From Sequential Equilibrium to Perfect Equilibrium: Revisit of Okada (1991)

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Abstract

There are two popular equilibrium concepts for finite extensive-form games, perfect equilibrium and sequential equilibrium. We find a relatively simple necessary and sufficient condition with which sequential equilibrium is perfect. We interpret the condition while referring to lexicographic domination proposed in Okada (1991). In particular, when each path includes at maximum two decision nodes in a game, regardless of the number of players in the game, any lexicographically undominated strategy combination is a perfect equilibrium, and any perfect equilibrium is a lexicographically undominated strategy combination. In addition, we indirectly discuss "perfect equilibrium" in games with uncountable actions via lexicographic domination, which is applicable to uncountable actions.

1 Introduction

Perfect equilibrium and sequential equilibrium are popular equilibrium concepts for finite extensive-form games. Selten (1975) proposed *perfect equilibrium* in which players are cautious about the possibility of future error. However, perfect equilibrium is difficult to calculate. To avoid this difficulty, Kreps & Wilson (1982) proposed a slightly weaker concept, *sequential equilibrium*,

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in which players select their best local strategies at each information set[1]. As a solution, we find a relatively simple necessary and sufficient condition with which sequential equilibrium is perfect. In other words, we find a procedure that directly eases the difficulty of finding perfect equilibrium. We call the condition From Sequential equilibrium To Perfect equilibrium (FSTP).

FSTP is the following condition: When there exist multiple best replies at information sets in sequential equilibrium, we can find a convergent sequence of completely-mixed strategy profiles in which each strategy profile justifies both the belief in the sequential equilibrium and the equilibrium local strategy at the information sets.

FSTP represents a veto by players who are indifferent among multiple choices at information sets in equilibrium. Players with multiple best local strategies can veto each equilibrium by selecting another local strategy outside of the equilibrium. In sequential equilibrium, even when all possible errors will be (relatively) negative for at least one of such players, they keep selecting the equilibrium local strategy. Perfect equilibrium does not allow such a sequential equilibrium, and thus we interpret the condition as a veto.

Although previous literature discussed the relation between sequential and perfect equilibrium, the necessary and sufficient condition was not found before this paper. For example, Okada (1991) pointed out a necessary requirement for sequential equilibrium to be perfect. However, the necessary requirement is enough for only limited situations because it does not require players to share a common idea for future errors (see Proposition 1). FSTP implies that each sequential equilibrium and perfect equilibrium coincide in generic settings. The original paper (Kreps & Wilson 1982) pointed out a similar equivalence, but our condition is more strict. Blume & Zame (1994) proved another strong equivalence that the whole sets of sequential equilibria and perfect equilibria coincide in any finite games with almost all assignments of payoffs to outcomes through algebraic geometry. This previous literature implies the existence of a simple difference between two concepts, and we finally catch the simple difference.

Perfect equilibrium requires that players share a common idea about future errors in standard settings, but we can relax this requirement in some situations. For example, consider a simple game in which each path includes two decision nodes. If players avoid lexicographically

[1]Later, sequential equilibrium is considered a basic concept for many refinements (see Govindan & Wilson (2008)).

dominated strategies proposed in Okada (1991), the strategy profile is a perfect equilibrium. In such a simple game, perfect equilibrium does not require a common idea about future errors.

Okada's (1991) lexicographically undominated strategy combination does not require a sequence of completely mixed strategy profiles. We apply it to a 3-player infinite-action game with two decision nodes in each path. As a result, we obtain a set of "perfect equilibrium" in the game. The example supports that we sometimes want and can find "perfect equilibrium" in settings with infinite actions. However, our approach in this paper can only discuss simple games. Our discussion leads to further investigation for finding a perfect equilibrium for infinite action games. As an example, we utilize our results to analyze a special alternative of completely mixed strategy profiles in Jinushi (2023a). Jinushi (2023b) combines these results with the perfect conditional equilibrium distribution in Myerson and Reny (2020) to propose a "perfect equilibrium" in games with uncountable signals.

2 FSTP in Finite Extensive-Form Games

In this section, we first formulate a standard finite extensive-form game. Then, we refer to the original definitions of perfect equilibrium and sequential equilibrium and consider alternative representations. Using alternative representations, we derive the necessary and sufficient condition with which sequential equilibrium is perfect. Although the difference has been discussed, to our knowledge, this is the first time that the exact condition has been derived. We interpret the condition while referring to a related concept in Okada (1991).

We employ the standard concepts and terminologies in the literature (e.g. Kuhn(1953), Selten (1975) and van Damme (1984))[2].

A n-player finite extensive-form game $\Gamma = (K, P, U, p, h)$ consists of following five elements:

1. The rooted game tree K consists of finite nodes including the origin $\emptyset$ and directed links towards terminal nodes. We denote the set of terminal(/non-terminal) nodes $A(/X)$. K represents a physical order in the game, and since a player makes a decision at each node in X, we call each node in X decision node. Each decision node $x \in X$ is directly connected to finite

[2] As Battigalli (1997) points out, these concepts are "by now standard", and so for the motivations of each requirement, please see Kuhn(1953), Selten (1975) and van Damme (1984).

directed links from x towards terminal nodes. These directed links from x represent alternatives players can select at each $x \in X$. A_x represents the set of alternatives at each x. The game begins at $\emptyset$. When a player picks $a_x \in A_x$ at $x \in X$, the next node connected to x via the link a_x is reached. We call each combination of nodes from the origin $\emptyset$ to $a \in A$ as play. We call each combination of nodes from $x \in X$ to $x' \in X$ as a path.

2. The player partition $P = (P_0, ..., P_n)$ is a partition of X. The set of players is $I = \{1, ..., n\}$, and $I^* = \{0, 1, ..., n\}$ where player 0 represents nature moves which follow an exogenous probability distribution. For any $i \in I^*$, at each decision node $x \in P_i$, player i picks an alternative.

3. The information partition $U = (U_0, ..., U_n)$ is a refinement of P. Each element $u \in U_i$ is called an information set. At each information set $u \in U_i$, player i selects a choice from the set of choices A_u. When player i makes a decision in $u \in U_i$, player i understands that player i is at a node in u but does not know the exact node $x \in u$. We require that, for any $x, x' \in u$, $|A_x| = |A_{x'}| = |A_u|$. For each $u \in U_0$, we assume that u is a singleton. We require that, in each play, a node from each information set can exist at a maximum of once.

4. Player 0 follows an exogenous completely mixed probability distribution p_u over A_u at each information set $u \in U_0$ s.t. $p_u(a_u) > 0$ for all $a_u \in A_u$. The probability assignment p is a combination of p_u for all $u \in U_0$.

5. The payoff function h: $A \to \mathbb{R}^n$ represents the players' payoff from $a \in A$.

We assume the following condition, called *perfect recall* defined in Kuhn (1953): For each player i, if $u, v \in U_i$, and if $x \in u$ comes after $a_v \in A_v$ at $y \in v$ in a path, any $x' \in u$ comes after a_v in any path including x'.

We denote a probability distribution b_{iu} over A_u as a local strategy for player $i \in I$ at an information set $u \in U_i$. A behavior strategy $b_i \in B_i$ is a combination of each local strategy for player i ($b_i = (b_{iu})_{u \in U_i}$).

When we want to replace only a behavior strategy from b_i to b'_i for $i \in I$, we denote b/b'_i. When we want to replace only a local strategy from b_{iu} to b'_{iu} at an information set $u \in U_i$, we denote b/b'_{iu}.

Since we consider a tree structure K, for each $x \in X - \{\emptyset\}$ (or $a \in A$), there is a single direct predecessor $pre(x) \in X$ (or $pre(a) \in X$). In addition, there is a single choice $ac(x) \in A_u$ at the information set u s.t. $pre(x) \in u$ which induces x at the node $pre(x)$. Then, there is a unique smallest finite set $Path(\emptyset, x)$ s.t. any pair of an information set and a choice (u, a_u) along the path from $\emptyset$ to $x \in X$ is in $Path(\emptyset, x)$. By using $Path(\emptyset, x)$, the realization probability of $x \in X$ for a given $b \in B$ is

$$\rho(x, b) = \prod_{\substack{(u,a_u) \in Path(\emptyset,x) \\ i \in I \ s.t. \ u \in U_i}} b_{iu}(a_u) \prod_{\substack{(u,a_u) \in Path(\emptyset,x) \\ s.t. \ u \in U_0}} p_u(a_u). \tag{1}$$

The realization probability of an information set u is $\rho(u, b) = \sum_{x \in u} \rho(x, b)$. By a similar process, we define the realization probability of an outcome $\rho(a, b)$ for any $a \in A$. For each $x \in X$, we define $\rho(a, b|x)$ by using the subset of the information sets and required choices along the path from $\emptyset$ to a and between x and a in the path. If $x \in X$ is not in the path from $\emptyset$ to $a \in A$, $\rho(a, b|x) = 0$.

When the realization probability of an information set $u \in U$ is 0 under b, the expected payoffs in the game are independent of b_{iu}. To consider players' rationality at each information set, we want to calculate the expected utility at each information set. For this purpose, we introduce a local belief $\rho_u(x)$ as the probability of each node $x \in u \in U_i$ the player i believes at u. For any $i \in I$, a belief ρ is a function from $u \in U_i$ to a local belief ρ_u.

Definition 1 *A strategy profile $b \in B$ is completely mixed iff $b_{iu}(a_u) > 0$ for all $a_u \in A_u$ and for all $u \in U$.*

When $b \in B$ is completely mixed, the realization probability of $x \in u$ at the information set u is decided uniquely because $\rho(u, b) > 0$. When there exists a unique belief s.t. $\rho_u(x) = \rho(u, x)/\rho(u, b)$ for each $u \in U$, such a belief is called a *consistent* belief with b. Kreps & Wilson (1982) extend this idea to construct a rational belief for any $b \in B$ with which the realization probability of some information sets is 0 in the following way:

Definition 2 *An assessment (b, ρ) is consistent if there exists a sequence of completely mixed strategy profiles and beliefs $(b^j, \rho^j) \to (b, \rho)$ where ρ^j is consistent with b^j.*

We denote CO as a mapping from $b \in B$ to a set of ρ s.t. (b, ρ) is consistent. For each sequence of completely mixed strategy profiles $b^j \to b$, there exists a unique sequence $\rho^j \in CO(b^j)$. This sequence may not converge to any points, but it always includes a convergent subsequence because the sequence is in compact space (Bolzano-Weierstrass Theorem). Hereafter, we consider such a subsequence and skip this explanation.

The ex ante expected payoff vector $H(b) = (H_1(b), ..., H_n(b))$ is

$$H(b) = \sum_{a \in A} \rho(a, b) h(a) \tag{2}$$

The expected payoff vector at an information set u is

$$H(b, u|\rho) = \sum_{x \in u} \rho_u(x) \sum_{a \in A} \rho(a, b|x) h(a). \tag{3}$$

When b is a completely mixed strategy profile, since $\rho(u, b) > 0$ for any $u \in U$, a consistent belief ρ is uniquely decided. When Γ is a perfect-information game, u includes only an element, and so a consistent belief ρ is uniquely decided. For such situations, we sometimes denote $H(b, u)$ instead of $H(b, u|\rho)$.

We want to mention a following basic characteristic of the expected payoff vector at an information set:

Lemma 1 *Consider a sequence of completely mixed strategy profiles and consistent beliefs s.t.* $(b^m, \rho^m) \to (b, \rho)$. *Then,* $H(b^m/pb_{iu}, u|\rho^m) \to H(b/pb_{iu}, u|\rho)$ *for any* $i \in I, u \in U_i$ *and* $pb_{iu} \in PB_{iu}$.

Proof: Since $\rho(x, b)$ and $\rho(a, b|x)$ are the product of the subset of $\{b_{iu}(a_u)|\forall i \in I, \forall u \in U_i, \forall a_u \in A_u\}$ (and path-specific constant values from the nature moves), $H(b^m/pb_{iu}, u|\rho) \to H(b/pb_{iu}, u|\rho)$ as $b^m \to b$. Since $\rho^m \to \rho$, $\rho^m_u(x) \to \rho_u(x)$ and so $H(b^m/pb_{iu}, u|\rho^m) \to H(b/pb_{iu}, u|\rho)$.

Definition 3 (Sequential Equilibrium (Kreps & Wilson (1982))) *An assessment* (b, ρ) *is sequential equilibrium iff there exists a sequence* $(b^m, \rho^m) \to (b, \rho)$ *s.t.* b^m *is a completely* *mixed strategy profile,* $\rho^m \in CO(b^m)$ *and* $\forall i \in I$, $\forall u \in U_i$ *and* $\forall b'_i \in B_i$,

$$H_i(b, u|\rho) \geqq H_i(b/b'_i, u|\rho) \tag{4}$$

The set of all sequential equilibrium strategy profiles $b \in B$ in Γ is denoted by $SE(\Gamma)$. The set of all belief parts ρ of sequential equilibria for $b \in SE(\Gamma)$ is denoted by $SEB(\Gamma, b)$.

From the main theorem of Hendon et al. (1996), there exists another following representation for sequential equilibrium:

Lemma 2 *An assessment* (b, ρ) *is sequential equilibrium iff there exists a sequence* $(b^m, \rho^m) \to (b, \rho)$ *s.t.* b^m *is a completely mixed strategy profile,* $\rho^m \in CO(b^m)$ *and* $\forall i \in I$, $\forall u \in U_i$ *and* $\forall b'_{iu} \in PB_{iu}$,

$$H_i(b, u|\rho) \geqq H_i(b/b'_{iu}, u|\rho) \tag{5}$$

The original definition requires that each behavior strategy b_i is optimal at each information set under (b, ρ). Then, each local strategy is optimal. Lemma 2 tells us that we can check each local strategy b_{iu} instead of b_i for finding sequential equilibrium.

Next, we define a perturbed game[3] $\hat{\Gamma} = (\Gamma, \eta)$ where Γ is a finite extensive-form game with perfect recall and a completely mixed perturbation $\eta_u : A_u \to \mathbb{R}_{++}$ s.t. $\sum_{a_u \in A_u} \eta_u(a_u) < 1$ $\forall u \in U$. In a perturbed game $\hat{\Gamma} = (\Gamma, \eta)$, each player selects each choice a_u with at least probability $\eta_u(a_u)$. The set of all strategy profiles in $\hat{\Gamma}$ is $\hat{B} = \{b \in B : b_{iu}(a_u) \geqq \eta_u(a_u)\ \forall i \in \{1, ..., n\}, \forall u \in U_i, \forall a_u \in A_u\}$.

Player i's best reply to b in $\hat{\Gamma}$ is

$$b^*_i(b) = \operatorname*{argmax}_{b'_i \in \hat{B}_i} H_i(b/b'_i) \tag{6}$$

and $b^* = (b^*_1(b^*), ..., b^*_n(b^*))$ is an equilibrium point in $\hat{\Gamma}$.

A sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, ...$ where $\hat{\Gamma}^k = (\Gamma, \eta^k)$ is a test sequence for Γ if $\max_{a_u \in A_u, u \in U}(\eta^k_u(a_u)) \to 0$ as $k \to \infty$. $b^* \in B$ of Γ is a limit equilibrium point of a test sequence $\hat{\Gamma}^k \to \Gamma$ if there exists a sequence of equilibrium points $\hat{b}^k$ of $\hat{\Gamma}^k$ which converges to b^* as $k \to \infty$.

[3]We completely follow the definition in Selten(1975).

From Selten (1975),

Definition 4 (Perfect Equilibrium) *b^* is a perfect equilibrium for Γ*
$\Leftrightarrow \exists$ a test sequence $\hat{\Gamma}^k \to \Gamma$ s.t. the limit equilibrium is b^*.

The set of all perfect equilibria $b \in B$ in Γ is denoted by $PE(\Gamma)$.

For the comparison between sequential equilibrium and perfect equilibrium, we claim the following:

Lemma 3 *b is a perfect equilibrium iff there exists a sequence $b^m \to b$ s.t. b^m is a completely mixed strategy profile, and $\forall i \in I$, $\forall u \in U_i$ and $\forall b'_i \in PB_i$,*

$$H_i(b^m/b_{iu}, u) \geqq H_i(b^m/b'_{iu}, u) \tag{7}$$

Proof of $\Rightarrow$: When $b \in PE(\Gamma)$, there exists a test sequence and a sequence of Nash equilibrium $\hat{b}^k \to b$. For the sake of contradiction, there is no subsequence of $\hat{b}^k$ which satisfies the condition (7). Then, b_{iu} allocates a positive probability to a suboptimal choice in any subsequence of $\hat{b}^k$. However, since $\hat{b}^k$ is Nash equilibrium in a perturbed game $\hat{\Gamma}^k$, and so the probability for any suboptimal strategy converges to 0 as $k \to \infty$.

Proof of $\Leftarrow$: If there exists a sequence $b^m \to b$ which satisfies the condition, each b_{iu} is optimal in b^m. Then, we can construct η^m for b^m s.t. $b^m_{iu}(a_u) = \eta^m_u(a_u)$ for each $a_u \in A_u - \mathrm{supp}(b_{iu})$ and $\frac{1}{m}b^m_{iu}(a_u) = \eta^m_u(a_u)$ for any $a_u \in \mathrm{supp}(b_{iu})$. Since $b^m \to b$, $\eta^m_u(a_u) \to 0$ for any $u \in U$ and $a_u \in A_u$.

From Lemma 1, 2 and 3, we get the following theorem explaining a necessary and sufficient condition with which sequential equilibrium is perfect.

Theorem 1 (From Sequential Equilibrium To Perfect Equilibrium) *For $b \in SE(\Gamma)$ in an incomplete-information game Γ, the following conditions A and B are equivalent.*

A: $b \in PE(\Gamma)$

B: There exist a belief system $\rho \in SEB(\Gamma, b)$, a sequence of completely-mixed strategy profiles and beliefs $(b''^k, \rho^k) \to (b, \rho)$ s.t. $\rho^k \in CO(b''^k)$, which satisfies, for any player $\forall i \in I$, at any information set $u \in U_i$ and each pure strategy profile $b' \in PB$ s.t. $H_i(b, u|\rho) = H_i(b/b'_{iu}, u|\rho)$, *the following condition:*

$$H_i(b''^k/b_{iu}, u|\rho^k) \geqq H_i(b''^k/b'_{iu}, u|\rho^k) \tag{8}$$

Proof of Theorem 1: $\Rightarrow$: From Lemma 3, if $b \in PE(\Gamma)$, there exist a sequence $b^k \to b$ and $\rho^k \to \rho$ s.t. $\rho^k \in CO(b^k)$ and (8) is satisfied for any $u \in U$. Since b^k is completely mixed and $\rho^k \in CO(b^k)$, from Lemma 1, $H_i(b^k/b'_{iu}, u|\rho^k) \to H_i(b/b'_{iu}, u|\rho)$. Then, from Lemma 3, $H_i(b/b_{iu}, u|\rho) \geqq H_i(b/b'_{iu}, u|\rho)$. Therefore, $\rho \in SEB(\Gamma, b)$ is satisfied.

$\Leftarrow$: Consider $b \in SE(\Gamma)$ and a sequence of completely-mixed strategy profiles and beliefs $(b'^k, \rho^k) \to (b, \rho)$ which satisfies the conditions in Theorem 1. If b'^k satisfies the conditions in Lemma 3, this proof would be done. At the limit, b becomes optimal because $b \in SE(\Gamma)$ and $\rho \in SEB(\Gamma, b)$. From Lemma 1, $H_i(b^k/b'_{iu}, u|\rho^k) \to H_i(b/b'_{iu}, u|\rho)$, and so when there exists only a single optimal choice in the information set u for b and ρ, when k is large enough, b_{iu} is the unique optimal local strategy at u in b'^k. If there exist multiple best choices, the required condition (8) (b_{iu} is weakly better than the other best choices in the sequence before the limit) is satisfied. Therefore, b'^k satisfies the conditions in Lemma 3.

Theorem 1 gives us a new insight into trembling-hand perfect equilibrium. Players have to be cautious only if the game is complicated[4]. For example, consider a perfect-information game and a sequential equilibrium with a single multi-best-reply information set u. At the information set u, if the player takes care of a (meaningful) impact on the payoff from any arbitrary possible error after u for each best choice, the player would pick a choice in the strategy profile of a perfect equilibrium. Some tie-breaking reasonings, like selecting a choice with max max possible outcome or max min possible outcome based on future errors, also select a choice in one of the strategy profiles in the set of perfect equilibrium if they have an impact on the player's payoff. Our result shows that, in simple settings, a trembling-hand perfect equilibrium can be explained by the optimization of each information set with a rough tie-breaking rule.

To understand when simple settings and the rough tie-breaking rules lead to perfect equilib-

[4]Here, we claim that players can play perfect equilibrium even if all players are not cautious in simple games. This is different from the discussion about whether the concept is robust or not. It is known that a trembling-hand perfect equilibrium is usually not robust from random fluctuations (e.g., Okada (1981) and Kohlberg & Mertens (1986)).

rium, we refer to Okada (1991)'s *lexicographically undominated strategy combination.*

Definition 5 (Proposition 2.4 of Okada (1991)) *In $b \in B$, b_{iu} lexicographically dominates $b'_{iu} \Leftrightarrow \exists$ some neighborhood $O \subset B$ of b such that*

$$H_i(b''/b_{iu}, u) > H_i(b''/b'_{iu}, u) \tag{9}$$

for any completely mixed strategy profile $b'' \in O$[5]*.*

A strategy profile b is called a lexicographically undominated strategy combination when each b_{iu} is not lexicographically dominated by any $b'_{iu} \in B_{iu}$ for any $i \in I$ and at any $u \in U_i$. A lexicographically undominated strategy combination does not require players to share a similar idea about the impact of the future error.

When an equilibrium is both sequential and lexicographically undominated, we call the equilibrium a lexicographically undominated sequential equilibrium. Okada (1991) points out the following necessary condition for perfect equilibrium:

Lemma 4 (Okada (1991)'s necessary condition) *Perfect equilibrium is always a lexicographically undominated sequential equilibrium.*

Okada (1991)'s condition is weaker than requirement B in Theorem 1. However, in the following paragraphs, we see that Okada's condition is sufficient for simple games which include a maximum of two decision nodes in each path.

For the preparation, we claim the following lemmas:

Lemma 5 *In Γ where each path includes a maximum of two decision nodes,*

$$H_i((1-\epsilon)b + \epsilon b''/b_{iu}, u) = (1-\epsilon)H_i(b/b_{iu}, u) + \epsilon H_i(b''/b_{iu}, u) \tag{10}$$

for any $b, b'' \in B'$, $i \in I$, $u \in U_i$ and $\epsilon \in (0, 1)$.

Proof: For the player i s.t. $\emptyset \in U_i$, (if $i \neq 0$), H_i at $\emptyset$ from each choice depends on only the local strategy at the following node. In any other information set u, for player j s.t. $u \in U_j$, H_j from each choice depends on ρ_u, and ρ_u depends on $b_{i,\emptyset}$ (or $p_\emptyset$ if $i = 0$).

[5] Here, O does not include b.

Lemma 6 *In Γ where each path includes a maximum of two decision nodes, if b_{iu} is not lexicographically dominated in b, there exists a sequence of completely mixed strategy profiles $b^k \to b$ s.t. for any $b'_{iu} \in B_{iu}$,*

$$H_i(b^k/b_{iu}, u) \geqq H_i(b^k/b'_{iu}, u). \tag{11}$$

Proof: For players $i, i' \in I$[6] and each pair $\emptyset \in U_i$ and $u \in U_{i'} - \{O\}$, we consider a two-player agent-normal-form game $\Gamma'(\emptyset, u)$ s.t. a pure strategy set is identical to a choice set $S_i = \{a_\emptyset \in A_\emptyset | \exists x \in u \text{ s.t. } a_\emptyset = ac(x)\}$ and $S_{i'} = A_u$ and the payoff from each outcome is identical.

Consider $b_{i',u}$ at u s.t. $\rho(u, b) > 0$. Then, there exist a player i's strategy σ_i s.t. $\sigma_i(s'_i) = b_{i\emptyset}(s'_i)/\sum_{s_i \in S_i} b_{i\emptyset}(s_i)$ and a player i''s strategy $\sigma_{i'} = b_{i'u}$. Since $b_{i'u}$ is not lexicographically undominated in the neighborhood of b in Γ, and $A_\emptyset - S_i$ does not have any impact on $H_{i'}$ in u, the player i''s mixed strategy $\sigma_{i'}$ is not lexicographically undominated around σ. Then, $\sigma_{i'}$ is not weakly dominated (hereafter dominated). In a two-player normal-form game, it is known that, if $\sigma_{i'}$ is not dominated, there exists a completely mixed $\hat{\sigma}_i$ with which $\sigma_{i'}$ is optimal (For example, see Appendix B of Pearce (1984)). Because of Lemma 5, $\sigma_{i'}$ is optimal for $(1-\epsilon)\sigma_i + \epsilon\hat{\sigma}_i$. Therefore, there exists a sequence of completely mixed strategy profiles $\sigma^k \to \sigma$ which justify the player i''s strategy $\sigma_{i'}$ in $\Gamma'(\emptyset, u)$. Any sequence $b^k_{i\emptyset}$ which satisfies $\sigma^k_i(s_i) = b^k_{i\emptyset}(s_i)/\sum_{s'_i \in S_i} b^k_{i\emptyset}(s'_i)$ satisfies the requirement in Lemma 6 for u s.t. $\rho(u, b) > 0$.

Second, consider an unreached information set u s.t. $\rho(u, b) = 0$. Since $b_{i'u}$ is not lexicographically undominated and $A_\emptyset - S_i$ does not have any impact on $H_{i'}$ in u, the player i''s mixed strategy $\sigma_{i'}$ is not dominated in $\Gamma'(\emptyset, u)$. Then, there exists a completely mixed strategy $\hat{\sigma}$ which justifies $\sigma_{i'}$. Then, any sequence $b^k_{i\emptyset}$ which satisfies $\sigma_i(s'_i) = b^k_{i\emptyset}(s'_i)/\sum_{s_i \in S_i} b^k_{i\emptyset}(s_i)$ satisfies the requirement in Lemma 6 for u s.t. $\rho(u, b) = 0$.

Third, for the optimality of $b_{i\emptyset}$, we consider a two-player normal-form game $\Gamma'(\emptyset)$ where an agent of player i at $\emptyset$ and an incomplete dictator D who selects the combination of local strategy at any $u \in U - \{O\}$. There exists a maximum of two decision nodes in each path, and so there is a bijection from a mixed strategy of the dictator in $\Gamma'(\emptyset)$ to the combination

[6] If either player is nature (player 0), we can apply the similar logic. This is because nature just follows p_u and does not change its behavior in the sequence. For simplicity, we skip the possibility.

of $b_{i'u}$ in Γ both of which give an identical outcome distribution against $b_{i\emptyset}$. σ_D is a mixed strategy that represents a combination of $b_{i'u}$ for all $i' \in I$ and $u \in U_{i'} - \{\emptyset\}$. Since $b_{i\emptyset}$ is not lexicographically undominated in Γ, the player i's (unique) mixed local strategy in $\Gamma'(\emptyset)$ which coincides to $b_{i\emptyset}$ is not lexicographically undominated, and so $b_{i\emptyset}$ is not dominated. From Pearce (1984) and Lemma 5, there exists a sequence of completely mixed strategy profiles $\sigma_D^k \to \sigma_D$ which justifies the player i's mixed strategy in $\Gamma'(\emptyset)$. Therefore, by using the bijection, there exists a sequence of completely mixed strategy profiles b^k which justify $b_{i\emptyset}$.

Then, we claim the following proposition:

Proposition 1 *When each path includes at maximum two decision nodes in Γ, a lexicographically undominated strategy combination is a perfect equilibrium.*

Proof: Consider a lexicographically undominated strategy combination b. In the following discussion, we ignore the possibility of player 0, because adding player 0 does not have an impact and we can apply a similar discussion, and assume that player 1 selects a decision at $\emptyset$.

First, we focus on $\emptyset$. Since $b_{1,\emptyset}$ is not lexicographically undominated, there exists a sequence of completely mixed strategy profiles $b^k \to b$ s.t. $H_1(b^k/b_{1,\emptyset}) \geqq H_1(b^k/b'_{1,\emptyset})$ for any $b'_{1,\emptyset} \in B_{1,\emptyset}$. Since $H_i(b^k/b'_{1,\emptyset})$ depends on b^k_{iu} s.t. $u \in U - \{\emptyset\}$ and does not depend on $b^k_{1,\emptyset}$, $b^k_{1,\emptyset}$ is optimal for any $b^k/b''_{1,\emptyset}$ where $b''_{1,\emptyset} \in B_{1,\emptyset}$. In the following paragraphs in this proof, without further explanation, we consider a sequence of completely mixed b^k s.t. $H_1(b^k/b_{1,\emptyset}) \geqq H_1(b^k/b'_{1,\emptyset})$ for any $b'_{1,\emptyset} \in B_{1,\emptyset}$.

Second, we focus on $u \in U - \{\emptyset\}$ and $i \in I$ s.t. $u \in U_i$. $H_i(b^k/b'_{iu}, u|\rho^k)$ depends on ρ^k_u consistent with $b^k_{1,\emptyset}$. Since each b_{iu} is not lexicographically dominated, there exists a sequence (b^k, ρ^k) s.t. $b^k \to b$ and $\rho^k \in CO(b^k)$ which justifies b_{iu}. Then, there exists the sequence of the ratios $b^k_{1,\emptyset}(ac(x))/\rho(u, b^k)$ for each $x \in u$. $SR(k, u, x)$ denotes the ratio.

Consider a sequence of completely mixed strategy profiles and beliefs $(b'^k, \rho'^k) \to (b, \rho)$ where $\rho'^k \in CO(b'^k)$. Then, from the sequence, we can calculate a sequence of the non-zero realization probabilities for each $u \in U - \{\emptyset\}$ which converges to $\rho(u, b)$. Denote the realization probability of u at kth element in the sequence by $\alpha(k, u)$. Since each choice is connected to a unique node and a unique information set (if they exist), we can disjointly decide the ratio of each choice at $\emptyset$ with which the sequence makes the realization probability converge to $\rho(x, b)$ and the ratio for the optimality is satisfied by the following way: For each $x \in u$, we decide $b^k_{1,\emptyset}(ac(x))$ s.t. $b^k_{1,\emptyset}(ac(x))/\alpha(k,u) = SR(k,u,x)$. Then, we get $b^k_{1,\emptyset} \to b_{1,\emptyset}$ and each b_{iu} is optimal to $b^k_{1,\emptyset}$. For any $i' \in I$ and $u \in U_{i'} - \{O\}$, $b^k_{iu} = b'^k_{iu}$ and so $b_{i,\emptyset}$ is optimal in the sequence. For any choice at $\emptyset$ conncected to an outcome $a \in A$, we set $b^k_{1,\emptyset}(ac(a)) = b'^k_{1,\emptyset}(ac(a))$. The constructed b^k satisfies $b^k \to b$, and each b_{iu} is optimal in the sequence, so b is perfect.

Proposition 1 means that when each path includes up to two decision nodes, if each player avoids lexicographically dominated strategies, the strategy combination is a perfect equilibrium. Since there exist a maximum of two players in each path, each player does not have to share idea about the possible error of the opponent with other players. It is well known that an undominated strategy combination in a normal-form game always satisfies normal-form perfection for two-player games. Compared to the well-known result, Proposition 1 allows any finite number of players, including more than two players.

A lexicographically undominated strategy combination does not require players to have a common idea about future errors. In other words, the settings above are the situations where cautious players play perfect equilibrium without a common idea about future errors. If a player who meets multiple best replies uses a rough tie-breaking rule, the same outcome appears.

3 Implication for Games with Infinite Actions

In this section, using Proposition 1, we briefly discuss “perfect equilibrium” in games with infinite uncountable actions. The original definition of perfect equilibrium cannot be applied to game with uncountable actions because completely mixed strategy profiles cannot be well defined in the setting. To avoid this problem, we utilize the following alternative definition of lexicographic dominance proposed in Okada (1991).

Definition 6 (Definition 2.2 of Okada (1991)) *In $b \in B$, b_{iu} lexicographically dominates b'_{iu} $\Leftrightarrow$ $\exists$ some neighborhood $O \subset B$ of b such that*

$$H_i(b''/b_{iu}, u) \geqq H_i(b''/b'_{iu}, u) \tag{12}$$

for any strategy profile $b'' \in O$ *with at least one strict inequality.*

We can apply this definition to games with uncountable actions because completely mixed strategy profiles are absent in this definition. We combine this definition and Proposition 1. When we consider a game with uncountable actions, if each path includes a maximum of two decision nodes, and if each player selects a lexicographically undominated local strategy, this strategy combination can be considered "perfect equilibrium" in such a game.

Consider a game with 3 players where Player 1 selects a number $a_{u1} \in A_{u1} = [2, 3] \cup [4, 5]$, and then if $a_{u1} \leqq 3$ Player 2 selects a number $a_{u2} \in A_{u2} = [2, 3]$, and otherwise Player 3 selects a number $a_{u3} \in A_{u3} = [4, 5]$. Player 2 and 3 observe Player 1's number before making their decision.

When both Player 1 and the next player select a unique number, all players get 1. If either Player 1 or the next player selects a noninteger, and if the opponent selects an integer, the former gets 0, and the other players get 1. Otherwise, all players get 0.

There is a type of lexicographically undominated strategy combination. Since any integer lexicographically dominates any noninteger, each player can select only integers in the undominated strategy combination. Player 1 can select any local strategies selecting only integers. In the information set after Player 1's integer, Player 2 or 3 selects the number Player 1 selected. In the other information sets, Player 2 and 3 can select any local strategy selecting only integers.

There are uncountably many other subgame-perfect equilibria where two players select a unique noninteger, but they are not plausible because selecting a noninteger is a vulnerable option for players.

In this paper, we propose a way to discuss Selten's (1975) perfection in games with uncountable actions while avoiding technical challenges that arise from uncountable actions. We subsequently examined a setting where Selten's (1975) perfection is demanded. Our approach can be applied only for games with a maximum of two decision nodes in each path. As a future endeavor, we aim to develop approaches for discussing Selten's (1975) perfection in more general settings.

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