 10T(\left\lfloor \frac{n}{5} \right\rfloor) + O(n).$$

We can then apply the Master Theorem. Since $a = 10$, $b = 5$, and $d = 1$, we have that $\frac{a}{b^d} = \frac{10}{5} > 1$ and thus the runtime is $\Theta(n^{\log_b a}) = \Theta(n^{\log_5 10})$

Problem 6. (4 points)

Suppose 13 cards from a standard deck of 52 cards are dealt evenly across 4 players. How many ways are there for each player to receive exactly one card of each rank?

Reminder: A standard deck of cards has 4 suits and 13 ranks (A, 2, 3,..., 10, J, Q, K).

(a) $4!$

(b) $(4!)^{13}$

(c) $P(13,4)^{13}$

(d) $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$

(e) $(4^{13})^4$

Solution: (b)
For each rank, we must distribute exactly one card to each player in order to satisfy the condition. This is a permutation, so there are $4P4 = 4!$ ways to distribute the cards of each rank. There are 13 ranks, and we must distribute the 4 cards of all ranks, so by the product rule, we have $(4!)^{13}$.

Problem 7. (4 points)

What is the Big-Theta runtime of the following recurrence?

$$T(n) = 8T\left(\frac{n}{2}\right) + 203n^3$$

(a) $\Theta(n^3)$

(b) $\Theta(8n^3)$

(c) $\Theta(n^4)$

(d) $\Theta(n^3 \log n)$

(e) $\Theta(n^{\log_2 3})$

Solution: (d)
This is a divide and conquer recurrence, so we apply the Master Theorem with $a = 8$, $b = 2$, and $d = 3$. Thus $a/b^d = 8/2^3 = 8/8 = 1$, so by the Master Theorem, the runtime is

$$T(n) = \Theta(n^d \log n) = \Theta(n^3 \log n).$$

Problem 8. (4 points)

Yili repeatedly rolls a pair of dice: one **unfair** 4-sided die and one **fair** 6-sided die. The 4-sided die is weighted such that it has a $\frac{2}{5}$ probability of rolling a 1. She keeps rolling the two dice until **both** of the following conditions are met in the same turn:

- The 4-sided die rolls a 1.
- The 6-sided die rolls a 3, 4, 5, or 6.

What is the expected number of turns it would take for Yili to stop, including the final turn?

(a) $\frac{3}{2}$

(b) 2

(c) $\frac{15}{4}$

(d) 6

(e) 12

Solution: (c)
Let C be the event that the 4-sided die rolls a 1, with $p(C) = \frac{2}{5}$. Let D be the event that the 6-sided die lands on a 3, 4, 5, or 6, with $p(D) = \frac{4}{6}$. Since C and D are independent, the probability that Yili stops on the first turn is $p(C \cap D) = p(C) \cdot p(D) = \frac{2}{5} \cdot \frac{4}{6} = \frac{2}{5} \cdot \frac{2}{3} = \frac{4}{15}$. The number of turns can be modeled by a random variable X with a geometric distribution that has parameter $p = \frac{4}{15}$. Therefore, the expected number of turns is $E(X) = \frac{1}{p} = \frac{1}{\frac{4}{15}} = \frac{15}{4}$.

Problem 9. (4 points)

How many strings of 8 upper-case English letters begin with GO or end with BLUE?

(a) 26^8

(b) $26^6 + 26^4$

(c) $26^6 \cdot 26^4$

(d) $26^6 + 26^4 - 26^2$

(e) $26^6 + 26^2 - 26^4$

Solution: (d) $26^6 + 26^4 - 26^2$
We use inclusion-exclusion. There are 26^6 strings of upper-case English letters that begin with GO (the first two characters are set as 'GO' and we have 26 possible choices for each of the last 6 characters). Likewise, there are 26^4 strings that end with BLUE (the last four characters are set as 'BLUE' and we have 26 possible choices for each of the first 2 characters). At this point we've double counted the 26^2 strings that both start with GO and end with BLUE (we only choose 2 characters in the string). Using inclusion-exclusion gives us a total of $26^6 + 26^4 - 26^2$ strings satisfy the condition.

Part A2: Multiple Answer Multiple Choice

Problem 10. (4 points)

Trent rolls two fair, 6-sided dice. One die is red and the other die is blue. Consider the following events:

- R: The red die is a 1.
- B: The blue die is a 6.
- S: The sum of the dice is 8.

Reminder: Two events are *mutually exclusive* when they cannot happen at the same time.

Which of the following statements are true?

(a) R and B are mutually exclusive.

(b) R and B are independent.

(c) R and S are mutually exclusive.

(d) R and S are independent.

(e) B and S are mutually exclusive.

Solution: (b), (c)

(a) False. We can have the red die roll a 1 and the blue die roll a 6 since the two are different dice.

(b) True. The two dice rolls do not impact each other. The roll of the red die has nothing to do with the roll of the blue die.

(c) True. If the red die is a 1, then the maximum the blue die can be is 6 (since we are rolling 6-sided dice). As a result, the maximum sum would be 7, and it is not possible that we have a sum of 8. Thus, R and S would be mutually exclusive.

(d) False. If the red die is 1, then sum can't be 8 (following the logic from part c). Similarly, we know that if the sum is 8, then the red die must have rolled a number that isn't 1. So, the two results are dependent on each other (when one is true, the probability of the other changes).

(e) False. B and S can both happen at the same time, namely when the blue die rolls 6 and the red die rolls 2, giving a sum of 8.

Problem 11. (4 points)

Which of the following graphs **must** be bipartite?

(a) A tree

(b) K_5

(c) C_8

(d) A simple graph with four vertices with degrees $2, 2, 3$, and 3

(e) W_4

Solution: (a),(c)

(a) Since a tree does not contain any cycles by definition, a tree does not contain any odd cycles. Thus, a tree must be bipartite.

(b) In K_5, every vertex is connected to every other vertex. This means that there exist multiple odd cycles: for example, a cycle that contains all 5 vertices. Thus, K_5 is not bipartite.

(c) C_8 is a single cycle with 8 vertices, which means that C_8 is two-colorable and therefore must be bipartite.

(d) In a simple graph with the given properties, two vertices connect to all other vertices in the graph since there are two vertices with degree 3. These two vertices are therefore connected to each other as well as a third vertex, forming an odd cycle with 3 vertices. Thus, this graph is not bipartite since it contains an odd cycle.

(e) In a wheel, the middle vertex is connected to every outer vertex. Thus, there exists an odd cycle that connects two outer vertices and the middle vertex, meaning that W_4 is not bipartite.

Problem 12. (4 points)

Let f, g be functions where

- $f(n) = 6n^4 + 4n + \log n + 4$
- $g(n) = n! + n^5$

Which of the following are true?

(a) $f(n)$ is $O(g(n))$

(b) $f(n)$ is $\Theta(g(n))$

(c) $f(n)$ is $\Omega(g(n))$

(d) $\dfrac{g(n)}{f(n)}$ is $O(n!)$

(e) $g(n) \cdot f(n)$ is $O(n!)$

Solution: (a), (d)
Note: $f(n)$ is $\Theta(n^4)$ and $g(n)$ is $\Theta(n!)$.

(a) **True:** we know that $n!$ will grow at a faster rate than polynomial terms (e.g., n^3), which $f(n)$ is dominated by

(b) **False:** $g(n)$ does not provide a lower bound for $f(n)$ and does not represent the exact performance value of $f(n)$ (it is not both $O(g(n))$ and $\Theta(g(n))$)

(c) **False:** since $n!$ grows at a faster rate than polynomials, it doesn't provide an accurate lower bound for $f(n)$

(d) **True:** when we have $\dfrac{g(n)}{f(n)}$, that is a smaller function than $g(n)$, so $n!$ is still an upper bound for this function.

(e) **False:** the fastest growing term in $g(n) \cdot f(n)$ will be $n! \cdot 6n^4$, which grows faster than $n!$

Part B: Short Answer

Questions 13-16: You DO NOT need to simplify your answer.
Question 17: You DO need to fully simplify your answer.

Problem 13. (6 points)

How many distinct permutations of the 10-letter string BOOKKEEPER are there?

Solution: $C(10,3) \cdot C(7,2) \cdot C(5,2) \cdot 3!$

Since some of the letters of BOOKKEEPER are the same, the answer is not given by the number of permutations of 10 letters. The word contains 2 Os, 2 Ks, and 3 Es. To determine the different strings that can be made by reordering the letters, first note that the three Es can be placed among the 10 positions in C(10, 3) different ways, leaving seven positions free. Then the two Os can be placed in C(7, 2) ways, leaving five positions free. The two Ks can be placed in C(5, 2) ways, leaving 3 positions free. The remaining letters are B, P, and R. The B can be placed in C(3,1) ways, the P in C(2,1) ways, and the R in C(1,1) ways in the remaining spot (or 3! for all three letters).

Thus, we get $C(10,3) \cdot C(7,2) \cdot C(5,2) \cdot 3!$ or $\dfrac{10!}{3!2!2!}$

Note: Students who directly used the Multinomial Theorem to directly calculate the factorial form of the answer also got full points, given that work was shown that there was understanding of the repeating letters and frequencies.

Note: There are several different orders the letters could be chosen in. For example, if one were to choose B first, we would get C(10,1), then choosing the 2 Os would be C(8, 2), and so on. The resulting number should be equal to 151200, different methods that got the same amount should receive the same amount of points.

Grading Guidelines:
Method 1: Multinomial Theorem
+1 for 10! in numerator
+1 for each term in denominator (3!, 2!, 2!)
+2 for justification/work (anything that indicates knowledge for repeating letters like showing letter counts/frequencies)
-1 for any extraneous terms
-0.5 for algebraic errors

Method 2: Choosing Spots
+1 for correctly choosing spots for repeat O's
+1 for correctly choosing spots for repeat K's
+1 for correctly choosing spots for repeat E's
+1 for using up the remaining spots

+2 for justification/work (anything that indicates knowledge for repeating letters like showing letter counts/frequencies)
-1 for any extraneous terms
-2 for incorrectly dividing by repeating letters (choosing spots already accounts for repeating letter overcount)

Common mistakes:

1. Incorrectly accounting for overcounting by doing substraction instead of division.
2. Not having sufficient justification, and leaving the final answer without explanation.
3. Having extraneous term in the denominator.
4. For Method 1, having $2 \times 2 \times 3$ as denominator instead of 2!2!3!
5. For Method 2, incorrectly doing division for overcounting. (The choosing spots method already accounts for the repeating letters, so there is no overcounting, and no division is needed.)
6. For Method 2, incorrectly using permutations instead of combinations.

Problem 14. (6 points)

Jasmine lives on an island in the archipelago Discretopia. Discretopia is a ring of n islands, with n bridges built all the way around the ring, making a cycle. Recently, there was an earthquake that damaged the bridges. Each bridge independently has a 10% chance of collapsing. What is the probability that Jasmine can still reach all of the islands in Discretopia?

Note: Your answer should be left in terms of n.

Solution: $0.9^n + n \cdot 0.9^{n-1} \cdot 0.1$

There are only two scenarios that will allow Jasmine to reach all the islands: if none of the bridges collapse (i.e., all bridges are intact), or if exactly one of the bridges collapses (i.e., all but one bridge is intact).

Case 1: None of the bridges collapse. The probability of a single bridge not collapsing is 0.9, so since the bridges collapse (or not) independently of each other, the probability of all n bridges not collapsing is 0.9^n.

Case 2: Exactly one bridge collapses. There are $\binom{n}{1}$ ways of selecting the bridge that collapses. The probability of this bridge collapsing is 0.1. The probability of all other $n-1$ bridges not collapsing is 0.9^{n-1}. Thus the probability of case 2 is $n \cdot 0.1 \cdot 0.9^{n-1}$.

Since the two events are mutually exclusive, the probability of either of the two events occurring is their sum. Thus the probability that Jasmine can still reach all the islands in Discretopia is $0.9^n + n \cdot 0.9^{n-1} \cdot 0.1$.

Alternate solution: You can also calculate the probability by treating this as a binomial distribution where a success refers to the event that a bridge does not collapse. The number of bridges that survive is a binomially distributed random variable, and the problem asks for the probability of exactly n successes or $n-1$ successes. In this case, the parameters for the binomial distribution are $p = 0.9$ and $n = n$:
Case 1: n successes $= \binom{n}{n} \cdot 0.9^n \cdot 0.1^0 = 0.9^n$
Case 2: $n-1$ successes $= \binom{n}{n-1} \cdot 0.9^{n-1} \cdot 0.1^1 = n \cdot 0.1 \cdot 0.9^{n-1}$

Likewise, you can define a success as a particular bridge collapsing, the binomially distributed random variable to be the number of bridges which *do* collapse, and find the probability that this value is 0 or 1. Noting that $\binom{n}{n} = \binom{n}{0} = 1$ and $\binom{n}{n-1} = \binom{n}{1} = n$, this gives the same result.

Grading guidelines:

- +1 Correctly identifies that there are 2 cases to calculate with appropriate justification, and gives the final answer as the sum of the probabilities of these cases.
- +2.5 Correctly calculates case of all bridges intact
- +1 Partial credit if answer to case 1 is 0.1^n
- +2.5 Correctly calculates case of all but one bridge intact
- +1.75 Partial credit if answer to case 2 has 2 out of 3 correct terms
- +1 Partial credit if answer to case 2 has 1 out of 3 correct terms

Common Mistakes:

1. Only considering either one of the two cases.
2. Not using n in the final answer.
3. Calculating case 2 as just 0.9^{n-1}.
4. Calculating case 1 as $1 - P$(one bridge collapses). More accurately, case 1 is $1 - P$(at least one bridge collapses), which includes the scenario in which 2 bridges collapse, 3 collapse, and so on. Thus this is not equal to the probability $n \cdot 0.1 \cdot 0.9^{n-1}$.

Problem 15. (6 points)

You are selecting an All Star Team from a set of 20 finalists, 7 of whom are wearing green shirts. If you select 9 people uniformly at random for your All Star Team, what is the probability that you end up with **at least** 1 green-shirted person on the team?

Solution: It's easiest to compute the probability of **less than 1** green shirt, and subtract that from 1. Any of the following answers are valid:

$$\frac{\binom{20}{9}-\binom{13}{9}}{\binom{20}{9}} = 1-\frac{\binom{13}{9}}{\binom{20}{9}} = \frac{P(20,9)-P(13,9)}{P(20,9)} = 1-\frac{P(13,9)}{P(20,9)} = 1-\frac{13!\cdot 11!}{4!\cdot 20!}$$

Alternate solution: If instead we use the sum rule, we will add up the probabilities of each case where exactly $1, 2, \cdots, 7$ green shirts are on the team. The denominator is still the same:

$$\frac{\binom{7}{1}\binom{13}{8}+\binom{7}{2}\binom{13}{7}+\binom{7}{3}\binom{13}{6}+\binom{7}{4}\binom{13}{5}+\binom{7}{5}\binom{13}{4}+\binom{7}{6}\binom{13}{3}+\binom{7}{7}\binom{13}{2}}{\binom{20}{9}}$$

Grading Guidelines:

- +2: Correctly uses complementary probability:

  $$1 - P(0 \text{ green shirts on the team})$$

  OR Correctly adds disjoint cases:

  $$P(1 \text{ green shirt}) + \cdots + P(7 \text{ green shirts})$$

- +1: Correct definition and use of E

  Note: If the solution is incorrect, explicit work indicating what definitions of E and S were used is necessary to receive partial credit for these items.

- +1: Correct computation of $|E|$ (given the definition)

  Note: Points for the correct computation of $|E|$ and $|S|$ are not awarded if the definition trivializes the computation.

- +1: Correct definition and use of S

- +1: Correct computation of $|S|$ (given the definition)

  Note: Writing $\binom{20}{9}$ on the page is not sufficient to receive this rubric item if no context or explanation is provided (either implicitly or explicitly).

Common Mistakes:

1. Choosing the team of no green-shirted people *with replacement*, e.g. $\left(\frac{13}{20}\right)^9$.

   This answer attempt gets a maximum 3 points.

2. Giving the size of the event, e.g. $\binom{20}{9} - \binom{13}{9}$, rather than the probability of the event.

   This answer attempt gets a maximum 5 points.

3. Inconsistency between the numerator and denominator as to whether the positions in the team are distinguishable. Similarly, drawing the numerator with replacement but the denominator without, or vice versa.

   This answer attempt gets a maximum 5 points.

4. Reasoning that "people are distinguishable so we must use permutation (not combination)." Permutations and combinations *both always* select from distinguishable items. The difference is that permutations select *into distinguishable positions* (like a line) while combinations select into indistinguishable positions (like a group).

   Points are not necessarily deducted for this mistake since $\dfrac{P(13,9)}{P(20,9)} = \dfrac{C(13,9)}{C(20,9)}$.

5. Conflating $1 - \dfrac{|E|}{|S|}$ with $\dfrac{|S| - |E|}{|S|}$, e.g. giving an answer of the form $1 - |E|$, $\dfrac{1 - |E|}{|S|}$, $|S| - P(\text{0 green shirts})$, or $\dfrac{|S| - P(\text{0 green shirts})}{|S|}$.

   This answer attempt gets a maximum 4 points.

6. Conflating the notation for combinations (sometimes with that of fractions). Correct notation is $C(n,k) = \binom{n}{k} = nCk$. Incorrect variations included $\left(\frac{n}{k}\right)$, $C\left(\frac{n}{k}\right)$, $\binom{k}{n}$, etc.

   Point deductions depend on the particular error and work shown. Generally a -0.5 point adjustment was made if the error was present in the final answer but meaning was clear from the work shown.

Problem 16. (6 points)

A binary string is **palindromic** if it reads the same backwards as forwards. For example,

- 1001 is a 4-bit palindromic binary string
- 01110 is a 5-bit palindromic binary string.

What is the probability that a randomly selected 11-bit binary string is palindromic?

Solution: The probability is 1/32. In a randomly selected string, the probability that a given pair of digits match is 1/2. Five such pairs must match if a binary string of length 11 is to be palindromic (namely, the first digit and last digit, the second digit and second-to-last digit, and so on). The probability that all four pairs match is $(1/2)^5 = 1/32$. Note that the middle digit would line up with itself, so we don't need to worry about it.

Alternate solution: We can construct a binary string of length 11 by specifying each of the first 5 digits, which uniquely determines the last 5 digits, and choosing 0 or 1 for the middle digit. This can be done $2^5 \cdot 2 = 2^6$ ways. There are 2^{11} distinct binary strings of length 11, so the probability that such a string, randomly selected, is palindromic, is $2^6/2^{11} = 1/2^5 = 1/32$.

Grading Guidelines:
+2: Correct probability that two bits match ($p = 1/2$)
+4: Correctly applies product rule to account for correct number of matching digits (i.e., p^4)
+3: Partial Credit: Applies product rule, but counts middle digit as well (i.e., p^5)
+2: Partial Credit: Uses product rule at least one place where it‘s needed (i.e., p^n, where n is not 4 or 5). This rubric item can be earned only if neither of the other product rule rubric items were earned.

Common Mistakes:

- Forgetting to divide by the sample space. Note that the question asked for a probability, not a total number.
- Counting the first 5 digits and the last 5 digits and not only one. By counting both, this overcounts the palindromic sequences by a factor of 2, since setting one forces the other to be set given that the string is palindromic.
- Forgetting to account for the middle digit. Note that although what value the middle digit is isn't restricted by the palindromic restriction, we still have to account for it, meaning we have to apply product rule using that digit as well.

Problem 17. (8 points)

Abby and Nihar both plan to volunteer this weekend. They will each volunteer one shift at one organization. They each select which organization to volunteer at with the probabilities outlined below.

Organization	Length of Volunteer Shift	Probability of Abby selecting this	Probability of Nihar selecting this
Humane Society	3 hours	1/3	1/6
Food Bank	6 hours	5/12	1/3
Big Brothers Big Sisters	10 hours	1/4	1/2

Let X be a random variable representing the total hours volunteered by Abby and Nihar this weekend. Find $E(X)$.

Express your answer as a single, **fully simplified** number.

Solution: $X = A + N$, where A is the number of hours Abby volunteers and N is the number of hours Nihar volunteers.
$\therefore \mathbb{E}[X] = \mathbb{E}[A + N] = \mathbb{E}[A] + \mathbb{E}[N]$ by Linearity of Expectation.
By definition of expectation:

$$\mathbb{E}[A] = \sum_{\text{all outcomes } s} P(s)A(s) = 1/3 \cdot 3 + 5/12 \cdot 6 + 1/4 \cdot 10 = 1 + 5/2 + 5/2 = 6$$

$$\mathbb{E}[N] = \sum_{\text{all outcomes } s} P(s)N(s) = 1/6 \cdot 3 + 1/3 \cdot 6 + 1/2 \cdot 10 = 1/2 + 2 + 5 = 7.5$$

So $\mathbb{E}[A] = 6$ and $\mathbb{E}[N] = 7.5$, so $\mathbb{E}[X] = 6 + 7.5 = 13.5$.

Grading Guidelines:
+2: Uses definition of expected value
+1.5: Correct expected hours for Abby
+1.5: Correct expected hours for Nihar
+2.5: Applies linearity of expectation to find total expected hours
+0.5: Simplifies final answer
-0.5: Arithmetic error

Part C: Free Response

Problem 18. (10 points)

Consider a non-standard 104-card deck with 8 suits ($\heartsuit$, $\diamondsuit$, $\clubsuit$, $\spadesuit$, $\bigcirc$, $\triangle$, $\bigstar$, $\blacksquare$) and the usual 13 ranks per suit (A, 2, 3, ..., 10, J, Q, K). The IAs play a game where a hand contains 6 cards each.

We define a *full mansion* as a 6-card hand that can be split into two sets of three cards each, where each set of three are of the same rank. The ranks of the two sets do **not** need to be different. (In other words, 6 cards of the same rank does qualify as a *full mansion*.)

For example, both of these hands are *full mansions*:

$$\{2\bigcirc, 2\clubsuit, 2\diamondsuit, K\diamondsuit, K\spadesuit, K\heartsuit\}$$

$$\{7\spadesuit, 7\bigstar, 7\clubsuit, 7\heartsuit, 7\triangle, 7\blacksquare\}$$

How many hands of 6 cards are a *full mansion*?

You do **not** need to simplify your answer.

Solution:
There are two cases for full mansions, (1) that there are 6 cards of one rank, and (2) that there are 3 cards of one rank and 3 cards of another rank. We must split into cases because the choices made in each case differ. Then we add the cases together by sum rule.

Case 1: 6 cards of one rank
Choose 1 rank out of 13, then choose the 6 cards (suits) of that rank out of 8 suits. Here are a few equivalent formats of that choice:

$$\binom{13}{1} \cdot \binom{8}{6} \equiv 13 \cdot \binom{8}{6} \equiv \frac{13(8!)}{2!6!}$$

Case 2: 3 cards of one rank, 3 cards of another rank
Choose 2 ranks out of 13, then choose 3 suits for first rank and 3 suits for other rank. Note that the order the ranks are chosen in does not matter so we can either write 13 choose 2 or $(13 \cdot 12)/2$. Here are a few equivalent formats of that choice:

$$\binom{13}{2} \cdot \binom{8}{3} \cdot \binom{8}{3} \equiv \frac{13 \cdot 12}{2} \cdot \binom{8}{3} \cdot \binom{8}{3} \equiv \frac{13 \cdot 12}{2} \cdot \binom{8}{3}^2 \equiv \frac{13 \cdot 12}{2} \cdot \left(\frac{8!}{3!5!}\right)^2$$

Final Solution: The total number of full mansions is...

$$\binom{13}{1}\binom{8}{6} + \binom{13}{2}\binom{8}{3}\binom{8}{3}$$

Alternate Case 1: choosing a card
We first choose any of the 104 cards in the deck, in 104 ways, then choose 5 more from the 7 with the same rank, in $\binom{7}{5}$ ways.

$$(\text{any card}, \{\text{same rank}, \ldots, \text{same rank}\})$$

However, this overcounts, since for a set of six cards, any of the 6 could have been chosen first. So, we need to divide by 6.

$$\frac{104 \cdot \binom{7}{5}}{6}$$

Alternate Case 2: choosing a card
We first choose any of the 104 cards in the deck, then choose 2 more from the 7 with the same rank, in $\binom{7}{2}$ ways. We get rid of all 8 cards with that rank, since in case 2, we don't want to get the same rank again. So, we pick one of the remaining $104 - 8 = 96$ cards, then choose 2 more from the 7 with the same rank, $\binom{7}{2}$ ways.

$$\begin{aligned}
&(\\
&\quad(\text{any card}, \{\text{same rank}, \text{same rank}, \text{same rank}\}),\\
&\quad(\text{any different rank card}, \{\text{same rank}, \text{same rank}, \text{same rank}\})\\
&)
\end{aligned}$$

Like the similar Alternate Case 1 above, each grouping of cards is overcounted, since any of the 3 cards could have been chosen first, so we have to divide by 3 for each grouping, so 3^2 in total. This also overcounts because we could swap the first and second grouping, so we have to divide by 2.

$$\frac{104 \cdot \binom{7}{2} \cdot 96 \cdot \binom{7}{2}}{3^2 \cdot 2}$$

Common Mistakes:

- Forgetting to divide by 2 in the case where there are two sets of the same rank, for instance writing $13 \cdot 12 \cdot \binom{8}{3}^2$ instead of $\binom{13}{2}\binom{8}{3}^2$. The reason we divide by 2 here is because we have two groups of the same size, so it doesn't matter in what order we pick the distinct ranks because they are both being assigned to groups of three cards.

- Choosing 6 cards by writing $\binom{8}{3}\binom{5}{3}$ in the case where all 6 cards have the same rank. Choosing the cards in this manner implies that we're forming two distinct groups of 3, however since all these cards are being put in the same group the correct way to choose 6 cards is $\binom{8}{6}$.

- Choosing a "first card" for each group rather than choosing the ranks a suits separately, and not dividing for over-counting correctly. The correct way to employ this method is shown in the alternate solution above. The key thing to note with this approach is that when we select an initial card (104), and then more cards of the same rank, ($C(7,5)$) for example, we over-count by the number of ways to pick a "first card" in the set. So in this instance, since we have 6 cards in the set, we over-count by a factor of 6. In the case where there are two distinct ranks, we still need to divide by 2 for the same reason specified in the first common mistake, in addition to 3^2 to account for the number of ways to pick the first card in each set of 3.

- Attempting to solve the problem without distinct cases. Most solutions which did this tended to multiply the number of ways to choose ranks for two different sets (such as 13^2 or $13 \cdot 12$), by the number of ways to choose two sets of cards of a given rank (such as $C(8,3)^2$ or $C(8,3)C(5,3)$). Unfortunately there is not a correct way to solve this problem without cases that we know of—the incorrect solutions mentioned above either under-counted/overcounted the number of ways to choose ranks for two sets, and subsequently incorrectly counted the number of ways to form two sets of cards of those ranks.

Grading Guidelines [10 Points]
+1.5 Splits into correct cases **and** adds the values from them
Same Rank Case:
+1.0 Chooses a rank (contains a factor of 13)
+1.0 Selects any 6 (possibly incorrect cards)

- If you picked more or less than 6, this point was not awarded.

+1.5 Choose 6 suits (contains a factor of $\binom{8}{6}$)
Different Rank Case:
+1.0 Picks 2 ranks (contains $13 \cdot 12$)
+1.5 Divides by 2 (order of ranks doesn't matter)

- Note the above two items are just $\binom{13}{2}$ so that would receive both points

+1.5 Chooses 3 cards of one rank (contains factor of $\binom{8}{3}$)
+1.0 Picks 3 cards from the other rank consistent with how first 3 were chosen

Problem 19. (10 points)

Suppose that Alicia selects one of three books. Each book has a different number of pages and a different probability that she selects it.

- Book 1 has 100 pages, and is selected with probability 2/7.
- Book 2 has 200 pages, and is selected with probability 4/7.
- Book 3 has 300 pages, and is selected with probability 1/7.

Alicia flips to a random page of the selected book, such that each page in the book is flipped to with equal probability.

Answer each question below, and **simplify your answer to a single, fully-simplified number.**

(a) What is the probability that Alicia flips to a page number that is **greater than** 150?

(b) Given that Alicia flips to a page number greater than 150, what is the probability that she has selected Book 2?

Solution:
Let B_1 be the event that she picks Book 1, B_2 be the event that she picks Book 2, and B_3 be the event that she picks Book 3.

(a) Let G be the event that she flips to a page number greater than 150. Based on the number of pages in each book, we know that $p(G|B_1) = 0$, $p(G|B_2) = 50/200 = 1/4$, and $p(G|B_3) = 150/300 = 1/2$. Using the law of total probability, we have:

$$\begin{aligned} p(G) &= p(G \cap B_1) + p(G \cap B_2) + p(G \cap B_3) \\ &= p(B_1) \cdot p(G|B_1) + p(B_2) \cdot p(G|B_2) + p(B_3) \cdot p(G|B_3) \\ &= \frac{2}{7} \cdot 0 + \frac{4}{7} \cdot \frac{1}{4} + \frac{1}{7} \cdot \frac{1}{2} \\ &= 0 + \frac{1}{7} + \frac{1}{14} \\ &= 3/14 \end{aligned}$$

(b) Apply Bayes' Theorem to find $p(B_2|G)$. We show the equation version of Bayes' below, but using a probability tree or the table method for Bayes' are equally valid. We can use our answer from part (a):

$$\begin{aligned} p(B_2|G) &= \frac{p(B_2) \cdot p(G|B_2)}{p(G)} \\ &= \frac{\frac{4}{7} \cdot \frac{1}{4}}{\frac{3}{14}} \\ &= \frac{\frac{1}{7}}{\frac{3}{14}} \\ &= 2/3 \end{aligned}$$

Note: Since $P(G \cap B_2)$ was already calculated above, it's also valid to apply the definition of conditional probability to state that $P(B_2|G) = \dfrac{P(G \cap B_2)}{P(G)}$ and cite work from part (a) rather than recompute the numerator.

Grading Guidelines:
Part (a): 5 points total

+1.5: Applies Law of Total probability (adding parts together)
+1: Correct calculation of $p(G \cap B_1) = p(B_1) \cdot p(G|B_1) = 0$
+1: Correct calculation of $p(G \cap B_2) = p(B_2) \cdot p(G|B_2) = 1/7$
+1: Correct calculation of $p(G \cap B_3) = p(B_3) \cdot p(G|B_3) = 1/14$
+0.5: Simplifies fully
0.5/1 awarded for partially calculating a term correctly

Part (b): 5 points total
+2: Uses Bayes' or conditional probability correctly
+1.25: Correct numerator $p(B_2) \cdot p(G|B_2) = 1/7$
+1.25: Correct denominator. $p(G) = 3/14$ or uses their answer from part a.
+0.5: Simplifies fully
0.5/1.25 awarded for partially calculating a term correctly

Common Mistakes

- Incorrect application of a form of Bayes' Theorem – For part (b), some attempted to use the form of Bayes' Thereom $p(B_2|G) = \frac{p(B_2)\cdot p(G|B_2)}{p(B_2)\cdot p(G|B_2)+p(\overline{B}_2)\cdot p(G|\overline{B}_2)}$. This equation is correct, however the calculation of the term $p(G|\overline{B}_2)$ is more nuanced, resulting in some incorrect calculations. Some had $p(\overline{B}_2) = \frac{3}{7}$ and $p(G|\overline{B}_2) = \frac{150}{300}$, however the latter term does not account for the fact that the complement of B_2 is $B_1 \cup B_3$, meaning that the possibility of selecting book 1 must also be taken into account. One method to account for this could be using law of total probability by finding $p(G|\overline{B}_2) = p(G|(B_1 \cup B_3)) = p(G|B_1)p(B_1|(B_1 \cup B_3)) + p(G|B_3)p(B_3|(B_1 \cup B_3))$, however this begins to look somewhat complex. It is worth noting that the expanded form of the denominator of Bayes' theorem is equivalent to using the Law of Total Probability to calculate $p(G)$. Hence, we can instead fully apply the Law of Total Probability as used in part (a) to calculate the denominator $p(G)$ more directly as $p(G) = p(G \cap B_1) + p(G \cap B_2) + p(G \cap B_3) = p(B_1) \cdot p(G|B_1) + p(B_2) \cdot p(G|B_2) + p(B_3) \cdot p(G|B_3)$.

- Off-by-one errors were common in counting the number of pages numbered above 150 in each book. 0.1 of a point was deducted for this mistake, and we were lenient about fully simplifying answers due to the numbers involved.

- For part (a), some students misstated the Law of Total Probability to claim that P(G) $= P(G|B_1) + P(G|B_2) + P(G|B_3)$.

  Solutions of this form get a maximum 0.5/5 for part (a).

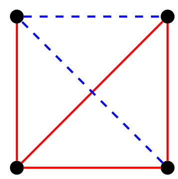
- For part (b), some students applied the formula $P(B_2|G) = \dfrac{|G \cap B_2|}{|G|}$, with $|G|$ as the number of pages numbered above 150 over all books and $|G \cap B_2|$ as the number of pages numbered above 150 in Book 2. The resulting answer is 1/4. However, this formula is only applicable if all elements of the sample space (i.e. pages) are equally likely. Since the procedure assigns different weights to different pages based on the length and likelihood of each book being chosen, this is not the case.

Solutions of this form get a maximum 0.5/5 for part (b).

Problem 20. (8 points)

We define a spanning tree of a graph G to be a connected subgraph that still has every vertex and contains no cycles (hence a tree). Then, let G be a graph in which each **edge** has been colored red or blue. G contains a **"monochromatic spanning tree"** if it contains a spanning tree in which all edges are the same color (i.e., all edges are red or all edges are blue.)

For example, the following red-blue coloring of K_4 *contains* a solid red monochromatic spanning tree:



Let $P(n)$ be the predicate

"Any possible red-blue coloring of K_n contains a monochromatic spanning tree."

Using induction, prove $\forall n \geq 2, P(n)$.

Solution:
Inductive Step: Assume $P(k)$ for some $k \geq 2$. That is, any possible red-blue coloring of K_k contains a monochromatic spanning tree. We will prove $P(k+1)$. Consider K_{k+1}, with edges colored either red or blue. Remove an arbitrary vertex v and its incident edges, which leaves us with K_k. By the inductive hypothesis, we know K_k has a monochromatic spanning tree. Without loss of generality, let's say the tree is blue. Now, we want to add v to K_k's tree to make it a spanning tree for K_{k+1}. Consider two cases:

1. A blue edge connects to v. We can choose to add this to the spanning tree, which gives us a blue spanning tree for K_{k+1}.
2. No blue edges connect to v, or in other words, all edges incident to v are red. These edges give a red spanning tree for K_{k+1}, since one can reach every vertex in K_k from v.

In either case, we have a monochromatic spanning tree for K_{k+1}.

Base Case: $P(2)$ is true. K_2 is its own spanning tree, and regardless of what color its singular edge is, the graph will be monochromatic, so K_2 contains a monochromatic spanning tree.

So by induction, $\forall n \geq 2$, $P(n)$ holds.

Grading Guidelines:
Base Case:

- +1 Correct base case

Inductive Step:

- +1 Makes a correct assumption and attempts to show a corresponding conclusion
- +2 Constructively applies inductive hypothesis to prove a monochromatic spanning tree exists in a subgraph of the given graph
- +2 Correctly shows that a monochromatic spanning tree exists when the remaining vertex **has an edge** with the same color as the subgraph's monochromatic spanning tree
- +2 Correctly shows that a monochromatic spanning tree exists when the remaining vertex **has no edge** with the same color as the subgraph's monochromatic spanning tree

Common Mistakes:

- Doing the inductive step additively, e.g. adding a vertex v to the K_{n-1} graph, and not proving that this process generates all possible 2-colored K_n graphs. In this case, it's pretty obvious that that can be done, but in general, it's a bad habit that should be avoided. The penalty for this was very small, though.
- Thinking you can decide the colors for the edges. Remember, you are proving this is true for **any** coloring, so you need to use an **arbitrarily colored** K_n.
- Forgetting that a monochromatic spanning tree must cover every vertex. In particular, students would often use the inductive hypothesis to show that there is a spanning tree in a subgraph, then not extend it to the remaining vertex, showing that there is a monochromatic tree, rather than a monochromatic **spanning** tree
- Using an extra base case, e.g. $P(3)$. Since the inductive hypothesis almost always only relied on the previous subgraph, then only one base case was necessary. However, we did not take off points for this.
- General confusing surrounding notation (which we also didn't take points off for). Remember that $P(n)$ is a predicate so it can either be true or false. K_n is a complete graph. Neither of these refer to the monochromatic spanning tree contained in the graph.