1. Technical vocab: ① proof by contrapositive
② "without loss of generality"
2. Recognize propositions for which proof by contrapositive might be helpful
3. Understand when it is and isn't valid to use "with loss of generality" in a proof.

9/14/23

L6: Proofs by Contrapositive -- ANSWERS

Disproofs (L5 cont'd)

Disproving a proposition is the same as ***proving*** its negation

Disprove: There exists an integer x for which $x > x^2 - 3x + 4$

Disproof:

- We will prove the negation: **"For all integers x, we have $x \leq x^2 - 3x + 4$."**
- Let x be an **arbitrary** integer.
- We have: $x^2 - 3x + 4 = (x^2 - 4x + 4) + x$
$= (x-2)^2 + x$
$\leq 0 + x$ (squares are always nonnegative)
$= x$

Some Proofs

Def: Int x is **"even"** if there exists an int k such that $x = 2k$.

Prove: For all integers x, if x is even, then x^2 is even.

Prove: For all integers x, if x^2 is even, then x is even.

Proof:

- Let x be an **arbitrary** integer.
- **Assume** that x is even.
- So there is an integer k with $x = 2k$.
- So $x^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
- Since k is an integer, $2k^2$ is also an integer
- So x^2 is even.

Not possible using "direct" proof methods...

② Proofs by contrapositive: Modify the proposition using logical equivalences to make it easier to prove it

Template: Proof by Contrapositive

逆否命题

Claim: If p, then q

contrapositive: If not q, then not p

Proof Template

We will prove the contrapositive: [state the contrapositive]
Assume not(q).

... (make some deductions) ...

Therefore, not(p).

Proof by Contrapositive

Def: Int x is **"odd"** if there exists an int k such that $x = 2k + 1$.

Prove: For all integers x, if x^2 is even, then x is even.

Proof:

- Let x be an **arbitrary** integer.
- We will prove the **contrapositive**: **"If x is odd, then x^2 is odd."** We actually proved this last lecture, but we'll recap:
- **Assume** that x is odd.
- So there is an integer k with $x = 2k + 1$.
- So $x^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
- Since k is an integer, $2k^2 + 2k$ is an integer.
- So x^2 is odd.

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Another Contrapositive

Def: Int x is **"even"** if there exists an int k such that $x = 2k$.

Def: Int x is **"odd"** if there exists an int k such that $x = 2k + 1$.

Prove: For all integers x, if $x^2 + 6x + 5$ is even, then x is odd.

Proof:

- Let x be an **arbitrary** integer.
- We will prove the **contrapositive**:
  "If x is even, then $x^2 + 6x + 5$ is odd."

- **Assume** that x is even
- So there is an integer k with $x = 2k$
- So $x^2 + 6x + 5 = (2k)^2 + 6(2k) + 5$
  $= 4k^2 + 12k + 5$
  $= 2(2k^2 + 6k + 2) + 1$
- Since k is an integer, $2k^2 + 6k + 2$ is an integer
- So $x^2 + 6x + 5$ is odd.

You Try It

Prove: For all integers x, if $5x^2 + 4$ is even, then x is even.

Def: Int x is **"even"** if there exists an int k such that $x = 2k$.

Def: Int x is **"odd"** if there exists an int k such that $x = 2k + 1$.

Proof:

- Let x be an **arbitrary** integer.
- We prove the **contrapositive**:
  "If x is odd, then $5x^2 + 4$ is odd."

- **Assume** that x is odd.
- So there is an integer k with $x = 2k + 1$.
- So $5x^2 + 4 = 5(2k+1)^2 + 4$
  $= 5(4k^2 + 4k + 1) + 4$
  $= 20k^2 + 20k + 9$
  $= 2(10k^2 + 10k + 4) + 1$
- Since k is an integer, $10k^2 + 10k + 4$ is an integer.
- Therefore $5x^2 + 4$ is odd.

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✲ Without Loss of generality: (Assume WLOG)
we can use it when: (1) There are several possibilities about the state of the world
(2) But these possibilities are completely symmetric
and the proof would look essentially the same under one possibility as the other

You Try It

Prove: For all integers x, y, if xy is even, then x is even or y is even.

Def: Int x is **"even"** if there exists an int k such that $x = 2k$.

Def: Int x is **"odd"** if there exists an int k such that $x = 2k + 1$.

Proof:

- Let x, y be **arbitrary** integers.
- We prove the **contrapositive**:
  "If x is odd and y is odd, then xy is odd."

- **Assume** that x is odd and y is odd.
- So there are integers j, k with $x = 2j + 1$ and $y = 2k + 1$.
- So $xy = (2j+1)(2k+1)$
  $= 4jk + 2j + 2k + 1$
  $= 2(2jk + j + k) + 1$
- Since j, k are integers, $2jk + j + k$ is an integer.
- Therefore xy is odd.

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A Fork in the Road

Def: Int x is **"even"** if there exists an int k such that $x = 2k$.

Def: Int x is **"odd"** if there exists an int k such that $x = 2k + 1$.

Prove: For all integers x, y, if $x + y$ is even, then x, y have the same parity
(meaning both are even or both are odd)

Proof:

- Let x, y be **arbitrary** integers.
- We will prove the **contrapositive**:
  "If x, y have different parities (one is even and the other is odd), **then $x + y$ is odd."**

- **Assume** that x, y have different parities.
- **Assume without loss of generality (WLOG)** that x is even and y is odd.
- So there are integers j, k with $x = 2j$ and $y = 2k + 1$.
- So $x + y = 2j + 2k + 1$
  $= 2(j + k) + 1$
- Since j, k are integers, $j + k$ is an integer
- So $x + y$ is odd.

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