

EECS 203: Discrete Mathematics
Fall 2023
Homework 5

Due **Thursday, October. 12**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$

Total Points: $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Induction Construction [16 points]

Let $P(n)$ be the statement that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer. In this problem, we will prove this statement via weak induction.

- (a) What is the statement $P(1)$?
- (b) Show that $P(1)$ is true, which is the base case for our inductive step.
- (c) In the base case we prove $P(1)$; what do you need to prove in the inductive step?
- (d) What is the inductive hypothesis for your proof?
- (e) Complete the inductive step, indicating where you used the inductive hypothesis.
- (f) Explain why this proof shows $P(n)$ is true for all positive integers n .

Solution:

- (a) $P(1)$: $1 \cdot 1! = (1+1)! - 1$.
- (b) For $P(1)$, LHS = 1
RHS = $2! - 1 = 2 \cdot 1 - 1 = 1$.
 \therefore LHS = RHS. $\therefore P(1)$ is true.
- (c) We need to prove that $P(k) \rightarrow P(k+1)$ for any integer n which is ≥ 1 .
- (d) The inductive hypothesis: Assume $P(k)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.
- (e) Let k be an arbitrary positive integer.
Assume $P(k)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$
Want to show: $P(k+1)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! = (n+1+1)! - 1$
Using $P(n)$ we know:
$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(1 + n+1) + 1 \\ &= (n+1+1) \cdot (n+1) \cdot n \cdot (n-1) \cdot \cdots \cdot 1 - 1 \\ &= (n+1+1)! - 1 \end{aligned}$$

Thus $P(k) \rightarrow P(k+1)$ for any integer n which is ≥ 1 .

(f) It is because that: (1) $P(1)$: $1 \cdot 1! = (1 + 1)! - 1$.
 (2) $P(k) \rightarrow P(k + 1)$ for any integer n which is ≥ 1 .
 Therefore $P(1) \rightarrow P(2) \rightarrow P(3) \cdots \rightarrow P(n)$
 Where n can be any positive integer.
 \therefore Since we know (1) is true and (2) is true, $P(n)$ is true for all positive integers n .

2. Base Two Blues [14 points]

Prove using mathematical induction that $\log_2(n) < n$ for every positive integer n . You may assume that the base-2 logarithm function is strictly increasing on its domain.

Fun Fact: $\log_b(n) < n$ is actually true for every positive real number n and arbitrary base $b > 1$, but we're asking you to prove this by induction for the special case where $b = 2$ and n is a positive integer.

Solution:

Let k be an arbitrary positive integer.

Assume $P(k)$: $\log_2 k < k$

Want to show: $P(k + 1)$: $\log_2 k + 1 < k + 1$

Base Case:

$P(0)$: $\log_2 1 < 1$

Since $\log_2 1 = 0 < 1$, base case is true.

Inductive Step:

Using the property of logarithm we know:

$$\begin{aligned}\log_2(k + 1) &= \log_2\left(\frac{k + 1}{k} \cdot k\right) \\ &= \log_2 \frac{k + 1}{k} + \log_2 k \\ &= \log_2\left(1 + \frac{1}{k}\right) + \log_2 k\end{aligned}$$

Since k is a positive integer, $k \geq 1$, $\frac{1}{k} \leq 1$, $\frac{1}{k} + 1 \leq 2$

$\therefore \log_2\left(\frac{1}{k} + 1\right) \leq 1$.

Using $P(n)$ we know: $\log_2 k < k$.

And Since $\log_2\left(\frac{1}{k} + 1\right) \leq 1$ and $\log_2 k < k$

$\log_2(k + 1) = \log_2\left(1 + \frac{1}{k}\right) + \log_2 k < k + 1$.

Then we have proved that $P(K) \rightarrow P(k + 1)$ for all positive integer k .

Conclusion: $\log_2(n) < n$ for every positive integer n .

3. Inductive Hypothesis [15 points]

Prove by weak induction that 6 divides $n^3 - n$ where n is a nonnegative integer. Don't include unneeded base cases.

Solution:

Let k be an arbitrary nonnegative integer.

Assume $P(k)$: $6 \mid (k^3 - k)$

Want to show: $P(k+1)$: $6 \mid [(k+1)^3 - (k+1)]$

Base Case:

$P(0)$: $6 \mid 0^3 - 0$

Since $0^3 - 0 = 0$, $6 \mid 0$, base case is true.

Inductive Step:

Since $P(k)$: $6 \mid (k^3 - k)$, for some integer m , $6m = (k^3 - k)$

Then

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\&= k^3 + 3k^2 + 2k \\&= (k^3 - k) + 3k^2 + 3k \\&= 6m + 3(k^2 + k) &= 6m + 3k \cdot (k+1)\end{aligned}$$

Since k is an integer, and integers consist of alternating odd numbers and even numbers, one of k and $k+1$ must be even. WLOG assume k is even, then for some integer p , $k = 2p$.

Then $(k+1)^3 - (k+1) = 6m + 3 \cdot 2p \cdot (k+1) = 6m + 6p \cdot (k+1) = 6[m + p(k+1)]$

Since p, m, k are integers, $p(k+1)$ is an integer, $[m + p(k+1)]$ is an integer.

Then $6 \mid [m + p(k+1)]$, i.e. $6 \mid [(k+1)^3 - (k+1)]$.

Therefore we have proved that $P(k) \rightarrow P(k+1)$ for any nonnegative integer k .

Conclusion: 6 divides $n^3 - n$ where n is a nonnegative integer.

4. Incorrect Strong Induction [14 points]

For each of the following **incorrect** strong induction proofs, note where the strong induction proof breaks down and is incorrect.

Hint: Consider where the inductive step breaks down.

- (a) Proving for every nonnegative integer n , $P(n): 3n = 0$.

Inductive Step:

Assume that $P(j): 3j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. We wish to show $P(k+1)$. We will rewrite $k+1 = a+b$ where a and b are nonnegative integers less than $k+1$. Thus, $3 \cdot (k+1) = 3 \cdot (a+b) = 3a + 3b = 0 + 0 = 0$, therefore $P(k+1)$ is proven.

Base Case: $P(0): 3 \cdot 0 = 0$

Since we have shown the basis step and the inductive step, we have proved for every nonnegative integer n , $P(n): 3n = 0$.

- (b) Proving that every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

Inductive Step:

Assume we can form cent values of j cents for all $3 \leq j \leq k$ using just 3-cent and 4-cent stamps. We wish to show we can form $k+1$ cents using just 3-cent and 4-cent stamps. We can form a $k+1$ cent value by replacing 1 3-cent stamp with 1 4-cent stamp or by replacing 2 4-cent stamps with 3 3-cent stamps.

Base Case:

We can form cent values of 3-cents using one 3-cent stamp and we can form cent values of 4-cents using one 4-cent stamp. This covers our two base cases.

Since we have shown the basis step and the inductive step, we have proved every cent value above 3 cents can be formed using just 3-cent and 4-cent stamps.

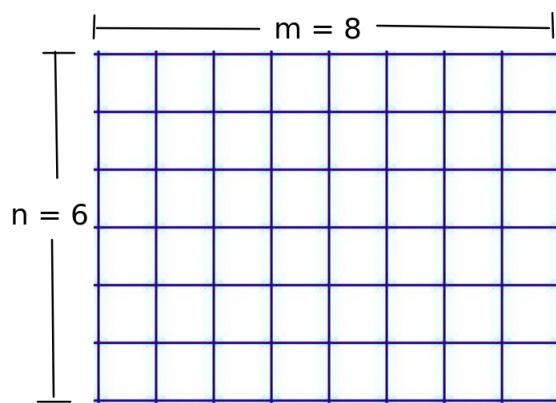
Solution:

- (a) The proof breaks down from the base case when we induce $P(0) \rightarrow P(1)$.
At this point, $k+1 = 1$, but we cannot find two integers a and b that are less than 1 while they add up to 1. If a, b are less than 1 and nonnegative, then they can only be both 0. Then they add to 0 but not 1.
Therefore the proof is incorrect.
- (b) The proof breaks down when we induce $(P(3) \wedge P(4)) \rightarrow P(5)$.
The inductive step states that we can form a $k+1$ cent value by replacing 1 3-cent

stamp with 1 4-cent or by replacing 2 4-cent stamps with 3 3-cent stamp. But at this point, we only have one 4-cent and can not apply the induction. Therefore the proof is incorrect.

5. Chopping Ice [15 points]

Claire doesn't have an ice tray, so she makes ice by freezing water into a rectangle and then dividing the rectangle into grid-aligned cells. She would like to divide her block of ice into n rows and m columns quickly, before the ice melts! See the image below for an example.



- State the number of cuts Claire needs to make to divide her ice block into $n \times m$ cells. One cut means splitting a single rectangle into two rectangles. In other words, you may NOT make a single cut across multiple pieces of ice. You may use n and/or m in your answer.
- Prove your answer from part (a).

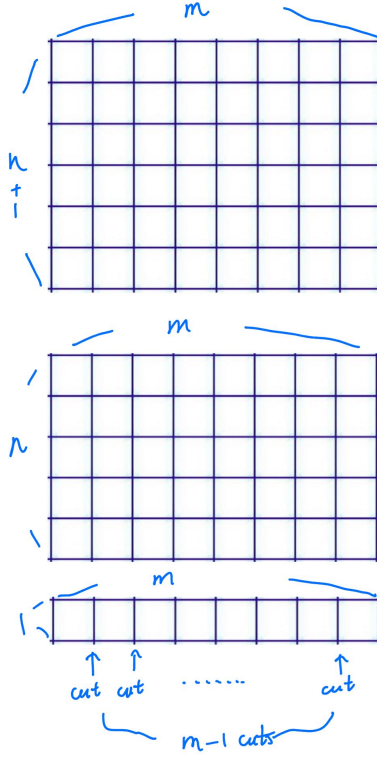
Solution:

- $(n - 1) + n \cdot (m - 1) = mn - 1$
 m, n are positive integers.

- Let m, n be an arbitrary positive integer. $k(m, n)$ is the number of cuts to make to divide her ice block into $n \times m$ cells.
 Assume $P(m, n)$: $k(m, n) = mn - 1$
 WLOG (due to symmetry, $k(m+1, n) = k(m, n+1)$), we want to show: $P(m, n+1)$:
 $k(m, n+1) = m(n+1) - 1$

Inductive Step:

We can first cut the $(n+1)$ row from the block. This requires 1 cut.
Then we get $n \times m$ block and $1 \times m$ block.



From the inductive hypothesis we know, to divide the $n \times m$ block, we need $k(m, n)$ cuts.

And to divide the $1 \times m$ block, since we can not make a single cut across multiple pieces of ice, we can only use $m - 1$ cuts.

\therefore In total, we need:

$$\begin{aligned}
 1 + k(m, n) + (m - 1) &= 1 + mn - 1 + m - 1 \\
 &= mn + m - 1 \\
 &= m(n + 1) - 1
 \end{aligned}$$

Base Case: To divide a $P(1, 1)$: $k(1, 1) = 1 \times 1 - 1 = 0$ is true.

Since $P(1, 1)$, $P(m, n) \rightarrow P(m, n+1)$, and due to symmetry $P(m, n) \rightarrow P(m+1, n)$, $P(m, n)$ is true for any positive integer m, n .

Conclusion: We need $mn - 1$ cuts to divide her ice block into $m \times n$ cells.

6. Pastry Recurrence [12 points]

A baker decorates a cookie in 2 minutes, a cupcake in 3 minutes, and a pie in 3 minutes. Let a_n denote the number of distinct ways the baker decorates pastries in exactly n minutes for $n \geq 0$ (where order matters).

- (a) Find a recurrence relation for a_n .
- (b) What are the initial conditions? Use the fewest initial conditions necessary.

Solution:

- (a) Case 1: The last pastry the baker decorates in n minutes is cookie.
Then before that, there are a_{n-2} ways.
Case 2: The last pastry the baker decorates in n minutes is cupcake.
Then before that, there are a_{n-3} ways.
Case 3: The last pastry the baker decorates in n minutes is pie.
Then before that, there are a_{n-3} ways.
 $\therefore a_n = a_{n-2} + 2a_{n-3}$. ($n \geq 3$)
- (b) Since a_n is valid only when $n \geq 0$, and we have a_{n-3} in our recurrence relation, we need to know all a_n where $n < 3$.
That is:
 $a_0 = 1$ (since the only choice is to do nothing)
 $a_1 = 1$ (since the only choice is to do nothing)
 $a_2 = 1$ (since the only choice is to decorate a cookie.)

7. Raven's Wrestlers [14 points]

Raven has n weeks to build her wrestling figure collection. Every week, Raven buys one item to add to her collection. There are 4 different types of things she can buy: Figures, T-shirts for her wrestlers to wear, Weapons for them to fight with, or Display Stands to show them off on her shelves.

- Her shelves can fit 2 Stands nicely, so when she buys a Display Stand, she will always buy a second one the next week to finish the shelf. Additionally, the week after buying the second Stand, she will buy something other than a Display Stand (they aren't as exciting to buy)

- When she buys a Figure, she gets very excited about it and wants to buy a new T-shirt for it to wear the following week.

Let a_n represent the number of ways Raven can buy items across the n weeks (where $n \geq 0$)

- (a) Find a recurrence relation for a_n .
- (b) Which terms would need to be defined with initial conditions (no need to find the value, just which terms)

Note 1: Buying the same items in a different order counts as a different way of buying items. We treat all items in a category as identical.

Note 2: on week n , Raven will not buy a Figure (because she knows she will miss buying a T-shirt) or a Stand (what a sad way to end the collection). This information is not needed for the simplest solutions, but some alternate solutions may need to know this.

Solution:

Groupwork

1. Grade Groupwork 4

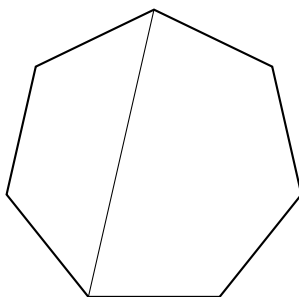
Using the solutions and Grading Guidelines, grade your Groupwork 4:

- Mark up your past groupwork and submit it with this one.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- Use the table below to calculate scores.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- What if my group changed?
 - If your current group submitted the same groupwork last time, grade it together.
 - If not, grade your version, which means submitting this groupwork assignment separately. You may discuss grading together.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 2												/20
Problem 3												/30
Total:												/50

2. Polly Gone [12 points]

A convex polygon is a 2D shape with all straight edges such that any line segment between any two non-adjacent vertices passes entirely through its interior (see example picture below). Show via induction that the sum of the interior angles of a convex polygon with n sides is $(n - 2) \cdot 180^\circ$. Don't include unneeded base cases.



Hint 1: It is helpful to know that a triangle's interior angles always sum to 180° . You may assume this is true for the problem.

Hint 2: In order to apply your inductive hypothesis to a convex polygon, you'll need to think of it in terms of smaller convex polygons. How can you make smaller polygons out of a big one?

Solution:

3. Running Recurrence [8 points]

An EECS 203 student goes to lecture everyday. On each day, she always chooses exactly one method of transportation, and always chooses to walk, bike, or take a bus. She also follows additional rules:

- She never walks two days in a row.
- If she takes the bus, she must have biked two days ago and walked a day ago.

Let a_n denote the number of ways she can go to EECS 203 lecture across n days for $n \geq 0$.

- Find a recurrence relation for a_n .
- What are the initial conditions? Use the fewest initial conditions necessary.

Solution:
