EECS 203: Discrete Mathematics
Fall 2023
Homework 6

Due **Friday, October 20**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $5 + 2$ Total Points: $100 + 33$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Set Me Free [18 points]

(a) For each of the following, determine if the statement is true or false. Justify your answers.

(i) $\emptyset \in \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

(ii) $\emptyset \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

(iii) $\emptyset \times \{0, 1\} \subseteq \emptyset$

(iv) $\{\{1, 2\}\} \subsetneq \{\{1, 2\}, \{2\}\}$

(b) Find the cardinality of $\{\emptyset, \emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$.

(c) (i) Find the cardinality of $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$.

(ii) Find $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$. Make sure it has the cardinality you wrote in (i).

Solution:

(a) (i) False.
$\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ has only two elements: $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$, $\emptyset$ is not an element.

(ii) True.
$\emptyset$ is a subset of every set. So it is also a subset of $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

(iii) True.
The Cartesian product of sets A and B is: $A \times B = \{(a, b)|a \in A \wedge b \in B\}$.
When A is $\emptyset$, $a \in A$ is false for any a. $\therefore$ $a \in A \wedge b \in B$ is also false.
$\therefore$ there is no such (a, b) that can be an elements of $\emptyset \times B$.
$\therefore \emptyset \times \{0, 1\} = \emptyset$
$\because \emptyset \subseteq \emptyset$
$\therefore \emptyset \times \{0, 1\} \subseteq \emptyset$.

(iv) True.
The only element in $\{\{1, 2\}\}$ is $\{1, 2\}$.
And this element is also in $\{\{1, 2\}, \{2\}\}$.
$\therefore \{\{1, 2\}\} \subseteq \{\{1, 2\}, \{2\}\}$.
Notice the element $\{2\}$, $\{2\} \in \{\{1, 2\}, \{2\}\}$, $\{2\} \notin \{\{1, 2\}\}$
$\therefore \{\{1, 2\}\} \subsetneq \{\{1, 2\}, \{2\}\}$.

(b) Since the two $\emptyset$ in $\{\emptyset, \emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$ are identical, we should remove one to simplify it.
And since the two $\emptyset$ in the element $\{\emptyset, \emptyset\}$ in $\emptyset$ in $\{\emptyset, \{\emptyset, \emptyset\}, \{\emptyset\}\}$ are identical, we

must remove one to simplify it.
After doing that, we have two idential elements $\{\emptyset\}$ in $\{\emptyset, \{\emptyset\}, \{\emptyset\}\}$, so we must remove one $\{\emptyset\}$ to simplify it.
$\therefore$ the simplest form of the original set is: $\{\emptyset, \{\emptyset\}\}$.
So the cardinality is 2 since there are two elements in the set.

(c) (i) There are 3 elements in $\{\emptyset, \{a\}, \{b, c\}\}$.
So there are $2^3 = 8$ elements in $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$.
$\therefore$ the cardinality of $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\})$ is 8.

(ii) $\mathcal{P}(\{\emptyset, \{a\}, \{b, c\}\}) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b, c\}\}, \{\{a\}, \{b, c\}\}, \{\emptyset, \{b, c\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{a\}, \{b, c\}\}\}$.

2. Is This Your Card(inality)? [18 points]

Suppose we define $A = \{m, a, r, c, h\}$, $B = \{a, r, t, i, c, h, o, k, e\}$, and $C = \{a, e, i, o, u\}$. Determine the cardinality of each of the following sets. You may leave large exponents unsimplified.

(a) $A \times B$

(b) $\mathcal{P}(A \cup B)$

(c) $\mathcal{P}((A \cap B) \times (B - C))$

Solution:

(a) $\because |A| = 5$, $|B| = 9$,
$\therefore |A \times B| = 5 \times 9 = 45$.

(b) $|A \cup B| = |A| + |B| - |A \cap B|$
$A \cap B = \{a, r, c, h\}, |A \cap B| = 4$
$\therefore |A \cup B| = 5 + 9 - 4 = 10$
$\therefore \mathcal{P}(|A \cup B|) = 2^{10} = 1024$

(c) $A \cap B = \{a, r, c, h\}$,
$B - C = \{r, t, c, h, k\}$,
$\therefore |(A \cap B) \times (B - C)| = 4 \times 5 = 20$.
$\therefore |\mathcal{P}((A \cap B) \times (B - C))| = 2^{20} = 1024 \times 1024 = 1048576$

3. Ice Cream Truck! [20 points]

There are 40 IAs for EECS 203 and they are all waiting in line for the ice cream truck! At the ice cream truck, every IA orders at least one flavor of ice cream, but they have the option to order more.

- 22 of them ordered chocolate ice cream
- 25 of them ordered strawberry ice cream
- 16 of them ordered vanilla ice cream

These numbers add up to more than 40, so we know that some IAs must have ordered multiple flavors. We also know:

- 12 IAs ordered chocolate ice cream and strawberry ice cream,
- 8 ordered strawberry ice cream and vanilla ice cream,
- 10 ordered vanilla cream and chocolate ice cream,
- and of these, some IAs ordered all three.

How many IAs ordered all three flavors? Show all your calculations.

Note: A Venn diagram is not acceptable justification for this question.

Solution:

We use A to represent the set of IAs who ordered chocolate ice cream;
and B: the set of IAs who ordered strawberry ice cream;
and C: the set of IAs who ordered vanilla ice cream;
Then the given information is that:
Above all, $|A \cup B \cup C| = 40$.
And $|A| = 22, |B| = 25, |C| = 16$.
And $|A \cap B| = 12, |B \cap C| = 8, |A \cap C| = 10$.
Since $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$,
$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| = 40 - 22 - 25 - 16 + 12 + 8 + 10 = 7$

4. S∪bset Fu∩ [20 points]

Let A, B, and C be sets. Prove that

$$(B - A) \cup (C - A) = (B \cup C) - A$$

by showing that each is a subset of the other.

Solution:

(a) We prove that $(B - A) \cup (C - A) \subseteq ((B \cup C) - A)$, that is, any element in $(B - A) \cup (C - A)$ is also in $(B \cup C) - A$.
Let x be an arbitrary element in the domain.
If $x\ in (B - A) \cup (C - A)$, $x \in (B - A) \vee x \in (C - A)$ due to definition of set union.
$\therefore (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A)$ due to definition of set substraction.
$\therefore (x \in B \wedge \neg x \in A) \vee (x \in C \wedge \neg x \in A)$
$\therefore \neg x \in A \wedge (x \in B \vee x \in C)$ due to distributive law.
$\therefore (x \in B \cup C) \wedge (x \notin A)$
$\therefore x \in ((B \cap C) - A)$ due to definition of set union.
$\therefore (B - A) \cup (C - A) \subseteq ((B \cup C) - A)$ due to definition of subset.

(b) We prove that $((B \cup C) - A) \subseteq (B - A) \cup (C - A)$, that is, any element in $(B \cup C) - A$ is also in $(B - A) \cup (C - A)$.
Let x be an arbitrary element in the domain.
If $x \in ((B \cup C) - A)$, $(x \in (B \cup C)) \wedge (x \notin A)$ due to definition of set substraction.
$\therefore ((x \in B) \vee (x \in C)) \wedge (\neg x \in A)$ due to definition of set union.
$\therefore (x \in B \wedge \neg x \in A) \vee (x \in C \wedge \neg x \in A)$ due to distributive law.
$\therefore (x \in (B - A)) \vee (x \in (C - A))$ due to definition of set substraction.
$\therefore ((B \cup C) - A) \subseteq (B - A) \cup (C - A)$ due to definition of subset.

Since $(B - A) \cup (C - A) \subseteq ((B \cup C) - A)$ and $((B \cup C) - A) \subseteq (B - A) \cup (C - A)$, $(B - A) \cup (C - A) = (B \cup C) - A$.

5. Subset Size Question [24 points]

Given that A, B, C are sets with $|A| = 13$, $|B| = 8$, $|C| = 10$, find the maximum *and* the minimum possible cardinalities of the following sets in the domain of integers.

(a) $\overline{A} \cap B$

(b) $A \cap (B \cup C)$

(c) $\mathcal{P}(B \times C)$

(d) $(A - B) \cap C$

Solution:

(a) From the property of intersection we know that $|\overline{A} \cap B| \leq |B|$, and since a set cannot have fewer than 0 element, $|\overline{A} \cap B| \geq 0$.
If $A \cap B = \emptyset$, then $\overline{A} \cap B = B$, then $|\overline{A} \cap B| = |A| \leq |B| = |B| = 8$.
If $A \cap B = B$, then $\overline{A} \cap B = \emptyset$, then $|\overline{A} \cap B| = |\emptyset| = 0$.
These two situations are all possible, $\therefore$ the maximum possible cardinality kf $\overline{A} \cap B$ is 9, and the minimum cardinality of $\overline{A} \cap B$ is 0.

(b)

(c) Since $|B| = 8, |C| = 10$, B, C cannot be $\emptyset$. Therefore $|\mathcal{P}(B \times C)|$ can only be $|B| \times |C| = 80$.

(d)

Groupwork

1. Grade Groupwork 5

Using the solutions and Grading Guidelines, grade your Groupwork 5:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/12
Problem 3												/8
Total:												/20

2. (Set)ting up a (Power)ful Proof [17 points]

(a) Suppose we want to prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$. What is the Inductive Hypothesis for your proof? *Hint: Consider what variable you should do induction on.*

(b) Prove by Induction that for any finite set S, it is true that $|\mathcal{P}(S)| = 2^{|S|}$.

Solution:

3. Out of the ordinary [16 points]

The (von Neumann) ordinals are a special kind of number. Each one is represented just in terms of sets. We can think of every natural number as an ordinal. We won't deal with it in this question, but there are also infinite ordinals that "keep going" after the natural numbers, which works because there are infinite sets.

The smallest ordinal, 0, is represented as $\emptyset$. Each ordinal after is represented as the set of all smaller ordinals. For example,

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

(a) What is the ordinal representation of 4, in terms of just sets?

(b) If the sets X and Y are ordinals representing the natural numbers x and y respectively, how can we tell if $x \leq y$ in terms of X and Y? Why does this work?

(c) If X is the ordinal representing the natural number x, what is the ordinal representation of $x + 1$? Why does this work?

Solution: