From Sequential Equilibrium to Perfect Equilibrium: Two Types of Perturbed Strategy Profiles

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Abstract

We analyze the necessary and sufficient condition with which sequential equilibrium (Kreps and Wilson 1982) is perfect (Selten 1975). This From-Sequential-equilibrium-To-Perfect-equilibrium (FSTP) condition consists of special types of strategy profiles (well-mixed strategy profiles) which are slightly weaker than completely mixed strategy profiles. Well-mixed strategy profiles are applicable to uncountable strategy sets. In addition, well-mixed strategy profiles enable us to check various rationalities based on completely mixed strategy profiles while requiring less data on payoffs of choices.

1 Introduction

Perfect equilibrium and sequential equilibrium are solution concepts for finite extensive-form games. In Selten's (1975) *perfect equilibrium*, players are cautious about the impact of errors in the following information sets. It is known that perfect equilibrium requires complicated calculations. To ease this difficulty[1], Kreps & Wilson (1982) propose *sequential equilibrium*. In sequential equilibrium, players select their best local strategies at each information set while holding a type of rational belief. The difference between these two concepts is small. For example, Blume & Zame (1994) prove that the whole sets of sequential equilibria and perfect equilibria coincide with each other in any finite games with almost all assignments of payoffs to outcomes. Previous literature implies the simple difference between two concepts. Our research fully characterizes the simple necessary and sufficient condition with which sequential equilibrium is perfect. We call the condition From Sequential equilibrium To Perfect equilibrium (FSTP)[2].

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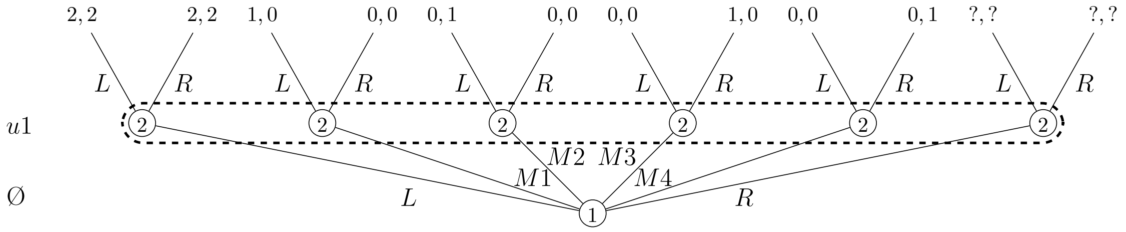
[1]Today, sequential equilibrium is considered as a fundamental concept of many solution concepts (see Govindan & Wilson (2008)).

[2]In Jinushi(2022a), we derive the FSTP condition based on completely mixed strategy profiles. This paper relaxes the condition by replacing completely mixed strategy profiles with *well-mixed strategy profiles* explained below.



Game 1: FSTP and a negligible choice

FSTP is the following condition; when there exist multiple best replies at information sets in sequential equilibrium, we can find a sequence of special types of strategy profiles in which each strategy profile justifies the equilibrium local strategy at the information sets. The strategy profile does not have to be completely mixed because the expected payoffs of each choice are in finite dimensions. The impact of a completely mixed strategy profile on the expected payoffs can be replicated by a special type of strategy profile (hereafter well-mixed strategy profile) based on finite outcomes. This condition (FSTP) is relatively easy to check.

In Game 1, we can intuitively see how FSTP allows us to ignore some choices when we distinguish whether a sequential equilibrium strategy profile is perfect or not. In Game 1, although we do not know the payoffs when player 1 selects R, if a strategy profile where Player 1 selects L and Player 2 selects a mixed strategy including both R and L is a sequential equilibrium strategy profile, we know that this profile is perfect. The reason is because if we consider a small perturbation where player 1 mistakenly selects M1-4, the combination of outcomes in the perturbation can offset any possible impact on the expected payoffs from the unknown node connected to Player 1's R. In other words, the calculation complexity derived from the cautious reasoning has an upper bound from the number of dimensions in the payoff vector.

A well-mixed strategy profile is a useful concept for general purposes including empirical analysis. For example, in some situations, we can get access only to the payoff data of limited choices. We cannot apply the original definitions of solutions concepts based on completely mixed strategy profiles. However, if the data is rich enough, we can apply these concepts while using well-mixed strategy profiles at least when we assume a sequential rationality, like demonstrated in the previous paragraph.

A well-mixed strategy profile can be applicable to games with uncountable strategy sets. This implies that various concepts based on completely mixed strategy profiles, such as perfect equilibrium and sequential equilibrium, become applicable to games with uncountable strategy sets via well-mixed strategy profiles. In Jinushi (2022b), we utilize this advantage for understanding the relation of solution concepts for games with countable and uncountable strategy sets.

2 FSTP in Finite Extensive-Form Games

In this section, we first formulate a standard finite extensive-form game. We define perfect equilibrium and sequential equilibrium. Then, we refer to FSTP (the necessary and sufficient condition with which sequential equilibrium is perfect) derived in Jinushi (2022a). In the next section, we refine this FSTP by introducing a well-mixed strategy profile.

We employ the standard concepts and terminologies in the literature (e.g. Kuhn(1953), Selten (1975) and van Damme (1984))[3]. The following formulation is identical to Jinushi (2022a).

A n-player finite extensive-form game $\Gamma = (K, P, U, p, h)$ consists of the following five elements:

1. The rooted game tree K consists of finite nodes including the origin $\emptyset$ and directed links towards terminal nodes. We denote the set of terminal(/non-terminal) nodes $A(/X)$. K represents a physical order in the game, and since a player makes a decision making at each node in X, we call each node in X decision node. Each decision node $x \in X$ is directly connected to finite directed links from x towards terminal nodes. These directed links from x represent alternatives players can select at each $x \in X$. A_x represents the set of alternatives at each x. The game begins at $\emptyset$. When a player picks $a_x \in A_x$ at $x \in X$, the next node connected to x via the link a_x is reached. Then, by repetition, eventually a terminal node is reached. We call each combination of nodes from the origin $\emptyset$ to $a \in A$ in such a process as a play. We call each combination of nodes from $x \in X$ to $x' \in X$ within a play as a path.

2. The player partition $P = (P_0, ..., P_n)$ is a partition of X. The set of players is $I = \{1, ..., n\}$, and $I^* = \{0, 1, ..., n\}$ where player 0 represents nature moves which follow an exogenous probability distribution. For any $i \in I^*$, at each decision node $x \in P_i$, player i picks an alternative.

3. The information partition $U = (U_0, ..., U_n)$ is a refinement of P. Each element $u \in U_i$ is called an information set. At each information set $u \in U_i$, player i selects a choice from the set of choices A_u. When player i makes a decision in $u \in U_i$, player i understands that player i is at a node in u but does not know the exact node $x \in u$. We require that, for any $x, x' \in u$, $A_x = A_{x'} = A_u$. For each $u \in U_0$, we assume that u is singleton. We require that, in each play, a node from each information set can exist at maximum once.

4. Player 0 follows an exogenous completely mixed probability distribution p_u over A_u at each information set $u \in U_0$ s.t. $p_u(a_u) > 0$ for all $a_u \in A_u$. The probability assignment p is a combination of p_u for all $u \in U_0$.

5. The payoff function h: $A \to \mathbb{R}^n$ represents the players' payoffs from $a \in A$.

We assume the following condition, called *perfect recall* in Kuhn (1953): For each player i, if $u, v \in U_i$, and if $x \in u$ comes after $a_v \in A_v$ at $y \in v$ in a path, any $x' \in u$ comes after a_v in any path including x'.

We denote a probability distribution b_{iu} over A_u as a local strategy for player $i \in I$ at an information set $u \in U_i$. A behavior strategy $b_i \in B_i$ is a combination of each local strategy

[3] As Battigalli (1997) points out, these concepts are "by now standard", and so for the motivations of each requirement, please see Kuhn(1953), Selten (1975) and van Damme (1984).

for player i ($b_i = (b_{iu})_{u \in U_i}$). A strategy profile is a combination of all players' behavior strategies $b = (b_i)_{i \in I}$.

When we want to replace only a local strategy from b_{iu} to b'_{iu} at an information set $u \in U_i$, we denote b/b'_{iu}.

We denote $\rho(x, b)$ as the realization probability of $x \in X$ uniquely determined by $b \in B$ and p[4]. The realization probability of an information set u is $\rho(u, b) = \sum_{x \in u} \rho(x, b)$. $\rho(a, b)$ for any $a \in A$ is the realization probability of an outcome a. $\rho(a, b|x)$ is the realization probability of a conditional on $x \in X$.

We introduce a local belief $\rho_u(x)$ as the probability of each node $x \in u \in U_i$ the player i believes at u. For any $i \in I$, a belief ρ is a function from $u \in U$ to a local belief ρ_u.

Definition 1 *For player $i \in I$ and $u \in U_i$, a local strategy b_{iu} is completely mixed, iff $b_{iu}(a_u) > 0$ for all $a_u \in A_u$. For player $i \in I$, a behavior strategy $b_i \in B_i$ is completely mixed iff $b_{iu}(a_u) > 0$ for all $a_u \in A_u$ and for all $u \in U_i$. We call a strategy profile $b \in B$ is completely mixed iff b_i is completely mixed for all $i \in I$.*

When $b \in B$ is completely mixed, the realization probability of $x \in u$ at the information set u is decided uniquely because $\rho(u, b) > 0$. When there exists a unique belief s.t. $\rho_u(x) = \rho(u, x)/\rho(u, b)$ for each $u \in U$, such a belief is called a *consistent* belief with b. Kreps & Wilson (1982) extends this idea to construct a type of rational beliefs for any $b \in B$, including non-completely mixed strategy profiles, by the following way:

Definition 2 *An assessment (b, ρ) is consistent if there exists a sequence of completely mixed strategy profiles and beliefs $(b^j, \rho^j) \to (b, \rho)$ where ρ^j is consistent with b^j.*

We denote CO as a mapping from $b \in B$ to a set of ρ s.t. (b, ρ) is consistent. For each sequence of completely mixed strategy profiles $b^j \to b$, there exists a unique sequence $\rho^k \in CO(b^k)$. This sequence may not converge to any points, but it always includes a convergent subsequence because the sequence is in compact space (from Bolzano-Weierstrass Theorem). Hereafter, we consider such a subsequence and skip this explanation.

The ex ante expected payoff vector $H(b) = (H_1(b), ..., H_n(b))$ is

$$H(b) = \sum_{a \in A} \rho(a, b) h(a) \tag{1}$$

The expected payoff vector at an information set u is

$$H(b, u|\rho) = \sum_{x \in u} \rho_u(x) \sum_{a \in A} \rho(a, b|x) h(a). \tag{2}$$

When b is a completely mixed strategy profile, since $\rho(u, b) > 0$ for any $u \in U$, a consistent belief ρ is uniquely decided. When Γ is a perfect-information game, u includes only an element, and so a consistent belief ρ is uniquely decided. In this paper, we focus on consistent

[4]For the details, please see Jinushi (2022a).

beliefs for strategy profiles, and when the consistent belief is uniquely decided, we sometimes denote $H(b,u)$ instead of $H(b,u|\rho)$.

It is known a following basic characteristic of the expected payoff vector at an information set (e.g. Jinushi (2022a)):

Lemma 1 *Consider a sequence of completely mixed strategy profiles and consistent beliefs s.t.* $(b^m,\rho^m) \to (b,\rho)$*. Then,* $H(b^m/pb_{iu},u|\rho^m) \to H(b/pb_{iu},u|\rho)$ *for any* $i \in I$*,*$u \in U_i$ *and* $pb_{iu} \in PB_{iu}$*.*

By combining the original definition of Kreps and Wilson (1982) and the theorem in p279 of Hendon et al. (1996), we define a sequential equilibrium:

Definition 3 *An assessment* (b,ρ) *is sequential equilibrium iff there exists a sequence* $(b^m,\rho^m) \to (b,\rho)$ *s.t.* b^m *is a completely mixed strategy profile,* $\rho^m \in CO(b^m)$ *and* $\forall i \in I$*,* $\forall u \in U_i$ *and* $\forall b'_{iu} \in PB_{iu}$*,*

$$H_i(b^m,u|\rho) \geqq H_i(b^m/b'_{iu},u|\rho) \tag{3}$$

The set of all sequential equilibrium strategy profiles $b \in B$ in Γ is denoted by $SE(\Gamma)$. The set of all belief parts ρ of sequential equilibria for $b \in SE(\Gamma)$ is denoted by $SEB(\Gamma,b)$.

Next, we define a perfect equilibrium. From Jinushi (2022a), the following definition is identical to the original definition of perfect equilibrium:

Definition 4 b *is perfect equilibrium iff there exists a sequence* $b^m \to b$ *s.t.* b^m *is a completely mixed strategy profile, and* $\forall i \in I$*,* $\forall u \in U_i$ *and* $\forall b'_i \in PB_i$*,*

$$H_i(b^m/b_{iu},u) \geqq H_i(b^m/b'_{iu},u) \tag{4}$$

Jinushi (2022a) derives the following necessary and sufficient condition with which sequential equilibrium is perfect by using completely mixed strategy profiles:

Theorem 1 (From Sequential Equilibrium To Perfect Equilibrium) *For* $b \in SE(\Gamma)$ *in an incomplete-information game* Γ*, the following conditions A and B are equivalent.*

A: $b \in PE(\Gamma)$

B: There exist a belief system $\rho \in SEB(\Gamma,b)$*, a sequence of completely-mixed strategy profiles and beliefs* $(b''^k,\rho^k) \to (b,\rho)$ *s.t.* $\rho^k \in CO(b''^k)$*, which satisfies, for any player* $\forall i \in I$*, at any information set* $u \in U_i$ *and each pure strategy profile* $b' \in PB$ *s.t.* $H_i(b,u|\rho) = H_i(b/b'_{iu},u|\rho)$*, the following condition:*

$$H_i(b''^k/b_{iu},u|\rho^k) \geqq H_i(b''^k/b'_{iu},u|\rho^k) \tag{5}$$

As Jinushi(2022a) explains, Theorem 1 means that perfect equilibrium does not require all players to be cautious of the identical future errors in simple settings. For example, if there exist at maximum two information sets in each path, if players select a lexicographically undominated strategy[5] proposed in Okada (1991), the equilibrium is perfect (Jinushi 2022a)[6].

In this paper, we relax this FSTP by replacing completely mixed strategy profiles with *well-mixed strategy profiles* we propose in Section 3. Our result shows that, in general settings, players do not have to be cautious about all possible future errors in perfect equilibrium. Compared to Jinushi(2022a), we analyze more general settings but require players to share the identical idea about future errors.

3 Well-Mixed Strategy Profiles

In this section, we propose a new concept, *well-mixed strategy profile*, which is a slightly weaker concept than a completely-mixed strategy profile. Since the expected payoff vector of each choice is in n-dimensional Euclidean space, the change in the expected payoff vector of each choice is the n-dimensional vector. Thus, if we consider an original strategy profile the support of which is wide enough, we can find a completely-mixed strategy profile with the same expected payoff vector. In other words, we are sometimes able to add a choice to the support of the original strategy profile while keeping the expected vector of each choice. In this section, we discuss the sufficient condition that a strategy profile is well mixed. Since well-mixed strategy profiles do not have to give positive probability to all choices in each information set, the application range includes uncountable strategy sets.

3.1 Perfect Information

In this subsection, we consider a perfect-information[7] finite extensive-form game Γ. Consider a node $x \in X$. As we discuss in the previous section, the impact of small perturbation on the payoff vector matters for whether a sequential equilibrium is perfect. Since H is in $\mathbb{R}^n$ at x, the change in the payoff vector derived from the small change in $b \in B$ is replicated by maximum n linearly-independent vectors based on n choices. If the number of choices connected to x is larger than $n+1$, the impact from the tiny probability on some choices is negligible in the sense that the combination of the other choices can completely offset the impact on the payoffs.

In a perfect-information game, all players select local strategies conditional on the correct and accurate history. In other words, the history does not have any impact on payoffs at each node. We point out the following basic characteristic of games with perfect information.

Lemma 2 *Consider a perfect information game Γ, $i \in I$, $b, b' \in B$, $u \in U_i$ s.t. $H(b,u) = H(b/b'_{iu}, u)$. $\Rightarrow H(b,u') = H(b/b'_{iu}, u')$ for all $u' \in U$.*

[5]This concept requires each player to avoid some type of choices which are not robust from perturbations. There is an explanation about lexicographically undominated strategies (in normal-form games) in Okada (1988). This concept does not necessarily require players to be cautious about all possible outcomes (see Definition 3.1. of Okada (1988)). Okada (1991) extends lexicographically undominated strategies to extensive-form games. Our results also show that players do not have to be cautious about all outcomes in perfect equilibrium but the logic is different from Okada's (1988,1991) approach.

[6]This result allows any finite players in the game and is different from Okada's (1988) Theorem 3.7 for 2-player normal-form games and normal-form perfection.

[7]A perfect-information game Γ is a game where any $u \in U$ is a singleton.

Proof: When $H(b,u) = H(b/b'_{iu}, u)$, at u, $\sum_{z\in Z} \rho(z, b|u)h(z) = \sum_{z\in Z} \rho(z, b/b'_{iu}|u)h(z)$. In each information set $u' \in U$ before u in any path from $\emptyset$ to z including u, $\rho(u, b|u') = \rho(u, b/b'_{iu}|u')$, and so $H(b, u') = H(b/b'_{iu}, u')$. For any other information set $u'' \in U$, $\rho(z, b/b'_{iu}|u'') = \rho(z, b/b'_{iu}|u'')$.

From Lemma 2, as long as H in u is unchanged, a change in b_{iu} does not have any impact on H in any $u' \in U$. In other words, if the change in the local strategy at u does not have an impact on H at u, the expected payoffs of each choice at each information set are unchanged.

Because of Lemma 2, we focus on a local strategy and H on u. There exists a convex combination of choices with ζ s.t.

$$b_{iu} = \sum_{\substack{pb_{iu}\in PB_{iu} \\ s.t.\ \mathrm{supp}(pb_{iu})\in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} pb_{iu} \tag{6}$$

and

$$\sum_{\substack{pb_{iu}\in PB_{iu} \\ s.t.\ \mathrm{supp}(pb_{iu})\in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} = 1. \tag{7}$$

By using this convex combination of pure local strategies, the expected payoff vector at u is

$$H(b, u|\rho) = \sum_{\substack{pb_{iu}\in PB_{iu} \\ s.t.\ \mathrm{supp}(pb_{iu})\in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} H(b/pb_{iu}, u|\rho). \tag{8}$$

In other words, the change in the local strategy at u does not change the expected payoffs of each choice at u. By using this characteristic, we define *negligible choice* as follows:

Definition 5 (Negligible Choice (Perfect Information)) *A choice $a_u \in A_u$ is negligible for $b \in B$ in a perfect-information game Γ, if*

1. $b_{iu}(a_u) = 0$
2. *There exists $b''_{iu} \in B_{iu}$ s.t.*
$$H(b/b''_{iu}, u) = H(b, u) \tag{9}$$
and $\mathrm{supp}(b''_{iu}) = \mathrm{supp}(b_{iu}) + \{a_u\}$.

When $a_u \in A_u$ is negligible, we can broaden the support of the local strategy while keeping the original payoff vectors because of the second requirement. If all $a'_u \in A_u - \mathrm{supp}(b_{iu})$ are negligible, since the expected payoffs of each choice do not depend on the local strategy at u, we can repeat adding choices to the support, and so there exists a completely mixed b'_{iu} with the identical H. In addition, for any $\epsilon \in (0, 1)$, $b''_{iu} = (1-\epsilon)b_{iu} + \epsilon b'_{iu}$ is completely mixed and $H(b, u) = H(b/b''_{iu}, u)$. Therefore, we can find a completely mixed strategy profile which is arbitrarily close to b with the identical H.

Next, we define *well-mixed strategy profile* as follows:

Definition 6 (Well-Mixed Strategy Profile (Perfect Information)) *In a perfect-information game Γ, $b \in B$ is a well-mixed strategy profile.*
$\Leftrightarrow$ For all $u \in U$, any $a_u \in A_u - \mathrm{supp}(b_{iu})$ is negligible.

Then,

Proposition 1 *In a perfect-information game Γ, if $b \in B$ is well-mixed, for any $\epsilon > 0$, there exists a completely mixed strategy profile $b' \in B$ s.t. $H(b,u) = H(b',u)$ and $|b_{iu}(a_u) - b'_{iu}(a_u)| < \epsilon$ for any $u \in U$ and for any $a_u \in A_u$.*

Proof: From the definition of negligible choice, there exists a completely mixed strategy profile $b' \in B$ s.t. $H(b,u) = H(b',u)$. Suppose $|b_{iu}(a_u) - b'_{iu}(a_u)| < M$ for any $a_u \in A_u$ but $M > 1 > \epsilon$. Then, there exists $b'' = (1 - \epsilon/M)b + \epsilon/Mb'$ which satisfies both conditions. If $\epsilon < M < 1$, there exists $b'' = (1-\epsilon)b + \epsilon b'$ which satisfies both conditions.

There is a simple sufficient condition that $b \in B$ is well mixed. To explain the sufficient condition, we first define a concept, *linearly-independent choice.*

Definition 7 (Linearly-Independent Choice (Perfect Information)) *Consider an information set $u \in U$, a strategy profile $b \in B$, and a nonempty set $LI_u \subseteq A_u$.*
A choice $a_u \in LI_u$ is linearly independent in LI_u at u with respect to b.
$\Leftrightarrow$ There exists $b'_{iu} \in PB_{iu}$ s.t.

1. *$b'_{iu}(a_u) = 1$*
2. *$H(b/b'_{iu}, u) \neq H(b,u)$*
   and there does not exist ζ s.t.
3. $$H(b/b'_{iu}, u) - H(b,u) = \sum_{\substack{pb_{iu} \in PB_{iu} \\ \mathrm{supp}(pb_{iu}) \in LI_u - \{a_u\}}} \zeta_{pb_{iu}} H(b/pb_{iu}, u). \tag{10}$$

Because the requirement 3 allows any linear combination in the right hand of the equation, for any $LI_u \subseteq A_u$, there exists at maximum n choices which are linearly independent in LI_u.

Definition 8 (Set of Linearly-Independent Choices (Perfect Information)) *Consider an information set $u \in U$, a strategy profile $b \in B$, and a nonempty set $LI_u \subseteq A_u$.*
LI_u is a set of linearly-independent choices respect to b $\Leftrightarrow$
All choices $a_u \in LI_u$ are linearly independent in LI_u at u with respect to b.

Lemma 3 *Consider a perfect-information game Γ, $b \in B$ and $u \in U$.*
There exists the maximum number of the elements $M_u \geqq 0$ s.t.

1. *$|LI_u| \leqq M_u \leqq n$ for any linearly-independent set $LI_u \subseteq A_u$*
2. *There exists a linearly-independent set $LI'_u \subseteq A_u$ s.t. $M_u = |LI'_u|$.*

Proof: Because H is in $\mathbb{R}^n$, the upper bound must be less than n for each $b \in B$ and $u \in U_i$ in a perfect-information game. M_u is uniquely decided by the set of $h(a)$ s.t. $\exists b' \in B$ s.t. $\sum_{x \in u} \rho(a, b'|x) > 0$.

Lemma 4 *Consider a perfect-information game Γ, $b \in B$, $u \in U$ and a set of linearly-independent choices $LI_u \subset \mathrm{supp}(b_{iu})$. If $|LI_u| = M_u$, all $a'_u \in A_u - \mathrm{supp}(b_{iu})$ are negligible.*

Proof: If there exist M_u linearly-independent choices in $LI_u \subset \mathrm{supp}(b_{iu})$, for any $a'_u \in A_u - \mathrm{supp}(b_{iu})$, we can always find $H(b/b'_{iu}) = \sum_{\substack{pb_{iu} \in PB_{iu} \\ \mathrm{supp}(pb_{iu}) \in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} H(b/pb_{iu}, u)$ where $b'_{iu}(a'_u) = 1$ and $\sum_{\substack{pb_{iu} \in PB_{iu} \\ \mathrm{supp}(pb_{iu}) \in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} = 0$[8]. Then, there exist $b''_{iu} \in B_{iu}$ and $\epsilon \in (0, 1)$ s.t. $b''_{iu} = (\epsilon b_{iu} + (1 - \epsilon)(b'_{iu} - \sum_{\substack{pb_{iu} \in PB_{iu} \\ \mathrm{supp}(pb_{iu}) \in \mathrm{supp}(b_{iu})}} \zeta_{pb_{iu}} pb_{iu}))$ s.t. $H(b, u) = H(b/b''_{iu})$ and $\mathrm{supp}(b''_{iu}) = \mathrm{supp}(b_{iu}) + \{a'_u\}$.

Then,

Theorem 2 *Consider a perfect-information game Γ and $b \in B$. For any $u \in U$, if there exists a set of linearly-independent choices $LI_u \subset \mathrm{supp}(b_{iu})$ s.t. $|LI_u| = M_u$ or $\mathrm{supp}(b_{iu}) = A_u$, b is well mixed.*

Proof: Since b_{iu} satisfies the requirement in Lemma 4 at each u or $\mathrm{supp}(b_{iu}) = A_u$, only negligible choices are outside of the support, and so b is well mixed.

For perfect-information games, our result implies the following. Firstly, players do not have to be cautious about all outcomes in perfect equilibrium. If all of them share a similar idea about the impact of future errors on the expected payoffs, it is enough for players to select perfect equilibrium. There is an upper-bound for the complexity of cautious reasoning based on completely mixed strategy profiles. Secondly, when we search for a perfect equilibrium, we can use well-mixed strategy profiles instead of completely mixed strategy profiles. Well-mixed strategy profiles require analysts to treat a subset of choices and would make analysis simpler than what we have to do by using completely mixed strategy profiles in broad situations. In the next section, we show these implications are valid in imperfect-information games.

3.2 Imperfect Information

In this subsection, we define well-mixed strategy profiles for imperfect-information games. Since u may include multiple decision nodes, a belief ρ matters. The logic in this subsection is essentially similar to the logic for perfect-information games. If the strategy profile $b \in B$ is well mixed enough, the other choices non-utilized in b are negligible.

When the realization probability of each information set is unchanged by the change in a local strategy, the expected payoffs of each choice satisfy the following characteristic:

Lemma 5 *Consider $b \in B$ and $b'_{iu} \in B_{iu}$ s.t. $\rho(u'', b) = \rho(u'', b/b'_{iu}) = \rho(u'', b/(\epsilon' b_{iu} + (1 - \epsilon')b'_{iu})) > 0$ $\forall u'' \in U$ and $\forall \epsilon' \in (0, 1)$. Then, for any $u' \in U$ and $\epsilon \in (0, 1)$,*

$$H(b/(\epsilon b_{iu} + (1 - \epsilon)b'_{iu}), u'|\rho'') = \epsilon H(b, u'|\rho) + (1 - \epsilon)H(b/b'_{iu}, u'|\rho') \tag{11}$$

[8] Since $LI_u \subset \mathrm{supp}(b_{iu})$, there exists $a_u \in \mathrm{supp}(b_{iu})$ but not in LI_u. Because $|LI_u| = M_u$, in the linear combination, by using a_u, we can construct a 0 vector. Then, we can adjust the total of $\zeta_{pb_{iu}}$.

where ρ, ρ', ρ'' are consistent beliefs for strategy profiles.

Proof: Denote $b'' = b/(\epsilon b_{iu} + (1-\epsilon)b'_{iu})$. From the definition,

$$H(b'', u'|\rho'') = \sum_{x \in u'} \rho''_{u'}(x) \sum_{a \in A} \rho(a, b''|x)h(a), \tag{12}$$

$$H(b, u'|\rho) = \sum_{x \in u'} \rho_{u'}(x) \sum_{a \in A} \rho(a, b|x)h(a), \tag{13}$$

$$H(b/b'_{iu}, u'|\rho') = \sum_{x \in u'} \rho'_{u'}(x) \sum_{a \in A} \rho(a, b/b'_{iu}|x)h(a). \tag{14}$$

Since each information set can appear only once at each path, for each x, either

$$\rho(x, b) = \rho(x, b/b'_{iu}) = \rho(x, b'') \tag{15}$$

or

$$\rho(a, b|x) = \rho(a, b/b'_{iu}|x) = \rho(a, b''|x) \tag{16}$$

is satisfied. Denote $X^1 \subseteq u'$ is the set of x satisfying the former and $u' - X^1 = X^2$ satisfies the latter.

We first focus on $x \in X^1$. Since (15) is satisfied, and because we consider b and b'_{iu} s.t. $\rho(u'', b) = \rho(u'', b/b'_{iu}) = \rho(u'', b/(\epsilon' b_{iu} + (1-\epsilon')b'_{iu}))\ \forall u'' \in U$,

$$\rho_{u'}(x) = \rho'_{u'}(x) = \rho''_{u'}(x). \tag{17}$$

In addition, since $\rho(a, b|x)$ is decided by a product of local strategies possibly including $b_{iu}(a_u)$ s.t. a_u in the path from x to a (if such a a_u exists) and does not depend $b_{iu}(a'_u)$ for any $a'_u \in A_u - \{a_u\}$, $\epsilon\rho(a, b|x) + (1-\epsilon)\rho(a, b/b'_{iu}|x) = \rho(a, b''|x)$. Therefore,

$$\sum_{x \in X^1} \rho''_{u'}(x) \sum_{a \in A} \rho(a, b''|x)h(a) = \sum_{x \in X^1} \rho''_{u'}(x) \sum_{a \in A} (\epsilon\rho(a, b|x) + (1-\epsilon)\rho(a, b/b'_{iu}|x))h(a). \tag{18}$$

Second, we focus on $x \in X^2$. Since $\rho(x, b)$ is decided by a product of local strategies possibly including $b_{iu}(a_u)$ s.t. a_u in the path from O to x (if such a a_u exists) and does not depend $b_{iu}(a'_u)$ for any $a'_u \in A_u - \{a_u\}$, $\epsilon\rho(x, b) + (1-\epsilon)\rho(x, b/b'_{iu}) = \rho(x, b'')$. Because $\rho(u'', b) = \rho(u'', b/b'_{iu}) = \rho(u'', b/(\epsilon' b_{iu} + (1-\epsilon')b'_{iu}))\ \forall u'' \in U$, $\epsilon\rho_{u'}(x) + (1-\epsilon)\rho'_{u'}(x) = \rho''_{u'}(x)$. Therefore,

$$\sum_{x \in X^2} \rho''_{u'}(x) \sum_{a \in A} \rho(a, b''|x)h(a) = \sum_{x \in X^2} (\epsilon\rho'_{u'}(x) + (1-\epsilon)\rho'_{u'}(x)) \sum_{a \in A} \rho(a, b''|x)h(a). \tag{19}$$

Then, by summing up (18) and (19), and because of (15) for $x \in X1$ and (16) for $x \in X2$,

$$H(b'', u'|\rho'') = \epsilon H(b, u'|\rho) + (1-\epsilon)H(b/(b'_{iu}), u'|\rho'). \tag{20}$$

In addition,

Lemma 6 *Consider $b \in B$ and $b'_{iu} \in B_{iu}$ s.t. $\rho(u'', b) = \rho(u'', b/b'_{iu}) = \rho(u'', b/(\epsilon' b_{iu} + (1 - \epsilon')b'_{iu})) > 0$ $\forall u'' \in U$ and $\forall \epsilon' \in (0, 1)$. Then, for any $u' \in U$, $\epsilon \in (0, 1)$, and $pb_{iu'} \in PB_{iu'}$*

$$H(b/(\epsilon b_{iu} + (1 - \epsilon)b'_{iu})/pb_{iu'}, u'|\rho'') = \epsilon H(b/pb_{iu'}, u'|\rho) + (1 - \epsilon)H(b/b'_{iu}/pb_{iu'}, u'|\rho') \quad (21)$$

where ρ, ρ', ρ'' are consistent beliefs for strategy profiles $b, b/b'_{iu}, b/(\epsilon b_{iu} + (1 - \epsilon)b'_{iu})$.

Proof:The proof is essentially identical to the proof for Lemma 5.

Lemma 5 and 6 explain that, as far as the realization of each information set is unchanged, we can separately consider the impact of the change in a local strategy on the expected payoffs for all information sets. This allows us to define linearly-independent choices for imperfect-information games as we did for perfect-information games.

We define a subset of each choice set at each information set s.t. each choice in each subset has the identical impact on the realization probability of each information set.

Definition 9 (Set of Choices with Identical Impact on Information) *Consider $i \in I$ and $u \in U_i$. A subset of the choice set $SA_u \subseteq A_u$ is a set of choices with identical impact on information if and onlyif,*

1. *For any $b, b' \in B$ s.t. $b_{i'u'}(a'_{u'}) = b'_{i'u'}(a'_{u'})$ for any $i' \in I$, $u' \in U_{i'}$ and $a'_{u'} \in A_{u'}$ s.t. $a'_{u'} \neq SA_u$ if $u = u'$, then*
$$\rho(u'', b) = \rho(u'', b') \quad (22)$$
for any $u'' \in U$.

2. *Any D s.t. $SA_u \subset D$ does not satisfy the condition 1 above.*

A_u might include more than one set of choices with identical impact on information. Therefore, we put an index $z \in \mathbb{N}$ on each of such a set SA^z_u. Z_u is the whole set of indexes of SA^z_u s.t. $\sum_{z \in Z_u} SA^z_u = A_u$ at u. Because of the condition 2, $SA^z_u \cap SA^{z'}_u = \emptyset$ for any $z \neq z'$. When we change a part of local strategy among choices in SA^z_u, the change in the expected payoffs is explained by a linear function (Lemma 5 and 6). For any b_{iu}, $SB^z(b_{iu})$ denotes the set of strategy profiles s.t. $b'_{iu} \in SB^z(b_{iu})$ if and only if $b_{iu}(a_u) = b'_{iu}(a_u)$ for all $a_u \in A_u - SA^z_u$. Therefore, if we replace b_{iu} by $b'_{iu} \in SB^z(b_{iu})$, the realization probability of each information set is unchanged. In addition, $\epsilon b_{iu} + (1 - \epsilon)b'_{iu} \in SB^z(b_{iu})$ for any $\epsilon \in (0, 1)$. We denote $b_{iu}(SA^z_u) = \sum_{a_u \in SA^z_u} b_{iu}(a_u)$. We define $PSB^z(b_{iu}) = \{b'_{iu} \in SB^z(b_{iu}) | \exists a_u \in SA^z_u$ *s.t.* $b'_{iu}(SA^z_u) = b'_{iu}(a_u)\}$.

Definition 10 (Linearly-Independent Choice (Imperfect Information)) *Consider an information set $u \in U$, an index $z \in Z_u$, a strategy profile $b \in B$ s.t. $\rho(u'', b) > 0$ $\forall u'' \in U$, and a nonempty set $LI^z_u \subseteq SA^z_u$.*
A choice $a_u \in LI^z_u$ is linearly independent in LI^z_u at u with respect to b.
$\Leftrightarrow$ There exists $b'_{iu} \in PSB^z(b_{iu})$ s.t.

1. *$a_u \in \mathrm{supp}(b'_{iu})$*

2. *There exist $u' \in U$ and $pb_{iu'} \in PB_{iu'}$ s.t. $H(b/b'_{iu}/pb_{iu'}, u') \neq H(b/pb_{iu'}, u')$*

3. *There does not exist ζ s.t. for all $u' \in U$ and $pb_{iu'} \in PB_{iu'}$,*

$$H(b/b'_{iu}/pb_{iu'}, u') - H(b/pb_{iu'}, u') = \sum_{\substack{psb_{iu} \in PSB^z(b_{iu}) \\ \exists a'_u \in LI^z_u - \{a_u\} \text{ s.t. } psb_{iu}(a'_u) > 0}} \zeta_{psb_{iu}} H(b/psb_{iu}, u'). \tag{23}$$

Definition 11 (Set of Linearly-Independent Choices (Imperfect Information)) *Consider an information set $u \in U$, an index $z \in Z_u$, a strategy profile $b \in B$, and a nonempty set $LI^z_u \subseteq SA^z_u$.*
LI^z_u is a set of linearly-independent choices respect to $b \Leftrightarrow$
All choices $a_u \in LI^z_u$ are linearly independent in LI^z_u at u with respect to b.

Lemma 7 *Consider an imperfect-information game Γ, $u \in U$, an index $z \in Z_u$ and $b \in B$ s.t. $\rho(u'', b) > 0$ $\forall u'' \in U$. There exists the maximum number of the elements for sets of linearly-independent choices*[9] *$M^z_u \geqq 0$ s.t.*

1. *$|LI^z_u| \leqq M^z_u \leqq n \sum_{u' \in U - \{u\}} |A'_u|$ for any linearly-independent set $LI^z_u \subseteq SA^z_u$*

2. *There exists a linearly-independent set $LI'^z_u \subseteq SA^z_u$ s.t. $M^z_u = |LI'^z_u|$.*

Proof: Because the expected payoffs of each choice are in $\mathbb{R}^n$ at each information set, and because we consider a linear combination in the right side of (23), the upper bound must be less than $n \sum_{u' \in U - \{u\}} |A'_u|$ for each $b \in B$ and $u \in U$. M^z_u is uniquely decided by b and the set of $h(a)$ s.t. $\exists b' \in SB^z(b_{iu})$ s.t. $\sum_{x \in u} \rho(a, b'|x) > 0$.

Theorem 3 *Consider $i \in I$ and $b \in B$ s.t. $\rho(u'', b) > 0$ $\forall u'' \in U$. If there exists only a single information set $u \in U_i$ s.t. $\exists a_u \in A_u - \text{supp}(b_{iu})$, and if, for each $z \in Z_u$, $SA^z_u \subseteq \text{supp}(b_{iu})$ or there exists a set of linearly-independent choices $LI^z_u \subset \text{supp}(b_{iu}) \cap SA^z_u$ s.t. $|LI^z_u| = M^z_u$. Then, for any neighborhood of b, denoted by O, there exists a completely mixed strategy profile $b' \in O$ s.t. $H(b'/pb_{iu'}, u') = H(b/pb_{iu'}, u')$ for all $u' \in U$ and $pb_{iu'} \in PB_{iu'}$.*

Proof: Because of Lemma 6, as long as the realization probability of each information set is unchanged, we can separately process each choice outside of $\text{supp}(b_{iu})$. For each $a_u \in A_u - \text{supp}(b_{iu})$, there exists $z \in Z_u$ s.t. $a_u \in SA^z_u$. Since a_u is not in LI^z_u, and since LI^z_u includes the maximum number of elements, $LI^z_u + \{a_u\}$ is not a set of linearly-independent choices. This implies, from the definition of linearly-independent choices, one of the following two cases. Consider $b'_{iu} \in PSB^z(b_{iu})$ s.t. $a_u \in \text{supp}(b'_{iu})$. In the first case, there does not exist $i' \in I$, $u' \in U_{i'}$ and $pb_{i'u'} \in PB_{i'u'}$ s.t. $H(b/b'_{iu}/pb_{iu'}, u') \neq H(b/pb_{iu'}, u')$. Then, for any $\epsilon \in (0, 1)$, because of Lemma 6, $b'' = b/(\epsilon b_{iu} + (1 - \epsilon)b'_{iu})$ satisfies $H(b''/pb_{i'u'}, u') = H(b/pb_{i'u'}, u')$ for any $u' \in U_{i'}$ and $pb_{i'u'} \in PB_{i'u'}$. In the second case, there exists ζ s.t. for all $u' \in U$ and

[9] The number is uniquely decided by Γ and b.

$pb_{iu'} \in PB_{iu'}$,

$$H(b/b'_{iu}/pb_{iu'}, u') - H(b/pb_{iu'}, u') = \sum_{\substack{psb_{iu} \in PSB^z(b_{iu}) \\ \exists a'_u \in \text{supp}(b_{iu}) \cap SA^z_u \ s.t. \ psb_{iu}(a'_u) > 0}} \zeta_{psb_{iu}} H(b/psb_{iu}, u') \quad (24)$$

where $\sum_{\substack{psb_{iu} \in PSB^z(b_{iu}) \\ \exists a'_u \in \text{supp}(b_{iu}) \ s.t. \ psb_{iu}(a'_u) > 0}} \zeta_{psb_{iu}} = 0$[10]. Then, there exists $\bar{\epsilon} > 0$ s.t. for any $\epsilon \in (\bar{\epsilon}, 1)$, $\epsilon b_{iu}(a'_u) - (1-\epsilon) \sum_{\substack{psb_{iu} \in PSB^z(b_{iu}) \\ \exists a'_u \in \text{supp}(b_{iu}) \cap SA^z_u \ s.t. \ psb_{iu}(a'_u) > 0}} \zeta_{psb_{iu}} psb_{iu}(a'_u) > 0 \ \forall a'_u \in \text{supp}(b_{iu}) \cap SA^z_u$ and there exists $b'''_{iu} \in SB^z(b_{iu})$ s.t.

$$b'''_{iu} = \epsilon b_{iu} + (1-\epsilon) \left(b'_{iu} - \sum_{\substack{psb_{iu} \in PSB^z(b_{iu}) \\ \exists a'_u \in \text{supp}(b_{iu}) \cap SA^z_u \ s.t. \ psb_{iu}(a'_u) > 0}} \zeta_{psb_{iu}} psb_{iu} \right), \quad (25)$$

and $b'''' = b/b'''_{iu}$ satisfies $H(b''''/pb_{i'u'}, u') = H(b/pb_{i'u'}, u')$ for any $u' \in U_{i'}$ and $pb_{i'u'} \in PB_{i'u'}$. Therefore, in either case, we can expand the support of b_{iu} while keeping the expected payoffs of each choice and the realization probability of each information set. We replace b by b'' or b'''', and we can repeat the same process for any negligible choices until b becomes completely mixed. In addition, in each step, when we construct b'' and b'''', we can take any $1-\epsilon$ close to 0 and so b'' and b'''' can be aribtrary close to b.

When there exist more than two information sets in which local strategies are not completely mixed, we need further technical discussions about the characteristic of the linearly-independent choices to understand the interactions between information sets. Although the expected payoffs of each choice are unchanged, the change in the belief can influence the maximum number of elements in sets of linearly-independent choices. There are following two useful lemmas:

Lemma 8 *Consider a set of linearly-independent choices LI^z_u respect to b s.t. $\rho(u'', b) > 0$ $\forall u'' \in U$. If a change happens in the belief but the change is tiny enough, each choice in the set of (ex-ante) linearly-independent choices LI^z_u is still linearly independent to each other.*

Proof: Consider a choice $a_u \in LI^z_u$. Because of the definition of linearly-independent choices, the expected payoffs of a_u are not in the subspace covered by the linear combination of the expected payoffs of the other choices in LI^z_u. Therefore, there is the minimum distance between the expected payoffs of a_u and each point in the subspace.

When the belief ρ_u at u slightly changes, the expected payoffs from each choice can change slightly, and the difference between the original expected payoffs and the new expected payoffs from each choice depends on the scale of the change in the belief. Then, the shortest distance between the original subspace and each element in the new subspace must be tiny if the change in the belief is tiny. In addition, the distance between the original expected payoffs of a_u and the new expected payoffs of a_u is tiny. Therefore, the new subspace based

[10] Since $LI^z_u \subset \text{supp}(b_{iu})$ s.t. $|LI^z_u| = M^z_u$, we can construct a 0 vector and so we can adjust the total.

on the linear combination of the other choices in LI_u^z cannot cover the new expected payoffs of a_u if the change in the belief is small enough.

Lemma 8 implies that there exists a neighborhood of b s.t. each linearly-independent choice in LI_u^z is still linearly independent for each other if we assume a consistent belief and b satisfies $\rho(u'', b) > 0\ \forall u'' \in U$. Hereafter, we denote such a neighborhood $LO(b)$.

Lemma 9 *Consider b s.t. $\rho(u'', b) > 0\ \forall u'' \in U$. After any tiny change in the belief, a set of ex-ante non-linearly independent choices can become a linearly-independent set. In addition, if it is possible to find another $b' \in LO(b)$ and $\rho' \in CO(b')$ with which the larger set of linearly-independent choices exists at an information set (M_u^z for b is smaller than M_u^z for b'), we can always find such b' by utilizing a single choice at each information set with any small probability.*

Proof: If there exists an outcome the payoff vector of which is not in the subspace based on the choices in LI_u^z, and if the outcome has a path from u via SA_u^z, we can give (additional) positive probability to the outcome by allocating (additional) positive probability to a single choice at each information set in the path (For this process, any tiny probability is enough.). If such an outcome does not exist, the payoff vector from each outcome is in the subspace, and any linear combination of such outcomes would be in the subspace based on LI_u^z.

Lemma 8 and 9 tell us that a tiny enough change in the belief would not decrease any M_u^z but can increase M_u^z. These characteristics imply that, when we change a local strategy by inserting a choice outside of $\mathrm{supp}(b_{iu})$, if the original b satisfies $\rho(u'', b) > 0\ \forall u'' \in U$, for consistent beliefs, each set of linearly-independent choices before the change is still a set of linearly-independent choices after the change. In other words, when $M_u^z < n \sum_{u' \in U - \{u\}} |A'_u|$, we are not sure whether M_u^z is the upper bound after a tiny change in the belief. Therefore, we introduce another number which counts the possibility. We denote that $M_u(b)$ is the maximum number of elements in sets of linearly-independent choices for $b \in B$.

Definition 12 (Robust Maximum Number of Linearly-Independent Choices) *Consider an imperfect-information game Γ, $u \in U$ and $b \in B$ s.t. $\rho(u'', b) > 0\ \forall u'' \in U$. $RM_u^z \geqq 0$ is the robust maximum number of the elements for sets of linearly-independent choices under b if and only if there exists a neighborhood of b denoted by O s.t. for any neighborhood of b s.t. $O' \subseteq O$, for any $pb \in PB$, and for any $\epsilon \in (0, 1)$ s.t. $((1-\epsilon)(b) + \epsilon pb) = b^\epsilon \in O'$, $RM_u^z \geqq M_u^z(b^\epsilon)$ and $RM_u^z = M_u^z(b)$.*

Theorem 4 *Consider $b \in B$ s.t. $\rho(u'', b) > 0\ \forall u'' \in U$ and for each $i \in I$, $u \in U_i$ and $z \in Z_u$, $SA_u^z \subseteq \mathrm{supp}(b_{iu})$ or there exists a set of linearly-independent choices $LI_u^z \subset \mathrm{supp}(b_{iu}) \cap SA_u^z$ s.t. $|LI_u^z| = RM_u^z$ where RM_u^z is the robust maximum number of linearly-independent choices under b. Then, there exists a completely mixed strategy profile b' s.t. $H(b'/pb_{iu'}, u') = H(b/pb_{iu'}, u')$ for all $u' \in U$ and $pb_{iu'} \in PB_{iu'}$. In addition, for any $\epsilon \in (0, 1)$, we can find such a completely mixed strategy profile b' satisfying*

$$\max_{i \in I, u \in U_i, a_u \in A_u} (|b'_{iu}(a) - b_{iu}(a)|) < \epsilon. \tag{26}$$

Proof: Because of Lemma 8, there exists a neighborhood $LO(b)$ in which any set of linearly-independent choices are still a set of linearly-independent choices. In $LO(b)$, there exists an open ball around b the diameter of which is denoted by $\epsilon > 0$. Consider $i \in I$ and $u \in U_i$. From the proof of Theorem 3, we can replace the local strategy from b_{iu} to completely mixed b'_{iu} while keeping the expected payoffs of each choice. In addition, from the proof of Theorem 3, we can make $\max_{a_u \in A_u}(|b_{iu}(a_u) - b'_{iu}(a_u)|)$ arbitrarily small. Then, we can pick b'_{iu} s.t. the maximum difference is smaller than $\epsilon/2$. Since $b/b'_{iu} \in LO(b)$, we can do the similar procedure to another $u' \in U$, and keep the difference smaller than $\epsilon/2$. We can repeat this process for all $u'' \in U$ until the strategy profile becomes completely mixed.

From Theorem 4, we define well-mixed strategy profile for imperfect-information games as follows:

Definition 13 (Well-Mixed Strategy Profile (Imperfect Information)) *b is well-mixed strategy profile iff*

1. $\rho(u'', b) > 0 \ \forall u'' \in U$
2. *For each $i \in I$, $u \in U_i$ and $z \in Z_u$, $SA_u^z \subseteq \mathrm{supp}(b_{iu})$ or there exists a set of linearly-independent choices $LI_u^z \subset \mathrm{supp}(b_{iu}) \cap SA_u^z$ s.t. $|LI_u^z| = RM_u^z$ where RM_u^z is the robust maximum number of linearly-independent choices under b*

Because of Theorem 4, if b is well mixed, in any neighborhood of b, there exists a completely mixed strategy profile b' s.t. the expected payoffs of each choice in b' are identical to the payoffs in b.

4 Applications

As an application of Theorem 1 and 4, consider Game 1 again. Suppose $b \in SE(\Gamma)$ s.t. $b_{1,\emptyset}(L) = 1$ and $b_{2,u1}(L) = 0.5, b_{2,u1}(R) = 0.5$. Since Player 2 at $u1$ cannot observe Player 1's action at $\emptyset$, and all choices induce $u1$, there exists a single set of choices with identical impact on information $SA_\emptyset$ s.t. $SA_\emptyset = A_\emptyset$. We consider a sequence of strategy profiles $b^k \to b$ and $\epsilon \in (0, 1)$ s.t. $b^k = b/((1-\epsilon^k)b_{1,\emptyset} + \epsilon^k b'_{1,\emptyset})$ where $b'_{1,\emptyset}(M) = \epsilon/4$ for any $M \in \{M1, ..., M4\}$ and $b'_{1,\emptyset}(L) = 1 - \epsilon$. Then, each b^k is well mixed because $\{M1, ..., M4\} \subset \mathrm{supp}(b'_{iu})$ is a set of linearly independent choices and the number of elements is 4 (2 players $\times$ 2 choices at $u1$). Since each local strategy in b is optimal in b^k and the consistent belief $\rho \in CO(b^k)$, from Theorem 1 and Theorem 4, b is perfect.

In addition, if we define sequential/perfect equilibrium by using a sequence of well-mixed strategy profiles instead of a sequence of completely mixed strategy profiles, we can apply these concepts to games with uncountable strategy sets. However, if there exist uncountable information sets, we need further discussions. Myerson and Reny (2020) proposes a new solution concept, perfect conditional equilibrium distribution, which coincides with sequential equilibrium in finite settings. This concept utilizes analytical tools, a perfect conditional ϵ-equilibrium and an admissible net of strategies and nature perturbations, which enables us to deal with the optimality in the uncountable information sets. In Jinushi (2022b), we apply an adjusted FSTP based on well-mixed strategy profiles to perfect conditional equilibrium distribution. By applying FSTP, we delete outcome distributions with weakly dominated choices from the set of perfect conditional equilibrium distributions as FSTP rejects a sequential equilibrium with weakly dominated choices.

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