Technical Vocab: proof by contradiction (归谬法)

9/19/23

Proof by Contradiction -- ANSWERS

Handout

Prove: There do not exist integers a, b such that $18a + 6b = 1$.

Seeking a contradiction, assume the negation:
"There exist integers a, b such that $18a + 6b = 1$."

- Dividing both sides by 6, we get $3a + b = \frac{1}{6}$
- Since a, b are integers, $3a + b$ is an integer
- **So $\frac{1}{6}$ is an integer.**

- This completes the contradiction: $1/6$ is not actually an integer. (explain contradiction if at all unclear)
- (optional concluding sentence) Assuming that these integers do exist led to contradiction. Thus, we have proved that there do **not** exist integers a, b such that $18a + 6b = 1$.

2

Template: Proof by Contradiction

②

Claim: p

①

Proof Template

Seeking a contradiction, assume: [state the negation of p]

... (make some deductions, eventually leading to a contradiction) ...

Common contradictions: a number is an **integer and is not an integer**; a number is both **even and odd**; a number is both **rational and irrational**.

Since *[restate contradictory statements]*, we have a contradiction. Assuming $\neg p$ led to a contradiction. Therefore, p must be true. (optional concluding sentence)

Special case: when the claim is an "if-then" statement

Claim: $a \to b$

Remember: the negation of $a \to b$ is
***a* and $\neg b$**

assume $\neg p$
$\to$ contradiction
$\to p$

Rational Reasoning

Definition:
- x is rational if there exist two integers a, b with $x = \frac{a}{b}$
- Otherwise, it is irrational

Prove:
For all **rational** numbers x and **irrational** numbers y, $x + y$ is **irrational**.

Proof:
- **Seeking a contradiction, we will assume the negation:**

"There exists a rational number x and an irrational number y for which $x + y$ is rational."

- Let a, b, c, d be integers such that $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$.
- So $y = (x + y) - x$
  $= \frac{c}{d} - \frac{a}{b}$
  $= \frac{cb - ad}{bd}$
- Since a, b, c, d are integers, $cb - ad$ and bd are both integers
- So y is rational.
- **This completes the contradiction** (y is both rational and irrational).

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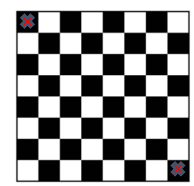
Tilings

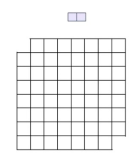
Definition: A **"tiling"** of a grid by 2x1 dominoes is a placement of dominoes on the grid so that every square has exactly one domino.

Proposition (negation of original):
There does not exist a way to **tile** an **8x8** grid **with the two opposite corners removed** with **2x1** dominoes.



Proof:
- **Seeking a contradiction, assume the negation:**
  "There exists a way to tile an 8x8 grid with the two opposite corners removed with 2x1 dominoes."

- Color the cells in a checkerboard pattern.
- Every domino covers one black square and one white square.
- We count 30 black squares remaining, so the tiling must use 30 dominos
- We count 32 white squares remaining, so the tiling must use 32 dominos
- **This completes the contradiction,** and so no such tiling exists



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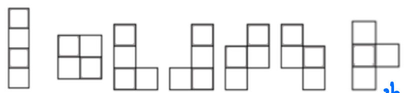
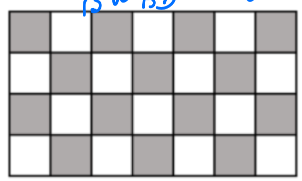
Tetris Tilings

Disprove:
You can place the 7 tetrominoes to perfectly tile this 4×7 grid.
You may rotate pieces if you like



2b2w ×6 + 3b1w or 3w1b ⇒ 15b13w / 15w13b

14b14w



Place each tetromino (possibly rotated) anywhere you like in the grid.

Color the squares of the tetromino by the grid square it overlaps.

If we assume that a tiling exists, how can we get a contradiction?

By arguing that the tetrominoes cover a different # of white squares vs. black squares in total

Infinite Primes

Theorem:
There are infinitely many primes.

Axiom:
Every integer $k \geq 2$ is a multiple of at least one prime number p.

Equivalent Proposition:
There does not exist a finitely-long list $\{p_1, \ldots, p_c\}$ containing all the prime numbers.

Proof:

- **Seeking a contradiction, assume the negation:**
  "There exists a finitely-long list $\{p_1, \ldots, p_c\}$ containing all the prime numbers."
- Let k be the number $p_1 \cdot \ldots \cdot p_c + 1$.
- Notice that k is not a multiple of any prime number $p_1, \ldots, p_c$, since when we divide k by any prime p_i we get a remainder of 1.
- This completes the contradiction, since k is not a multiple of any prime, contradicting our axiom.

12

Irrationality of $\sqrt{2}$ (Strategizing)

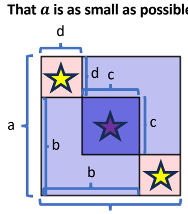
Definition:
- x is rational if there exist two integers a, b with $x = \frac{a}{b}$
- Otherwise, it is irrational

Proposition:
$\sqrt{2}$ is irrational.

What equivalent proposition will we use in our proof? **There exist integers a, b with $a^2 = 2b^2$.**

What special property will we assume about the integers in this equivalent proposition?

That a is as small as possible



What's the relationship between c and d? $c^2 = 2d^2$

How can we write these in terms of a and b?

$c = 2b - a$

$d = a - b$

Irrationality of $\sqrt{2}$ (Main Proof)

Proof:

- **Seeking a contradiction, assume the negation:** **"There exist ints a, b with $a^2 = 2b^2$."**

- Let a, b be ints with $a^2 = 2b^2$, with a as small as possible.
- Consider the ints $c = 2b - a$, $d = a - b$
- We have
$$\begin{aligned} c^2 &= (2b - a)^2 \\ &= 4b^2 - 4ab + a^2 \\ &= 4b^2 - 4ab + a^2 + (a^2 - 2b^2) \quad \text{(since } a^2 = 2b^2\text{)} \\ &= 2b^2 - 4ab + 2a^2 \\ &= 2(b^2 - 2ab + a^2) \\ &= 2(b - a)^2 \\ &= 2d^2 \end{aligned}$$
- Also, we have
$$\begin{aligned} c &= 2b - a \\ &< 2a - a \quad \text{(since } b < a\text{)} \\ &= a. \end{aligned}$$
- This is a contradiction: we found new ints c, d with $c < a$ and $c^2 = 2d^2$, but we had assumed that a was as small as possible.

if $a^2 = 2b^2$
then $(2a)^2 = 2(2b)^2$
$(3a)^2 = 2(3b)^2$
⋮

∴ If $a^2 = 2b^2$ there exists a smallest pair satisfying the identity
∴ let a be as small as possible
and then we found there is always a pair that is smaller, as long as $a^2 = 2b^2$
∴ contradiction.