Inflation in String Theory

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Abstract

We give an introduction to string inflation which is an attempt to investigate inflation models based on the string theory. We give a brief review on inflationary cosmology, compactification in string theory and moduli stabilization as well as how to derive inflation models in the framework of string theory. The whole discussion is for the sake of the detailed analysis on the right-handed sneutrino inflation scenario in TypeIIB theory.

Contents

1 Introduction

Many physicists believe that inflation of the universe has actually occurred. One reason for this belief is that inflation can solve the three puzzles of big-bang cosmology, flatness problem, horizon problem and monopole problem. The other reason is that the quantum fluctuation at the beginning of inflation can naturally become the origin of structure formation and that it is incredibly compatible with the CMB observation. In spite of this belief, no one is sure what exactly was going on; how the inflation started and ended. Therefore a lot of people are working on the inflation model building, but we have some troubles related to the research. First of all there are too many inflation models which are compatible with the observation and experiment, and therefore we cannot decide which model is the "truth". Although the future development of observation helps us to eliminate some of the inflation models surviving now, it would be better getting down to theoretical approach for the determination of the inflation model. The second difficulty is that most of the models cannot control the UV physics regardless of the sensitivity. In a certain class of inflation models called large field inflation models, say chaotic inflation which is one of the leading candidates for the inflation model, the field value of the inflaton must change beyond Planck scale which makes us suspicious about the approximation of effective theory below the Planck scale.

Here comes the research on string inflation. String inflation stands for inflation theory based on string theory, that is, building inflation models on assumption that string theory is correct. In this case there are obviously differences from building inflation models just based on effective field theory. The first difference comes from the fact that string theory is an "all-in-one package"; once we choose a package (= a set of a compactification manifold and local sources such as D-branes, O-planes and fluxes) the particle content and their interactions are completely determined. The theory must contain standard model, cause inflation and be consistent with post inflationary evolution of the universe. This results in a bunch of strong constraints on the theory and thus the set of inflation models based on string theory is likely to be much smaller than that of the effective field theory approach. (Note here that this statement is not obvious and stable but just a conjecture to be studied more.) This difference accommodates the first problem that there are too many inflation models mentioned in above paragraph. The other difference is that using string theory we can transcend Planck scale to some extent and cover the inflation models which are unavailable to those sticking to low energy effective field theory approach. Recalling the suspicion about the over Planck excursion of inflaton field in large field inflation models in the context of effective field theory, string theory has the potential to make it clear because the scalar field can be translated into the language of geometry in ten-dimensional spacetime. This clearly stands a chance against the second problem about inflation model building. This is how the set of inflation models based on string theory is totaly different from that of effective field theory. The research on string inflation is nothing but to clarify the boundary between the two sets; compatible and incompatible models with string theory, that is, to determine whether each inflation model is eligible for a string model or not.

What kind of "gain" can we expect from the study of string inflation? It depends on the "truth", which has roughly three possibilities. The first possibility is that string theory is correct and inflation theory cannot be dealt with the low energy effective field theory. In this case there is no choice but study the string inflation in order to unravel the mystery of the early universe, and the motivation for string inflation is self-explanatory. The second possibility is that string theory is correct and inflation can be described by the effective field theory. Even in this case it is a better approach to assume string theory because we can focus on a tiny part of the models which includes the truth. Moreover in both cases, if inflation model is determined by the experiment it must be a kind of circumstantial evidences that string theory is correct. This is far from a decisive proof but it is a big achievement that we can limit the candidates for the "package" of string theory which describes the real world. The third possibility is that string theory is not correct and inflation can be dealt with effective field theory. This is the most exciting case because we can experimentally exclude the string theory by proving that inflation has occurred in a way which cannot be accomplished by the string theory. (Of course there are other possibilities that string theory is incorrect and effective theory cannot deal with inflation, in this case what we can do is only to wait for new theory to be created.)

That is the motivation for the string inflation. Although the gain by the study of string inflation looks attractive, it is very difficult to achieve it. Actually even a single string inflation model which is consistent with the standard model and observation has never established yet. In this situation, the first thing we can do is to pick up a inflation model and investigate whether the model can be generated in the frame work of string theory. We therefore investigated in [1] whether a model suggested by [10] in the perspective of bottom-up approach can be realized in the string theory without any theoretical and phenomenological inconsistency. This dissertation is devoted to the comprehensive explanation of the work [1].

Organisation of this paper

To begin with, we would like to inform the readers that the main theme of this paper is given in section 6, and that other sections are just for the preliminary study providing background knowledge of the discussion in the main section. Those who are familiar with inflationary cosmology and string theory, especially basics of compactification in TypeII theory, can jump to section 6 from the very start, and refer to the equations in the former sections when it is necessary.

- In section 2 we first give a short review on inflationary cosmology, and then describe some of the representative inflation models and their difficulties which ought to be solved by string inflation. Note that explanation and calculation are not as pedagogical as ordinary reviews or textbooks, so it should be used as a reminder.
- In section 3 we present basic facts and equations which are related to string theory (excluding compactification) and used in the later section.
- In section 4 we review compactification and dimensional reduction in string theory, focusing on TypeIIB theory which will be used in the main section.
- In section 5 we provide a brief review on moduli stabilization, and describe mechanisms for Kähler moduli stabilization in TypeIIB theory proposed so far.
- In section 6, the main section, we start by the overview of string inflation models in subsection 6.1, and then give a detailed explanation of a string inflaiton model studied in the work [1] by the author.

Section 2, 3, 4 (and 5) are written to fix our notation and to remind me in the future of what I have studied in the last few months. I believe these sections are useful also for beginners of string phenomenology.

2 Review of inflation

In this section we review the basics of inflationary cosmology, focusing on the topics relevant to string inflation model building in later sections. Until subsection 2.6 in which we show some of the representative examples of inflation models, the discussion is so basic that one can find similar discussion everywhere in the street. In subsubsection 2.7, we describe three difficulties one would face during the attempt to build inflation models.

Some of the discussion and organization are greatly inspired by the first half of ref.[4].

2.1 Formulation of 4D uniform isotropic universe

Einstein suggested that we should consider the metric $g_{\mu\nu}$ of the spacetime manifold as the dynamical variables. The action as a functional of metric which respects diff symmetry is written as

$$S[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}(R-2\Lambda)+\mathcal{L}_{\rm m}(\phi)\right], \tag{2.1}$$

where R is Ricci scalar of the space time, Λ is a constant called **cosmological constant** and $\mathcal{L}_{\rm m}$ is lagrangian for other fields denoted as ϕ. Conceptually we would like to treat not only metric but also geometry or topology of the spacetime as dynamical variable and sum over it when we operate path integral, but we do not know how exactly it should be formulated so the subsequent discussion is merely about local structure of the spacetime.

Functional derivative with respect to metric gives us Einstein equations,

$$G_{\mu\nu} = M_{pl}^{-2}T_{\mu\nu}. \tag{2.2}$$

Here $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$ is the Einstein tensor ($R_{\mu\nu}$ is Ricc tensor, R is Ricc scalar and Λ is cosmological constant), and $T_{\mu\nu}$ is the energy momentum tensor (EMT) of matter fields which is defined as

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L}_{\rm m})}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial(\sqrt{-g}\mathcal{L}_{\rm m})}{\partial {g^{\mu\nu}}_{,\alpha}}\right]. \tag{2.3}$$

To study the time development of the whole universe around us, we assume that the theory is uniform and isotropic. This assumption is so strong that it severely restricts the form of metric as follows;

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -dt^2 + a(t)^2\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right), \tag{2.4}$$

where the function of time $a(t)$ is called scale factor and the constant K stands for the curvature of the spacetime. The metric of this form is called **Friedmann Robertson Walker (FRW) metric**. We often assume without any remark that the metric of the universe is FRW metric at least classically.

Usually we further assume that the particles in the universe are recognized as perfect fluids, and thus EMT has a form

$$T_{\mu\nu} = \rho u_\mu u_\nu + (\rho + p) g_{\mu\nu}, \tag{2.5}$$

where $\rho(t)$ is the density of the universe and $p(t)$ is the pressure (both are functions of time because of the assumption of uniformity).

Then Einstein equations of FRW metric and perfect fluids are reduced to two independent equations;

$$H^2 = \frac{\rho}{3M_{pl}^2} - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{2.6}$$

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6M_{\mathrm{Pl}}^2} + \frac{\Lambda}{3}, \tag{2.7}$$

where $H(t) := \dot{a}/a$ is Hubble parameter. The first equation is called **Friedmann equation**, while the second equation is **acceleration equation**. From these equations one can derive **fluid equation**,

$$\dot{\rho} = -3H(\rho + p). \tag{2.8}$$

We have three variables $a(t), \rho(t), p(t)$, but we have only two equations. Therefore we need one more equation to solve them. Then we assume the linear relation between ρ and p ;

$$p = w\rho, \tag{2.9}$$

where w is a constant. We call it **radiation**, **matter** and **dark energy** when w is $1/3, 0$ and -1 respectively[1].

[1] Comments on each component are in order. First, recall that we can derive that pressure of massless particle is proportionate to one thirds of its energy density. So we call it radiation when $p = \rho/3$. We can deal with massive particles who have relativistic velocity as radiation. As for matter, it does not mean just massive particles but massive particles which are totally stationary. Not to mention it is an approximate state but valid well when we consider non-relativistic particles. When the universe is dominated by dark energy, the universe expands very rapidly. Although it makes sense that negative pressure pushes space outward, to any rational person substances with negative pressure is weird because we do not know such things on earth.

In the case of the flat universe ($K = 0$) and zero cosmological constant ($\Lambda = 0$) (both are preferable from the observation), the solution of Einstein equations (2.6,2.8) is

$$\rho(t) = \rho_0 a(t)^{-3(1+w)},\ a(t) = a_0 (t/t_0)^{\frac{2}{3(1+w)}}, \tag{2.10}$$

where we assumed $w \neq -1$. When $w = -1$,

$$\rho(t) = \rho_0,\ a(t) = a_0 e^{H_0 t}. \tag{2.11}$$

This type of solution is called **de Sitter universe**, and as you can see the scale factor grows exponentially with a constant rate H_0. This solution is important when we consider inflation.

The table below is the summary of the result.

name	pressure	energy density $\rho(t)$	scale factor $a(t)$
radiation	$p = \frac{1}{3}\rho$	$\propto a^{-4}$	$\propto t^{1/2}$
matter	$p = 0$	$\propto a^{-3}$	$\propto t^{2/3}$
dark energy	$p = -\rho$	const.	$\propto e^{H_0 t}$

(2.12)

We have discussed one component universe so far. It is easy to extend it to the multi-component case, that is, the case where $\rho = \sum_i \rho_i, p = \sum_i p_i, p_i = w_i \rho_i$. However in many cases, it works to take only one dominant component into account. When the universe is dominated by radiation, matter or dark energy, we say that the universe is in the **radiation dominated era** (RD), **matter dominated era** (MD) and **dark energy dominated era** (DD) respectively.

2.2 Definition and motivation of inflation

2.2.1 Definition of inflation

The definition of inflation is accelerating expansion of the universe, more accurately we say that the universe is in a epoch of inflation when the scale factor is accelerating, i.e.

$$\ddot{a} > 0. \tag{2.13}$$

This is obviously equivalent to

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0. \tag{2.14}$$

Since $1/aH$ represents comoving Hubble radius, this inequality implies that comoving Hubble radius is decreasing during inflation.

From Eq. (2.7), the third equivalent definition of inflation is that the universe is filled with substance which satisfies

$$p < -\frac{1}{3}\rho. \tag{2.15}$$

Next we would like to define what the inflation model is. To this end we need a index which expresses how much inflation expands the universe in total, so we define the **e-fold number** N_e as follows;

$$N_e := \ln[a(t_e)/a(t_i)], \tag{2.16}$$

where t_i and t_e is the time when inflation starts and ends respectively. If the universe is de Sitter during inflation (i.e. $a(t) \propto e^{H_{\text{inf}}t}$), the e-fold number can be written as

$$N_e = H_{\text{inf}}(t_e - t_i). \tag{2.17}$$

It is known that e-fold must be larger than about 60 in order for the cosmological parameters to match with observation. In the language of Eq. (2.17), inflation must last enough longer than the Hubble time.

Note that sometimes e-fold is defined as $N_e(t) := \ln[a(t_e)/a(t)]$ which express how much the universe expands from time t to the end of inflation t_e. To distinguish the difference, e-fold number N_e in Eq. (2.16) is sometime referred to as total e-fold number N_{tot}.

We say a model is qualified as a inflation model if the model brings about inflation at the beginning and the inflation lasts long enough to make e-fold larger than 60 but eventually ends and leaves radiation dominant universe. We will present a framework of successful inflation models called slow-roll inflation in subsection 2.4, and some examples of inflation models in subsection 2.6. Before that, we will explain the motivation for inflation in the next subsubsection.

2.2.2 Motivation for inflation

There are roughly two motivations. One is to solve the problems arising from big-bang theory; flatness problem, horizon problem and monopole problem. We will explain each of them in the next subsection. The other motivation for inflation is that it naturally generate an initial density perturbation as a seed of the structure formation of the universe. Actually, the spectrum of the density perturbation fits incredibly well with the observational data from the CMB.

2.3 Solving three puzzles of Big-Bang cosmology by inflation

In this section, we explain the three problems of Big-Bang cosmology and see how inflation solves them.

2.3.1 Flatness problem

What is flatness problem?

From Friedmann equation (2.6) with zero cosmological constant, we obtain

$$K = a(t)^2 H(t)^2 \left[\Omega(t) - 1\right], \tag{2.18}$$

where $\Omega := \rho/\rho_c$ is **density parameter** and $\rho_c := 3M_{\rm Pl}^2 H(t)^2$ is **critical density**. Furthermore we define curvature parameter;

$$\Omega_K(t) := \Omega(t) - 1 = \frac{K}{a(t)^2 H(t)^2}. \tag{2.19}$$

Observation tells us that the present value of curvature parameter is consistent with zero of the order 0.01, more precisely

$$-0.0133 < \Omega(t_{\rm now}) < 0.0084 \quad (95\%\ \text{CL}), \tag{2.20}$$

when we assume ΛCDM model. Now let us go back to the past and see what kind of initial condition realize the present value. We assume that the universe is dominated by matter up until now from the equivalent time, we obtain

$$\Omega(t_{eq}) = \Omega(t_{now}) \frac{a^2 H^2(t_{now})}{a^2 H^2(t_{eq})} = \Omega(t_{now}) \frac{a(t_{eq})}{a(t_{now})} \sim 0.01 \times \frac{3 \times 10^{-4}}{1}. \tag{2.21}$$

In the second equality we used $H^2 \propto \rho \propto a^{-3}$ when matter dominates. Before equality time the universe is dominated by radiation components, so similar calculation gives

$$\Omega(t_{BBN}) = \Omega(t_{eq}) \frac{a^2 H^2(t_{eq})}{a^2 H^2(t_{BBN})} = \Omega(t_{eq}) \frac{a(t_{BBN})^2}{a(t_{eq})^2} \sim 10^{-18}, \tag{2.22}$$

$$\Omega(t_{Pl}) = \Omega(t_{eq}) \frac{a^2 H^2(t_{eq})}{a^2 H^2(t_{Pl})} = \Omega(t_{eq}) \frac{a(t_{Pl})^2}{a(t_{eq})^2} \sim 10^{-62}, \tag{2.23}$$

here $t_{Pl} := 1/M_{Pl} \sim 5 \times 10^{-44}$s and hence $a(t_{Pl}) \sim 10^{-32}$. These results are obviously fine-tuned initial condition, because too small $\Omega_K(t^*)$ means the energy density of the universe at the time t^* is too close to the critical density. Unless we have a reasonable explanation for the value we need to admit that this is a problem and this is the flatness problem.

Solving flatness problem

Inflation gives a clear explanation for the present value of curvature. The curvature parameter at the end of the inflation is evaluated as

$$\Omega(t_e) = \Omega(t_i)\frac{a^2H^2(t_i)}{a^2H^2(t_e)} \simeq \Omega(t_i)\frac{a(t_i)^2}{a(t_e)^2} = \Omega(t_i)e^{-2N}. \tag{2.24}$$

In the second equality we used $H(t) \simeq H_{\text{inf}}(\text{constant})$ during inflation. Therefore even if the initial value of curvature parameter is of the order unity or even larger, the value at the end of the inflation is very small because e-fold is larger than 60. Thus flatness problem is solved.

2.3.2 Horizon problem

Next problem is what is called horizon problem. Roughly speaking, the problem is that we do not have any explanation for such isotropic CMB which is out of causality.

Preliminary

To explain horizon problem we need to define particle horizon and event horizon.

The particle horizon is defined as

$$l_{PH}(t) := a(t)\int_0^{r_H} dr = a(t)\int_0^t \frac{dt'}{a(t')} = a(t)\int_0^{a(t)} \frac{da'}{a'^2H}. \tag{2.25}$$

This quantity expresses the maximum length the photon who set off at time $t = 0$ can reach until the time t, in other words the length of causality. If the space is not expanding of course $l_H(t)$ is nothing but t, but the expanding space disturbs our intuition. As we summarized in table (5.17), the scale factor evolves as $a(t) \propto t^{1/2}$(RD), $t^{2/3}$(MD) and e^{Ht} (DD). Therefore we can evaluate the particle horizon corresponding to each era;

$$l_{PH}(t) = \begin{cases} 2t & \text{(RD)}, \\ 3t & \text{(MD)}, \\ (e^{Ht}-1)/H & \text{(DD)}. \end{cases} \tag{2.26}$$

In terms of comoving coordinate, we define comoving particle horizon as $L_{PH}(t) := l_{PH}(t)/a(t)$

and obtain

$$L_{PH}(t) = \begin{cases} 2t^{1/2} & \text{(RD)}, \\ 3t^{1/3} & \text{(MD)}, \\ (1-e^{-Ht})/H & \text{(DD)}, \end{cases} \tag{2.27}$$

here we used $a(t=0)=1$. This means that in radiation or matter dominated universe the causal range goes to infinity as time progresses, although the speed of expansion is slower than stationary universe. This is intuitively understandable because the expansion is decelerated during both radiation and matter dominated era, and thus the photon can reach eventually everywhere. On the other hand, in de-Sitter universe, since the space is expanding acceleratingly there is a limit of photon's reach; $(1-e^{-Ht})/H \to 1/H$.

We define one more quantity called event horizon;

$$l_{EH}(t) := a(t)\int_t^\infty \frac{dt'}{a(t')}, \quad L_{EH}(t) := l_{EH}(t)/a(t). \tag{2.28}$$

This represents the causal length which the photon departed at the time t can reach until $t=\infty$. Obviously $L_{EH}(t)$ is infinite unless the expansion of the universe is accelerating. In de-Sitter universe L_{EH} can be finite;

$$L_{EH}(t) = e^{-Ht}/H \tag{2.29}$$

This implies that in accelerating universe the causal region is given by a finite value for a given time t and it gets smaller and smaller as time progresses.

What is horizon problem?

First let us mention the observational fact that our universe is isotropic according to WMAP or Planck observation. They have shown that the temperature of CMB is around $2K$ and that its fluctuation, or the direction dependence is as small as 10^{-5}. Therefore we have to admit the isotropy of the universe at least in this sense.

Let us check whether the isotropy is compatible with the Big-Bang cosmology or not. First we evaluate the particle horizon at the last-scattering time t_{LS} assuming that the history of the universe is Big-Bang without inflaiton type.

$$l_{PH}(t_{LS}) = a(t_{LS})\int_0^{a(t_{LS})} \frac{da}{a^2 H} \simeq a(t_{LS})H_0^{-1}\int_0^{a(t_{LS})} a^{-1/2}da \simeq H_0^{-1}a(t_{LS})^{3/2}. \tag{2.30}$$

Here we assumed that universe is dominated by matter components from the start of the universe to the last-scattering time, and we used $H \simeq H_0 a^{-3/2}$ in the second equality. Whereas the size of the last-scattering sphere at the time t_{LS} is

$$l_{SP}(t_{LS}) \simeq H_0^{-1} a(t_{LS}). \tag{2.31}$$

Therefore the number of causally disconnected region on the sphere is estimated as

$$l_{SP}^2 / l_{PL}^2(t_{LS}) \simeq a(t_{LS})^{-1} \simeq 10^3. \tag{2.32}$$

This is completely incompatible with the CMB observation. Thus horizon problem have emerged.

Solving horizon problem

If we assume the inflationary cosmology, the calculation of particle horizon should be modified as follows;

$$\begin{aligned} l_{PH}(t_{LS}) &= a(t_{LS}) \left[\int_{a(t_i)}^{a(t_e)} \frac{da}{a^2 H} + \int_{a(t_e)}^{a(t_{LS})} \frac{da}{a^2 H} \right] \\ &\gtrsim \frac{a(t_{LS})}{H_{\inf}} \int_{a(t_i)}^{a(t_e)} \frac{da}{a^2} = \frac{a(t_{LS})}{a_e H_{\inf}} (e^N - 1). \end{aligned} \tag{2.33}$$

In the second inequality we dropped the second term and assumed that the universe is de-Sitter space during inflaiton. If the e-fold number N is large enough we can realize $l_{PH} > l_{SP}$. Actually $N > 60$ is sufficient condition for this, and then horizon problem is solved.

2.3.3 Monopole problem

What is horizon problem?

Although the standard model of particle physics is a successful model, there are some experimental facts it cannot explain and some theoretical subtlety like fine-tuning problem. In such situation, there have been a lot of effort to build models beyond the standard model and one of the compelling model is Grand Unified Theory, dubbed GUT, in which, starting from the gauge theory whose gauge group G_{GUT} is large enough to embed the gauge group of the standard model $G_{SM} := SU(3) \times SU(2) \times U(1)$, G_{GUT} is spontaneously broken to standard model gauge group in some high energy scale beyond TeV scale. In SUSY model, the energy scale of the spontaneous break, or the unification scale is expected to be about 10^{15} GeV due to the calculation of gauge coupling running of standard model.

When the gauge group G_{GUT} is broken to standard model gauge group G_{SM}, topological defects are generated and may disturb the success of Big-Bang cosmology. Here let us calculate the energy density of such topological defects and see the effect on the Big-Bang cosmology. Since the universe is dominated by radiation component from the start to the GUT scale $T_{GUT} \simeq 10^{15}$GeV, the present number density of monopole is estimated as

$$n_{MP}(t_{now}) = n_{MP}(t_{GUT})a(t_{GUT})^3 \sim \frac{3a(t_{GUT})^3}{4\pi l_{PH}(t_{GUT})^3} \simeq \frac{3a(t_{GUT})^3}{32\pi t_{GUT}^3}. \tag{2.34}$$

Further calculations are necessary. Friedmann equation gives the relation between t_{GUT} and $T_{GUT} \simeq 10^{15}$ GeV,

$$\frac{\pi^2}{30} g_{GUT} T_{GUT}^4 = \rho(t_{GUT}) = 3M_{\rm Pl}^2 H(t_{GUT})^2 = \frac{3M_{\rm Pl}^2}{4t_{GUT}^2}, \tag{2.35}$$

where g_{GUT} is the degree of freedom of GUT, and we assume it to be of the order 10^3. Moreover, from the entropy conservation $S(t) \propto a(t)^3 g T^3 = \text{(constant)}$, we obtain

$$a(t_{GUT})^3 = a(t_{now})^3 \frac{g_{now} T_{now}^3}{g_{GUT} T_{GUT}^3}. \tag{2.36}$$

Thus we get

$$\begin{aligned} n_{MP}(t_{now}) &= \frac{3}{32\pi} \frac{g_{now} T_{now}^3}{g_{GUT} T_{GUT}^3} \times \left(\frac{90}{4\pi^2}\right)^{-3/2} g_{GUT}^{3/2} T_{GUT}^6 / M_{\rm Pl}^3 \\ &\simeq 0.01 g_{now} g_{GUT}^{1/2} \left(\frac{T_{GUT}}{M_{\rm Pl}}\right)^3 T_{now}^3 \simeq \left(\frac{10^{15}}{2.43 \times 10^{18}}\right)^3 (2.7 \times 10^{-13}\text{GeV})^3 \\ &\simeq 10^{-48}\text{GeV}^3 \end{aligned} \tag{2.37}$$

Assuming that the mass of monopole is of the order of GUT scale, $m_{MP} \sim 10^{15}$GeV, the present energy density parameter of the monopole can be estimated as

$$\Omega_{MP}(t_{now}) := \frac{m_{MP} n_{MP}(t_{now})}{\rho_c(t_{now})} \simeq \frac{10^{15}\text{GeV} \times 10^{-48}\text{GeV}^3}{10^{-47}\text{GeV}^4} = 10^{14}, \tag{2.38}$$

which is obviously insane. This is the monopole problem. Of course, if one does not assume GUT, monopole problem is no problem. However, high energy physics often produces stable particles which deteriorate Big-Bang cosmology like monopole here, and that can cause similar problems. Therefore monopole problem can be generalized to **unwanted relics problem**, and we need to solve it unless we found a true high energy model which does not produce unwanted relics.

Solving monopole problem

The solution of the monopole problem by inflation is very simple. If we assume that inflation occurs after the production of unwanted relics, we can dilute the energy density by the accelerating expansion of the space. To discuss quantitatively, let us modify the calculation of the energy density parameter of monopole we have just done above in the case of Big-Bang cosmology without inflation.

We need not to change the calculation of (2.34) and (2.35), but the entropy does not conserve now because of the inflation (inflaton decay) and thus the calculation of scale factor $a(t_{GUT})$ has to be modified. Since the inflation occurs after the production of relics, the scale factor is bounded from above;

$$a(t_{GUT}) < a(t_i) = e^{-N} a(t_e) < e^{-N} \tag{2.39}$$

Therefore the energy density parameter is also bounded from above;

$$\begin{aligned}\Omega_{MP}(t_{now}) &< \frac{m_{MP}}{\rho_c} \frac{3}{32\pi} e^{-3N} \times \left(\frac{90}{4\pi^2}\right)^{-3/2} g_{GUT}^{3/2} T_{GUT}^6 / M_{\rm Pl}^3 \\ &\simeq \frac{(10^{15}{\rm GeV})^4}{10^{-47}{\rm GeV}^4} \times \left(\frac{10^{15}}{2.43 \times 10^{18}}\right)^3 \times 10^5 \times e^{-3N} \\ &\simeq 10^{100} \times e^{-3N} < e^{-80}\end{aligned} \tag{2.40}$$

In the last inequality we used $N > 60$. This clearly shows that the dilution via inflation is effective enough, and thus monopole problem is solved. Similarly even if a theory produces some stable particles before inflation, the energy density is diluted enough during inflation and there is no unwanted relic problem. Therefore when considering a high energy theory and its relics in terms of cosmology, one need to worry about only the stable particles (or topological defects) generated after inflation.

2.4 Slow-roll inflation

In this section we present a framework for inflation models. Although it is not difficult to build a model which brings about inflation, however, it is a herculean task to end inflation. To stifle inflation it is better to assume that inflation is dictated by dynamical fields, therefore quintessential inflation models have fields called **inflaton** who are in charge of inflation and in many models they are scalar fields. For simplicity we assume one real scalar field ϕ is an inflaton and see how inflation works.

Let us first start from the following action;

$$S[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \tag{2.41}$$

where ϕ is a real scalar field and $V(\phi)$ is effective potential of ϕ. We assume that the universe is dominated by ϕ and thus we neglect the contribution of other fields to the dynamics of $g_{\mu\nu}$ and ϕ. Functional derivative with respect to $g_{\mu\nu}$ yields the Friedmann equation[2] ;

$$3M_{\rm Pl}^2H^2 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi), \tag{2.42}$$

and inflaton equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\Delta\phi = -V'(\phi). \tag{2.43}$$

As the universe expands, the terms with spacial derivatives become smaller because of the scale factor in the denominators, therefore it is reasonable to assume that these terms are negligible and that $\phi = \phi(x)$ is a function of only time t, i.e. $\phi := \phi(t)$, indicating the homogeneity of the space. Here we introduce **slow-roll condition**;

$$\dot{\phi}^2 \ll V, \quad \ddot{\phi} \ll \frac{dV}{d\phi}, \tag{2.44}$$

[2]The stress-energy tensor is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L}_{\rm m})}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial(\sqrt{-g}\mathcal{L}_{\rm m})}{\partial g^{\mu\nu}{}_{,\alpha}}\right] = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi - g_{\mu\nu}V,$$

and thus it is perfect fluids form if we neglect the space dependence. Then density and pressure are

$$\rho = T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi), \quad p = a^{-2}T_{ii} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + a^{-2}\left((\partial_i\phi)^2 - \frac{1}{2}(\nabla\phi)^2\right),$$

here space-derivative terms are supposed to be neglected and the pressure density ratio is

$$w := \frac{p}{\rho} = \frac{K - V}{K + V},$$

where $K := (1/2)\dot{\phi}^2$ is the kinetic term. Therefore if $K \ll V$ then $w \simeq -1$ and thus inflaiton occurs. This is a heuristic derivation of slow-roll inflation. What about the acceleration equation (**??**)

which implies that the motion of inflaton is very slow compared with the height of the potential, and that the acceleration is much smaller than the steepness of the inflaton potential. To apply the slow-roll condition to the dynamics of inflation is called slow-roll approximation. We will see that under the slow-roll condition inflation is successful and that it is one of the easiest sufficient condition for inflation. Under these approximation, Friedmann equation (2.42) and equation of motion for ϕ (2.43) are reduced to

$$3M_{\mathrm{Pl}}^2H^2 = V, \quad 3H\dot{\phi} = -\frac{dV}{d\phi}. \tag{2.45}$$

It is shown that in this case inflation occurs successfully, see also the last footnote.

Before moving on to an example, we define useful indexes for the slow-roll condition, which are dubbed **slow-roll parameters**;

$$\epsilon := \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_{\mathrm{Pl}}^2, \quad \eta := \frac{V''}{V}M_{\mathrm{Pl}}. \tag{2.46}$$

Note that ϵ and η are functions of ϕ and thus functions of time t, we do not explicitly write the dependence and their names include parameter though. It is easy to show that slow-roll condition is sufficient condition of $\epsilon \ll 1$, $\eta \ll 1$. Therefore if slow-roll parameters are as large as unity the slow-roll approximation is invalid, and this is the usage of slow-roll parameters as one can see more in the next subsection.

There is another definition of slow-roll parameters based on the Hubble $H(t)$;

$$\epsilon_H := -\frac{\dot{H}}{H^2}, \quad \eta_H := \frac{\dot{\epsilon}_H}{H\epsilon_H}. \tag{2.47}$$

We add the subscript $_H$ as reminder of the difference. It is a trivial task to show that $\epsilon = \epsilon_H$ and $\eta = \eta_H$ under slow-roll approximation.

Example: chaotic inflation

To demonstrate slow-roll inflation, we take chaotic inflation as an example and show how it works as an inflation model. Chaotic inflation is a model, proposed first by Linde [**?**], where the potential consists of only the mass term;

$$V = \frac{1}{2}m\phi^2. \tag{2.48}$$

Let us first check whether this model can be a slow-roll inflation model. Intuitively speaking, inflaton oscillates around the origin when the field value is small, while when the field value is large, despite of the larger steepness of the potential, Hubble friction term decelerates the inflaton and slow-roll inflation seems realized. For quantitative discussion, let us calculate the slow-roll parameters. Substituting the potential (2.75) into the definition of them (2.46) we get

$$\epsilon = \eta = \frac{2M_{\mathrm{Pl}}^2}{\phi^2}, \tag{2.49}$$

which implies that if ϕ is somewhat larger than M_{Pl} slow-roll condition can be satisfied. Conversely if ϕ is as small as M_{Pl} inflaton cannot slow-roll. Then we determine the field value of inflaton at the end of the inflation;

$$\epsilon|_{end} = \eta|_{end} = 1 \quad \rightarrow \quad \phi(t_{end}) = \sqrt{2}M_{\mathrm{Pl}} \tag{2.50}$$

Next we determine the initial value of inflaton field so that e-fold is as large as 60. Follow the calculation below;

$$\begin{aligned} 60 = N = \ln[a(t_e)/a(t_i)] = \int_{t_i}^{t_e} H dt = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_e}^{\phi_i} \frac{1}{\sqrt{2\epsilon}} \frac{d\phi}{M_{\mathrm{Pl}}} \\ = \frac{1}{4M_{\mathrm{Pl}}^2}(\phi_i^2 - \phi_e^2), \quad \rightarrow \quad \phi(t_i) \simeq 15M_{\mathrm{Pl}} \end{aligned} \tag{2.51}$$

As in this model, inflation models in which inflaton transcends Planck scale are called **large field inflation models**, and others are called **small field inflation models**. This classification is important when we take the validity of effective field theory seriously because large field inflation models smell invalid in this point of view.

Next thing we can do is to solve the equations (2.45) assuming slow-roll approximation, and the result is

$$\phi(t) = \phi(t_i) - \frac{2}{3}mM_{\mathrm{Pl}}(t - t_i), \tag{2.52}$$

$$a(t) = a_i \exp\left(\frac{1}{4M_{\mathrm{Pl}}^2}(\phi(t)^2 - \phi(t_i)^2)\right). \tag{2.53}$$

From this result we can estimate the time and scale factor during inflation (Of course we already know that $a_e = e^{60}a_i$ since the second equation is almost equivalent to the calculation of e-fold above.). Now we understand almost completely what is going on during the inflation.

In subsection 2.6 we illustrate various kinds of inflation models and analyze in more detail by calculating observational quantities based on the next section and comparing them with the observed value.

2.5 Observational quantities on inflation

In this subsection we define and calculate several quantities which can be determined by observation, which includes Power spectrum $\mathcal{P}_\zeta$, tensor to scalar ratio r, spectral index (or tilt) n_s, running n_s'. This is crucial study for inflation model research because the comparing observational data with theoretical predictions enables us to exclude some class of models or to determine some of the parameters in the model. Our objective is to give definitions and results of them and hence the discussion or calculation is somewhat skipped. See [4] or [5] for more detailed discussion.

There are other observational quantities related to inflation such as non-gausianity f_{NL}, isocurvature, magnetic force and so on, but we do not discuss them in this paper.

2.5.1 Density perturbation from inflaton

The most important and basic observable in inflationary cosmology is the density perturbation $\mathcal{P}_\phi$ which represents a spacial fluctuation of inflaton ϕ. The definition is given by

$$\mathcal{P}_\phi := \left\langle |\phi|^2 \right\rangle . \tag{2.54}$$

The calculation is not difficult but it is beyond a scope of this paper and to type them is really painstaking job, and thus we just present the result;

$$\mathcal{P}_\phi = \left(\frac{H}{2\pi} \right)^2 , \tag{2.55}$$

where H is Hubble constant during inflation.

The corresponding observational value is, on the other hand, given not as the fluctuation of inflaton itself but as that of density or curvature of the universe. The curvature perturbation ζ is defined as the fluctuation of the scale factor at the uniform density slice;

$$a(t, \mathbf{x}) = a(t) e^{\zeta(t, \mathbf{x})} \tag{2.56}$$

and its perturbation is defined as

$$\mathcal{P}_\zeta = \left\langle |\zeta|^2 \right\rangle , \tag{2.57}$$

and observation has revealed that

$$1.91 \times 10^{-5} = \delta_H(k^*) := \frac{4}{25} \mathcal{P}_\zeta(k). \tag{2.58}$$

We would now like to connect observation $\mathcal{P}_\zeta$ with theoretical prediction $\mathcal{P}_\phi$. In flat slice $t_f(t, \mathbf{x}) = t + \delta t(t, \mathbf{x})$ in which $\zeta = 0$, we obtain

$$\begin{aligned} -\zeta(t, \mathbf{x}) &= \delta \ln a(t_f, \mathbf{x}) - \delta \ln a(t, \mathbf{x}) \\ &= \ln a(t, \mathbf{x}) - \ln a(t_f, \mathbf{x}) = -\partial_t(\ln a)\delta t = -H(t)\delta t(t, \mathbf{x}). \end{aligned} \tag{2.59}$$

In uniform density slice, on the other hand, the inflaton perturbation vanishes and hence

$$\delta\phi(t_f, \mathbf{x}) \simeq \delta\phi(t_f, \mathbf{x}) - \delta\phi(t, \mathbf{x}) = \phi(t) - \phi(t_f) = -\dot{\phi}(t)\delta t(t.\mathbf{x}), \tag{2.60}$$

where $\phi(t, x) = \phi(t) + \delta\phi(t, \mathbf{x})$. Comparing these relations, we obtain

$$\zeta(t, \mathbf{x}) = -\frac{H(t)}{\dot{\phi}(t, \mathbf{x})}\delta\phi(t, \mathbf{x}). \tag{2.61}$$

Therefore it follows that

$$\begin{aligned} \mathcal{P}_\zeta &= \left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_\phi = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \\ &= \frac{1}{12\pi^2 M_{\rm Pl}^6}\left(\frac{V^3}{V'^2}\right) = \frac{1}{24\pi^2 M_{\rm Pl}^4}\left(\frac{V}{\epsilon}\right), \end{aligned} \tag{2.62}$$

here we have used slow-roll approximation and slow-roll parameter.

As a result, a constraint on inflaton potential is now available,

$$\left|\frac{V^{3/2}}{M_{\rm Pl}^3 V'}\right| \simeq 5 \times 10^{-4}, \quad \text{or} \quad V^{1/4} \simeq 3 \times 10^{-2}\epsilon^{1/4} M_{\rm Pl} \tag{2.63}$$

This constraint is called COBE normalization after the first experiment.

2.5.2 Tensor to scalar ratio: r

Let us next calculate the fluctuation of tensor field $h_{ij}(t, \mathbf{x})$ (i,j=1,2,3) defined as

$$ds^2 = a(\eta)^2 \left[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j\right]. \tag{2.64}$$

The gauge symmetry considered, h_{ij} has two physical degree of freedom, called cross mode and plus mode, and the result of perturbation calculation is

$$\mathcal{P}_h \simeq 2 \times \frac{4}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2, \tag{2.65}$$

where $2\times$ means that the contribution of both modes are summed.

We are now ready for the definition and theoretical calculation of tensor to scalar ratio r;

$$r := \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = \frac{8}{M_{\rm Pl}^2}\left(\frac{\phi}{H}\right)^2 \simeq 16\epsilon \tag{2.66}$$

We have used Eq.(2.62) and Eq.(2.65).

Planck observation has set an upper bound to r;

$$r \lesssim 0.2, \quad \text{or} \quad \epsilon < 0.015 \tag{2.67}$$

which greatly restricts inflation models.

On top of that, Eq.(2.63) gives rise to the following information,

$$V_{\rm inf}^{1/4} \simeq 2.4 \times 10^{16} \left(\frac{r}{0.24}\right)^{1/4} \text{GeV}, \tag{2.68}$$

which means the potential energy during inflation is about 100 times smaller than the Planck scale. In addition, using the relation $3M_{\rm Pl}^2 H^2 = V$, we obtain

$$H_{\rm inf} \simeq 1.4 \times 10^{14} \left(\frac{r}{0.24}\right)^{1/2} \text{GeV}. \tag{2.69}$$

Therefore the observation of r is incredibly significant to inflation model building.

2.5.3 Spectral index (tilt) n_s

The definition of spectral index is

$$n_s - 1 := \frac{d\ln \mathcal{P}_\zeta}{d\ln k} \tag{2.70}$$

Observation have revealed

$$n_s(k^*) = 0.968 \pm 0.012, \tag{2.71}$$

within 1σ. We would like to extract a constraint on inflaton potential from this data. To this end, we first envision $d\ln k = d\ln(aH) = Hdt = -3H^2 d\phi/V' = -V d\phi/M_{\rm Pl}^2 V'$, and then obtain

$$n_s - 1 = -M_{\rm Pl}^2 \frac{V'}{V}\frac{d\ln \mathcal{P}_\zeta}{d\phi} \simeq -M_{\rm Pl}^2 \left[\frac{3V'^2}{V^2} - \frac{2V''}{V}\right] = -6\epsilon + 2\eta, \tag{2.72}$$

in the second equality we used Eq.(2.62) and next the definition of slow-roll parameters. Thus it follows

$$0.032 \pm 0.012 \simeq 6\epsilon - 2\eta, \tag{2.73}$$

which greatly limits inflation models.

2.5.4 Running spectral index n_s'

The definition and calculation is very similar to the previous subsubsection. The equations below themselves are self-explanatory;

$$\begin{aligned} n_s' := \frac{d^2 \ln \mathcal{P}_\zeta}{d(\ln k)^2} &\simeq -M_{\rm Pl}^2 \frac{V'}{V}\frac{d}{d\phi}\left[-M_{\rm Pl}^2 \frac{V'}{V}\frac{d}{d\phi}\right](3\ln V - 2\ln V') \\ &= M_{\rm Pl}^4\left[8\frac{V'^2V''}{V} - 6\left(\frac{V'}{V}\right)^4 - 2\frac{V'V'''}{V^2}\right] \\ &= 16\epsilon\eta - 24\epsilon^2 - 2\zeta, \end{aligned} \tag{2.74}$$

where $\zeta := M_{\rm Pl}^4 V'V'''/V^2$ is an additional slow-roll parameter.

This running n_s' is too small to see by observation, indeed it is a quadratic quantity with respect to slow-roll parameters.

2.6 Examples of inflation model

In this subsection, we review representative examples of inflation model.

2.6.1 Chaotic inflation

Chaotic inflation model is the most successful model of inflation if the tensor to scalar ratio r is larger than about 0.1, and we have already introduced as an example in subsection 2.4. Before explaining the advantages of chaotic inflation, we briefly review the mechanism of chaotic inflation. ϕ is a canonically normalized real scalar field and acts as an inflaton, and its potential consists of only the mass term;

$$V = \frac{1}{2}m^2\phi^2. \tag{2.75}$$

By calculating slow-roll parameters ($\epsilon = \eta = \frac{2M_{\rm Pl}^2}{\phi^2}$) and e-fold number ($60 \lesssim N_e = \frac{\phi_i^2 - \phi_e^2}{4M_{\rm Pl}^2}$), we have already estimated the initial and final field value of ϕ;

$$\phi(t_{ini}) = \sqrt{2N_e}M_{\rm Pl}, \quad \phi(t_{end}) = \sqrt{2}M_{\rm Pl}. \tag{2.76}$$

Let us next evaluate the observational values predicted in chaotic inflation model, and see whether they are consistent with the experiment. As for the COBE normalization (2.63),

$$
\begin{aligned}
3\times 10^{-2} M_{\rm Pl} \simeq \left(\frac{V}{\epsilon}\right)^{1/4} &= \sqrt{\frac{m}{2M_{\rm Pl}}}\,\phi_i = \sqrt{N_e\frac{m}{M_{\rm Pl}}}\,M_{\rm Pl}, \\
\Longrightarrow \quad \frac{m}{M_{\rm Pl}} &\simeq \frac{(3\times 10^{-2})^2}{N_e} \simeq 1.5\times 10^{-5}\left(\frac{60}{N_e}\right).
\end{aligned}
\tag{2.77}
$$

Thus we have determined the inflaton mass of chaotic inflation. The question that why the mass is as small as 10^{-5} compared with Planck mass has to be studied.

We can predict tensor to scalar ratio r, Eq.(2.66), and tilt n_s, Eq.(2.72) as follows;

$$
r = 16\epsilon = \frac{32M_{\rm Pl}^2}{\phi^2} \simeq 0.13\left(\frac{60}{N_e}\right), \quad 1-n_s = 6\epsilon - 2\eta = \frac{8M_{\rm Pl}^2}{\phi^2} \simeq 0.03\left(\frac{60}{N_e}\right). \tag{2.78}
$$

This is consistent with the experiment at the time of writing. It is notable that chaotic inflation predicts relatively large tensor to scalar ratio. We can also predict the potential and Hubble scale at the beginning of inflation according Eq.(2.68) and Eq.(2.69);

$$
V_{\rm inf}^{1/4} \simeq 2\times 10^{16}{\rm GeV}\left(\frac{60}{N_e}\right)^{1/4}, \quad H_{\rm inf} \gtrsim 1\times 10^{14}{\rm GeV}\left(\frac{60}{N_e}\right)^{1/2} \tag{2.79}
$$

What remains to be written is a comment on one of the advantages of chaotic inflation related to initial condition problem in closed universe. Naturally the initial condition must be around $M_{\rm Pl}$ and must not be tuned. As one can see in Eq.(2.76), the initial condition satisfies this criterion. On top of that, it does not conflict with closed universe. Recall the Friedmann equation with positive curvature term;

$$
H^2 = \frac{\rho}{3M_{\rm Pl}^2} - \frac{K}{a^2} \tag{2.80}
$$

As a initial condition we assume $\rho \sim M_{\rm Pl}^4$ and $K \sim \mathcal{O}(1)$. If inflaton does not slow-roll at the beginning, ρ is decreases faster than curvature term and thus the universe shrinks immediately. This is a predicament. However, inflaton in chaotic inflation does slow-roll if the field value is larger than $\sqrt{2}M_{\rm Pl}$, and therefore there is no such problem.

We generalize the chaotic inflation model. Let us first consider the following potential,

$$
V \propto \phi^p, \tag{2.81}
$$

where $p \in \mathbf{R}$ is prameter of the model. This model is called power law inflation. The same analysis as in the case of chaotic inflation works again. To begin with let us calculate the slow-roll parameters and e-folding number;

$$\epsilon = \frac{p^2}{2\phi^2}, \quad \eta = \frac{p(p-1)}{\phi^2}, \quad N_e = \frac{\phi_i^2 - \phi_e^2}{2p}, \tag{2.82}$$

here we omit $M_{\rm Pl}$ because dimensional analysis can always restore it. Since $\eta > \epsilon \iff p > 2$, the end of inflation is defined by $1 = \eta = p(p-1)/\phi_e^2$ if we assume $p > 2$, and then the initial field value of inflaton is $\phi_i^2 = \phi_e^2 + 2pN_e$. Then, the observation of curvature perturbation Eq.(2.63) determines the coefficient of ϕ^p in the potential V.

Let us next evaluate the tensor to scalar ratio r and the tilt n_s;

$$r = 16\epsilon = 8\frac{p^2}{\phi_i^2} = \frac{8p}{2N_e + p - 1}, \quad 1 - n_s = 6\epsilon - 2\eta = \frac{p(p+2)}{\phi_i^2} = \frac{p+2}{2N_e + p - 1} \tag{2.83}$$

The observation ($r < 0.2$, $1 - n_s = 0.032 \pm 0.012$) rules out $p \gtrsim 3$, if N_e is not much larger than 60. The chaotic inflation ($p = 2$) is a favored power-law inflation model.

Let us next consider adding a quartic term $\lambda\phi^4$ to the potential of the chaotic inflation,

$$V = \frac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{2.84}$$

As one can anticipate from the discussion so far, observational results sets a strong constraint on the value of the coefficient; the numerical analysis in [6] shows that $\lambda \lesssim 10^{-13}$ (the full calculation is too convoluted to explain here). We therefore have to control the higher terms well when we build chaotic inflation models.

2.6.2 Natural inflation

Natural inflation is one of successful inflation models in which single real scalar field ϕ feels a potential of the form of

$$V = \Lambda^4(1 - \cos(\phi/f)), \tag{2.85}$$

where the two parameters Λ and f are called dynamical scale and decay constant respectively. In large decay constant limit, natural inflation is reduced to chaotic inflation with mass $m^2 = \Lambda^4/f^2$ (and additional quartic coupling $\lambda = \Lambda^4/f^4$). Rough matching of theory with observation can be done as follows; from the experience in the chaotic inflation model, we know that $10^{-10}M_{\rm Pl}^2 \simeq m^2 \simeq \Lambda^4/f_2$ and $10^{-13} \gtrsim \lambda \simeq \Lambda^4/f^4 \simeq 10^{-10}M_{\rm Pl}^2/f^2$, and then it follows that

$$f \gtrsim \mathcal{O}(10)M_{\rm Pl} \tag{2.86}$$

in order to be a successful model. This result is supported also by the numerical analysis in [6], we do not discuss further though.

In the context of string inflation, natural inflation (and its generalization like N-flation [7]) plays a fundamental roll when we discuss (string) axion inflation.

2.6.3 inflation model in supergravity

In string inflation research, it is often important to accommodate inflation models in supergravity. In this subsubsection, we try to build a simple inflation model and it turns out that realizing inflation in supergravity is so difficult that we need a special mechanism.

First attempt

The first apparent difficulty one would face when building an inflation model in supergravity is the over all factor $e^{K/M_{\rm Pl}^2}$ in the scalar potential. Therefore it is often difficult to make a large field inflation model like chaotic inflation model. To avoid this one may assume tha Kähler potential is logarithmic form, and we first try this idea below.

We consider a supergravity model with one chiral superfield ϕ whose Kähler potential and superpotensital are given by

$$\begin{cases} K(\phi,\phi^*)/M_{\rm Pl}^2 = p\ln\left(1+\frac{|\phi|^2}{M_{\rm Pl}^2}\right), \\ W(\phi) = M\phi^2 + c. \end{cases} \tag{2.87}$$

$p \in \mathbf{R}$ and $c \in \mathbf{C}$ are just parameters and we can take M to be positive real number by rotating the phase of ϕ (and rescaling fixes the coefficient in the logarithm). We investigate whether (one direction of) ϕ can be an inflaton. The lagrangian for the scalar field ϕ is given by

$$\sqrt{-g}\mathcal{L} = \sqrt{-g}\left[-K_{\phi\phi^*}\partial_\mu\phi^*\partial^\mu\phi - V(\phi,\phi^*)\right]. \tag{2.88}$$

Let us investigate what the potential is like. In large $|\phi|^2$ limit, the potential becomes

$$V \simeq \frac{(p+2)^2}{p}M^2M_{\rm Pl}^2\left(\frac{|\phi|^2}{M_{\rm Pl}^2}\right)^{3+p}, \tag{2.89}$$

and thus the stabilization of the potential requires $p=-2$ or $p>0$. It turns out that $p=-2$ case does not work, so we do not take that choice. For the flatness of the inflaton potential smaller p would be better, so we simply take $p=1$ (in this case the Kähler potential is nothing but a that of CP_1). Then the potential becomes

$$V = \left(1+\frac{|\phi|^2}{M_{\rm Pl}^2}\right)\left[\ M^2|\phi|^2\left\{4+9\left(\frac{|\phi|^2}{M_{\rm Pl}^2}\right)+9\left(\frac{|\phi|^2}{M_{\rm Pl}^2}\right)^2\right\}\right.$$
$$\left.+\left(-1+3\frac{|\phi|^2}{M_{\rm Pl}^2}\right)M\left(c\phi^{*2}+c^*\phi^2\right)+|c|^2M_{\rm Pl}^2\left(-3+\frac{|\phi|^2}{M_{\rm Pl}^2}\right)\ \right] \tag{2.90}$$

At first sight this potential is too steep to cause slow-roll inflation (recall that the severe upper bound for the $\lambda\phi^4$ coupling in chaotic inflation), and thus our first attempt to realize simple chaotic like inflation in supergravity seems to have failed.

However, ϕ is not a canonical real scalar field but a non-canonical complex field, so the naive estimation may not work. To discuss exactly, we parametrize ϕ using two real scalar field r and θ as,

$$\phi =: re^{i\theta/M_{\rm Pl}}, \tag{2.91}$$

and thus the lagrangian can be written by

$$\mathcal{L} = -\frac{1}{\left(1+\frac{r^2}{M_{\rm Pl}^2}\right)^2}\left[\partial_\mu r\partial^\mu r+\left(\frac{r}{M_{\rm Pl}}\right)^2\partial_\mu\theta\partial^\mu\theta\right]-V. \tag{2.92}$$

From the shape of the potential it is reasonable to assume that r is fixed at the vacuum value and consider only the dynamics of θ. We define $\chi := \frac{\sqrt{2}r/M_{\rm Pl}}{1+(r/M_{\rm Pl})^2}\theta$ because it looks canonical for a given value of r. Then the lagrangian takes the form same as the natural inflation;

$$\mathcal{L} = -\frac{1}{2}\partial\chi\partial\chi-\Lambda^4\cos\left(\frac{\chi}{f}+\text{const.}\right)+\text{const.},$$
$$\Lambda^4 := 2M|c|r^2\left(1+\frac{r^2}{M_{\rm Pl}^2}\right)\left(-1+3\frac{r^2}{M_{\rm Pl}^2}\right),\quad f := \frac{r/M_{\rm Pl}}{\sqrt{2}(1+(r/M_{\rm Pl}^2)^2)}. \tag{2.93}$$

However this f cannot surpass the Planck scale while the fitting to the observation requires f to be larger than Planck scale as we have seen in the study of natural inflation, which means the model we have considered so far is dead.

We have here learned that in order to get access to flat potential in SUGRA, independence of inflaton in the Kähler potential is more important than to avoid exponential form.

KYY model

There is a beautiful solution to this difficulty invented by Kawasaki Yamaguchi Yanagida [8].

Let us first assume that in energy scale higher than TeV and lower than Planck scale there are two singlet chiral multiplets Φ and X whose matter parities are odd, with the superpotential

$$W = MX\Phi, \tag{2.94}$$

where M is a real parameter, and the Kähler potential

$$K = \frac{1}{2}(\Phi + \Phi^*)^2 + X^* X + \text{higher}. \tag{2.95}$$

Note that Φ has a shift symmetry $\mathrm{Im}\Phi \to \mathrm{Im}\Phi + \text{const}$, so that scalar field $\phi := \sqrt{2}\mathrm{Im}\Phi$ has flat enough potential and plays a roll as an inflaton. The scalar potential of supergravity $V = e^{K/M_{\mathrm{Pl}}^2}(K^{-1}|DW|^2 - 3|W|^2/M_{\mathrm{Pl}}^2)$ tells us that X and $\mathrm{Re}\Phi$ are no less heavy than Planck scale, so we obtain the low energy effective potential for ϕ by substituting $X = 0$ and $\Phi = 0 + i\phi/\sqrt{2}$ into V;

$$V(\phi) = \frac{1}{2}M^2\phi^2. \tag{2.96}$$

This is nothing but KYY mechanism. Thus ϕ can be a inflaton of chaotic inflation with the parameter value $M \sim 10^{13}$ GeV,

Generalization

We can easily generalize the KYY model [9]. If we assume

$$K = \frac{1}{2}(\Phi + \Phi^*)^2 + X^* X, \quad W = Xf(\Phi), \tag{2.97}$$

then we obtain

$$V = |f(\phi/\sqrt{2})|^2. \tag{2.98}$$

That's all.

2.6.4 Right-handed sneutrino inflation

Among the most attractive models for inflation is right-handed sneutrino inflation model suggested in refs. [10]. The beauty of this model is that it passes various phenomenological tests such as small (left-handed) neutrino mass, baryon asymmetry and gravitino problem.

The set-up

First of all we assume the MSSM with three right-handed sneutrinos (gauge singlet chiral multiplets) $N_{i=1\sim 3}$ as particle theory. MSSM has matter parity in order to avoid the proton decay, and we assume that N_i is odd under that parity[3] which leads to the superpotential of right-handed sneutrino of the form

$$W = \frac{1}{2}M_{ij}N_iN_j + h_{i\alpha}N_iL_\alpha H_u \tag{2.99}$$

where $M_{ij=1\sim 3}$ is complex symmetric 3 times 3 matrix, $h_{i\alpha} \in \mathbf{C}$ is Yukawa coupling, $L_{\alpha=1\sim 3}$ is lepton doublets and H_u is up-type Higgs doublets. We redefine N_i to $N'_i := U_i{}^jN_j$ where $U_i{}^j$ is an unitary matrix, so that M_{ij} becomes

$$M_{ij} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_2 \\ 0 & M_3 & 0 \end{pmatrix} \tag{2.100}$$

where $M_{i=1,2,3} \in \mathbf{R}$. Assuming that the Kähler is given by

$$K = N_1^\dagger N + N_2^\dagger N_2 + \frac{1}{2}(N_3^\dagger + N_3)^2, \tag{2.101}$$

and that

$$4 \times 10^8\text{GeV} \lesssim M_1 \ll |M_2| \simeq |M_3| \simeq 10^{13}\text{GeV} \tag{2.102}$$

we can explain inflation, small neutrino mass and baryon asymmetry of the universe at the same time as follows.

Inflation and reheating

We can easily see that $\sqrt{2}\text{Im}N_3 =: \phi$ can be an inflaton by virtue of the KYY model explained in the last subsubsection assuming an appropriate initial condition; $\phi_i \sim M_{\text{Pl}} \gg$ (other fields). That is because in this case the potential becomes chaotic type; $V \simeq (1/2)M^2\phi^2$ where $M := (1/2)(M_2 + M_3) \simeq 10^{13}\text{GeV}$.

After the inflation, the inflaton ϕ will decay into MSSM particles through the Yukawa interaction. The reheating temperature T_{RH} can be estimated as

$$\begin{aligned} 3M_{\text{Pl}}^2H^2 = \frac{??\pi^2}{45}g_*T_{\text{RH}}^4, \quad H \sim \Gamma \sim \frac{h^2}{8\pi}M \\ \rightarrow \quad T_{\text{RH}} \sim g_*^{-1/4}\sqrt{\frac{h^2}{8\pi}MM_{\text{Pl}}} \gtrsim 10^{13}\text{GeV}. \end{aligned} \tag{2.103}$$

[3]If we assume N_i to have the even matter parity, the superpotential must have the form $W \propto N_iH_uH_d$ which has nothing to do with neither neutrino mass or baryogenesis. Therefore we choose the odd case where the existence of N_i plays a crucial role of generating both neutrino mass and baryon number density of the universe.

Here we assumed h is of the order 0.1 for the left-handed neutrino mass to be around 0.01eV.

Baryogenesis by thermal leptogenesis

After the reheating universe is full of relativistic MSSM particles and N_1. Since N_1 is much heavier than other particles it dominates the universe after some cooling down $T \sim M_1$. After that the usual thermal leptogenesis occurs. The dominant N_1 decays into MSSM particles by the Yukawa interaction and generate the lepton number and the lepton number is partially converted to baryon number by the sphaleron process.

neutrino mass by see-saw mechanism

Generating small left-handed neutrino mass is a trivial task. By integrating out N_i we get neutrino mass;

$$(m_\nu)_{\alpha\beta} := h_{i\alpha} M_{ij}^{-1} h_{j\beta} \langle H_u \rangle^2 \sim 0.1\text{eV} \times \left(\frac{h}{1}\right)^2, \tag{2.104}$$

where we used $\langle H_u \rangle = 246\text{GeV}$ and $M_{ij} \sim 10^{13}\text{GeV}$.

Gravitino problem

The relatively high reheating temperature in Eq. (2.103) suggests a large abundance of the gravitino [11];

$$Y_{3/2} := \frac{n_{3/2}}{s} \sim 10^{-9} \left(\frac{T_{\text{RH}}}{10^{13}\text{GeV}}\right). \tag{2.105}$$

We consider the following two cases and discuss the thermal history of the universe in terms of gravitino and LSP. (i) Gravitino = LSP case; we have to be careful because such light gravitino easily overclose the universe. For a given mass of gravitino, reheating temperature is limited from above [11]. For the thermal leptogenesis, we need a reheating temperature that is higher than 10^9GeV. Therefore the mass of the gravitino has to be smaller than $\mathcal{O}(10)$eV.

(ii) Gravitino $\neq$ LSP case; We assume that the gravitino is not LSP and thus it decays into LSP. Since the coupling between gravitino and LSP is so weak (Planck surpressed), the gravitino is long enough life time to survive until BBN. This and the fact that LSP must be heavier than a TeV by the collider experiment lead to the over production of LSP. What we can do to avoid this problem is to introduce couplings which violate matter parity and to make LSP decay into other particles.

2.7 General difficulties in inflation models

In this subsection, we describe general difficulties we often encounter when we build inflation models.

2.7.1 Lyth bound

Consider a single field slow-roll inflation and calculate the e-folding number;

$$N = \ln[a(t_e)/a(t_i)] = \int_{t_i}^{t_e} H dt \simeq \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi \simeq \int_{\phi_i}^{\phi_e} \frac{1}{\sqrt{2\epsilon}} \frac{d\phi}{M_{\text{Pl}}} \lesssim \frac{2\sqrt{2}}{\sqrt{r_{min}}} \frac{\Delta\phi}{M_{\text{Pl}}}, \tag{2.106}$$

where $\Delta\phi := |\phi_i - \phi_e|$. In the last inequality we have used the relation $r = 16\epsilon$ derived in subsubsection 2.5.2.

According to the definition of the inflation, e-fold N must be larger than about 60, then we reach to the conclusion that

$$\frac{\Delta\phi}{M_{\text{Pl}}} \gtrsim \sqrt{\frac{r_{\text{min}}}{8}} 60 \sim 3\sqrt{\frac{r_{\text{min}}}{0.01}}. \tag{2.107}$$

This is called **Lyth bound** and means that the field range of a inflaton has to exceed the Plank scale to earn sufficiently large e-folding number unless the tensor to scalar ratio is very small.

2.7.2 UV sensitivity of inflation models

There are subtleties of inflation models based on low energy effective field theory in which higher dimensional terms are omitted just by assumption. We illustrate this idea in this subsubsection, following the ref.[2] by Liam McAllister. Let us consider a dimension $\Delta > 4$ correction term $\delta V = V_0(\phi/\Lambda)^{\Delta-4}$ to the theory of canonically normalized real scalar field ϕ which is supposed to be an inflaton with renormalizable potential V;

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - (V + \delta V). \tag{2.108}$$

The correction to the slow-roll parameter η is estimated as

$$\delta\eta = M_{\text{Pl}}^2 \left| \frac{\delta V''}{V + \delta V} \right| \sim \left(\frac{M_{\text{Pl}}}{\phi} \right)^2 \left(\frac{\phi}{\Lambda} \right)^{\Delta-4}. \tag{2.109}$$

Assuming $\Lambda \leq M_{\text{Pl}}$, a notable conclusion follows. In the case of large field inflation, in which the inflaton experiences a over Planck excursion, $\phi > M_{\text{Pl}}$, above equation indicates $\delta\eta > 1$

which is disastrous for a inflation model. Therefore we have to control the infinite number of higher dimensional terms for a successful large field inflation. Even in small inflation models where the field value of the inflaton does not exceed a Planck scale, $\phi < M_{\rm Pl}$, one have to control not only the renormalizable terms ($\Delta \leq 4$) but also $\Delta \leq 6$ operators. Thus it is implied that inflation models are really sensitive to the UV physics and what we can do in the effective theory approach is only to assume that higher dimensional operators are absent or small enough without any reasonable explanation. This is obviously a strong motivation for string inflation which can study the UV physics.

2.7.3 Eta problem in supergravity

It was suggested in example of subsubsection 2.6.3, that in SUGRA it is difficult to invent a flat inflaton potential, in other words to build a model whose slow-roll parameter η is smaller than unity. We show here that that is one of the general obstacles in inflation model building in the frame work of supergravity.

Consider a supergravity model containing a chiral multiplet ϕ which plays the roll of an inflaton. The lagrangian relevant to the scalar component of ϕ is given by

$$\begin{aligned} \mathcal{L} &= -K_{\phi^*\phi}\partial_\mu\phi^*\partial^\mu\phi - V(\phi,\phi^*), \\ V &= e^K\left(K^{\phi\phi^*}|D_\phi W|^2 + \cdots - 3|W|^2\right), \end{aligned} \tag{2.110}$$

where $\cdots$ in the potential represents F-terms of other chiral multiplets the model has. Expanding the Kähler potential around a reference point $\phi = 0$ we obtain

$$K(\phi,\phi^*) = K(0,0) + K_{\phi\phi^*}(0,0)\phi\phi^* + \cdots . \tag{2.111}$$

We define canonical field $\phi_c := \sqrt{K_{\phi,\phi^*(0,0)}}\phi$ and rewrite the lagrangian into

$$\mathcal{L} \simeq -\partial_\mu\phi_c^*\partial^\mu\phi_c - V(0,0)\left(1 + \frac{\phi_c^*\phi_c}{M_{\rm Pl}^2} + \cdots\right), \tag{2.112}$$

then the mass of ϕ_c can be read off as

$$m = \frac{V(0,0)}{M_{\rm Pl}^2} + \cdots . \tag{2.113}$$

This indicates that the slow-roll parameter η is estimated as

$$\eta \equiv \frac{V''}{V}M_{\rm Pl}^2 = 1 + \cdots , \tag{2.114}$$

which is not small and ϕ_c cannot cause inflation. This is the **eta-problem** in supergravity inflation models.

To avoid this problem we need to blindside the discussion above. A loophole one can notice that the leading term in Eq.(2.114) might be cancelled by other terms thanks to a certain peculiar form of the Kähler potential, like shift symmetry. In subsubsection 2.6.3 we have learned KYY model which is an example of this possibility.

3 Brief review of string theory

In this section, we remind readers of basic facts and equations about super string theory for the discussion in later sections. In subsection3.1, we write down the effective SUGRA actions in TypeIIA and TypeIIB string theory and explain their properties. Subsection3.2 is devoted to a brief review on D-branes. We basically assume flat space-time throughout this section and leave the topic of compactification to the next section.

3.1 Effective SUGRA action

The local supersymmetry in ten-dimensional space-time is so stringent that the effective field theory below the string scale M_s is immediately determined. In this section, we will see the ten-dimensional effective SUGRA action of TypeII string theory and their properties which are relevant to the subsequent sections. Although we mainly work on TypeIIB theory and might not use TypeIIA theory in the later sections of this paper, we discuss not only TypeIIB SUGRA effective action but also TypeIIA effective action because both theories have much in common and can be learned at the same time.

3.1.1 TypeIIA SUGRA action

The massless fields in TypeIIA theory are graviton G_{MN}, Kalb-Ramond field B_{MN} and dilaton Φ for NS-NS sector and anti-symmetric tensors (p-forms) C_1, C_3 and C_5 for R-R sector. TypeIIA string theory has $\mathcal{N} = (1,1)$ supersymmetry, meaning two supersymmetry with opposite chirality. The effective SUGRA action is given by

$$
\begin{aligned}
S_{\text{IIA}} &= S_{NS} + S_R + S_{CS} + S_f + S_{local}, \\
S_{NS} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[e^{-2\Phi}\left(R + 4\partial_M\Phi\partial^M\Phi - \frac{1}{12}H_3^2\right)\right], \\
S_R &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[-\frac{1}{2}F_0^2 - \frac{1}{2\cdot 2!}\widetilde{F}_2^2 - \frac{1}{2\cdot 4!}\widetilde{F}_4^2\right], \\
S_{CS} &= -\frac{1}{4\kappa_{10}^2}\int B_2\wedge\widetilde{F}_4\wedge\widetilde{F}_4,
\end{aligned}
\tag{3.1}
$$

where S_f represents terms which contain fermion fields, and S_{local} comes from local sources such as D-branes and O-planes which is discussed in subsection 3.2. Here F_0 is introduced just as a constant for the convenience in the later discussion. The overall coefficient κ_{10} is determined as $\kappa_{10}^{-2} := 4\pi l_s^{-8}$. The definitions of field strengthes are

$$H_3 := dB_2 \tag{3.2}$$

$$\widetilde{F}_2 := dC_1 - F_0 \wedge B_2 \tag{3.3}$$

$$\widetilde{F}_4 := dC_3 - B_2 \wedge dC_1 + \frac{1}{2} F_0 \wedge B_2 \wedge B_2. \tag{3.4}$$

The action in Eq.(3.1) is not canonical in terms of graviton fields G_{MN} because of the factor $e^{-2\Phi}$ in front of the Ricci scalar R, so we would like to define G^E_{MN} as

$$G^E_{MN} := e^{-\widetilde{\Phi}/2} G_{MN}, \quad \widetilde{\Phi} := \Phi - \langle \Phi \rangle \tag{3.5}$$

and render the action canonical form. After some manipulation one finds

$$S_{NS} = \frac{1}{2\kappa_{10E}^2} \int d^{10}x \sqrt{-G^E} \left[R^E - \partial_\mu \Phi \partial^\mu \Phi - \frac{g_s}{12} e^{\Phi} H_3^2 \right], \tag{3.6}$$

where $\kappa_{10E}^{-2} := \kappa_{10}^{-2} g_s^{-2} = 4\pi/(g_s^2 l_s^8)$. Here we can see ten-dimensional version of Einstein-Hilbert term so we call this frame **Einstein frame** and add the superscript E. We often omit this superscript (and even the subscript in κ_{10}) unless it is ambiguous.

3.1.2 TypeIIB SUGRA action

The massless fields in TypeIIB theory are graviton G_{MN}, B-field B_{MN} and dilaton Φ for NS-NS sector and anti-symmetric tensors C_0, C_2 and C_4 for R-R sector. C_4 satisfies the self-duality condition. There are two 10D SUSY which have the same chirality, that is, this theory has $\mathcal{N} = (2,0)$ supersymmetry. The effective action for massless bosonic fields is given by

$$S_{\text{IIB}} = S_{NS} + S_R + S_{CS} + S_{local} \tag{3.7}$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left(R + 4\partial_M \Phi \partial^M \Phi - \frac{1}{2\cdot 3!} H_3^2 \right)$$

$$S_R = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[-\frac{1}{2} F_1^2 - \frac{1}{2\cdot 3!} \widetilde{F}_3^2 - \frac{1}{4\cdot 5!} \widetilde{F}_5^2 \right] \tag{3.8}$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge \widetilde{F}_3, \tag{3.9}$$

The definitions of field strengths are

$$\widetilde{F}_3 := F_3 - C_0 \wedge H_3, \quad F_3 := dC_2$$

$$\widetilde{F}_5 := dC_4 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3. \tag{3.10}$$

In the Einstein frame, $e^{\widetilde{\Phi}/2}G_{MN} \to G_{MN}$, we can write the action as

$$
\begin{aligned}
&S_{NS} + S_R \\
&= \frac{1}{2\kappa_{10E}^2}\int d^{10}x\sqrt{-G}\left(R - \frac{\partial_M\tau\partial^M\bar{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{g_s}{12\mathrm{Im}\tau}G_3\cdot\overline{G}_3 - \frac{g_s^2}{4\cdot 5!}\widetilde{F}_5^2\right), \qquad (3.11)\\
&S_{\mathrm{CS}} = \frac{1}{2\kappa_{10}^2}\int C_4\wedge\frac{G_3\wedge\overline{G}_3}{4i\mathrm{Im}\tau}, \qquad (3.12)
\end{aligned}
$$

where

$$
\tau := C_0 + ie^{-\Phi}, \quad G_3 := F_3 - \tau H_3. \qquad (3.13)
$$

The massless 4-form C_4 in TypeIIB superstring theory has a self-duality condition;

$$
\widetilde{F}_5 = *\widetilde{F}_5 \qquad (3.14)
$$

However the action (3.9) or (3.12) does not reflect this property, instead we have to impose the self-duality condition after deriving a equation of motion from the action. Therefore the action (3.9) or (3.12) cannot be used for the path integral quantization, it should be used only for the derivation of equation of motions.

3.2 D-brane

In this subsection, we give a brief account of basic concepts and equations about D-branes for later discussion.

3.2.1 Brief introduction to D-branes

Dp-brane is a object spanning p-dimensional space ((p+1) in space-time) on which open strings can have their ends. If we consider a flat space-time superstring theory with a Dp-brane, then there exists open string sector whose boundary condition is given by

$$
\mathrm{N}: \partial_\sigma X^{a=0\sim p}(\tau,\sigma)|_{\sigma=0,l} = 0, \quad \mathrm{D}: X^{i=p+1\sim 9}(\tau,\sigma)|_{\sigma=0,l} = 0, \qquad (3.15)
$$

where N and D stand for Neumann and Dirichlet boundary condition respectively. The massless states are expressed in terms of unbroken $SO(9-p)$ Lorentz group and consist of a vector A_a ($\leftrightarrow \psi^a_{-1/2}|0\rangle_{\mathrm{NS}}$) and 9-p real scalars ϕ^i ($\leftrightarrow \psi^i_{-1/2}|0\rangle_{\mathrm{NS}}$) from NS-sector and a spinor λ_α ($\leftrightarrow |\mathbf{8}_C\rangle_{\mathrm{R}}$) from R-sector.

Dirichlet boundary condition in R-sector flips the sign of the GSO projection and thus TypeIIB theory admits only Dp-branes with odd p (even Dirichlet conditions), while TypeIIA admits even Dp-branes.

If there are N Dp-branes, we introduce Chan-Paton factor $A = 1 \sim N$ to label which Dp-branes the both ends of a open string stick to. The massless states from them have $U(N)$ gauge symmetry if all the N Dp-branes are coincident, and if it is not it breaks into a subgroup of $U(N)$.

D-brane effective action

If the energy scale is lower than the string scale $M_s := \alpha'^{-1/2}$, the string excitation modes can be integrated out and we can use the effective action for the massless fields just as the closed string sectors. In this paper we merely present the result for the action of the bosonic massless fields A_a and ϕ^i and skip the several techniques to derive it. The entire Dp-brane action can be split into two part; the DBI-action $S_{\rm DBI}$ and the Chern-Simons action $S_{\rm CS}$.

The DBI action corresponds to the kinetic terms and is given by

$$S_{\rm DBI} = -g_s T_p \int d^{p+1}\sigma e^{-\Phi} \sqrt{-det(G_{ab} + \mathcal{F}_{ab})} \tag{3.16}$$

where $G_{ab}(a, b = 0 \sim p)$ is the pullback of metric G_{MN} in string frame onto the Dp-brane world-volume and $\mathcal{F}_{ab}$ is the gauge invariant field strength defined as

$$\mathcal{F}_{ab} := (2\pi\alpha')f_{ab} + B_{ab}, \tag{3.17}$$

where B_{ab} is the pullback of B_{MN} and f_{ab} is the field strength of the gauge field A_a[4]. The coefficient T_p is Dp-brane tension and can be determined by the string amplitude computations as

$$T_p = \frac{1}{g_s(2\pi)^p \alpha'^{(p+1)/2}} = \frac{2\pi}{g_s l_s^{p+1}}, \tag{3.18}$$

where $l_s := 2\pi\sqrt{\alpha'}$ is the string length. Note that in weak string coupling limit $g_s \ll 1$ the tension is large which means Dp-brane is heavy and the dynamics is suppressed.

By expanding the DBI action in α', the leading term turns out to be a Yang-Mills action

[4]The gauge transformation here is B_2 gauge transformation, $B_2 \to B_2 + d\Lambda_1$ plus A_1 shift; $A_1 \to A_1 + (1/2\pi\alpha')\Lambda_1$. This shift is necessary because the action of worldsheet with boundary contains the terms $S = \int_\Sigma B_2 - 2\pi\alpha' \int_{\partial\Sigma} A_1$.

for the gauge field A_a in (p+1)-dimensional space-time spanned by the Dp-brane;

$$\begin{aligned} S_{\mathrm{DBI}} &= -g_s T_p \int d^{p+1}\sigma e^{-\Phi}\sqrt{-G}\sqrt{1+\frac{(2\pi\alpha')^2}{2}tr(f_{ab}f^{ab})+\cdots} \\ &= -g_s T_p \int d^{p+1}\sigma \sqrt{-G} e^{-\Phi}\frac{(2\pi\alpha')^2}{4}tr(f_{ab}f^{ab})+\cdots \\ &= -\frac{(2\pi\alpha')^2 T_p}{4}\int d^{p+1}\sigma\sqrt{-G}\ e^{(p-7))\widetilde{\Phi}/4}tr(f_{ab}f^{ab})+\cdots \end{aligned} \tag{3.19}$$

Here in the last equality we have moved from string frame to the Einstein frame. The trace tr is added for the generalization to the non-abelian case.

The Chern-Simon action appears if there are background n-form fields C_n and takes the form

$$S_{\mathrm{CS}} = \mu_p \int_{Dp} e^{\mathcal{F}_2}\wedge\sum_n C_n, \tag{3.20}$$

where the overall factor μ_p is determined to be equal to $g_s T_p$ up to phase factor. This is the topological couplings between branes and n-form fields and is investigated more in the next subsubsection, and after that we will focus on TypeIIA and TypeIIB cases in subsubsection 3.2.3.

3.2.2 Brane charges and coupling with p-form gauge field

We will see in this subsubsection that if the theory has p-form gauge field A_p there are spacially (p-1)-dimensional object D_{p-1} which has electric coupling with A_p and spacially (D-p-3)-dimensional object $\widetilde{D}_{D-p-3}$ which has magnetic coupling, and confirm our notation related to the topic. Of course one can regard them as D_{p-1}-branes and D_{D-p-3}-branes respectively. One can find that $p=1$ is an analogue of the familiar electro-magnetic theory with point particle.

Let us first suppose that there is a p-form gauge field A_p with an action $S=S(F_{p+1})$ in D dimensional space-time. We introduce a dual gauge field $\widetilde{A}_{D-p-2}$ and its field strength $\tilde{F}_{D-p-1}$ though the equations below;

$$\delta S =: \int \tilde{F}_{D-p-1}\wedge\delta F_{p+1}, \quad \tilde{F}_{D-p-1} =: d\tilde{A}_{D-p-2} \tag{3.21}$$

And we can obtain an action for the dual field via Legendre transformation;

$$S'(\tilde{F}) := S(F) - \int \tilde{F}\wedge F \tag{3.22}$$

Since the functional derivative of S' yields $\delta S' = -\int \delta\tilde{F} \wedge F$, we can see that dual of dual gauge field is equivalent to an original gauge field up to the sign[5].

Let us next see the charges of p-form A_p generated by spacially p and (D-p-3)-dimensional objects, D_p and $\widetilde{D}_{D-p-3}$. The first one is

$$S_e = q_e \int_{D_{p-1}} A_p = q_e \int \delta_{D-p}(D_{p-1}) \wedge A_p. \tag{3.23}$$

We call this action electric coupling and q_e electric charge. The other is a magnetic coupling and charge;

$$S_m = (-)^{D(p-1)+1} q_m \int_{\widetilde{D}_{D-p-3}} \widetilde{A}_{D-p-2} = (-)^{D(p-1)+1} q_m \int \delta_{p+2}(\widetilde{D}_{D-p-3}) \wedge \tilde{A}_{D-p-2}. \tag{3.24}$$

The equation of motion for A_p from the action $S_{tot} := S(F_{p+1}) + S_e$ is given by

$$d\tilde{F}_{D-p-1} = q_e \delta_{D-p-2}(D_{p-1}), \tag{3.25}$$

and the equation of motion from the action $\widetilde{S}_{tot} := S(\tilde{F}_{D-p-2}) + S_m$ is

$$dF_{p+1} = q_m \delta_{p+2}(\widetilde{D}_{D-p-3}). \tag{3.26}$$

By integrating this in a (p+2)-dimensional region M_{p+2} which intersects with $\widetilde{D}_{D-p-3}$ at a point, we obtain

$$\int_{M_{p+2}} dF_{p+1} = q_m \int_{M_{p+2}} \delta_{p+2}(\widetilde{D}_{D-p-3}) \quad \rightarrow \quad \oint_{\partial M_{p+1}} F_{p+1} = q_m \tag{3.27}$$

One can interpret this relation as a generalization of Gauss's theorem in 4D classical electromagnetism.

Dirac's quantization condition

We show that gauge invariance of the coupling between D-branes and gauge fields requires the charge conservation law and Dirac's quantization condition.

Consider the electric coupling between A_p and D_{p-1} and its infitestimal gauge transformation ($A_p \rightarrow A_p + d\Lambda_{p-1}$);

$$S_e = q_e \int_{D_{p-1}} A_p \quad \Rightarrow \quad \delta S_e = q_e \int_{D_{p-1}} d\Lambda_{p-1} = q_e \int_{\partial D} \Lambda. \tag{3.28}$$

[5] In this footnote, we consider a case $S(F) = (1/2g^2)\int *F \wedge F$ and see its dual theory as a pedagogical example. Simple calculation gives $\tilde{F} = (1/g^2) * F$ and $S'(\tilde{F}) = (1/2g_m^2)\int *\tilde{F} \wedge \tilde{F}$ with $g_m := g^{-1}$. Note that dual transformation changes strong coupling into weak coupling or vice vasa.

Therefore the gauge invariant requires that ∂D must vanish except for the infinity. This is a generalization of **charge conservation law** in 4D electromagnetism.

Consider next the invariance under large gauge transformation To this end we rewrite the action using the continual deformation of D_{p-1} to D'_{p-1};

$$S_D = q_e \int_D A_p = q_e \int_D A_p - q_e \int_{D'} A_p + S_{D'} = S_{D'} + q_e \int_{D-D'} A_p = S_{D'} + q_e \int_{M_1} F_{p+1}, \tag{3.29}$$

where M_1 is one of a (p+1)-dimensional objects which satisfies $\partial M_1 = D - D'$. If we take a different M_2, we obtain

$$S_D = S_{D'} + q_e \int_{M_2} F_{p+1}, \tag{3.30}$$

Eq. (3.29) must be equal to Eq. (3.30) up to $2\pi\mathbf{Z}$, because the action is used as e^{iS} in the theory. Thus we get

$$q_e \oint_{M_1-M_2} F \in 2\pi\mathbf{Z}. \tag{3.31}$$

for any M_1, M_2, which means

$$q_e \oint_C F \in 2\pi\mathbf{Z}. \tag{3.32}$$

for any closed (p+1)-dimensional object C. This is **flux quantization condition**.

$$q_e q_m \in 2\pi\mathbf{Z}. \tag{3.33}$$

This is **Dirac's quantization condition**.

3.2.3 D-branes in TypeII string theory

Type II string theories contain various kinds of antisymmetric tensor in their massless spectrum. As we have seen in the last subsubsection, the existence of p-form field admits spacially (p-1)-dimensional objects and (D-p-3)-dimensional objects which have electric charge and magnetic charge respectively. It is known that some assumptions require the existence of all the D-branes which have the corresponding charges against the gauge fields.

The table below is the summary of required D-branes in Type II string theory.

theory	gauge field	electric coupling	magnetic coupling
common	B_2	F1-brane	NS5-brane
Type IIA	C_1	D0-brane	D6-brane
	C_3	D2-brane	D4-brane
Type IIB	C_0	D-instanton	D7-brane
	C_2	D1-brane	D5-brane
	C_4	D3-brane	D3-brane

(3.34)

The object which electrically couples with Kalb-Ramond field B_2 is nothing but the fundamental string. When we regard fundamental string as one the branes, we call it fundamental one-brane, or F1-brane for short. While the object which magnetically couples with B_2 is called NS5-brane. D(-1)-brane is called D-instanton because it is a "point" in the space-time.

In Eq.(3.28) we derived the D-brane generalized charge conservation law of D-branes, however the action there was simplest case and in full action of type II theory the situation is more complicated. Therefore in this subsubsection, we formulate the full equations of motion and Bianchi identities with sources for p-forms in TypeIIA and TypeIIB theory using the dual-notation of [14]. In this subsubsection we omit the tilde on the p-form field strengthes to simplify the notation.

Type IIA

We start from the effective SUGRA action (3.6). In the discussion here, we neglect the fermion fields for simplicity, but it is known that the same results can be derived even when we deal with the fermion fields.

The equations of motion for RR-forms C_3 and C_1 is given by

$$d(*F_4) = -H_3 \wedge F_4, \quad d(-*F_2) = -H_3 \wedge (*F_4), \tag{3.35}$$

and the Bianch identities are

$$dF_2 = -F_0 \wedge H_3, \quad dF_4 = -H_3 \wedge F_2 \tag{3.36}$$

We rewrite these equations into more useful form. First we define corresponding dual fields

$$F_{10} := *F_0, \quad F_8 := -*F_2, \quad F_6 := *F_4. \tag{3.37}$$

this definition is equivalent to the definition in Eq.(3.21) up to sign (signs are determined for the later convenience.). Then equations of motions and Bianch identities are given by just a single equation;

$$dF_{\rm even} = -H_3 \wedge F_{\rm even}, \quad F_{\rm even} := \sum_{n=0}^{5} F_{2n} \tag{3.38}$$

In $n = 0$ case, we regard the right hand side as zero and obtain $dF_0 = 0$ and thus F_0 is not a dynamical variable but just a constant $F_0 = m$. The duality relation (3.37) can be represented by

$$*F_{\rm even} = -\mathcal{T} F_{\rm even} \tag{3.39}$$

where $\mathcal{T}(\cdots)$ is a transpose of forms defined as

$$\mathcal{T}(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}) := dx^{\mu_n} \wedge \cdots \wedge dx^{\mu_1}. \tag{3.40}$$

We can solve the equations (3.38);

$$F_{\rm even} = e^{-B_2} \wedge (m + dC_{\rm odd}) \tag{3.41}$$

Next let us discuss B_2 and $H_3 := B_2$. Bianch identity is simple,

$$dH_3 = 0, \tag{3.42}$$

and the equation of motion is given by

$$d\left(e^{-2\phi} * H_3\right) = F_0 \wedge *F_2 + *F_4 \wedge F_2 - \frac{1}{2} F_4 \wedge F_4 \tag{3.43}$$

$$\Rightarrow \quad dH_7 = -\frac{1}{2}\left(\mathcal{T} F_{\rm even} \wedge F_{\rm even}\right)|_8, \tag{3.44}$$

in the second equation we defined the dual of H_3 as $H_7 := e^{-2\phi} * H_3$ and $|_8$ means extraction of only 8-forms.

The field strengths defined above are invariant under the following gauge transformation;

$$\delta B_2 = d\Lambda_B, \quad \delta C_{\rm odd} = \Lambda_B \wedge (F_0 + dC_{\rm odd}) + d\Lambda_{\rm even}, \tag{3.45}$$

note that $\delta C_{\rm odd}$ is dependent on Λ_B.

Preliminaries are over. Let us make gauge fields coupled to currents on branes,

$$J_8^{F1}, \quad J_4^{NS5}, \quad J_{\rm odd} := \sum_{n=1}^{5} J_{2n-1}^{D(10-2n)}, \tag{3.46}$$

by adding these currents on the right hand side of Eq.(3.38,3.42,3.44);

$$\begin{cases} dH_3 = J_4^{NS5}, \\ dH_7 = -\frac{1}{2}\left(\mathcal{T}F_{\rm even}\wedge F_{\rm even}\right)|_8 + J_8^{F1}, \\ dF_{\rm even} = -H_3\wedge F_{\rm even} + J_{\rm odd} \end{cases} \tag{3.47}$$

As we do when we derive current conservation of electromagnetism, we take derivatives of these equations and obtain

$$\begin{cases} dJ_4^{NS5} = 0, \\ dJ_8^{F1} = \left(J_{\rm odd}\wedge \mathcal{T}F_{\rm even}\right)|_9, \\ dJ_{\rm odd} = J_4^{NS5}\wedge F_{\rm even} + J_{\rm odd}\wedge H_3. \end{cases} \tag{3.48}$$

The first equation is an ordinary conservation law of NS5 current, which means that NS5-branes cannot have boundaries. The situation is different when it comes to other currents. The second equation shows that F1-branes can have endpoints unless all the D-brane currents are zero. The third equation can be interpreted in a similar way. For example, if we have a D6-brane, the third equation tells us that D4-branes can have boundaries on the D6-brane and the second equation shows that F1-branes can have endpoints on it.

Type IIB

In type IIB theory we have three RR fields C_0, C_2, C_4 instead of $C_{\rm odd}$ in Type IIA, but the discussion is almost the same as that in Type IIA theory. First we define dual fields as

$$F_9 := *F_1, \quad F_7 := *F_3 \tag{3.49}$$

and $F_{\rm odd} := \sum_{n:{\rm odd}} F_n$. Note that the self-duality condition for C_4 is embedded in the equation $*F_{\rm odd} = -\mathcal{T}F_{\rm odd}$. Then the equations of motion and Bianch identities are written as

$$\begin{cases} dH_3 = J_4^{NS5}, \\ dH_7 = -\frac{1}{2}\left(\mathcal{T}F_{\rm odd}\wedge F_{\rm odd}\right)|_8 + J_8^{F1}, \\ dF_{\rm odd} = -H_3\wedge F_{\rm odd} + J_{\rm even}, \end{cases} \tag{3.50}$$

and conservation lows derived from equations above are

$$\begin{cases} dJ_4^{NS5} = 0, \\ dJ_8^{F1} = (J_{\text{even}} \wedge \mathcal{T} F_{\text{odd}})\,|_9, \\ dJ_{\text{even}} = F_{\text{odd}} \wedge J_4^{NS5} - H_3 \wedge J_{\text{even}}. \end{cases} \tag{3.51}$$

The interpretation of these equations are quite the same as the TypeIIA case.

4 Compactification and dimesional reduction

Super string theory is constructed in ten-dimensional space-time but our world seems to be in four-dimensional spacetime as long as all the experiments have shown so far. To handle this mismatch, we suppose our world is depicted by a theory around local minimum in the configuration space where six out of the ten dimensional space is compact and too small to affect the remaining four dimensional theory in which we live, if energy scale is lower than TeV. To consider the string theory around such vacua where 6D space is compact is called **compactification**.

Compactification breaks some of (often all of) the supersymmetry as well as the Poincare symmetry in the 10-dimensional spacetime. However for both theoretical and phenomenological reasons, we would like to possess minimum amount of supersymmetry in four dimensional theory after the compactification. We therefore consider the background solution for compactification focusing on the SUSY preserving condition in subsection 4.1. We will see that we can preserve desired fraction of supersymmetry if we take what is called Calabi-Yau 3-fold, and that whether local sources such as flux, D-branes and O-planes exist or not makes a big difference.

In subsection4.2, we consider the fluctuation around such vacua, figure out the 4D effective matter content, and determine the 4D effective action by integrating out the heavy 6D effects, the procedure of which is called **dimensional reduction**.

In subsection4.3, we concentrate on TypeIIB theory compactified on a Calabi-Yau 3-fold and see its 4D massless fields and effective action.

4.1 Background solutions for compactified TypeII theory

We in this section seek for background solution of compactified TypeII theory. In subsubsection 4.1.1, we focus on SUSY preserving condition and the discussion there is mainly based on the review paper on compactification [16] by M. Grana. In subsubsection 4.1.2, on the other hand, we do not pay much attention to SUSY, and instead exhibit a famous example of warped background solution for TypeIIB with non-vanishing flux and branes, which was studied by Giddings, Kachru and Polchinski in [21].

4.1.1 How many SUSY does compactification preserve?

String theory and its 10D effective SUGRA in flat space-time enjoy supersymmetry. In TypeIIB case, what is there is 10D $\mathcal{N} = (2, 0)$ SUSY (in TypeIIA it is $\mathcal{N} = (1, 1)$). Let us now consider what the fate of this SUSY is when the theory is compactified on a certain compact space X_6 with local sources. Compactification breaks some of the supersymmetry, often all of it. However in order to build realistic and controllable models, we would like to preserve a part of the supersymmetry. In this subsubsection we first consider the conditions for the conservation of supersymmetry. What follows is the discussion on supersymmetric background without and with fluxes. Whether we should really leave supersymmetry unbroken is discussed in the last part of this subsubsection. Until then it is supposed that breaking into 4D $\mathcal{N}=1$ SUSY is the best outcome for phenomenology.

Condition for 4D supersymmetry in TypeIIB

It is known that 4D spacetimes which admit maximal symmetry[6] are Minkovski M_4, de-Sitter dS_4 and anti-de-Sitter AdS_4, and the most general 10D metric consistent with 4D maximal symmetry is given by

$$ds^2 = G_{MN}dx^M dx^N = e^{2A(y)}\widetilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n, \tag{4.1}$$

where $\widetilde{g}_{\mu\nu}$ is either Minkovski, de-Sitter or anti-de-Sitter metric, $g_{mn}(y)$ is an arbitrary 6D metric and the scalar function $A(y)$ on X_6 is called **warp factor**.

The necessary condition for unbroken SUSY is the vanishing vacuum expectation value of SUSY variation of fermion fields;

$$0 = \left\langle \delta_\epsilon \psi^i_{M;A} \right\rangle = (\nabla_M)_A{}^B \epsilon^i_B + \frac{1}{8} H_{MNP}(\Gamma^{NP})_A{}^B (\mathcal{P})^i{}_j \epsilon^j_B + \frac{1}{16} e^\Phi \sum_{n:\mathrm{odd}} [\not{F}_{\mathrm{odd}}|_n \Gamma_M (\mathcal{P}_n \epsilon)^i]_A \tag{4.2}$$

$$0 = \left\langle \delta_\epsilon \lambda^i_A \right\rangle = \left(\not{\partial}\Phi + \frac{1}{2}\not{H}_3 \mathcal{P} \right)_A{}^B \epsilon^i_B + \frac{1}{8} e^\Phi \sum_{n:\mathrm{odd}} (-)^n (5-n) [\not{F}_{\mathrm{odd}}|_n (\mathcal{P}_n \epsilon)^i]_A \tag{4.3}$$

where $\epsilon^{i=1,2}_{A=1\sim 32}$ is 10D SUSY parameter, $\psi^i_{M;A}$ gravitino, λ^i_A dilatino, and $\mathcal{P} = -\sigma^3, \mathcal{P}_n = \sigma^1$ for even $(n+1)/2$ and $\mathcal{P}_n = i\sigma^2$ for odd $(n+1)/2$. Here we omit the fermion terms because the vacuum expectation values of fermions in maximally symmetric 4D space-time vanish. Slashes mean of course Feynman slashes, do not forget the factor $1/n!$ when we contract n indices, i.e. $\not{F}_n := (1/n!)F_{M_1\cdots M_n}\Gamma^{M_1\cdots M_n}$ for n-form field F_n. We have used the democratic formulation as in subsubsection 3.2.3.

[6] A n-dimensional manifold is said to be maximally symmetric if it has the same number of symmetries as ordinary Euclidean space, i.e. $12n(n+1)$ linearly independent Killing vectors. This is equivalent with to be both homogeneous and isotropic. It is also possible to prove that a manifold is maximally symmetric if and only if the Riemann tensor obeys the relation $R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

Supersymmetric background without flux

In the absence of flux, the SUSY condition Eqs.(4.2,4.3) have rather simple forms and we can derive strong necessary conditions for unbroken SUSY. Taking M to be μ in Eq.(4.2), it requires

$$
\begin{aligned}
0 = \nabla_\mu \epsilon_A &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{NP}\Gamma_{NP}\right)\epsilon_A \\
&= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{\nu\rho}\Gamma_{\nu\rho} + \frac{1}{4}\omega_\mu{}^{np}\Gamma_{np}\right)\epsilon_A \\
&= \widetilde{\nabla}_\mu \epsilon_A + \frac{1}{2}\left[(\gamma_\mu\gamma_5)_\alpha{}^\beta \otimes (\Gamma^m\nabla_m A)_a{}^b\right]\epsilon_{(\beta,b)}. \qquad (4.4)
\end{aligned}
$$

In the last equality we have decomposed the 10D spinor parameter A into 4D and 6D correspondences ($\alpha = 1 \sim 4$, $a = 1 \sim 8$). From this we obtain the following integrability condition,

$$
[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu]\epsilon = -\frac{1}{2}(\nabla_m A)(\nabla^m A)\gamma_{\mu\nu}\epsilon. \qquad (4.5)
$$

On the other hand the left-hand side becomes

$$
[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu]\epsilon = \frac{1}{4}R_{\mu\nu\rho\sigma}(\widetilde{g})\gamma^{\rho\sigma}\epsilon = \frac{R(\widetilde{g})}{24}\gamma_{\mu\nu}\epsilon \qquad (4.6)
$$

Then it follows

$$
\frac{R(\widetilde{g})}{12} + \nabla_m A \nabla^m A = 0. \qquad (4.7)
$$

It can be proven that the only possible constant value of $(\nabla A)^2$ on a compact manifold is zero. Therefore we obtain

$$
A(y) = \text{const.}, \quad R(\widetilde{g}) = 0. \qquad (4.8)
$$

The first equality implies that the 10D space-time is nor warped and the second equality means that the 4D space-time is neither dS_4 nor AdS_4 but Minkovski M_4. Thus we can conclude that in the absence of flux, the 4D maximally symmetric background cannot be warped and necessarily Minkowski (in terms of 4D) if it is supersymmetric.

Let us next take $M = m$ in Eq.(4.2).

$$
0 = \nabla_m \epsilon_A = \xi_\alpha^+ \otimes [\nabla_m \eta^+]_a + \xi_\alpha^- \otimes [\nabla_m \eta^-]_a, \qquad (4.9)
$$

in the second equality we have decomposed the TypeIIB SUSY spinor; $\epsilon^{i=1,2}_{A=(\alpha,m)} = \xi^+_\alpha \otimes \eta^+_a + \xi^-_\alpha \otimes \eta^-_a$.[7] Therefore unbroken 4D SUSY requires

$$[\nabla_m \eta^\pm]_a = 0, \tag{4.10}$$

in other words compact space X_6 need to have covariantly constant spinor. This is a stringent constraint on X_6 because general six dimensional manifolds have $SO(6) \simeq SU(4)$ holonomy and thus possesses no covariantly constant spinors and breaks all the supersymmetries. Therefore we need to prepare "good-shaped" 6D manifold as a compact space to preserve some of supersymmetries.

Let us consider T^6 which has trivial holonomy as an example. In this case, we can have full covariantly constant spinors and thus all the SUSY is left unbroken. 10D TypeIIB SUGRA has 2×16 SUSY parameters, so in the language of 4D theory it is a $\mathcal{N} = 8$ SUSY. It is little too many SUSY for a realistic model so we would like to break some of them but preserve some of them. Here comes the Calabi-Yau 3-folds. It allows at most $SU(3)$ holonomy and thus leaves at least one fourth of supersymmetries unbroken. Thus a 4D theory compactified on a general Calabi-Yau 3-fold has $2 \times 16 \div 4 = 2 \times 4$ SUSY parameters and is a $\mathcal{N} = (2, 0)$ SUSY theory.

We summarize the discussion so far. The starting point is to assume the TypeIIB SUGRA on 10D space-time (M_{10}) with 4D maximal symmetry and 6D compactness. Then unbroken SUSY conditions with vanishing fluxes (and no local sources) determine the structure of background space-time as follows;

$$M_{10} = M_4 \times X_6 \tag{4.11}$$

$$ds^2 = G_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n \tag{4.12}$$

where M_4 is 4D Minkovski space-time, X_6 is 6D compact space with vanishing Ricci scalar, often supposed to be Calabi-Yau 3-fold (CY_3) to preserve one fourth of SUSY, and g_{mn} is metric on X_6. Of course this is a solution for the equation of motion of TypeIIB SUGRA action with vanishing flux and local sources, $S^{local} = 0$.

Even in the Calabi-Yau case, $\mathcal{N} = 2$ SUSY (in TypeII theory) is a little too many because the theory cannot contain chiral fermions which are the necessary ingredients for standard model. In the next part, We discuss the compactification with flux and see flux can break $\mathcal{N} = 2$ SUSY down to $\mathcal{N} = 1$ or even completely.

[7] In the TypeIIA case, two SUSY parameters have opposite chirality and thus $\epsilon^{i=1}_{A=(\alpha,m)} = \xi^+_\alpha \otimes \eta^+_a + \xi^-_\alpha \otimes \eta^-_a$ and $\epsilon^{i=2}_{A=(\alpha,m)} = \xi^+_\alpha \otimes \eta^-_a + \xi^+_\alpha \otimes \eta^-_a$ are the appropriate decomposition.

Supersymmetric background under the existence of flux

Once we turn on the fluxes, the conclusion just above can be avoided and warped space-time theories with $\mathcal{N}=1$ SUSY can be constructed. The solutions for Eqs.(4.2, 4.3) are classified well but as one can easily anticipate the derivation is complex mainly because the unbroken SUSY conditions are messy unless fluxes vanish. We just refer the reader to the references [17, 18]. We present one example of warped solutions with flux in the next subsubsection.

Supersymmetric copmactification or non-supersymmetric?

We often consider supersymmetric (4D $\mathcal{N}=1$) compactification since it has several advantages from both bottom-up and top-down point of view as follows:

- Based on the perspective of top-down approach, suppersymmetric compactification is preferred because they are under better control. As an example it can be shown that non-suppersymmetric configurations often contain tachyon and thus unstable, while suppersymmetric configuration does not have tachyon. The other example of theoretical better control is that suppersymmetric theories enjoy massless multiplets and non-renormalizable theorem.

- In the point of view of bottom-up approach, we do not want more than $\mathcal{N}=1$ SUSY in the language of 4D theory because the theory cannot contain chiral fermions which are essential ingredients for standard model. On top of that, the Higgs mass naturalness problem requires supersymmetry breaking at around TeV scale. If there is $\mathcal{N}=1$ SUSY above TeV scale and the running of gauge couplings are controlled by d.o.f. in MSSM's, all the the standard model gauge couplings unify their values at around 10^{16}GeV which strongly suggests the grand unification theory. Therefore we would like to identify the compactification scale M_{KK} with the GUT scale M_{GUT} or some higher scale and keep $\mathcal{N}=1$ SUSY over the TeV scale, i.e. we prefer keeping SUSY at compactification.

However, there are three cases which avoid this choice of scenario and are still possible. (i) The first one is high scale SUSY breaking (not by compactification). This is getting more likely because the collider experiments have never discovered the existence of SUSY. In this case, we need to solve the Higgs naturalness problem in some ways, the anthropic principle can always be a solution though. (ii) The second is low scale SUSY breaking by compactification. In this case the miracle of GUT in MSSM should be abandoned. This possibility is easy to be tested by future experiment because we can see the effect from KK modes or string excitation modes in TeV scale. (iii) The last one is the combination of (i) and (ii), high scale SUSY breaking by compactification. This is a nightmare because we lose the good control of supersymmetric effective theory after the compactification. Besides, Higgs fine-tuning problem need to be solved by some mechanism.

Since there remain these possibilities, it is dangerous to stick to the supersymmetric conpactification at GUT scale and SUSY breaking at TeV scale. Nevertheless, we concentrate on that scenario in this paper (especially in subsection 6.2 which contains a main theme of this paper) as a first step to investigate the string inflation models.

4.1.2 Flux compactification in TypeIIB

In this subsubsection, we provide an example of warped background solution for TypeIIB with non-vanishing flux and branes, based on the original paper [21]. Note that discussion here does not depend on whether SUSY is broken or not.

To get started, recall that the TypeIIB SUGRA action in Einstein frame is given by Eq.(3.12);

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(R - \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\text{Im}\tau)^2} - \frac{g_s}{12\text{Im}\tau} G_3 \cdot \overline{G}_3 - \frac{g_s^2}{4 \cdot 5!} \widetilde{F}_5^2 \right) \\ + S_{\text{CS}} + S_{local}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} S_{\text{CS}} &= \frac{g_s^2}{2\kappa_{10}^2} \int C_4 \wedge \frac{G_3 \wedge \overline{G}_3}{4i\text{Im}\tau}, \\ \widetilde{F}_5 &:= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3, \\ G_3 &:= F_3 - \tau H_3, \quad \tau := C_0 + ie^{-\Phi} \\ \kappa_{10}^2 &= \frac{1}{2} g_s^2 (2\pi)^7 \alpha'^4 = \frac{1}{4\pi} g_s^2 l_s^8, \end{aligned}$$

and S_{local} consists of terms from local sources. Recall also that $\widetilde{F}_5$ is self-dual $*\widetilde{F}_5 = +\widetilde{F}_5$.

We would like to consider the compactification under a metric ansatz which respects 4D maximal symmetry as in the case of Eq.(4.1);

$$ds^2 = G_{MN} dx^M dx^N = e^{2A(y)} g_{\mu\nu}(x) dx^\mu dx^n + e^{-2A(y)} \widetilde{g}_{mn}(y) dy^m dy^n \tag{4.14}$$

Here the notation is slightly different from that in the last subsubsection just for the later convenience. We further set the ansatz for 5-form fields;

$$\widetilde{F}_5 = (1+*)\left[d\alpha(y)\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\right], \tag{4.15}$$

where $\alpha(y)$ is a function of local coordinates y_m on X_6. This is the most general form of 5-form flux which respects 4D maximal symmetry and the self-duality condition $*\widetilde{F}_5 = +\widetilde{F}_5$ and thus its equation of motion, $d*\widetilde{F}_5 = +d\widetilde{F}_5$. The Bianch identity remains to be studied.

g-EoM

Under the ansatz of Eqs.(4.14,4.15), we would like to determine the allowed background solutions for 3-form fluxes and the arbitrary functions $A(y)$ and $\alpha(y)$ in the ansatz by calculating the equations of motion for each field.

Let us first see the equation of motion for metric G_{MN}. It turns out that the equation leads to a strong constraint on the displacement of local sources. Just take the functional derivative with respect to G^{MN} and find 10D Einstein equation

$$R_{MN} - \frac{1}{2}G_{MN}R = \kappa_{10}^2 T_{MN}, \tag{4.16}$$

where T_{MN} is a energy momentum tensor defined as

$$T_{MN} := -\frac{2}{\sqrt{-G}}\frac{\delta S_{IIB}}{\delta G^{MN}}. \tag{4.17}$$

For later convenience we also define

$$\begin{aligned} T_{MN}^{sugra} &:= -\frac{2}{\sqrt{-G}}\frac{\delta}{\delta G^{MN}}\left(S_{NS}+S_R+S_{CS}\right), \\ T_{MN}^{local} &:= -\frac{2}{\sqrt{-G}}\frac{\delta S_{local}}{\delta G^{MN}}. \end{aligned} \tag{4.18}$$

and then the relation $T_{MN} = T_{MN}^{sugra} + T_{MN}^{local}$ is obvious. We take the trace of Eq.(4.16), obtain $R - 5R = \kappa_{10}^2 T \Rightarrow R = -(1/4)\kappa_{10}^2 T$, substitute this back to the original equation and find

$$R_{MN} = \kappa_{10}^2\left(T_{MN} - \frac{1}{8}g_{MN}T\right). \tag{4.19}$$

Let us take the 4D component of Eq.(4.19). After the tedious calculation of R_{MN} and T_{MN}^{sugra}, we obtain

$$(\text{LHS}) = R_{\mu\nu} = -\eta_{\mu\nu}e^{4A}\widetilde{\nabla}^2 A = -\frac{1}{4}\eta_{\mu\nu}\left(\widetilde{\nabla}^2 e^{4A} - e^{-4A}\partial_m e^{4A}\partial^m e^{4A}\right), \tag{4.20}$$

$$(\text{RHS}) = -\eta_{\mu\nu}\left(\frac{G_{mnp}\overline{G}^{mnp}}{48\text{Im}\tau} + \frac{1}{4}e^{-8A}\partial_m\alpha\partial^m\alpha\right) + \kappa_{10}^2\left(T_{\mu\nu}^{local} - \frac{1}{8}\eta_{\mu\nu}T^{local}\right). \tag{4.21}$$

Then it follows that

$$\widetilde{\nabla}^2 e^{4A} = e^{2A}\frac{G_{mnp}\overline{G}^{mnp}}{12\mathrm{Im}\tau} + e^{-6A}\left[\partial_m\alpha\partial^m\alpha + \partial_m e^{4A}\partial^m e^{4A}\right] + \frac{\kappa_{10}^2}{2}e^{2A}\left(T^m_m - T^\mu_\mu\right)^{local}. \quad (4.22)$$

The left-hand side represents how the space is warped, the first term on the right-hand side is the strength of 3-form flux, the second is the combination of 5-form flux and warped factor, and the last term is the effect of local sources. Integration over the compact space X_6 gives us the result;

$$0 = \int_{X_6} e^{2A}\frac{|G_3|^2}{12\mathrm{Im}\tau} + \int_{X_6}\left[(\partial\alpha)^2 + (\partial e^{4A})^2\right] + \frac{\kappa_{10}^2}{2}\int_{X_6} e^{2A}\left(T^m_m - T^\mu_\mu\right)^{local}. \quad (4.23)$$

The first and second terms are non-negative, so if there is no local sources (vanishing third term) then $G_3 \equiv \partial\alpha \equiv \partial A \equiv 0$, which means no flux and no warped factor. Therefore the stress energy tensor form local sources need to be negative if one require for nonzero flux and warped space-time.

Let us calculate the stress energy tensor of a p-brane wrapped on (p-3)-cycle Σ_{p-3} in the compact space. The Dp-brane action in string frame was given in Eq.(3.16) and Eq.(3.20) but we rewrite it in Einstein frame;

$$S_p = -\int_{M_4\times\Sigma_{p-3}} d^{p+1}\sigma\ \mathcal{T}_p\sqrt{-det[G_{ab}+\cdots]} + \mu_p\int_{M_4\times\Sigma_{p-3}} C_{p+1}, \quad (4.24)$$

where $\mathcal{T}_p$ is the Einstein frame tension determined as $\mathcal{T}_p := |\mu_p|g_s e^{(p-3)\widetilde{\Phi}/4} = |\mu_p|g_s^{-(p+1)/4}e^{(p-3)\Phi/4}$ for Dp-brane. Under the ansatz of Eq.(4.14), some calculations yield the following result;

$$(T^m_m - T^\mu_\mu)^{local} = (7-p)\mathcal{T}_p\delta(\Sigma_{p-3}). \quad (4.25)$$

This implies we need negative tension $\mathcal{T}_p < 0$ objects unless we consider $p > 7$. Although the tension of Dp-brane is positive, $\mathcal{T}_p > 0$, it is known that string theory has other objects whose tensions are negative, O-planes. It is also known that the charge of an Op-plane is (-2^{p-5}) times the charge of a Dp-brane; $\mathcal{T}_{Op} = -2^{p-5}\mathcal{T}_p < 0$.

As a conclusion here, the Eq.(4.23) implies that if we have non-vanishing flux or warped metric, negative tension objects are required to exist in the theory.

$\widetilde{\mathbf{F}}_5$-Bianch

Let us next consider the Bianch identity of 5-form field $\widetilde{F}_5$ (recall Eq.(3.50));

$$d\widetilde{F}_5 = H_3 \wedge F_3 + l_s^4\rho_6^3, \quad (4.26)$$

where the last term is 6-form source term from spacially 3-dimensional objects (denoted as J_6 in Eq.(3.50)). Just integrate this over the compact space and obtain

$$Q^3 + \frac{1}{l_s^4}\int_{X_6} H_3 \wedge F_3 = 0, \tag{4.27}$$

where $Q^3 := \int_{X_6} \rho_6^3$ is total D_3-brane charge. This equality says that the total D_3-brane charge and the 3-form flux must be in equilibrium. One can regard this as a constraint on 3-form flux.

Under the ansatz of Eq.(4.15), Eq.(4.26) becomes

$$\widetilde{\nabla}^2\alpha = ie^{2A}\frac{G_{mnp}(*_6 G^{mnp})}{12\mathrm{Im}\tau} + 2e^{-6A}\partial_m\alpha\partial^m e^{4A} + e^{2A}l_s^4\rho^3. \tag{4.28}$$

This is just for the future reference.

BPS-like condition

We have so far considered the $g_{\mu\nu}$-EoM and $\widetilde{F}_5$-Bianch. These two equations and one more condition give us a class of warped background solutions. That one more condition is the following inequality;

$$\frac{1}{4}(T^m_m - T^\mu_\mu)^{local} \geq \mathcal{T}_3\rho^3. \tag{4.29}$$

What are some of the sources which satisfy the condition? D3-branes and O3-planes saturate the inequality. Anti-D3-branes satisfy it but do not saturate. D5-branes satisfy if they are wrapping on collapsed cycles. It is known that D7-branes also satisfy even when the first-order correction terms are taken into account. What does not satisfy the condition includes O5-planes and anti-D3-branes.

Solution

Subtract Eq.(4.28) from Eq.(4.22);

$$\begin{aligned}\widetilde{\nabla}^2(e^{4A}-\alpha) = \frac{e^{2A}}{6\mathrm{Im}\tau}|iG_3 - *_6G_3|^2 + e^{-6A}|\partial(e^{4A}-\alpha)|^2 \\ + 2\kappa_{10}^2 e^{2A}\left(\frac{1}{4}(T^m_m - T^\mu_\mu)^{local} - \mathcal{T}_3\rho^3\right)\end{aligned} \tag{4.30}$$

After integrated over the compact space, this equation leads to strong constraints on the solutions as follows. First, 3-form G_3 is imaginary self dual;

$$*_6G_3 = +iG_3 \tag{4.31}$$

Second, there is a relation between the warp-factor and 5-form flux[8];

$$e^{4A} = \alpha \tag{4.32}$$

Finally, BPS-like condition must be saturated;

$$\frac{1}{4}(T^m_m - T^\mu_\mu)^{local} = \mathcal{T}_3\rho^3 \tag{4.33}$$

What remains to be studied are equations of motion for other fields. g_{mn}-EoM can be extracted from Eq.(4.19);

$$\widetilde{R}_{mn} = \kappa_{10}^2 \left[\frac{\partial_{(m}\tau\partial_{n)}\bar{\tau}}{4(\mathrm{Im}\tau)^2} + \left(\widetilde{T}^7_{mn} - \frac{1}{8}\widetilde{g}_{mn}\widetilde{T}^7 \right) \right] \tag{4.34}$$

The equation of motion for axio-dilaton is

$$\widetilde{\nabla}^2\tau = \frac{\widetilde{\nabla}\tau \cdot \widetilde{\nabla}\bar{\tau}}{i\mathrm{Im}\tau} - \frac{4\kappa_{10}^2(\mathrm{Im}\tau)^2}{\sqrt{-G}}\frac{\delta S^{local}}{\delta\bar{\tau}}. \tag{4.35}$$

The 3-forms must obey the Bianchi identity

$$dF_3 = dH_3 = 0, \tag{4.36}$$

and in this case their equation of motion,

$$d\Lambda + \frac{i}{\mathrm{Im}\tau} d\tau \wedge \mathrm{Re}\Lambda = 0, \quad \Lambda := e^{4A} *_6 G_3 - \alpha G_3 \tag{4.37}$$

is automatically satisfied because $\Lambda \equiv 0$.

Do not forget that α and A must satisfy their equations of motion although one of them is sufficient if the relation $e^{4A} = \alpha$ is maintained. Especially, from the α-EoM, the total D3 charge must vanish;

$$\frac{1}{l_s^4}\int_{X_6} H_3 \wedge F_3 = Q^3 \tag{4.38}$$

Comment on SUSY breaking

We have so far in this subsubsection never considered the SUSY breaking. The above conclusion holds even if SUSY is broken or unbroken. We will here state that the above solution

[8]We have used that α does not have constant by definition, and the constant term in A can be eliminated by 4D coordinate transformation.

leaves supersymmetry under some condition. Imaginary self-dual G_3 generally consists of $H^{(2,1)}$ (primitive) and $H^{(0,3)}$. We would like to state that if G_3 does not contain (0,3)-component (and compact space is Calabi-Yau 3-fold), 4D $\mathcal{N}=1$ SUSY remains unbroken. This statement is confirmed directly by calculating the variation of fermions (Eq.(4.2) for gravitino and Eq.(4.3) for dilatino), but we omit that discussion in this paper. However in subsubsection 4.3.2, we will see that this result is consistent with the calculation in terms of 4D effective field theory.

4.2 4D massless modes and dimensional reduction

In this subsection we turn our attention from background solutions to the fluctuation around them and see what kind of and how many 4D massless fields come out of each 10D field. We assume $M_{10} = M_4 \times X_6$, where M_4 is 4D Minkovski space-time and we take X_6 to be CY_3 to preserve SUSY.

4.2.1 Basic strategy

In this subsubsection we show the basic strategy of how to figure out the 4D massless fields and how to derive their effective action, based on the discussion in [12, 15].

Consider 10D massless field $\Phi(x,y)$ under the control of an action of the form

$$S[\Phi] = \int_{M_4} d^4x \int_{X_6} d^6y \sqrt{-g}\ \overline{\Phi} \Box \Phi. \tag{4.39}$$

Under the ansatz of $M_{10} = M_4 \times X_6$, $\Box$ can be written as $\Box_x + \Delta_y$. To acquire the the 4D effective action, we first expand $\Phi(x,y)$ in the complete set of functions of on X_6;

$$\Phi(x,y) = \sum_k \phi_k(x) \cdot \psi_k(y), \tag{4.40}$$

where $\psi_k(y)$ is orthonormal eigen function of the operator Δ_y;

$$\Delta_y \psi_k(y) = -\lambda_k^2 \psi_k(y), \quad \int_{X_6} d^6y \sqrt{g|_y}\ \overline{\psi}_k(y) \psi_l(y) = \delta_{k,l}. \tag{4.41}$$

Note here that the eigen values of an hermite operator are non-negative. Then the action becomes

$$S = \int d^4x \sqrt{-g|_x} \sum_k \overline{\phi}_k \left(\Box_x - \lambda_k^2 \right) \phi_i. \tag{4.42}$$

Therefore in the viewpoint of 4D theory, there are infinite number of fields $\phi_k(x)$ which have mass of λ_k. As usual we can integrate out the heavier modes than the energy scale one is thinking. Especially the fields corresponding to zero modes are massless and always significant.

Example: S^1 compactification

As a pedagogical example, we consider a 5D theory on $M_4 \times S^1$ and derive its effective action. Especially, we would like to estimate the mass of massive fields concretely. Let $x^{M=0\sim 4} = (x^{\mu=0\sim 3}, y := x^4)$ be a coordinate of the space time $M_4 \times S^1$ and R be a radius of S^1; $y \in [0, R)$. The action is given by

$$\begin{aligned} S[\phi] :&= \int d^4x dy \sqrt{-g} \left[-\partial_M \overline{\phi} \partial^M \phi \right] \\ &= \int_{\mathcal{M}^{(1,3)}} d^4x \sqrt{-g|_x} \int_{S^1} dy \sqrt{g|_y}\, \overline{\phi}(x,y) \left[-\partial_t^2 + \nabla_x^2 + \nabla_y^2 \right] \phi(x,y). \end{aligned} \tag{4.43}$$

In the second equality we have performed partial integral and divided the metric into 4D and S^1 parts. To get the 4D effective action we expand $\phi(x,y)$ in y coordinate;

$$\phi(x,y) = \sum_{n=0}^{\infty} \phi_n(x) \psi_n(y), \tag{4.44}$$

where $\psi_n(y)$ is a complete set of scalar field on S^1 which satisfies

$$\nabla_y^2 \psi_n(y) = -\lambda_n^2 \psi_n, \quad \int_{S^1} dy \overline{\psi}_n(y) \psi_m(y) = \delta_{n,m}. \tag{4.45}$$

Specifically they are $\psi_n(y) = e^{i2\pi n y/R}/\sqrt{2\pi R}$, $\lambda_n = 2\pi n/R$ and the expansion is just a familiar Fourier expansion. Substituting this expanded form into the action, we obtain

$$\begin{aligned} S[\phi] &= \int d^4x dy \sum_{n,m} \overline{\phi}_n(x) \overline{\psi}_n(y) \left[-\partial_t^2 + \nabla_x^2 - \left(\frac{2\pi m}{R} \right)^2 \right] \phi_m(x) \psi_m(y) \\ &= \sum_{n=0}^{\infty} \int d^4x\, \overline{\phi}_n(x) \left[-\partial_t^2 + \nabla_x^2 - \left(\frac{2\pi n}{R} \right)^2 \right] \phi_n(x). \end{aligned} \tag{4.46}$$

The effective mass of $\phi_n(x)$ is $m_n = 2\pi n/R$ and thus integrating out $\phi_{n\geq 1}$ produces effective theory whose energy scale is lower than $M_{\mathrm{KK}} := 2\pi/R$. This energy scale is called **Kaluza-Klein scale**.

This is the procedure of **dimensional reduction**. In short, the procedure is to expand fields in eigen functions in the compact space, to perform the internal space integration and to integrate out the effect of heavy fields. In the case of free theory as above, we can just neglect the heavier field. On the other hand in the case of interacting theory, we need to of course be careful not to forget the interactions among lighter fields or corrections to them which are induced by heavy fields. However we often just truncate the heavy fields and do not actually integrate out and this operation is called **dimensional truncation**. Is this correct? If the interaction between light mode ϕ_l and heavy mode ϕ_h is of the form $\phi_l\phi_h^2$ or $\phi_l^2\phi_h$, then induced interaction is suppressed by the mass of heavy mode. Such effects are called **Kaltza-Klein correction** and should be studied but basically small enough. We encounter danger when there is interaction of the form $\phi_l\phi_h$. However, 10D theory we start from often consists of only massless fields and thus such interaction does not exist from the very first.

What follows is investigation of 4D massless fields from every kinds of 10D fields which are relevant to the subsequent sections.

4.2.2 scalar field

This is the simplest case. One 10D scalar field $\Phi(x,y)$ yields one 4D scalar field $\phi(x)$ because harmonic function on compact space X_6 is only constant;

$$\Phi(x,y) = \sum_k \phi_k(x) \cdot \psi_k(y) = \phi(x) \cdot 1 + (\text{massive}) \tag{4.47}$$

That is all for the compactification of scalar field.

4.2.3 spinor field

10D spinors can be expanded as follows;

$$(16) \to (2,4) + (2',\bar{4}), \quad \Psi_A(x,y) = \psi_{L,\alpha}(x) \cdot \chi_a(y) + \psi_{R,\alpha(x)} \cdot \chi_a(y) \tag{4.48}$$

Thus 10D spinors yield 4D spinors, more precisely, $\dim\ker \not{D}_6$ left-handed spinors and $\dim\ker \not{D}_6^\dagger$ right-handed spinors. We do not pay much attention to fermion fields because there remains at least one SUSY in Calabi-Yau compactification.

There is one thing notable. The index theorem tells us about the 4D chirality;

$$\mathrm{ind}\not{D}_6 = \dim\ker \not{D}_6 - \dim\ker \not{D}_6^\dagger. \tag{4.49}$$

However we do not discuss this topic in this paper. See for example [15] for more.

4.2.4 p-form field

Consider 10D p-form field A_p and the field strength $F_{p+1} := dA_p$ with an action

$$S[A_p] = -\frac{1}{2}\int_{M_{10}} F_{p+1} \wedge *F_{p+1}. \tag{4.50}$$

Do not forget that this theory has gauge symmetry; $A_p \to A_p + d\Lambda_{p-1}$. The equation of motion is

$$d^* dA_p = 0. \tag{4.51}$$

If we take a gauge choice of $d^* A_p = 0$, the equation of motion can be written as[9]

$$\Delta_{10} A_p = 0, \quad \Delta_{10} := dd^* + d^* d. \tag{4.52}$$

Under the assumption that $M_{10} = M_4 \times X_6$, Δ_{10} can be decomposed as

$$\Delta_{10} = \Delta_4 + \Delta_6. \tag{4.53}$$

Let us next expand A_p in (n,m)-forms[10] $\alpha_{(n,m)}$ in compact space X_6;

$$A_p(x,y) = \sum_i \sum_{n,m} a^i_{p-n-m}(x) \wedge \alpha^i_{(n,m)}(y) \tag{4.54}$$

The 4D (p-n-m)-form field a^i_{p-n-m} is massless if and only if $\alpha^i_{(n,m)}(y)$ is a harmonic form; $\Delta_6 \alpha^i_{(n,m)} = 0$. Hodge decomposition law tells us that harmonic forms are in one to one correspondence with cohomology group ($\mathrm{Harm}^{(n,m)}_{\bar{\partial}}(X_6) \simeq H^{(n,m)}_{\bar{\partial}}(X_6)$), and thus the number of 4D massless (p-n-m)-form is $h^{n,m}(X_6)$. (Note that $(n', m') \neq (n, m)$ can also generate 4D (p-n-m)-form if $n' + m' = n + m$.)

4.2.5 Rarita-Schwinger field

There are two types of decomposition in terms of the representation of $SO(1,3)$

$$\Psi_{MA}(x,y) = \begin{cases} \psi_{\mu\alpha}(x) \cdot \chi_a(y) \\ \psi_\alpha(x) \cdot \chi_{ma}(y) \end{cases} \tag{4.55}$$

[9] This is a generalization of Lorentz gauge in 4D; $\partial^\mu A_\mu = 0$. Under this gauge choice, the equation of motion, $\partial^\mu F_{\mu\nu} = 0$ can be written as $\partial^\nu \partial_\nu A_\mu = 0$.

[10] Here we assume X_6 to be a complex manifold and thus q-forms on it can be decompose into (q-r,r)-forms.

As for the first type, 4D gravitinos are generated if there are killing spinors on X_6, $\not{D}_6\chi_a = 0$. In $X_6 = CY_3$ case, there is one killing spinor and thus one 10D gravitino yields one 4D gravitino. In the second type of decomposition, we obtain 4D spinor as many as zero modes of the form $\not{D}_6\chi_{ma} = 0$. As in the spinor case, we do not study this type much because supersymmetry tells us that what we need to do is only investigating the bosonic sector.

4.2.6 spin2-field

Candelas and delaOssa [19] showed that the metric moduli in Calabi-Yau compactification can be classified into two parts; $h^{1,1}$ Kähler moduli which correspond to the deformation of $g_{i\bar{j}}$ and $h^{2,1}$ complex structure moduli corresponding to the deformation of g_{ij} (and $g_{\bar{i}\bar{j}}$). We review on this in this subsubsection.

Our starting point is an Einstein Hilbert action for 10D metric field $G_{MN}(x, y)$;

$$S[G_{MN}] = \frac{1}{2\kappa_{10E}^2} \int d^{10}x\sqrt{-G}\ R(G), \quad \kappa_{10E}^2 = \frac{1}{2} g_s^2 (2\pi)^7 \alpha'^4 = \frac{1}{4\pi} g_s^2 l_s^8. \tag{4.56}$$

We consider fluctuation around $M_4 \times CY_3$ background

$$G_{MN}(x, y) = \begin{cases} G_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \\ G_{ij} = g_{ij} + \delta g_{ij} \\ G_{i\bar{j}} = g_{i\bar{j}} + \delta g_{i\bar{j}} \end{cases} \tag{4.57}$$

Here we have neglected the non-diagonal fluctuation like $\delta g_{\mu i}$ roughly because the CY_3 has no harmonic 1-form and thus such fluctuations end up with massive states.

Since $\delta g_{\mu\nu}(x, y)$ is a scalar field in terms of the compact space, the corresponding massless state is just a single 4D graviton $\delta g_{\mu\nu}(x)$, with action of the form;

$$\begin{aligned} S &= \frac{1}{2\kappa_{10E}^2} \int d^4x \int_{X_6} d^6y\sqrt{-G}\ R(G) \\ &= \frac{1}{2\kappa_{10E}^2} \int d^4x\sqrt{-g} \int_{X_6} d^6y\sqrt{g_{|y}}\ R(G) + \cdots \\ &\supset \frac{1}{2\kappa_{10E}^2} \int d^4x\sqrt{-g} R(g_{\mu\nu}) \int_{X_6} d^6y\sqrt{g_{|y}} + \cdots \\ &= \frac{1}{2\kappa_{10E}^2} \int d^4x\sqrt{-g}\ \mathcal{V}_6 R(g_{\mu\nu}) + \cdots\ , \end{aligned} \tag{4.58}$$

here $g_{\mu\nu}(x) := \eta_{\mu\nu} + \delta g_{\mu\nu}(x)$, and $\cdots$ represents interactions including massive states. In the second equality, we have split $\sqrt{-G}$ into the product of $\sqrt{-g}$ and $\sqrt{g|_y}$ where g and $g|_y$

is a determinant of matrices restricted to 4×4 and 6×6 components of the original G_{MN} respectively. Here we have omitted two types of terms; fluctuation of off-diagonal modes and y-dependence of $\sqrt{-g}$, both of which include massive states. When moving on to the third line, we neglected kinetic terms for metrics on X_6, that is $R(g|_y)$, which we will investigate soon, and we thus used $\supset$. In the last equality we have defined

$$\mathcal{V}_6 := \int_{X_6} d^6 y \sqrt{g|_y}, \tag{4.59}$$

which stand for the volume of the compact space X_6. One need here to be careful that $\mathcal{V}_6$ may depend on x and the x-dependence does have 4D massless fields and thus we cannot neglect it easily. Therefore we have to move from this frame (what we called 4D **reduction frame** in [1]) to 4D **Einstein frame**;

$$\frac{\mathcal{V}_6}{\langle \mathcal{V}_6 \rangle} g_{\mu\nu} =: g^{4E}_{\mu\nu} \tag{4.60}$$

Then we finally get 4D Einstein-Hilbert term,

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4 x \sqrt{-g^{4E}}\; R(g^{4E}_{\mu\nu}) \tag{4.61}$$

here we defined (4D) Planck scale;

$$M_{\rm Pl}^2 := \kappa_{10E}^{-2} \langle \mathcal{V}_6 \rangle = \frac{4\pi \widetilde{\langle \mathcal{V}_6 \rangle}}{g_s^2 l_s^2}, \qquad \widetilde{\langle \mathcal{V}_6 \rangle} := \langle \mathcal{V}_6 \rangle / l_s^6. \tag{4.62}$$

One 10D graviton gives birth to a 4D graviton whose coupling is determined by the vacuum expectation value of the size of the internal space.

complex structure moduli

Let us next focus on the deformation of $\delta g_{\bar{i}\bar{j}}$. It can be shown that the corresponding massless fields are $h^{2,1}$ complex scalars $z^{\alpha = 1 \sim h^{2,1}}$ defined as

$$\delta g_{\bar{i}\bar{j}}(x,y) = -\frac{1}{|\Omega|^2} \overline{\Omega}_{\bar{i}}{}^{kl} \chi_{\alpha,kl\bar{j}}(y) \cdot z^\alpha(x) + (\text{massive}), \quad |\Omega|^2 := \frac{i}{\mathcal{V}_6} \int_{X_6} \Omega \wedge \overline{\Omega} \tag{4.63}$$

where Ω is a holomorphic 3-form on X_6 and $\chi_{\alpha,ij\bar{k}}$ is a coordinate form of the basis of the harmonic (2,1)-forms χ_α on X_6;

$$\chi_\alpha = \frac{1}{2} \chi_{\alpha,ij\bar{k}} dy^i \wedge dy^j \wedge d\bar{y}^{\bar{k}}. \tag{4.64}$$

The action is given by

$$
\begin{aligned}
S &\supset \frac{1}{2\kappa_{10E}^2}\int d^4x\sqrt{-g}\int d^6y\sqrt{g_6}g^{i\bar{l}}g^{j\bar{k}}\partial^\mu(\delta g_{ij})\partial_\mu(\delta g_{\bar{k}\bar{l}}) \\
&= -\frac{i}{2\kappa_{10E}^2}\int d^4x\sqrt{-g}\partial_\mu(z^\alpha)\partial^\mu(\bar{z}^{\bar{\beta}})\frac{2}{|\Omega|^2}\int_{X_6}\chi_\alpha\wedge\overline{\chi}_{\bar{\beta}}, \\
&= -\frac{\mathcal{V}_6}{\kappa_{10E}^2}\int d^4x\sqrt{-g}\left(-\frac{\int\chi_\alpha\wedge\overline{\chi}_{\bar{\beta}}}{\int\Omega\wedge\overline{\Omega}}\right)\partial_\mu(z^\alpha)\partial^\mu(\bar{z}^{\bar{\beta}}) \\
&= -M_{\rm Pl}^2\int d^4x\sqrt{-g}\ G_{\alpha\bar{\beta}}\partial_\mu(z^\alpha)\partial^\mu(\bar{z}^{\bar{\beta}}),
\end{aligned}
\tag{4.65}
$$

in the first equality, we extracted only massless modes using Eq.(4.63), and in the last equality we defined $G_{\alpha\bar{\beta}}$ as

$$
G_{\alpha\bar{\beta}} := -\frac{\int_{X_6}\chi_\alpha\wedge\overline{\chi}_{\bar{\beta}}}{\int_{X_6}\Omega\wedge\overline{\Omega}}
\tag{4.66}
$$

In the Calabi-Yau compactification 4D effective theory should have at least one SUSY and thus there must be a corresponding Kähler potential. It is found to be

$$
K = -\ln\left[-i\int_{X_6}\Omega\wedge\overline{\Omega}\right].
\tag{4.67}
$$

One can easily check $G_{\alpha\bar{\beta}} = \partial_\alpha\bar{\partial}_{\bar{\beta}}K$ using the relation

$$
\frac{\partial\Omega}{\partial z^\alpha} = k_\alpha(z,\bar{z})\Omega + \chi_\alpha.
\tag{4.68}
$$

Kähler moduli

Consider next the deformation of $\delta g_{i\bar{j}}$. One can see that the corresponding massless fields are $h^{1,1}$ real scalar fields $v^{a=1\sim h1,1}(x)$ defined as

$$
\delta g_{i\bar{j}}(x,y) = v^a(x)\cdot(\omega_a)_{i\bar{j}}(y) + (\text{massive})
\tag{4.69}
$$

where ω_a is a basis of harmonic (1,1)-form on X_6.

The action is determined as follows;

$$
\begin{aligned}
S &\supset \frac{1}{2\kappa_{10E}^2}\int d^4x\sqrt{-g}\int d^6y\sqrt{g|_y}g^{i\bar{l}}g^{k\bar{j}}\partial^\mu(\delta g_{i\bar{j}})\partial_\mu(\delta g_{k\bar{l}}) \\
&= \frac{1}{2\kappa_{10E}^2}\int d^4x\sqrt{-g}\left(\frac{3}{2}\int_{X_6}\omega_a\wedge *\omega_b\right)\partial_\mu v^a\partial^\mu v^b \\
&= -\frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\ G_{ab}\partial_\mu v^a\partial^\mu v^b,
\end{aligned}
\tag{4.70}
$$

In the last equality we have defined metric G_{ab} as

$$G_{ab} = \frac{3}{2\mathcal{V}_6}\int_{X6} \omega_a \wedge *\omega_b = -\frac{3}{2}\left(\frac{\mathcal{K}_{ab}}{\mathcal{K}} - \frac{3}{2}\frac{\mathcal{K}_a\mathcal{K}_b}{\mathcal{K}^2}\right) \tag{4.71}$$

where

$$\begin{aligned} \mathcal{K}_{abc} &:= \int \omega_a \wedge \omega_b \wedge \omega_c, \quad \mathcal{K}_{ab} := \int \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc}v^c, \\ \mathcal{K}_a &:= \int \omega_a \wedge J \wedge J = \mathcal{K}_{abc}v^b v^c, \quad \mathcal{K} := \int_{X6} J \wedge J \wedge J = \mathcal{K}_{abc}v^a v^b v^c. \end{aligned} \tag{4.72}$$

Here J is a Kähler form on X_6, $J = v^a\omega_a = g_{i\bar{j}}dy^i d\bar{y}^{\bar{j}}$, and in the second equality in Eq.(4.71) we used the following relation

$$\mathcal{V}_6 = \int_{X_6} \sqrt{g|_y} d^6 y = \frac{1}{3!}\int_{X6} J \wedge J \wedge J = \frac{1}{6}\mathcal{K} \tag{4.73}$$

In sharp contrast to complex structure moduli, Kähler moduli are not complex scalar but real scalar as 4D fields and hence the metric cannot be expressed by Kähler potential without combining them with their complex partner. To do so, we need to wait until the next subsection.

4.3 Dimensional reduction in TypeII theory

In this subsection, we summarize the 4D massless spectrum in TypeIIB theory compactified on Calabi-Yau 3-fold, following the review paper by Grimm and Louis [20]. In subsubsection 4.3.1 we investigate the massless fields without local sources or orientifolding, and superpotential for moduli in subsubsection 4.3.2. Massless spectrum in the case of the orientifold compactification is studied in subsubsection 4.3.3.

4.3.1 4D massless modes in TypeIIB compactified on CY_3

Before getting started, we summarize our notation for the basis of harmonic forms, i.e. cohomology group in the table below;

cohomology group	dimension	basis
$H^{(1,1)}$	$h^{1,1}$	$\omega_{a=1\sim h^{1,1}}$
$H^{(2,1)}$	$h^{2,1}$	$\chi_{\alpha=1\sim h^{2,1}}$
$H^{(2,1)}$	$h^{2,1}$	$\overline{\chi}_{\bar{\alpha}=1\sim h^{2,1}}$
$H^{(1,1)}$	$h^{1,1}$	$\overline{\omega}^{a=1\sim h^{1,1}}$
$H^{(3,0)} \oplus H^{(2,1)}$	$1+h^{2,1}$	$\alpha_{K=0\sim h^{2,1}}$
$H^{(2,1)} \oplus H^{(0,3)}$	$h^{2,1}+1$	$\beta^{K=0\sim h^{2,1}}$

(4.74)

The basis of $H^{(3)}$ are taken to be symplectic;

$$\int_{X_6} \alpha_K \wedge \beta^L = \delta_K^L \tag{4.75}$$

Now, let us see the massless spectrum of TypeIIB theory compacftified on CY_3. First we consider NSNS-sector, which consists of graviton G_{MN}, Kalb-Ramond field B_{MN} and dilaton Φ. From what we have learned in the last subsection, their 4D massless fields can be read of as follows;

$$\begin{aligned}
&\Phi(x,y) = \phi(x) \\
&G_{MN}(x,y) = \begin{cases} g_{\mu\nu}(x,y) = g_{\mu\nu}(x) \\ g_{i\bar{j}}(x,y) = iv^a(x)\cdot(\omega_a)_{i\bar{j}}(y) \\ g_{ij}(x,y) = i\bar{z}^{\bar{\alpha}}(x)\cdot\left(\dfrac{(\overline{\chi}_{\bar{\alpha}})_{i\bar{k}\bar{l}}\Omega^{\bar{k}\bar{l}}{}_j}{|\Omega|^2}\right)(y) \end{cases} \\
&B_2(x,y) = B_2(x) + b^a(x)\cdot\omega_a(y)
\end{aligned} \tag{4.76}$$

Φ spews out one scalar field ϕ, 10D graviton gives birth to one 4D graviton $g_{\mu\nu}$, $h^{1,1}$ real scalar v^a (Kähler moduli) and $h^{2,1}$ complex scalar z^a (complex structure moduli). Kalb-Ramond field yields one 4D 2-form B_2 which is dual to one real scalar field in 4D, and $h^{1,1}$ real scalar field b^a.

Let us next list up the 4D massless fields from RR-sector in TypeIIB theory.

$$
\begin{aligned}
&C_0(x,y) = C_0(x) \\
&C_2(x,y) = C_2(x) + \ c^a(x)\cdot\omega_a(y) \\
&C_4(x,y) = C_4(x) + \widetilde{\rho}_2^a(x)\cdot\omega_a(y) + \widetilde{V}_{1K}(x)\cdot\beta^K(y) + V_1^K(x)\cdot\alpha_K(y) + \rho_a(x)\cdot\overline{\omega}^a(y)
\end{aligned}
\tag{4.77}
$$

We skip the explanation because it is self-explanatory, except for the self-duality of $C_4(x,y)$. Self-duality of $C_4(x,y)$ eliminates half the degree of freedom in the equation above. We choose V_1^K and ρ_a as surviving fields which are dual of $\widetilde{V}_{1K}$ and $\widetilde{\rho}_2^a$ respectively. Note that C_4 is a not dynamical variable in 4-dimensional theory.

These massless fields must constitute 4D $\mathcal{N}=2$ SUGRA. Corresponding multiplets are summarized in the following table;

gravity multiplet	1	$(g_{\mu\nu}, V_1^0)$
vector multiplet	$h^{2,1}$	$(V_1^\alpha, z^\alpha, \bar{z}^{\bar{\alpha}})$
hypermultiplet	$h^{1,1}$	(v^a, b^a, c^a, ρ_a)
tensor multiplet	1	(B_2, C_2, ϕ, C_0)

(4.78)

4.3.2 Superpotential for moduli

Here we will derive the superpotential for complex structure moduli induced by the 3-form flux in TypeIIB, which plays the crucial roll when we discuss the moduli stabilization in section 5.

To begin, let us recall that TypeIIB SUGRA action Eq.(4.13) contains a term

$$
S_{\mathrm{IIB}} \supset -\frac{g_s}{2\kappa_{10E}^2}\int d^4x\sqrt{-g}\int d^6y\sqrt{g|_y}\,\frac{G_{mnl}\overline{G}^{mnl}}{12\mathrm{Im}\tau} \tag{4.79}
$$

We define

$$
G_3^\pm := \frac{1}{2}\left(G_3 \pm i *_6 G_3\right), \tag{4.80}
$$

then it follows

$$
G_3 = G_3^+ + G_3^-, \quad *_6 G_3^\pm = \mp i G_3^\pm, \tag{4.81}
$$

which means G_3 splits into anti-imaginary self dual (AISD) part G_3^+ and imaginary self dual (ISD) part G_3^-. This notation changes the form of the above action as

$$S_{\rm IIB} \supset \int d^4x\sqrt{-g}\left[-\frac{-g_s}{2\kappa_{10E}^2{\rm Im}\tau}\int_{X_6} G_3^+\wedge *_6\overline{G_3^+} - \frac{-g_s}{4\kappa_{10E}^2{\rm Im}\tau}\int_{X_6} G_3\wedge\overline{G}_3\right]. \tag{4.82}$$

The second term is topological, and the first term is the potential term we have been looking for[11];

$$\begin{aligned} V &= \frac{-g_s}{2\kappa_{10E}^2{\rm Im}\tau}\int_{X_6} G_3^+\wedge *_6\overline{G^+}_3 \\ &= \frac{ig_s}{2\kappa_{10E}^2{\rm Im}\tau}\frac{\int G_3\wedge\overline{\Omega}\int\overline{G}_3\wedge\Omega + G^{\alpha\bar\beta}\int G_3\wedge\chi_\alpha\int\overline{G}_3\wedge\overline{\chi}_{\bar\beta}}{\int\Omega\wedge\overline{\Omega}} \end{aligned} \tag{4.83}$$

In the second equality, we have expanded G_3^+ by Ω and $\overline{\chi}_{\bar\alpha}$[12].

It can be shown by direct calculation [21] that this potential can be derived by substituting the Kähler potential and what is called Gukov-Vafa-Witten superpotential [22]

$$W_{\rm GVW} = c\int G_3\wedge\Omega, \tag{4.84}$$

into the potential formula, $V = e^{K/M_{\rm Pl}^2}(K_{i\bar j}^{-1}|D_iW|^2 - 3|W|^2/M_{\rm Pl}^2)$.

The overall coefficient c is not discussed much in the literatures except for [23], but it is desirable to know the value as precisely as possible for the quantitative discussion on phenomenology. It is easily determined just by the direct calculation. Here we will write down only a part of the calculation which is sufficient for the determination of c; the potential formula above contains a term of the form

$$\begin{aligned} V &\supset e^{K/M_{\rm Pl}^2}G^{\alpha\bar\beta}(\partial_\alpha W)(\overline{\partial_{\bar\beta}W}) \\ &\supset \frac{e^{K_0/M_{\rm Pl}^2}|c|^2/M_{\rm Pl}^2}{-i\int\Omega\wedge\overline{\Omega}\left\langle\widetilde{\mathcal{V}}_6\right\rangle^2 2{\rm Im}\tau}G^{\alpha\bar\beta}\int G_3\wedge\chi_\alpha\int\overline{G}_3\wedge\overline{\chi}_{\bar\beta}, \end{aligned} \tag{4.85}$$

where ∂_α stands for the derivative with respect to the canonicalized complex structure moduli $z^\alpha\times M_{\rm Pl}$ which has mass dimension one (see also Eq.(4.65)), and

$$K = -2M_{\rm Pl}^2\ln\left[\widetilde{\mathcal{V}}_6\right] - M_{\rm Pl}^2\ln\left[-i\int\Omega\wedge\Omega\right] - M_{\rm Pl}^2\ln\left[-i(\tau-\bar\tau)\right] + K_0. \tag{4.86}$$

---

[11]We have to be careful that we need to move to 4D Einstein frame from reduction frame (recall Eq.(4.60); $\frac{\mathcal{V}_6}{\langle\mathcal{V}_6\rangle}g_{\mu\nu} =: g_{\mu\nu}^{4E}$). However we do not do that apparently because it does not change the result.

[12]Note here that Ω and $\bar\chi^\alpha$ are ISD and $\overline{\Omega}$ and χ_α are AISD, and that Ω is (3,0)-form but is different from the basis of (3,0)-form $\alpha_{K=0}$ because the relation Eq.(4.68) has to be kept.

To determine the coefficient c, we would like to compare Eq.(4.85) with the second term in Eq.(4.83). Before that we need to pay attention to mass dimensions. We adopt the convention that the $\int_X G \wedge \Omega$ part in (4.84) has been formulated completely dimensionless, using ℓ_s to render the Calabi–Yau coordinates y^m dimensionless $\underline{y}^m$. The quantization condition

$$\frac{1}{\ell_s^2}\int F_3 \in \mathbf{Z}, \qquad \frac{1}{\ell_s^2}\int H_3 \in \mathbf{Z} \tag{4.87}$$

in the literature like [21] therefore turns into $\int F^{(3)} = n \in$ and $\int H^{(3)} = m \in$ in the dimensionless coordinate setting. Thus, $\int_X G \wedge \Omega = \sum_i (n_i - \tau m_i)\Pi_i$, with integer flux quanta $n_i, m_i \in$ and period integrals Π_i's. In this case, the potential derived in Eq.(4.83) must be rewritten as

$$V = \frac{i g_s l_s^4}{2\kappa_{10E}^2 \mathrm{Im}\tau} \frac{\int G_3 \wedge \overline{\Omega} \int \overline{G}_3 \wedge \Omega + G^{\alpha\bar{\beta}} \int G_3 \wedge \chi_\alpha \int \overline{G}_3 \wedge \overline{\chi}_{\bar{\beta}}}{\int \Omega \wedge \overline{\Omega}}. \tag{4.88}$$

The difference is only the factor of l_s^4 to make the internal space dimensionless.

Now we are good to compare Eq.(4.85) with Eq.(4.88) and obtain the desired result;

$$|c|^2 = e^{-K_0/2M_{\mathrm{Pl}}^2} \frac{g_s M_{\mathrm{Pl}}^2}{\kappa_{10E}^2} \left\langle \widetilde{\mathcal{V}}_6 \right\rangle^2 = e^{-K_0/2M_{\mathrm{Pl}}^2} \frac{g_s^3 M_{\mathrm{Pl}}^6}{4\pi}, \tag{4.89}$$

here we used the result of Eq.(4.62), $M_{\mathrm{Pl}}^2 := \kappa_{10E}^{-2} \left\langle \widetilde{\mathcal{V}}_6 \right\rangle l_s^6 = \frac{4\pi \widetilde{\langle \mathcal{V}_6 \rangle}}{g_s^2 l_s^2}$, $\widetilde{\langle \mathcal{V}_6 \rangle} := \langle \mathcal{V}_6 \rangle / l_s^6$. Note that c has mass dimension 3 as it should be in our notation and depends on the choice of Käler potential for which we defined K_0.

SUSY breaking

We will here discuss the SUSY preserving condition deduced from the Gukov-Vafa-Witten superpotential Eq.(4.84). Simple calculation gives

$$D_\alpha W = c \int G_3 \wedge \chi_\alpha, \quad D_\tau W = \frac{-1}{\tau - \bar{\tau}} \int \overline{G}_3 \wedge \Omega \tag{4.90}$$

Thus, the necessary condition for unbroken SUSY is that G_3 is imaginary self-dual, which have been already imposed in the background solution discussed in subsection 4.1.2.

In the case of no-scale model in which superpotential does not depend on Kähler moduli, one SUSY preserving condition is added;

$$D_\rho W \propto W = c \int G_3^- \wedge \Omega. \tag{4.91}$$

It says that $G_3 = G_3^-$ must be (2,1)-primitive for supersymmetric vacua, that is vanishing (0,3)-component is required.

4.3.3 TypeII theory on Calabi-Yau orientifolds

Type II string theory compactified on Calabi-Yau 3-fold yields 4D $\mathcal{N}=2$ SUGRA action in general. Orientifolds break some of supersymmetry and we obtain $\mathcal{N}=1$ SUGRA action. Therefore orientifolding model is preferred in the context of string phenomenology, and we will see the surviving 4D massless fields in TypeIIB orientifold in this subsubsection.

Orientifold projection is to project out the states whose $(-)^{F_L}\Omega_p\sigma$ eigen values are negative, where F_L is the fermion number in the left-moving sector, Ω_p is the world sheet parity transformation operator and σ is a internal symmetry which acts on the compact space and is a involution ($\sigma^2 = id$). The actions of $(-)^{F_L}$ and Ω_p to the TypeIIB massless fields are summarized in the table below;

	Φ	G	B	C_0	C_2	C_4
$(-)^{F_L}$	$+$	$+$	$+$	$-$	$-$	$-$
Ω_p	$+$	$+$	$-$	$-$	$+$	$-$

(4.92)

As for the left-moving fermion number $(-)^{F_L}$, the sign is positive for the NSNS-sector and negative for the RR-sector. This is because the vacuum state of NS-sector is fermionic and the massless states are first excited states raised by fermionic operators, while the massless states of RR-sector are vacuum states which are fermionic too. The signs by Ω_p are determined by whether the left-sector and right-sector of the state is symmetric or anti-symmetric. A heads up is needed since the action of Ω_p to the RR-sectors generates an additional -1 factor due to the fermionic property of the states.

The action of σ is a little more complicated to explain. The set of fixed points under σ are called O_p-planes, where p is the dimension of the set as a manifold. In order not to break the 4D space-time symmetry, O_p-plane must span the whole 4D space-time and thus wrap a (p-3)-cycle on the internal space, just like D-branes.

A set of O_p-planes one can put in the theory are limited and that limit depends on the theory. Here is the summary of the possible orietifolds;

$$\begin{cases} \text{Type IIA with O6-plane:} & \sigma J = -J, \quad \sigma\Omega = \bar{\Omega}, \\ \text{Type IIB with O3/O7-plane:} & \sigma J = J, \quad \sigma\Omega = -\Omega, \\ \text{Type IIB with O5/O9-plane:} & \sigma J = J, \quad \sigma\Omega = \Omega. \end{cases} \tag{4.93}$$

In TypeIIB theory, the involution σ is holomorphic ($\sigma J = J$). Therefore $H^{(p,q)}$ is decomposed into $H^{(p,q)}_+ \oplus H^{(p,q)}_-$ according to the eigen value of σ. Here is the table of corresponding basis.

cohomology group	dimension	basis	cohomology group	dimension	basis
$H_+^{(1,1)}$	$h_+^{1,1}$	$\omega_{\hat{a}}$	$H_-^{(1,1)}$	$h_-^{1,1}$	$\omega_{\check{a}}$
$H_+^{(2,1)}$	$h_+^{2,1}$	$\chi_{\hat{\alpha}}$	$H_-^{(2,1)}$	$h_-^{2,1}$	$\chi_{\check{\alpha}}$
$H_+^{(2,1)}$	$h_+^{2,1}$	$\overline{\chi}_{\hat{\overline{\alpha}}}$	$H_-^{(2,1)}$	$h_-^{2,1}$	$\overline{\chi}_{\check{\overline{\alpha}}}$
$H_+^{(1,1)}$	$h_+^{1,1}$	$\overline{\omega}^{\hat{a}}$	$H_-^{(1,1)}$	$h_-^{1,1}$	$\overline{\omega}^{\check{a}}$

(4.94)

Notice that basis of positive eigen value is labeled by indices with hat $\hat{}$, while negative is labeled by checked $\check{}$ indices.

The table below is the summary of surviving 4D massless fields;

	O3/O7		O5/O9	
gravity multiplet	1	$g_{\mu\nu}$	1	$g_{\mu\nu}$
vector multiplet	$h_+^{2,1}$	$V_1^{\hat{\alpha}}$	$h_-^{2,1}$	$V_1^{\check{\alpha}}$
chiral multiplet	$h_-^{2,1}$	$z^{\check{\alpha}}$	$h_+^{2,1}$	$z^{\hat{\alpha}}$
	$h_+^{1,1}$	$(v^{\hat{a}}, \rho_{\hat{a}})$	$h_+^{1,1}$	$(v^{\hat{a}}, c^{\hat{a}})$
	$h_-^{1,1}$	$(b^{\check{a}}, c^{\check{a}})$	$h_-^{1,1}$	$(b^{\check{a}}, \rho_{\check{a}})$
	1	(ϕ, C_0)	1	(ϕ, C_2)

(4.95)

We explain the table in order. In both cases, dilaton ϕ and graviton $g_{\mu\nu}$ survive and the Kähler moduli v_a is projected to $h_+^{1,1}$. Complex structure moduli z^α are projected to $h_-^{2,1}$ in O3/O7 case since σ change the sign of Ω, and in O5/O9 case they are projected to $h_+^{1,1}$. As for the Kalb-Ramond field, there is no longer $B_2(x)$ after the projection and b^a is projected to $h_-^{1,1}$ because of the negative sign from world sheet parity action. When it comes to RR-sector, the surviving multiplets are self-explanatory in O3/O7 case, but in O5/O9 case everything is opposite.

4.3.4 dimensional reduction of branes action

We will discuss in this subsubsection, the 4D effective action from D-brane action which we have seen in subsection 3.2. The result is used in section 5 and 6 for the Kähler moduli stabilization.

Recall the DBI-action for Dp-brane wrapped on a (p-3)-cycle Σ_{p-3} in X_6, which is given in Eq.(3.19). We can do the procedure of dimensional reduction as follows;

$$\begin{aligned} S_{\rm DBI} &= -g_s T_p \int d^{p+1}\sigma e^{-\Phi}\sqrt{-det[G_{ab}+(2\pi\alpha')f_{ab}+B_{ab}]} \\ &= -\int_{M_4} d^4x\sqrt{-g}\frac{(2\pi\alpha')^2 T_p}{4}\int_{\Sigma_{p-3}} d^{p-3}y\sqrt{g|}e^{(p-7)\widetilde{\Phi}/4}tr(f_{ab}f^{ab})+\cdots \\ &= -\int_{M_4} d^4x\sqrt{-g}\frac{(2\pi\alpha')^2 T_p}{4}{\rm Vol}(\Sigma_{p-3})e^{(p-7)\widetilde{\phi}/4}tr(f_{\mu\nu}f^{\mu\nu})+\cdots . \end{aligned} \tag{4.96}$$

In the second equality we moved from string frame to Einstein frame, and in the third equality we moved from Einstein frame to 4D Einstein frame and we have defined

$$\mathrm{Vol}(\Sigma_{p-3}) := \int_{\Sigma_{p-3}} d^{p-3}y\sqrt{g|}. \tag{4.97}$$

Therefore we effectively have gauge theory on M_4 with the gauge coupling

$$g_{\rm YM}^{-2} := (2\pi\alpha')^2 T_p {\rm Vol}(\Sigma_{p-3})e^{(p-7)\widetilde{\phi}/4}. \tag{4.98}$$

Non-perturbative effect

Consider N_c coincident D7-branes wrapping a 4-cycle Σ_4 in X_6. The 4D effective action is given by

$$S_{D7} \supset -\int_{M_4} d^4x\sqrt{-g}\ \frac{1}{4{g_{\rm YM}^{(7)}}^2}tr(f_{\mu\nu}f^{\mu\nu}), \tag{4.99}$$

where the gauge coupling is

$${g_{\rm YM}^{(7)}}^{-2} = (2\pi\alpha')^2 T_p {\rm Vol}(\Sigma_4) = \frac{T}{4\pi}, \tag{4.100}$$

where T is the Kähler modulus for the overall volume of X_6.

Now recall the superpotential from gaugino condensation in $SU(N_c)$ pure super Yang-Mills theory;

$$W \sim \Lambda^3 = \mu^3 \exp\left(-\frac{8\pi^2}{g_{\rm YM}(\mu)^2 N_c}\right), \tag{4.101}$$

where μ is the scale one is considering, and now we take it to be the Kaluza-Klein scale $M_{\rm KK}$. Thus we obtain the desired result;

$$W_{\rm np}(T,z) = A(z)\exp\left(-\frac{2\pi}{N_c}T\right). \tag{4.102}$$

Keep in mind that the over all factor A may depend on the complex structure moduli z because there must be a threshold correction to the gauge coupling, $g_{\rm YM}^{-2} = T/4\pi + b \ln M(z)/\Lambda$, where Λ is a cutoff scale and M is a new physics mass scale which depends on the complex structure.

This superpotential from the non-perturbative effect plays an important roll when we discuss the Kähler moduli stabilization in the next section.

Estimation of A

As long as we know, the superpotential is inevitable when we apply TypeII theory to the cosmology because the Kähler moduli violate the phenomenology unless they are stabilized by the superpotential from the non-perturbative effects. We would therefore like to know the value of A in (4.102) as precisely as possible, for quantitative discussion of moduli stabilization in the next section. In the discussion below, we estimate A up to order one factor by comparing 4D $\mathcal{N}=1$ SUGRA action with gaugino condensation (denoted as V_1), with the SUGRA scalar potential V_2 with $W = Ae^{2\pi T/N}$ and $K/M_{\rm Pl}^2 = -3(T+\bar{T})$.

Let us begin with V_1. Here is a portion of SUGRA lagrangian taken from Wess bagger [24];

$$\sqrt{-g_4}^{-1}\mathcal{L}_4 \supset$$
$$e^{K/M_{\rm Pl}^2}\left[3\left|\frac{W_0}{M_{\rm Pl}}\right|^2 - K^{\bar{j}i}\left[D_i W_0 - H_{,i}e^{-\frac{K}{2M_{\rm Pl}^2}}2\,{\rm tr}[\lambda\lambda]_{4,h}\right]\left[\overline{D_j W_0} - \overline{H_{,j}}e^{-\frac{K}{2M_{\rm Pl}^2}}2\,{\rm tr}[\bar{\lambda}\bar{\lambda}]_{4,h}\right]\right], \tag{4.103}$$

where W_0 is superpotential which does not depend on T, and the gauge kinetic function H is given by $T/16\pi$ in our context. The gaugino condensation in rigid supersymmetry is evaluated as

$$\frac{\langle 2{\rm tr}_N[\lambda\lambda]_h\rangle}{32\pi^2} = N\Lambda_h^3, \tag{4.104}$$

by instanton calculation, where Λ_h is dynamical scale related to the scale reference μ by

$$\Lambda_h^{3N} = \mu^{3N}e^{-\frac{8\pi^2}{g_{\rm YM}^2(\mu)}+i\theta}. \tag{4.105}$$

We now take μ to be $M_{\rm KK}$, and using the relation Eq.(4.100) we obtain

$$\langle {\rm tr}_N[\lambda\lambda]_h\rangle = 16\pi^2 N\Lambda_h^3 \simeq 16\pi^2 N M_{\rm KK}^3 e^{-\frac{2\pi}{N}T}. \tag{4.106}$$

Focusing on the cross term in Eq.(4.103) the potential for T contains the term such as

$$V_1 \supset e^{K/2M_{\rm Pl}^2}(K^{T\bar{T}}K_{\bar{T}}\overline{W_0})2\pi N M_{\rm KK}^3 e^{-2\pi T/N}. \tag{4.107}$$

On the other hand, substituting $W = Ae^{-2\pi T/N} + W_0$ into the SUGRA potential $V = e^{K/M_{\rm Pl}^2}\,(K^{T\bar{T}}|D_T W|^2 - 3|W|^2)$ with $K = -3\ln(T+\bar{T}) + K_0$, we obtain another form of the potential for T as a cross term with W_0;

$$V_2 \supset e^{K/M_{\rm Pl}^2}(K^{T\bar{T}}K_{\bar{T}}\overline{W_0})A\frac{2\pi}{N}e^{-2\pi T/N}. \tag{4.108}$$

Comparing V_1 and V_2, we get the result

$$A \simeq aN^2\,\langle {\rm Re}T\rangle^{3/2}\,M_{\rm KK}^3 \simeq M_{\rm Pl}^3\frac{a\;N^2}{[4\pi{\rm Re}\,\langle T\rangle]^{3/2}}, \tag{4.109}$$

with

$$a = \sqrt{e^{-K_0/M_{\rm Pl}^2}} \sim \left[\int \Omega \times \sqrt{2/g_s}\right], \tag{4.110}$$

in the second equality, we assumed $K_0 = -\ln\left[-i\int\Omega\wedge\overline{\Omega}\right] - \ln[-i(\tau-\bar{\tau})]$.

5 Moduli stabilization

4D effective theory from string theory contains many kinds of moduli fields as we have seen in the last section. Such moduli cause phenomenological flaw especially in the context of thermal history of the universe, and such problems are called cosmological moduli problem. The detail is so complicated that we do not discuss it and just present the solution for it. See for example [30] for the more explanation of moduli problem. To avoid the problem, what we need to do is only to give them large enough mass so that they cannot spoil the successful cosmology. In this section, we discuss how to achieve it in TypeIIB string theory.

5.1 Kähler moduli stabilization in Type IIB

In Type IIB theory, it is automatical to stabilize complex structure moduli if the theory has fluxes, the nontrivial vacuum expectation values of 3-form fields. This is one of the biggest differences from heterotic string and is a great advantage to realize realistic model in string theory.

On the other hand, stabilizing Kahler moduli is a non-trivial task. Let us first make sure that superpotential of cannot depend on the Kähler moduli in perturbation theory. The reason is simple. Since $T = \cdots + iC_4$, the shift symmetry of C_4 ($C_4 \to C_4$+const.) and the holomorphy of the superpotential forbid any terms of the form T^q. Therefore what we can do to stabilize the Kähler potential is consider non-perturbative corrections to the superpotential or α' corrections to the Kähler potential. In this sense, there are roughly two leading candidates for the scenario to stabilize the Kahler moduli, called KKLT and LVS; the first one considers non-perturbative effects to the superpotential and the later α' corrections to the Kähler potential. We see the basic idea of the both in the following subsubsections.

In the discussion throughout this subsection, we neglect fields except for the Kähler moduli just for simplicity although the interplay is of course important to build complete string inflation models.

5.1.1 KKLT scenario

In this subsubsection, we review on the idea of KKLT [31] mechanism, in which Kähler moduli in TypeIIB theory are stabilized by the non-perturbative effect of D7-branes.

Consider a Kähler modulus T with the Kähler potential

$$K = -3\ln(T + \bar{T}). \tag{5.1}$$

If the superpotential does not depend on the Kähler modulus, then the potential is completely flat and it ends with the destruction of cosmology, i.e. moduli problem. However, assuming that N_c D7-branes are wrapped on the same 4-cycle, the non-perturbative effect of gaugino condensation yields a superpotential as we have derived in Eq.(4.102);

$$W = Ae^{-\frac{2\pi}{N_c}T} + W_0, \tag{5.2}$$

where A is mass dimension three coefficient which depends on complex structure moduli and N_c comes form gaugino condensation in $SU(N_c)$ super Yang-Mills theory. We add a constant W_0 for contribution form other fields which are supposed to stay at their vacuum values.

Let us calculate the potential in this set-up. Plotting the potential with some choice of parameters (A, N_c, W_0) tells us that $\langle T \rangle$ runs away in usual cases but in rare cases T is stabilized. We would like to figure out what exactly the "rare cases" are, by analytic calculation. One can easily see that $\partial_T V = 0$ is equivalent to $D_T W = 0$ and determine the vacuum value of T;

$$W_0 = -Ae^{-\frac{2\pi}{N_c}\mathrm{Re}\langle T \rangle}\left(1 + \frac{4\pi}{3N_c}\mathrm{Re}\,\langle T \rangle\right), \tag{5.3}$$

here we assumed $\mathrm{Im}\,\langle T \rangle = 0$. Therefore the value of W_0 must be exponentially small compared with A to obtain sufficiently large $\mathrm{Re}T$ (large volume is necessary to be compatible with gauge unification). In other words, we need a fine-tuning of W_0 to stabilize the Kähler modulus. What we would like to investigate next is the vacuum expectation value of the potential because we are concerned with the cosmological constant.

This is the **KKLT mechanism**, in which Kähler moduli are stabilized by the superpotential of non-perturbative effect with fine-tuningly small constant, and the vacuum value of the potential is always negative. Note that T has to be large enough for the control of the Kähler potential is available and that $\frac{2\pi}{N}T$ also has to be large enough for the approximation of neglecting higher instanton corrects to remain valid.

uplifting to de-Sitter vacuum by $\bar{D}3$-branes

One of the defects in the KKLT scenario is obviously that it ends with the anti-de-Sitter vacuum. It has shown that it is possible to uplift the potential to give the de-Sitter potential just by adding anti-D3-branes to the system [31]. The existence of anti-D3-branes breaks the supersymmetry and its tension gives a following positive contribution to the scalar potential;

$$\delta V = \frac{D}{(T + T^*)^3} \tag{5.4}$$

Adding this correction, the potential of Kähler modulus gives rise to a de-Sitter vacuum with enough high barrier if we tune W_0 and D to small values again. Whether we can derive these small value in a realistic set-up is of course the next problem to be considered.

This is, unfortunately, not the end of the story. The above discussion is just on the toy model which has only one Kähler modulus. Besides, in realistic type IIB string model, the standard model sectors made by D7-branes also get the correction from anti-D3-branes which results in a disaster of phenomenology.

One more comment is on supersymmetry breaking. The D-term correction breaks the supersymmetry and it is shown in [31] that inflation models based on KKLT plus D-term correction of D-branes leads to a heavy gravitino and make the phenomenology disastrous.

5.1.2 Racetrack scenario

Racetrack scenario (suggested already in [31], refocused in [32] in the context of susy breaking and its phenomenological results) is a natural extension of KKLT scenario which leads to not anti-de-Sitter vacuum but de-Sitter or Minkovski vacuum without the help of $\overline{D3}$-branes and without SUSY breaking. In this scenario we assume that Kähler moduli have non-perturbative correction terms in the superpotential from gaugino condensation of $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ super Yang-Mills theory;

$$\begin{aligned} W &= A_1 e^{-\frac{2\pi}{N_1}T} + A_2 e^{-\frac{2\pi}{N_2}T} + W_0 \\ &= \frac{M_{\mathrm{Pl}}^3}{[(4\pi)\mathrm{Re}\,\langle T\rangle]^{3/2}} \left(a_1 N_1^2 e^{-\frac{2\pi}{N_1}T} + a_2 N_2^2 e^{-\frac{2\pi}{N_2}T} \right) + W_0, \end{aligned} \tag{5.5}$$

We further assume $h^{(1,1)} = 1$ just for simplicity. As we have estimated the order of the prefactors A_i to be about $N_i^2 M_{\mathrm{Pl}}^3/[(4\pi)\mathrm{Re}\,\langle T\rangle]^{3/2}$ in subsubsection 4.3.4, we rewrite it as in the second line of the equation above where a_1, a_2 are dimensionless and may depend on the complex structure moduli. Kähler potential is supposed to be the no-scale form (neglecting α' ot g_s corrections);

$$K = -3\ln(T + \bar{T}). \tag{5.6}$$

This set-up yields a supersymmetric Minkovski (or de-Sitter) vacuum if we tune some of the parameters in it.

Let discuss what the parameteres $(N_1, N_2, a_1, a_2, W_0)$ should be to stabilize T well, following the discussion of section 4.1 in Ref. [1]. First we assume $a_1 = 1$ and define $r := a_2/a_1$ because what is relevant to the shape of the potential is not both a_1 and a_2 but only their

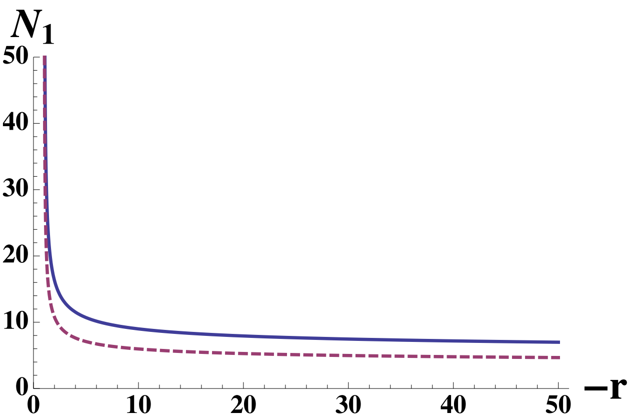


Figure 1: This is a contour plot of the vacuum value $\langle T\rangle$ on the $(-r, N_1)$ plane; the curve above (solid) is the contour of $\langle T\rangle = 25$, and the one below (dashed) that of $\langle T\rangle = 10$.

ratio r. Next we set a simple ansatz that $N_1 = N_2 + 1$ Thus we reduce the independent parameters into (N_1, r, W_0). The vacuum value of T is determined by $\partial V/\partial T|_{T=\langle T\rangle} = 0$, and we assume for the Minkovski vacuum (or de-Sitter vacuum with a small cosmological constant) that $V|_{T=\langle T\rangle} = 0$. The sufficient condition for these two equations turns out to be $\langle D_T W\rangle = 0$ and $\langle W\rangle = 0$. We use the first equation to determine the value of $\langle T\rangle$;

$$\langle T\rangle = \frac{N_1(N_1-1)}{2\pi}\ln\left[-r\frac{N_1-1}{N_1}\right]. \tag{5.7}$$

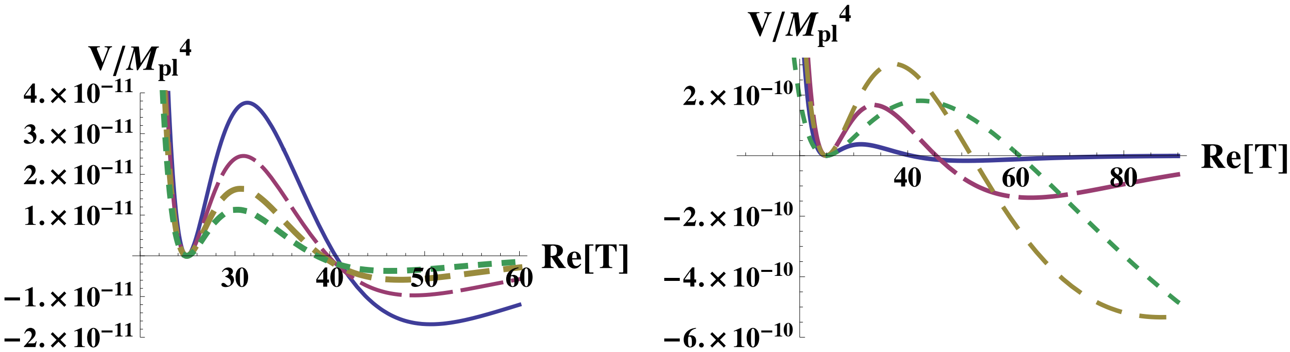
The second equation is for fixing the value of W_0;

$$\begin{aligned} W_0 &= -\frac{M_{\rm Pl}^3}{[(4\pi)\mathrm{Re}\,\langle T\rangle]^{3/2}}\left(a_1 N_1^2 e^{-\frac{2\pi}{N_1}\langle T\rangle} + a_2 N_2^2 e^{-\frac{2\pi}{N_2}\langle T\rangle}\right) \\ &= -\frac{M_{\rm Pl}^3}{[(4\pi)\mathrm{Re}\,\langle T\rangle]^{3/2}}\langle a_1\rangle N_1\left(-\frac{\langle a_1\rangle}{\langle a_2\rangle}\frac{N_1}{N_2}\right)^{N_2}, \end{aligned} \tag{5.8}$$

and we call this relation "**Kallosh-Linde tuning**". We impose one more condition to the remaining parameters (N_1, r) that

$$25 = \frac{N_1(N_1-1)}{2\pi}\ln\left[-r\frac{N_1-1}{N_1}\right], \tag{5.9}$$

because we would like to unify the gauge coupling constants at the value suggested by the calculation of running gauge couplings in MSSM.



(a) (b)

Figure 2: The potential $V_{\text{racetrack}}(T)$ for a couple of different choices of N_1 and $r := \langle a_2/a_1 \rangle$ that lead to $\langle T \rangle = 25$.
Panel (a): the curves shown in solid-blue, long dashed-red, dashed-yellow and dotted-green lines are for parameters $(r, N_1) = (-1.05, 68)$, $(-1.06, 62)$, $(-1.07, 57)$ and $(-1.08, 53)$, respectively. The smaller the N_1 is, the larger the value of r and the height of the potential barrier become.
Panel (b): the curves shown in solid-blue, long dashed-red, dashed-yellow and dotted-green lines are for parameters $(r, N_1) = (-1.05, 68)$, $(-1.02, 118)$, $(-1.007, 238)$ and $(-1.002, 626)$ respectively. The second (negative energy) minimum deepens for larger r.

By virtue of all the assumptions above, once we fix the N_1, we get $N_2 = N_1 - 1$ just as an optimistic choice, r is determined by the relation $\langle T \rangle = 25$ and W_0 is suggested by the Kallosh-Linde tuning $\langle W \rangle = 0$. Figure 1 shows how the required value of r changes for different values of N_1 in order to achieve a given value of Re $\langle T \rangle$.

What we would like to investigate next is what the potential looks like for a given choice of N_1. To that end, we plotted the potential and the outcome is shown in Figure 2. Comments on Figure 2 are in order.

First, we can see that the potential is in good shape when $(N_1, r) = (68, -1.05)$; the Kähler potential is stabilized by high barrier without a jeopardy of tunneling out to the AdS vacuum. In this case the value of W_0 is about $-0.01 M_{\text{Pl}}^3$ which means that nearly 1% tuning is necessary for the scenario to be successful, and that we need to discuss whether there is a flux choice to satisfy this tuning or not when we consider the theory as a whole.

Second, we can see the high sensitivity of the potential to the value of r, which implies that another tuning is necessary to stabilize the Kähler moduli in racetrack scenario. For the height of the potential to be large, r must be tuned near -1 (and in this case N_1 must be as large as $\mathcal{O}(100)$ for $\langle T \rangle = 25$). We call this tuning "**hidden tuning**" of the racetrack scenario. It is difficult for now to judge this tuning can be possible or not in a priori study.

The last comment is on the height of the potential. Even in the case of the most optimistic choice $(N_1, r) = (68, -1.05)$, the potential height is $V \simeq (10^{-2\sim-3} M_{\rm Pl})^4$ and thus it is likely that the Kähler moduli are distabilized during the inflation whose energy scale is of the order $V_{\rm inf} \simeq 10^{16}$GeV as suggested by the BICEP II experiment. We leave the discussion regarding this point of view to section 7 where concrete string inflation models are discussed.

5.1.3 Large volume scenario (LVS)

Since the α' corrections are comparable with the instanton effects in type IIB orientifold compactification , it is preferable to take them into account. Fortunately it is shown in [33] that the inclusion of the corrections helps us with the fine-tuning smallness of W_0 to obtain the large volume Kähler moduli stabilization as follows, and it is called **large volume scenario**, or **LVS** for short.

The basic idea of the scenario is following. Consider We first assume the non-perturbative corrections to some of the Kähler moduli as in the KKLT scenario;

$$W = \sum_i A_i e^{-\frac{2\pi}{N_i} T_i} + W_0. \tag{5.10}$$

The Kähler potential with first order α' correction is given by [25]

$$K = -2 \ln \left[\mathcal{V} + \frac{\xi}{2 g_s^{3/2}} \right] + K^{(\rm cpx)}, \quad e^K = \frac{e^{K^{(\rm cpx)}}}{\mathcal{V}^2} + \mathcal{O}(1/\mathcal{V}^3), \tag{5.11}$$

where $K^{(\rm cpx)}$ represents the factors which come from the complex structure moduli. To take this α' correction into account is the main difference from KKLT, which leads to a large volume vacuum as we see below. Under the Kähler potential and superpotential above, the scalar potential takes the following form in the $\mathcal{V} \to \infty$ limit;

$$V \sim |A|^2 \frac{\tau_s^{1/2}}{\mathcal{V}} e^{-\frac{4\pi}{N}\tau_s} - |W_0 A^*| \frac{\tau_s}{\mathcal{V}^2} e^{-\frac{2\pi}{N}\tau_s} + |W_0|^2 \frac{\xi}{\mathcal{V}^3}, \tag{5.12}$$

where τ_s represents the Kähler moduli which can be independent of the over all volume $\mathcal{V}$. For the detailed discussion and calculation, see the original paper [33]. First it can be read off that the direction $\mathcal{V} \sim e^{\frac{2\pi}{N}\tau_s}$ seems light compared with the other directions because in this case the positive and negative terms in the potential can be cancelled each other. Keeping this relation, let us consider the limit $\mathcal{V} \to \infty$ and find that the second term is the dominant and that V goes to zero from negative. On the other hand, when we consider the small $\mathcal{V}$ limit, V goes to $+\infty$ if ξ is positive. These observations are convincing enough to conclude that there exists a anti-de-Sitter vacuum for any given parameters (W_0 need not be tuned!). Note that this vacuum spontaneously breaks the supersymmetry. It can be seen in a simple way; at the vacuum V is estimated as $\mathcal{O}(\mathcal{V}^{-3})$ while $e^K|W|^2 \sim \mathcal{O}(\mathcal{V}^{-2})$, and thus $DW \neq 0$.

To lay on the idea of the scenario more concretely, let us consider a toy model compactified on $CP^4_{(1,1,1,6,9)}$ which has two Kähler moduli, T_b and T_s. The Kähler potential is

$$K = -2\ln\left[\frac{1}{9\sqrt{2}}(\tau_b^{3/2} - \tau_s^{3/2}) + \frac{\xi}{2g_s^{3/2}}\right], \tag{5.13}$$

where $\tau_b := \mathrm{Re}T_b$, $\tau_b := \mathrm{Re}T_b$ and $\xi := -\frac{\chi(X_6)\zeta_3}{2(2\pi)^3} \in \mathbf{R}$ is constant which is positive because the Eular number is negative in this case. We further assume that N D7-branes are wrapping the 4-cycle corresponding to T_s (and that T_b corresponds to the over all volume of X_6) and thus the superpotential is;

$$W = Ae^{-\frac{2\pi}{N}T_s} + W_0, \tag{5.14}$$

where W_0 is supposed to be a constant. It turns out that the potential for τ_b and τ_s are stabilized at not so small value like realistic $\langle\tau_b\rangle = \mathcal{O}(10)$ without tuning W_0 to be very small compared with A. To see this without tedious plotting of the potential, let us again take a look at the structure of the potential. Rough calculation gives

$$V \sim |A|^2\frac{\tau_s^{1/2}}{\tau_b^{3/2}}e^{-\frac{4\pi}{N}\tau_s} - (W_0A)\frac{\tau_s}{\tau_b^3}e^{-\frac{2\pi}{N}\tau_s} + |W_0|^2\frac{\xi}{\tau_b^{9/2}}, \tag{5.15}$$

where we omit $\mathcal{O}(1)$ coefficients. First it can be read off that the direction $\tau_b^{3/2} \sim e^{\frac{2\pi}{N}\tau_s}$ seems light again. Keeping this relation, let us consider the limit $\tau_b \to \infty$ and find that the second term is the dominant and that V goes to zero from negative, while V goes to $+\infty$ in small τ_b limit. Thus there exists a anti-de-Sitter vacuum without tuning the value of W_0. Full calculation is necessary for skeptical person, but we skip that.

Let us get away from the toy model and consider generic cases and consider in what kind of cases LSV is successful. For LVS to work, we need more than one Kähler modulus and the volume must have the form

$$\mathcal{V} = c_b\tau_b^{3/2} - f^{(3/2)}(\tau_s^i), \tag{5.16}$$

where c_b is a positive constant, $f^{(3/2)}$ is a homogeneous polynomial of the degree $3/2$ in other Kähler moduli $\{\tau_s^i\}_{i=1\sim h^{1,1}-1}$. This type of Calabi-Yau 3-dold is called "Swiss-cheese" Calabi-Yau manifolds, because it has a lot of $(h^{1,1}-1)$ small holes on it and looks like swiss cheese. We need to assume further that the euler number is negative for the positive contribution to the volume in the Kähler potential; $\xi > 0$. The potential induced from the Kähler potential $K = -2\ln(\mathcal{V} + \xi/2g_s^{3/2})$ and the superpotential $W = \sum_i^{h^{(1,1)}-1} A_i e^{-\frac{2\pi}{N_i}T_i} + W_0$ can have a stable vacuum in large volume regime for an arbitrary given value of W_0.

Having solved the tuning W_0 problem, there remains, however, anti-de-Sitter vacuum problem, so we need to uplift the potential by some mechanism like putting anti-D3 branes.

5.1.4 Summary of Kähler moduli stabilization in TypeIIB

Discouraged by the supersymmetry breaking and the limited choice of Calabi-Yau 3-folds, we prefer the racetrack scenario to LVS, in my opinion.

The table below is the summary of the result.

	KKLT	KKLT+$\overline{D3}$	Racetrack	LVS	LVS+$\overline{D3}$
Restriction to CY_3					
SUSY at vacuum	preserved	broken	preserved	broken	broken
Cosmological const.	$V<0$	$V \geq 0$	$V \geq 0$	$V<0$	$V \geq 0$
Fine-tuning	$\lvert W_0 \rvert$	$\lvert W_0 \rvert \& D$	$\lvert W_0 \rvert \& r$	none	D

(5.17)

: The Calabi-Yau must contain a stuck of N_1 D7-branes and another stuck of N_2 D7-branes.

: The euler number must be negative and the volume as a function of Kähler moduli must has the form of Eq. (5.13).

Note that this result is valid under the assumption that other fields are heavier than Kähler moduli. Especially susy breaking and the sign of the potential value at the vacuum are very sensitive to the lighter fields and we need to reconsider unless we focus on Kähler moduli inflation.

6 String inflation models

In this section we will briefly review on string inflation models proposed so far, and then take a closer look at one of them, based on the work [1] by the author.

In subsection 6.1, we explain the basic strategies for building inflation models in string theory and provide a brief overview of string inflation models proposed so far. In subsection 6.2 we concentrate on a string inflation model in [1] and review the work in detail.

6.1 Overview

6.1.1 Assumptions and ingredients for string inflation models

Before giving an overview of string inflation models, we discuss the hierarchy among the related energy scale, classify the models into three parts and clarify our standpoint. There are four energy scale we need to consider; $M_s, M_{KK}, M_{\rm Pl}$ and $M_{\rm inf}$;

- The first one is the string scale defined as $M_s := 1/\sqrt{\alpha'}$ which represents the mass of string excitation modes, where α' is the unique parameter in string theory. We have no idea what the value is other than it must be so high that we cannot see the effect of string excitation by any experiment on earth.

- $M_{KK} := 2\pi/R$ is the Kaluza-Klein scale and R is the typical size of internal space which is in principle determined by the dynamics of internal space itself and thus changeable during inflation. After the inflation the value must be higher than TeV scale to be compatible with the experimental fact that there is no sign of KK-modes so far.

- $M_{\rm Pl}$ is the Planck scale and is the coefficient of 4D Einstein-Hilbert term. As is well known the value is about 2.43×10^{18}GeV. Recall the procedure of dimensional reduction in subsubsection 4.2.6, especially Eq.(4.62), and that Planck scale is determined by the two energy scales above;

$$M_{\rm Pl}^2 := \kappa_{10E}^{-2}\langle \mathcal{V}_6\rangle = \frac{4\pi\langle \mathcal{V}_6\rangle}{g_s^2 l_s^8} = \frac{4\pi R^6}{g_s^2(2\pi\sqrt{\alpha'})^8} \sim M_s^8 M_{KK}^{-6} \tag{6.1}$$

- $M_{\rm inf} := V_{\rm inf}^{1/4}$ is the inflation scale. The value is of course changeable during inflation. It is notable that the value can be determined by the observation and has an upper bound from above; $M_{\rm inf} \leq 10^{16}$GeV.

These are the four energy scale relevant to string inflation models and the hierarchy among them is crucial when building models. Therefore let us classify models according to the order of them. Although the permutation of four distinct objects is $4! = 24$, models are turned out to fall into only four category because of the relation Eq.(6.1), observational implications and the T-duality of TypeII string theory. To begin with let us consider the two cases $M_s < M_{\rm Pl}$(case I) or $M_s > M_{\rm Pl}$(case II). From the relation Eq.(6.1) $M_{KK} < M_s$ is equivalent to $M_s < M_{\rm Pl}$, thus case I and case II can be written as

$$\begin{cases} \text{case I:} \quad M_{KK} < M_s < M_{\rm Pl}, \\ \text{case II:} \quad M_{\rm Pl} < M_s < M_{KK}. \end{cases} \tag{6.2}$$

Using T-duality $M'_{KK} \simeq M_s^2/M_{KK}$, one can see that case II is dual to case I, so it is sufficient to take only case I into account (we need to both TypeIIA and TypeIIB theory to cover all the possibility of TypeII theory). What is remaining is $M_{\rm inf}$ and finally we have reached three cases of hierarchy (a), (b) and (c) described below;

$$\text{(a)}:M_{\rm inf} < M_{KK} < \text{(b)}:M_{\rm inf} < M_s < \text{(c)}:M_{\rm inf} < M_{\rm Pl} \simeq 10^{18}\text{GeV}. \tag{6.3}$$

Here, due to the upper bound $M_{\rm inf} \lesssim 10^{16}$GeV we neglect the case (d); $M_{\rm Pl} \simeq 10^{18}\text{GeV} < M_{\rm inf}$. The case (a) is the most easiest because we can handle inflation only with 4D effective field theory; energy scale lower than M_s means SUGRA approximation is available and being lower than M_{KK} means we can integrate KK modes out. The case (b) is more cumbersome than case (a) because we cannot rely on 4D effective theory. Rather we need to begin from 10D SUGRA action and seek for a mechanism which causes expansion of our four-dimensional space-time but makes extra six-dimension smaller and eventually takes us to the case (a). The last case (c) is the most difficult, and the author still does not understand what should we do in this case. Anyway the case(b) and (c) are not dealt with in this paper, and most of the existing models is classified as the case (a).

Actually, one more energy scale is important for string inflation model, the SUSY breaking scale $M_{\rm SUSY}$. However we do not include it in the classification of string inflation models in this paper and regard it as a model dependent energy scale. As we have explained in the last part of subsubsection 4.1.1, it is preferable to preserve minimum amount of SUSY (4D $\mathcal{N} = 1$ SUSY) below the Kaluza-Klein scale for realization of realistic particle models. Therefore we assume that $M_{\rm SUSY}$ is lower than the inflation scale $M_{\rm inf}$ and thus rely on supersymmetry when we analyze the model, so do most of the works.

Protocol for string inflation model building

Here is the protocol for building inflation models from string theory.

- Choose which types of super string theory we use; Hetero, TypeI, TypeIIA or TypeIIB. Since it is known that the standard model is likely to be generated by intersecting D-branes in TypeII string theory, we often take TypeIIB or TypeIIA theory as a starting point, it is of course not our obligation, though.

- Set energy scales. As has been explained above, the inequality of related energy scale can be fixed to $M_{KK} < M_s < M_{\rm Pl}$ in TypeII theory. Then all the possibility falls into three parts; (a), (b), and (c). We often take (a) just because it is easy and thus rely on 4D effective action.

- Choose the topology and geometry of internal six-dimensional space as a background solution during inflation. We usually adopt Calabi-Yau 3-fold because it preserves minimum amount of supersymmetry which is preferable for phenomenological discussion.

- Put local sources like fluxes, D-branes and O-planes as a background. We sometimes use their motion as the cause of inflation, and sometimes do not pay attention to the dynamics of them at all and deal with them as stationary objects during inflation The later case is justified well if we take the case (a) in which $M_{\rm inf} < M_{\rm KK} < M_s$, because the energy scale of such dynamics is about M_s or $M_{\rm KK}$. Note that we have to care about whether the configuration is stable or not and about the back-reaction from them.

- Derive the 4D effective action and investigate whether the phenomenology is consistent with the observation and experiment.

6.1.2 Overview of string inflation models

We will here survey the representative examples of string inflation models. We do not present them in detail due to the lack of time and in order not to make this paper too long. See the review [2] and references there in. In the list below, we classify and name the models in terms of what causes the inflation.

- Brane-inflation:
  The motion of branes put in the theory is expressed by scalar fields and can cause inflation. For example slow-roll behavior is possible by the motion of D3-brane in a

warped flux compactification. Inflation models driven by branes in un-warped geometries are also proposed. These efforts have shown that slowly-moving D-branes suffer from the eta-problem. To avoid this, relativistically moving D-branes were used to cause inflation. Relying on DBI action, such models are called DBI-inflation.

- Axion inflation:
  Axions in string theory are most attractive candidate for inflaton because they enjoy shift symmetry, which is helpful to have flat potential, by definition. Due to the non-perturbative effects, such shift symmetry is broken to discreet symmetry and axions feel potential which is similar to that in natural inflation. However, natural inflation has to be have over Planck decay constant to earn e-folding number and it is difficult to achieve it by single string axion. To avoid this, multi-field natural inflation, dubbed N-flation, is being constructed in string theory. It is natural to get many axions in string theory as one can expect from the discussion in subsection 4.3. Another possible axion inflation is achieved by making use of the monodromy. Such string inflation models are called axion monodromy inflation, and have been studied well.

- Kähler moduli inflation:
  Kähler moduli, which are stabilized by non-perturbative effects, can also be inflaton, and a lot of models like racetrack inflation are proposed so far. This is somewhat related to axion inflation because some of the axions and Kähler moduli are in the same supersymmetric multiplet as we have seen in subsection 4.3

- complex structure inflation:
  Complex structure moduli, which are often fixed still by flux compactification, are capable of causing inflation. In this paper, we focus on this scenario and the discussion is going to start just from the next sentence.

6.2 Complex structure moduli inflation

In this section we review the work [1] in detail. In that work, the authors study the possibility that the inflaton may be the complex structure moduli in TypeIIB theory. This possibility is inspired by the bottom-up works [10] in which the inflaton is right-handed sneutrino which can be identified with complex structure moduli in TypeIIB theory. We will first briefly review [10] in subsubsection 6.2.1. In subsubsection 6.2.2, we will actually try to construct the model and investigate whether the complex structure moduli can have flat enough potential to be an inflaton taking a mirror quintic as an internal space. What follows is the consistency checks of the scenario. Focusing on the Kähler moduli T, complex structure moduli z and dilaton τ, we first estimate their masses and study their phenomenology in subsubsection 6.2.3. We next check the consistency of complex structure inflation with the Kähler moduli stabilization in the racetrack scenario in subsubsection 6.2.4. We summarize our result in subsubsection 6.2.5.

6.2.1 Right-handed sneutrino inflaiton revisited

Among the most attractive models for inflation is right-handed sneutrino inflation model suggested in refs. [10]. The beauty of this model is that it passes various phenomenological tests such as neutrino mass, baryon asymmetry and gravitino problem. We have already discussed the model as one of the examples of inflation models in subsubsection 2.6.4, but we revisit it to discuss how to realize the model in string theory.

Especially, baryon asymmetry can be generated through leptogenesis thanks to the decay of right-handed neutrino. Leptogenesis is a mechanism in which lepton number generated after the inflation by the decay of right-handed neutorino (because their Yukawa couplings violate the lepton number and can be asymmetric in terms of lepton number creating process) is transformed to baryon number by the Sphaleron process (a non-perturbative effect in the standard model which preserves B-L symmetry but breaks B+L symmetry). So, inflaton decay after the inflation can naturally generate a present baryon number in the right-handed sneutrino inflation model. In usual inflation models where inflaton is NOT right-handed sneutrino, we need to impose conditions on the mass of right-handed neutrinos and their Yukawa couplings and reheating temperature to be consistent with the observed value of the present baryon number density and the constraints on the mass of left-handed neutrinos which we assume are generated by see-saw mechanism. However, in right-handed sneutrino inflation model, we can explain the present baryon number and the mass of left-handed neutrinos by natural and wide-ranged choice of the parameters. This is one of the main reason why the right-handed sneutrino is plausible for a realistic inflation model.

On top of that, this model can be embedded in the framework of supergravity without facing the η problem due to the shift symmetry of one of right-handed sneutrino (chiral multiplets) in the Kähler potential. See the last paper in refs. [10] or subsubsection 2.6.4 in this paper for the specific setup.

We would like to study whether this bottom-up successful model can be realized in the top-down approach, string theory. Then, the first question one might first come up with is “What corresponds to right-handed sneutrions in string theory?”. The answer is already known thanks to the works of T. Watari [29] in which it is shown that complex structure moduli in TypeIIB (or F-theory) can have mass terms and Yukawa couplings with standard model particles. Thus we can identify complex structure moduli in TypeIIB with right-handed sneutrinos in MSSM.

In conclusion, inflation models driven by complex structure moduli are plausible in the view point of bottom-up approach and we study the possibility of derivation of such models from string theory in the subsequent subsubsections.

6.2.2 Realization in string framework; mirror quintic example

In this subsubsection, we construct the right-handed sneutrino inflation model in TypeIIB string theory. We deal with the supersymmetric 4D effective field theory as we have explained in subsection 4.3, and focus on the dynamics of only complex structure z, dilaton τ and Kähler moduli T. Recall that their Kähler potential is given by

$$K/M_{\rm Pl}^2 = -2\ln[\mathcal{V}_6] - \ln[-i(\tau-\bar{\tau})] - \ln\left[-i\int_{X_6}\Omega\wedge\overline{\Omega}\right]. \tag{6.4}$$

In the case of the single Kähler modulus, the first term becomes

$$-2\ln[\mathcal{V}_6] = -3\ln(T+\bar{T}) + \text{const.}. \tag{6.5}$$

Dilaton and complex structure moduli have non-zero potential due to 3-form fluxes as depicted by the Gukov-Vafa-Witten superpotential;

$$W_{\rm GVW} = c\int_{X_6} G_3\wedge\Omega, \quad c = M_{\rm Pl}^3/\sqrt{4\pi}. \tag{6.6}$$

For a realistic model, Kähler moduli need to have potential too. We adopt racetrack scenario and assume that there are a stuck of N_1 D7-branes and a stuck of N_2 D7-branes wrapping on 4-cycle in the compact space X_6, and thus the full superpotential is

$$W^{(\rm tot)} = W^{(T)} + W^{(\rm cpx)} \tag{6.7}$$

$$W^{(T)}(T,z) = A_1(z)e^{-\frac{2\pi}{N_1}T} + A_2(z)e^{-\frac{2\pi}{N_2}T}, \tag{6.8}$$

$$W^{(\rm cpx)}(z,\tau) = W_{\rm GVW}, \tag{6.9}$$

Recall that $A_{i=1,2}$ are estimated as $A_i \simeq a_i N_i^2 M_{\rm Pl}^2/(4\pi{\rm Re}\,\langle T\rangle)^{3/2}$, where a_i are of the order unity and depend on complex structure moduli by the threshold correction. The full lagrangian of the theory we are considering is

$$\mathcal{L} = -K_{T\bar{T}}\partial_\mu T\partial^\mu \bar{T} - K_{z\bar{z}}\partial_\mu z\partial^\mu \bar{z} - K_{\tau\bar{\tau}}\partial_\mu \tau\partial^\mu \bar{\tau} - V, \tag{6.10}$$

$$V = e^K\left(K^{T\bar{T}}|D_T W^{(\mathrm{tot})}|^2 + K^{z\bar{z}}|D_z W^{(\mathrm{tot})}|^2 + K^{\tau\bar{\tau}}|D_T W^{(\mathrm{tot})}|^2 - 3|W^{(\mathrm{tot})}|^2\right). \tag{6.11}$$

It is possible that there are correction terms to the Kähler potential (6.4) generated in the presence of fluxes. However, corrections will be small, relatively by $\alpha' G^2 \sim 1/\langle\omega\rangle^3 \ll 1$ in the regime of our interest in this article (Re $\langle T\rangle \approx \mathcal{O}(10)$ or somewhat larger). We do not pay attention to the correction terms in the Kähler potential for this reason.

We would like some of complex structure moduli to have shift symmetry in Kähler moduli for a qualification as an inflaton. It of course depends on the choice of the compact space X_6. Therefore, as a test study, we take an example of mirror quintic who has a unique complex structure modulus, and see the specific form of its Kähler potential and superpotential, following the discussion and even notation in [37].

Effective theory on Mirror-Quintic

The mirror-quintic Calabi–Yau 3-fold X_{mq} can be regarded as a crepant resolution of $\mathbf{Z}_5 \times \mathbf{Z}_5 \times \mathbf{Z}_5$ orbifold of a geometry given by

$$\left\{[X_1 : \cdots : X_5] \in CP^4 \mid \sum_{i=1}^{5}(X_i)^5 - 5\psi\prod_{i=1}^{5} X_i = 0\right\}. \tag{6.12}$$

Thus the number of complex structure moduli of X_{mq} is one and the complex structure modulus z can be written by the parameter ψ in the definition of X_{mq};

$$z := \frac{1}{2\pi i}\ln\left[\frac{1}{(5\psi)^5}\right] \tag{6.13}$$

Since the (2,1)-hodge number of mirror quintic is one, $h^{2,1} = 1$, the basis of $H^{(3)}(X_{\mathrm{mq}})$ can be written by four elements, in other words $b^3 = 4$. We use the same notation as in subsection 4.3 and write the basis as $(\alpha_{K=0,1}, \beta^{K=0,1})$. Let (A^K, B_K) be the dual basis of homology $H_{(3)}(X_{\mathrm{mq}})$;

$$\int_{A^K}\alpha_L = \delta^K_L, \quad \int_{B_K}\beta^L = \delta^L_K \tag{6.14}$$

Then it follows that

$$A^K \cap B_L = \delta^K_L, \quad A^K \cap A^L = B_K \cap B^L = 0. \tag{6.15}$$

We define the cycle integral of holomorphic three form $\Omega(z) \in H^{(3,0)}$ as

$$\Pi_1 := \int_{B^0} \Omega, \quad \Pi_2 := \int_{B^1} \Omega, \quad \Pi_3 := \int_{A^0} \Omega, \quad \Pi_4 := \int_{A^1} \Omega. \tag{6.16}$$

The period integral is expressed for a symplectic basis of $H_3(X_{\rm mq})$ as follows

$$\Pi := \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -\frac{5}{2}z^2 - \frac{11}{2}z + \frac{25}{12} \\ \frac{5}{6}z^3 + \frac{25}{12}z - i\frac{\chi(X_{\rm mq})\zeta_3}{(2\pi)^3} \end{pmatrix} + \mathcal{O}(e^{2\pi i n z})_{n\geq 1}, \tag{6.17}$$

where ζ_3 is meant to be $z(3) = 1.202\cdots$, and $\chi(X_{\rm mq}) = 2(1 + h^{1,1} - h^{2,1}) = 200$ is the topological Euler number of the mirror-quintic Calabi–Yau 3-fold $X_{\rm mq}$. Here we omit all the calculation and just refer the reader to the ref. [37].

Then the Kähler potential of the complex structure modulus (chiral multiplet) z is given by

$$\begin{aligned} K^{(\rm cpx)}/M_{\rm Pl}^2 &= -\ln\left(-i\int_{X_{\rm mq}} \Omega \wedge \overline{\Omega}\right) = -\ln\left(-i\overline{\Pi}\cdot\Sigma\cdot\Pi\right) \\ &= -\ln\left(\frac{5}{6}[(z-\bar{z})/i]^3 + \frac{50}{\pi^3}\zeta_3 + \mathcal{O}(e^{-2\pi{\rm Im}(z)})\right). \end{aligned} \tag{6.18}$$

where

$$\Sigma := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.19}$$

Note that complex structure modulus has shift symmetry in the "large complex structure region", $|\psi| \gg 1$, i.e. ${\rm Im}[z] \gg 1$.

Let us next investigate the form of superpotential $W_{\rm GVW}$. To this end, let us first remind ourselves that 3-form fluxes in Type IIB, $G_3 = F_3 - \tau H_3$, are characterized by the choice of integers $m_{i=1,2,3,4}$ and $n_{i=1,2,3,4}$ defined as

$$\begin{aligned} m_{i=1,2} &= \frac{1}{l_s^2}\int_{A^{K=0,1}} F_3, \quad m_{i=3,4} = \frac{1}{l_s^2}\int_{B_{K=0,1}} F_3, \\ n_{i=1,2} &= \frac{1}{l_s^2}\int_{A^{K=0,1}} H_3, \quad n_{i=3,4} = \frac{1}{l_s^2}\int_{B_{K=0,1}} H_3. \end{aligned} \tag{6.20}$$

Then what we need to do is only the straightforward calculation;

$$
\begin{aligned}
W_{\mathrm{GVW}}/c &= \int_{X_6} G_3 \wedge \Omega = \left[\int_{A^K} G_3 \int_{B_K} \Omega - \int_{B_K} G_3 \int_{A^K} \Omega \right] \\
&= \sum_{i=1}^{4} (m_i - \tau n_i) \Pi_i(z) = \left(f_R^{(3)} - \tau f_{NS}^{(3)} \right)
\end{aligned}
\tag{6.21}
$$

where $f_R^{(3)}(z)$ and $f_{NS}^{(3)}(z)$ are the polynomials of degree of three in z whose coefficients consist of RR-flux, m_i and NSNS-flux n_i respectively;

$$
\begin{aligned}
f_R^{(3)}(z) = & \frac{100m_1\pi^3 - 48m_2\pi^3 + 200m_3\pi^3 - 1200im_4\zeta_3}{48\pi^3} \\
& + i\frac{132m_1 + 240m_3 - 50m_4}{48\pi} z + \frac{30m_1 + 60m_3}{48\pi^2} z^2 + i\frac{5m_4}{48\pi^3} z^3, \\
f_{NS}^{(3)}(z) = & \frac{100n_1\pi^3 - 48n_2\pi^3 + 200n_3\pi^3 - 1200in_4\zeta_3}{48\pi^3} \\
& + i\frac{132n_1 + 240n_3 - 50n_4}{48\pi} z + \frac{30n_1 + 60n_3}{48\pi^2} z^2 + i\frac{5n_4}{48\pi^3} z^3.
\end{aligned}
\tag{6.22}
$$

Monodromy on Mirror-Quintic

Changing the phase of the parameter z by 2π, we come back to the same point in the complex structure moduli space of the mirror-quintic. In terms of z this results in a shift

$$
z \longrightarrow z' = z + 1. \tag{6.23}
$$

This is a subgroup of approximate shift symmetry ($z \to z' = z + \text{const.}$) in the Kähler potential. The period integral (6.17) does change under the shift (6.23), but the change is in the form of

$$
\Pi(z) \longrightarrow \Pi(z+1) = M_\infty \cdot \Pi(z), \quad M_\infty := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -8 & -5 & 1 & 0 \\ 5 & -3 & -1 & 1 \end{pmatrix}, \tag{6.24}
$$

and is regarded as monodromy transformation of a integral symplectic basis of $H_{(3)}(X_{\mathrm{mq}})$.

The complex structure for z and for $z' = z + 1$ are regarded the same physically, only in the absence of 3-form fluxes in Type IIB compactification. For a given choice of fluxes,

$\{m_i, n_i\}$, two choices of complex structure, z and z' are not equivalent in physics, because $W = c\sum_i (m_i - \tau n_i)\Pi_i(z)$ is not the same as that for $\Pi_i(z')$, i.e.

$$W = c\sum_{i=1}^{4}(m_i - \tau n_i)\Pi_i(z) = c\sum_{i=1}^{4}(m'_i - \tau n'_i)\Pi_i(z'), \tag{6.25}$$

where $m'_i = M_\infty^{-1T} m_i$ and $n'_i = M_\infty^{-1T} n_i$. One could get the period integral $\Pi_i(z')$ back to $\Pi_i(z)$ by the monodromy transformation, but the flux quanta in the new symplectic integral basis $\{m_i, n_i\}$ are not the same as before. Due to this monodromy mechanism on the moduli space of complex structure moduli, its covering space is a more appropriate moduli space for physics in the presence of flux on X_6.

Real part of z as an inflaton?

In the large complex structure region $\mathrm{Im}(z) \gg 1$, the kinetic term is approximately

$$\mathcal{L} \simeq \frac{3M_{\mathrm{Pl}}^2}{(z-\bar{z})^2}|\partial z|^2 - V. \tag{6.26}$$

This means that it is better to parametrize the complex z-plane by two real fields σ and φ as

$$z = ie^{2\sigma/M_{\mathrm{Pl}}}(1 + i2\varphi/M_{\mathrm{Pl}}), \tag{6.27}$$

so that the kinetic term is close to the canonical one,

$$\mathcal{L} \simeq -3\left[(\partial\sigma)^2 + (\partial\varphi + 2(\partial\sigma)\varphi/M_{\mathrm{Pl}})^2\right] - V. \tag{6.28}$$

We assume that σ is fixed at its vacuum value and focus on the dynamics of σ. This seems valid because the superpotential W_{GVW} for the period integral in (6.17) is approximately a polynomial in z that is at most cubic, it is a combination of exponential functions in σ and polynomial in φ, and thus the potential V gives heavier mass to σ than φ. Therefore we suspect that σ is an inflaton if we choose the values of 3-form fluxes.

What we need to do then is to estimate the slow-roll parameters ϵ and η for φ, and seek for conditions on fluxes for successful inflation (ϵ and η must be smaller enough than unity for some initial value of φ, and after it rolls down it must become as large as unity to end the inflation). Let us first begin with $\epsilon = (1/2)(V_\varphi/V)^2 M_{\mathrm{Pl}}^2$. To evaluate this, we have to know what V is like as a function of φ. Keeping in mind that σ is supposed to be fixed and that the overall factor of V is irrelevant to ϵ, we can perform the rough calculation below;

$$V \sim |W|^2 \sim |f_{R/NS}^{(3)}|^2. \tag{6.29}$$

Since $f^{(3)}_{R/NS}$ is a third degree polynomial of $z \propto \varphi$, V is a sixth degree polynomial of φ. Then let us simply assume that $V \propto \varphi^6 + r\varphi$, where r is a constant determined by the choice of the flux. In this case, ϵ can be estimated as

$$\epsilon \simeq \frac{1}{2}\left(\frac{V_\varphi}{V}\right)^2 M_{\rm Pl}^2 = \frac{1}{2\varphi^2}\left(\frac{6\varphi + 5r}{\varphi + r}\right)^2, \tag{6.30}$$

which means that ϵ goes to 0 and ∞ as φ becomes large and small respectively, for any value of r. We can easily come to the same conclusion in the case that V is an arbitrary sixth degree polynomial of φ. Therefore it is likely that in a wide range of flux choices ϵ can be smaller than unity at a given initial value and ϵ grows up as the inflaton rolls. We can see that η can also go well. In conclusion it is possible for the real part of z, φ is an inflaton for some choice of flux. Note that this is nothing but the virtue of the shift symmetry in the Kähler potential.

Next we calculate the e-fold number N_e and estimate the initial value of φ, especially required field range $\Delta\varphi$ for $N_e > 60$. Although the detail of course depends on the choice of flux, we can estimate that

$$N_e = \int_{\varphi_i}^{\varphi_e} \frac{1}{\sqrt{2\epsilon}} \frac{d\varphi}{M_{\rm Pl}} \quad \Rightarrow \quad |\varphi_i| \sim \Delta\varphi > \mathcal{O}(1) M_{\rm Pl} \tag{6.31}$$

to get larger than 60 N_e. This conclusion is supported by the discussion of Lyth bound in subsubsection 2.7.1.

Comments and Conclusion

All the properties described so far are not specific to the mirror-quintic Calabi–Yau 3-fold, but hold true for most of Calabi–Yau 3-folds. "*All the properties*" include the approximate shift symmetry in the Kähler potential in the large complex structure moduli region in the absence of flux, the monodromy group action on the moduli space and the flux quanta, rational number coefficients in the period integral (like (6.17)) determined by topology of X_6 and the small corrections due to flux controlled by $1/[\mathrm{Re}\,\langle T\rangle]^{3/2}$. Under a shift symmetry in Kähler potential, it is somewhat plausible to conclude that the slow-roll parameters can be small, inflation can naturally end, and $\Delta\phi > \mathcal{O}(1)M_{\rm Pl}$ is necessary to earn e-folding number. Such an idea of exploiting an approximate shift symmetry and monodromy in the complex structure moduli for inflation in TypeIIB (or F-theory) has been presented in some literatures. This is a mirrorversion of the same set of ideas exploited in axion monodromy inflation [35]. In this sense, the use of complex structure moduli for string inflation is another variation of the same theme that has been pursued for the last decade.

6.2.3 Consistency check1; Estimate the masses of moduli

In this subsubsection and the next, we will be devoted to some consistency checks of the scenario that complex structure moduli in TypeIIB theory is an inflaton while Kähler moduli are stabilized by racetrack mechanism. Let us first estimate the masses of moduli T and z, and discuss the phenomenology lead by the result. We neglect dilaton τ in this subsubsection because its mass is expected to be comparable to the mass of complex structure moduli.

Now let's get started. We first split the moduli into their vacuum value and fluctuations around it;

$$T = \langle T \rangle + \delta T, \quad z = \langle z \rangle + \delta z. \tag{6.32}$$

Then the lagrangian (excluding the dynamics of dilaton τ) is given by

$$\begin{aligned}\mathcal{L} =& - \langle K_{T\bar{T}} \rangle |\partial(\delta T)|^2 - \langle K_{z\bar{z}} \rangle |\partial(\delta z)|^2 \\ &- \left(\overline{\delta T}, \overline{\delta z}\right) \cdot \begin{pmatrix} \langle \partial^2 V/\partial \bar{T} \partial T \rangle & \langle \partial^2 V/\partial \bar{T} \partial z \rangle \\ \langle \partial^2 V/\partial \bar{z} \partial T \rangle & \langle \partial^2 V/\partial \bar{z} \partial z \rangle \end{pmatrix} \cdot \begin{pmatrix} \delta T \\ \delta z \end{pmatrix} \\ =& - |\partial(\widetilde{\delta T})|^2 - |\partial(\widetilde{\delta z})|^2 - \left(\overline{\widetilde{\delta T}}, \overline{\widetilde{\delta z}}\right) \cdot \langle M^2 \rangle \cdot \begin{pmatrix} \delta T \\ \delta z \end{pmatrix}\end{aligned} \tag{6.33}$$

where we have defined the canonical fields

$$\widetilde{\delta T} := (K_{T\bar{T}})^{1/2} \delta T, \quad \widetilde{\delta z} := (K_{z\bar{z}})^{1/2} \delta z, \tag{6.34}$$

and physical mass squared matrix

$$M^2 := \begin{pmatrix} K_{T\bar{T}}^{-1} \dfrac{\partial^2 V}{\partial \bar{T} \partial T} & K_{T\bar{T}}^{-1/2} K_{z\bar{z}}^{-1/2} \dfrac{\partial^2 V}{\partial \bar{T} \partial z} \\ K_{T\bar{T}}^{-1/2} K_{z\bar{z}}^{-1/2} \dfrac{\partial^2 V}{\partial \bar{z} \partial T} & K_{z\bar{z}}^{-1} \dfrac{\partial^2 V}{\partial \bar{z} \partial z} \end{pmatrix} =: \begin{pmatrix} M_{\bar{T}T}^2 & M_{\bar{T}z}^2 \\ M_{\bar{z}T}^2 & M_{\bar{z}z}^2 \end{pmatrix}. \tag{6.35}$$

We would like to estimate each component of this mass squared matrix. If we assume the validness of the rough evaluation of $V \simeq e^K |W|^2 \simeq e^K |cf(z)|^2$, where $f(z)$ is defined as $W_{\rm GVW} = c \int G_3 \wedge \Omega = c \sum_{i=1}^{2(1+h^{2,1})} (m_i - \tau n_i) \Pi_i(z) = c \left(f_R(z) - \tau f_{NS}(z)\right) =: cf(z)$, then the mass of complex structure moduli is

$$\begin{aligned}
\langle M_{\bar{z}z}^2 \rangle &= \left\langle K_{z\bar{z}}^{-1} \frac{\partial^2 V}{\partial \bar{z} \partial z} \right\rangle \simeq \left(\frac{\int \chi \wedge \overline{\chi}}{\int \Omega \wedge \overline{\Omega}} M_{\mathrm{Pl}}^2 \right)^{-1} \frac{1}{(\mathrm{Re}\,\langle T \rangle)^3 \,\langle \mathrm{Im}\tau \rangle \int \Omega \wedge \overline{\Omega}} \frac{|c|^2 |f'(z)|^2}{M_{\mathrm{Pl}}^2} \\
&\simeq \frac{|f'(z)|^2}{\int \chi \wedge \overline{\chi}} \frac{M_{\mathrm{Pl}}^2}{4\pi g_s^{-1} (\mathrm{Re}\,\langle T \rangle)^3} \simeq \frac{|f'(z)|^2}{\int \chi \wedge \overline{\chi}} \frac{\left\langle \widetilde{\mathcal{V}}_6 \right\rangle}{l_s^2 (\mathrm{Re}\,\langle T \rangle)^3} \simeq M_{\mathrm{KK}}^2 \left(\frac{M_{\mathrm{KK}}}{M_s} \right)^4 . \qquad (6.36)
\end{aligned}$$

In the last approximate equality we have assumed $|f'(z)|^2 / \int \chi \wedge \overline{\chi} \simeq 1$ and used the relation $M_{\mathrm{KK}} = \langle \mathcal{V}_6 \rangle^{1/6} = \left\langle \widetilde{\mathcal{V}}_6 \right\rangle^{1/6} l_s^{-1} \simeq (\mathrm{Re}\,\langle T \rangle)^{1/4} l_s^{-1}$. This means the mass of complex structure moduli $M_{\bar{z}z}$ is $(M_{\mathrm{KK}}/M_s)^2$ times as large as the Kaltza-Klein mass scale M_{KK}. This is consistent with the complex structure moduli stabilization and inflaton mass.

What we do next is to estimate the mass of Kähler modulus T. To this end, let us recall the estimation of A in subsubsection 4.3.4, the result of which is written down as Eq.(4.109); $A \simeq a(z) N^2 M_{\mathrm{Pl}}^2 / (4\pi \langle \mathrm{Re}T \rangle)^{3/2}$.

$$\begin{aligned}
\langle M_{\bar{T}T}^2 \rangle &= \left\langle K_{T\bar{T}}^{-1} \frac{\partial^2 V}{\partial \bar{T} \partial T} \right\rangle \\
&\simeq \left(\frac{3}{4 \langle \mathrm{Re}T \rangle^2} M_{\mathrm{Pl}}^2 \right)^{-1} \frac{1}{\langle \mathrm{Re}T \rangle^3 \langle \mathrm{Im}\tau \rangle \int \Omega \wedge \overline{\Omega}} \left(\frac{2\pi}{N} \right)^2 |A|^2 |e^{-2\pi T/N}|^2 M_{\mathrm{Pl}}^{-2} \\
&\simeq \frac{|a(z)|^2}{g_s \int \Omega \wedge \overline{\Omega}} N^2 |e^{-2\pi T/N}|^2 \frac{M_{\mathrm{Pl}}^2}{\langle \mathrm{Re}T \rangle^4} \simeq \frac{4\pi N^2 |e^{-2\pi T/N}|^2}{g_s^2} \left(\frac{M_{\mathrm{KK}}}{M_s} \right)^8 M_{\mathrm{KK}}^2 . \qquad (6.37)
\end{aligned}$$

Here in the last equality we used $|a(z)|^2 / (g_s \int \Omega \wedge \overline{\Omega}) \simeq 1$ which is reasonable from Eq.(4.110). This result suggests that the mass of the Kähler modulus is as large as or somewhat smaller than M_{KK}, which is consistent with the assumption of Kähler moduli stabilization.

In the calculation above we assume V to be $e^K |W|^2$, but in the original paper [1] it is assumed that $V \sim e^K |K_{T\bar{T}}^{-1} \partial_T W|^2$ in the calculation. The difference is the factor $(\frac{2\langle \mathrm{Re}T \rangle}{3} \frac{2\pi}{N})^2$, which is of the order of unity if we take $\langle \mathrm{Re}T \rangle = 25$ and $N = 67$ as we have done in subsubsection 5.1.2. Therefore we do not care the difference.

What remains here is the estimation of the mixing mass $M_{\bar{z}T}^2$, which does not vanish by virtue of the z dependence of A in $W^{(T)} = Ae^{-2\pi T/N}$. Its value can be estimated as follows;

$$\begin{aligned}
\langle M_{\bar{z}T}^2 \rangle &= \left\langle K_{T\bar{T}}^{-1/2} K_{z\bar{z}}^{-1/2} \frac{\partial^2 V}{\partial \bar{z} \partial z} \right\rangle \\
&\simeq \langle \mathrm{Re}T \rangle \sqrt{\frac{\int \Omega \wedge \overline{\Omega}}{\int \chi \wedge \overline{\chi}}} M_{\mathrm{Pl}}^2 \frac{1}{\langle \mathrm{Re}T \rangle^3 \langle \mathrm{Im}\tau \rangle \int \Omega \wedge \overline{\Omega}} \left| \frac{a'(z)}{a(z)} \right| \frac{2\pi}{N} \frac{|A|^2}{M_{\mathrm{Pl}}^2} |e^{-2\pi T/N}|^2 \\
&\simeq g_s M_{\mathrm{KK}}^2 \left(\frac{M_{\mathrm{KK}}}{M_s} \right)^{12} \left| \frac{a'(z)}{a(z)} \right| \frac{N^3 |e^{-2\pi T/N}|^2}{8\pi} \sqrt{\frac{\int \Omega \wedge \overline{\Omega}}{\int \chi \wedge \overline{\chi}}}.
\end{aligned} \tag{6.38}$$

If we assume $\langle M_{\bar{z}T} \rangle = M_{\mathrm{KK}}(M_{\mathrm{KK}}/M_s)^6$, $M_{\mathrm{KK}} = M_{\mathrm{GUT}} = 10^{16}\mathrm{GeV}$ and $(M_{\mathrm{KK}}/M_s) = 10^{-2\sim-1}$, then the mixing mass is of the order of $10^{16+6\times(-2\sim-1)=4\sim10}\mathrm{GeV}$. This means that Kähler moduli can decay into standard model particle through the mass mixing with complex structure moduli. This is a good news for phenomenology because even if the mass of Kähler moduli is light enough to cause the cosmological moduli problem, that decay via mixing with complex structure moduli helps us to avoid the problem. In the context of Kähler moduli inflation which we do not discuss much in this paper, this mixing mass enables Kähler moduli, the inflaton, to decay into standard model particles and cause reheating process. Therefore this mixing mass is a one of the main positive result in the paper [1].

In summary, using the order estimation of prefactors c and A in the superpotential, we have estimated the moduli masses and showed that they take reasonable value for the scenario to work. Especially, the not-small mixing mass has the potential to play a crucial roll in Kähler moduli inflation models.

6.2.4 Consistency check2; Kähler moduli stabilization

In our scenario of complex structure moduli inflation, one direction of complex structure moduli plays the roll of inflaton and other moduli are stabilized by non-perturbative effects. Kähler moduli are stabilized by racetrack mechanism. Then what we are going to do in this subsubsection is to investigate the consistency between racetrack mechanism and inflation driven by complex structure moduli. The difficulty of the consistency is apparent because we need to fine-tune the superpotential to make Kähler moduli stabilized as we have seen in subsubsection 5.1.2, while inflaton rolls during inflation and changes its value more than Planck scale and thus the value of superpotential also changes to some extent, which may violate the fine-tuning. We therefore in this subsubsection evaluate the allowed range of motion of the superpotential $W^{(\mathrm{cpx})}(z,\tau)$ to keep the Kähler moduli stabilized.

To study the constraint from volume destabilization, one should use the full scalar potential (6.11), but it is messy, complicated, and even worse, depends very much on the choice of Calabi–Yau geometry for compactification. Thus it is difficult to draw a definite conclusion without specifying the internal space X_6. We instead try to acquire an implication which does not depend on the choice of X_6. To this end, we consider the following lagrangian

$$\begin{aligned}
\mathcal{L} &= -K_{T\bar{T}}\partial_\mu T\partial^\mu \bar{T} - V_{\text{racetrack}}, \\
V_{\text{racetrack}} &= e^K\left(K^{T\bar{T}}|D_T W^{(\text{tot})}|^2 - 3|W^{(\text{tot})}|^2\right),
\end{aligned} \tag{6.39}$$

where

$$\begin{aligned}
K &= -3\ln(T+\bar{T}), \\
W^{(\text{tot})} &= W^{(T)} + W^{(\text{cpx})} = \frac{M_{\text{Pl}}^3}{[(4\pi)\text{Re}\,\langle T\rangle]^{3/2}}\left(\sum_{i=1,2} a_i N_i^2 e^{-\frac{2\pi}{N_i}T}\right) + W^{(\text{cpx})}.
\end{aligned} \tag{6.40}$$

The differences between this and the full theory in Eq.(6.11) with given values of τ and z are the overall factor $e^{K(\text{cpx})}$ and F-terms of z and τ in the potential.

The parameters in Eq.(6.40) are taken to be the most optimistic as follows (see subsubsection 5.1.2 for the reason);

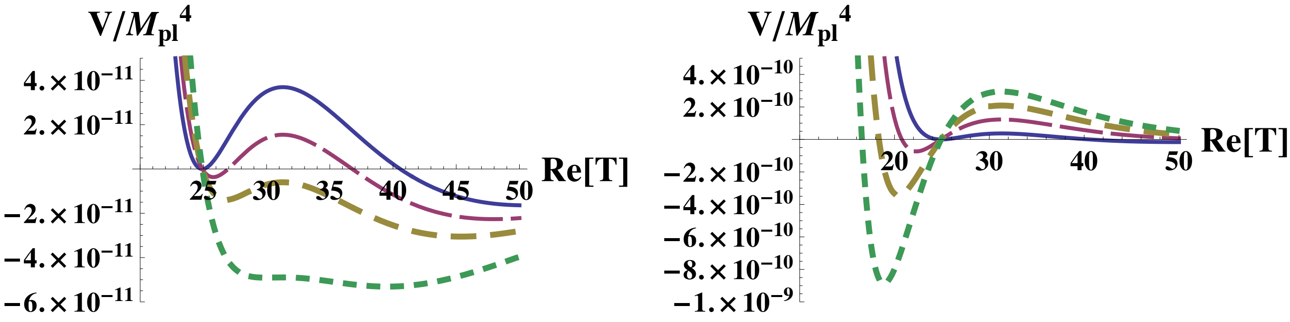
$$a_1 = 1, \quad r := a_2/a_1 = -1.05041, \quad N_1 = N_2 + 1 = 68. \tag{6.41}$$

In addition, $W^{(\text{cpx})}$ is tuned so that $\left\langle W^{(\text{tot})}\right\rangle = 0 \iff \left\langle W^{(\text{cpx})}\right\rangle = -\left\langle W^{(T)}\right\rangle$ and thus the vacuum is Minkovski, $\langle V\rangle = 0$. It is not forbidden to assume this tuning, but the difference comes into effect during inflation by complex structure moduli because z is not on the vacuum and thus $W^{(\text{cpx})} \neq \left\langle W^{(\text{cpx})}\right\rangle$. We thus focus on δW defined as

$$W^{(\text{cpx})} = -\left\langle W^{(T)}\right\rangle + \delta W \tag{6.42}$$

and investigate how much $|\delta W|$ can becomes large without destabilizing the Kähler modulus.

Note that a_i also depends on the complex structure moduli and thus changeable during inflation. Therefore we have two independent changeable parameters $\delta W/a_1$ and $r = a_2/a_1$ in terms of the shape of the potential $V_{\text{racetrack}}$. Although it is desirable to study the deformation of $V_{\text{racetrack}}(T)$ for the complex two-dimensional parameter space of deformation, $(\delta W/a_1, a_2/a_1)$, we will carry out the study only along three real 1-parameter deformations in the parameter space in the following three cases; (i) variation of $\delta W/a_1$ with fixed r, (ii) variation of r with fixed δW, and (iii) variation of both $\delta W/a_1$ and r.



(a) (b)

Figure 3: The potential $V_{\text{racetrack}}(T)$ for various values of $\delta W/a_1$; the vacuum parameters are used here, and the other deformation parameter a_2/a_1 is held fixed at its vacuum value in this figure.
Panel (a): the four curves from top to bottom (solid-blue, long dashed-red, dashed-yellow and dotted-green) correspond to the deformation parameters $\delta W/(a_1 10^{-4} M_{\text{Pl}}^3) = 0$, -1.41, -2.82 and -5.64, respectively.
Panel (b): the potential for the deformation $\delta W/(a_1 10^{-4} M_{\text{Pl}}^3) = 0$, 5.64, 11.3 and 17.0 are shown in the solid, long dashed, dashed and dotted curves, respectively.

(i) variation of $W^{(\text{cpx})}(z)/a_1$

Figure 3 (a, b) shows how $V_{\text{racetrack}}(T)$ changes as we change the value of $(\delta W/a_1)$ while keeping a_2/a_1 fixed at the vacuum value $r = \langle a_2/a_1 \rangle$. Let us first focus on the Panel (a). Starting from the vacuum value $\delta W/a_1 = 0$ (i.e., $W^{(\text{cpx})} = -\langle W \rangle^{(T)} \simeq -0.00122 \times a_1 M_{\text{Pl}}^3$) and adding deformation δW in the negative real-valued region, we see that the barrier in $V_{\text{racetrack}}(T)$ almost disappears by the time $\delta W/a_1 \simeq -3 \times 10^{-4} M_{\text{Pl}}^3$. Looking next at the Panel (b), we see that the minimum of the potential becomes deeper and deeper as we deform $\delta W/a_1$ into positive real-valued region. Thinking of the full theory Eq.(6.11), however, F-terms of z and τ uplift the potential, and the deformation from the Minkovski vacuum must not lead to negative vacuum. Note that the uplifting correction from the F-terms is run-away shape in terms of T. Thus if we deform $\delta W/a_1$ as much as in Panel (b) in the full theory, then the barrier height of the racetrack potential is likely to disappear.

For these reasons, and in this meaning, we see that the volume stabilization in $V_{\text{racetrack}}(T)$ remains to be reliable, as long as the value of $W^{(\text{cpx})}$ differs from its vacuum value $-\left\langle W^{(T)} \right\rangle$ within the range

$$-3 \times 10^{-4} M_{\text{Pl}}^3 \lesssim \delta W/a_1 \lesssim 5 \times 10^{-4} M_{\text{Pl}}^3. \tag{6.43}$$

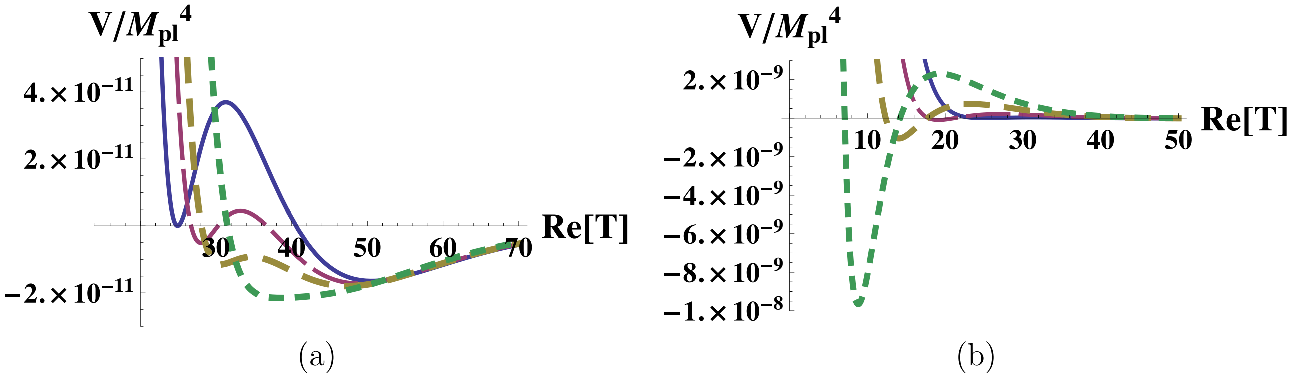


Figure 4: The potential $V_{\text{racetrack}}(T)$ with $W^{(\text{tot})}$ in (**??**) for various values of a_2/a_1; the value of $W^{(\text{cpx})}(z)$ is fixed at W_0 in this figure.
Panel (a): the four curves (solid, long dashed, dashed and dotted) are for $a_2/a_1 \simeq -1.0504$, -1.0530, -1.0550 and -1.0600, respectively.
Panel (b): the parameter value is set at $a_2/a_1 = -1.0504$, -1.0450, -1.0400 and -1.0350 in the curves drawn in the solid, long dashed, dashed and dotted lines, respectively. The potential $V_{\text{racetrack}}(T)$ drawn in the solid line in (a) and (b) are the same, the one at the vacuum $a_2/a_1 = \langle a_2/a_1 \rangle$.

(ii) variation of $a_2/a_1(z)$

Figure 4 shows how $V_{\text{racetrack}}(T)$ changes when the value of a_2/a_1 is different from its vacuum value $r = \langle a_2/a_1 \rangle \simeq -1.05041$, while the value of $W^{(\text{cpx})}(z)$ somehow remains to be $-\left\langle W^{(T)} \right\rangle$, i.e. $\delta W/a_1 \equiv 0$. We can use the numerical results in the figure, to set a limit

$$-1.0550 \lesssim a_2/a_1 \lesssim -1.0400 \tag{6.44}$$

for the same reason as in the analysis of changing $\delta \widetilde{W}^{(\text{cpx})}_{\text{eff.}}/a_1$. The potential $V_{\text{racetrack}}(T)$ is highly sensitive to the value of a_2/a_1, when the value of $W^{(\text{cpx})}(z)/a_1$ is held fixed, and there is not much room around the vacuum value $\langle a_2/a_1 \rangle \simeq -1.05041$. This result is not hard to imagine from the discussion in subsubsection 5.1.2, because of the high sensitivity of the potential $V_{\text{racetrack}}(T)$ on the vacuum value $\langle a_2/a_1 \rangle$.

If we replace the vacuum parameters in (6.41) by more negative $r = \langle a_2/a_1 \rangle$ and smaller N_1, we expect larger range of deformation in a_2/a_1 from the new vacuum value will be allowed, than in (6.44); this comes at the cost of limiting the energy density during inflation from above, however.

(iii) variation of both $\delta W/a_1$ and $a_2/a_1(z)$

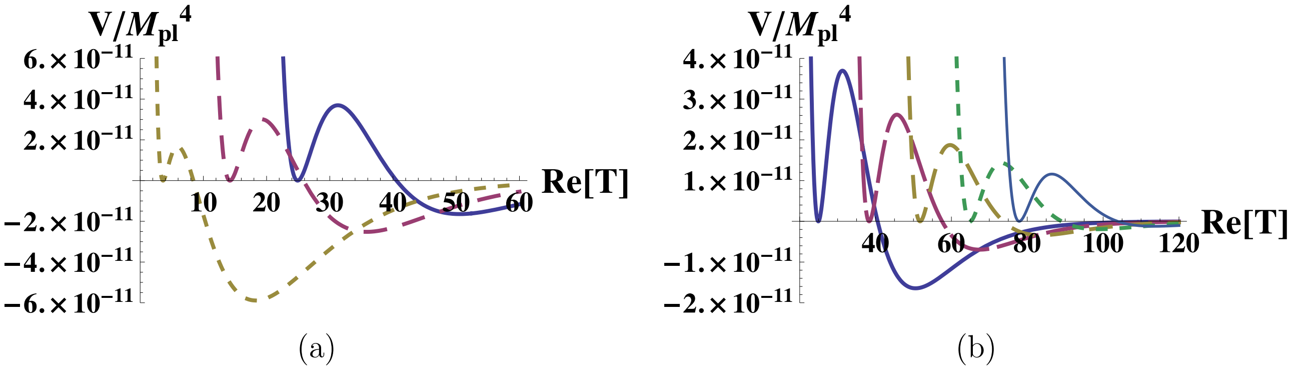


Figure 5: The potential $V_{\text{racetrack}}(T)$ with different sets of deformation parameters $(\delta W/a_1, a_2/a_1)$ satisfying the relation (6.45).
Panel (a): $V_{\text{racetrack}}(T)$ is drawn for the deformation parameters $(\delta W/(a_1 10^{-2} M_{\text{Pl}}^3), a_2/a_1) = (0, -1.0504)$, $(-0.20, -1.0350)$ and $(-0.76, -1.0200)$ in the solid-blue, dashed-red and dotted-yellow line, respectively, after multiplied by 1, 10^{-1} and 10^{-2}, respectively; this means, for example, that the local maximum of the potential for $(-0.76, -1.0200)$ is about $2\times10^{-9}\times M_{\text{Pl}}^4$.
Panel (b): the solid-blue, long dashed-red, dashed-yellow, dotted-green and thin solid-blue lines show the potential $V_{\text{racetrack}}(T)$ for the sets of deformation parameters $(\delta W/(a_1 10^{-2} M_{\text{Pl}}^3), a_2/a_1) = (0, -1.0504)$, $(0.88, -1.07)$, $(1.13, -1.09)$, $(1.18, -1.11)$ and $(1.21, -1.13)$, respectively, multiplied by a factor 1, 10, 10^2, 10^3 and 10^4, respectively; this means, for example, that the local maximum of $V_{\text{racetrack}}(T)$ is about $10^{-15}\times M_{\text{Pl}}^4$ for the deformation $(1.21, -1.13)$.

We have so far searched only along two real-valued 1-parameter deformations in the phenomenological parameter space $(\delta W/a_1, a_2/a_1)$ of $V_{\text{racetrack}}(T)$. This leaves a room for some combination of changes in $\delta W/a_1$ and in a_2/a_1 so that the potential barrier remains in $V_{\text{racetrack}}(T)$. There is no obvious strategy to look for such a coordinated changes, or to claim their absence. We just try one more, a 1-parameter deformation in $\delta W/a_1$ and a_2/a_1 so that $V_{\text{racetrack}}(T)$ remains to have vanishing energy at the local minimum around $T \sim \langle T\rangle \simeq 25$ (the energy density of the full potential (6.11) is positive); this corresponds to focus on a subspace of $(\delta W/a_1, a_2/a_1)$ satisfying a relation $W^{(\text{tot})}(\langle T\rangle) = 0$, that is

$$\frac{M_{\text{Pl}}^3}{[(4\pi)\text{Re}\,\langle T\rangle]^{3/2}}\left(N_1^2 e^{-\frac{2\pi}{N_1}\langle T\rangle} + \frac{a_2}{a_1}N_2^2 e^{-\frac{2\pi}{N_2}\langle T\rangle}\right) - \left\langle W^{(T)}\right\rangle/a_1 + \delta W/a_1 = 0. \tag{6.45}$$

As in Figure 5 (a), for the choice of a_2/a_1 closer to zero than the vacuum value $\langle a_2/a_1\rangle \simeq -1.0504$ [i.e., $\delta W/a_1 \le 0$], the potential barrier $V_{\text{racetrack}}(T)$ becomes higher (read the caption carefully), but the second minimum (at larger value of T) also gets deeper more rapidly. The

Kähler moduli field T may remain marginally stable in $V_{\rm racetrack}(T)$ for $a_2/a_1 = -1.035$, but it will not be against quantum tunneling for a value even closer to zero. Thus, this destabilization argument sets a limit in the deformation satisfying (6.45) as follows:

$$-0.20 \times 10^{-2} M_{\rm Pl}^3. \lesssim \delta W/a_1, \qquad a_2/a_1 \lesssim -1.035. \tag{6.46}$$

With the coordinated deformation in (6.45), certainly the destabilization limit above has been relaxed a bit from (6.43, 6.44), but not much.

As one can see in Figure 5 (b), for deformation in the other direction, however, the second minimum in the potential $V_{\rm racetrack}(T)$ becomes less and less pronounced for even more negative value of a_2/a_1 relatively to the vacuum value -1.0504. There may be no danger of destabilization for such a change, and the lower bound on a_2/a_1 disappears. This considerably relaxes the constraint $-1.0550 \lesssim a_2/a_1$ obtained earlier (only the $W^{(\rm cpx)} \leq 0$ region is probed under the relation (6.45)). The potential barrier height is also reduced considerably at the same time, however, for such a deformation in $(\delta W/a_1, a_2/a_1)$; see the caption of the figure. In order to support inflation at high energy scale, that may not be appropriate.

Summary and implication

We have found in the discussion above that there are strong constrains on the deformation parameters $(\delta W/a_1, a_2/a_1)$ in $V_{\rm racetrack}$. The constraint on a_2/a_1 is stringent, but we do not know this is avoidable or not. We therefore could not draw any decisive conclusion from this. On the other hand, the restriction $|\delta W/a_1| \lesssim 10^{-2} M_{\rm Pl}^3$, even in the conspire case (iii), suggests a strong statement that the motion of inflaton must not change the value of the superpotential not to destabilize the Kähler moduli. To see how strong this is, we take again the example of mirror quintic. In that case, the superpotential as a function of inflaton φ is of the form (see Eq.(6.21))

$$W_{\rm GVW} = c\left(f_R^{(3)} - \tau f_{NS}^{(3)}\right)(z = ie^{2\sigma/M_{\rm Pl}}(1 + i2\varphi/M_{\rm Pl})), \tag{6.47}$$

where $f_{R/NS}^{(3)}(z)$ is given in Eq.(6.22). Then the superpotential is third degree polynomial of φ whose coefficients are of the order of

$$\frac{1}{\sqrt{4\pi}} \mathcal{O}(10^{-2\sim 1})(\text{flux}) e^{2\times(1\sim 3)\langle\sigma\rangle/M_{\rm Pl}} \tag{6.48}$$

in Planck unit. Thus it is difficult to earn e-folding number (which means $\Delta\varphi > \mathcal{O}(1) M_{\rm Pl}$) while suppressing $|\delta W|$ to be smaller than $10^{-2} M_{\rm Pl}^3$. We need at least special mechanism to avoid this problem.

6.2.5 Summary and conclusion

In this subsection 6.2, we have investigated the possibility that inflation is driven by the complex structure moduli in TypeIIB theory which can be identified with the right-handed sneutrino in MSSM, while stabilizing the Kähler moduli by racetrack mechanism. We have seen that the complex structure moduli can have flat enough potential as an inflaton due to the shift symmetry in the Kähler potential. We have also evaluated the masses of the moduli and found that the values are not dangerous for phenomenology. Especially, the not-small mixing mass between the Kähler moduli and complex structure moduli has an advantage in reheating process. In spite of all these positive results, we have shown that it is difficult to make the scenario consistent with the Kähler moduli stabilization by racetrack mechanism, because the superpotential of the complex structure moduli must not change more than $10^{-2}M_{\rm Pl}^3$ while inflaton must move more than $\mathcal{O}(1)M_{\rm Pl}$ to earn e-folding number. In conclusion inflation by complex structure moduli, i.e. right-handed sneutrino inflation from TypeIIB string theory, is not possible unless we have a special mechanism to avoid the problem.

7 Conclusion

We have seen how to derive and test inflation models in the framework of string theory. We have especially focused on the right-handed sneutrino inflation in TypeIIB theory and found both the positive results, the success in translating the control of the inflaton potential into the language of string theory, and the estimation of the masses of inflaton and moduli which turned out to be consistent with the scenario, and the negative result, the insight from numerical calculation that inflation driven by complex structure moduli and Kähler moduli stabilization by racetrack mechanism seems incompatible. We need to consider totally different scenarios or to invent some tricky mechanism to remove the incompatibility. Here is the list of the possibilities;

1. We cannot predicate the inflation in the language of 4D effective field theory.

2. 4D effective field theory from string theory works but complex structure inflation is nonsense, i.e. inflaton is other particles such as Kähler moduli or axions.

3. Complex structure inflation is true but other mechanism than racetrack which stabilizes the Kähler moduli is needed.

4. There is a vacuum setup where we can tune the 3-form fluxes so that the superpotential of the complex structure moduli does not change, and thus the problem is solved.

5. Other possibility the author have missed is the truth.

The fourth possibility is the easiest to be investigated, so why not? That is because the author has to study other topics before being totally absorbed in this topic.

In summary, string inflation has A LOT to be studied and far more endeavor is required to achieve something testable.

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