Title

Ryoma Senda[1], Yoshiaki Takata[2], and Hiroyuki Seki[1]

[1] Graduate School of Informatics, Nagoya University,
Furo-cho, Chikusa, Nagoya 464-8601, Japan
ryoma.private@sqlab.jp, seki@i.nagoya-u.ac.jp
[2] Graduate School of Engineering, Kochi University of Technology,
Tosayamada, Kami City, Kochi 782-8502, Japan
takata.yoshiaki@kochi-tech.ac.jp

Abstract.

1 Introduction

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $[n] = \{1, \cdots, n\}$ for $n \in \mathbb{N}$. For a set A, let $\mathscr{P}(A)$ be the power set of A, let A^* and A^ω be the sets of finite and infinite words over A, respectively. We denote $A^+ = A^* \setminus \{\varepsilon\}$ and $A^\infty = A^* \cup A^\omega$. For a word $\alpha \in A^\infty$ over a set A, let $\alpha(i) \in A$ be the i-th element of α ($i \geq 0$), $\alpha(i : j) = \alpha(i)\alpha(i+1)\cdots\alpha(j-1)\alpha(j)$ for $i \geq j$ and $\alpha(i :) = \alpha(i)\cdots$ for $i \geq 0$. Let $\langle u, w\rangle = u(0)w(0)u(1)w(1)\cdots \in A^\infty$ for words $u, w \in A^\infty$ and $\langle B, C\rangle = \{\langle u, w\rangle \mid u \in B, w \in C\}$ for sets $B, C \subseteq A^\infty$. By $|\beta|$, we mean the cardinality of β if β is a set and the length of β if β is a finite sequence. For a function $f : A \to B$ from a set A to a set B, let $f(w) = f(w(0))f(w(1))\ldots$ for a word $w \in A^\infty$ and let $f(L) = \{f(w) \mid w \in L\}$ for a set $L \subseteq A^\infty$ of words. Let fst be the function such that $\mathit{fst}((a, b)) = a$ for any pair (a, b). Let id be the identity function; i.e., $\mathit{id}(a) = a$ for any a.

In this paper, disjoint sets $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ denote a (finite) input alphabet, an output alphabet and a stack alphabet, respectively, and $\Sigma = \Sigma_{\mathbb{i}} \cup \Sigma_{\mathbb{o}}$. For a set Γ, let $\mathtt{Com}(\Gamma) = \{\mathtt{pop}, \mathtt{skip}\} \cup \{\mathtt{push}(z) \mid z \in \Gamma\}$ be the set of stack commands over Γ.

2.1 Transition Systems

Definition 1. *A transition system (TS) is* $\mathcal{S} = (S, s_0, A, E, \to_{\mathcal{S}}, c)$ *where*

- S *is a (finite or infinite) set of states,*
- $s_0 \in S$ *is the initial state,*
- A, E *is (finite or infinite) alphabets such that* $A \cap E = \emptyset$*,*
- $\to_{\mathcal{S}} \subseteq S \times (A \cup E) \times S$ *is a set of transition relation, written as* $s \to^a s'$ *if* $(s, a, s') \in \to_{\mathcal{S}}$ *and*

– $c : S \to [n]$ *is a coloring function where* $n \in \mathbb{N}$.

An element of A is an observable label and an element of E is an internal label. A *run* of TS $\mathcal{S} = (S, s_0, A, E, \to_{\mathcal{S}}, c)$ is a pair $(\rho, w) \in S^\omega \times (A \cup E)^\omega$ that satisfies $\rho(0) = s_0$ and $\rho(i) \to^{w(i)} \rho(i+1)$ for $i \geq 0$. Let $\min_{\inf} : S^\omega \to [n]$ be a minimal coloring function such that $\min_{\inf}(\rho) = \min\{m \mid$ there exist an infinite number of $i \geq 0$ such that $c(\rho(i)) = m\}$. We call $\mathcal{S}$ deterministic if $s \to^a s_1$ and $s \to^a s_2$ implies $s_1 = s_2$ for all $s, s_1, s_2 \in S$ and $a \in A \cup E$.

For $w \in (A \cup E)^\omega$, let $\mathit{ef}(w) = a_0 a_1 \cdots \in A^\infty$ be the sequence obtained from w by removing all symbols belonging to E. Note that $\mathit{ef}(w)$ is not always an infinite sequence even if w is an infinite sequence. We define the *language* of $\mathcal{S}$ as $L(\mathcal{S}) = \{\mathit{ef}(w) \in A^\omega \mid$ there exists a run (ρ, w) such that $\min_{\inf}(\rho)$ is even$\}$. For $m \in \mathbb{N}_0$, we call $\mathcal{S}$ an m-TS if for every run (ρ, w) of $\mathcal{S}$, w contains no consecutive subsequence $w' \in E^{m+1}$.

Consider two TSs $\mathcal{S}_1 = (S_1, s_{01}, A_1, E_1, \to_{\mathcal{S}_1}, c_1)$ and $\mathcal{S}_2 = (S_2, s_{02}, A_2, E_2, \to_{\mathcal{S}_2}, c_2)$ and a function $\sigma : (A_1 \cup E_1) \to (A_2 \cup E_2)$. We call $R \subseteq S_1 \times S_2$ a *σ-bisimulation relation* from $\mathcal{S}_1$ to $\mathcal{S}_2$ if R satisfies the followings:

(1) $(s_{01}, s_{02}) \in R$.
(2) For any $s_1, s_1' \in S_1$, $s_2 \in S_2$, and $a_1 \in A_1 \cup E_1$, if $s_1 \to_{\mathcal{S}_1}^{a_1} s_1'$ and $(s_1, s_2) \in R$, then $\exists s_2' \in S_2 : s_2 \to_{\mathcal{S}_2}^{\sigma(a_1)} s_2'$ and $(s_1', s_2') \in R$.
(3) For any $s_1 \in S_1$, $s_2, s_2' \in S_2$, and $a_2 \in A_2 \cup E_2$, if $s_2 \to_{\mathcal{S}_2}^{a_2} s_2'$ and $(s_1, s_2) \in R$, then $\exists s_1' \in S_1$, $\exists a_1 \in A_1 \cup E_1 : \sigma(a_1) = a_2$ and $s_1 \to_{\mathcal{S}_1}^{a_1} s_1'$ and $(s_1', s_2') \in R$.
(4) If $(s_1, s_2) \in R$, then $c_1(s_1) = c_2(s_2)$.

We say $\mathcal{S}_1$ is *σ-bisimilar* to $\mathcal{S}_2$ if there exists a σ-bisimulation relation from $\mathcal{S}_1$ to $\mathcal{S}_2$. We call R a bisimulation relation if R is a id-bisimulation relation. We say $\mathcal{S}_1$ and $\mathcal{S}_2$ are bisimilar if $\mathcal{S}_1$ is id-bisimilar to $\mathcal{S}_2$.

The following lemma can be proved by definition.

Lemma 2. *If* $\mathcal{S}_1 = (S_1, s_{01}, A_1, E_1, \to_{\mathcal{S}_1}, c_1)$ *is σ-bisimilar to* $\mathcal{S}_2 = (S_2, s_{02}, A_2, E_2, \to_{\mathcal{S}_2}, c_2)$ *for a function* $\sigma : (A_1 \cup E_1) \to (A_2 \cup E_2)$ *that satisfies* $a \in A_1 \Leftrightarrow \sigma(a) \in A_2$ *for any* $a \in A_1 \cup E_1$, *then* $\sigma(L(\mathcal{S}_1)) = L(\mathcal{S}_2)$.

3 Pushdown Transducers, Automata and Games

3.1 Pushdown Transducers

Definition 3. *A pushdown transducer (PDT) over finite alphabets* $\Sigma_{\mathtt{i}}$, $\Sigma_{\mathtt{o}}$ *and* Γ *is* $\mathcal{T} = (P, p_0, z_0, \Delta)$ *where P is a finite set of states,* $p_0 \in P$ *is the initial state,* $z_0 \in \Gamma$ *is the initial stack symbol and* $\Delta : P \times \Sigma_{\mathtt{i}} \times \Gamma \to P \times \Sigma_{\mathtt{o}} \times \mathtt{Com}(\Gamma)$ *is a finite set of deterministic transition rules having one of the following forms:*

– $(p, a, z) \to (q, b, \mathtt{pop})$ *(pop rule)*
– $(p, a, z) \to (q, b, \mathtt{skip})$ *(skip rule)*
– $(p, a, z) \to (q, b, \mathtt{push}(z))$ *(push rule)*

where $p, q \in P$, $a \in \Sigma_{\mathbb{i}}$, $b \in \Sigma_{\mathbb{o}}$ *and* $z \in \Gamma$.

For a state $p \in P$ and a finite sequence representing stack contents $u \in \Gamma^*$, (p, u) is called a *configuration* or *instantaneous description (abbreviated as ID)* of PDT $\mathcal{T}$. Let $ID_{\mathcal{T}}$ denote the set of all IDs of $\mathcal{T}$. Let $\Rightarrow_{\mathcal{T}} \subseteq ID_{\mathcal{T}} \times \Sigma_{\mathbb{i}} \cdot \Sigma_{\mathbb{o}} \times ID_{\mathcal{T}}$ be the transition relation of $\mathcal{T}$ that satisfies the following conditions. For $u \in \Gamma^+$ and $\mathtt{com} \in \mathtt{Com}(\Gamma)$, let us define $\mathtt{upds}(u, \mathtt{com})$ as $\mathtt{upds}(u, \mathtt{com}) = u(1:)$, $\mathtt{upds}(u, \mathtt{skip}) = u$ and $\mathtt{upds}(u, \mathtt{push}(z')) = z'u$.

For two IDs $(p, u), (q, u') \in ID_{\mathcal{T}}$, $a \in \Sigma_{\mathbb{i}}$ and $b \in \Sigma_{\mathbb{o}}$, $((p, u), ab, (q, u')) \in \Rightarrow_{\mathcal{T}}$, written as $(p, u) \Rightarrow^{ab}_{\mathcal{T}} (q, u')$, if there exist a rule $(p, a, z) \to (q, b, \mathtt{com}) \in \Delta$ such that $z = u(0)$ and $u' = \mathtt{upds}(u, \mathtt{com})$. If $\mathcal{T}$ is clear from the context, we abbreviate $\Rightarrow^{ab}_{\mathcal{T}}$ as $\Rightarrow^{ab}$. That is, there is no transition from an ID with empty stack. We define a run and the language $L(\mathcal{T}) \subseteq (\Sigma_{\mathbb{i}} \cdot \Sigma_{\mathbb{o}})^\omega$ of PDT $\mathcal{T}$ as those of deterministic 0-TS $(ID_{\mathcal{T}}, (q_0, z_0), \Sigma_{\mathbb{i}} \cdot \Sigma_{\mathbb{o}}, \emptyset, \Rightarrow_{\mathcal{T}}, c)$ where $c(s) = 2$ for all $s \in ID_{\mathcal{T}}$. In this paper, we assume that every run of PDT never reach an ID whose stack is empty. Let **PDT** be the class of PDT.

Example 4. Let us consider PDT $\mathcal{T} = (\{p\}, p, z, \Delta)$ over $\{0, 1\}, \{a, b\}$ and $\{z\}$ where $\Delta = \{(p, 0, z) \to (p, a, \mathtt{skip}), (p, 1, z) \to (p, b, \mathtt{push}(z))\}$ We can see a pair of sequences $(\rho, w) \in ID^\omega_{\mathcal{T}} \times (\{0, 1\} \cdot \{a, b\})^\omega$ where $\rho = (p, z)(p, z)(p, zz)(p, zz)(p, zzz)(p, zzz) \cdots$ and $w = (0a1b)^\omega$ is a run of $\mathcal{T}$. Also, we can check $L(\mathcal{T}) = (\{0a\} \cup \{1b\})^\omega$.

3.2 Pushdown Automata

Definition 5. *A nondeterministic pushdown automata (NPDA) over finite alphabets* $\Sigma_{\mathbb{i}}$, $\Sigma_{\mathbb{o}}$ *and* Γ *is* $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, z_0, \delta, c)$ *where* Q, $Q_{\mathbb{i}}, Q_{\mathbb{o}}$ *are finite sets of states that satisfy* $Q = Q_{\mathbb{i}} \cup Q_{\mathbb{o}}$ *and* $Q_{\mathbb{i}} \cap Q_{\mathbb{o}} = \emptyset$, $q_0 \in Q_{\mathbb{i}}$ *is the initial state,* $z_0 \in \Gamma$ *is the initial stack symbol,* $c : Q \to [n]$ *is a coloring function where* $n \in \mathbb{N}$ *is the number of priorities and* $\delta : Q \times \Sigma \times \Gamma \to \mathscr{P}(Q \times \mathtt{Com}(\Gamma))$ *is a finite set of transition rules, having one of the following forms:*

- $(q_{\mathbb{x}}, a_{\mathbb{x}}, z) \to (q_{\overline{\mathbb{x}}}, \mathtt{com})$ *(input/output rules)*
- $(q_{\mathbb{x}}, \tau, z) \to (q'_{\mathbb{x}}, \mathtt{com})$ *(τ rules)*

where $(\mathbb{x}, \overline{\mathbb{x}}) \in \{(\mathbb{i}, \mathbb{o}), (\mathbb{o}, \mathbb{i})\}$, $q_{\mathbb{x}}, q'_{\mathbb{x}} \in Q_{\mathbb{x}}, q_{\overline{\mathbb{x}}} \in Q_{\overline{\mathbb{x}}}, a_{\mathbb{x}} \in \Sigma_{\mathbb{x}}$, $z \in \Gamma$ *and* $\mathtt{com} \in \mathtt{Com}(\Gamma)$.

We define $ID_{\mathcal{A}} = Q \times \Gamma^*$ and a transition relation $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times (\Sigma \cup \{\tau\}) \times ID_{\mathcal{A}}$ as $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$ iff there exist a rule $(p, a, z) \to (q, \mathtt{com}) \in \delta$ and a sequence $u \in \Gamma^*$ such that $z = u(0)$ and $u' = \mathtt{upds}(u, \mathtt{com})$. We write $(q, u) \vdash^a_{\mathcal{A}} (q', u')$ iff $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$. We write $\vdash^a_{\mathcal{A}}$ as $\vdash^a$ if $\mathcal{A}$ is clear from comtext. We define a run and the language $L(\mathcal{A})$ of $\mathcal{A}$ as those of TS $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, z_0), \Sigma, \{\tau\}, \vdash_{\mathcal{A}}, c')$ where $c'((q, u)) = c(q)$ for every $(q, u) \in ID_{\mathcal{A}}$. We call a PDA $\mathcal{A}$ deterministic if $\mathcal{S}_{\mathcal{A}}$ is deterministic. We call $\mathcal{A}$ an m-NPDA (or m-DPDA when $\mathcal{A}$ is deterministic) if $\mathcal{S}_{\mathcal{A}}$ is an m-TS. We abbreviate 0-NPDA (0-DPDA) as NPDA (DPDA). Let **DPDA** and **NPDA** be the classes of DPDA and NPDA, respectively.

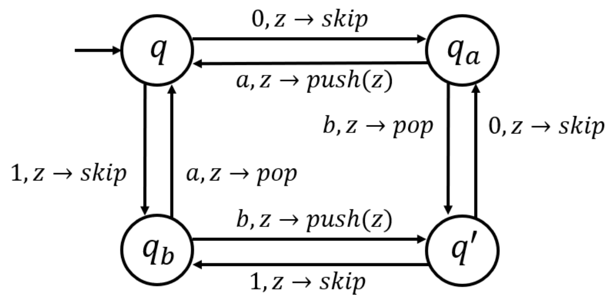

Fig. 1. States and transitions of $\mathcal{A}$. (A label $a, b \to c$ from q to q' means $(q, a, b) \to (q', c) \in \delta$.)

Example 6. Let us consider DPDA $\mathcal{A} = (\{q, q', q_a, q_b\}, \{q, q'\}, \{q_a, q_b\}, q, z, \delta, c)$ over $\{0, 1\}, \{a, b\}$ and $\{z\}$ where $c(q') = 1$, $c(s) = 2$ for $s = q, q_a, q_b$ and δ is defined as Fig. 1.

We can see a pair of sequences $(\rho, w) \in ID^{\omega}_{\mathcal{T}} \times (\{0,1\} \cdot \{a,b\})^{\omega}$ defined in Example 4, where $\rho = (q,z)(q,z)(q,zz)(q,zz)(q,zzz)(q,zzz)\cdots$ and $w = (0a1b)^{\omega}$, is a run of $\mathcal{A}$. However, the sequence $w_1 = (0a1a)^{\omega}$ and $w_2 = 0b(0a1b)^{\omega}$ are not in $L(\mathcal{A})$ because the run (ρ_1, w_1) visits q' infinitely and the input sequence w_2 forces a stack of $\mathcal{A}$ empty by reading $0b$ first. We call $0a$ and $1b$ good sequences and $0b$ and $1a$ bad sequences. For a sequence $w \in (\{0,1\} \cdot \{a,b\})^{\infty}$, let $\#_g(w)$ and $\#_b(w)$ be the number of good and bad sequences appearing in w, respectively. We can check $L(\mathcal{A}) = \{w \in (\{0,1\} \cdot \{a,b\})^{\omega} \mid \#_b(w)$ is finite and $\#_g(w') - \#_b(w') \geq 0$ for all subsequence $w' = w(0:m)$ of w for all $m \in \mathbb{N}_0\}$. For PDT $\mathcal{T}$ defined in Example 4, we can check $L(\mathcal{T}) \subseteq L(\mathcal{A})$ because $L(\mathcal{T}) = (\{0a\} \cup \{1b\})^{\omega}$ contains no sequence that includes a bad sequence.

The following lemma states that the class of language recognized by m-DPDA and 0-DPDA is the same for a fixed m.

Lemma 7. *For a given m-DPDA $\mathcal{A}$, we can construct a 0-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$*

A main idea to prove is that we define a stack alphabet Γ' of $\mathcal{A}'$ as a m-tuple of the stack alphabet Γ of $\mathcal{A}$, that is, $\Gamma' = \Gamma^m$. We can simulate m-steps consecutive push rules (or pop rules) of $\mathcal{A}$ as 1-step push (or pop) rule of $\mathcal{A}'$.

3.3 Pushdown Games

Definition 8. *A pushdown game of DPDA $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, z_0, \delta, c)$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ is $\mathcal{G}_{\mathcal{A}} = (V, V_{\mathbb{i}}, V_{\mathbb{o}}, E, C)$ where $V = Q \times \Gamma^*$ is the set of vertices with $V_{\mathbb{i}} = Q_{\mathbb{i}} \times \Gamma^*, V_{\mathbb{o}} = Q_{\mathbb{o}} \times \Gamma^*$, $E \subseteq V \times V$ is the set of edges defined as $E = \{(v, v') \mid v \vdash^a v'$ for some $a \in \Sigma_{\mathbb{i}} \cup \Sigma_{\mathbb{o}}\}$ and $C : V \to [n]$ is the coloring function such that $C((q, u)) = c(q)$ for all $(q, u) \in V$.*

The game starts with some $(q_0, z_0) \in V_{\mathtt{i}}$. When the current vertex is $v \in V_{\mathtt{i}}$, Player I chooses a successor $v' \in V_{\mathtt{o}}$ of v as the next vertex. When the current vertex is $v \in V_{\mathtt{o}}$, Player II chooses a successor $v' \in V_{\mathtt{i}}$ of v. Formally, a finite or infinite sequence $\rho \in V^{\infty}$ is *valid* if $\rho(0) = (q_0, z_0)$ and $(\rho(i-1), \rho(i)) \in E$ for every $i \geq 1$. A *play* of $\mathcal{G}_{\mathcal{A}}$ is an infinite and valid sequence $\rho \in V^{\omega}$. Let PL be the set of plays. A play $\rho \in PL$ is *winning* for Player I iff $\min\{m \in [n] \mid$ there exist an infinite number of $i \geq 0$ such that $c(\rho(i)) = m\}$ is even.

By the definition of $\mathcal{G}_{\mathcal{A}}$, the following lemma holds.

Lemma 9. *Let $f_1 : PL \to (Q \times \mathtt{Com}(\Gamma))^{\omega}$ and $f_2 : PL \to \Sigma^{\omega}$ be the functions defined as follows. For every play $\rho = (q_0, u_0)(q_1, u_1) \cdots \in PL$ of $\mathcal{G}_{\mathcal{A}}$,*

- *$f_1(\rho) = (q_0, \mathtt{com}_0)(q_1, \mathtt{com}_1) \cdots \in (Q \times \mathtt{Com})^{\omega}$ where $u_{i+1} = \mathtt{upds}(u_i, \mathtt{com}_i)$ for all $i \geq 0$ and*
- *$f_2(\rho) = w$ where $\rho(i) \vdash^{w(i)} \rho(i+1)$ for all $i \geq 0$.*

Then, f_1 and f_2 are well-defined and both of f_1 and f_2 are injections.

[Walukiewucz, 2001] proved that we can construct a PDT $\mathcal{T}$ that gives a winning strategy of $\mathcal{G}_{\mathcal{A}}$, that is, $L(\mathcal{T}) = \{f_1(\rho) \mid \rho$ is winning for Player I $\}$.

Theorem 10. *[Walukiewucz, 2001] If player I has a winning strategy of $\mathcal{G}_{\mathcal{A}}$, we can construct a PDT $\mathcal{T}$ over $Q_{\mathtt{i}} \times \mathtt{Com}(\Gamma), Q_{\mathtt{o}} \times \mathtt{Com}(\Gamma)$ and a stack alphabet Γ' that gives a winning strategy of $\mathcal{G}_{\mathcal{A}}$. That is, $\rho \in PL$ is winning for Player I iff $f_1(\rho) \in L(\mathcal{T})$.*

By Lemma 9, a winning strategy can be also given as the set of sequences $w \in \Sigma^{\omega}$ such that the play ρ is winning where $f_2(\rho) = w$. Thus, we can obtain the following lemma in a similar way to Theorem 10.

Lemma 11. *If player I has a winning strategy of $\mathcal{G}_{\mathcal{A}}$, we can construct a PDT $\mathcal{T}$ over $\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}$ and Γ' that gives a winning strategy of $\mathcal{G}_{\mathcal{A}}$. That is, $\rho \in PL$ is winning for Player I iff $f_2(\rho) \in L(\mathcal{T})$.*

4 Realizability problems for PDA and PDT

For a specification S and an implementation I, we write $I \models S$ if $L(I) \subseteq L(S)$.

Definition 12. *Realizability problem* $\textsc{Real}$*$(\mathcal{S}, \mathcal{I})$ for a class of specifications $\mathcal{S}$ and of implementations $\mathcal{I}$: For a specification $S \in \mathcal{S}$, is there an implementation $I \in \mathcal{I}$ such that $I \models S$?*

Example 13. By Examples 4 and 6, $L(\mathcal{T}) \subseteq L(\mathcal{A})$ holds for PDT $\mathcal{T}$ and DPDA $\mathcal{A}$ defined in the examples. Thus, $\mathcal{T} \models \mathcal{A}$ holds.

Theorem 14. $\textsc{Real}$(**DPDA**, **PDT**) *is decidable.*

Proof. Let $\mathcal{A}$ be a given DPDA. By Lemma 11, we can construct a PDT $\mathcal{T}$ such that ρ is winning play of $\mathcal{G}_{\mathcal{A}}$ iff $f_2(\rho) \in L(T)$. By the definition of f_2, $\rho(i) \vdash^{w(i)} \rho(i+1)$ holds for all $i \geq 0$ for $\rho \in PL$ such that $f_2(\rho) = w$. Then, $w \in L(\mathcal{A})$ holds, and thus $\mathcal{T} \models \mathcal{A}$. Hence, we can say $\mathcal{T} \models \mathcal{A}$ iff player I has a winning strategy for the game $\mathcal{G}_{\mathcal{A}}$. Because there is an algorithm for constructing $\mathcal{T}$ in [Walukiewucz, 2001], Real(**DPDA**, **PDT**) is decidable.

Theorem 15. Real(**NPDA**, **PDT**) *is undecidable.*

Proof. We prove the theorem by a reduction from the universality problem of NPDA, which is undecidable. For a given NPDA $\mathcal{A} = (Q, Q_{\mathtt{i}}, Q_{\mathtt{o}}, q_0, z_0, \delta, c)$ over $\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}$ and Γ, we can construct an NPDA $\mathcal{A}' = (Q \times [2], Q \times \{1\}, Q \times \{2\}, q_0, z_0, \delta', c')$ over $\Sigma'_{\mathtt{i}}, \Sigma'_{\mathtt{o}}$ and Γ where $\Sigma'_{\mathtt{i}} = \Sigma_{\mathtt{i}} \cup \Sigma_{\mathtt{o}}$, $\Sigma'_{\mathtt{o}}$ is an arbitrary (nonempty) alphabet, $c'((q,1)) = c'((q,2)) = c(q)$ for all $q \in Q$ and δ' satisfies that $((q,1), a, z) \to ((q',2), \mathtt{com}) \in \delta$ iff $(q, a, z) \to (q', \mathtt{com}) \in \delta'$, and $((q',2), b, z) \to ((q',1), \mathtt{skip}) \in \delta'$ for all $b \in \Sigma'_{\mathtt{o}}$ and $z \in \Gamma$.

We show $L(\mathcal{A}) = (\Sigma'_{\mathtt{i}})^\omega$ iff there exists $\mathcal{T}$ such that $\mathcal{T} \models \mathcal{A}$. By the construction of $\mathcal{A}'$, $L(\mathcal{A}') = \langle L(\mathcal{A}), (\Sigma'_{\mathtt{o}})^\omega \rangle$ holds. If $L(\mathcal{A}) = (\Sigma'_{\mathtt{i}})^\omega$, then $L(\mathcal{A}') = \langle (\Sigma'_{\mathtt{i}})^\omega, (\Sigma'_{\mathtt{o}})^\omega \rangle$ and thus $\mathcal{T} \models \mathcal{A}$ holds for every $\mathcal{T}$. Assume that $L(\mathcal{A}) \neq (\Sigma'_{\mathtt{i}})^\omega$. Then, there exists a word $w \in (\Sigma'_{\mathtt{i}})^\omega$ such that $w \notin L(\mathcal{A})$. For any PDT $\mathcal{T}$ and any $u \in (\Sigma'_{\mathtt{i}})^\omega$, there is $v \in (\Sigma'_{\mathtt{o}})^\omega$ such that $\langle u, v \rangle \in L(\mathcal{A}')$. On the other hand, $\langle w, v \rangle \notin L(\mathcal{A}')$ holds for any $v \in (\Sigma'_{\mathtt{o}})^\omega$. Hence, $\mathcal{T} \not\models \mathcal{A}'$ holds for any PDT $\mathcal{T}$. This completes the reduction and the realizability problem for PDT and NPDA is undecidable.

5 Register Pushdown Transducers and Register Pushdown Automata

5.1 Data words and registers

We assume a countable set D of *data values*. For finite alphabets $\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}$, an infinite sequence $(a_1^{\mathtt{i}}, d_1)(a^{\mathtt{o}}, d_1') \cdots \in ((\Sigma_{\mathtt{i}} \times D) \cdot (\Sigma_{\mathtt{o}} \times D))^\omega$ is called a *data word*. We let $\mathtt{DW}(\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}, D) = ((\Sigma_{\mathtt{i}} \times D) \cdot (\Sigma_{\mathtt{o}} \times D))^\omega$.

For $k \in \mathbb{N}_0$, a mapping $\theta : [k] \to D$ is called an *assignment* (of data values to k registers). Let Θ_k denote the collection of assignments to k registers. We assume $\perp \in D$ as the initial data value and let $\theta_\perp^k \in \Theta_k$ be the initial assignment such that $\theta_\perp^k(i) = \perp$ for all $i \in [k]$.

We denote $\mathtt{Tst}_k = \mathscr{P}([k] \cup \{\mathtt{top}\})$ and $\mathtt{Asgn}_k = \mathscr{P}([k])$ where $\mathtt{top} \notin \mathbb{N}$ is a unique symbol that represents a stack top value. $\mathtt{Tst}_k$ is the set of guard conditions. For $\mathtt{tst} \in \mathtt{Tst}_k$, $\theta \in \Theta_k$ and $d, e \in D$, we denote $(\theta, d, e) \models \mathtt{tst}$ if $(\theta(i) = d \Leftrightarrow i \in \mathtt{tst})$ and $(e = d \Leftrightarrow \mathtt{top} \in \mathtt{tst})$ hold. In the definitions of register pushdown transducer and automaton in the next section, the data values d and e correspond to an input data value and a stack top data value, respectively. $\mathtt{Asgn}_k$ is the set of assignment conditions. For $\mathtt{asgn} \in \mathtt{Asgn}_k$, $\theta, \theta' \in \Theta_k$ and $d \in D$, let $\theta[\mathtt{asgn} \leftarrow d]$ be the assignment θ' such that $\theta'(i) = d$ for $i \in \mathtt{asgn}$ and $\theta'(i) = \theta(i)$ for $i \notin \mathtt{asgn}$.

5.2 Register pushdown transducers

Definition 16. *A k-register pushdown transducer (k-RPDT) over finite alphabets $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and an infinite set D of data values is $\mathcal{T} = (P, p_0, \Delta)$ where P is a finite set of states, $p_0 \in P$ is the initial state, $\Delta : P \times \Sigma_{\mathbb{i}} \times \mathtt{Tst}_k \to P \times \Sigma_{\mathbb{o}} \times \mathtt{Asgn}_k \times [k] \times \mathtt{Com}([k])$ is a finite set of deterministic transition rules.*

D is used as a stack alphabet. For $u \in D^+$, $\theta' \in \Theta_k$ and $\mathtt{com} \in \mathtt{Com}([k])$, let us define $\mathtt{upds}(u, \theta', \mathtt{com})$ as $\mathtt{upds}(u, \theta', \mathtt{pop}) = u(1:)$, $\mathtt{upds}(u, \theta', \mathtt{skip}) = u$ and $\mathtt{upds}(u, \theta', \mathtt{push}(j')) = \theta'(j')u$. Let $ID_{\mathcal{T}} = P \times \Theta_k \times D^*$ and $\Rightarrow_{\mathcal{T}} \subseteq ID_{\mathcal{T}} \times ((\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D)) \times ID_{\mathcal{T}}$ be the transition relation of $\mathcal{T}$ such that $((p, \theta, u), (a, d^{\mathbb{i}})(b, d^{\mathbb{o}}), (q, \theta', u')) \in \Rightarrow_{\mathcal{T}}$ iff there exists a rule $(p, a, \mathtt{tst}) \to (q, b, \mathtt{asgn}, j, \mathtt{com}) \in \Delta$ that satisfies the following conditions: $(d^{\mathbb{i}}, u(0), \theta) \models \mathtt{tst}$, $\theta' = \theta[\mathtt{asgn} \leftarrow d^{\mathbb{i}}]$, $\theta'(j) = d^{\mathbb{o}}$ and $u' = \mathtt{upds}(u, \theta', \mathtt{com})$, and we write $(p, \theta, u) \Rightarrow_{\mathcal{T}}^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})} (q, \theta', u')$. If $\mathcal{T}$ is clear from the context, we abbreviate $\Rightarrow_{\mathcal{T}}^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})}$ as $\Rightarrow^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})}$.

A run and the language $L(\mathcal{T})$ of $\mathcal{T}$ are those of deterministic 0-TS $(ID_{\mathcal{T}}, (q_0, \theta_{\perp}^k, \perp), (\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D), \emptyset, \Rightarrow_{\mathcal{T}}, c)$ where $c(s) = 2$ for all $s \in ID_{\mathcal{T}}$. In this paper, we assume that every run of RPDT never reach an ID whose stack is empty. Let **RPDT**$[k]$ be the class of k-RPDT and **RPDT**$= \bigcup_{k \in \mathbb{N}_0}$**RPDT**$[k]$.

Example 17. Let us consider 1-RPDT $\mathcal{T} = (\{p, p'\}, p, \Delta)$ over $\{a\}, \{b\}$ and D where $\Delta = \{(p, a, \{1, \mathtt{top}\}) \to (p', b, \{1\}, 1, \mathtt{push}(1)), (p', a, \{1, \mathtt{top}\}) \to (p', b, \emptyset, 1, \mathtt{skip}), (p', a, \emptyset) \to (p, b, \{1\}, 1, \mathtt{push}(1))\}$ Let $(\rho, w) \in ID_{\mathcal{T}}^{\omega} \times ((\{a\} \times D) \cdot (\{b\} \times D))^{\omega}$ be a pair of sequences where $\rho = (p, [\perp], \perp)(p', [d_1], d_1\perp)(p', [d_1], d_1\perp)(p', [d_2], d_2d_1\perp)(p', [d_2], d_2d_1\perp)\cdots$, where $[d] \in \Theta_1$ is the assignment such that $[d](1) = d$, and $w = (a, d_1)(b, d_1)(a, d_1)(b, d_1)(a, d_2)(b, d_2)(a, d_2)(b, d_2)\cdots$, then (ρ, w) is a run of $\mathcal{T}$.

5.3 Register pushdown automata

Definition 18. *A nondeterministic k-register pushdown automaton (k-NRPDA) over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and D is $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, \delta, c)$, where*

- *Q is a finite set of states,*
- *$Q_{\mathbb{i}} \cup Q_{\mathbb{o}} = Q, Q_{\mathbb{i}} \cap Q_{\mathbb{o}} = \emptyset$,*
- *$q_0 \in Q$ is the initial state, and*
- *$\delta : Q \times (\Sigma \cup \{\tau\}) \times \mathtt{Tst}_k \to \mathscr{P}(Q \times \mathtt{Asgn}_k \times \mathtt{Com}([k]))$ is a transition function having one of the forms:*
  - *$(q_{\mathbb{x}}, a_{\mathbb{x}}, \mathtt{tst}) \to (q_{\overline{\mathbb{x}}}, \mathtt{asgn}, \mathtt{com})$ (input/output rule)*
  - *$(q_{\mathbb{x}}, \tau, \mathtt{tst}) \to (q'_{\mathbb{x}}, \mathtt{asgn}, \mathtt{com})$ (τ rule)*

  where $(\mathbb{x}, \overline{\mathbb{x}}) \in \{(\mathbb{i}, \mathbb{o}), (\mathbb{o}, \mathbb{i})\}$, $q_{\mathbb{x}}, q'_{\mathbb{x}} \in Q_{\mathbb{x}}, q_{\overline{\mathbb{x}}} \in Q_{\overline{\mathbb{x}}}, a_{\mathbb{x}} \in \Sigma_{\mathbb{x}}$, $\mathtt{tst} \in \mathtt{Tst}_k$, $\mathtt{asgn} \in \mathtt{Asgn}_k$ and $\mathtt{com} \in \mathtt{Com}([k])$.
- *$c : Q \to [n]$ where $n \in \mathbb{N}$ is the number of priorities.*

Let $ID_{\mathcal{A}} = Q \times \Theta_k \times D^*$. We define the transition relation $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times ((\Sigma \cup \{\tau\}) \times D) \times ID_{\mathcal{A}}$ as $((q,\theta,u),(a,d),(q',\theta',u')) \in \vdash_{\mathcal{A}}$, written as $(q,\theta,u) \vdash^{(a,d)} (q',\theta',u')$, iff there exists a rule $(p,a,\mathtt{tst}) \to (q,\mathtt{asgn},\mathtt{com}) \in \delta$ such that $(d,u(0),\theta) \models \mathtt{tst}$, $\theta' = \theta[\mathtt{asgn} \leftarrow d]$ and $u' = \mathtt{upds}(u,\theta',\mathtt{com})$. We write $\vdash_{\mathcal{A}}^{(a,d)}$ as $\vdash^{(a,d)}$ if $\mathcal{A}$ is clear from the context. For $s, s' \in ID_{\mathcal{A}}$ and $w \in ((\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D))^m$, we write $s \vdash^{w} s'$ if there exists $\rho \in ID_{\mathcal{A}}^{m+1}$ such that $\rho(0) = s, \rho(m) = s'$, and $\rho(0) \vdash^{w(0)} \ldots \vdash^{w(m-1)} \rho(m)$.

A run and the language $L(\mathcal{A})$ of k-DRPDA $\mathcal{A}$ are those of TS $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, \theta_{\perp}^{k}, \perp), \Sigma \times D, \{\tau\} \times D, \Rightarrow_{\mathcal{A}}, c')$ where $c'((q,\theta,u)) = c(q)$ for all $(q,\theta,u) \in ID_{\mathcal{A}}$. We call an $\mathcal{A}$ deterministic, or k-DRPDA, if $\mathcal{S}_{\mathcal{A}}$ is deterministic. We call an $\mathcal{A}$ (m,k)-NRPDA (or an (m,k)-DRPDA when $\mathcal{A}$ is deterministic) if $\mathcal{S}_{\mathcal{A}}$ is an m-TS. We abbreviate $(0,k)$-NRPDA ($(0,k)$-DPDA) as k-NRPDA (k-DRPDA).

5.4 PDA simulating RPDA

Let $\sigma : ((\Sigma \cup \{\tau\}) \times D) \to (\Sigma \cup \{\tau\})$ be the function defined as $\sigma((a,d)) = a$ for any $a \in \Sigma \cup \{\tau\}$ and $d \in D$. In this subsection, we show that we can construct an NPDA $\mathcal{A}'$ from a given k-NRPDA $\mathcal{A}$ over $\Sigma_{\mathbb{i}}$, $\Sigma_{\mathbb{o}}$, D such that $\sigma(L(\mathcal{A})) = L(\mathcal{A}')$.

Let Φ_k be the set of *equivalence relations* over the set of $2k+1$ symbols $X_k = \{\mathtt{x}_1, \mathtt{x}_2, \ldots, \mathtt{x}_k, \mathtt{x}'_1, \mathtt{x}'_2, \ldots, \mathtt{x}'_k, \mathtt{x}_{\mathtt{top}}\}$. We write $a \equiv_\phi b$ and $a \not\equiv_\phi b$ to mean $(a,b) \in \phi$ and $(a,b) \notin \phi$, respectively, for $a, b \in X_k$ and $\phi \in \Phi_k$. Intuitively, each $\phi \in \Phi_k$ represents the equality and inequality among the data values in the registers and the stack top, as well as the transition of the values in the registers between two assignments. Two assignments θ, θ' and a value d at the stack top satisfy ϕ, denoted as $\theta, d, \theta' \models \phi$, if and only if for $i, j \in [k]$,

$$
\begin{aligned}
&\mathtt{x}_i \equiv_\phi \mathtt{x}_j \Leftrightarrow \theta(i) = \theta(j), && \mathtt{x}_i \equiv_\phi \mathtt{x}_{\mathtt{top}} \Leftrightarrow \theta(i) = d,\\
&\mathtt{x}_i \equiv_\phi \mathtt{x}'_j \Leftrightarrow \theta(i) = \theta'(j), && \mathtt{x}'_j \equiv_\phi \mathtt{x}_{\mathtt{top}} \Leftrightarrow \theta'(j) = d,\\
&\mathtt{x}'_i \equiv_\phi \mathtt{x}'_j \Leftrightarrow \theta'(i) = \theta'(j).
\end{aligned}
$$

Let $\phi_\perp \in \Phi_k$ be the equivalence relation satisfying $a \equiv_{\phi_\perp} b$ for any $a, b \in X_k$.

For $\mathtt{tst} \subseteq [k] \cup \{\mathtt{top}\}$ and $\mathtt{asgn} \subseteq [k]$, define a subset $\Phi_k^{\mathtt{tst},\mathtt{asgn}}$ of Φ_k as:

$$
\begin{aligned}
\Phi_k^{\mathtt{tst},\mathtt{asgn}} = \{\phi \in \Phi_k \mid\ & (\forall i \in \mathtt{tst} : \ \ \forall j \in [k] \cup \{\mathtt{top}\} : j \in \mathtt{tst} \Leftrightarrow \mathtt{x}_i \equiv_\phi \mathtt{x}_j),\\
& (\forall i \in \mathtt{asgn} : \forall j \in [k] \cup \{\mathtt{top}\} : j \in \mathtt{tst} \Leftrightarrow \mathtt{x}_j \equiv_\phi \mathtt{x}'_i),\\
& (\forall i, j \in \mathtt{asgn} : \mathtt{x}'_i \equiv_\phi \mathtt{x}'_j),\\
& (\forall i \in [k] \setminus \mathtt{asgn} : \mathtt{x}_i \equiv_\phi \mathtt{x}'_i)\}.
\end{aligned}
$$

For $j \in [k]$, define $\Phi_k^{=,j} = \{\phi \in \Phi_k \mid \mathtt{x}_{\mathtt{top}} \equiv_\phi \mathtt{x}_j,\ \forall i \in [k] : \mathtt{x}_i \equiv_\phi \mathtt{x}'_i\}$. By definition, $\theta, e, \theta' \models \phi$ for $\phi \in \Phi_k^{\mathtt{tst},\mathtt{asgn}}$ iff $\theta, d, e \models \mathtt{tst}$ and $\theta' = \theta[\mathtt{asgn} \leftarrow d]$ for some $d \in D$. Similarly, $\theta, e, \theta' \models \phi$ for $\phi \in \Phi_k^{=,j}$ iff $\theta' = \theta$ and $\theta(j) = e$.

Let $\odot$ and $\odot_{\mathtt{T}}$ be binary relations over Φ_k defined as:

$$
\begin{aligned}
\phi_1 \odot \phi_2 \ &:\Leftrightarrow\ \mathtt{x}'_i \equiv_{\phi_1} \mathtt{x}'_j \Leftrightarrow \mathtt{x}_i \equiv_{\phi_2} \mathtt{x}_j \text{ for } i, j \in [k].\\
\phi_1 \odot_{\mathtt{T}} \phi_2 \ &:\Leftrightarrow\ \phi_1 \odot \phi_2 \text{ and } \mathtt{x}'_i \equiv_{\phi_1} \mathtt{x}_{\mathtt{top}} \Leftrightarrow \mathtt{x}_i \equiv_{\phi_2} \mathtt{x}_{\mathtt{top}} \text{ for } i \in [k].
\end{aligned}
$$

Below we will define the *composition* of two equivalence relations, and $\phi_1 \odot \phi_2$ means that ϕ_1 and ϕ_2 are composable. For $\phi \in \Phi_k$ and $\Phi' \subseteq \Phi_k$, let $\phi \odot \Phi' = \{\phi' \in \Phi' \mid \phi \odot \phi'\}$ and $\phi \odot_{\mathtt{T}} \Phi' = \{\phi' \in \Phi' \mid \phi \odot_{\mathtt{T}} \phi'\}$. By definition, $\phi \odot_{\mathtt{T}} \Phi_k^{\mathtt{tst},\mathtt{asgn}}$ consists of at most one equivalence relation for any $\phi \in \Phi_k$, $\mathtt{tst} \subseteq [k] \cup \{\mathtt{top}\}$, and $\mathtt{asgn} \subseteq [k]$. Similarly, $\phi \odot \Phi_k^{=,j}$ consists of exactly one equivalence relation for any $\phi \in \Phi_k$ and $j \in [k]$.

For $\phi_1, \phi_2 \in \Phi_k$ with $\phi_1 \odot \phi_2$, the *composition* $\phi_1 \circ \phi_2$ of them is the equivalence relation in Φ_k that satisfies the followings:

$$\begin{aligned}
\mathtt{x}_i \equiv_{\phi_1} \mathtt{x}_j \Leftrightarrow \mathtt{x}_i \equiv_{\phi_1 \circ \phi_2} \mathtt{x}_j &\quad \text{for } i, j \in [k] \cup \{\mathtt{top}\},\\
\mathtt{x}'_i \equiv_{\phi_2} \mathtt{x}'_j \Leftrightarrow \mathtt{x}'_i \equiv_{\phi_1 \circ \phi_2} \mathtt{x}'_j &\quad \text{for } i, j \in [k],\\
(\exists l \in [k] : \mathtt{x}_i \equiv_{\phi_1} \mathtt{x}'_l \wedge \mathtt{x}_l \equiv_{\phi_2} \mathtt{x}'_j) \Leftrightarrow \mathtt{x}_i \equiv_{\phi_1 \circ \phi_2} \mathtt{x}'_j &\quad \text{for } i \in [k] \cup \{\mathtt{top}\},\ j \in [k].
\end{aligned}$$

By definition, if $\theta_1, d_1, \theta_2 \models \phi_1$ and $\theta_2, d_2, \theta_3 \models \phi_2$, then $\theta_1\, d_1, \theta_3 \models \phi_1 \circ \phi_2$. By definition, $\circ$ is associative.

Let $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, \delta, c)$ be a k-NRPDA over $\Sigma_{\mathbb{i}}$, $\Sigma_{\mathbb{o}}$, and D. We construct a PDA $\mathcal{A}' = (Q', Q'_{\mathbb{i}}, Q'_{\mathbb{o}}, q'_0, \phi_\bot, \delta', c')$ over $\Sigma_{\mathbb{i}}$, $\Sigma_{\mathbb{o}}$, and Φ_k, where $Q' = Q \times \Phi_k$, $Q'_{\mathbb{i}} = Q_{\mathbb{i}} \times \Phi_k$, $Q'_{\mathbb{o}} = Q_{\mathbb{o}} \times \Phi_k$, $q'_0 = (q_0, \phi_\bot)$, $c'((q, \phi)) = c(q)$ for any $q \in Q$ and $\phi \in \Phi_k$, and for any $(q, \phi_2) \in Q'$, $a \in \Sigma \cup \{\tau\}$, and $\phi_1 \in \Phi_k$, $\delta'((q, \phi_2), a, \phi_1)$ is the smallest set satisfying the following inference rules:

$$\frac{\delta(q, a, \mathtt{tst}) \ni (q', \mathtt{asgn}, \mathtt{skip}),\ \phi_1 \odot \phi_2,\ \phi_3 \in \phi_2 \odot_{\mathtt{T}} \Phi_k^{\mathtt{tst},\mathtt{asgn}}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_2 \circ \phi_3), \mathtt{skip})} \tag{1}$$

$$\frac{\delta(q, a, \mathtt{tst}) \ni (q', \mathtt{asgn}, \mathtt{pop}),\ \phi_1 \odot \phi_2,\ \phi_3 \in \phi_2 \odot_{\mathtt{T}} \Phi_k^{\mathtt{tst},\mathtt{asgn}}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_1 \circ \phi_2 \circ \phi_3), \mathtt{pop})} \tag{2}$$

$$\frac{\delta(q, a, \mathtt{tst}) \ni (q', \mathtt{asgn}, \mathtt{push}(j)),\ \phi_1 \odot \phi_2,\ \phi_3 \in \phi_2 \odot_{\mathtt{T}} \Phi_k^{\mathtt{tst},\mathtt{asgn}},\ \phi_4 \in \phi_3 \odot \Phi_k^{=,j}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_4), \mathtt{push}(\phi_2 \circ \phi_3))} \tag{3}$$

Let $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, \theta_\bot, \bot), \Sigma \times D, \{\tau\} \times D, \vdash_{\mathcal{A}}, c_{\mathcal{A}})$ and $\mathcal{S}_{\mathcal{A}'} = (ID_{\mathcal{A}'}, (q'_0, \phi_\bot), \Sigma, \{\tau\}, \vdash_{\mathcal{A}'}, c_{\mathcal{A}'})$ be the TSs that represent the semantics of $\mathcal{A}$ and $\mathcal{A}'$, respectively. Below we show that $\mathcal{S}_{\mathcal{A}}$ is σ-bisimilar to $\mathcal{S}_{\mathcal{A}'}$. We define an intermediate TS $\mathcal{S}_{\mathcal{A}}^{\mathrm{aug}} = (ID_{\mathcal{A}}^{\mathrm{aug}}, (q_0, \theta_\bot, (\bot, \theta_\bot)), \Sigma \times D, \{\tau\} \times D, \vdash_{\mathcal{A}^{\mathrm{aug}}}, c'_{\mathcal{A}})$ where $ID_{\mathcal{A}}^{\mathrm{aug}} = Q \times \Theta_k \times (D \times \Theta_k)^*$, $c'_{\mathcal{A}}((q, \theta, v)) = c_{\mathcal{A}}((q, \theta, \mathit{fst}(v)))$ for any $(q, \theta, v) \in ID_{\mathcal{A}}^{\mathrm{aug}}$, and $\vdash_{\mathcal{A}^{\mathrm{aug}}}$ is defined as follows: $(q, \theta, v) \vdash_{\mathcal{A}^{\mathrm{aug}}}^{(a,d)} (q', \theta', v')$ if and only if $\delta(q, a, \mathtt{tst}) \ni (q', \mathtt{asgn}, \mathtt{com})$, $d, \mathit{fst}(v(0)), \theta \models \mathtt{tst}$, $\theta' = \theta[\mathtt{asgn} \leftarrow d]$, and $v' = v(1{:})$, v, or $(\theta'(j'), \theta')v$ if $\mathtt{com} = \mathtt{pop}$, $\mathtt{skip}$, or $\mathtt{push}(j')$, respectively. Obviously, $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{A}'}$ are bisimilar; the smallest relation $R \subseteq ID_{\mathcal{A}} \times ID_{\mathcal{A}}^{\mathrm{aug}}$ satisfying $((q, \theta, \mathit{fst}(v)), (q, \theta, v)) \in R$ for every $(q, \theta, v) \in ID_{\mathcal{A}}^{\mathrm{aug}}$ is a bisimulation relation between $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{A}}^{\mathrm{aug}}$. We can show the following lemma.

Lemma 19. *$\mathcal{S}_{\mathcal{A}}^{\mathrm{aug}}$ is σ-bisimilar to $\mathcal{S}_{\mathcal{A}'}$.*

Proof. Let $R \subseteq ID_{\mathcal{A}}^{\mathrm{aug}} \times ID_{\mathcal{A}'}$ be the smallest relation that satisfies for every $q \in Q$, $\theta_0, \ldots, \theta_n \in \Theta_k$, $d_0, \ldots, d_{n-1} \in D$, and $\phi_0, \ldots, \phi_n \in \Phi_k$, $((q, \theta_n, (d_{n-1}, \theta_{n-1})$

$\ldots(d_1,\theta_1)(d_0,\theta_0)),((q,\phi_n),\phi_{n-1}\ldots\phi_1\phi_0)) \in R$ if $\forall i \in [n] : \theta_{i-1},d_{i-1},\theta_i \models \phi_i$ and $\theta_\perp,\perp,\theta_0 \models \phi_0$. We can show that R is a σ-bisimulation relation from $\mathcal{S}^{\mathrm{aug}}_{\mathcal{A}}$ to $\mathcal{S}_{\mathcal{A}'}$.

Assume that $((q,\theta_n,v),((q,\phi_n),u)) \in R$ for $v = (d_{n-1},\theta_{n-1})\ldots(d_0,\theta_0)$ and $u = \phi_{n-1}\ldots\phi_0$. Because $\forall i \in [n] : \theta_{i-1},d_{i-1},\theta_i \models \phi_i$ and $\theta_\perp,\perp,\theta_0 \models \phi_0$, $\forall i \in [n] : \phi_{i-1} \odot \phi_i$. If $(q,\theta_n,v) \vdash^{(a,d)}_{\mathcal{A}^{\mathrm{aug}}} (q',\theta',v')$, then there exist $\mathtt{tst}$, $\mathtt{asgn}$, $\mathtt{com}$, and d such that $\delta(q,a,\mathtt{tst}) \ni (q',\mathtt{asgn},\mathtt{com})$, $\theta_n,d,d_{n-1} \models \mathtt{tst}$, $\theta' = \theta[\mathtt{asgn} \leftarrow d]$, and $v' = v(1{:})$, v, or $(\theta'(j'),\theta')v$ if $\mathtt{com} = \mathtt{pop}$, $\mathtt{skip}$, or $\mathtt{push}(j')$, respectively. There exists $\phi' \in \phi_n \odot_{\mathtt{T}} \Phi^{\mathtt{tst},\mathtt{asgn}}_k$ because $\theta_{n-1},d_{n-1},\theta_n \models \phi_n$ and $\theta_n,d,d_{n-1} \models \mathtt{tst}$. Thus, by definition, $\delta'((q,\phi_n),a,\phi_{n-1}) \ni ((q',\phi''),\mathtt{com}')$ where $\mathtt{com}' = \mathtt{pop}$ and $\phi'' = \phi_{n-1} \circ \phi_n \circ \phi'$ if $\mathtt{com} = \mathtt{pop}$, $\mathtt{com}' = \mathtt{skip}$ and $\phi'' = \phi_n \circ \phi'$ if $\mathtt{com} = \mathtt{skip}$, and $\mathtt{com}' = \mathtt{push}(\phi_n \circ \phi')$ and $\phi'' \in \phi' \odot \Phi^{=,j'}_k$ if $\mathtt{com} = \mathtt{push}(j')$. Therefore, $((q,\phi_n),u) \vdash^{a}_{\mathcal{A}'} ((q',\phi''),u')$ where $u' = u(1{:})$, u, or $(\phi_n \circ \phi')u$ for $u = \phi_{n-1}\ldots\phi_0$ if $\mathtt{com}' = \mathtt{pop}$, $\mathtt{skip}$, or $\mathtt{push}(\phi_n \circ \phi')$, respectively. Because $\theta_{n-1},d_{n-1},\theta_n \models \phi_n$ and $\theta_n,d_{n-1},\theta' \models \phi'$, $\theta_{n-1},d_{n-1},\theta' \models \phi_n \circ \phi'$. In the case of $\mathtt{com} = \mathtt{pop}$, $\theta_{n-2},d_{n-2},\theta' \models \phi_{n-1} \circ \phi_n \circ \phi'$ because $\theta_{n-2},d_{n-2},\theta_{n-1} \models \phi_{n-1}$ and $\theta_{n-1},d_{n-1},\theta' \models \phi_n \circ \phi'$. In the case of $\mathtt{com} = \mathtt{push}(j')$, $\theta',\theta'(j'),\theta' \models \phi''$ because $\phi'' \in \Phi^{=,j'}_k$. Hence, $((q',\theta',v'),((q',\phi''),u')) \in R$ in any case.

On the other hand, if $((q,\phi_n),u) \vdash^{a}_{\mathcal{A}'} ((q',\phi''),u')$, then $\delta'((q,\phi_n),a,\phi_{n-1}) \ni ((q',\phi''),\mathtt{com}')$ where $\mathtt{com}' = \mathtt{pop}$, $\mathtt{skip}$, or $\mathtt{push}(\phi''')$ if $u' = u(1{:})$, u, or $\phi'''u$, respectively. By definition, there exist $\mathtt{tst}$, $\mathtt{asgn}$, $\mathtt{com}$, and $\phi' \in \phi_n \circ \Phi^{\mathtt{tst},\mathtt{asgn}}_k$ such that $\delta(q,a,\mathtt{tst}) \ni (q',\mathtt{asgn},\mathtt{com})$, and $\mathtt{com} = \mathtt{pop}$ and $\phi'' = \phi_{n-1} \circ \phi_n \circ \phi'$ if $\mathtt{com}' = \mathtt{pop}$, $\mathtt{com} = \mathtt{skip}$ and $\phi'' = \phi_n \circ \phi'$ if $\mathtt{com}' = \mathtt{skip}$, or $\mathtt{com} = \mathtt{push}(j')$, $\phi'' \in \phi' \odot \Phi^{=,j'}_j$ and $\phi''' = \phi_n \circ \phi'$ if $\mathtt{com}' = \mathtt{push}(\phi''')$. Because $\theta_{n-1},d_{n-1},\theta_n \models \phi_n$ and $\phi' \in \phi_n \odot_{\mathtt{T}} \Phi^{\mathtt{tst},\mathtt{asgn}}_k$, there exists $d \in D$ satisfying $\theta_n,d,d_{n-1} \models \mathtt{tst}$ and $\theta_n,d_{n-1},\theta' \models \phi'$ for $\theta' = \theta_n[\mathtt{asgn} \leftarrow d]$. Therefore, $(q,\theta_n,v) \vdash^{(a,d)}_{\mathcal{A}^{\mathrm{aug}}} (q',\theta',v')$ where $v' = v(1{:})$, v, or $(\theta'(j'),\theta')v$ if $\mathtt{com} = \mathtt{pop}$, $\mathtt{skip}$, or $\mathtt{push}(j')$, respectively. We can show that $((q',\theta',v'),((q',\phi''),u')) \in R$ in the same way as the last paragraph.

By Lemmas 2 and 19, we obtain the following theorem.

Theorem 20. *For a given k-NRPDA $\mathcal{A}$, we can construct an NPDA $\mathcal{A}'$ such that $\sigma(L(\mathcal{A})) = L(\mathcal{A}')$.*

6 Realizability problems for RPDA and RPDT

6.1 Visibly RPDA

Let **DRPDA** and **NRPDA** be the unions of k-DRPDA and k-NRPDA for all $k \in \mathbb{N}_0$, respectively. Let $\mathtt{Com}_v = \{\mathtt{pop},\mathtt{skip},\mathtt{push}\}$ and $v : \mathtt{Com}([k]) \to \mathtt{Com}_v$ be the function such that $v(\mathtt{push}(j)) = \mathtt{push}$ for $j \in [k]$ and $v(\mathtt{com}) = \mathtt{com}$ otherwise. We say that an k-DRPDA $\mathcal{A}$ over $\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}$ and D stack visibly manipulates its stack (or a *stack visibly* RPDA) if there exists a function $\mathtt{vis} : \Sigma \to \mathtt{Com}_v$ such that every rule $(q,a,\mathtt{tst}) \to (q',\mathtt{asgn},\mathtt{com})$ of $\mathcal{A}$ satisfies $\mathtt{vis}(a) = v(\mathtt{com})$. Also, we say that $\mathcal{A}$ is a *test visibly* RPDA if there exists a function $\mathtt{vis}_t : \Sigma \to \mathtt{Tst}_k$ such that every rule $(q, a, \mathtt{tst}) \to (q', \mathtt{asgn}, \mathtt{com})$ of $\mathcal{A}$ satisfies $\mathtt{vis}_t(a) = \mathtt{tst}$. If $\mathcal{A}$ is stack visibly and test visibly RPDA, we call $\mathcal{A}$ visibly RPDA. Let **DRPDAv** be the union of visibly k-DRPDA for all $k \in \mathbb{N}_0$, respectively.

6.2 Finite actions

For $k \in \mathbb{N}_0$, we define the set of finite input actions as $A^{\mathbb{i}}_k = \Sigma_{\mathbb{i}} \times \mathtt{Tst}_k$ and the set of finite output actions as $A^{\mathbb{o}}_k = \Sigma_{\mathbb{o}} \times \mathtt{Asgn}_k \times [k] \times \mathtt{Com}([k])$. A sequence $w = (a^{\mathbb{i}}_0, d^{\mathbb{i}}_0)(a^{\mathbb{o}}_0, d^{\mathbb{o}}_0) \cdots \in \mathtt{DW}(\Sigma_{\mathbb{i}} \times \mathtt{Com}_v, \Sigma_{\mathbb{o}} \times \mathtt{Com}_v, D)$ is *compatible* with a sequence $\overline{a} = (a^{\mathbb{i}}_0, \mathtt{tst}_0)(a^{\mathbb{o}}_0, \mathtt{asgn}_0, j_0, \mathtt{com}_0) \cdots \in (A^{\mathbb{i}}_k \cdot A^{\mathbb{o}}_k)^\omega$ iff there exists a sequence $(\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^\omega$, called a *witness*, such that $\theta_0 = \theta^k_\perp$, $u_0 = \perp$, $(\theta_i, d^{\mathbb{i}}_i, u_i(0)) \models \mathtt{tst}_i, \theta_{i+1} = \theta_i[\mathtt{asgn}_i \leftarrow d^{\mathbb{i}}_i], \theta_{i+1}(j_i) = d^{\mathbb{o}}_i$ and $u_{i+1} = \mathtt{upds}(u_i, \theta_{i+1}, \mathtt{com}_i)$. Let $\mathtt{Comp}(\overline{a}) = \{w \in \mathtt{DW}(\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}, D) \mid w$ is compatible with $\overline{a}$ $\}$. For a specification $S \subseteq \mathtt{DW}(\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}, D)$, we define $W_{S,k} = \{\overline{a} \mid \mathtt{Comp}(\overline{a}) \subseteq S\}$.

Theorem 21. *For a specification $S \subseteq \mathtt{DW}(\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}, D)$, the following statements are equivalent.*

- *There exists a k-RPDT $\mathcal{T}$ such that $L(\mathcal{T}) \subseteq S$.*
- *There exists a PDT $\mathcal{T}'$ such that $L(\mathcal{T}') \subseteq W_{S,k}$.*

Proof. Assume k-RPDT $\mathcal{T}$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and D satisfies $L(\mathcal{T}) \subseteq S$. Then we construct PDT $\mathcal{T}'$ over $A^{\mathbb{i}}_k, A^{\mathbb{o}}_k$ and $\{z\}$ such that $(q, a, \mathtt{tst}) \to (q', \mathtt{asgn}, j, \mathtt{com})$ iff $(q, (a, \mathtt{tst}), z) \to (q', (\mathtt{asgn}, j, \mathtt{com}), \mathtt{skip})$. For $\overline{a} \in L(\mathcal{T}')$, every $w \in \mathtt{Comp}(\overline{a})$ has a witness and thus $w \in L(\mathcal{T})$ holds. By the assumption $L(\mathcal{T}) \subseteq S$, $\mathtt{Comp}(\overline{a}) \subseteq S$ holds and thus $\overline{a} \in W_{S,k}$. Hence, we obtain $L(\mathcal{T}') \subseteq W_{S,k}$.

Next, assume an PDT $\mathcal{T}'$ over $A^{\mathbb{i}}_k, A^{\mathbb{o}}_k$ and Γ satisfies $L(\mathcal{T}') \subseteq W_{S,k}$. By the assumption, every $\overline{a} \in L(\mathcal{T}')$ satisfies $\mathtt{Comp}(\overline{a}) \subseteq S$. We construct a k-RPDT $\mathcal{T}$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and D such that $(q, a, \mathtt{tst}) \to (q', \mathtt{asgn}, j, \mathtt{com})$ iff $(q, (a, \mathtt{tst}), z) \to (q', (\mathtt{asgn}, j, \mathtt{com}), \mathtt{com}')$ for some $z \in \Gamma$ and $\mathtt{com}' \in \mathtt{Com}(\Gamma)$. For $w \in L(\mathcal{T})$, the compatible sequence $\overline{a} \in (A^{\mathbb{i}}_k \cdot A^{\mathbb{o}}_k)^\omega$ satisfies $\overline{a} \in L(\mathcal{T}')$. Because $\mathtt{Comp}(\overline{a}) \subseteq S$ holds for $\overline{a} \in L(\mathcal{T}')$, $w \in S$ holds. Hence, we obtain $L(\mathcal{T}) \subseteq S$.

For a data word $w \in \mathtt{DW}(\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}, D)$ and a sequence $\overline{a} \in (A^{\mathbb{i}}_k \cdot A^{\mathbb{o}}_k)^\omega$ such that for each $i \geq 0$, there exists $a \in \Sigma$ and we can write $w(i) = (a, d)$ and $\overline{a}(i) = (a, \mathtt{tst})$ if i is even and $w(i) = (a, d)$ and $\overline{a}(i) = (a, \mathtt{asgn}, j, \mathtt{com})$ if i is odd, we define $w \otimes \overline{a} \in \mathtt{DW}(A^{\mathbb{i}}_k, A^{\mathbb{o}}_k, D)$ as $w \otimes \overline{a}(i) = (\overline{a}(i), d)$ where $w(i) = (a, d)$.

6.3 Decidability and undecidability of realizability problems

Lemma 22. $L_k = \{w \otimes \overline{a} \mid w \in \mathtt{Comp}(\overline{a})\}$ *is definable as the language of a $(2, k+2)$-DRPDA.*

Proof sketch. Let $(2, k+2)$-DRPDA $\mathcal{A}_1 = (Q_1, Q^{\mathbb{i}}_1, Q^{\mathbb{o}}_1, q^0_1, \delta_1, c_1)$ over $A^{\mathbb{i}}_k, A^{\mathbb{o}}_k$ and D where $Q_1 = \{p, q\} \cup (\mathtt{Asgn}_k \times [k] \times \mathtt{Com}([k])) \cup [k]$, $Q^{\mathbb{i}}_1 = \{p\}$, $Q^{\mathbb{o}}_1 = Q_1 \setminus Q^{\mathbb{i}}_1$,

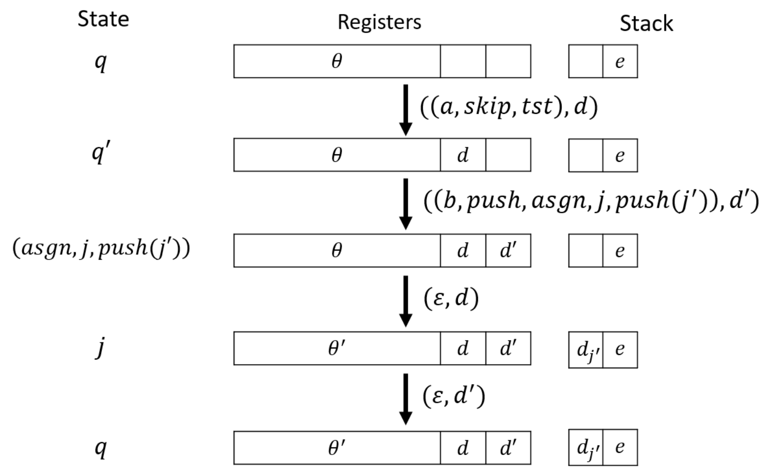

Fig. 2. An example of transitions of $\mathcal{A}_k$

$q_1^0 = p$, $c_1(s) = 2$ for every $s \in Q$ and δ_1 consists of rules of the form

$$(p, (a_{\mathsf{i}}, \mathtt{tst}), \mathtt{tst} \cup \mathtt{tst}') \to (q, \{k+1\}, \mathtt{skip}) \tag{4}$$

$$(q, (a_{\mathsf{o}}, \mathtt{asgn}, j, \mathtt{com}), \mathtt{tst}'') \to ((\mathtt{asgn}, j, \mathtt{com}), \{k+2\}, \mathtt{skip}) \tag{5}$$

$$((\mathtt{asgn}, j, \mathtt{com}), \tau, \{k+1\} \cup \mathtt{tst}'') \to (j, \mathtt{asgn}, \mathtt{com}) \tag{6}$$

$$(j, \tau, \{j, k+2\} \cup \mathtt{tst}'') \to (p, \emptyset, \mathtt{skip}) \tag{7}$$

for all $(a_{\mathsf{i}}, \mathtt{tst}) \in A_k^{\mathsf{i}}$, $(a_{\mathsf{o}}, \mathtt{asgn}, j, \mathtt{com}) \in A_k^{\mathsf{o}}$, $\mathtt{tst}' \subseteq \{k+1, k+2\}$ and $\mathtt{tst}'' \in \mathtt{Tst}_{k+2}$. We can show $L(\mathcal{A}_k) = L_k$ by checking $w \otimes \overline{a} \in L(\mathcal{A}_k) \Leftrightarrow w \in \mathtt{Comp}(\overline{a})$ by the induction on the length of $w \otimes \overline{a}$.

Lemma 23. *For a specification $\mathcal{S}$ defined by some visibly k'-DRPDA, $L_{\overline{S},k} = \{w \otimes \overline{a} \mid w \in \mathtt{Comp}(\overline{a}) \cap \overline{S}\}$ is definable as the language of a $(4, k+k'+4)$-DRPDA.*

Proof sketch. Let $L_{\overline{S}} = \{w \otimes \overline{a} \mid w \in \overline{S}, \overline{a} \in (A_k^{\mathsf{i}} \cdot A_k^{\mathsf{o}})^\omega\}$. We can construct a visibly k'-DRPDA $\mathcal{A}_2 = (Q_2, Q_2^{\mathsf{i}}, Q_2^{\mathsf{o}}, q_2^0, \delta_2, c_2)$ over $A_k^{\mathsf{i}}, A_k^{\mathsf{o}}$ and D such that $L(\mathcal{A}_2) = L_{\overline{S}}$. Let $\mathcal{A}_1$ be the $(2, k+2)$-DRPDA such that $L(\mathcal{A}_1) = L_k$, which is given in Lemma 22. Because $L_{\overline{S},k} = L_{\overline{S}} \cap L_k$, it is enough to show that we can construct a $(4, k+k'+4)$-DRPDA $\mathcal{A}$ over $A_k^{\mathsf{i}}, A_k^{\mathsf{o}}$ and D such that $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.

To construct $\mathcal{A}$, we use the facts that $c_1(q)$ is even for every $q \in Q_1$ and δ_1 consists of several groups of three consecutive rules having the following forms:

$$(q_1, a, \mathtt{tst}_1) \to (q_2, \mathtt{asgn}_1, \mathtt{skip}) \tag{5'}$$

$$(q_2, \tau, \mathtt{tst}_2) \to (q_3, \mathtt{asgn}_2, \mathtt{com}_1) \tag{6'}$$

$$(q_3, \tau, \mathtt{tst}_3) \to (q_4, \mathtt{asgn}_3, \mathtt{skip}). \tag{7'}$$

Note that $\mathtt{vis}(a) = v(\mathtt{com}_1)$ always holds for every consecutive rules. $(5'), (6')$ and $(7')$ correspond to $(5), (6)$ and (7), respectively, and (4) is also converted to three consecutive rules like $(5')$-$(7')$ by adding dummy τ rules.

We let $k_1 = k + 2$ and $k_2 = k'$. We construct $(4, k_1 + k_2 + 2)$-DRPDA $\mathcal{A} = (Q_{\mathring{\imath}} \cup Q_{\Phi} \cup \{q_0\}, Q_{\mathring{\imath}} \cup \{q_0\}, Q_{\Phi}, q_0, \delta, c)$ where $Q_{\mathring{\imath}} = Q_1^{\mathring{\imath}} \times Q_2^{\mathring{\imath}} \times [5]$, $Q_{\Phi} = Q_1^{\Phi} \times Q_2^{\Phi} \times [5]$. c is defined as $c(q_0) = 1$ and $c((q_1, q_2, i)) = c_2(q_2)$ for all $(q_1, q_2, i) \in Q$. δ has first τ rule $(q_0, \tau, [k] \cup \{\mathtt{top}\}) \to ((q_0^1, q_0^2, 1), \mathtt{push}(1))$. For all rules

$$(q_1, a, \mathtt{tst}_1) \to (q_2, \mathtt{asgn}_1, \mathtt{skip}) \in \delta_1 \tag{8}$$

$$(q_2, \tau, \mathtt{tst}_2) \to (q_3, \mathtt{asgn}_2, \mathtt{com}_1) \in \delta_1 \tag{9}$$

$$(q_3, \tau, \mathtt{tst}_3) \to (q_4, \mathtt{asgn}_3, \mathtt{skip}) \in \delta_1 \tag{10}$$

$$(q, a, \mathtt{tst}) \to (q', \mathtt{asgn}, \mathtt{com}) \in \delta_2 \tag{11}$$

(note that $v(\mathtt{com}_1) = \mathtt{vis}(a) = v(\mathtt{com})$ always hold) for $a \in A_k^{\mathring{\imath}} \cup A_k^{\Phi}$, let $\mathtt{tst}^{+k_1} = \{i + k_1 \mid i \in \mathtt{tst}\} \cup \{\mathtt{top} \mid \mathtt{top} \in \mathtt{tst} \setminus [k_1]\}$, $\mathtt{asgn}^{+k_1} = \{i + k_1 \mid i \in \mathtt{asgn}\}$ and $\mathtt{com}^{+k_1} = \mathtt{push}(j + k_1)$ if $\mathtt{com} = \mathtt{push}(j)$ and $\mathtt{com}^{+k_1} = \mathtt{com}$ otherwise, then δ consists of the rules

$$((q_1, q, 1), \tau, \mathtt{tst}[0, k_1 + k_2 + 2] \cup \{\mathtt{top}\}) \to ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathtt{pop}) \tag{12}$$

$$((q_1, q, 2), \tau, \mathtt{tst}[0, k_1 + k_2 + 2] \cup \{\mathtt{top}\}) \to ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathtt{push}(k_1 + k_2 + 1)) \tag{13}$$

$$((q_1, q, 3), a, \mathtt{tst}_1 \cup ((\mathtt{tst}^{+k_1} \setminus \{\mathtt{top}\}) \cup \mathtt{Top})) \to ((q_2, q', 4), \mathtt{asgn}_1 \cup \mathtt{asgn}^{+k_1}, \mathtt{com}^{+k_1}) \tag{14}$$

$$((q_2, q', 4), \tau, ((\mathtt{tst}_2 \setminus \{\mathtt{top}\}) \cup \mathtt{Top}') \cup \mathtt{tst}[k_1, k_1 + k_2 + 1]) \to ((q_3, q', 5), \mathtt{asgn}_2, \mathtt{com}_1) \tag{15}$$

$$((q_3, q', 5), \tau, \mathtt{tst}_3 \cup \mathtt{tst}[k_1, k_1 + k_2 + 2]) \to ((q_4, q', 1), \mathtt{asgn}_3, \mathtt{skip}) \tag{16}$$

for all $\mathtt{tst}[i, j] \in \mathtt{Tst}_j \setminus \mathtt{Tst}_i$ (we assume $\mathtt{Tst}_0 = \emptyset$) where $\mathtt{Top} = \{k_1 + k_2 + 2\}$ ($\mathtt{Top}' = \{k_1 + k_2 + 1\}$) if $\mathtt{top} \in \mathtt{tst}$ ($\mathtt{top} \in \mathtt{tst}_2$) and $\mathtt{Top} = \emptyset$ ($\mathtt{Top}' = \emptyset$) otherwise.

Fig. 3 indicates an example of transitions from $(q_1, q, 1)$ to $(q_4, q', 1)$ with updating contents of registers and stack. The first to k_1-th registers simulate the registers of $\mathcal{A}_1$, $(k_1 + 1)$-th to $(k_1 + k_2)$-th register simulate the registers of $\mathcal{A}_2$ and $(k_1 + k_2 + 1)$-th and $(k_1 + k_2 + 2)$-th registers is for keeping the first and second stack top contents, respectively. The stack contents of $\mathcal{A}$ simulates that of $\mathcal{A}_1$ and $\mathcal{A}_2$ by restoring contents of stacks of $\mathcal{A}_1$ and $\mathcal{A}_2$ alternately.

The transition rules (12) and (13) is for restoring two stack data values from the top to $(k_1 + k_2 + 1)$-th and $(k_1 + k_2 + 2)$-th registers. The transition rule (14) is for updating states, registers and stacks by simulating the rules (8) and (11). Because the first to k_2-th registers of $\mathcal{A}_2$ is simulated by $(k_1 + 1)$-th to $(k_1 + k_2)$-th registers of $\mathcal{A}$, we use $\mathtt{tst}^{+k_1}$, $\mathtt{asgn}^{+k_1}$ and $\mathtt{com}^{+k_1}$ instead of $\mathtt{tst}$, $\mathtt{asgn}$ and $\mathtt{com}$, and replace $\mathtt{top}$ in $\mathtt{tst}^{+k_1}$ to $k_1 + k_2 + 2$. The transition rules (15) and (16) simulates the rules (9) and (10), respectively.

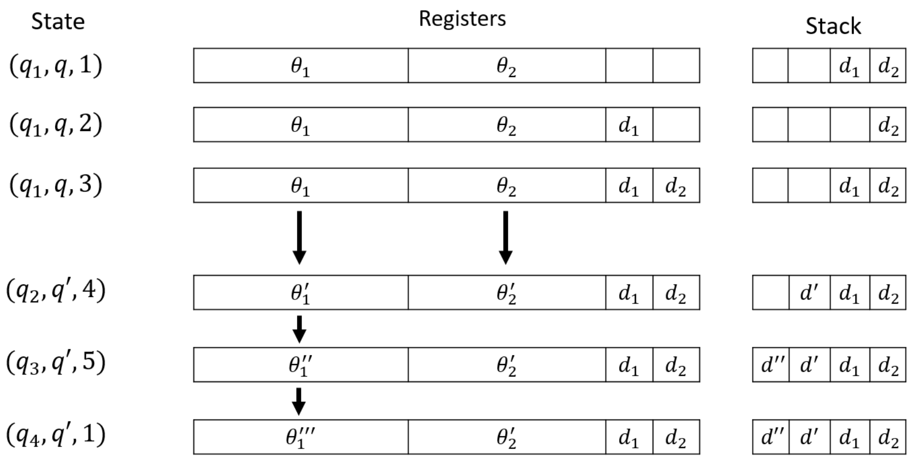

Fig. 3. An example of transitions of $\mathcal{A}$

We can show $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ by checking $w \in L(\mathcal{A})$ iff $w \in L(\mathcal{A}_1)$ and $w \in L(\mathcal{A}_2)$ by the induction on the length of w.

Lemma 24. $W_{S,k} = \overline{\mathtt{Lab}(L_{\overline{S},k})}$.

Proof. For every $\overline{a} \in (A_k^{\mathsf{i}} A_k^{\mathsf{o}})^\omega$, $\overline{a} \notin W_{S,k} \Leftrightarrow \mathtt{Comp}(\overline{a}) \not\subseteq S \Leftrightarrow \exists w.w \in \mathtt{Comp}(\overline{a}) \cap \overline{S} \Leftrightarrow \exists w.w \otimes \overline{a} \in L_{\overline{S},k} \Leftrightarrow \overline{a} \in \mathtt{Lab}(L_{\overline{S},k})$. Thus, $W_{S,k} = \overline{\mathtt{Lab}(L_{\overline{S},k})}$ holds.

Theorem 25. *For all $k \geq 0$,* Real(**DRPDAv**, **RPDT**$[k]$) *is decidable.*

Proof. By Lemma 23, $L_{\overline{S},k}$ is definable by a $(4, k + k' + 4)$-DRPDA. By Lemma 20, $\mathtt{Lab}(L_{\overline{S},k})$ is definable by a 4-DPDA. Thus, $W_{S,k} = \overline{\mathtt{Lab}(L_{\overline{S},k})}$ is also definable by a 4-DPDA $\mathcal{A}_f$ by Lemma 24. By Theorem 21, we can check Real(**DPDA**, **PDT**) for $W_{S,k}$. By Lemma 7, we can construct 0-DPDA $\mathcal{A}_f'$ such that $L(\mathcal{A}_f') = L(\mathcal{A}_f)$. We can check Real(**DPDA**, **PDT**) for $\mathcal{A}_f'$, which is shown to be decidable in Theorem 14, instead of checking Real(**DRPDAv**, **RPDT**$[k]$).

Theorem 26. *For all $k \geq 0$,* Real(**NRPDA**, **RPDT**$[k]$) *is undecidable.*

Proof. We can easily reduce the problem from Real(**NPDA**, **PDT**), whose undecidability is proved in Theorem 15.

7 Conclusion

References

A Appendix

A.1 A proof of Lemma 7

Lemma 7. *For a given m-DPDA $\mathcal{A}$, we can construct a 0-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$*

Proof. For a given m-DPDA $\mathcal{A}$, we can construct an $2m$-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$ and $\mathcal{A}'$ has no skip rule by replacing every skip rule $(q, a, z) \to (q', \mathtt{skip})$ of $\mathcal{A}$ to a pair of push and pop rules $(q, a, z) \to (q'', \mathtt{push}(z')), (q, \tau, z') \to (q', \mathtt{pop})$ of $\mathcal{A}'$ for $a \in \Sigma \cup \{\tau\}$. Thus, we show the lemma for m-DPDA $\mathcal{A}$ that has no skip rule by the induction on m. The case $m = 0$ is obvious. For an arbitrary m, m-DPDA $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, z_0, \delta, c)$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ can be converted to an $(m-1)$-DPDA $\mathcal{A}'$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ^2 such that $L(\mathcal{A}) = L(\mathcal{A}')$. Let $\mathcal{A}' = (Q \cup (Q \times \Gamma), Q_{\mathbb{i}} \cup (Q_{\mathbb{i}} \times \Gamma), Q_{\mathbb{o}} \cup (Q_{\mathbb{o}} \times \Gamma), (q_0, z_0), (z_0, z_0), \delta', c')$ such that $c'(q) = c(q), c'((q, a)) = c(q)$ for all $q \in Q, a \in \Sigma$ and

- $(q, a, z_1) \to (q', \mathtt{pop}) \in \delta$ iff $(q, a, (z_1, z_2)) \to ((q', z_2), \mathtt{pop})$, $((q, z_1), a, (z_c, z_c')) \to (q', \mathtt{skip}) \in \delta'$ for all $z_c, z_c' \in \Gamma$.
- $(q, a, z_1) \to (q', \mathtt{skip}) \in \delta$ iff $(q, a, (z_1, z_2)) \to (q', \mathtt{skip})$, $((q, z_1), a, (z_c, z_c')) \to ((q', z_1), \mathtt{skip}) \in \delta'$ for all $z_c, z_c' \in \Gamma$.
- $(q, a, z_1) \to (q', \mathtt{push}(z')) \in \delta$ iff $(q, a, (z_1, z_2)) \to ((q', z'), \mathtt{skip})$, $((q, z_1), a, (z_c, z_c')) \to (q', \mathtt{push}((z', z_1))) \in \delta'$ for all $z_c, z_c' \in \Gamma$.

for $a \in \Sigma$, and

- $(q, a, z_1) \to (q', \mathtt{pop}), (q', b, z_2) \to (q'', \mathtt{pop}) \in \delta$ iff $(q, x, (z_1, z_2)) \to (q'', \mathtt{pop})$, $((q, z_1), x, (z_2, z_c)) \to ((q'', z_c), \mathtt{pop}) \in \delta'$ for all $z_c \in \Gamma$.
- $(q, a, z_1) \to (q', \mathtt{push}(z')) \in \delta', (q', b, z') \to (q'', \mathtt{pop}) \in \delta$ iff $(q, x, (z_1, z_c)) \to (q'', \mathtt{skip})$, $((q, z_1), x, (z_c, z_c')) \to ((q'', z_1), \mathtt{skip}) \in \delta'$ for all $z_c, z_c' \in \Gamma$.
- $(q, a, z_1) \to (q', \mathtt{push}(z')), (q', b, z') \to (q'', \mathtt{push}(z'')) \in \delta$ iff $(q, x, (z_1, z_c)) \to (q'', \mathtt{push}((z'', z')))$, $((q, z_1), x, (z_c, z_c')) \to ((q'', z''), \mathtt{push}(z', z_1)) \in \delta'$ for all $z_c, z_c' \in \Gamma$.

where $a, b \in \Sigma \cup \{\tau\}$, $x = a$ if $a \in \Sigma$ and $x = b$ otherwise. As the definition of $\mathcal{A}'$, the ID $(q, z_1 z_2 z_3 \cdots z_n)$ of $\mathcal{A}$ corresponds to an ID $(q, (z_1, z_2) \cdots (z_{n-1}, z_n))$ of $\mathcal{A}'$ if n is even and an ID $((q, z_1), (z_2, z_3) \cdots (z_{n-1}, z_n))$ if n is odd. We can check $L(\mathcal{A}) = L(\mathcal{A}')$ by the induction on the length of a sequence $w \in L(\mathcal{A})$.

A.2 A full proof of Lemma 22

Lemma 22. $L_k = \{w \otimes \overline{a} \mid w \in \mathtt{Comp}(\overline{a})\}$ *is definable as the language of a* $(2, k+2)$*-DRPDA.*

Proof. Let $(2, k+2)$-DRPDA $\mathcal{A}_1 = (Q_1, Q_1^{\mathtt{i}}, Q_1^{\mathtt{o}}, q_1^0, \delta_1, c_1)$ over $A_k^{\mathtt{i}}, A_k^{\mathtt{o}}$ and D where $Q_1 = \{p, q\} \cup (\mathtt{Asgn}_k \times [k] \times \mathtt{Com}([k])) \cup [k]$, $Q_1^{\mathtt{i}} = \{p\}$, $Q_1^{\mathtt{o}} = Q_1 \setminus Q_1^{\mathtt{i}}$, $q_1^0 = p$, $c_1(s) = 2$ for every $s \in Q$ and δ_1 consists of rules of the form

$$(p, (a_{\mathtt{i}}, \mathtt{tst}), \mathtt{tst} \cup \mathtt{tst}') \to (q, \{k+1\}, \mathtt{skip}) \tag{4}$$

$$(q, (a_{\mathtt{o}}, \mathtt{asgn}, j, \mathtt{com}), \mathtt{tst}'') \to ((\mathtt{asgn}, j, \mathtt{com}), \{k+2\}, \mathtt{skip}) \tag{5}$$

$$((\mathtt{asgn}, j, \mathtt{com}), \tau, \{k+1\} \cup \mathtt{tst}'') \to (j, \mathtt{asgn}, \mathtt{com}) \tag{6}$$

$$(j, \tau, \{j, k+2\} \cup \mathtt{tst}'') \to (p, \emptyset, \mathtt{skip}) \tag{7}$$

for all $(a_{\mathtt{i}}, \mathtt{tst}) \in A_k^{\mathtt{i}}$, $(a_{\mathtt{o}}, \mathtt{asgn}, j, \mathtt{com}) \in A_k^{\mathtt{o}}$, $\mathtt{tst}' \subseteq \{k+1, k+2\}$ and $\mathtt{tst}'' \in \mathtt{Tst}_{k+2}$.

We show $L(\mathcal{A}_k) = L_k$. For this proof, we redefine compatibility for finite sequences $w \in ((\Sigma_{\mathtt{i}} \times D) \cdot (\Sigma_{\mathtt{o}} \times D))^*$ and $\overline{a} \in (A_k^{\mathtt{i}} \cdot A_k^{\mathtt{o}})^*$. We show the following claim.

Claim. Let $n \in \mathbb{N}_0$ and $w \otimes \overline{a} = ((a_0^{\mathtt{i}}, \mathtt{skip}), d_0^{\mathtt{i}})((a_0^{\mathtt{o}}, \mathtt{asgn}_0, j_0, \mathtt{com}_0), d_0^{\mathtt{o}}) \cdots \in ((A_k^{\mathtt{i}} \times D) \cdot (A_k^{\mathtt{o}} \times D))^*$ whose length is $2n$ and $\rho = (\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^*$ whose length is $n+1$ and $(\theta_0, u_0) = (\theta_\perp^k, \perp)$. Then, ρ is a witness of the compatibility between w and $\overline{a}$ iff $(p, \theta_0', u_0) \vdash^{w \otimes \overline{a}(0:1)(\tau, d_0^{\mathtt{i}})(\tau, d_0^{\mathtt{o}})} (\theta_1', u_1) \vdash^{w \otimes \overline{a}(2:3)(\tau, d_1^{\mathtt{i}})(\tau, d_1^{\mathtt{o}})} \ldots \vdash^{w \otimes \overline{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathtt{i}})(\tau, d_{n-1}^{\mathtt{o}})} (p, \theta_n', u_n)$ where $\theta_i' \in \Theta_{k+2}$ $(i \in [n])$ satisfies $\theta_i'(j) = \theta_i(j)$ for $j \in [k]$.

(Proof of the claim) We show the claim by induction on n. The case of $n = 0$ is obvious. We show the claim for arbitrary $n > 0$ with the induction hypothesis.

We first show left to right. By the induction hypothesis, $(p, \theta_0', u_0) \vdash^{w \otimes \overline{a}(0:1)(\tau, d_0^{\mathtt{i}})(\tau, d_0^{\mathtt{o}}) \cdots w \otimes \overline{a}(2n-4:2n-3)(\tau, d_{n-2}^{\mathtt{i}})(\tau, d_{n-2}^{\mathtt{o}})} (p, \theta_{n-1}', u_{n-1})$ holds. By the assumption, because ρ is the witness, (a) $\theta_{n-1}, d_{n-1}^{\mathtt{i}}, u_{n-1}(0) \models \mathtt{tst}_{n-1}$, (b) $\theta_n = \theta_{n-1}[\mathtt{asgn}_{n-1} \leftarrow d_{n-1}^{\mathtt{i}}]$, (c) $\theta_n(j_{n-1}) = d_{n-1}^{\mathtt{o}}$ and (d) $u_n = \mathtt{upds}(u_{n-1}, \theta_n, \mathtt{com}_{n-1})$ hold. By the condition (a), $\mathcal{A}_k$ can do a transition $(p, \theta_{n-1}', u_{n-1}) \vdash^{w \otimes \overline{a}(2n-2)} (q, \theta_{n-1}^1, u_{n-1})$ for unique $\theta_{n-1}^1 \in \Theta_{k+2}$ by the rule $(p, (a_{n-1}^{\mathtt{i}}, \mathtt{tst}_{n-1}), \mathtt{tst}_{n-1} \cup \mathtt{tst}') \to (q, \{k+1\}, \mathtt{skip})$ of the form (4). We can also say $(q, \theta_{n-1}^1, u_{n-1}) \vdash^{w \otimes \overline{a}(2n-1)} ((\mathtt{asgn}_{n-1}, j_{n-1}, \mathtt{com}_{n-1}), \theta_{n-1}^2, u_{n-1})$ by the rule of the form (5). Note that $\theta_{n-1}^2(j) = \theta_{n-1}(j)$ if $j \in [k]$, $\theta_{n-1}^2(k+1) = d_{n-1}^{\mathtt{i}}$ and $\theta_{n-1}^2(k+2) = d_{n-1}^{\mathtt{o}}$. $((\mathtt{asgn}_{n-1}, j_{n-1}, \mathtt{com}_{n-1}), \theta_{n-1}^2, u_{n-1}) \vdash^{(\tau, d_{n-1}^{\mathtt{i}})} (j_{n-1}, \theta_{n-1}^3, u_n)$ is also valid transition of $\mathcal{A}_k$ of the form (6) by the conditions (b) and (d) where $\theta_3^{n-1}(j) = \theta_n(j)$ for $j \in [k]$ and $\theta_{n-1}^3(k+2) = d_{n-1}^{\mathtt{o}}$. By the condition (c), $\theta_3^{n-1}(j_{n-1}) = \theta_3^{n-1}(k+1) = d_{n-1}^{\mathtt{o}}$ holds. Thus, a transition $(j_{n-1}, \theta_{n-1}^3, u_n) \vdash^{(\tau, d_{n-1}^{\mathtt{o}})} (p, \theta_n', u_n)$ is valid with the rule of the form (7). In conclusion, $(p, \theta_{n-1}', u_{n-1}) \vdash^{w \otimes \overline{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathtt{i}})(\tau, d_{n-1}^{\mathtt{o}})} (p, \theta_n', u_n)$ holds, and with the induction hypothesis, we obtain the left to right of the claim.

Next, we prove right to left. By the assumption, $(p, \theta_{n-1}', u_{n-1}) \vdash^{w \otimes \overline{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathtt{i}})(\tau, d_{n-1}^{\mathtt{o}})} (p, \theta_n', u_n)$ holds. By checking four transition rules that realize the above transition relation, we can obtain that $\rho(n-1), \rho(n), w \otimes \overline{a}(2n-2)$ and $w \otimes \overline{a}(2n-1)$ satisfies the conditions (a) to (d) described in the previous paragraph. Thus, by the induction hypothesis, we obtain ρ is a witness of the compatibility between w and $\overline{a}$.

(end of the proof of the claim)

By the claim, $w \otimes \overline{a} \in L_k \Leftrightarrow$ there exists a witness $(\theta_0, u_0)(\theta_1, u_1)\cdots \in (\Theta_k \times D^*)^\omega$ of w and $\overline{a} \Leftrightarrow$ there exists a run $(p, \theta'_0, u_0) \vdash^{w\otimes\overline{a}(0:1)(\tau,d_0^{\mathbbm{i}})(\tau,d_0^{\mathbbm{o}})} (\theta'_1, u_1) \vdash^{w\otimes\overline{a}(2:3)(\tau,d_1^{\mathbbm{i}})(\tau,d_1^{\mathbbm{o}})} \cdots$ of $\mathcal{A} \Leftrightarrow w \otimes \overline{a} \in L(\mathcal{A}_k)$ holds for all $w \otimes \overline{a} \in \mathtt{DW}(A_k^{\mathbbm{i}}, A_k^{\mathbbm{o}}, D)$.

A.3 A full proof of Lemma 23

Lemma 23. *For a specification $\mathcal{S}$ defined by some visibly k'-DRPDA, $L_{\overline{S},k} = \{w\otimes\overline{a} \mid w \in \mathtt{Comp}(\overline{a})\cap\overline{S}\}$ is definable as the language of a $(4, k+k'+4)$-DRPDA.*

Proof. Let $L_{\overline{S}} = \{w\otimes\overline{a} \mid w \in \overline{S}, \overline{a} \in (A_k^{\mathbbm{i}}\cdot A_k^{\mathbbm{o}})^\omega\}$. We can construct a visibly k'-DRPDA $\mathcal{A}_2 = (Q_2, Q_2^{\mathbbm{i}}, Q_2^{\mathbbm{o}}, q_2^0, \delta_2, c_2)$ over $A_k^{\mathbbm{i}}, A_k^{\mathbbm{o}}$ and D such that $L(\mathcal{A}_2) = L_{\overline{S}}$. Let $\mathcal{A}_1$ be the $(2, k+2)$-DRPDA such that $L(\mathcal{A}_1) = L_k$, which is given in Lemma 22. Because $L_{\overline{S},k} = L_{\overline{S}} \cap L_k$, it is enough to show that we can construct a $(4, k+k'+4)$-DRPDA $\mathcal{A}$ over $A_k^{\mathbbm{i}}, A_k^{\mathbbm{o}}$ and D such that $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.

To construct $\mathcal{A}$, we use the facts that $c_1(q)$ is even for every $q \in Q_1$ and δ_1 consists of several groups of three consecutive rules having the following forms:

$$(q_1, a, \mathtt{tst}_1) \to (q_2, \mathtt{asgn}_1, \mathtt{skip}) \tag{5'}$$

$$(q_2, \tau, \mathtt{tst}_2) \to (q_3, \mathtt{asgn}_2, \mathtt{com}_1) \tag{6'}$$

$$(q_3, \tau, \mathtt{tst}_3) \to (q_4, \mathtt{asgn}_3, \mathtt{skip}). \tag{7'}$$

Note that $\mathtt{vis}(a) = v(\mathtt{com}_1)$ always holds for every consecutive rules. $(5'), (6')$ and $(7')$ correspond to $(5), (6)$ and (7), respectively, and (4) is also converted to three consecutive rules like $(5')$-$(7')$ by adding dummy τ rules.

We let $k_1 = k+2$ and $k_2 = k'$. We construct $(4, k_1 + k_2 + 2)$-DRPDA $\mathcal{A} = (Q_{\mathbbm{i}} \cup Q_{\mathbbm{o}} \cup \{q_0\}, Q_{\mathbbm{i}} \cup \{q_0\}, Q_{\mathbbm{o}}, q_0, \delta, c)$ where $Q_{\mathbbm{i}} = Q_1^{\mathbbm{i}} \times Q_2^{\mathbbm{i}} \times [5]$, $Q_{\mathbbm{o}} = Q_1^{\mathbbm{o}} \times Q_2^{\mathbbm{o}} \times [5]$. c is defined as $c(q_0) = 1$ and $c((q_1, q_2, i)) = c_2(q_2)$ for all $(q_1, q_2, i) \in Q$. δ has first τ rule $(q_0, \tau, [k] \cup \{\mathtt{top}\}) \to ((q_0^1, q_0^2, 1), \mathtt{push}(1))$. For all rules

$$(q_1, a, \mathtt{tst}_1) \to (q_2, \mathtt{asgn}_1, \mathtt{skip}) \in \delta_1 \tag{8}$$

$$(q_2, \tau, \mathtt{tst}_2) \to (q_3, \mathtt{asgn}_2, \mathtt{com}_1) \in \delta_1 \tag{9}$$

$$(q_3, \tau, \mathtt{tst}_3) \to (q_4, \mathtt{asgn}_3, \mathtt{skip}) \in \delta_1 \tag{10}$$

$$(q, a, \mathtt{tst}) \to (q', \mathtt{asgn}, \mathtt{com}) \in \delta_2 \tag{11}$$

(note that $v(\mathtt{com}_1) = \mathtt{vis}(a) = v(\mathtt{com})$ always hold) for $a \in A_k^{\mathbbm{i}} \cup A_k^{\mathbbm{o}}$, let $\mathtt{tst}^{+k_1} = \{i + k_1 \mid i \in \mathtt{tst}\} \cup \{\mathtt{top} \mid \mathtt{top} \in \mathtt{tst} \setminus [k_1]\}$, $\mathtt{asgn}^{+k_1} = \{i + k_1 \mid i \in \mathtt{asgn}\}$ and $\mathtt{com}^{+k_1} = \mathtt{push}(j + k_1)$ if $\mathtt{com} = \mathtt{push}(j)$ and $\mathtt{com}^{+k_1} = \mathtt{com}$ otherwise, then δ

consists of the rules

$$((q_1, q, 1), \tau, \mathtt{tst}[0, k_1 + k_2 + 2] \cup \{\mathtt{top}\}) \to ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathtt{pop}) \tag{12}$$

$$((q_1, q, 2), \tau, \mathtt{tst}[0, k_1 + k_2 + 2] \cup \{\mathtt{top}\}) \to ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathtt{push}(k_1 + k_2 + 1)) \tag{13}$$

$$((q_1, q, 3), a, \mathtt{tst}_1 \cup ((\mathtt{tst}^{+k_1} \setminus \{\mathtt{top}\}) \cup \mathtt{Top})) \to ((q_2, q', 4), \mathtt{asgn}_1 \cup \mathtt{asgn}^{+k_1}, \mathtt{com}^{+k_1}) \tag{14}$$

$$((q_2, q, 4), \tau, ((\mathtt{tst}_2 \setminus \{\mathtt{top}\}) \cup \mathtt{Top}') \cup \mathtt{tst}[k_1, k_1 + k_2 + 1]) \to ((q_3, q', 5), \mathtt{asgn}_2, \mathtt{com}_1) \tag{15}$$

$$((q_3, q', 5), \tau, \mathtt{tst}_3 \cup \mathtt{tst}[k_1, k_1 + k_2 + 2]) \to ((q_4, q', 1), \mathtt{asgn}_3, \mathtt{skip}) \tag{16}$$

for all $\mathtt{tst}[i, j] \in \mathtt{Tst}_j \setminus \mathtt{Tst}_i$ (we assume $\mathtt{Tst}_0 = \emptyset$) where $\mathtt{Top} = \{k_1 + k_2 + 2\}$ ($\mathtt{Top}' = \{k_1 + k_2 + 1\}$) if $\mathtt{top} \in \mathtt{tst}$ ($\mathtt{top} \in \mathtt{tst}_2$) and $\mathtt{Top} = \emptyset$ ($\mathtt{Top}' = \emptyset$) otherwise.

Fig. 3 indicates an example of transitions from $(q_1, q, 1)$ to $(q_4, q', 1)$ with updating contents of registers and stack. The first to k_1-th registers simulate the registers of $\mathcal{A}_1$, $(k_1 + 1)$-th to $(k_1 + k_2)$-th register simulate the registers of $\mathcal{A}_2$ and $(k_1 + k_2 + 1)$-th and $(k_1 + k_2 + 2)$-th registers is for keeping the first and second stack top contents, respectively. The stack contents of $\mathcal{A}$ simulates that of $\mathcal{A}_1$ and $\mathcal{A}_2$ by restoring contents of stacks of $\mathcal{A}_1$ and $\mathcal{A}_2$ alternately.

The transition rules (12) and (13) is for restoring two stack data values from the top to $(k_1 + k_2 + 1)$-th and $(k_1 + k_2 + 2)$-th registers. The transition rule (14) is for updating states, registers and stacks by simulating the rules (8) and (11). Because the first to k_2-th registers of $\mathcal{A}_2$ is simulated by $(k_1 + 1)$-th to $(k_1 + k_2)$-th registers of $\mathcal{A}$, we use $\mathtt{tst}^{+k_1}$, $\mathtt{asgn}^{+k_1}$ and $\mathtt{com}^{+k_1}$ instead of $\mathtt{tst}$, $\mathtt{asgn}$ and $\mathtt{com}$, and replace $\mathtt{top}$ in $\mathtt{tst}^{+k_1}$ to $k_1 + k_2 + 2$. The transition rules (15) and (16) simulates the rules (9) and (10), respectively.

For two assignments $\theta_1 \in \Theta_{k_1}$ and $\theta_2 \in \Theta_{k_2}$, let $[\theta_1, \theta_2, d, d'] \in \Theta_{k_1+k_2+2}$ be the assignment such that $[\theta_1, \theta_2, d, d'](i) = \theta_1(i)$ if $i \in [k_1]$, $[\theta_1, \theta_2, d, d'](i) = \theta_2(i)$ if $k_1 + 1 \leq i \leq k_2$, $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 1) = d$ and $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 2) = d'$. To prove $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$, we show the following claim.

Claim. For all $n \in \mathbb{N}_0$ and $w \in ((A^{\mathtt{i}}_k \cup A^{\mathtt{c}}_k) \times D)^n$, there exists sequences of transitions $(q^1_0, \theta^1_0, u^1_0) \vdash^{w(0)(\tau,d_0)(\tau,d'_0)}_{\mathcal{A}_1} (q^1_1, \theta^1_1, u^1_1) \vdash^{w(1)(\tau,d_1)(\tau,d'_1)}_{\mathcal{A}_1} \cdots \vdash^{w(n-1)(\tau,d_{n-1})(\tau,d'_{n-1})}_{\mathcal{A}_1} (q^1_n, \theta^1_n, u^1_n)$ and $(q^2_0, \theta^2_0, u^2_0) \vdash^{w(0)}_{\mathcal{A}_2} (q^2_1, \theta^2_1, u^2_1) \vdash^{w(1)}_{\mathcal{A}_2} \cdots \vdash^{w(n-1)}_{\mathcal{A}_2} (q^2_n, \theta^2_n, u^2_n)$ iff $(q_0, \theta^{k_1+k_2+2}_\perp, \perp) \vdash^{(\tau,\perp)}_{\mathcal{A}} ((q^1_0, q^2_0, 1), [\theta^1_0, \theta^2_0, d^1_0, d^2_0], \langle u^1_0, u^2_0 \rangle) \vdash^{(\tau,u^1_0(0))(\tau,u^2_0(0))w(0)(\tau,d_0)(\tau,d'_0)}_{\mathcal{A}} ((q^1_1, q^2_1, 1), [\theta^1_1, \theta^2_1, d^1_1, d^2_1], \langle u^1_1, u^2_1 \rangle) \vdash^{(\tau,u^1_1(0))(\tau,u^2_1(0))w(1)(\tau,d_1)(\tau,d'_1)}_{\mathcal{A}} \cdots \vdash^{(\tau,u^1_{n-1}(0))(\tau,u^2_{n-1}(0))w(n-1)(\tau,d_{n-1})(\tau,d'_{n-1})}_{\mathcal{A}} ((q^1_n, q^2_n, 1), [\theta^1_n, \theta^2_n, d^1_n, d^2_n], \langle u^1_n, u^2_n \rangle)$ holds where $b \in \{1, 2\}$, $i \in [n]$, $\theta^b_0 = \theta^{k_b}_\perp$, $u^b_0 = \perp$, $q^b_i \in Q_b$, $\theta^b_i \in \Theta_{k_b}$, $u^b_i \in D^*$ and $d_{i-1}, d'_{i-1} \in D$.

(Proof of the claim) We show the claim by the induction on n. The case $n = 0$ is obvious.

We first show left to right. By induction hypothesis, $(q_0, \theta^{\mathcal{A}}_{\perp}, \perp) \vdash^{(\tau,\perp)}_{\mathcal{A}} ((q^1_0, q^2_0, 1), [\theta^1_0, \theta^2_0, d^1_0, d^2_0], \langle u^1_0, u^2_0\rangle) \vdash^{(\tau,u^1_0(0))(\tau,u^2_0(0))w(0)(\tau,d_0)(\tau,d'_0)}_{\mathcal{A}} \cdots \vdash^{(\tau,u^1_{n-2}(0))(\tau,u^2_{n-2}(0))w(n-2)(\tau,d_{n-2})(\tau,d'_{n-2})}_{\mathcal{A}} ((q^1_{n-1}, q^2_{n-1}, 1), [\theta^1_{n-1}, \theta^2_{n-1}, d^1_{n-1}, d^2_{n-1}], \langle u^1_{n-1}, u^2_{n-1}\rangle)$ holds. Also, by the assumption, $(q^1_{n-1}, \theta^1_{n-1}, u^1_{n-1}) \vdash^{w(n-1)}_{\mathcal{A}_1} (q', \theta', u^1_{n-1}) \vdash^{(\tau,d)}_{\mathcal{A}_1} (q'', \theta'', u^1_n) \vdash^{(\tau,d')}_{\mathcal{A}_1} (q^1_n, \theta^1_n, u^1_n)$ and $(q^2_{n-1}, \theta^2_{n-1}, u^2_{n-1}) \vdash^{w(n-1)}_{\mathcal{A}_2} (q^2_n, \theta^2_n, u^2_n)$ for some $q', q'' \in Q_1, \theta', \theta'' \in \Theta_{k_1}$ and $d, d' \in D$, and let the following be the rules used in these transitions.

$$(q_{n-1}, (a, v(\texttt{com}_1)), \texttt{tst}_1) \to (q', \texttt{asgn}_1, \texttt{skip}) \in \delta_1 \tag{8'}$$
$$(q', \tau, \texttt{tst}_2) \to (q'', \texttt{asgn}_2, \texttt{com}_1) \in \delta_1 \tag{9'}$$
$$(q'', \tau, \texttt{tst}_3) \to (q_n, \texttt{asgn}_3, \texttt{skip}) \in \delta_1 \tag{10'}$$
$$(q_{n-1}, (a, v(\texttt{com})), \texttt{tst}) \to (q_n, \texttt{asgn}, \texttt{com}) \in \delta_2 \tag{11'}$$

We can check the follows.

$$\begin{aligned}
&((q^1_{n-1}, q^2_{n-1}, 1), [\theta^1_{n-1}, \theta^2_{n-1}, d^1_{n-1}, d^2_{n-1}], \langle u^1_{n-1}, u^2_{n-1}\rangle) \vdash^{(\tau,u^1_{n-1}(0))(\tau,u^2_{n-1}(0))}_{\mathcal{A}}\\
&((q^1_{n-1}, q^2_{n-1}, 3), [\theta^1_{n-1}, \theta^2_{n-1}, u^1_{n-1}(0), u^1_{n-1}(0)], \langle u^1_{n-1}, u^2_{n-1}\rangle) \vdash^{w(n-1)}_{\mathcal{A}}\\
&((q^1_{n-1}, q^2_{n-1}, 4), [\theta^1_n, \theta', u^1_{n-1}(0), u^1_{n-1}(0)], \langle u^1_{n-1}, u^2_n\rangle) \vdash^{(\tau,d_{n-1})}_{\mathcal{A}}\\
&((q^1_{n-1}, q^2_{n-1}, 5), [\theta^1_n, \theta'', u^1_{n-1}(0), u^1_{n-1}(0)], \langle u^1_n, u^2_n\rangle) \vdash^{(\tau,d'_{n-1})}_{\mathcal{A}}\\
&((q^1_n, q^2_n, 1), [\theta^1_n, \theta^2_n, u^1_{n-1}(0), u^1_{n-1}(0)], \langle u^1_n, u^2_n\rangle)
\end{aligned}$$

Thus, the right side of the claim holds. In a similar way, we can also show right to left.

(end of the proof of the claim)

By the claim, $w \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2) \Leftrightarrow$ there exists runs $(q^1_0, \theta^1_0, u^1_0) \vdash^{w(0)(\tau,d_0)(\tau,d'_0)}_{\mathcal{A}_1} (q^1_1, \theta^1_1, u^1_1) \vdash^{w(1)(\tau,d_1)(\tau,d'_1)}_{\mathcal{A}_1} \cdots$ and $(q^2_0, \theta^2_0, u^2_0) \vdash^{w(0)}_{\mathcal{A}_2} (q^2_1, \theta^2_1, u^2_1) \vdash^{w(1)}_{\mathcal{A}_2} \cdots$ that satisfies the minimum number appearing in the sequence $q^1_0, q^1_1, \cdots$ infinitely is even. $\Leftrightarrow$ there exists a run $(q_0, \theta^{\mathcal{A}}_{\perp}, \perp) \vdash^{(\tau,\perp)}_{\mathcal{A}} ((q^1_0, q^2_0, 1), [\theta^1_0, \theta^2_0, d^1_0, d^2_0], \langle u^1_0, u^2_0\rangle) \vdash^{(\tau,u^1_0(0))(\tau,u^2_0(0))w(0)(\tau,d_0)(\tau,d'_0)}_{\mathcal{A}} ((q^1_1, q^2_1, 1), [\theta^1_1, \theta^2_1, d^1_1, d^2_1], \langle u^1_1, u^2_1\rangle) \vdash^{(\tau,u^1_1(0))(\tau,u^2_1(0))w(1)(\tau,d_1)(\tau,d'_1)}_{\mathcal{A}} \cdots$ that satisfies the minimum number appearing in the sequence $(q^1_0, q^2_0, 1), (q^1_0, q^2_0, 2) \cdots, (q^1_1, q^2_1, 1), \cdots$ infinitely is even. $\Leftrightarrow w \in L(\mathcal{A})$ holds.