From Sequential Equilibrium to Perfect Equilibrium: Revisit of Okada (1991)

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Abstract

Perfect equilibrium and sequential equilibrium are popular equilibrium concepts for finite extensive-form games. We find a relatively simple necessary and sufficient condition making sequential equilibrium perfect. We interpret the condition while referring to Okada's (1991) lexicographic domination. When each path includes at maximum two decision nodes, any lexicographically undominated strategy combination is a perfect equilibrium. In addition, we indirectly discuss "perfect equilibrium" in games with uncountable actions via lexicographic domination, which is applicable to uncountable actions.

1 Introduction

Perfect equilibrium and sequential equilibrium are popular equilibrium concepts for finite extensive-form games. Selten (1975) proposed *perfect equilibrium*, in which players are cautious about the possibility of future error. However, perfect equilibrium is difficult to calculate. To avoid this difficulty, Kreps & Wilson (1982) proposed a slightly weaker concept, *sequential equilibrium*, in which players select their best local strategies at each information set.We find a relatively simple necessary and sufficient condition making sequential equilibrium perfect. In other words, we find

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a procedure that supports finding perfect equilibrium. We call this condition From Sequential equilibrium To Perfect equilibrium (FSTP).

FSTP implies that perfect equilibrium does not require a common idea about future errors in some situations. For example, consider a simple game in which each path includes two decision nodes (hereafter a short game). If players avoid Okada's (1991) lexicographically dominated strategies, the strategy profile is a perfect equilibrium.

In the process of analyzing short games, we found the linearity of utility function (Lemma 2). It looks valid only in a short game, but this linearity exists in general settings and we analyze it in another project Jinushi (2024).

Okada's (1991) lexicographically undominated strategy combination does not require a sequence of completely mixed strategy profiles. In Appendix A.6, we apply it to a 3-player short-game with infinite actions. As a result, we obtain a set of "perfect equilibria" in the infinite-action game. Our discussion leads to further investigation for finding a perfect equilibrium for infinite-action games.

2 FSTP in Finite Extensive-Form Games

In this section, we first formulate a standard finite extensive-form game. Then, we define perfect equilibrium and sequential equilibrium. We derive the necessary and sufficient condition with which sequential equilibrium is perfect. Although the difference has been discussed[1], to our knowledge, this is the first time that the exact condition has been derived. We interpret the condition while referring to a related concept in Okada (1991).

We employ the standard concepts and terminologies in the literature (e.g. Kuhn(1953), Selten (1975) and van Damme (1984)).

A n-player finite perfect-recall extensive-form game $\Gamma = (K, P, U, p, h)$ consists of the following five elements: the game tree K, the player partition $P = (P_0, ..., P_n)$, the information partition $U = (U_0, ..., U_n)$, the nature's probability distribution p, and the payoff function h. K consists of non-terminal nodes in the finite set X including the origin $\emptyset$, terminal nodes in the finite set Z and directed links towards terminal nodes. A_u represents the set of choices at each u.

[1] See Kreps & Wilson (1982), Blume & Zame (1994) and Okada (1991).

We call each combination of nodes from the origin $\emptyset$ to $z \in Z$ a play. We call each combination of nodes from $x \in X$ to $x' \in X$ a path. In each play, a node from each information set can exist at a maximum of once. Player 0 follows an exogenous completely mixed probability distribution p_u over A_u at each information set $u \in U_0$ such that $p_u(a_u) > 0$ for all $a_u \in A_u$. u is a singleton if $u \in U_0$. The payoff function h: $Z \to \mathbb{R}^n$ represents the players' payoff from $z \in Z$.

We denote a probability distribution b_{iu} over A_u as a local strategy for player $i \in I$ at an information set $u \in U_i$. When there exists $a_u \in A_u$ such that $b_{iu}(a_u) = 1$, we call b_{iu} a pure local strategy. A behavior strategy $b_i \in B_i$ is a combination of each local strategy for player i ($b_i = (b_{iu})_{u \in U_i}$). A pure behavior strategy $b_i \in PB_i$ is a combination of each pure local strategy for player i. b/b'_{iu} is a strategy profile following b'_{iu} at u and all other local strategy remain unchanged from b.

For a given strategy profile, the realization probability of each $x \in X$ and $z \in Z$, denoted by $\rho(x, b)$ and $\rho(z, b)$, is uniquely determined. The realization probability of an information set u is $\rho(u, b) = \sum_{x \in u} \rho(x, b)$. $\rho(z, b|x)$ is the conditional realization probability of $z \in Z$ after the players reached x. If $x \in X$ is not in the path from $\emptyset$ to $z \in Z$, then $\rho(z, b|x) = 0$.

We introduce a local belief $\rho_u(x)$ as the probability of each node $x \in u \in U_i$ the player i believes at u. For any $i \in I$, a belief ρ is a function from $u \in U_i$ to a local belief ρ_u.

Definition 1 *A strategy profile $b \in B$ is completely mixed iff $b_{iu}(a_u) > 0$ for all $a_u \in A_u$ and for all $u \in U$.*

When $b \in B$ is completely mixed, the realization probability of $x \in u$ and the belief, $\rho_u(x) = \rho(x, b)/\rho(u, b)$, at each information set are uniquely determined. Such a belief is called a *consistent* belief with b. Kreps & Wilson (1982) extend this idea to construct a rational belief for any $b \in B$ in the following way:

Definition 2 *An assessment (b, ρ) is consistent if there exists a sequence of completely mixed strategy profiles and beliefs $(b^j, \rho^j) \to (b, \rho)$ where ρ^j is consistent with b^j.*

We denote CO as a mapping from $b \in B$ to a set of ρ such that (b, ρ) is consistent. For each sequence of completely mixed strategy profiles $b^j \to b$, there exists a unique sequence

$\rho^j \in CO(b^j)$. This sequence may not converge to any point, but it always includes a convergent subsequence because the sequence is in a compact space (Bolzano-Weierstrass Theorem). Hereafter, we consider such a subsequence and skip this explanation.

The ex ante expected payoff vector $H(b) = (H_1(b), ..., H_n(b))$ is

$$H(b) = \sum_{z \in Z} \rho(z, b) h(z) \tag{1}$$

The expected payoff vector at an information set u is

$$H(b, u|\rho) = \sum_{x \in u} \rho_u(x) \sum_{z \in Z} \rho(z, b|x) h(z). \tag{2}$$

When b is a completely mixed strategy profile, since $\rho(u, b) > 0$ for any $u \in U$, a consistent belief ρ is uniquely decided. For such situations, we sometimes denote $H(b, u)$ instead of $H(b, u|\rho)$.

We claim the following basic characteristic of the expected payoff vector at an information set:

Lemma 1 *Consider a sequence of completely mixed strategy profiles and consistent beliefs s.t.* $(b^m, \rho^m) \to (b, \rho)$. *Then,* $H(b^m/pb_{iu}, u|\rho^m) \to H(b/pb_{iu}, u|\rho)$ *for any* $i \in I$,$u \in U_i$ *and* $pb_{iu} \in PB_{iu}$.

Proof:See Appendix A.1.

Hendon et al. (1996) shows the following representation for sequential equilibrium:

Definition 3 *An assessment* (b, ρ) *is a sequential equilibrium if and only if there exists a sequence* $(b^m, \rho^m) \to (b, \rho)$ *s.t.* b^m *is a completely mixed strategy profile,* $\rho^m \in CO(b^m)$ *and for all* $i \in I$*, for all* $u \in U_i$ *and for all* $b'_{iu} \in PB_{iu}$*,*

$$H_i(b, u|\rho) \geqq H_i(b/b'_{iu}, u|\rho) \tag{3}$$

$SE(\Gamma)$ denotes the set of all sequential equilibria strategy profiles $b \in B$ in Γ. The set of beliefs ρ satisfying sequential equilibria for $b \in SE(\Gamma)$ is denoted by $SEB(\Gamma, b)$.

The following representation for perfect equilibrium is useful for comparison[2].

Definition 4 *A strategy profile b is perfect equilibrium if and only if there exists a sequence $b^m \to b$ s.t. b^m is a completely mixed strategy profile for all $i \in I$, for all $u \in U_i$ and for all $b'_{iu} \in PB_{iu}$,*

$$H_i(b^m/b_{iu}, u) \geqq H_i(b^m/b'_{iu}, u) \tag{4}$$

The set of all perfect equilibria $b \in B$ in Γ is denoted by $PE(\Gamma)$.

From Lemma 1, Definition 3 and Definition 4, we get the following theorem.

Theorem 1 (From Sequential Equilibrium To Perfect Equilibrium) *For $b \in SE(\Gamma)$ in an incomplete-information game Γ, the following conditions A and B are equivalent.*

A: $b \in PE(\Gamma)$

B: There exist a belief system $\rho \in SEB(\Gamma, b)$, a sequence of completely-mixed strategy profiles and beliefs $(b''^k, \rho^k) \to (b, \rho)$ s.t. $\rho^k \in CO(b''^k)$, which satisfies, for any player $\forall i \in I$, at any information set $u \in U_i$ and each pure strategy profile $b' \in PB$ suchthat $H_i(b, u|\rho) = H_i(b/b'_{iu}, u|\rho)$, the following condition:

$$H_i(b''^k/b_{iu}, u|\rho^k) \geqq H_i(b''^k/b'_{iu}, u|\rho^k) \tag{5}$$

Proof:See Appendix A.2.

Theorem 1 provides a new insight into trembling-hand perfect equilibrium. Players have to be cautious only if the game is complicated. For example, consider a perfect-information game and a sequential equilibrium with a single multi-best-reply information set u. At u, if the player takes care of an impact on the payoff from any arbitrary possible error after u, the player would pick a choice in the strategy profile of a perfect equilibrium. Some tie-breaking reasonings, like selecting a choice with the max max possible outcome based on future errors, also select a choice in one of the strategy profiles in the set of perfect equilibria if they impact the player's payoff.

[2]Selten (1975) defines perfect equilibrium as a limit of Nash equilibria in a sequence of perturbed agent-normal-form games converging to the original agent-normal-form game. If and only if this definition is satisfied, the sequence of completely mixed strategy profiles makes all local strategies in the limit of strategy profiles optimal.

In simple settings, a trembling-hand perfect equilibrium can be explained by the optimization of each information set with a rough tie-breaking rule.

To understand when simple settings and the individual tie-breaking rules lead to perfect equilibrium, we refer to Okada (1991)'s *lexicographically undominated strategy combination.*

Definition 5 (Proposition 2.4 of Okada (1991)) *In $b \in B$, b_{iu} lexicographically dominates $b'_{iu} \Leftrightarrow \exists$ some neighborhood $O \subset B$ of b such that*

$$H_i(b''/b_{iu}, u) > H_i(b''/b'_{iu}, u) \tag{6}$$

for any completely mixed strategy profile $b'' \in O$[3]*.*

A strategy profile b is called a lexicographically undominated strategy combination when each b_{iu} is not lexicographically dominated by any $b'_{iu} \in B_{iu}$ for any $i \in I$ and at any $u \in U_i$. A lexicographically undominated strategy combination does not require players to share a similar idea about the impact of future errors.

In the following paragraphs, we show that this lexicographically undominated strategy combination is necessary and sufficient for trembling-hand perfection in short games. For the preparation, we claim the following lemmas:

Lemma 2 *In Γ where each path includes a maximum of two decision nodes,*

$$H_i((1-\epsilon)b + \epsilon b''/b_{iu}, u) = (1-\epsilon)H_i(b/b_{iu}, u) + \epsilon H_i(b''/b_{iu}, u) \tag{7}$$

for any $b, b'' \in B'$, $i \in I$, $u \in U_i$ and $\epsilon \in (0, 1)$.

Proof: See Appendix A.3.

Lemma 3 *In Γ where each path includes a maximum of two decision nodes, if b_{iu} is not lexicographically dominated in b, there exists a sequence of completely mixed strategy profiles $b^k \to b$ such that for any $b'_{iu} \in B_{iu}$,*

$$H_i(b^k/b_{iu}, u) \geqq H_i(b^k/b'_{iu}, u). \tag{8}$$

[3] Here, O does not include b.

Proof: See Appendix A.4.

In conclusion, we obtain the following proposition:

Proposition 1 *When each path includes at maximum two decision nodes in* Γ*, a lexicographically undominated strategy combination is a perfect equilibrium.*

Proof: See Appendix A.5.

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A Online Appendix

A.1 Proof of Lemma 1

Since $\rho(x,b)$ and $\rho(z,b|x)$ are the product of the subset of $\{b_{iu}(a_u)|\forall i \in I, \forall u \in U_i, \forall a_u \in A_u\}$ (and path-specific constant values from the Player 0's moves), $H(b^m/pb_{iu},u|\rho) \to H(b/pb_{iu},u|\rho)$ as $b^m \to b$. Since $\rho^m \to \rho$, $\rho^m_u(x) \to \rho_u(x)$ and so $H(b^m/pb_{iu},u|\rho^m) \to H(b/pb_{iu},u|\rho)$.

A.2 Proof of Theorem 1

$\Rightarrow$: From Definition 4, if $b \in PE(\Gamma)$, there exist a sequence $b^k \to b$ and $\rho^k \to \rho$ s.t. $\rho^k \in CO(b^k)$ and (5) is satisfied for any $u \in U$. Since b^k is completely mixed and $\rho^k \in CO(b^k)$, from Lemma 1, $H_i(b^k/b'_{iu},u|\rho^k) \to H_i(b/b'_{iu},u|\rho)$. Then, from Definition 4, $H_i(b/b_{iu},u|\rho) \geqq H_i(b/b'_{iu},u|\rho)$. Therefore, $\rho \in SEB(\Gamma,b)$ is satisfied.

$\Leftarrow$: Consider $b \in SE(\Gamma)$ and a sequence of completely-mixed strategy profiles and beliefs $(b'^k,\rho^k) \to (b,\rho)$ which satisfies the conditions in Theorem 1. If b'^k satisfies the conditions in Definition 4, this proof would be done. At the limit, b becomes optimal because $b \in SE(\Gamma)$ and $\rho \in SEB(\Gamma,b)$. From Lemma 1, $H_i(b^k/b'_{iu},u|\rho^k) \to H_i(b/b'_{iu},u|\rho)$, and so when there exists only a single optimal choice in the information set u for b and ρ, when k is large enough, b_{iu} is the unique optimal local strategy at u in b'^k. If there exist multiple best choices, the required condition (5) (b_{iu} is weakly better than the other best choices in the sequence before the limit) is satisfied. Therefore, b'^k satisfies the conditions in Definition 4.

A.3 Proof of Lemma 2

Proof: For the player i s.t. $\emptyset \in U_i$, (if $i \neq 0$), H_i at $\emptyset$ from each choice depends on only the local strategy at the following node. In any other information set u, for player j s.t. $u \in U_j$, H_j from each choice depends on ρ_u, and ρ_u depends on $b_{i,\emptyset}$ (or $p_\emptyset$ if $i=0$).

A.4 Proof of Lemma 3

For players $i,i' \in I$[4] and each pair $\emptyset \in U_i$ and $u \in U_{i'} - \{O\}$, we consider a two-player agent-normal-form game $\Gamma'(\emptyset,u)$ such that a pure-strategy set is identical to a choice set $S_i = \{a_\emptyset \in A_\emptyset | \exists x \in u \; s.t. \; a_\emptyset \text{ induces } x \text{ in the game tree } K\}$ and $S_{i'} = A_u$ and the payoff from each outcome is identical.

[4]If either player is nature (player 0), we can apply the similar logic. This is because nature just follows p_u and does not change its behavior in the sequence. For simplicity, we skip the possibility.

Consider $b_{i',u}$ at u s.t. $\rho(u,b) > 0$. Then, there exist a player i's strategy σ_i s.t. $\sigma_i(s_i') = b_{i\emptyset}(s_i')/\sum_{s_i \in S_i} b_{i\emptyset}(s_i)$ and a player i''s strategy $\sigma_{i'} = b_{i'u}$. Since $b_{i'u}$ is not lexicographically undominated in the neighborhood of b in Γ, and $A_\emptyset - S_i$ does not have any impact on $H_{i'}$ in u, the player i''s mixed strategy $\sigma_{i'}$ is not lexicographically dominated around σ. Then, $\sigma_{i'}$ is not weakly dominated (hereafter dominated). In a two-player normal-form game, it is known that, if $\sigma_{i'}$ is not dominated, there exists a completely mixed $\hat{\sigma}_i$ with which $\sigma_{i'}$ is optimal (For example, see Appendix B of Pearce (1984)). Because of Lemma 2, $\sigma_{i'}$ is optimal for $(1-\epsilon)\sigma_i + \epsilon\hat{\sigma}_i$. Therefore, there exists a sequence of completely mixed strategy profiles $\sigma^k \to \sigma$ which justify the player i''s strategy $\sigma_{i'}$ in $\Gamma'(\emptyset, u)$. Any sequence $b^k_{i\emptyset}$ which satisfies $\sigma^k_i(s_i) = b^k_{i\emptyset}(s_i)/\sum_{s_i' \in S_i} b^k_{i\emptyset}(s_i')$ satisfies the requirement in Lemma 3 for u s.t. $\rho(u,b) > 0$.

Second, consider an unreached information set u s.t. $\rho(u,b) = 0$. Since $b_{i'u}$ is lexicographically undominated and $A_\emptyset - S_i$ does not have any impact on $H_{i'}$ in u, the player i''s mixed strategy $\sigma_{i'}$ is not dominated in $\Gamma'(\emptyset, u)$. Then, there exists a completely mixed strategy $\hat{\sigma}$ which justifies $\sigma_{i'}$. Then, any sequence $b^k_{i\emptyset}$ which satisfies $\sigma_i(s_i') = b^k_{i\emptyset}(s_i')/\sum_{s_i \in S_i} b^k_{i\emptyset}(s_i)$ for all $s_i \in S_i$ satisfies the requirement in Lemma 3 for u s.t. $\rho(u,b) = 0$.

Third, for the optimality of $b_{i\emptyset}$, we consider a two-player normal-form game $\Gamma'(\emptyset)$ where an agent of player i at $\emptyset$ and an incomplete dictator D who selects the combination of local strategy at any $u \in U - \{O\}$. There exists a maximum of two decision nodes in each path, and so there is a bijection from a mixed strategy of the dictator in $\Gamma'(\emptyset)$ to the combination of $b_{i'u}$ in Γ both of which give an identical outcome distribution against $b_{i\emptyset}$. σ_D is a mixed strategy that represents a combination of $b_{i'u}$ for all $i' \in I$ and $u \in U_{i'} - \{\emptyset\}$. Since $b_{i\emptyset}$ is lexicographically undominated in Γ, the player i's (unique) mixed local strategy in $\Gamma'(\emptyset)$ which coincides to $b_{i\emptyset}$ is lexicographically undominated, and so $b_{i\emptyset}$ is not dominated. From Pearce (1984) and Lemma 2, there exists a sequence of completely mixed strategy profiles $\sigma^k_D \to \sigma_D$ which justifies the player i's mixed strategy in $\Gamma'(\emptyset)$. Therefore, by using the bijection, there exists a sequence of completely mixed strategy profiles b^k which justify $b_{i\emptyset}$.

A.5 Proof of Proposition 1

Consider a lexicographically undominated strategy combination b. In the following discussion, we ignore the possibility of player 0, because adding player 0 does not have an impact and we can apply a similar discussion, and assume that player 1 selects a decision at $\emptyset$. In the following proof, $ac(x)$ denotes the unique action $a_\emptyset \in A_\emptyset$ such that $a_\emptyset$ induces $x \in X$ in the game tree K. $ac(z)$ implies the unique action $a_\emptyset \in A_\emptyset$ such that $a_\emptyset$ induces $z \in Z$ in the game tree K.

First, we focus on $\emptyset$. Since $b_{1,\emptyset}$ is not lexicographically undominated, there exists a sequence of completely mixed strategy profiles $b^k \to b$ s.t. $H_1(b^k/b_{1,\emptyset}) \geqq H_1(b^k/b'_{1,\emptyset})$ for any $b'_{1,\emptyset} \in B_{1,\emptyset}$. Since

$H_i(b^k/b'_{1,\emptyset})$ depends on b^k_{iu} s.t. $u \in U - \{\emptyset\}$ and does not depend on $b^k_{1,\emptyset}$, $b^k_{1,\emptyset}$ is optimal for any $b^k/b''_{1,\emptyset}$ where $b''_{1,\emptyset} \in B_{1,\emptyset}$. In the following paragraphs in this proof, without further explanation, we consider a sequence of completely mixed b^k s.t. $H_1(b^k/b_{1,\emptyset}) \geqq H_1(b^k/b'_{1,\emptyset})$ for any $b'_{1,\emptyset} \in B_{1,\emptyset}$.

Second, we focus on $u \in U - \{\emptyset\}$ and $i \in I$ s.t. $u \in U_i$. $H_i(b^k/b'_{iu}, u|\rho^k)$ depends on ρ^k_u consistent with $b^k_{1,\emptyset}$. Since each b_{iu} is not lexicographically dominated, there exists a sequence (b^k, ρ^k) s.t. $b^k \to b$ and $\rho^k \in CO(b^k)$ which justifies b_{iu}. Then, there exists the sequence of the ratios $b^k_{1,\emptyset}(ac(x))/\rho(u, b^k)$ for each $x \in u$. $SR(k, u, x)$ denotes the ratio.

Consider a sequence of completely mixed strategy profiles and beliefs $(b'^k, \rho'^k) \to (b, \rho)$ where $\rho'^k \in CO(b'^k)$. Then, from the sequence, we can calculate a sequence of the non-zero realization probabilities for each $u \in U - \{\emptyset\}$ which converges to $\rho(u, b)$. Denote the realization probability of u at kth element in the sequence by $\alpha(k, u)$. Since each choice is connected to a unique node and a unique information set (if they exist), we can disjointly decide the ratio of each choice at $\emptyset$ with which the sequence makes the realization probability converge to $\rho(x, b)$ and the ratio for the optimality is satisfied by the following way: For each $x \in u$, we decide $b^k_{1,\emptyset}(ac(x))$ s.t. $b^k_{1,\emptyset}(ac(x))/\alpha(k, u) = SR(k, u, x)$. Then, we get $b^k_{1,\emptyset} \to b_{1,\emptyset}$ and each b_{iu} is optimal to $b^k_{1,\emptyset}$. For any $i' \in I$ and $u \in U_{i'} - \{O\}$, $b^k_{iu} = b'^k_{iu}$ and so $b_{i,\emptyset}$ is optimal in the sequence. For any choice at $\emptyset$ conncected to an outcome $z \in Z$, we set $b^k_{1,\emptyset}(ac(z)) = b'^k_{1,\emptyset}(ac(z))$. The constructed b^k satisfies $b^k \to b$, and each b_{iu} is optimal in the sequence, so b is perfect.

A.6 Implication for Games with Infinite Actions

In this section, using Proposition 1, we briefly discuss "perfect equilibrium" in games with uncountable actions. The original definition of perfect equilibrium cannot be applied to game with uncountable actions because completely mixed strategy profiles cannot be well defined in the setting. To avoid this problem, we utilize the following alternative definition of lexicographic dominance proposed in Okada (1991).

Definition 6 (Definition 2.2 of Okada (1991)) *In $b \in B$, b_{iu} lexicographically dominates b'_{iu} $\Leftrightarrow$ $\exists$ some neighborhood $O \subset B$ of b such that*

$$H_i(b''/b_{iu}, u) \geqq H_i(b''/b'_{iu}, u) \tag{9}$$

for any strategy profile $b'' \in O$ with at least one strict inequality.

We can apply this definition to games with uncountable actions because completely mixed strategy profiles are absent in this definition. We combine this definition and Proposition 1. When we consider a game with uncountable actions, if each path includes a maximum of two decision nodes, and if each player selects a lexicographically undominated local strategy, this strategy combination can be considered "perfect equilibrium" in such a game.

Consider a game with 3 players where Player 1 selects a number $a_{u1} \in A_{u1} = [2,3] \cup [4,5]$, and then if $a_{u1} \leqq 3$ Player 2 selects a number $a_{u2} \in A_{u2} = [2,3]$, and otherwise Player 3 selects a number $a_{u3} \in A_{u3} = [4,5]$. Player 2 and 3 observe Player 1's number before making their decision.

When both Player 1 and the next player select a unique number, all players get 1. If either Player 1 or the next player selects a noninteger, and if the opponent selects an integer, the former gets 0, and the other players get 1. Otherwise, all players get 0.

There is a type of lexicographically undominated strategy combination. Since any integer lexicographically dominates any noninteger, each player can select only integers in the undominated strategy combination. Player 1 can select any local strategies selecting only integers. In the information set after Player 1's integer, Player 2 or 3 selects the number Player 1 selected. In the other information sets, Player 2 and 3 can select any local strategy selecting only integers.

There are uncountably many other subgame-perfect equilibria where two players select a unique noninteger, but they are not plausible because selecting a noninteger is a vulnerable option for players.

In this paper, we propose a way to discuss Selten's (1975) perfection in games with uncountable actions while avoiding technical challenges that arise from uncountable actions. We subsequently examined a setting where Selten's (1975) perfection is demanded. Our approach can be applied only for games with a maximum of two decision nodes in each path. As a future endeavor, we aim to develop approaches for discussing Selten's (1975) perfection in more general settings.