sineFit

sineFit is a function to detect the parameters of a noisy sine function.

1 Syntax:

SineParams=sineFit(x,y)

Optional: SineParams=sineFit(x,y,0), no graphic displayed.

Input:

x and y values: y = offs + amp * sin(2pi * f * x + phi) + noise

Output:

SineParams(1):offset (offs)
SineParams(2): amplitude (amp)
SineParams(3): frequency (f)
SineParams(4): phaseshift (phi)
SineParams(5): Mean Squared Error (MSE)

2 Minimum requirements:

- Length of input: tested for more than 0.1 periods
- Sampling rate: more than 2.0 samples per period, see also chapter 9
- Total number of samples: 4 or more

Required toolbox: none

3 Goal of sineFit:

The goal is to find the original noise free sine. But in all cases another sine fits better the noisy sine. In most of the cases the the result of SineFit is close to the original sine.

4 Method:

This is a brief and not exact description of the program flow.

- Estimate the offset by the mean of all y values.
- Build the FFT with heavy zero padding.
- Take the frequency, amplitude and phase of the largest FFT peak.
  If the frequency is at the Nyquist limit or the period is less than one, add extra frequencies for evaluation.
- Take those values as initial values for the regressions.
- Take the resulting MSE as rating.
- Exclude results above Nyquist frequency.
- Depending on the number of samples and the MSE, set a limit for an accepted amplitude in relation to the FFT amplitude.
- If the amplitude from regression is higher than the accepted amplitude, take the FFT parameters.

5 Processing time

The mean calculation time is on my PC 13 ms with a maximum of 2400 ms.

The regression with ‘fminsearch’ uses about 80% of the total time, while the FFT requires only about 2% of the time.

6 Evaluations:

Sine curves tested:

100000 random sine curves were created with following properties:

- Offset: 0 to 10
- Amplitude: 0.1 to 10
- Frequency: 0.001 to 100 Hz
- Phase shift: 0 to 2π
- Number of samples per period: 2.01*) to 30
- Minimum number of all samples: 4
- Signal to noise ratio: 0 to 100 dB (not applied on offset).
  Low SNR are overrepresented, for 1 to 100 periods, see Fig. 2.
- x=0 within observed range +- ½ period
- Out of the 100 000 test patterns
  40000 have 0.1 to 1 periods
  30000 have 1 to 10 periods
  30000 have 10 to 100 periods

*) In order to detect the amplitude and phase, you need more than 2 samples per period, see chapter 9!

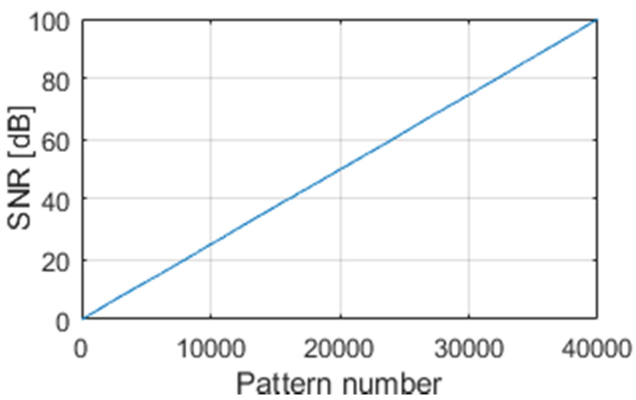

Fig. 1 For 0.1 to 1 period linear SNR distribution

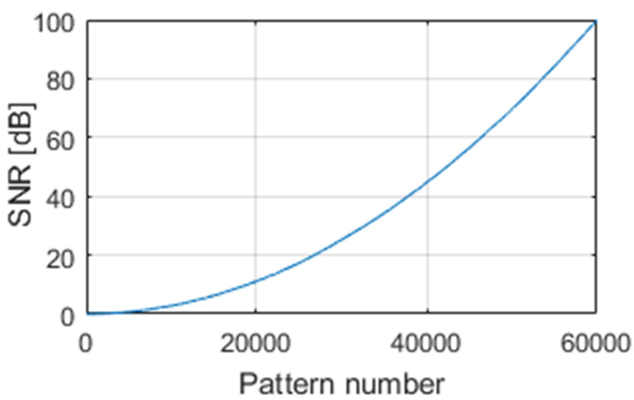

Fig. 2 For 1 to 100 periods Overrepresentation of low SNR (1/3 of all test patterns have a SNR below 11 dB)

6.1 Definition of 10% deviation:

The deviation is the difference between parameters of the fit result and the original noise free sine.

- The amplitude or the frequency is faulty, if the deviation is more than 10% compared to the noise free sine.
- The offset can be zero and is considered to be wrong, if it deviates by more than 10% of the original offset and more than 10% of the original amplitude.
- The phase is wrong if it deviates by more than 10% of 2π.

If any the four parameters breaks the 10% limit, the results is declared not to match the original sine.

6.2 Definition of 1% deviation:

Like above, but replace 10% with 1%.

7 Results:

Table 1 shows the results of sineFit.

The four sections of the table are:

- 0.1 to 0.5 periods
- 0.5 to 1 period
- 1 to 10 periods
- 10 to 100periods

"100%" in the table 1 means all parameters of the results deviate by less than 10% (respectively 1%) to the original sine.

With a fraction of a period, first line of

	Number of periods	Samples per periods	Total number of samples	Signal to noise [dB]	Deviation <10%	Deviation < 1%	Deviation original fit (<1%)	Number of tests
1	0.1 to 1		4 to 30	0 to 100	75%	56%	56%	40000
2	0.1 to **0.5**		4 to 30	0 to 100	62%	42%	43%	17808
3	0.1 to 0.5		4 to 30	**94** to 100	**100%**	98%	99%	1144
4	0.1 to 0.5		4 to 30	**99.8** to 100	100%	**100%**	100%	54
5	0.5 to 1		4 to 30	0 to 100	86%	67%	67%	22192
6	0.5 to 1		**6** to 30	**38** to 100	**100%**	98%	98%	12940
7	0.5 to 1		**5** to 30	**60** to 100	100%	**100%**	100%	8662
8	1 to 10	2.01 to 30		0 to 100	89%	56%	56%	30000
9	1 to 10	**2.5** to 30		**20** to 100	**100%**	95%	95%	16309
10	1 to 10	2.5 to 30		**41** to 100	100%	**100%**	0%	10606
11	10 to 100	2.01 to 30		0 to 100	99%	71%	29%	30000
12	10 to 100	**2.1** to 30		**7** to 100	**100%**	91%	91%	22198
13	10 to 100	2.1 to 30		**28** to 100	100%	**100%**	100%	14187

Table 1, 75% results of 40000 test patterns detect the original sine within an error of 10% compared to the noise free sine and 56% match the original within the 1% limit.

Out of those test patterns, very low numbers of periods have the largest deviations. In line 2 we see that only 62% of the evaluations are within 10% deviation. We get only good results with nearly noise free sine curves (SNR>99.8 dB), see line 4.

If we have a little more periods (0.5 to 1) the good results with less than 10% deviation go up to 86%, see 5th line.

Finally, with many periods, line 12, we need only 28 dB SNR to detect all original sine curve parameters with a deviation better than 1%.

The column "Deviation original fit" is the percentage of detected good results, if the parameters of the noise free sine are used as initial values for the fitting process. I.e. if we have then a deviation, another sine than the original sine fits the noisy sine better. Since nearly all values of this column are equal to the values of the column "Deviation < 1%", the sineFit process cannot detect all original sine using this fitting method. The same holds for the 10% errors.

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Table 1: Percentage of good results compared to the original sine.
Test set: 100000 noisy sine curves having 0.1 to 100 periods.

The number of undetected original sines depends mainly on the SNR, see Fig. 3.

Fig. 3 gives an idea about the required SNR for reliable results. E.g. for around 10 periods you need an SNR of at least 26 dB and all results with more than 4 samples and more than 2.1 samples per period will very probably detect the original sine within an error of 1%.
With another test pattern set, you will get similar curves.

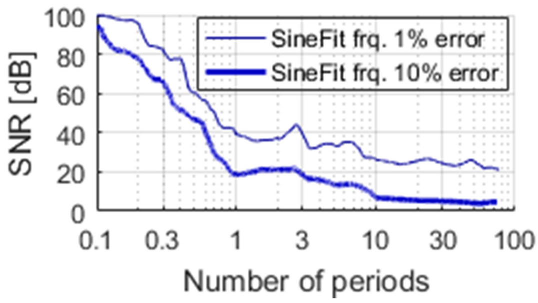

Fig. 3 Required SNR for detecting the original sine

Number of periods: 0.1 to 100
Number of samples: >4
Samples per period > 2.1

8 sineFit compared to FFT

In most cases with

- Number of periods: >1
- Number of samples: >10
- Number samples per period: >2.5
- SNR: >0

the FFT frequency is close to the original frequency.

Many failures of the FFT in respect to the expected frequency happen with samples rates below 3 per period. The following example (*Fig. 4*) is a noise free sine with 2.4 samples per period and 2.5 periods. The FFT (*Fig. 5*) has its maximum at the nyquist frequency (2.4 Hz) while the expected frequency is 2.0 Hz.

SineFit calculates the correct parameters, see result (blue line) in *Fig. 4* on top of the original sine (green line).

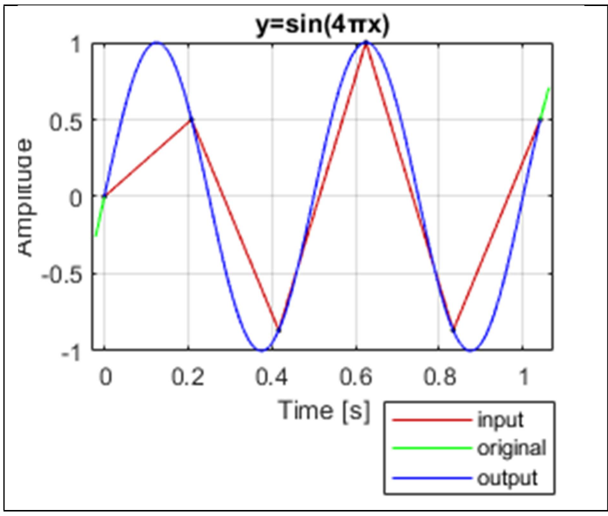

Fig. 4 Sine with low sample rate

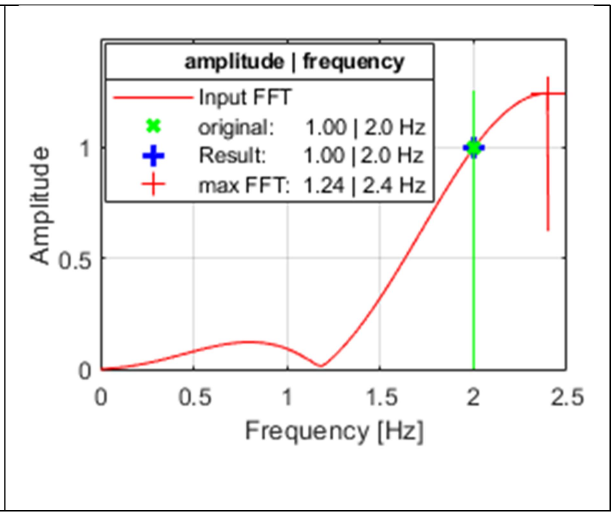

Fig. 5 Corresponding FFT

Table 2 compares sineFit with FFT.
The first line of the table is the global result for all 100000 test patterns. The total number of noisy sines that are detected with 10% (1%) deviation on all 100000 test patterns is with sineFit: 87% (60%). Taking only the FFT peak parameters, we gather much less good results: 59% (26%).
With fractions of a period, 2^{nd} line, the FTT results are very bad.
With 1 to 10 periods, only the number of 1% deviations are significantly lower with the FFT.
From 10 to 100 the results of sineFit and FFT are similar.
Of course, you may improve the FFT result, if you know you have more than 4 or 5 periods. E.g., use a Hann window.
Unfortunately, about 90% of the detection failures of the FFT include wrong detected amplitudes.

	sineFit results			**FFT results**			
Number of periods	**Devia-tion < 10%**	**Devia-tion < 1%**	**Frequency deviation < 10% (<1%)**	**Devia-tion < 10%**	**Devia-tion < 1%**	**Frequency deviation < 10% (>1%)**	**Number of tests**
0.1 to 100	87%	60%	91%	59%	26%	66%	100000
0.1 to 1	75%	56%	79%	8%	0%	16%	40000
1 to 10	88%	56%	99% (88%)	86%	18%	98% (76%)	30000
10 to 100	99%	71%	100%(~100%)	99%	69%	100% (~100%)	30000

Table 2: sineFit compared to FFT, SNR 0 to 100 dB

Again, with other Fourier transformations, esp. window technics, you may get better results. In addition, the FFT by itself is much faster than sineFit. If speed is crucial and 10% errors are good enough, you may consider using only the FFT. In this case, you need to know, that you have more than about five periods.

9 About samples per period

With exactly 2 samples per period the amplitude or the phase cannot be determined. With a phase=0 and 2 samples per period, only the DC-level and frequency is known. The amplitude may seem to be zero, see Fig. 6, red dots.

9.1 Time limited sine function

Since the number of samples is an integer number, the sample rate and the number of periods are dependent on each other.

Ns: Number of samples, <u>integer number!</u>

SpP: Number of samples per period, more than 2.0

Np: Number of periods.

Following equations hold:

$$Ns = Np * SpP \quad \text{with SpP} > 2 \text{ and Ns} = \ 4, 5, 6, \ldots \tag{1}$$

The constraints are:

More than 2 samples per period with a minimum of 4 samples in total.

This means SpP is restricted by the duration of the sine and the integer value of Ns.

For example, with a sine limited to 2.5 periods, following sampling SpP are allowed:

$$SpP = \frac{Ns}{Np} = \ \frac{6}{2.5}, \ \frac{7}{2.5}, \ \cdots \tag{2}$$

Fig. 6 shows the first two sample rates acc. Equation (2) and in addition the not allowed 2 samples per period. The first sample of the next period would be for all sample rates at y,x=0, 2.5, see black circle in Fig. 6.

If you want to determine the amplitude, you must have more than two samples per period. In the case of Equation (2) this is at least 6/2.5=2.4 samples per period. All patterns used to test sineFit meet this constraint.

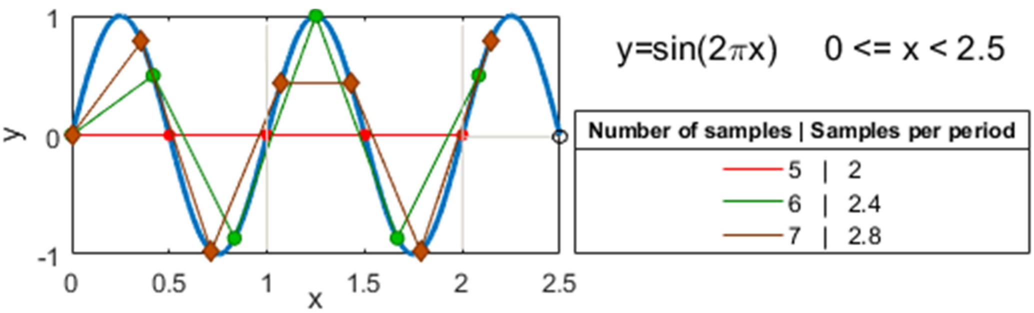

Fig. 6 Sine curve with 2.5 periods, sampled with different sample rates

In Fig. 7 we see what happens if we have only 3 samples in a sine (green and red line). E.g. for any amplitude larger than the initial amplitude we can find a sine curve that fit the three samples. Both sine curves, green and blue line in Fig. 7, go through the three points.

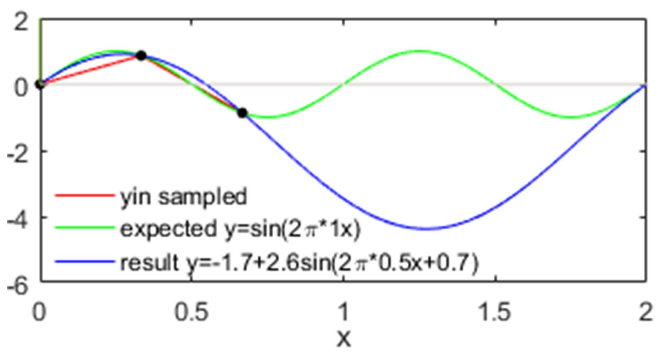

Fig. 7 Two sine curves with the same 3 samples

To determine the four unknown parameters of a sine you need at least four samples

10 About offset

The offset of an ordinary unlimited sine is zero and equal to its mean value. This holds also for full periods. With other time-limited sine curves, the mean value does not represent the offset.

The first estimate of the mean value (*meanSamples*) is the average of all y-samples:

$$meanSamples = \frac{1}{N_s} \sum_{1}^{N_s} y_i \tag{3}$$

y_i: y-values of samples
N_s: Number of samples

The mean value of a time limited sine curve without a DC-value (*meanSine*) is:

$$meanSine = \frac{A}{(2\pi \mathrm{f}(\mathrm{x}_{end}\text{-}\mathrm{x}_1))} \int_{x_1}^{x_{end}} \sin(2\pi \mathrm{f}\mathrm{x} + \varphi) dx \tag{4}$$

A: Amplitude of sine
x_1: Start of sine (time of first sample)
x_{end}: End of sine (time of last sample)
f: Frequency
φ: Phase shift

We get the values A, f and φ from the FFT.
With *meanSine* and *meanSamples* we calculate a better estimation for the offset (*offsetEstimate*):

$$offsetEstimate = meanSamples - meanSine \tag{5}$$

Example, Fig. 8:

$$y_{sine} = 1 + 2\sin(2\pi 1x + 3.1) \tag{6}$$

Offset of sine =1
meanSamples = 0.62 (eq. (3))

With parameters from y_{sine}:

- meanSine = -0.35 (eq. (4))
- offsetEstimate = 0.97 (eq. (5))

With parameters from FFT:

- meanSine = -0.30
- offsetEstimate = 0.93

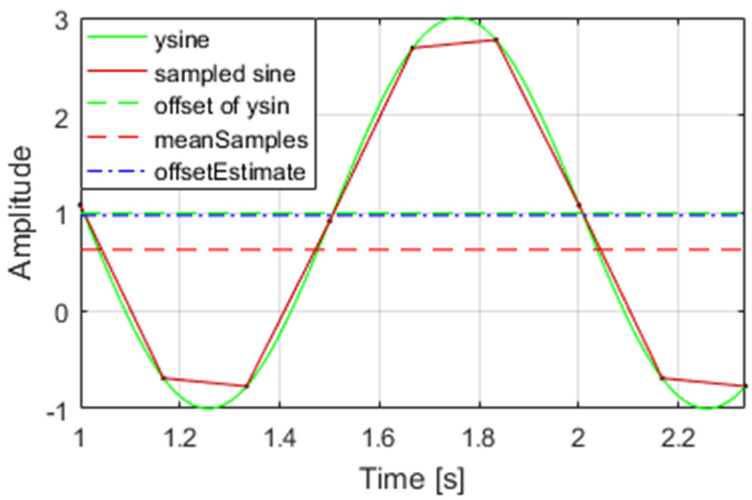

Fig. 8 Offset values for time limited sine

If you set the number of samples to 100, you get:
meanSamples = 0.57

With parameters from y_{sine}:

- meanSine = -0.43
- offsetEstimate = 1.0

With parameters from FFT:

- meanSine = -0.39
- offsetEstimate = 0.97

This offset estimation improves very little the sineFit results and is therefore not included in sineFit. However, it improves a lot the results taken from the FFT and the separate calculated offset.

For Fig. 9 we take noise free sine curves . All have an offset of zero and an amplitude of one.
The blue curve represents the mean values of the samples.
The green curve is built up according eq.(5) with the parameters of the clean sine.
The red curve is more realistic in our case, since we gather amplitude, frequency and phase from the FFT .
At up to 0.4 periods the values we gain from the FFT are very bad and the improvement is marginal. Mainly the amplitude is detected much too low. However, from 0.4 periods on the red line is closer to the expected zero line.

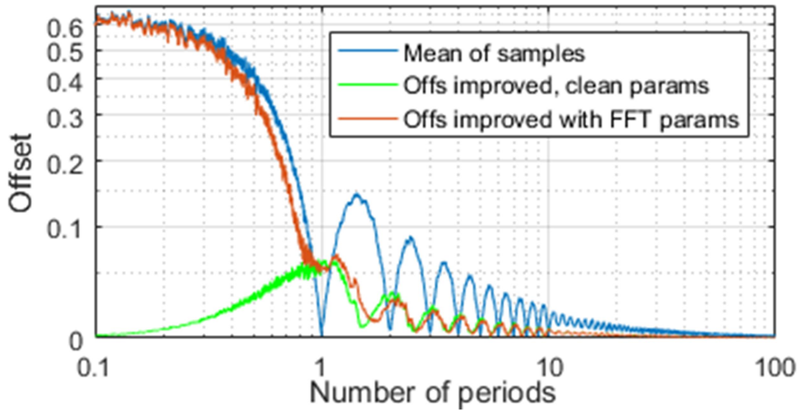

Fig. 9 Offset of clean sine curves

The most realistic experiment is with the test set of 100000 noisy sine curves. Fig. 10 demonstrates the improvement (green line). Below 0.5 periods the improvement is low, since the detected amplitudes, frequencies and phases from the FFT are far off from the expected values. In a small region around 0.9 periods, the pure mean values are better. Most improvement is between 0.6 and 3.5 periods.

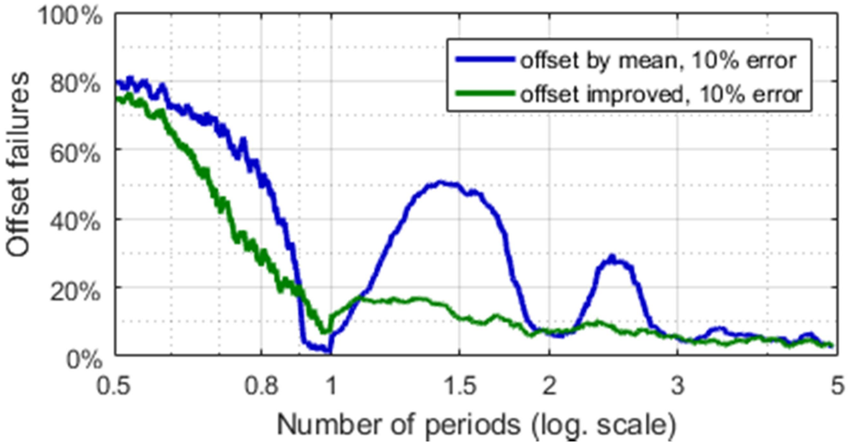

Fig. 10 Offset failures for max. 10% error, 0.5 to 5 periods, SNR 0 to 100

An attempt to exclude the offset correction at around 0.9 periods fails. The variation of the determined number of periods is too large.
The number of periods (Np) is calculated by this formula:

$$Np = (\underbrace{x_{end} - x_1}_{\text{total duration}} + \underbrace{x_2 - x_1}_{\text{time within 2 samples}})f \tag{7}$$

x_1: Start of sine (time of first sample)
x_2: time of second sample
x_{end}: End of sine (time of last sample)
f: Frequency of FFT peak

In the observed range of 0.9 to 1.1 periods, the frequency of the noisy sine curves detected by the FFT is up to +- 25% off from the expected frequency (all values nearly linear distributed, some extreme values already excluded,). Therefore, the number of periods, eq.(7), becomes too uncertain.