EECS 203 Exam 1 Review

Day 1

Today's Review Topics

- Propositional Logic
- Predicates and Quantifiers
- Proof Methods
  - Direct Proof
  - Proof by Contrapositive
  - Proof by Contradiction
  - Proof by Cases

Propositional Logic

Cheat Sheet Suggestions

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 2 De Morgan's Laws.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Cheat Sheet Suggestions

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Compound Proposition	Expression in English
¬p	"It is not the case that p"
p∧q	"Both p and q"
p∨q	"p or q (or both)"
p⊕q	"p or q (but not both)"
p→q	"if p then q" "p implies q"
p↔q	"p if and only if q"

Quick Recap

- Proposition - declarative statement that is either true or false
- $p \rightarrow q$
  - Logically equivalent to $\neg p \vee q$
  - Converse: $q \rightarrow p$
  - Contrapositive: $\neg q \rightarrow \neg p$
  - Inverse: $\neg p \rightarrow \neg q$
  - The original implication and the contrapositive have the same truth value, while the converse and inverse have the same truth values.
- Tautology - compound proposition that is always true
- Contradiction - compound proposition that is always false
- Satisfiable/Consistent - some assignment of truth values that make the compound proposition true
- How many propositions does a truth table with 256 rows have?

Truth Tables

If we have 2 propositions, how many rows will there be in the truth table?

If we have 5 propositions, how many rows will be in the truth table?

If we have n propositions, how many rows will be in the truth table?

Solution

2^2, 2^5, 2^n

Which of the following expressions is a contradiction?

(a) $(p \wedge q) \leftrightarrow (p \wedge r)$

(b) $(p \wedge q) \wedge T \wedge (\neg q \vee \neg p)$

(c) $(r \rightarrow q) \rightarrow (p \wedge \neg p)$

(d) $F \vee ((\neg\neg p \rightarrow q) \leftrightarrow \neg r)$

(e) $(q \wedge \neg q) \leftrightarrow (r \wedge \neg r)$

Solution

b, because we can perform DeMorgan's on $\neg q \lor \neg p \equiv \neg(q \land p)$ which is a contradiction with $p \land q$. Also for any truth values we use for p and q, the compound proposition is false

Given:

- c : school is canceled
- s : it snows two feet
- t : the temperature is -40 degrees

Which of the following is a propositional logic translation of the sentence:

"School will be canceled whenever the temperature is -40 degrees or it snowed two feet."

(a) $(s \wedge t) \rightarrow c$

(b) $(s \vee t) \rightarrow c$

(c) $\neg c \leftrightarrow \neg(s \vee t)$

(d) $c \rightarrow (s \wedge t)$

(e) $c \rightarrow (s \vee t)$

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"

"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Solution: (b)

"p whenever q" means that if q happens, then p must also happen; however, p can happen even if q does not happen. So "p whenever q" $\equiv q \to p$.

Suppose we have the following premises:

(i) If you are in Ann Arbor and it is not winter, then it is not snowing $[(a \wedge \neg w) \rightarrow \neg s]$

(ii) If you are not in Ann Arbor, then you are on vacation $[\neg a \rightarrow v]$

(iii) It is snowing $[s]$

(iv) If you are not enrolled in school then it is not the case that either you are on vacation or it is winter $[\neg e \rightarrow \neg(v \vee w)]$

Which is **NOT** a valid conclusion?

(A) You are on vacation or it is winter $[v \vee w]$

(B) You are not in Ann Arbor and it is winter $[\neg a \wedge w]$

(C) You are not in Ann Arbor or it is winter $[\neg a \vee w]$

(D) You are enrolled in school $[e]$

Solution

Solution: B. Combining premises (i) and (iii), we have that $\neg(a \wedge \neg w)$, i.e., $\neg a \vee w$, is correct. Combining $\neg a \vee w$ and premise (ii), we have $v \vee w$ is correct. Thus, e is correct based on $v \vee w$ and premise (iv).

Show that $(p \wedge q) \to r$ is **not** logically equivalent to $(p \to r) \wedge (q \to r)$.

TABLE 2 **The Truth Table for the Conjunction of Two Propositions.**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 5 **The Truth Table for the Conditional Statement $p \to q$.**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution

Let $p = T$, $q = F$, and $r = F$. Then

$$\begin{aligned} (p \wedge q) \to r &\equiv (T \wedge F) \to F \\ &\equiv F \to F \\ &\equiv T \end{aligned}$$

and

$$\begin{aligned} (p \to r) \wedge (q \to r) &\equiv (T \to F) \wedge (F \to F) \\ &\equiv F \to T \\ &\equiv F \end{aligned}$$

The two compound propositions give different truth values for these inputs, therefore they are **not** logically equivalent.

Alternate Solution

Truth table method:

p	q	r	$(p \wedge q)$	$(p \to r)$	$(q \to r)$	$(p \wedge q) \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The last two columns differ on the 4th line (and also the 6th line), therefore the two compound propositions are **not** logically equivalent.

Show that $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology. You can use truth tables or logical equivalences.

TABLE 2 **The Truth Table for the Conjunction of Two Propositions.**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 **The Truth Table for the Disjunction of Two Propositions.**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 5 **The Truth Table for the Conditional Statement $p \to q$.**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table Solution

$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Predicates and Quantifiers

Cheat Sheet Suggestions

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Cheat Sheet Suggestions

TABLE 1 Quantifications of Two Variables.

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

It's true that:

– $\forall x\, [P(x) \land Q(x)] \equiv [\forall x\, P(x)] \land [\forall x\, Q(x)]$

But it's not true that:

– $\forall x\, [P(x) \lor Q(x)] \equiv [\forall x\, P(x)] \lor [\forall x\, Q(x)]$

Likewise, it's true that:

– $\exists x\, [P(x) \lor Q(x)] \equiv [\exists x\, P(x)] \lor [\exists x\, Q(x)]$

But it's not true that:

– $\exists x\, [P(x) \land Q(x)] \equiv [\exists x\, P(x)] \land [\exists x\, Q(x)]$

Problem 6. (4 points)

Let $S(x, y)$ be the statement that "person x is shorter than person y". If Atreya is taller than Nouman but shorter than twins Eric and Paul (who are the same height), which of the following is true?

(a) S(Atreya, Nouman)

(b) S(Eric, Eric)

(c) S(Eric, Paul)

(d) S(Nouman, Eric)

(e) S(Paul, Nouman)

Solution

d, we know from the statement that Nouman is shorter than Atreya who is shorter than the twins Eric and Paul

Small note on translations

When we translate a sentence such as “Someone in this class is going to ace the exam” to proposition logic, we use $\exists x(C(x) \land A(x))$, where C(x) is x is in this class and A(x) is x is going to ace the exam. We do not want to use the $\rightarrow$ here, because for a person that isn’t a student, the implication would be true, which is not what we want.

When we translate a sentence such as “Everyone in this class is going to ace the exam” to proposition logic, we use $\forall x(C(x) \rightarrow A(x))$, where C(x) is x is in this class and A(x) is x is going to ace the exam. We do not want to use the $\land$ here, because the translation would give us false for those not in the class, even though those people do not matter.

Let $H(x,t)$ be the statement that "person x is happy at time t". Translate the following sentence:

"All the time someone is happy, but no one is happy all the time."

a) $\forall t \exists x H(x,t) \wedge \neg \exists x \forall t H(x,t)$

b) $\forall t \exists x H(x,t) \rightarrow \neg \exists x \forall t H(x,t)$

c) $\exists x \forall t H(x,t) \wedge \neg \forall t \exists x H(x,t)$

d) $\exists x \forall t H(x,t) \rightarrow \neg \forall t \exists x H(x,t)$

Solution

A, these are two separate statements connected by the “but”, which is equivalent to an “and” statement. The quantifiers then fall into place.

Let $L(x, y)$, $C(x, y)$, and $R(x, y)$ be the statements "x eats lunch with y", "x has a class with y", and "x is roommates with y" respectively. The domain for x and y is students at the University of Michigan.

Translate the following expressions of quantifiers, logical connectives, and predicates into English in the clearest way possible.

(a) $\forall x \forall y((C(x, y) \wedge R(x, y)) \rightarrow L(x, y))$

(b) $\exists x \forall y(((x \neq y) \wedge C(x, y)) \rightarrow \neg L(x, y))$

(c) $\forall x \exists y((x \neq y) \wedge (C(x, y) \vee R(x, y)) \wedge \neg L(x, y))$

Solution:

(a) All students who have a class together and are roommates will eat lunch together.

(b) There exists at least one student who does not eat lunch with any other students in any of their classes. (Alternately, there exists a student who, if they are in a class with another student, they do not eat lunch with that other student)

(c) Every student has another roommate or classmate with whom they don't eat lunch. (Alternately, For all students, there exists a different student with whom they either share a class or a room, but with whom they do not eat lunch.

13. Rewrite each of the following statements so that the negation appears before the predicates

(a) $\neg\forall x(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z))$

(b) $\neg(\exists x\exists y\neg P(x,y) \wedge \forall x\forall y Q(x,y))$

Solution:

(a) $\exists x(\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x,y,z))$

(b) $\forall x \forall y P(x,y) \vee \exists x \exists y \neg Q(x,y)$

Choose the true statements from the following if the domain of discourse is $\mathbb{R}$.

(a) $\forall x \forall y \exists z (x^2 + y^2 = z^2)$

(b) $\forall x [(x > 4) \to |x - 4| \geq 1]$

(c) $\forall x \exists y P(x, y) \to \exists y \forall x P(x, y)$ for all predicates $P(x, y)$

(d) $\exists y \forall x P(x, y) \to \forall x \exists y P(x, y)$ for all predicates $P(x, y)$

Solution: (a), (d)

(a) is true because for every x and y, $x^2 + y^2 \geq 0$ and we can use $z = \sqrt{x^2 + y^2}$.

(b) is not true. 4.5 as a counterexample.

(c) is also not true.

(d) is true. Note that while $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$ are not logically equivalent, the latter does imply the former. This is because if we have a single y value that works for every x, we can use that same value for each x in turn. However, if each x has a y value that works, it might be a different y for each one.

5 Minute Break

https://paveldogreat.github.io/WebGL-Fluid-Simulation/

Proof Methods

Proofs Overview

- Direct Proof - Prove p → q by showing that **if p is true, then q must also be true.**
- Proof by Contraposition - Prove p → q by showing that **if not q, then not p.**
  - Assume not q and arrive at not p
- Proof by Contradiction
  - Prove p by **assuming ¬p and arriving at a contradiction, therefore proving p is true (can think of this as ¬-intro from natural deduction)**
  - Prove p → q by **assuming p and ¬q and arriving at a contradiction, therefore ¬(p and ¬q) is true which is equivalent to saying p → q is true**

Overview Cont.

- Proof by Cases
  - Prove that a predicate is true by **separating into all possible cases and showing that the predicate is true in each individual case.**
  - Proof by cases is similar to the idea of $\lor$ - elimination.

NOTE: Proof by Induction will not be covered in Exam 1

Proof Methods Table

$p \to q$	Assumptions	Want to Reach
Direct Proof	p	q
Proof By Contrapositive	$\neg q$	$\neg p$
Proof By Contradiction	$p \land \neg q$	F

Proving + Disproving Quantified Statements

	Prove	**Disprove**
$\forall x P(x)$	Show that **arbitrary** x **satisfies** P(x)	Find a **counterexample** x which does not satisfy P(x)
$\exists x P(x)$	Find an **example** x which satisfies P(x)	Show that an **arbitrary** x **does not satisfy** P(x)

NOTE: The above does not show proof by example. Proof by example is **never** valid.

WLOG

Without Loss of Generality (WLOG) – used when the same argument can be made for multiple cases

Example: Show that if x and y are integers and both $x \cdot y$ *and* $x+y$ are even, then both x and y are even.

Proof: Use a proof by contraposition. Suppose x and y are not both even. Then, one or both are odd. Without loss of generality, assume that x is odd. Then $x = 2m + 1$ for some integer k.

Case 1: y is even. Then $y = 2n$ for some integer n, so $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd.

Case 2: y is odd. Then $y = 2n + 1$ for some integer n, so $x \cdot y = (2m + 1)(2n + 1) = 2(2m \cdot n + m + n) + 1$ is odd.

Prove that if n is an odd integer, then n^2 is odd.

Direct Proof Solution

Prove that if n is an odd integer, then n^2 is odd.

$p = \text{Odd}(n)$

$q = \text{Odd}(n^2)$

Direct Proof of $p \rightarrow q$:

1) Let n be odd; $\text{odd}(n) \rightarrow n = 2k + 1$ for some arbitrary $k \in \mathbb{Z}$
2) $n^2 = (2k + 1)^2$
3) $= 4k^2 + 4k + 1$
4) $= 2\ (2k^2 + 2k) + 1$
5) Since this is of the form 2(some integer) + 1, then we conclude $\text{odd}(n^2)$
6) Therefore, since we started by assuming p and were able to conclude q, then $p \rightarrow q$.

Prove that if $a \cdot b < 0$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $(a / b) < 0$.

Proof by Cases Solution

Prove that if $a \cdot b < 0$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a / b < 0$.

If $a \cdot b < 0$, then a and b must be of opposite signs and $a, b \neq 0$ (since then $a \cdot b = 0$)

Case 1: $a > 0$, $b < 0$

Then a / b would be + / - which would divide to become a negative number.

Case 2: $a < 0$, $b > 0$

Then a / b would be - / + which would divide to become a negative number.

In all (both) cases $a / b < 0$, therefore we have proven our implication that $a \cdot b < 0 \rightarrow a / b < 0$.

Prove that if $n = ab$, where a and b are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Proof by Contraposition Solution

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

$p : n = ab$

$q : a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

1) Start by assuming not q, that $a > \sqrt{n}$ and $b > \sqrt{n}$ (by De Morgan's Law)
2) Multiply the two inequalities (able to do this since left side is > right side for both inequalities)
3) $ab > \sqrt{n} \cdot \sqrt{n} \equiv ab > n$, so $ab \neq n$ (this is $\neg p$)
4) Therefore we have reached $\neg p$ and have shown that $\neg q \rightarrow \neg p$
5) By using proof by contraposition, we have now shown that $p \rightarrow q$

Prove that if $3n + 2$ is odd, then n is odd.

Proof by Contradiction Solution

Prove that if 3n + 2 is odd, then n is odd.

p = odd(3n + 2)

q = odd(n)

1) Assume odd(3n + 2), then assume ¬odd(n) ≡ even(n)
2) If n is even, then n = 2k, $k \in \mathbb{Z}$.
3) 3(2k) + 2 = 6k + 2 = 2(3k + 1)
4) This shows even(3n + 2) which is a contradiction with our first assumption, therefore our second assumption (¬odd(n)) must have been false and odd(n) must be true.
5) Therefore, $p \rightarrow q$ by proof by contradiction

Prove or Disprove: For all rational numbers x and y, x^y is also rational.

Prove/Disprove For-all Statement Solution

For all rational numbers x and y, x^y is also rational.

Recall that roots are applied to numbers raised to fractions. We can use this to our advantage in coming up with a counterexample.

Disproof by counterexample: let $x=2$, $y=\frac{1}{2}$

- x and y are both rational numbers.
- However, $x^y = 2^{1/2} = \sqrt{2}$, which is not rational.
- Thus, our counterexample disproves the statement.

Prove or Disprove: There exists an integer n such that $4n^2 + 8n + 16$ is prime

Prove/Disprove There-exists Statement Solution

There exists an integer n such that $4n^2 + 8n + 16$ is prime.

Notice that $4n^2 + 8n + 16$ is divisible by 4, no matter what integer can be plugged in. Therefore, it cannot be prime, so we know to disprove this statement.

Disproof of an Exists Statement:

- Let x be an arbitrary integer.
- Then, we have the expression $y = 4x^2 + 8x + 16$ (abbreviate y for less writing)
- We can factor out a 4 to get $y = 4(x^2 + 2x + 4)$.
- $x^2 + 2x + 4$ is an integer, and because we have written y as 4 times (some integer), y is divisible by 4.

Prove/Disprove There-exists Statement Solution (cont.)

- Now we have 2 cases: $y = 4$ and $y \neq 4$
  1. $y = 4$: y is not prime, because it (4) has a factor of 2
  2. $y \neq 4$: y is not prime, because it is divisible by 4 (a factor that is not equal to y, as y is not 4)
- In all cases y is not prime. Therefore, there does not exist any integer such that $4n^2 + 8n + 16$ is prime.

Which of the following describe the proof method(s) used to show the following statement? Mark all that apply.

Statement: If x is rational and y is irrational, then $x + y$ is irrational.
Proof: Assume that x is rational, y is irrational, and $x + y$ is rational. Notice that $y = (x + y) - x$. Since both $x + y$ and x are rational, and the difference of two rational numbers is also rational, this means that y is rational. But we assumed y was irrational. So it must be the case that whenever x is rational and y is irrational, $x + y$ is irrational.

(a) Proof by contrapositive

(b) Proof by cases

(c) Proof by contradiction

(d) Direct Proof

(e) Exhaustive proof Proving all cases possible

Solution

C, we assume p and not q and arrive at a contradiction

Identify the mistakes in the following proof, multiple answers

We prove that $0 = 2$ as follows.

S1. We have $4x^2 = 4x^2$.

S2. Rewriting the left and right hand sides, we get $(-2x)^2 = (2x)^2$.

S3. Taking the square root, we get $-2x = 2x$.

S4. Adding $x^2 + 1$ on both sides gives $-2x + x^2 + 1 = 2x + x^2 + 1$.

S5. By algebra, this can be written as $(x - 1)^2 = (x + 1)^2$.

S6. Taking the square root, we get $x - 1 = x + 1$.

S7. Subtracting $x - 1$ on both sides, we get $x - 1 - (x - 1) = x + 1 - (x - 1)$, i.e., $0 = 2$.

Solution: The mistake was made in steps 3 and 6. $a^2 = b^2$ does not imply that $a = b$ and so $(-2x)^2 = (2x)^2$ does not imply that $-2x = 2x$. Similarly, $(x-1)^2 = (x+1)^2$ does not imply that $x - 1 = x + 1$. The problem only asks to select the step where first error is made, therefore the correct answer is S3.

Good luck studying!