Practice Exam 2
QUESTIONS PACKET
EECS 203
Fall 2023

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

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*****MAKE SURE YOU HAVE PROBLEMS 1 - 19 IN THIS BOOKLET.*****

General Instructions

You have 120 minutes to complete this exam. You should have two exam packets.

- **Questions Packet:** Contains ALL of the questions for this exam, worth 90 points total. There are 9 Multiple Choice questions (4 points each), 4 Short Answer questions (5 or 6 points each), and 4 Free Response questions (8 points each). You may do scratch work on this part of the exam, but only work in the Answers Packet will be graded.
- **Answers Packet:** Write all of your answers in the Answers Packet, including your answers to multiple choice questions. **For free response questions, you must show your work! Answers alone will receive little or no credit.**
- You may bring **one** 8.5” by 11” note sheet, front and back, created by you.
- You may **NOT** use any other sources of information, including but not limited to electronic devices (including calculators), textbooks, or notes.
- After completing the exam, sign the Honor Code on the front of the Answers Packet.
- You must turn in both parts of this exam.
- **You are not to discuss the exam until the solutions are published.**

Part A: Multiple Answer Multiple Choice

For the following questions, select all the options that apply. **This could be all answers, no answers, or anything in between.**

Problem 1. (4 points)

Which of the following statements are logically equivalent to this statement:

"If it's Friday, Regan teaches discussion today."

(a) "If it's not Friday, Regan does not teach discussion today."

(b) "If Regan teaches discussion today, it's Friday."

(c) "If Regan does not teach discussion today, it's not Friday."

(d) "Regan does not teach discussion today or it's Friday."

(e) "It's not Friday or Regan teaches discussion today."

Solution:

(c), (e)

(a) This is the inverse of the original, which is equivalent to the converse, but not the original. For example, this doesn't say what happens when it is Friday, so Regan might not teach discussion on Friday. However, the original statement says Regan has to teach discussion.

(b) This is the converse of the original, which is not equivalent. It's different for the same reasons as (a). For another example, the original doesn't say what happens on Thursdays. So, it's possible that Regan teaches discussion on Thursday. However, the converse says that can't happen, since teaching discussion means it's Friday, not Thursday.

(c) This is the contrapositive of the original statement. The negation of "it's Friday" is "it's not Friday", and the negation of "Regan teaches discussion today" is "Regan does not teach discussion today." So, the contrapositive of the original is "If Regan does not teach discussion today, it's not Friday."

(d) This is implication breakout on the converse, which is equivalent to the converse, but not the original. This is because the negation of "Regan teaches discussion today" is "Regan does not teach discussion today," so the implication breakout is "Regan does not teach discussion today or it's Friday". We can see that it's different for the same reasons as (a) and (b).

(e) This is the application of implication breakout on the original, so it's equivalent. The negation of "it's Friday" is "it's not Friday", so applying implication breakout gives us "It's not Friday or Regan teaches discussion today."

Problem 2. (4 points)

Which one of these proof methods would allow us to prove the following statement:
For an integer n, n^2 is odd if and only if n is odd.

(a) Consider the integer 3. 3 is odd and $3^2 = 9$ is also odd.

(b) List out all the possible odd numbers and show that all of their squares are odd as well.

(c) Let n be odd and show that n^2 is also odd. Let n^2 be even and show that n is even.

(d) Let n be even and show that n^2 is also even. Let n be odd and show that n^2 is also odd.

(e) Assume n^2 is odd and n is even and arrive at a contradiction.

Solution:

(d)

(a) This will not prove the statement *for all integers* n.

(b) You will never be able to finish exhausting all of the odd numbers.

(c) This proves the "if" direction twice, so it is not sufficient.

(d) "Let n be even and show that n^2 is also even." proves the "only if" direction of this statement by contraposition. "Let n be odd and show that n^2 is also odd." proves the "if" direction of this statement directly. Therefore, this is sufficient for the proof of this biconditional statement.

(e) This only proves the "only if" direction of this proof, by contradiction. Therefore it is not sufficient.

Problem 3. (4 points)

Which of the following are equivalent to $\forall x \neg \exists y \forall z [P(x,y,z) \to Q(x,y,z)]$?

(a) $\neg \exists x \exists y \forall z \quad [P(x,y,z) \to Q(x,y,z)]$

(b) $\forall x \forall y \forall z \quad [P(x,y,z) \to Q(x,y,z)]$

(c) $\forall x \exists y \neg \exists z \; [P(x,y,z) \to Q(x,y,z)]$

(d) $\forall x \forall y \exists z \; \neg[\neg P(x,y,z) \vee Q(x,y,z)]$

(e) $\forall x \forall y \exists z \quad [P(x,y,z) \wedge \neg Q(x,y,z)]$

Solution:

(a), (d), (e)
This is a question about applying DeMorgans. Recall that we can apply DeMorgans law to distribute a negation one "layer" at a time. We can "reverse" the distribution of the negation over the first $\forall$ to get choice (a).

We can also distribute the negation over the $\exists y \forall z$ to conclude

$$\forall x \neg \exists y \forall z [A(x,y,z) \to B(x,y,z)] \equiv \forall x \forall y \exists z \neg [A(x,y,z) \to B(x,y,z)]$$

Using the definition of implies,

$$\forall x \forall y \exists z \neg [A(x,y,z) \to B(x,y,z)] \equiv \forall x \forall y \exists z \neg [\neg A(x,y,z) \vee B(x,y,z)]$$

so choice (d) is correct. Applying DeMorgans one more time,

$$\forall x \forall y \exists z \neg [\neg A(x,y,z) \vee B(x,y,z)] \equiv \forall x \forall y \exists z [A(x,y,z) \wedge \neg B(x,y,z)]$$

which is choice (e).

We can intuit (b) and (c) are incorrect, because we cannot reach them by applying our logical equivalence laws (DeMorgans or definition of implies). Instead, in part (b), if we tried to apply DeMorgans, we would flip the $\forall z$ to a $\exists z$, and in part (c) we would convert the $\exists y$ to a $\forall y$. Note that these observations are **not proof** that the statements are not equivalent. We could prove that they are not equivalent by providing a counterexample, but we omit that here.

Problem 4. (4 points)

Let $P(x)$ and $Q(x)$ be predicates over the domain of integers. $a \oplus b$ is the XOR symbol, and $a \oplus b$ means a or b is true, but not both. Suppose we know that the following statement is true:

$$\forall x[P(x) \oplus Q(x)]$$

Which of the statements below **must** also be true?

(a) $\forall x[P(x) \lor Q(x)]$

(b) $\exists x[\neg Q(x) \land P(x)]$

(c) $\neg\exists x[P(x) \land Q(x)]$

(d) $\forall x[P(x)] \oplus \forall x[Q(x)]$

(e) $\exists x[P(x)] \lor \exists x[Q(x)]$

Solution:

From the given true statement, we know that every integer must either have $P(x)$ be true, or $Q(x)$ be true, but not both. Note, for each integer, it is not guaranteed that the same predicate is true, we just are able to guarantee that only one is true, yet we can not draw conclusions about which predicate.

(a) True. This is because we know that for every integer, that if one of the predicates is true, than the statement $P(x) \lor Q(x)$ is true.

(b) False. This is because we can not guarantee which predicate is true. It is possible that given $\forall x[P(x) \oplus Q(x)]$ that every single integer has $Q(x)$ true and $P(x)$ false, meaning there does not exist an integer that has $Q(x)$ false and $P(x)$ true, which is what we need to satisfy the and statement.

(c) True. This is because we know that every integer does not make both predicates true. This means we can not find an integer that makes both predicates true.

(d) False. Quantifiers do not distribute over $\oplus$. This is making the claim that every integer either satisfies $P(x)$ or every integer satisfies $Q(x)$ but not both. We can not guarantee this as one integer may satisfy $P(x)$ while another satisfies $Q(x)$. This means our given statement is still true, however the two quantified statements will both be false.

(e) True. For this to be true, we need to guarantee that there exists some integer that either makes $P(x)$ true or $Q(x)$ true. This can be guaranteed since we know that every integer must make $P(x)$ or $Q(x)$ true.

Problem 5. (4 points)

Which of the following are propositions?

(a) For all integers x, if x is odd, then x^2 is even

(b) $p \to q \equiv \neg p \lor q$, where p and q are propositions

(c) x is a power of 5

(d) Is mayonnaise an instrument?

(e) $\exists x[(x \in \mathbb{Z}) \land (x^3 = 27)]$

Solution:

(a), (b), (e)

(a) This statement is a false proposition, the square of an odd number is always odd

(b) This is a true proposition, these two statements are equivalent based on the definition of implies

(c) This is a predicate that depends on the unspecified variable x

(d) This is a question, not a declaration about the world, so it is not a proposition s

(e) This is a true proposition, satisfied by $x = 3$

Problem 6. (4 points)

Which of the following proof outlines could be used to prove the given statement:

For all real numbers x, if x^3 is irrational, then x is irrational.

(a) **Direct Proof:**
Let x be an arbitrary real number.
Assume that x is irrational and show that x^3 must be irrational.

(b) **Proof by Contradiction:**
Let x be an arbitrary real number.
Assume that x^3 is irrational and assume that x is rational. Show that this leads to a contradiction.

(c) **Proof by Contrapositive:**
Let x be an arbitrary real number.
Assume that x^3 is rational and show that if x^3 is rational, then x must be rational.

(d) **Proof by Contrapositive:**
Let x be an arbitrary real number.
Assume that x is rational, and show that if x is rational, then x^3 must be rational.

(e) **Proof by Example:**
Find a specific value of x where x^3 is irrational and x is irrational.

Solution:

(b), (d)

(a) Though a direct proof is possible, it should be completed by first assuming that x^3 is irrational and showing that x is irrational rather than the other way around. The provided method proves the converse of the statement, which is not equivalent to the original.

(b) This approach is correct since it correctly negates the proposition, and states that we should show that the negation is a contradiction.

(c) This is an incorrect proof by contrapositive. The conrapositive of $p \to q$ is $\neg q \to \neg p$ rather than $\neg p \to \neg q$. As such, once again, we would be proving the converse/inverse.

(d) This approach is correct since it correctly finds the contrapositive, which is if x is rational, then x^3 is rational.

(e) This is an incorrect approach since the proof must be completed with an arbitrary x when showing that the statement is true for all x.

Problem 7. (4 points)

For $x, y, z \in \mathbb{R}$, which of the following are true?

(a) $\exists x \forall y (xy = 0)$

(b) $\exists x \forall y (xy = 1)$

(c) $\forall x \exists y (y^2 = x)$

(d) $\forall x \exists y (x^2 = y)$

(e) $\forall z \exists x \exists y (\frac{x}{y} = z)$

Solution:

(a), (d), (e)

(a) There exists a real number x such that when you multiply it by any other real number, the product is 0.
TRUE: x = 0

(b) There exists a real number x such that for all real numbers y, the product xy = 1.
FALSE: Each number will have exactly one inverse such that the product = 1, so there cannot be a single real number x that is the inverse of every real number.

(c) For all real numbers x, there exists a real number y such that $y^2 = x$.
FALSE: Not every real number has a real square root, i.e. negative numbers.

(d) For every real number x, there exists a real number y such that $x^2 = y$.
TRUE: Every real number has a square.

(e) For every real number z, there exists real numbers x and y such that $\frac{x}{y} = z$.
TRUE: Any real number can be represented by the ratio of two real numbers. Example: z can always be represented by $\frac{x}{y}$ when x = z and y = 1.

Problem 8. (4 points)

Let the domain of x and y be the **integers**. Which of the following are true?

(a) $\exists x \exists y\ [x^2 + y^2 = 0]$

(b) $\exists x \forall y\ [y = 3x]$

(c) $\forall y \exists x\ [y = 3x]$

(d) $\exists x \forall y\ [x + y^2 = 7]$

(e) $\forall y \exists x\ [x + y^2 = 7]$

Solution:

(a), (e)

(a) This statement is TRUE. We are looking for two integer values such that $x^2 + y^2 = 0$. Since our quantifiers are both "there-exists", all we need to do is find two values that make this equation true. We can set $x = 0$ and $y = 0$, which would mean $x^2 + y^2 = 0^2 + 0^2 = 0$. Thus, this statement is true with the domain of all integers.

(b) This statement is FALSE. We are looking for an integer value for x that will make $3x = y$ true for every single value of y. However, each integer value multiplied by 3 produces a unique value. For instance, let us suppose $x = 4$. Then $3x = 12$, and if we wanted to say $3x = y$ ($y = 3x$), then we would know this statement was true for $y = 12$. For any other value of y, we can't say that $3x = y$ when $x = 4$, since y can be any integer, ex. 1, 1000, or 203, all of which would not be equal to $3x$ when $x = 4$.

(c) This statement is FALSE. If this statement was true, then for every single integer value of y, we'd be able to find an x that made it true, Consider the case where $y = 1$. Then, $x = \frac{1}{3}$. However, if x is in the domain of integers, $x = \frac{1}{3}$ is an invalid value. Thus, we have shown a counterexample that makes this statement false. Another way of looking at this statement is interpreting it as "every integer y is a multiple of 3", which we know is not true. Of note, if the domain were real numbers, this would be true.

(d) This statement is FALSE. 7 is a constant value, and since this answer requires us to find an x that works for every value of y, it wouldn't be valid. This is because the value of y^2 is constantly changing based on the value y takes on in the domain so we would be trying to add a constant and dynamic value to get a constant value. There is no single value of x that would make this equation true for all values of y.

(e) This statement is TRUE. We are looking to check if every value of y has a value of x that will make this equation true. Since the square of any integer is also an integer, that means that we are looking for a value of x such that $x +$ some arbitrary integer $= 7$. So, to find an x that satisfies the equation, we solve this equation for x by rewriting the equation as $x = 7 - y^2$. Since integers are closed under multiplication (squaring) and subtraction, we know that for every possible integer value of y, we can always find a corresponding integer value for x that makes the equation true.

Problem 9. (4 points)

Emily X, Emily Y, and Emily Z walk into a room. One is a Professor, one is a GSI, and one is an IA.

- Emily X says: "Neither Y nor Z are professors."
- Emily Y says: "I'm not an IA."
- Emily Z says: "I've never heard of EECS 203."

It turns out all the Emilys are lying. Which Emily is the professor and which is the GSI?

(a) Professor: X, GSI: Y

(b) Professor: Y, GSI: X

(c) Professor: X, GSI: Z

(d) Professor: Z, GSI: X

(e) Professor: Z, GSI: Y

Solution:

(d)
Emily Y must be an IA in order to be lying.

Now, Emily X must be either a professor or a GSI. If Emily X is a professor, then neither Y nor Z are professors, making the first claim true, so Emily X must be a GSI.

That means Emily Z must be the Professor.

Part B: Short Answer

For the following questions, keep the answer brief. If there are intermediate steps involved, you would need to show work and justification to get full credit.

Problem 10. (6 points)

Prove that if $5n^2 - 2$ is odd, then n is odd.
Note: You cannot use the lemmas "even + odd = odd", "even $\cdot$ even= even", etc. without proving it.

Solution:

Solution I: Contrapositive
We will prove the contrapositive: if n is even, then $5n^2 - 2$ is even.
Assume n is even. Then there is an integer k so that $n = 2k$. Therefore, $5n^2 - 2 = 20k^2 - 2 = 2(10k^2 - 1)$. Letting $l = 10k^2 - 1$, we see that $5n^2 - 2 = 2l$. Hence, $5n^2 - 2$ is even and the claim holds.

Solution II: Contradiction
Seeking a proof by contradiction, assume $5n^2 - 2$ is odd and that n is even. By definition of even, $n = 2k$ and thus if we plug it into $5n^2 - 2$, we get $5(2k)^2 - 2$. This becomes $5(4k^2) - 2$ which becomes $2(5(2k^2) - 1)$. $5(2k^2) - 1$ can be represented by the integer j. So, $3n^2 + 3$ can be represented as $2j$, and thus is even. This completes our contradiction, and therefore if $5n^2 - 2$ is odd, then n is odd.

Grading Guidelines:
Contrapositive
+1 Writing the correct contrapositive
+2 Correctly applying even/odd definition to n
+2 Correctly applying even/odd definition to $5n^2 - 2$
+1 Concluding that $5n^2 - 2$ is even
Contradiction
+1 Assuming $5n^2 - 2$ is odd for contradiction
+2 Correctly applying even/odd definition to n
+2 Correctly applying even/odd definition to $5n^2 - 2$
+1 Concluding that $5n^2 - 2$ is even and deriving a contradiction

Problem 11. (5 points)

Let $P(x, y)$ be the predicate "x is taller than y", defined on the domain of all people. What is the truth value of the following proposition? Briefly explain your answer by explaining the meaning of the proposition in English.

$$\exists x \exists y \exists z\, [P(x, y) \wedge P(y, z) \wedge P(z, x)]$$

Solution:

False

Translated to English, this statement says that there are three people (not necessarily distinct) such that person 1 is taller than person 2, person 2 is taller than person 3, and person 3 is taller than person 1. Trying to figure out who among the three is tallest quickly reveals the contradictory nature of this statement. For example, if person 1 is taller than person 2 and person 2 is taller than person 3, then person 1 is taller than person 3. But the statement also says that person 3 is taller than person 1. This shows that no such 3 people can exist, so this proposition is false.

Grading Guidelines:

- Explanation / Translation:
  - 4 points for correctly translating the proposition into English
  - 3 points for minor translation errors
  - 2 points for major translation errors
  - 1 point for incorrect translation but at least one part of the translation correct
- 1 point for correct truth value

Common Mistakes:

- Forgetting to state a truth value for the proposition
- Assuming x, y, z are distinct
- Imprecise English translation, like "there is a group of three people where each is taller than one other and shorter than one other" or "each is taller than two others."
- Scoping issues, like "there exists a person y who x is taller than, there exists a person z who y is taller than, and there exists a person x who z is taller than."

Problem 12. (6 points)

Let the domain of x be all people. Let $N(x)$ be the statement, "x is a nice person". Let $T(x)$ be the statement, "x drinks tea".

Translate each of the following English statements into logical statements.

(a) All nice people drink tea.

(b) Everyone who drinks tea is nice.

(c) There is a nice person who drinks tea.

(d) Not everybody is nice, but everybody drinks tea.

Solution:

(a) $\forall x(N(x) \to T(x))$

(b) $\forall x(T(x) \to N(x))$

(c) $\exists x(T(x) \land N(x))$

(d) $\neg\forall x N(x) \land \forall x T(x)$ or
$\exists x \neg N(x) \land \forall x T(x)$ or
$\exists x \forall y(\neg N(x) \land T(y))$

Grading Guidelines:

- 1.5 points for each part
- Full credit for a correct translation
- 0.5 points partial credit for a translation with error (ex. correct quantifiers and correct connective, but incorrect scoping, missing quantifiers).

Common Mistakes:

- Using wrong logical connective (Ex. $\implies$ vs $\land$).
- Incorrect scoping for quantifier. Ex. for part (c), $(\exists x T(x)) \land N(x)$
- Part (d): Combining statements incorrectly such as $\neg\forall x[N(x) \land T(x)]$
- Part (d): Saying $\forall x[T(x) \land (N(x) \lor \neg N(x))]$. We already know $N(x) \lor \neg N(x)$, as $N(x)$ has to be either true or false, so that simplifies to $\forall x T(x)$

- Part (d): Putting the negation symbol and quantifier in the wrong order by saying $\forall x \neg N(x) \wedge T(x)$ or $\neg \exists x N(x) \wedge \forall x T(x)$
- Part (d): Neglecting $N(x)$ by just writing $\forall x T(x)$

Problem 13. (5 points)

Claim: "The sum of two non-negative integers is always non-negative."

Suppose we want to prove the above claim using proof by contradiction. Complete the following sentence that would begin such a proof.

"Seeking contradiction, assume that ..."

Notes:

- Your answer *can* use the phrase "non-negative", but should **not** contain the word "not" or any other negation.
- Complete the given sentence. Do **not** complete the full proof.

Solution:

... there are two non-negative integers whose sum is negative.

NOTE: the original statement can be interpreted as $\forall x \forall y[(x \geq 0 \wedge y \geq 0) \rightarrow (x+y \geq 0)]$
Therefore, the negation can be expressed as: $\exists x \exists y[(x \geq 0 \wedge y \geq 0) \wedge (x+y<0)]$

Alternate solutions:
... there exist two non-negative integers whose sum is negative.
... there exists a negative number which is the sum of two non-negative integers.
... there exists a sum of two non-negative integers which is negative.

- +5 for fully correct solution (expressed in any correct combination of English and logic)
- +2 (Partial) Explicitly states some form of "there exists" ("there are")
- +2 (Partial) Correctly simplifies negation of inner statement (negates "non-negative" to get "negative")
- +1 (Partial) Set up to imply "there exists" without explicitly stating it (automatically given if item 2 is given).

Common Mistakes:

"The sum of two non-negative integers is always negative"
"The sum of two non-negative integers is negative"

These are wrong as they are "for-all" statements, and therefore the quantifier was not negated correctly in these responses.

"The sum of two non-negative integers can be negative"
"The sum of two non-negative integers is sometimes negative"

These are incorrect as there is no indication of a counterexample; this is still a "for-all" statement with some condition that is not specified. "Can be" and "sometimes" are vague and do not add to the overall meaning of the sentence.

"The sum of two non-negative integers is always positive"
"The sum of two non-negative integers is never non-negative"
"The sum of two non-negative integers is never negative"

Not only are these all "for-all" statements, they also do not correctly or fully negate and simplify the negation of the right-hand side of the implies statement.

Part C: Free Response

Problem 14. (9 points)

p	q	r	a	b	c
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	F
F	T	F	T	T	F
F	F	T	F	T	T
F	F	F	F	T	T

Use the truth table for the compound propositions a, b, and c given above to answer the following question.

For each unknown proposition, a, b, and c, find an expression for the proposition as a compound proposition using p, q, and/or r. Note the following requirements:

- You may use **only** $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, and parentheses in each expression.
- You may use p, q, and r **at most once** in each expression.

Solution:

$a \equiv \neg p \wedge q$
$\equiv \neg(p \vee \neg q)$
$\equiv \neg(q \rightarrow p)$
To get this solution, we can look for patterns. For example, we see that a is only true when p is F and q is T.

$b \equiv \neg p \vee q \vee \neg r$
$\equiv \neg(p \wedge r) \vee q$
$\equiv (p \wedge r) \rightarrow q$
$\equiv p \rightarrow (q \vee \neg r)$
$\equiv p \rightarrow (r \rightarrow q)$
$\equiv (r \rightarrow q) \vee \neg p$
$\equiv r \rightarrow (p \rightarrow q)$
$\equiv (p \wedge r) \rightarrow q$

$\equiv r \to (\neg p \lor q)$
$\equiv (p \to q) \lor \neg r$
There are multiple ways to get a correct answer (including using De Morgan's law or other logical equivalence rules on the given statements to get equivalent ones, there may be more valid statements that the ones listed). The patterns we notice are that b is only F when p is T, q is F, and r is T. Otherwise, it is T. We could think of a few symbols that result in more T than F, such as $\lor$, and then add negation symbols to fit our table.

$c \equiv p \leftrightarrow q$
$\equiv \neg p \leftrightarrow \neg q$
We see that c is only true when p and q hold the same truth values. $\leftrightarrow$ is a way to show this.

Grading Guidelines: 3 points for each expression, with partial credit as follows:

- +3 points for Correct
- +1.5 points for Correct but violates one or both requirements +1 point for Incorrect expression, but partially correct justification such as citing a line from the truth table.

Common Mistakes:

- In part a, mixing up the $\lor$ and $\land$ operators and answering $\neg p \lor q$ instead of $\neg p \land q$
- For parts a and c, the r term was not a part of the expression. A common mistake was adding an extraneous r term. For part a, common incorrect solutions were $\neg p \land q \lor r$ and $(\neg p \land q) \to r$. For part c, common incorrect solutions were $(p \leftrightarrow q) \to r$
- Another common mistake was writing the negation of the correct statement. In part a, this would look like $q \to p$. For part b, the common incorrect solution was $p \land \neg q \land r$, and for part c, the common incorrect solution was $\neg(p \leftrightarrow q)$
- Primarily for part c, another common mistake was using variables more than once and violating the rules of the problem. These solutions were often logically equivalent, but not fully correct because of the problem's requirements: $(p \land q) \lor (\neg p \land \neg q)$, $(p \to q) \land (q \to p)$

Problem 15. (7 points)

Let a, b, c be consecutive integers with $a < b < c$. Prove that the sum $a+b+c$ is a multiple of 3.

Note: For example, 3,4,5 are consecutive integers, but 3,4,6 are not. An integer x is a multiple of an integer y if and only if there is an integer k with $x = ky$.

Solution:

Assume consecutive integers a, b, c. Since a, b, c are consecutive, we can write $b = a + 1$ and $c = a + 2$. So $a + b + c = a + (a + 1) + (a + 2) = 3a + 3 = 3(a + 1)$. Since a is an integer, $a + 1$ is an integer. Therefore, $a + b + c$ is a multiple of 3.

Grading Guidelines:

- 3 points for writing $b = a + 1$ and $c = a + 2$ or similar
- 3 points for regrouping $a + b + c$ as 3 times an integer
- 1 points for applying the definition of “multiple of 3” to this expression to conclude the proof

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- Full credit for any correct alternate proof, even if overcomplicated.

Common Mistakes:

- Attempt to prove by example, such as letting $a = 3$, $b = 4$, and $c = 5$, or showing a few ”cases” for the values of a, b, and c. To prove a for all statement, a, b, and c must be arbitrary.
- Assuming the conclusion $a + b + c = 3k$ is true and then doing algebra on that. The conclusion should not be assumed to be true; instead, a direct proof should assume the premises and use those to lead to the conclusion. **Remember that to prove $p \to q$ by direct proof we assume p and then show q.**
- Dividing $a + b + c$ by 3. Integers are not closed under division so we want to avoid division.

- Creating variables without indication of where they came from. For instance, writing just $k + k + 1 + k + 2$ by itself without justification and then proceeding is unclear.

- *Not a mistake* – Creating cases where a is odd and a is even. While this is not incorrect and would receive full points, it is unnecessary and makes the proof take more time.

Problem 16. (8 points)

Let x and y be integers. Prove that if $x^2 + y^2 - 3x^2y$ is even, then x and y are both even.

For this question only, you may use the following 8 properties about odd and even numbers without proof.

- Odd + Odd = Even
- Odd + Even = Odd
- Even + Even = Even
- Odd $\times$ Odd = Odd
- Odd $\times$ Even = Even
- Even $\times$ Even = Even
- Odd^2 is Odd
- Even^2 is Even

Solution:

This can be proven in a variety of ways. The most common solution was a Proof by Contraposition, the second most common solutions was a direct proof, and some students did Proof by Contradiction (which closely resembles the contraposition proof, as you'll see below).

Solution 1: Proof by Contraposition

We will prove the contrapositive of the given statement: "If $\neg$ (x and y are even), then $\neg$(x$^2 + y^2 - 3x^2y$ is even)" which simplifies to "If x or y is odd, then $x^2 + y^2 - 3x^2y$ is odd". Here 'or' refers to inclusive OR. We can prove this statement using Proof by cases, where the individual cases are :

- **Case 1:** x is odd and y is even

  Since x is odd, x^2 is odd. Since y is even, y^2 is even as well. $-3x^2y$ is the product of an odd number, with a product of an odd and even number, resulting in an even number. The resulting sum is of an odd number, even number, and even number, which means $x^2 + y^2 - 3x^2y$ is odd.

- **Case 2:** x is even and y is odd

  Since x is even, x^2 is even. Since y is odd, y^2 is odd as well. $-3x^2y$ is the product of an odd number, with a product of an even and odd number, resulting in an even number. The resulting sum is of an even number, odd number, and even number, which means $x^2 + y^2 - 3x^2y$ is odd.

- **Case 3:** x and y are both odd

  Since x is odd, x^2 is odd. Since y is odd, y^2 is odd as well. $-3x^2y$ is the product of an odd number, with a product of an odd and odd number, resulting in an odd number. The resulting sum is of an odd number, odd number, and odd number, which means $x^2 + y^2 - 3x^2y$ is odd.

In all 3 cases, we proved that $x^2 + y^2 - 3x^2y$ is odd, so it must be odd. Since we proved the contrapositive, the original statement is true as well.

Solution 2: Direct proof
We will use 4 cases: x and y both odd; x odd and y even; x even and y odd; x and y both even.

- **Case 1:** x and y are both odd
  x^2 and y^2 are both odd as $odd^2 = odd$. Then, $3x^2y$ is also odd as this is (odd)(odd)(odd)=(odd)(odd)=odd. odd + odd - odd = even - odd = odd, therefore $x^2 + y^2 - 3x^2y$ is odd.
  Thus, "$x^2 + y^2 - 3x^2y$ is even" is false, while "x and y are both even" is also false
  $F \to F = T$

- **Case 2:** x is odd, y is even
  since $odd^2 = odd$ and $even^2 = even$, x^2 and y^2 are odd and even, respectively. Then, $3x^2y$ = (odd)(odd)(even)=(odd)(even)=even. odd + even - even = odd - even = odd, therefore $x^2 + y^2 - 3x^2y$ is odd.

Thus, “$x^2 + y^2 - 3x^2y$ is even” is false, while “x and y are both even” is also false
$F \rightarrow F = T$

- **Case 3:** x is even, y is odd
  since $even^2 = even$ and $odd^2 = odd$, x^2 and y^2 are even and odd, respectively. Then, $3x^2y$ = (odd)(even)(odd)=(even)(odd)=even. even + odd - even = odd - even = odd, therefore $x^2 + y^2 - 3x^2y$ is odd.
  Thus, “$x^2 + y^2 - 3x^2y$ is even” is false, while “x and y are both even” is also false
  $F \rightarrow F = T$

- **Case 4:** x and y are both even
  x^2 and y^2 are both even as $even^2 = even$. Then, $3x^2y$ is also even as this is (odd)(even)(even)=(even)(even)=even. even + even - even = even - even = even, therefore $x^2 + y^2 - 3x^2y$ is even.
  Thus, “$x^2 + y^2 - 3x^2y$ is even” is true, while “x and y are both even” is also true
  $T \rightarrow T = T$

In all 4 cases, the desired implies statement evalutes to true, so it must be true in general: if $x^2 + y^2 - 3x^2y$ is even, then x and y are both even.

Solution 3: Proof by Contradiction
Repeating what we want to prove, just to have it easily visible here:

“If $x^2 + y^2 - 3x^2y$ is even, then x and y are both even.”

Seeking contradiction, assume $x^2 + y^2 - 3x^2y$ is even and x and y are not both even. That is, assume at least one of x or y is odd. There are 3 cases to consider: x and y both odd; x even and y odd; x odd and y even.

- **Case 1:** x and y both odd.

Using the 8 properties,

$$\begin{aligned} x^2 + y^2 - 3x^2y &= \text{odd}^2 + \text{odd}^2 - \text{odd} \cdot \text{odd}^2 \cdot \text{odd} \\ &= \text{odd} + \text{odd} - \text{odd} \cdot \text{odd} \cdot \text{odd} \\ &= \text{even} - \text{odd} \cdot \text{odd} \\ &= \text{even} - \text{odd} \\ &= \text{odd} \end{aligned}$$

So $x^2 + y^2 - 3x^2y$ is odd. But we are given that $x^2 + y^2 - 3x^2y$ is even, so we have a contradiction.

- **Case 2:** x even y odd.
  Using the 8 properties,

$$\begin{aligned} x^2 + y^2 - 3x^2y &= \text{even}^2 + \text{odd}^2 - \text{odd} \cdot \text{even}^2 \cdot \text{odd} \\ &= \text{even} + \text{odd} - \text{odd} \cdot \text{even} \cdot \text{odd} \\ &= \text{odd} - \text{even} \cdot \text{odd} \\ &= \text{odd} - \text{even} \\ &= \text{odd} \end{aligned}$$

  So $x^2 + y^2 - 3x^2y$ is odd. But we are given that $x^2 + y^2 - 3x^2y$ is even, so we have a contradiction.

- **Case 3:** x odd y even.
  Using the 8 properties,

$$\begin{aligned} x^2 + y^2 - 3x^2y &= \text{odd}^2 + \text{even}^2 - \text{odd} \cdot \text{odd}^2 \cdot \text{even} \\ &= \text{odd} + \text{even} - \text{odd} \cdot \text{odd} \cdot \text{even} \\ &= \text{odd} - \text{odd} \cdot \text{even} \\ &= \text{odd} - \text{even} \\ &= \text{odd} \end{aligned}$$

  So $x^2 + y^2 - 3x^2y$ is odd. But we are given that $x^2 + y^2 - 3x^2y$ is even, so we have a contradiction.

In each case, we have reached a contradiction, so our assumption that "x and y are not both even" must be false. Therefore, when $x^2 + y^2 - 3x^2y$ is even, x and y must both be even.

Grading Guidelines:

Contrapositive
+2 correct contrapositive statement
+2 per correct case

Direct proof
+2 per correct case

Contradiction
+2 correct contradictive statement
+2 per correct case

Common Mistakes:

- Stating the contrapositive as "if both x and y are odd, then $x^2 + y^2 - 3x^2y$ is odd". This left out the 2 cases where one of x and y are odd, with the other being even.
- Proving the converse: "If x and y are both even, then $x^2 + y^2 - 3x^2y$ is even. Note that the converse and original statements are not logically equivalent.

Problem 17. (8 points)

Prove or disprove the following claim:

"For all positive integers a, b, and d, if $d|a$ or $d|b$, then $d|a^2b$".

Solution:

Let a, b, and d be arbitrary positive integers. Suppose $d|a$ or $d|b$.

Case 1: $d|a$, so $a = kd$ where k is an integer.
This means $a^2b = kd \times ab = (kab)d$, and kab is an integer, so $d|a^2b$

Case 2: $d|b$, so $b = kd$ where k is an integer.
This means $a^2b = a^2kd = (a^2k)d$, and a^2k is an integer, so $d|a^2b$.

In either case, we have that $d|a^2b$, so this must be true regardless of which premise is true.

Since we proved that $(d|a \vee d|b) \to d|a^2b$ for arbitrary a, b, and d, the claim holds for all positive integers.

Alternate solution: Let a, b, and d be arbitrary positive integers. Suppose $d|a$ or $d|b$.

Case 1: $d|a$, so $\frac{a}{d} = x$ where x is an integer. Then $\frac{a^2b}{d} = axb$, which must be an integer since it is a product of integers. Thus we have that $d|a^2b$.

Case 2: $d|b$, so $\frac{b}{d} = x$ where x is an integer. Then $\frac{a^2b}{d} = a^2x$, which must be an integer since it is a product of integers. Thus we have that $d|a^2b$.

In either case, we have that $d|a^2b$, so this must be true regardless of which premise is true.

Since we proved that $(d|a \vee d|b) \to d|a^2b$ for arbitrary a, b, and d, the claim holds for all positive integers.

Grading Guidelines:

+2 Split into correct cases
+2 Identify either that $a = kd$ or $\frac{a}{d} = x$ and similar for case 2
+2 Correct mathematical evaluation when plugging in a or x
+2 Correct conclusion when plugged into a^2b or similar

Common Mistakes:

1. Incorrectly interprets the prompt as:

   "For all positive integers a, b, and d, if $d|a$ **and** $d|b$, then $d|a^2b$"

   This is a much weaker claim. It can be proved using similar logic to the original but doesn't need cases. A correct proof of this claim substitutes $a = dp$ and $b = dq$, so $a^2b = d(d^2p^2q)$ where d^2p^2q is an integer. Therefore $d|a^2b$.

   These solutions achieve a maximum 4 points if otherwise fully correct.

2. Analyzes three cases defined as: (1) $d|a$; (2) $d|b$; (3) $d|a$ and $d|b$

   This isn't really a mistake, but it's important to note that case (3) is completely extranneous and can be removed without affecting the validity of the proof. Note that no penalty is directly incurred by including this case, but all of the work you show there is fair game for other point deductions if it includes errors.

   These solutions achieve a maximum 9 points (full credit) if fully correct.

3. Analyzes three cases defined as: (1) d divides a but not b; (2) d divides b but not a; (3) d divides both a and b

   Again, this isn't exactly a mistake, since it correctly defines three mutually exclusive and exhaustive cases. Each case can be analyzed using a similar approach to the one given for the cases in our solution. However, you are making more work for yourself and writing a longer proof.

   These solutions achieve a maximum 9 points (full credit) if fully correct.

4. Reverses definition of divisibility, resulting in the claim

   "For all positive integers a, b, and d, if $a|d$ or $b|d$, then $a^2b|d$."

This is a significantly simpler claim than the one actually given in the prompt, and is easily disproved by giving a counterexample (e.g. $a = 2$, $b = 1$, $d = 2$).

These solutions achieve a maximum 1 point if a correct disproof by counterexample is given.

5. Assumes the conclusion $d|a^2b$ in one or more cases.

   Several students wrote "Assume $d|a^2b$" in their proofs (which is incorrect), but then were able to separately arrive at the correct justifications for why this must be true (without actually using that assumption). In this case, we deduct 0.25 points.

   In solutions where $d|a^2b$ is assumed in such a way that the rest of the proof is trivialized, up to 8 points may be deducted on a case-by-case basis, depending on the extent to which the assumption trivializes the proof.

   These solutions achieve a maximum 8.75 points if otherwise fully correct.

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