EECS 203: Discrete Mathematics
Fall 2023
Homework 11

Due **Tuesday, December 5**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $9 + 2$ Total Points: $100 + 18$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

Reminder: Make sure to leave your answers in combination, permutation, or factorial form and **not** simplified.

1. Bayes Easy [12 points]

Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.

(a) What is the probability that someone who tests positive has the genetic disease?

(b) What is the probability that someone who tests negative does not have the disease?

Solution:

Let D be the event that a randomly chosen person has the rare genetic disease. We are told that $P(D) = 0.0001$ and therefore $P(\overline{D}) = 0.9999$. Let A be the event that a randomly chosen person tests positive for the disease. We are told that $P(A \mid D) = 0.999$ ("true positive") and that $P(A \mid \overline{D}) = 0.0002$ ("false positive"). From these we can conclude that $P(\overline{A} \mid D) = 0.001$ ("false negative") and $P(\overline{A} \mid \overline{D}) = 0.9998$ ("true negative").

(a) We are asked for $P(D \mid P)$. We use Bayes' theorem:

$$\begin{aligned} P(D \mid A) &= \frac{P(A \mid D)P(D)}{P(A \mid D)P(D) + P(A \mid \overline{D})P(\overline{D})} \\ &= \frac{0.999 \cdot 0.0001}{0.999 \cdot 0.0001 + 0.0002 \cdot 0.9999} \\ &= \frac{555}{1666} \approx 33.3\% \end{aligned}$$

(b) We are asked for $P(\overline{D} \mid \overline{P})$. Using Bayes' theorem again we have

$$\begin{aligned} P(\overline{D} \mid \overline{A}) &= \frac{P(\overline{A} \mid \overline{D})P(\overline{D})}{P(\overline{A} \mid \overline{D})P(\overline{D}) + P(\overline{A} \mid D)P(D)} \\ &= \frac{0.9998 \cdot 0.9999}{0.9998 \cdot 0.9999 + 0.0001 \cdot 0.001} \\ &= 0.9999998999 \approx 1 \end{aligned}$$

Draft Grading Guidelines [12 points]

+2 correctly identifies $P(D)$ and $P(\overline{D})$
+2 correctly identifies $P(A \mid D)$
+1 correctly computes $P(\overline{A} \mid D)$
+2 correctly identifies $P(A \mid \overline{D})$
+1 correctly computes $(P(\overline{A} \mid \overline{D})$
+2 correct answer for (a)
+2 correct answer for (b)

2. Spaced Out [12 points]

A space probe heading to Mars sends messages back to Earth using bit strings. Suppose that it sends a '1' one-third of the time and a '0' two-thirds of the time. However, the communication channel is noisy—when a 1 is sent, it is possible that noise interferes, causing Earth to receive a 0 and vise versa. Probabilities of different situations are listed:

- When a 0 is sent, the probability that it is received correctly is 0.6.
- When a 0 is sent, the probability that it is received incorrectly (as a 1) is 0.4.
- When a 1 is sent, the probability that it is received correctly is 0.8.
- When a 1 is sent, the probability that it is received incorrectly (as a 0) is 0.2.

(a) Suppose Earth received a '0'. What is the probability that the probe sent '0'?

(b) The space probe then transmits the same bit as part (a) again, and Earth receives '0' a second time. What is the probability the probe sent a 0? You can assume that the event of a bit getting corrupted is independent of any other bit getting corrupted.

Solution:

(a) Let R_0 be the event that a 0 was received, S_0 be the event that a 0 was sent, and S_1 be the even a 1 was sent. Note that $S_1 = \overline{S_0}$. We are told that $P(S_1) = \frac{1}{3}$, $P(S_0) = \frac{2}{3}$, $P(R_0 \mid S_0) = 0.6$, and $P(R_0 \mid S_1) = 0.2$. Bayes' theorem tells us:

$$\begin{aligned} P(S_0 \mid R_0) &= \frac{P(R_0 \mid S_0) \cdot P(S_0)}{P(R_0 \mid S_0) \cdot P(S_0) + P(R_0 \mid S_1) \cdot P(S_1)} \\ &= \frac{0.6 \cdot \frac{2}{3}}{0.6 \cdot \frac{2}{3} + 0.2 \cdot \frac{1}{3}} \\ &= \frac{\frac{6}{15}}{\frac{7}{15}} = \frac{6}{7}. \end{aligned}$$

(b) We take the same setup as part (a), and we still have that $P(S_0) = \frac{2}{3}$ and $P(S_1) = \frac{1}{3}$. However, this time the probability we receive 0 twice given that '00' was sent is $(0.6)^2$, and is $(0.2)^2$ given that '11' was sent. So by Bayes' theorem we get

$$\begin{aligned} P(S_0 \mid R_0) &= \frac{(0.6)^2 \cdot \frac{2}{3}}{(0.6)^2 \cdot \frac{2}{3} + (0.2)^2 \cdot \frac{1}{3}} \\ &= \frac{18}{19} \approx 0.94737. \end{aligned}$$

Alternatively, we can interpret our answer from part (a) as our new probability for S_0, and instead only focus on the second bit that was sent. So using Bayes' theorem we get

$$\begin{aligned} P(S_0 \mid R_0) &= \frac{0.6 \cdot \frac{6}{7}}{0.6 \cdot \frac{6}{7} + 0.2 \cdot \frac{1}{7}} \\ &= \frac{18}{19}. \end{aligned}$$

Note that by Bayes' theorem the denominator in this second solution is still equal to $P(R_0)$, and since this probability changes with the information given in part (a), our value of $P(R_0)$ needs to be updated from (a) as well.

Draft Grading Guidelines [12 points]

Part a:

+1 correctly identifies $P(S_0) = \frac{1}{3}$ and $P(\overline{S_0}) = P(S_1) = \frac{2}{3}$
+3 correctly identifies $P(R_0 \mid S_0)$, $P(R_0 \mid S_1)$, $P(S_0 \mid R_0)$
+1 selects correct form of Bayes' theorem
+1 correct final answer

Part b:

+3 identifies that $P(S_0)$ remains the same (if using first method), or that we can use part (a) (if using second method)
+2 correct values for $P(R_0 \mid S_0)$ and $P(R_0 \mid S_1)$
+1 correct final answer

3. Aye Aye Esti-matey [8 points]

Give the tightest big-O estimate of the following functions:

(a) $g(n) = (3^n) \cdot (n^2 + \log n) \cdot (2n^4 + n) + (4n + n!) \cdot (1000^{n+1000} + n^n)$

(b) $f(n) = (n^2 + n \log n) \cdot \left(4n + \sum_{i=1}^{10} n^i\right)$

Note: $\sum_{i=1}^{10} n^i = n^1 + n^2 + n^3 + \cdots + n^{10}$

Solution:

(a) Look for the fastest growing term in the function. Multiplying the fastest growing terms of the first product, we have $3^n \cdot n^2 \cdot n^4$. Then multiply the fastest growing terms of the second product $n! \cdot n^n$. Since $n! \cdot n^n$ grows faster than $3^n \cdot n^2 \cdot n^4$, we have $g(n) \in O(n! \cdot n^n)$.

(b) n^2 grows fastest in the first sum, and n^{10} grows the fastest in the second sum. Hence, $f(n) \in O(n^2 \cdot n^{10}) = O(n^{12})$.

Draft Grading Guidelines [8 points]

For each part:

+2 correct answer
+0.5 partial credit: applies product/sum rule correctly at least once
+2 correct justification (can receive along side the partial credit above)

4. Al Gore, It Him [12 points]

Give a big-O estimate for the number of operations, where an operation is an addition or a multiplication, used in this segment of an algorithm (ignoring comparisons used to test the conditions in the **while** or **for** loop).

Hint: Your estimates may use more than one variable.

(a)

$t \leftarrow 0$
for $i := 1$ to n **do**
  for $j := 1$ to m **do**
    $t \leftarrow t + i + j$
  end for
end for

(b)

$t \leftarrow 0$
for $i := 1$ to n **do**
  $t \leftarrow t \cdot 2$
end for
for $j := 1$ to m **do**
  $t \leftarrow t + j$
end for

(c)

$t \leftarrow 0$
$i \leftarrow 1$
while $i \leq n$ **do**
  $t \leftarrow t - i$
  $i \leftarrow i \cdot 3$
end while

Solution:

(a) Since (a) is a nested for loop, we will multiply the number of iterations of the outer loop by the number of iterations of the inner loop. Here we compute the runtime to be $O(n \cdot m)$.

(b) With for loops that are not nested, we add the number of iterations of each loop to get a runtime of $O(n + m)$.

(c) Let x represent the number of iterations of the while loop. Then after x iterations of the while loop, i will equal 3^x. Then since we continue iterating through the while loop until $i > n$, then we solve for the smallest integer x for which $3^x > n$. This occurs when $x = \lceil \log_3 n \rceil$, so the code segment is $O(\log n)$.

Draft Grading Guidelines [12 points]

For each part:

+2 correct complexity
+2 correct explanation

5. Breakout Room [12 points]

In a class with 34 students there are 6 breakout rooms, with 3, 3, 4, 7, 8, and 9 students in each room, respectively.

(a) Suppose we pick a room at random, and consider X to be the random variable defined by the number of people in that room. What is the expected value of X?

(b) Now suppose we pick one of the students at random. Let Y be the random variable defined by the number of people in that student's room. What is the expected value of Y?

Solution:

(a) In this case each room is equally likely to be chosen with probability $\frac{1}{6}$. So

$$E(X) = \frac{1}{6}(3+3+4+7+8+9) = \frac{17}{3}.$$

(b) With this set up it is no longer equally likely for each room to be chosen. So

$$\begin{aligned} E(Y) &= 3 \cdot P(Y=3) + 4 \cdot P(Y=4) + 7 \cdot P(Y=7) + 8 \cdot P(Y=8) + 9 \cdot P(Y=9) \\ &= 3 \cdot \frac{6}{34} + 4 \cdot \frac{4}{34} + 7 \cdot \frac{7}{34} + 8 \cdot \frac{8}{34} + 9 \cdot \frac{9}{34} \\ &= \frac{114}{17} \approx 6.706. \end{aligned}$$

Draft Grading Guidelines [12 points]

Part a:

+2 correct formula
+2 correct computation

Part b:

+2 identifies the rooms have different probabilities of being selected
+2 correct expected value formula
+2 correct computation

6. Rolling Dice [12 points]

You roll a fair six-sided die 12 times. Find the probability that:

(a) Exactly two rolls come up as a 6.

(b) Exactly two rolls come up as a 4, given that the first four rolls each came up as 3.

(c) At least two rolls come up as a 6.

Solution:

(a) We apply the formula for a binomial distribution with $p = \frac{1}{6}$, $n = 12$, and $k = 2$. If X is the number of rolls that come up as a 6: $P(X = 2) = \binom{12}{2}(\frac{1}{6})^2(\frac{5}{6})^{10} = \binom{12}{2}\frac{5^{10}}{6^{12}} \approx 0.296$.

(b) Since these events are independent, then the first four rolls being determined reduces to the previous problem, but with $n = 8$, so $P(X = 2) = \binom{8}{2}(\frac{1}{6})^2(\frac{5}{6})^6 = \binom{8}{2}\frac{5^6}{6^8} \approx 0.260$.

Alternate soln using the conditional probability formula:
Let E be the event where where exactly 2 rolls out of the 12 come up as 4.
Let F be the event that the first four rolls are a 3.
So, $P(E|F) = P(E \cap F)/P(F)$.
$P(E \cap F) = (\frac{1}{6})^4\binom{8}{2}(\frac{1}{6})^2(\frac{1}{6})^6$. The first four rolls have to be a 4, so the probability of this is $(\frac{1}{6})^4$. Out of the 8 rolls left, we choose 2 rolls that are a 4, and multiply with the corresponding probability, $(\frac{1}{6})^2$. The remaining 6 rolls can be any number but 4. So the proabbility of that is $(\frac{1}{6})^6$.
$P(F) = (\frac{1}{6})^3$
Hence $P(E|F) = \binom{8}{2}(\frac{1}{6})^2(\frac{5}{6})^6$

(c) We can find the probability that less than two rolls come up as 6, and subtract that from 1:

$$\begin{aligned}
P(X \geq 2) &= 1 - (P(X = 0) + P(X = 1)) \\
&= 1 - \left[\binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11}\right] \\
&= 1 - \left[\left(\frac{5}{6}\right)^{12} + 12\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11}\right] \\
&\approx 1 - 0.381 \\
&= 0.619.
\end{aligned}$$

Draft Grading Guidelines [12 points]

Parts a and b:

+2 identifies correct parameters for binomial distribution
+1 correct answer

Part c:

+2 uses difference rule
+1 correct expression for $P(X = 0)$
+1 correct expression for $P(X = 1)$
+2 correct answer

7. More Dice [10 points]

Suppose Emily is rolling a pair of standard dice until the dice roll sums to 8 three times. What is the probability that they will roll more than 4 times?

For example, some sequences of rolls include:

$$(1,2), (4,6), \underline{(4,4)}, \underline{(5,3)}, (2,5), (2,3), \underline{(6,2)}$$
$$\underline{(4,4)}, (2,1), (4,5), (6,6), \underline{(5,3)}, (4,3), (5,5), \underline{(4,4)}$$

Solution:

In our sample space we treat the dice as distinguishable. There are 5 rolls that sum to 8: $(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, $(6,2)$. So there is a $\frac{5}{36}$ chance that a roll will sum to 8.

Note that at least three rolls will be required to stop. If the rolling stops on the n-th roll, then that roll must sum to 8. This can be conceptualized as Bernoulli trials with 2 successes in the first $n-1$ rolls and the n-th roll being successful.

The probability that they stop on the third roll is $\frac{5}{36}\binom{2}{2}\left(\frac{5}{36}\right)^2\left(\frac{31}{36}\right)^0$.

The probability that they stop on the fourth roll is $\frac{5}{36}\binom{3}{2}\left(\frac{5}{36}\right)^2\left(\frac{31}{36}\right)^1$.

The probability that the rolling ends after the 4th roll, is 1 minus the probability that it ends on some roll before the fifth. So our final probability is

$$1 - \frac{5^3}{36^3}\left[\binom{2}{2}\left(\frac{31}{36}\right)^0 + \binom{3}{2}\cdot\left(\frac{31}{36}\right)^1\right].$$

If you're curious, this evaluates to

$$1 - \frac{5^3}{36^3}\left[1 + \frac{31}{12}\right] = \frac{554497}{559872} \approx 0.9903996.$$

Draft Grading Guidelines [10 points]

+2 attempts to apply binomial distribution
+2 correct probability of stopping after three rolls
+2 correct probability of stopping after four rolls
+2 uses complement rule
+2 correct final answer

8. Mystery Boba [10 points]

Isabel loves to get bubble tea on campus. On any given day, there is 15% chance she gets taro milk tea, 25% chance she gets a matcha latte, 40% chance she gets passion fruit tea, and 20% chance she doesn't get any bubble tea. Assume Isabel has a maximum of one bubble tea every day.

(a) In a 7-day week, what is the probability that Isabel gets 2 taro milk teas, 1 matcha latte, and 2 passion fruit teas (in any order)?

(b) In a 7-day week, what is the probability that Isabel gets exactly 4 passion fruit teas?

(c) Over 14-days, what is the expected number of taro milk teas Isabel gets?

Solution:

(a) Since Isabel would only get 5 bubble teas, she must not get bubble tea 2 times during the week. First, consider the probability that Isabel gets 2 taro milk teas, 1 matcha latte, 2 passion fruit teas, and no bubble tea twice in that order:

$$P(T, T, M, P, P, N, N) = (0.15)^2(0.25)^1(0.4)^2(0.2)^2$$

This is only the probability of one sequence and since we want the probability of any sequence, we multiply this probability by the number of ways Isabel could've had 2 taro milk teas, 1 matcha latte, 2 passion fruit teas, and no bubble tea twice in a week. We have 7 days to work with and we choose 2 for taro, then choose 1 for matcha from the remaining, then choose 2 for passion fruit from the remaining, and finally choose 2 days to not get bubble tea from the remaining. We arrive at the following multinomial coefficient:

$$\binom{7}{2,1,2,2} = \binom{7}{2}\binom{5}{1}\binom{4}{2}\binom{2}{2} = \frac{7! \cdot 5! \cdot 4!}{2! \cdot 5! \cdot 1! \cdot 4! \cdot 2! \cdot 2!} = \frac{7!}{2! \cdot 2! \cdot 2!}$$

(Note this fraction of factorials can also be achieved using division rule.) Thus, the final probability is:

$$\binom{7}{2,1,2,2}(0.15)^2(0.25)^1(0.4)^2(0.2)^2 = \frac{7!}{2!\cdot 2!\cdot 2!}\cdot(0.15)^2(0.25)^1(0.4)^2(0.2)^2$$

(b) We can think of this in terms of a Bernoulli trial and binomial distribution where the probability of success is the same as the probability of Isabel getting passion fruit tea. Using the fact that we want to succeed 4 times, we can get the following probability:

$$P(X=4)=\binom{7}{4}(0.4)^4(1-0.4)^3$$

(c) We can think of this as a binomial distribution where each day she either gets a taro milk tea (0.15 probability) or she does not get taro milk tea (0.85 probability). The expected number of successes out of 14 trials with a 0.15 probability of success each time is:

$$E(\text{number of taro milk teas}) = 14\cdot 0.15 = 2.1 \text{ taro milk teas}$$

Grading Guidelines [10 points]

Part a:

+2 contains correct multinomial coefficient (any form is acceptable)
+1 contains correct probability of one arrangement
+1 reaches the correct overall probability (those two things multiplied together)

Part b:

+1 recognizes binomial distribution
+1 correct probability for success and failure (0.4 and 0.6 respectively)
+1 correct final probability

Part c:

+3 correct answer

9. The 101 Dalmations Binary Ballet [12 points]

Consider a binary sequence of length 14 selected at random. What is the expected number of times 101 appears in the sequence? For example, it appears 4 times in the string

$$\overline{10\underline{1}}\underline{01}0000\overline{10\underline{1}}\underline{01}.$$

Solution:

For $1 \leq k \leq 12$, let $I_k = 1$ if there is a 101 starting at position k (i.e. the k-th place is 1, the $k+1$-st place is 0, and the $k+2$-nd place is 1), and let $I_k = 0$ otherwise. If X is the random variable mapping binary sequences to the number of times 101 appears in the sequence, then

$$X = I_1 + I_2 + \cdots + I_{11} + I_{12}.$$

So by linearity of expectation,

$$E(X) = \sum_{k=1}^{12} E(I_k) = \sum_{k=1}^{12} \frac{1}{8} = \frac{12}{8} = \frac{3}{2}.$$

Draft Grading Guidelines [12 points]

+3 sets up indicator variables
+3 applies linearity of expectation
+3 obtains $E(I_k) = P(I_k = 1) = \frac{1}{8}$
+3 correct final answer

Groupwork

1. Grade Groupwork 10

Using the solutions and Grading Guidelines, grade your Groupwork 10:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/15
Problem 3												/15
Total:												/30

2. I Am Speed [10 points]

Suppose we have two algorithms, $\mathcal{A}$ and $\mathcal{B}$. Suppose that on inputs of size n, algorithm $\mathcal{A}$ runs in time $\Theta(n^2)$, while algorithm $\mathcal{B}$ runs in time $\Theta(n^3)$. Show that there exists some n_0 such that for any $n > n_0$, algorithm $\mathcal{A}$ will take less time to run than algorithm $\mathcal{B}$ on inputs of size n.
Note: you may find it useful for your notation to let the runtime of $\mathcal{A}$ on inputs of size n be denoted as $f_{\mathcal{A}}(n)$, and similarly for algorithm $\mathcal{B}$ as $f_{\mathcal{B}}(n)$.

Solution:

Because the run-time of $\mathcal{A}$ is $\Theta(n^2)$, it is also $O(n^2)$. From the definition, there exist $k_{\mathcal{A}}, C_{\mathcal{A}}$ such that for all $n > k_{\mathcal{A}}$, we know $f_{\mathcal{A}}(n) \leq C_{\mathcal{A}} \cdot n^2$.

Because the run-time of B is $\Theta(n^3)$, it is also $\Omega(n^3)$. From the definition, there exist $k_{\mathcal{B}}, C_{\mathcal{B}}$ such that for all $n > k_{\mathcal{B}}$, we know $f_{\mathcal{B}}(n) \geq C_{\mathcal{B}} \cdot n^3$.

We will choose now $n_0 = \max\left\{k_{\mathcal{A}}, k_{\mathcal{B}}, \frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\right\}$.

Consider an arbitrary input size n, which is greater than n_0. Because $n > n_0 \geq k_{\mathcal{A}}$ and $n > n_0 \geq k_{\mathcal{B}}$, we can apply what we know from the above big-O and big-Ω statements for n, that is, $f_{\mathcal{A}}(n) \leq C_{\mathcal{A}} \cdot n^2$, and also $f_{\mathcal{B}}(n) \geq C_{\mathcal{B}} \cdot n^3$. So we have

$$\begin{aligned}
f_{\mathcal{A}}(n) &\leq C_{\mathcal{A}} \cdot n^2 && \text{(Big-O statement from } \mathcal{A}\text{)} \\
&= C_{\mathcal{A}} \cdot \frac{C_{\mathcal{B}}}{C_{\mathcal{B}}} \cdot n^2 && \text{(Multiply by 1)} \\
&= C_{\mathcal{B}} \cdot \frac{C_{\mathcal{A}}}{C_{\mathcal{B}}} \cdot n^2 && \text{(Algebra)} \\
&\leq C_{\mathcal{B}} \cdot n_0 \cdot n^2 && (n_0 \geq \tfrac{C_{\mathcal{A}}}{C_{\mathcal{B}}}) \\
&< C_{\mathcal{B}} \cdot n \cdot n^2 && (n > n_0) \\
&= C_{\mathcal{B}} \cdot n^3 \\
&\leq f_{\mathcal{B}}(n) && \text{(Big-}\Omega\text{ statement from } \mathcal{B}\text{)}
\end{aligned}$$

This tells us that $f_{\mathcal{A}}(n) \leq f_{\mathcal{B}}(n)$, which means algorithm $\mathcal{A}$ takes less time on inputs of size n than algorithm $\mathcal{B}$.

Draft Grading Guidelines [10 points]

(i) +1 uses fact that $f_{\mathcal{A}} \in O(n^2)$
(ii) +1 uses fact that $f_{\mathcal{B}} \in \Omega(n^3)$
(iii) +3 correct value of n_0
(iv) +2 attempts to show $f_{\mathcal{A}}(n) \leq f_{\mathcal{B}}(n)$ for some arbitrary $n > n_0$
(v) +3 correctly shows $f_{\mathcal{A}}(n) \leq f_{\mathcal{B}}(n)$

3. GameStop or GameRoll? [8 points]

In a game of repeated die rolls, a player is allowed to roll a standard die up to n times, where n is determined prior to the start of the game. On any roll except the last, the player may choose to either keep that roll as their final score, or continue rolling in hopes of a higher roll later on. If the player rolls all n times, then on the n-th roll the player must keep that roll as their final score. A player always acts to maximize their expected final score. Finally, let V_n denote the final score in a game with a max of n rolls allowed.

(a) Compute $E(V_2)$ with justification.

(b) Compute $E(V_3)$ with justification.

(c) Find the smallest n such that $E(V_n) \geq 5$.

Solution:

(a) The expected value of any single die roll is 3.5, therefore, by extension, the expected value of the final of 2 die rolls is also 3.5. This means that since we are always trying to maximize final score, then if we roll higher than 3.5 on the first die roll, we should stop rolling. The probability of rolling higher than 3.5 on the first die roll is equal to the probability of rolling either a $4, 5$, or 6. The probability of rolling $4, 5$, or 6 equals $\frac{3}{6} = \frac{1}{2}$. The expected value of stopping after the first roll is then $\frac{4+5+6}{3} = 5$. Now when $n = 2$, if we did not stop after the first roll, then the expected value of the remainder of the game is simply equal to the expected value of a single die roll, or just 3.5. Note that this happens with probability $1 - \frac{1}{2} = \frac{1}{2}$. To finally combine these two cases, we see $E(V_2) = \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 3.5 = 4.25$.

(b) When $n = 3$, we treat the game like a single die roll followed by a game where $n = 2$. We saw by part (a) that $E(V_2) = 4.25$. With this information, we now will only stop after the first die roll when we roll either 5 or 6 (higher than 4.25). This occurs with probability $\frac{2}{6} = \frac{1}{3}$ and has an expected value of $\frac{5+6}{2} = 5.5$. Then the remainder of the game has expected value 4.25 and we will enter the second game with probability $1 - \frac{1}{3} = \frac{2}{3}$. Therefore, $E[V_3] = \frac{1}{3} \cdot 5.5 + \frac{2}{3} \cdot 4.25 = \frac{14}{3} \approx 4.67$.

(c) We continue the process of finding $E(V_n)$ until $E(V_n) \geq 5$.

$$E(V_4) = \frac{1}{3} \cdot 5.5 + \frac{2}{3} \cdot \frac{14}{3} = \frac{89}{18} \approx 4.94 \tag{1}$$

$$E(V_5) = \frac{1}{3} \cdot 5.5 + \frac{2}{3} \cdot \frac{89}{18} = \frac{277}{54} \approx 5.13 \tag{2}$$

Therefore, $n = 5$ is the smallest n such that $E(v_n) \geq 5$.

Grading Guidelines [8 points]

Part a:

(i) +1 correct use of the expected value of single die roll
(ii) +1 correct calculation of expected value for stopping after first roll
(iii) +1 correct calculation of expected value for stopping after second roll
(iv) +1 correct combining of the two expected values

Part b:

(v) +1 references the value from part (a)
(vi) +1 correct combining of the expected value of the first roll with that of second and third

Part c:

(vii) +1 correct answer for $E(V_4)$ (based on previous work)
(viii) +1 correct answer for $E(V_5)$ (based on previous work)