FA23

EECS 203 Discussion 2

Introduction to Logic

Admin Notes:

Forms:

- Two beginning of semester surveys on Canvas.
  - **Due:** Thursday, Sept. 14th @11:59 pm
- Exam Date Confirmation Survey
  - **Due:** Tuesday, Sept. 19th @11:59 pm
  - Fill this out even if you don't have an exam conflict!
- They are each worth a few points so fill them out!

Homework:

- Assignment 0 was due Sept. 5th
- Homework/Groupwork 1 was due Sept. 7th
- Homework/Groupwork 2 should be released! It will be due **Sept. 14th**
  - **Don't forget to match pages!**
  - Please note as soon as you press submit you've successfully submitted by the deadline, you can still match the pages with no rush, that doesn't add to your submission time.
- Groupwork:
  - It can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
  - Discussion section is a great place to find a group!
  - There is also a pinned Piazza thread for searching for homework groups.

Propositions

Important Truth Tables

p	q	$p \rightarrow q$	$p \leftrightarrow q$	$p \wedge q$	$p \vee q$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	T	F	F

Notes:

- The only time **p → q is false** is when we have **T → F**

Problem:

1. Finding Truth Values of Compound Propositions

For each compound proposition, find its truth value when $p = T$, $q = F$, $r = F$, $s = F$, $t = T$, $u = F$, and $v = F$

a) $(q \to \neg p) \vee (\neg p \to \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

Solution:

1. Finding Truth Values of Compound Propositions

For each compound proposition, find its truth value when $p = T$, $q = F$, $r = F$, $s = F$, $t = T$, $u = F$, and $v = F$

a) $(q \to \neg p) \lor (\neg p \to \neg q)$

b) $(p \lor \neg t) \land (p \lor \neg s)$

c) $(p \to r) \lor (\neg s \to \neg t) \lor (\neg u \to v)$

d) $(p \land r \land s) \lor (q \land t) \lor (r \land \neg t)$

Solution:

a) $(q \to \neg p) \lor (\neg p \to \neg q)$
$$\equiv (F \to \neg T) \lor (\neg T \to \neg F)$$
$$\equiv (F \to F) \lor (F \to T)$$
$$\equiv T \lor T$$
$$\equiv T$$

b) $(p \lor \neg t) \land (p \lor \neg s)$
$$\equiv (T \lor \neg T) \land (T \lor \neg F)$$
$$\equiv (T \lor F) \land (T \lor T)$$
$$\equiv T \land T$$
$$\equiv T$$

c) $(p \to r) \lor (\neg s \to \neg t) \lor (\neg u \to v)$
$$\equiv (T \to F) \lor (\neg F \to \neg T) \lor (\neg F \to F)$$
$$\equiv (T \to F) \lor (T \to F) \lor (T \to F)$$
$$\equiv F \lor F \lor F$$
$$\equiv F$$

d) $(p \land r \land s) \lor (q \land t) \lor (r \land \neg t)$
$$\equiv (T \land F \land F) \lor (F \land T) \lor (F \land \neg T)$$
$$\equiv (T \land F \land F) \lor (F \land T) \lor (F \land F)$$
$$\equiv F \lor F \lor F$$
$$\equiv F$$

Problem:

2. English to Logic Translation I

Let p, q, and r be the propositions defined as follows.

- p: Grizzly bears have been seen in the area.
- q: Hiking is safe on the trail.
- r: Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives (including negations).
***Reminder:** $\wedge$ denotes "and", $\vee$ denotes "or", $\leftrightarrow$ denotes"if and only if", and $\neg$ denotes "not".

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe

Solution:

2. English to Logic Translation I

Let p, q, and r be the propositions defined as follows.

- p: Grizzly bears have been seen in the area.
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a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe

Solution:

a) $r \wedge \neg p$

b) $\neg p \wedge q \wedge r$

c) $r \rightarrow (q \leftrightarrow \neg p)$

d) $\neg q \wedge \neg p \wedge r$

Problem:

3. Logic to English Translation

Consider the following propositions:

- g: you can graduate
- m: you owe money to the university
- r: you have completed the requirements of your major
- b: you have an overdue library book

Translate the following statement to English: $g \rightarrow (r \wedge \neg m \wedge \neg b)$

Solution:

3. Logic to English Translation

Consider the following propositions:

- g: you can graduate
- m: you owe money to the university
- r: you have completed the requirements of your major
- b: you have an overdue library book

Translate the following statement to English: $g \rightarrow (r \wedge \neg m \wedge \neg b)$

Solution:
You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book.
Equivalently, if you can graduate, then you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book.

Logical Equivalences

Logical Equivalence

- **Logical Equivalence:** Two compound propositions are logically equivalent if they have the same truth value for any input truth values
  - **Example:** p and ¬¬p

p	¬p	¬¬p
T	F	T
F	T	F

***have same truth table → logically equivalent!**

Other Logical Equivalences

De Morgan's And/Or Laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Identity Laws:

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

Implication Breakout:

$$p \rightarrow q \equiv \neg p \vee q$$

Commutative Laws:

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

Associative Laws:

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

Distributive Laws:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

The Contrapositive

Take an "if, then" statement $\mathbf{p \to q}$. We define the **contrapositive** of this statement as $\mathbf{\neg q \to \neg p}$. An implies statement and its contrapositive are logically equivalent…

$$\mathbf{p \to q \equiv \neg q \to \neg p}$$

Example:

- If I am teaching, then my materials are prepared.
- Contrapositive: If my materials are NOT prepared, then I am NOT teaching.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Equivalence Tables (Rosen 1.3)

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Tautologies & Contradictions

Tautology, Contradiction, Satisfiability

- **Tautology:** A compound proposition that is **always true** regardless of its input values
  - **Example:** $p \lor \neg p$
- **Contradiction:** A compound proposition that is **always false** regardless of its input values
  - **Example:** $p \land \neg p$
- **Satisfiable:** A compound proposition is satisfiable if it **can be true** (there is at least one set of inputs that makes the proposition true)
  - **Example:** $p \land q$

Notes:

- **Tautology:** A compound proposition that is **always true** regardless of its input values

Problem:

4. Tautologies

a) Determine whether $[\neg p \land (p \to q)] \to \neg q$ is a tautology.

b) Show that this conditional statement is a tautology by using truth tables.

$$[p \land (p \to q)] \to q$$

Solution:

4. Tautologies

a) Determine whether $[\neg p \wedge (p \to q)] \to \neg q$ is a tautology.

b) Show that this conditional statement is a tautology by using truth tables.

$$[p \wedge (p \to q)] \to q$$

Solution:

a) This is not a tautology. It is saying that knowing that the hypothesis of a conditional statement is false allows us to conclude that the conclusion is also false, and we know that this is not valid reasoning. To show that it is not a tautology, we need to find truth assignments for p and q that make the entire proposition false. Since this is possible only if the conclusion is false, we want to let q be true; and since we want the hypothesis to be true, we must also let p be false. It is easy to check that if, indeed, p is false and q is true, then the conditional statement is false. Therefore, it is not a tautology.

b) As seen in the truth table, all combinations of boolean assignments result in the statements being true. Therefore, it is a tautology.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Of note, we can also show this statement is equivalent to true (a tautology) by our logical equivalences.

$[p \wedge (p \to q)] \to q$

$\equiv [p \wedge (\neg p \vee q)] \to q$ Implication Breakout

$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \to q$ Distributive Law

$\equiv [F \vee (p \wedge q)] \to q$ $(p \wedge \neg p \equiv F$, contradiction)

$\equiv [(p \wedge q) \to q]$ Identity Law

$\equiv \neg(p \wedge q) \vee q$ Implication Breakout

$\equiv (\neg p \vee \neg q) \vee q$ De Morgan's

$\equiv \neg p \vee (\neg q \vee q)$ Associative

$\equiv \neg p \vee T$ $(\neg q \vee q \equiv T$, tautology)

$\equiv T$ Domination Law

Note: We will not ask you to be familiar with these logical equivalence laws, but for homework and exams, we will not ask you to do a line by line logical equivalence proof (you can just use a truth table).

Predicates & Quantifiers

Predicates & Quantifiers

- **Predicate:** A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified.
  - **Example:** **$x > 10$ -** predicate that depends on parameter x
- **Universal Quantifier:** Denoted by $\forall$ and translated as **"for all"**, it specifies that the following propositional function is true for all possible parameters in the domain.
  - **Example:** Let x be a positive integer. $\forall x \; [x > 0]$
- **Existential Quantifier:** Denoted by $\exists$ and translated as **"there exists"**, it specifies that the following propositional function is true for at least one parameter in the domain.
  - **Example:** Let x be an integer. $\exists x \; [x = 3]$

Notes:

- $\forall$ - "for all"
- $\exists$ - "there exists"

Problem:

5. Truth Values of Predicates

Let $Q(x)$ be the statement "$x + 1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(-1)$

b) $Q(1)$

c) $\exists x Q(x)$

d) $\forall x Q(x)$

e) $\exists x \neg Q(x)$

f) $\forall x \neg Q(x)$

Solution:

5. Truth Values of Predicates

Let $Q(x)$ be the statement "$x + 1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(-1)$

b) $Q(1)$

c) $\exists x Q(x)$

d) $\forall x Q(x)$

e) $\exists x \neg Q(x)$

f) $\forall x \neg Q(x)$

Solution:

a) True. Substitute $x = -1$ into the predicate, and check if the the inequality holds.

b) False. Substitute $x = 1$ into the predicate, and check if the the inequality holds. In this case, the inequality does not hold.

c) True. To prove an exists statement is true, we just need to find one element of the domain for which the statement holds. Let's consider $x = -10$. $Q(-10)$ is the statement "$-10 + 1 > 2(-10)$". This is true since $-10 + 1 = -11 > -20 = 2(-10)$. There are many other values of x for which the statement holds as well (any integer less than 1 will work).

d) False. To prove a for all statement is true, we need to show $Q(x)$ is true for all x in the domain. Or we could prove it is false by finding a counterexample. In this case, we can provide a counterexample $x = 1$. Now we have proved this statement is false.

e) True. To prove this statement is true, we only need to find one example where $Q(x)$ is false. We have found this example in part (b).

f) False. To prove this to be false, we only need one example to show $Q(x)$ is true, and we can use examples such as $x = 0$ or $x = -1$.

Quantifiers Continued

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers to quantify a compound proposition P(x,y). In nested quantifiers, order matters…
  - **P(x,y):** some statement about x and y
  - **Example:** $\forall x \exists y\, P(x,y)$ is different from $\exists y \forall x\, P(x,y)$
    - **$\forall x \exists y\, P(x,y)$:** "For all x, there exists y such that…"
    - **$\exists y \forall x\, P(x,y)$:** "There exists y such that for all x…"
- **De Morgan's Laws for Quantifiers:**
  - $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
  - $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Problem:

6. Quantifiers and Negations

Express the negation of each of these statements in terms of quantifiers without using the negation symbol.

a) $\exists x(-4 < x \leq 1)$

b) $\forall z \exists x \exists y(x^3 + y^3 = z^3)$

Solution:

6. Quantifiers and Negations

Express the negation of each of these statements in terms of quantifiers without using the negation symbol.

a) $\exists x(-4 < x \leq 1)$

b) $\forall z \exists x \exists y(x^3 + y^3 = z^3)$

Solution:

a) $\forall x((x \leq -4) \vee (x > 1))$

b) $\exists z \forall x \forall y(x^3 + y^3 \neq z^3)$

Problem:

7. Quantified Statement Counterexamples

Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Solution:

7. Quantified Statement Counterexamples

Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Solution:

a) There are many counterexamples. If $x = 2$, then there is no y among the integers such that $2 = 1/y$, since the only solution of this equation is $y = 1/2$. Even if we were working in the domain of real numbers, $x = 0$ would provide a counterexample, since $0 = 1/y$ for no real number y.

b) We can rewrite $y^2 - x < 100$ as $y^2 < 100 + x$. Since squares can never be negative, no such y exists if x is, say, -200. This x provides a counterexample.

c) This is not true, since sixth powers are both squares and cubes. Trivial counterexamples would include $x = y = 0$ and $x = y = 1$, but we can also take something like $x = 27$ and $y = 9$, since $27^2 = 3^6 = 9^3$.

Problem:

8. Quantifier Translations I

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

Solution:

8. Quantifier Translations I

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

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c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

Solution: Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people.

a) This means that everyone has the property of being not perfect: $\forall x \neg P(x)$. Alternatively, we can write this as $\neg\exists x P(x)$, which says that there does not exist a person who is perfect.

b) This is just the negation of "Everyone is perfect": $\neg\forall x P(x)$. Alternatively we could write $\exists x \neg P(x)$ (i.e. there exists someone who is not perfect).

c) If someone is your friend, then that person is perfect: $\forall x(F(x) \implies P(x))$. Note the use of conditional statement with universal quantifiers.

d) We do not have to rule out your having more than one perfect friend. Thus we have simply $\exists x(F(x) \land P(x))$. Note the use of conjunction with existential quantifiers.

e) The expression is $\forall x(F(x) \land P(x))$. Note that here we did use a conjunction with the universal quantifier, but the sentence is not natural (who could claim this?). Because $\forall$ distributes over $\land$, we can also split up the expression into two quantified statements and write $(\forall x F(x)) \land (\forall x P(x))$. To avoid confusion, one could instead write $(\forall x F(x)) \land (\forall y P(y))$, where the domain of both x and y is all people.

f) This is a disjunction. The expression is $(\neg\forall x F(x)) \lor (\exists x \neg P(x))$. Again, to avoid confusion, one could instead write $(\neg\forall x F(x)) \lor (\exists y \neg P(y))$, where the domain of both x and y is all people.