Groupwork

1. Grade Groupwork 2

Using the solutions and Grading Guidelines, grade your Groupwork 2:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
  - If your current group submitted the same groupwork last time, grade it together.
  - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1	+2	+1	+2	+1	+2	+1	+2	+2				13 /13
Problem 2	+3	+1	+2	+1	+2	+1	+2					12 /12
Total:										filcl		25 /25

Previous Group Homework 2(1): Implication Inception [13 points]

Consider two propositions, A and B.

(a) Prove via a truth table that $A \equiv [(A \to B) \to A]$.

(b) Consider the compound proposition

$$\underbrace{((A \to B) \to A) \to B \ldots}_{\text{203 letters}}$$

which has 203 total letters in it, alternating between A and B. Fill in the rest of the following truth table. You don't need to manually make all 203 columns of the truth table to solve this, so try to find a pattern and think of a shortcut.

A	B	$\underbrace{((A \to B) \to A) \to B \ldots}_{\text{203 letters}}$
T	T	
T	F	
F	T	
F	F	

(c) Consider the following truth table. This is similar to the truth table in part (b), but it contains each iteration of the compound proposition from 1 to 203 letters.

A	B	$\underbrace{A}_{\text{1 letter}}$	$\underbrace{A \to B}_{\text{2 letters}}$	$\underbrace{(A \to B) \to A}_{\text{3 letters}}$	...	$\underbrace{((A \to B) \to A) \to B \ldots}_{\text{203 letters}}$
T	T	T				
T	F	T				
F	T	F				
F	F	F				

What is the total number of cells that will be T among all 4 rows and 203 columns? (Note that the two initial A and B columns do *not* count as columns that should be counted. However, the $\underbrace{A}_{\text{1 letter}}$ column and all following columns do count.)

Solution:

(a) Truth Table:

A	B	$A \to B$	$(A \to B) \to A$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

+2

and so we have proved that $((A \to B) \to A) \equiv A$ +1

(b) We have proved that $((A \to B) \to A) \equiv A$
Therefore $(((A \to B) \to A) \to B) \equiv (A \to B)$
and $((((A \to B) \to A) \to B) \to A) \equiv ((A \to B) \to A) \equiv A$.
We can see that it returns to the original proposition and thus forms a cycle because
+2 the next letter is always A and then B, continuing.
If we replace components from the inside out to be its logical equivalence and recursively perform this operation, we can find replace all the propositions by either A or $(A \to B)$. And by doing this, we can derive that propositions ended with A

+1

is equivalent to A, and those ended with B is equivalent to $(A \to B)$.
And since propositions ended with A have odd letters, and propositions ended with B have even letters (vice versa), the 203-letter-proposition is ended with A and has the same truth value with the A column.
$\therefore$

We did that correctly and detailedly!

+2

A	B	$\underbrace{((A \to B) \to A) \to B \dots}_{\text{203 letters}}$
T	T	T
T	F	T
F	T	F
F	F	F

(c) In question(b) we have justified that propositions ended with A are logically equivalent to A and have odd letters, while propositions ended with B are logically equivalent to $A \to B$ and have even letters (vice versa), the 203-letter-proposition is ended with A and has the same truth value with the A column.
+1 $\therefore$ In the Table, odd number columns have the truth value of $TTFF$, the same as that of A; even number columns have the truth value of $TFTT$, the same as $(A \to B)$.
+2 $\therefore$ There are 102 $TTFF$s and 101 $TFTT$s in the 203 columns.
$\therefore$ $102 \times 2 + 101 \times 3 = 505$ Ts.

Correct!

+2

A	B	$\underbrace{A}_{\text{1 letter}}$	$\underbrace{A \to B}_{\text{2 letters}}$	$\underbrace{(A \to B) \to A}_{\text{3 letters}}$	...	$\underbrace{((A \to B) \to A) \to B \dots}_{\text{203 letters}}$
T	T	T	T	T		T
T	F	T	F	T		T
F	T	F	T	F		F
F	F	F	T	F		F

Previous Group Homework 2(2): Functionally Complete [12 points]

A logical operator (or a set of logical operators) is considered to be *functionally complete* if it can be used to make any truth table.

(a) One set of functionally complete logical operators is $\{\vee, \neg\}$. In other words, we can use the $\vee$ and $\neg$ operators to make any truth table. Let's test this out with an example! Consider two propositions p and q. Write a compound proposition that is logically equivalent to $p \wedge q$ by only using p, q, $\vee$, $\neg$, and parentheses.

(b) Now, let's consider a new logical operator: NAND. The symbol for NAND is $\overline{\wedge}$. Below is the truth table for NAND. *(If you take EECS 370, you will get to use NAND even more!)*

p	q	$p \barwedge q$
T	T	F
T	F	T
F	T	T
F	F	T

Let's start trying to figure out whether $\barwedge$ is functionally complete. Is it possible to write a proposition that is logically equivalent to $\neg p$ by only using p and $\barwedge$? If so, write the proposition. If not, explain why it is impossible to do so.

(c) Is it possible to write a compound proposition that is logically equivalent to $p \vee q$ by only using p, q, $\barwedge$, and parentheses? If so, write the compound proposition. If not, explain why it is impossible to do so.

(d) Based on parts (a), (b), and (c), is $\barwedge$ functionally complete? Why or why not?

Solution:

(a) $\neg(\neg p \vee \neg q) \equiv (p \wedge q)$ ✓

+3

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

✓

(b) $(p \barwedge p) \equiv \neg p$ ✓

+1

We did a nice job!

+2

p	$\neg q$	$p \barwedge p$
T	F	F
F	T	T

✓

(c) $(p \vee q) \equiv [(p \barwedge p) \barwedge (q \barwedge q)]$ ✓

+1

+2

p	q	$p \vee q$	$p \barwedge p$	$q \barwedge q$	$(p \barwedge p) \barwedge (q \barwedge q)$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	T	T	F	T
F	F	F	T	T	F

✓

+1

(d) Yes. We are given that $\{\vee, \neg\}$ is functionally complete, and through (b) and (c) we have found equivalences of the two operators by using $\barwedge$. Therefore by substitution, we can say that $\barwedge$ is functionally complete. +2

2. Bézout's Identity [10 points]

In number theory, there's a simple yet powerful theorem called Bézout's identity, which states that for any two integers a and b (with a and b not both zero) there exist two integers r and s such that $ar+bs = \gcd(a,b)$. Use Bézout's identity to prove the following statements (you may assume all variables are integers):

(a) If $d \mid a$ and $d \mid b$, then $d \mid \gcd(a,b)$.

(b) If $a \mid bc$ and $\gcd(a,b) = 1$, then $a \mid c$.

Note: gcd is short for "greatest common divisor," so the value of $gcd(a,b)$ is the largest integer that evenly divides a and b. You won't need to apply this definition, just know that $\gcd(a,b)$ is an integer.

Solution:

(a) Assume $d \mid a$ and $d \mid b$.
So there exists an int x such that $a = dx$, and there exist an int y such that $b = dy$.
Through Bézout's identity we can state that there exists two integers r, s such that $ar + bs = \gcd(a,b)$.
Substitute a and b:

$$\begin{aligned} (dx)r + (dy)s &= \gcd(a,b) \\ dxr + dys &= \gcd(a,b) \\ d(xr + ys) &= \gcd(a,b) \end{aligned}$$

Since x, r, y, s are all integers, $xr + ys$ is an integer.
Therefore we have proved that $d \mid \gcd(a,b)$.

(b) Assume $a \mid bc$ and $\gcd(a,b) = 1$.
So there exists an int x such that $bc = ax$.
Through Bézout's identity, we can state that there exists an integer r and an integer s such that $ar + bs = \gcd(a,b)$.
So:

$$\begin{aligned} ar + bs &= 1 \\ car + cbs &= c \\ car + axs &= c \\ a(cr + xs) &= c \end{aligned}$$

Since c, r, x, s are integers, $cr + xs$ is an integer.
Therefore we proved that $a \mid c$.

3. High Five! [20 points]

Prove the following fun numerical facts:

(a) If a 5-digit integer is divisible by 4, its last two digits are also divisible by 4. For example, 40156 is divisible by 4, and so is 56.

(b) If a 5-digit integer is divisible by 3, the sum of the digits of that integer is also divisible by 3. For example, 33762 is divisible by 3, and so is $3 + 3 + 7 + 6 + 2 = 21$.

Hint: Think about how you can represent the digits of an integer. For instance, if a is a 2 digit number, then $a = a_1a_2 = a_1 \cdot 10 + a_2 \cdot 1$ (fill in the blanks).

Solution:

(a) Let x be an arbitrary 5-digit integer which can be divided by 4.
Since x is a 5-digit integer, let n_1, n_2, n_3, n_4, n_5 be the 5 digits of x, which means $x = 10^4 \cdot n_1 + 10^3 \cdot n_2 + 10^2 \cdot n_3 + 10 \cdot n_4 + n_5$ ($n_1 \in [1, 9]$; $n_2, n_3, n_4, n_5 \in [0, 9]$, and all of them are integers)
and $10 \cdot n_4 + n_5$ is the last two digits of x.
Also, since $4|x$, let $x = 4n$, n is an integer whose value depends on x
Then we have:

$$\begin{aligned} 4n &= 10^4 \cdot n1 + 10^3 \cdot n_2 + 10^2 \cdot n_3 + 10 \cdot n_4 + n_5 \\ 4n &= 4 \cdot 2500 \cdot n_1 + 4 \cdot 250 \cdot n_2 + 4 \cdot 25 \cdot n_3 + 10 \cdot n_4 + n_5 \\ 10 \cdot n_4 + n_5 &= 4(n - 2500 \cdot n_1 + 250 \cdot n_2 + 25 \cdot n_3) \end{aligned}$$

Since $n1, n2, n3, n$ are integers, $(n - 2500 \cdot n_1 + 250 \cdot n_2 + 25 \cdot n_3)$ is an integer. Therefore we prove that $4 \mid (10 \cdot n_4 + n_5)$.

(b) Let x be an arbitrary 5-digit integer which can be divided by 3.
Since x is a 5-digit integer, let n_1, n_2, n_3, n_4, n_5 be the 5 digits of x, which means $x = 10^4 \cdot n_1 + 10^3 \cdot n_2 + 10^2 \cdot n_3 + 10 \cdot n_4 + n_5$ ($n_1 \in [1, 9]$; $n_2, n_3, n_4, n_5 \in [0, 9]$, and all of them are integers)
And $n_1 + n_2 + n_3 + n_4 + n_5$ is the sum of the 5 digits.
Also, since $3|x$, let $x = 3n$, n is an integer whose value depends on x.

Then we have:

$$
\begin{aligned}
3n &= 10^4 \cdot n1 + 10^3 \cdot n_2 + 10^2 \cdot n_3 + 10 \cdot n_4 + n_5 \\
3n &= 9999 \cdot n_1 + 999 \cdot n_2 + 99 \cdot n_3 + 9 \cdot n_4 + n_1 + n_2 + n_3 + n_4 + n_5 \\
n_1 + n_2 + n_3 + n_4 + n_5 &= 3 \cdot (3333 \cdot n_1 + 333 \cdot n_2 + 33 \cdot n_3 + 3 \cdot n_4)
\end{aligned}
$$

Since n_1, n_2, n_3, n_4 are all integers, $(3333 \cdot n_1 + 333 \cdot n_2 + 33 \cdot n_3 + 3 \cdot n_4)$ is an integer. Therefore we prove that $3 \mid (n_1 + n_2 + n_3 + n_4 + n_5)$.