Groupwork

1. Grade Groupwork 5

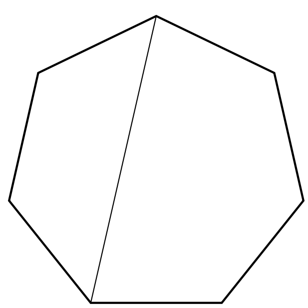
Using the solutions and Grading Guidelines, grade your Groupwork 5:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2	+4	+4	+4									12 /12
Problem 3	+2	+2	+2	+1	+1							8 /8
Total:												20 /20

Previous Groupwork 5(1): Polly Gone [12 points]

A convex polygon is a 2D shape with all straight edges such that any line segment between any two non-adjacent vertices passes entirely through its interior (see example picture below). Show via induction that the sum of the interior angles of a convex polygon with n sides is $(n-2)\cdot 180°$. Don't include unneeded base cases.



Hint 1: It is helpful to know that a triangle's interior angles always sum to 180°. You may assume this is true for the problem.

Hint 2: In order to apply your inductive hypothesis to a convex polygon, you'll need to think of it in terms of smaller convex polygons. How can you make smaller polygons out of a big one?

Solution:

Good job with ease!

Let $k(n)$ = the sum of the interior angles of a convex polygon with n sides
$P(n) : k(n) = (n-2) \cdot 180°$
Let K be an arbitrary integer in the domain that $k \geq 4$.
Assume: $P(3) \wedge P(4) \wedge P(5) \wedge ... \wedge P(k-1)$
Want to prove: $P(k)$

Base Case:
(Triangle) $n = 3$, the sum of the interior angles is 180°
$k(3) = (3-2) \times 180° = 1 \times 180° = 180°$

+4

Inductive Step:
Use a straight line to divide the k-sides polygon into a triangle and a $(k-1)$-sides polygon.
Using the inductive assumption we know that the k-sided polygan should have $(k-2) \times 180°$ degrees for its interior angles.

Tus the sum of the interior angles is

+4 +4

$$180° + (k-1-2) \times 180° = 180° + (k-3) \times 180° \quad = 180° \times (1 + k - 3) = 180(k-2)°$$

Therefore $P(k)$ is true for every integer $k \geq 4$.

Previous Groupwork 5(2): Running Recurrence [8 points]

An EECS 203 student goes to lecture everyday. On each day, she always chooses exactly one method of transportation, and always chooses to walk, bike, or take a bus. She also follows additional rules:

- She never walks two days in a row.
- If she takes the bus, she must have biked two days ago and walked a day ago.

Let a_n denote the number of ways she can go to EECS 203 lecture across n days for $n \geq 0$.

(a) Find a recurrence relation for a_n.

(b) What are the initial conditions? Use the fewest initial conditions necessary.

Solution:

It is not that easy, but we made that right!

(a) The recurrence relation is:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$$

week n week $n-1$ week $n-2$ week $n-3$

$a_n =$ W: bi $(= a_{n-2})$ + bs $\longrightarrow$ w $\longrightarrow$ bi $(= a_{n-4})$
+ bi $(= a_{n-1})$
+ bs $\longrightarrow$ w $\longrightarrow$ bi $(= a_{n-3})$

$\therefore a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$

There are 3 cases for the week n.

+2

Case 1: In week n, she goes to school by bike. This choice has no restrction, so in the weeks before, the number of ways she could go to lecture is a_{n-1}. +2
Case 2: In week n, she goes to school by bus. This means that in week $n-1$, she walked to school and in week $n-2$, she went to lecture by bike. In the weeks before, the number of ways she could go to lecture is a_{n-3}. +2
Case 3: In week n, she walks to school. This means that in week $n-1$, she can only went to lecture by bike or by bus. In the case she went to lecture by bike, the number of ways she could go to lecture before is a_{n-2}. And in the case she wen to lecture by bus, we apply the same logic as in case 2 and get that before the week, the number of ways she could go to lecture is a_{n-4}.

(b) Since for a_n, $n \geq 0$, and in our recurrence relation there is a_{n-4}. We need $n-4 \geq 0$ for the recurrence relation. So the cases where $0 \leq n < 4$ should be initial conditions.
That is:
$a_0 = 1$ (no way)
$a_1 = 2$ (can only go by bike or on foot)
$a_2 = 3$ (walk bike, bike walk, bike bike)
$a_3 = 6$ (walk bike walk, walk bike bike, bike bike walk, bike walk bike, bike bike bike, bike walk bus)

+2

2. (Set)ting up a (Power)ful Proof [17 points]

(a) Suppose we want to prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$. What is the Inductive Hypothesis for your proof? *Hint: Consider what variable you should do induction on.*

(b) Prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$.

Solution:

(a) for $S(k)$ be an arbitrary set s.t. $|S(k)| = k$. Assume that $\mathcal{P}(S(k)) = 2^k$.

(b) Let k be an arbitrary nonnegative integer.
Let $S(k)$ be an arbitrary set s.t. $|S(k)| = k$, i.e. $S(k)$ has k elements.
Assume: $\mathcal{P}(S(k)) = 2^k$.
Want to show: $\mathcal{P}(S(k+1)) = 2^{k+1}$.

Base Case:
$k = 0.S(k) = \emptyset = \{\}$
The only subset of $S(k)$ is $\emptyset$.
$\therefore \mathcal{P}(S(k)) = 1 = 2^0$ is true.

Inductive Step:
Mark the elements in $S(k)$ as $a_1, a_2, ..., a_k$, i.e., $S(k) = \{a_1, a_2, ..., a_k\}$.
And mark the new element in $S(k+1)$ as a_{k+1}, i.e., $S(k) = \{a_1, a_2, ..., a_{k+1}\}$.
Then all the new subsets created by a_{k+1} is exactly a subset of $S(k)$ plus the element a_{k+1}.
And every subset plus an element of a_{k+1} is a included in $\mathcal{P}(S(k+1))$.
$\therefore$ The number of elements in $\mathcal{P}(S(k+1))$ is $|\mathcal{P}(S(k))| + |\mathcal{P}(S(k))| = 2|\mathcal{P}(S(k))| = 2 \times 2^k = 2^{k+1}$
$\therefore |\mathcal{P}(S(k+1))| = 2^{k+1}$
$\therefore$ we have proved that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$.

3. Out of the ordinary [16 points]

The (von Neumann) ordinals are a special kind of number. Each one is represented just in terms of sets. We can think of every natural number as an ordinal. We won't deal with it in this question, but there are also infinite ordinals that "keep going" after the natural numbers, which works because there are infinite sets.

The smallest ordinal, 0, is represented as $\emptyset$. Each ordinal after is represented as the set of all smaller ordinals. For example,

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

(a) What is the ordinal representation of 4, in terms of just sets?

(b) If the sets X and Y are ordinals representing the natural numbers x and y respectively, how can we tell if $x \leq y$ in terms of X and Y? Why does this work?

(c) If X is the ordinal representing the natural number x, what is the ordinal representation of $x + 1$? Why does this work?

Solution:

(a) $4 = \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

(b) We can tell $x \leq y$ by comparing the cardinality of X and Y.
i.e., if $|X| \leq |Y|$, then $x \leq y$.
This works because every ordinal is represented as the set of all smaller ordinals than it, so it has more elements than ordinals smaller than it.

(c) It is $\mathcal{P}(X)$.
Since every ordinal is represented as the set of all smaller ordinals than it,
$x+1$ is represented as $\{0, 1, 2, 3, ..., x\}$, in which $0, 1, 2, 3, ...x-1$ are all the subsets of X but X itself, and plus X itself, they form a set of all the subsets of X, which is also defined as the Power Set of X, i.e. $\mathcal{P}(X)$.
This can be proved by induction.
Assume $x = \{0, 1, ..., x-1\} = \mathcal{P}(X-1)$
Base Case:
$0 = \emptyset, 1 = \{0\} = \{\emptyset\} = \mathcal{P}(0)$
Inductive Step:
$x + 1 = \{0, 1, ..., x-1, x\}$, $0, 1, ...x-1$ is all the elements in $\mathcal{P}(X-1)$ which is all the subsets of the ordinal of $x - 1$, and by plusing x, we have all the subsets of x in the set.
$\therefore$ it is $\mathcal{P}(X)$.