

# Title

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**Abstract.** The realizability problem for a given specification  $\mathcal{S}$  is to decide whether there exists an implementation satisfying  $\mathcal{S}$ . However the problem is important in the field of reactive synthesis, the problem has not been studied yet when specification and implementation are given by pushdown computational models. This paper investigates the realizability problem for the cases that specification and implementation are given by a pushdown automaton (PDA) and a pushdown transducer (PDT), and a register pushdown automata (RPDA) and a register pushdown transducer (RPDT).

## 1 Introduction

Reactive synthesis is a method of synthesizing a system that satisfies a given specification representing the input-output relation of the system. A specification is formally a subset of an infinite alternate sequences of input and output symbols. When an environment or a user gives inputs  $i_0, i_1, \dots$  in this order to a system, the latter is required to emit a sequence of outputs  $o_0, o_1, \dots$  such that  $(i_0, o_0)(i_1, o_1) \dots \in S$ . The realizability problem is to decide whether for a given specification  $S$ , there exists a reactive system satisfying  $S$ , and if exists, to generate such a system (called an implementation of  $S$ ).

Studies on reactive synthesis has its origin in 1960s and has been one of central topics in formal methods, as well as model checking [11]. Among them, Büchi and Landweber [8] show EXPTIME-completeness of the problem when a specification is given by a finite  $\omega$ -automaton (a finite automaton on infinite words). Pnueli and Rosner [28] show 2EXPTIME-completeness of the problem when a specification is given by an LTL formula. We can find an excellent tutorial and survey of the previous studies on the synthesis problem in [2]. The standard approach to the problem is as follows. When a specification is given by an LTL formula  $\varphi$ , we translate  $\varphi$  to an equivalent nondeterministic  $\omega$ -automaton and then convert it to an equivalent deterministic  $\omega$ -automaton  $\mathcal{A}$ . Next, we convert  $\mathcal{A}$  to a tree automaton (or equivalently, a parity game)  $\mathcal{B}$  by separating the input and output streams. Then, we test whether  $L(\mathcal{B}) = \emptyset$  (or equivalently, there is a winning strategy for player I in  $\mathcal{B}$ ). The answer to the problem is affirmative

if and only if  $L(\mathcal{B}) \neq \emptyset$ , and any  $t \in L(\mathcal{B})$  (or any winning strategy for  $\mathcal{B}$ ) is an implementation of the specification.

Classical models such as a finite automaton cannot deal with objects from an infinite set (called data values). However, when we add an ability of manipulating data values to such a classical model, the model easily becomes Turing-complete and basic problems become undecidable. Register automaton (RA) is an extension of finite automaton by adding limited ability of manipulating data values [21, 27, 30]. An RA has a finite number of registers for storing data values taken from an input word and it can compare the contents of its registers with the current input data value to determine the next transition. RA inherits some good properties from finite automaton such as the closure under some language operations. The membership and emptiness are decidable for RA while the universality is undecidable. With the increase of interest in RA, the realizability problem for register models has been investigated recently [16, 23, 19, 22]. A specification is given by an RA and an implementation is represented by a register transducer (abbreviated as RT).

Pushdown automaton (PDA) is a simple model for recursive programs. Model checking algorithms for pushdown systems (PDA without input) has been extensively studied [20, 36, 6, 17, 18]. Extensions by adding registers are also done for PDA [12, 26, 33], called register pushdown automaton (RPDA). Since many real-world programs are recursive and also manipulate data values, it is important to investigate the realizability problem for PDA/RPDA.

This paper first extends the realizability problem to PDA and pushdown transducer. A pushdown transducer (PDT) is a deterministic PDA with output. PDT serves as a model of a recursive program that emits outputs according to the inputs given from its environment. The main difficulties to solve the realizability problem in this setting come from the facts that the class of languages recognized by nondeterministic PDA (NPDA) (i.e., context-free languages) does not have the closure properties under some set operations and some relevant decision problems such as the universality are undecidable for PDA. To avoid these difficulties, this paper mainly considers deterministic PDA (DPDA). (Note that, PDT is deterministic by definition.) In section 4, we show that the realizability problem for DPDA is decidable while the problem is undecidable for NPDA. The former is proved by using the well-known property that the (two-players zero-sum parity) pushdown game is decidable and a winning strategy of the game can be constructed as a PDT [36].

Then, the paper moves to the realizability problem for RPDA and register PDT. In section 5, we introduce RPDA and register PDT (RPDT). Both RPDA and RPDT read a data word, which is an infinite sequence of a pair  $(a, d)$  of a symbol  $a$  from a finite alphabet and a data value  $d$  from an infinite set. We show that the projection of the language recognized by a nondeterministic RPDA to the finite alphabet can be recognized by a nondeterministic PDA. Note that we assume the freshness of input data values [35] for RPDA.

In section 6, we discuss the realizability problem for deterministic RPDA (DRPDA) and RPDT. Our approach is to reduce the problem for DRPDA to

the problem for DPDA. In this reduction, we want to convert a given DRPDA to a DPDA that recognizes the projection to the finite alphabet of the former language by using the method in section 5. For this purpose, we further assume that a given DPDA has the visibility on the guard condition inherited from the DRPDA as well as the visibility of stack operation (see [1] for the visibility of stack operation). Finally, we show that the realizability problem is decidable for visibly DRPDA and RPDT.

**Related work** RA is frequently used as computation models for querying structured data such as XML documents and graph databases [25, 24]. LTL with the freeze quantifier ( $LTL\downarrow$ ) [14, 15] is an extension of LTL by adding the ability of memorizing and comparing data values.  $LTL\downarrow$  has the strong relationship with two-way alternate extension of RA. Two-variable first-order logic on data words [3](also see [7]) is another famous logic for data values, whose expressive power is incomparable with  $LTL\downarrow$ . Other extensions of RA are found in [4, 9, 13, 5]. Nominal automaton [4] is an extension of finite automaton by using nominal sets, which are infinite sets having finite orbits of group actions and equivariant functions among them.

When considering the realizability for RA, the difficulty is to identify the number of registers needed for implementing the specification, i.e., the number of registers of RT. If the upperbound of the registers of RT is not given as an input, the realizability problem is undecidable for both of nondeterministic and universal RA [19]. If the upperbound of the registers is *a priori* known, the problem (called the bounded synthesis problem) is shown to be decidable in EXP-TIME for universal RA [22]. In [19], it is shown that the bounded synthesis problem remains undecidable for nondeterministic RA (NRA) [19] and becomes decidable in 2EXP-TIME for a subclass called test-free NRA.

Extensions by adding registers are also done for PDA [12, 26, 33] and context-free languages [10, 31, 32], called register pushdown automata (RPDA) and register context-free grammar (RCFG), respectively. The expressive powers of RPDA and RCFG are the same. RPDA is a natural model for recursive programs with data values while RCFG has an advantage such that explicit representation of pushdown stack is not needed. For other extensions of PDA and verification of them, see [34, 37, 29].

## 2 Preliminaries

Let  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$  and  $[n] = \{1, \dots, n\}$  for  $n \in \mathbb{N}$ . For a set  $A$ , let  $\mathcal{P}(A)$  be the power set of  $A$ , let  $A^*$  and  $A^\omega$  be the sets of finite and infinite words over  $A$ , respectively. We denote  $A^+ = A^* \setminus \{\varepsilon\}$  and  $A^\infty = A^* \cup A^\omega$ . For a word  $\alpha \in A^\infty$  over a set  $A$ , let  $\alpha(i) \in A$  be the  $i$ -th element of  $\alpha$  ( $i \geq 0$ ),  $\alpha(i : j) = \alpha(i)\alpha(i+1)\cdots\alpha(j-1)\alpha(j)$  for  $i \leq j$  and  $\alpha(i :) = \alpha(i)\cdots$  for  $i \geq 0$ . Let  $\langle u, w \rangle = u(0)w(0)u(1)w(1)\cdots \in A^\infty$  for words  $u, w \in A^\infty$  and  $\langle B, C \rangle = \{\langle u, w \rangle \mid u \in B, w \in C\}$  for sets  $B, C \subseteq A^\infty$ . By  $|\beta|$ , we mean the cardinality of  $\beta$  if  $\beta$  is a set and the length of  $\beta$  if  $\beta$  is a finite sequence. For a function  $f : A \rightarrow B$  from a set  $A$  to a set  $B$ , let  $f(w) = f(w(0))f(w(1))\cdots$  for

a word  $w \in A^\infty$  and let  $f(L) = \{f(w) \mid w \in L\}$  for a set  $L \subseteq A^\infty$  of words. Let  $fst$  and  $scnd$  be the functions such that  $fst((a, b)) = a$  and  $scnd((a, b)) = b$  for any pair  $(a, b)$ . Let  $id$  be the identity function; i.e.,  $id(a) = a$  for any  $a$ .

In this paper, disjoint sets  $\Sigma_i, \Sigma_o$  and  $\Gamma$  denote a (finite) input alphabet, an output alphabet and a stack alphabet, respectively, and  $\Sigma = \Sigma_i \cup \Sigma_o$ . For a set  $\Gamma$ , let  $\text{Com}(\Gamma) = \{\text{pop}, \text{skip}\} \cup \{\text{push}(z) \mid z \in \Gamma\}$  be the set of stack commands over  $\Gamma$ .

## 2.1 Transition Systems

**Definition 1.** A transition system (TS) is  $\mathcal{S} = (S, s_0, A, E, \rightarrow_{\mathcal{S}}, c)$  where

- $S$  is a (finite or infinite) set of states,
- $s_0 \in S$  is the initial state,
- $A, E$  are (finite or infinite) alphabets such that  $A \cap E = \emptyset$ ,
- $\rightarrow_{\mathcal{S}} \subseteq S \times (A \cup E) \times S$  is a transition relation, written as  $s \rightarrow^a s'$  if  $(s, a, s') \in \rightarrow_{\mathcal{S}}$  and
- $c : S \rightarrow [n]$  is a coloring function where  $n \in \mathbb{N}$ .

An element of  $A$  is an observable label and an element of  $E$  is an internal label. A run of TS  $\mathcal{S} = (S, s_0, A, E, \rightarrow_{\mathcal{S}}, c)$  is a pair  $(\rho, w) \in S^\omega \times (A \cup E)^\omega$  that satisfies  $\rho(0) = s_0$  and  $\rho(i) \xrightarrow{w(i)} \rho(i+1)$  for  $i \geq 0$ . Let  $\min_{\text{inf}} : S^\omega \rightarrow [n]$  be the minimal coloring function such that  $\min_{\text{inf}}(\rho) = \min\{m \mid \text{there exist an infinite number of } i \geq 0 \text{ such that } c(\rho(i)) = m\}$ . We call  $\mathcal{S}$  deterministic if  $s \xrightarrow{a} s_1$  and  $s \xrightarrow{a} s_2$  implies  $s_1 = s_2$  for all  $s, s_1, s_2 \in S$  and  $a \in A \cup E$ .

For  $w \in (A \cup E)^\omega$ , let  $ef(w) = a_0 a_1 \dots \in A^\infty$  be the sequence obtained from  $w$  by removing all symbols belonging to  $E$ . Note that  $ef(w)$  is not always an infinite sequence even if  $w$  is an infinite sequence. We define the *language* of  $\mathcal{S}$  as  $L(\mathcal{S}) = \{ef(w) \in A^\omega \mid \text{there exists a run } (\rho, w) \text{ such that } \min_{\text{inf}}(\rho) \text{ is even}\}$ . For  $m \in \mathbb{N}_0$ , we call  $\mathcal{S}$  an  $m$ -TS if for every run  $(\rho, w)$  of  $\mathcal{S}$ ,  $w$  contains no consecutive subsequence  $w' \in E^{m+1}$ .

Consider two TSs  $\mathcal{S}_1 = (S_1, s_{01}, A_1, E_1, \rightarrow_{\mathcal{S}_1}, c_1)$  and  $\mathcal{S}_2 = (S_2, s_{02}, A_2, E_2, \rightarrow_{\mathcal{S}_2}, c_2)$  and a function  $\sigma : (A_1 \cup E_1) \rightarrow (A_2 \cup E_2)$ . We call  $R \subseteq S_1 \times S_2$  a  $\sigma$ -bisimulation relation from  $\mathcal{S}_1$  to  $\mathcal{S}_2$  if  $R$  satisfies the followings:

- (1)  $(s_{01}, s_{02}) \in R$ .
- (2) For any  $s_1, s'_1 \in S_1$ ,  $s_2 \in S_2$ , and  $a_1 \in A_1 \cup E_1$ , if  $s_1 \xrightarrow{a_1}_{\mathcal{S}_1} s'_1$  and  $(s_1, s_2) \in R$ , then  $\exists s'_2 \in S_2 : s_2 \xrightarrow{\sigma(a_1)}_{\mathcal{S}_2} s'_2$  and  $(s'_1, s'_2) \in R$ .
- (3) For any  $s_1 \in S_1$ ,  $s_2, s'_2 \in S_2$ , and  $a_2 \in A_2 \cup E_2$ , if  $s_2 \xrightarrow{a_2}_{\mathcal{S}_2} s'_2$  and  $(s_1, s_2) \in R$ , then  $\exists s'_1 \in S_1$ ,  $\exists a_1 \in A_1 \cup E_1 : \sigma(a_1) = a_2$  and  $s_1 \xrightarrow{a_1}_{\mathcal{S}_1} s'_1$  and  $(s'_1, s'_2) \in R$ .
- (4) If  $(s_1, s_2) \in R$ , then  $c_1(s_1) = c_2(s_2)$ .

We say  $\mathcal{S}_1$  is  $\sigma$ -bisimilar to  $\mathcal{S}_2$  if there exists a  $\sigma$ -bisimulation relation from  $\mathcal{S}_1$  to  $\mathcal{S}_2$ . We call  $R$  a bisimulation relation if  $R$  is an  $id$ -bisimulation relation. We say  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are bisimilar if  $\mathcal{S}_1$  is  $id$ -bisimilar to  $\mathcal{S}_2$ .

The following lemma can be proved by definition.

**Lemma 2.** If  $\mathcal{S}_1 = (S_1, s_{01}, A_1, E_1, \rightarrow_{\mathcal{S}_1}, c_1)$  is  $\sigma$ -bisimilar to  $\mathcal{S}_2 = (S_2, s_{02}, A_2, E_2, \rightarrow_{\mathcal{S}_2}, c_2)$  for a function  $\sigma : (A_1 \cup E_1) \rightarrow (A_2 \cup E_2)$  that satisfies  $a \in A_1 \Leftrightarrow \sigma(a) \in A_2$  for any  $a \in A_1 \cup E_1$ , then  $\sigma(L(\mathcal{S}_1)) = L(\mathcal{S}_2)$ .

### 3 Pushdown Transducers, Automata and Games

#### 3.1 Pushdown Transducers

**Definition 3.** A pushdown transducer (PDT) over finite alphabets  $\Sigma_i$ ,  $\Sigma_o$  and  $\Gamma$  is  $\mathcal{T} = (P, p_0, z_0, \Delta)$  where  $P$  is a finite set of states,  $p_0 \in P$  is the initial state,  $z_0 \in \Gamma$  is the initial stack symbol and  $\Delta : P \times \Sigma_i \times \Gamma \rightarrow P \times \Sigma_o \times \text{Com}(\Gamma)$  is a finite set of deterministic transition rules having one of the following forms:

- $(p, a, z) \rightarrow (q, b, \text{pop})$  (pop rule)
- $(p, a, z) \rightarrow (q, b, \text{skip})$  (skip rule)
- $(p, a, z) \rightarrow (q, b, \text{push}(z))$  (push rule)

where  $p, q \in P$ ,  $a \in \Sigma_i$ ,  $b \in \Sigma_o$  and  $z \in \Gamma$ .

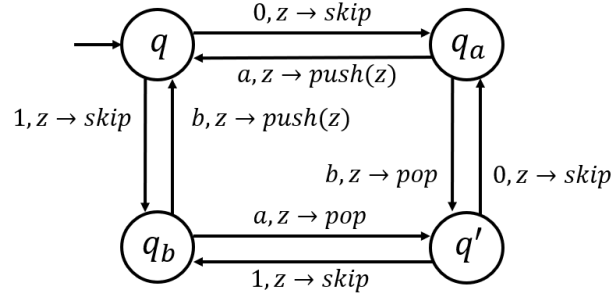
For a state  $p \in P$  and a finite sequence representing stack contents  $u \in \Gamma^*$ ,  $(p, u)$  is called a *configuration* or *instantaneous description* (abbreviated as *ID*) of PDT  $\mathcal{T}$ . Let  $ID_{\mathcal{T}}$  denote the set of all IDs of  $\mathcal{T}$ . For  $u \in \Gamma^+$  and  $\text{com} \in \text{Com}(\Gamma)$ , let us define  $\text{upds}(u, \text{com})$  as  $\text{upds}(u, \text{com}) = u(1 :)$ ,  $\text{upds}(u, \text{skip}) = u$  and  $\text{upds}(u, \text{push}(z')) = z'u$ .

For two IDs  $(p, u), (q, u') \in ID_{\mathcal{T}}$ ,  $a \in \Sigma_i$  and  $b \in \Sigma_o$ ,  $((p, u), ab, (q, u')) \in \Rightarrow_{\mathcal{T}}$ , written as  $(p, u) \Rightarrow_{\mathcal{T}}^{ab} (q, u')$ , if there exist a rule  $(p, a, z) \rightarrow (q, b, \text{com}) \in \Delta$  such that  $z = u(0)$  and  $u' = \text{upds}(u, \text{com})$ . If  $\mathcal{T}$  is clear from the context, we abbreviate  $\Rightarrow_{\mathcal{T}}^{ab}$  as  $\Rightarrow^{ab}$ . We will use similar abbreviations for the other models defined later. Note that there is no transition from an ID with empty stack. We define a run and the language  $L(\mathcal{T}) \subseteq (\Sigma_i \cdot \Sigma_o)^\omega$  of PDT  $\mathcal{T}$  as those of deterministic 0-TS  $(ID_{\mathcal{T}}, (q_0, z_0), \Sigma_i \cdot \Sigma_o, \emptyset, \Rightarrow_{\mathcal{T}}, c)$  where  $c(s) = 2$  for all  $s \in ID_{\mathcal{T}}$ . In this paper, we assume that no run of PDT reaches an ID whose stack is empty. Let **PDT** be the class consisting of PDT.

*Example 4.* Let us consider PDT  $\mathcal{T} = (\{p\}, p, z, \Delta)$  over  $\{0, 1\}, \{a, b\}$  and  $\{z\}$  where  $\Delta = \{(p, 0, z) \rightarrow (p, a, \text{skip}), (p, 1, z) \rightarrow (p, b, \text{push}(z))\}$ . We can see a pair of sequences  $(\rho, w) \in ID_{\mathcal{T}}^\omega \times (\{0, 1\} \cdot \{a, b\})^\omega$  where  $\rho = (p, z)(p, z)(p, zz)(p, zz)(p, zzz)(p, zzz) \dots$  and  $w = (0a1b)^\omega$  is a run of  $\mathcal{T}$ . Also,  $L(\mathcal{T}) = (\{0a\} \cup \{1b\})^\omega$ .

#### 3.2 Pushdown Automata

**Definition 5.** A nondeterministic pushdown automata (NPDA) over finite alphabets  $\Sigma_i$ ,  $\Sigma_o$  and  $\Gamma$  is  $\mathcal{A} = (Q, Q_i, Q_o, q_0, z_0, \delta, c)$  where  $Q, Q_i, Q_o$  are finite sets of states such that  $Q = Q_i \cup Q_o$  and  $Q_i \cap Q_o = \emptyset$ ,  $q_0 \in Q_i$  is the initial state,  $z_0 \in \Gamma$  is the initial stack symbol,  $c : Q \rightarrow [n]$  is the coloring function where  $n \in \mathbb{N}$  is the number of priorities and  $\delta : Q \times \Sigma \times \Gamma \rightarrow \mathcal{P}(Q \times \text{Com}(\Gamma))$  is a finite set of transition rules having one of the following forms:



**Fig. 1.** States and transitions of  $\mathcal{A}$ . (A label  $a, b \rightarrow c$  from  $q$  to  $q'$  means  $(q, a, b) \rightarrow (q', c) \in \delta$ .)

- $(q_{\mathbb{x}}, a_{\mathbb{x}}, z) \rightarrow (q_{\mathbb{x}}, \text{com})$  (input/output rules)
- $(q_{\mathbb{x}}, \tau, z) \rightarrow (q'_{\mathbb{x}}, \text{com})$  ( $\tau$  rules)

where  $(\mathbb{x}, \bar{\mathbb{x}}) \in \{(\mathbb{i}, \mathbb{o}), (\mathbb{o}, \mathbb{i})\}$ ,  $q_{\mathbb{x}}, q'_{\mathbb{x}} \in Q_{\mathbb{x}}$ ,  $q_{\bar{\mathbb{x}}} \in Q_{\bar{\mathbb{x}}}$ ,  $a_{\mathbb{x}} \in \Sigma_{\mathbb{x}}$ ,  $z \in \Gamma$  and  $\text{com} \in \text{Com}(\Gamma)$ .

We define  $ID_{\mathcal{A}} = Q \times \Gamma^*$  and the transition relation  $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times (\Sigma \cup \{\tau\}) \times ID_{\mathcal{A}}$  as  $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$  iff there exist a rule  $(p, a, z) \rightarrow (q, \text{com}) \in \delta$  and a sequence  $u \in \Gamma^*$  such that  $z = u(0)$  and  $u' = \text{upds}(u, \text{com})$ . We write  $(q, u) \vdash_{\mathcal{A}}^a (q', u')$  iff  $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$ . We write  $\vdash_{\mathcal{A}}^a$  as  $\vdash^a$  if  $\mathcal{A}$  is clear from context. We define a run and the language  $L(\mathcal{A})$  of  $\mathcal{A}$  as those of TS  $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, z_0), \Sigma, \{\tau\}, \vdash_{\mathcal{A}}, c')$  where  $c'((q, u)) = c(q)$  for every  $(q, u) \in ID_{\mathcal{A}}$ . We call a PDA  $\mathcal{A}$  deterministic if  $\mathcal{S}_{\mathcal{A}}$  is deterministic. We call  $\mathcal{A}$  an  $m$ -NPDA (or  $m$ -DPDA when  $\mathcal{A}$  is deterministic) if  $\mathcal{S}_{\mathcal{A}}$  is an  $m$ -TS. We abbreviate 0-NPDA (0-DPDA) as NPDA (DPDA). Let **DPDA** and **NPDA** be the classes of DPDA and NPDA, respectively.

*Example 6.* Let us consider DPDA  $\mathcal{A} = (\{q, q', q_a, q_b\}, \{q, q'\}, \{q_a, q_b\}, q, z, \delta, c)$  over  $\{0, 1\}$ ,  $\{a, b\}$  and  $\{z\}$  where  $c(q') = 1$ ,  $c(s) = 2$  for  $s = q, q_a, q_b$  and  $\delta$  is defined as in Fig. 1. We can see a pair of sequences  $(\rho, w) \in ID_{\mathcal{T}}^{\omega} \times (\{0, 1\} \cdot \{a, b\})^{\omega}$  in Example 4, where  $\rho = (q, z)(q, z)(q, zz)(q, zz)(q, zzz)(q, zzz) \cdots$  and  $w = (0a1b)^{\omega}$ , is also a run of  $\mathcal{A}$ . However,  $w_1 = (0a1a)^{\omega}$  and  $w_2 = 0b(0a1b)^{\omega}$  are not in  $L(\mathcal{A})$  because the run  $(\rho_1, w_1)$  visits  $q'$  infinitely and the input sequence  $w_2$  forces a stack of  $\mathcal{A}$  empty by reading  $0b$  first. We call  $0a$  and  $1b$  push subsequences and  $0b$  and  $1a$  pop subsequences. For a sequence  $w \in (\{0, 1\} \cdot \{a, b\})^{\omega}$ , let  $\#_{\text{push}}(w)$  and  $\#_{\text{pop}}(w)$  be the number of push and pop subsequences appearing in  $w$ , respectively. We can check  $L(\mathcal{A}) = \{w \in (\{0, 1\} \cdot \{a, b\})^{\omega} \mid \#_{\text{pop}}(w) \text{ is finite and } \#_{\text{push}}(w') - \#_{\text{pop}}(w') \geq 0 \text{ for every subsequence } w' = w(0 : m) \text{ of } w \text{ for all } m \in \mathbb{N}_0\}$ . For PDT  $\mathcal{T}$  defined in Example 4, we know  $L(\mathcal{T}) \subseteq L(\mathcal{A})$  because  $L(\mathcal{T}) = (\{0a\} \cup \{1b\})^{\omega}$  contains no sequence that includes a pop subsequences.

The following lemma states that the class of language recognized by  $m$ -DPDA and 0-DPDA are the same for a fixed  $m$ .

**Lemma 7.** *For a given  $m$ -DPDA  $\mathcal{A}$ , we can construct a 0-DPDA  $\mathcal{A}'$  such that  $L(\mathcal{A}) = L(\mathcal{A}')$*

**Proof sketch** We define the stack alphabet  $\Gamma'$  of  $\mathcal{A}'$  as  $\Gamma' = \Gamma^m$ . We can simulate  $m$ -steps by consecutive push rules (or pop rules) of  $\mathcal{A}$  as a single step by a push (or pop) rule of  $\mathcal{A}'$ .

### 3.3 Pushdown Games

**Definition 8.** *A pushdown game of DPDA  $\mathcal{A} = (Q, Q_{\mathfrak{i}}, Q_{\mathfrak{o}}, q_0, z_0, \delta, c)$  over  $\Sigma_{\mathfrak{i}}, \Sigma_{\mathfrak{o}}$  and  $\Gamma$  is  $\mathcal{G}_{\mathcal{A}} = (V, V_{\mathfrak{i}}, V_{\mathfrak{o}}, E, C)$  where  $V = Q \times \Gamma^*$  is the set of vertices with  $V_{\mathfrak{i}} = Q_{\mathfrak{i}} \times \Gamma^*$ ,  $V_{\mathfrak{o}} = Q_{\mathfrak{o}} \times \Gamma^*$ ,  $E \subseteq V \times V$  is the set of edges defined as  $E = \{(v, v') \mid v \vdash^a v' \text{ for some } a \in \Sigma_{\mathfrak{i}} \cup \Sigma_{\mathfrak{o}}\}$  and  $C : V \rightarrow [n]$  is the coloring function such that  $C((q, u)) = c(q)$  for all  $(q, u) \in V$ .*

The game starts with some  $(q_0, z_0) \in V_{\mathfrak{i}}$ . When the current vertex is  $v \in V_{\mathfrak{i}}$ , Player II chooses a successor  $v' \in V_{\mathfrak{o}}$  of  $v$  as the next vertex. When the current vertex is  $v \in V_{\mathfrak{o}}$ , Player I chooses a successor  $v' \in V_{\mathfrak{i}}$  of  $v$ . Formally, a finite or infinite sequence  $\rho \in V^\omega$  is *valid* if  $\rho(0) = (q_0, z_0)$  and  $(\rho(i-1), \rho(i)) \in E$  for every  $i \geq 1$ . A *play* of  $\mathcal{G}_{\mathcal{A}}$  is an infinite and valid sequence  $\rho \in V^\omega$ . Let  $PL$  be the set of plays. A play  $\rho \in PL$  is *winning* for Player I iff  $\min\{m \in [n] \mid \text{there exist an infinite number of } i \geq 0 \text{ such that } c(\rho(i)) = m\}$  is even.

By the definition of  $\mathcal{G}_{\mathcal{A}}$ , the following lemma holds.

**Lemma 9.** *Let  $f_1 : PL \rightarrow (Q \times \text{Com}(\Gamma))^\omega$  and  $f_2 : PL \rightarrow \Sigma^\omega$  be the functions defined as follows. For every play  $\rho = (q_0, u_0)(q_1, u_1) \cdots \in PL$  of  $\mathcal{G}_{\mathcal{A}}$ ,*

- $f_1(\rho) = (q_0, \text{com}_0)(q_1, \text{com}_1) \cdots \in (Q \times \text{Com})^\omega$  where  $u_{i+1} = \text{upds}(u_i, \text{com}_i)$  for all  $i \geq 0$  and
- $f_2(\rho) = w$  where  $\rho(i) \vdash^{w(i)} \rho(i+1)$  for all  $i \geq 0$ .

*Then,  $f_1$  and  $f_2$  are well-defined and both of  $f_1$  and  $f_2$  are injections.*

[36] proved that we can construct a PDT  $\mathcal{T}$  that gives a winning strategy of  $\mathcal{G}_{\mathcal{A}}$ , that is,  $L(\mathcal{T}) = \{f_1(\rho) \mid \rho \text{ is winning for Player I}\}$ .

**Theorem 10.** [36] *If player I has a winning strategy of  $\mathcal{G}_{\mathcal{A}}$ , we can construct a PDT  $\mathcal{T}$  over  $Q_{\mathfrak{i}} \times \text{Com}(\Gamma), Q_{\mathfrak{o}} \times \text{Com}(\Gamma)$  and a stack alphabet  $\Gamma'$  that gives a winning strategy of  $\mathcal{G}_{\mathcal{A}}$ . That is,  $\rho \in PL$  is winning for Player I iff  $f_1(\rho) \in L(\mathcal{T})$ .*

By Lemma 9, a winning strategy can be also given as the set of sequences  $w \in \Sigma^\omega$  such that the play  $\rho$  is winning where  $f_2(\rho) = w$ . Thus, we can obtain the following lemma in a similar way to Theorem 10.

**Lemma 11.** *If player I has a winning strategy of  $\mathcal{G}_{\mathcal{A}}$ , we can construct a PDT  $\mathcal{T}$  over  $\Sigma_{\mathfrak{i}}, \Sigma_{\mathfrak{o}}$  and  $\Gamma'$  that gives a winning strategy of  $\mathcal{G}_{\mathcal{A}}$ . That is,  $\rho \in PL$  is winning for Player I iff  $f_2(\rho) \in L(\mathcal{T})$ .*

## 4 Realizability problems for PDA and PDT

For a specification  $S$  and an implementation  $I$ , we write  $I \models S$  if  $L(I) \subseteq L(S)$ .

**Definition 12.** *Realizability problem  $\text{REAL}(\mathcal{S}, \mathcal{I})$  for a class of specifications  $\mathcal{S}$  and of implementations  $\mathcal{I}$ : For a specification  $S \in \mathcal{S}$ , is there an implementation  $I \in \mathcal{I}$  such that  $I \models S$ ?*

*Example 13.* By Examples 4 and 6,  $L(\mathcal{T}) \subseteq L(\mathcal{A})$  holds for PDT  $\mathcal{T}$  and DPDA  $\mathcal{A}$  defined in the examples. Thus,  $\mathcal{T} \models \mathcal{A}$  holds.

**Theorem 14.**  $\text{REAL}(\text{DPDA}, \text{PDT})$  is decidable.

**Proof.** Let  $\mathcal{A}$  be a given DPDA. By Lemma 11, we can construct a PDT  $\mathcal{T}$  such that  $\rho$  is a winning play of  $\mathcal{G}_{\mathcal{A}}$  iff  $f_2(\rho) \in L(\mathcal{A})$ . By the definition of  $f_2$ ,  $\rho(i) \vdash^{w(i)} \rho(i+1)$  holds for all  $i \geq 0$  where  $\rho \in PL$  such that  $f_2(\rho) = w$ . Then,  $w \in L(\mathcal{A})$  holds, and thus  $\mathcal{T} \models \mathcal{A}$ . Hence,  $\mathcal{T} \models \mathcal{A}$  holds iff player I has a winning strategy for the game  $\mathcal{G}_{\mathcal{A}}$ . Because there is an algorithm for constructing  $\mathcal{T}$  in [36],  $\text{REAL}(\text{DPDA}, \text{PDT})$  is decidable.

**Theorem 15.**  $\text{REAL}(\text{NPDA}, \text{PDT})$  is undecidable.

**Proof.** We prove the theorem by a reduction from the universality problem of NPDA, which is undecidable. For a given NPDA  $\mathcal{A} = (Q, Q_i, Q_o, q_0, z_0, \delta, c)$  over  $\Sigma_i, \Sigma_o$  and  $\Gamma$ , we can construct an NPDA  $\mathcal{A}' = (Q \times [2], Q \times \{1\}, Q \times \{2\}, q_0, z_0, \delta', c')$  over  $\Sigma'_i, \Sigma'_o$  and  $\Gamma$  where  $\Sigma'_i = \Sigma_i \cup \Sigma_o$ ,  $\Sigma'_o$  is an arbitrary (nonempty) alphabet,  $c'((q, 1)) = c'((q, 2)) = c(q)$  for all  $q \in Q$  and  $\delta'$  satisfies that  $((q, 1), a, z) \rightarrow ((q', 2), \text{com}) \in \delta$  iff  $(q, a, z) \rightarrow (q', \text{com}) \in \delta$ , and  $((q', 2), b, z) \rightarrow ((q', 1), \text{skip}) \in \delta'$  for all  $b \in \Sigma'_o$  and  $z \in \Gamma$ .

We show  $L(\mathcal{A}) = (\Sigma'_i)^\omega$  iff there exists  $\mathcal{T}$  such that  $\mathcal{T} \models \mathcal{A}$ . By the construction of  $\mathcal{A}'$ ,  $L(\mathcal{A}') = \langle L(\mathcal{A}), (\Sigma'_o)^\omega \rangle$  holds. If  $L(\mathcal{A}) = (\Sigma'_i)^\omega$ , then  $L(\mathcal{A}') = \langle (\Sigma'_i)^\omega, (\Sigma'_o)^\omega \rangle$  and thus  $\mathcal{T} \models \mathcal{A}$  holds for every  $\mathcal{T}$ . Assume that  $L(\mathcal{A}) \neq (\Sigma'_i)^\omega$ . Then, there exists a word  $w \in (\Sigma'_i)^\omega$  such that  $w \notin L(\mathcal{A})$ . For any PDT  $\mathcal{T}$  and any  $u \in (\Sigma'_i)^\omega$ , there is  $v \in (\Sigma'_o)^\omega$  such that  $\langle u, v \rangle \in L(\mathcal{A}')$ . On the other hand,  $\langle w, v \rangle \notin L(\mathcal{A}')$  holds for any  $v \in (\Sigma'_o)^\omega$ . Hence,  $\mathcal{T} \not\models \mathcal{A}'$  holds for any PDT  $\mathcal{T}$ . This completes the reduction and the realizability problem for PDT and NPDA is undecidable.

## 5 Register Pushdown Transducers and Register Pushdown Automata

### 5.1 Data words and registers

We assume a countable set  $D$  of *data values*. For finite alphabets  $\Sigma_i, \Sigma_o$ , an infinite sequence  $(a_1^\sharp, d_1)(a^\circ, d'_1) \cdots \in ((\Sigma_i \times D) \cdot (\Sigma_o \times D))^\omega$  is called a *data word*. We let  $\text{DW}(\Sigma_i, \Sigma_o, D) = ((\Sigma_i \times D) \cdot (\Sigma_o \times D))^\omega$ .



For  $k \in \mathbb{N}_0$ , a mapping  $\theta : [k] \rightarrow D$  is called an *assignment* (of data values to  $k$  registers). Let  $\Theta_k$  denote the collection of assignments to  $k$  registers. We assume  $\perp \in D$  as the initial data value and let  $\theta_\perp^k \in \Theta_k$  be the initial assignment such that  $\theta_\perp^k(i) = \perp$  for all  $i \in [k]$ .

We denote  $\mathbf{Tst}_k = \mathcal{P}([k] \cup \{\mathbf{top}\})$  and  $\mathbf{Asgn}_k = \mathcal{P}([k])$  where  $\mathbf{top} \notin \mathbb{N}$  is a unique symbol that represents a stack top value.  $\mathbf{Tst}_k$  is the set of guard conditions. For  $\mathbf{tst} \in \mathbf{Tst}_k$ ,  $\theta \in \Theta_k$  and  $d, e \in D$ , we denote  $(\theta, d, e) \models \mathbf{tst}$  if  $(\theta(i) = d \Leftrightarrow i \in \mathbf{tst})$  and  $(e = d \Leftrightarrow \mathbf{top} \in \mathbf{tst})$  hold. In the definitions of register pushdown transducer and automaton in the next section, the data values  $d$  and  $e$  correspond to an input data value and a stack top data value, respectively.  $\mathbf{Asgn}_k$  is the set of assignment conditions. For  $\mathbf{asgn} \in \mathbf{Asgn}_k$ ,  $\theta, \theta' \in \Theta_k$  and  $d \in D$ , let  $\theta[\mathbf{asgn} \leftarrow d]$  be the assignment  $\theta'$  such that  $\theta'(i) = d$  for  $i \in \mathbf{asgn}$  and  $\theta'(i) = \theta(i)$  for  $i \notin \mathbf{asgn}$ .

## 5.2 Register pushdown transducers

**Definition 16.** A  $k$ -register pushdown transducer ( $k$ -RPDT) over finite alphabets  $\Sigma_i, \Sigma_o$  and an infinite set  $\Gamma$  of data values is  $\mathcal{T} = (P, p_0, z_0, \Delta)$  where  $P$  is a finite set of states,  $p_0 \in P$  is the initial state,  $\Delta : P \times \Sigma_i \times \mathbf{Tst}_k \times \Gamma \rightarrow P \times \Sigma_o \times \mathbf{Asgn}_k \times [k] \times \mathbf{Com}(\Gamma \times [k])$  is a finite set of deterministic transition rules.

For  $u \in (\Gamma \times D)^+$ ,  $\theta' \in \Theta_k$  and  $\mathbf{com} \in \mathbf{Com}(\Gamma \times [k])$ , let us define  $\mathbf{upds}(u, \theta', \mathbf{com})$  as  $\mathbf{upds}(u, \theta', \mathbf{pop}) = u(1 :)$ ,  $\mathbf{upds}(u, \theta', \mathbf{skip}) = u$  and  $\mathbf{upds}(u, \theta', \mathbf{push}((z, j')) = (z, \theta'(j'))u$ . Let  $ID_{\mathcal{T}} = P \times \Theta_k \times (\Gamma \times D)^*$  and  $\Rightarrow_{\mathcal{T}} \subseteq ID_{\mathcal{T}} \times ((\Sigma_i \times D) \cdot (\Sigma_o \times D)) \times ID_{\mathcal{T}}$  be the transition relation of  $\mathcal{T}$  such that  $((p, \theta, u), (a, d^\sharp)(b, d^\circ), (q, \theta', u')) \in \Rightarrow_{\mathcal{T}}$  iff there exists a rule  $(p, a, \mathbf{tst}, z) \rightarrow (q, b, \mathbf{asgn}, j, \mathbf{com}) \in \Delta$  that satisfies the following conditions:  $(d^\sharp, \mathbf{scnd}(u(0)), \theta) \models \mathbf{tst}$ ,  $\theta' = \theta[\mathbf{asgn} \leftarrow d^\sharp]$ ,  $\theta'(j) = d^\circ$ ,  $z = \mathbf{fst}(u(0))$  and  $u' = \mathbf{upds}(u, \theta', \mathbf{com})$ , and we write  $(p, \theta, u) \Rightarrow_{\mathcal{T}}^{(a, d^\sharp)(b, d^\circ)} (q, \theta', u')$ .

A run and the language  $L(\mathcal{T})$  of  $\mathcal{T}$  are those of deterministic 0-TS  $(ID_{\mathcal{T}}, (q_0, \theta_\perp^k, (z_0, \perp)), (\Sigma_i \times D) \cdot (\Sigma_o \times D), \emptyset, \Rightarrow_{\mathcal{T}}, c)$  where  $c(s) = 2$  for all  $s \in ID_{\mathcal{T}}$ . In this paper, we assume that no run of RPDT reaches an ID whose stack is empty. Let  $\mathbf{RPDT}[k]$  be the class of  $k$ -RPDT and  $\mathbf{RPDT} = \bigcup_{k \in \mathbb{N}_0} \mathbf{RPDT}[k]$ .

*Example 17.* Let us consider 1-RPDT  $\mathcal{T} = (\{p\}, p, \Delta)$  over  $\{a\}, \{b\}$  and  $\{z\}$  where  $\Delta = \{(p, a, \{1, \mathbf{top}\}) \rightarrow (p, b, \emptyset, 1, \mathbf{skip}), (p, a, \emptyset) \rightarrow (p, b, \{1\}, 1, \mathbf{push}(1))\}$ . Let  $(\rho, w) \in ID_{\mathcal{T}}^\omega \times ((\{a\} \times D) \cdot (\{b\} \times D))^\omega$  be a pair of sequences where  $\rho = (p, [\perp], \perp)(p, [d_1], d_1 \perp)(p, [d_1], d_1 \perp)(p, [d_2], d_2 d_1 \perp)(p, [d_2], d_2 d_1 \perp) \cdots$ ,  $[d] \in \Theta_1$  is the assignment such that  $[d](1) = d$ , and  $w = (a, d_1)(b, d_1)(a, d_1)(b, d_1)(a, d_2)(b, d_2)(a, d_2)(b, d_2) \cdots$ , then  $(\rho, w)$  is a run of  $\mathcal{T}$ .

## 5.3 Register pushdown automata

**Definition 18.** A nondeterministic  $k$ -register pushdown automaton ( $k$ -NRPDA) over  $\Sigma_i, \Sigma_o$  and  $D$  is  $\mathcal{A} = (Q, Q_i, Q_o, q_0, \delta, c)$ , where

- $Q$  is a finite set of states,
- $Q_{\mathfrak{i}} \cup Q_{\mathfrak{o}} = Q, Q_{\mathfrak{i}} \cap Q_{\mathfrak{o}} = \emptyset$ ,
- $q_0 \in Q$  is the initial state, and
- $\delta : Q \times (\Sigma \cup \{\tau\}) \times \mathbf{Tst}_k \rightarrow \mathcal{P}(Q \times \mathbf{Asgn}_k \times \mathbf{Com}([k]))$  is a transition function having one of the forms:
  - $(q_{\mathfrak{x}}, a_{\mathfrak{x}}, \mathbf{tst}) \rightarrow (q_{\mathfrak{x}}, \mathbf{asgn}, \mathbf{com})$  (input/output rule)
  - $(q_{\mathfrak{x}}, \tau, \mathbf{tst}) \rightarrow (q'_{\mathfrak{x}}, \mathbf{asgn}, \mathbf{com})$  ( $\tau$  rule)
 where  $(\mathfrak{x}, \bar{\mathfrak{x}}) \in \{(\mathfrak{i}, \mathfrak{o}), (\mathfrak{o}, \mathfrak{i})\}$ ,  $q_{\mathfrak{x}}, q'_{\mathfrak{x}} \in Q_{\mathfrak{x}}, q_{\bar{\mathfrak{x}}} \in Q_{\bar{\mathfrak{x}}}, a_{\mathfrak{x}} \in \Sigma_{\mathfrak{x}}, \mathbf{tst} \in \mathbf{Tst}_k$ ,  $\mathbf{asgn} \in \mathbf{Asgn}_k$  and  $\mathbf{com} \in \mathbf{Com}([k])$ .
- $c : Q \rightarrow [n]$  where  $n \in \mathbb{N}$  is the number of priorities.

Let  $ID_{\mathcal{A}} = Q \times \Theta_k \times D^*$ . We define the transition relation  $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times ((\Sigma \cup \{\tau\}) \times D) \times ID_{\mathcal{A}}$  as  $((q, \theta, u), (a, d), (q', \theta', u')) \in \vdash_{\mathcal{A}}$ , written as  $(q, \theta, u) \vdash_{\mathcal{A}}^{(a, d)} (q', \theta', u')$ , iff there exists a rule  $(p, a, \mathbf{tst}) \rightarrow (q, \mathbf{asgn}, \mathbf{com}) \in \delta$  such that  $(d, u(0), \theta) \models \mathbf{tst}$ ,  $\theta' = \theta[\mathbf{asgn} \leftarrow d]$  and  $u' = \mathbf{upds}(u, \theta', \mathbf{com})$ . For  $s, s' \in ID_{\mathcal{A}}$  and  $w \in ((\Sigma_{\mathfrak{i}} \times D) \cdot (\Sigma_{\mathfrak{o}} \times D))^m$ , we write  $s \vdash^w s'$  if there exists  $\rho \in ID_{\mathcal{A}}^{m+1}$  such that  $\rho(0) = s, \rho(m) = s'$ , and  $\rho(0) \vdash^{w(0)} \dots \vdash^{w(m-1)} \rho(m)$ .

A run and the language  $L(\mathcal{A})$  of  $k$ -DRPDA  $\mathcal{A}$  are those of TS  $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, \theta_{\perp}^k, \perp), \Sigma \times D, \{\tau\} \times D, \vdash_{\mathcal{A}}, c')$  where  $c'((q, \theta, u)) = c(q)$  for all  $(q, \theta, u) \in ID_{\mathcal{A}}$ . We call  $\mathcal{A}$  deterministic, or  $k$ -DRPDA, if  $\mathcal{S}_{\mathcal{A}}$  is deterministic. We call  $\mathcal{A}$  an  $(m, k)$ -NRPDA (or an  $(m, k)$ -DRPDA when  $\mathcal{A}$  is deterministic) if  $\mathcal{S}_{\mathcal{A}}$  is an  $m$ -TS. We abbreviate  $(0, k)$ -NRPDA ( $(0, k)$ -DPDA) as  $k$ -NRPDA ( $k$ -DRPDA).

#### 5.4 Visibly RPDA

Let **DRPDA** and **NRPDA** be the unions of  $k$ -DRPDA and  $k$ -NRPDA for all  $k \in \mathbb{N}_0$ , respectively. Let  $\mathbf{Com}_v = \{\mathbf{pop}, \mathbf{skip}, \mathbf{push}\}$  and  $v : \mathbf{Com}([k]) \rightarrow \mathbf{Com}_v$  be the function such that  $v(\mathbf{push}(j)) = \mathbf{push}$  for  $j \in [k]$  and  $v(\mathbf{com}) = \mathbf{com}$  otherwise. We say that a  $k$ -DRPDA  $\mathcal{A}$  over  $\Sigma_{\mathfrak{i}}, \Sigma_{\mathfrak{o}}$  and  $D$  visibly manipulates its stack (or a *stack-visibly* RPDA) if there exists a function  $\mathbf{vis} : \Sigma \rightarrow \mathbf{Com}_v$  such that every rule  $(q, a, \mathbf{tst}) \rightarrow (q', \mathbf{asgn}, \mathbf{com})$  of  $\mathcal{A}$  satisfies  $\mathbf{vis}(a) = v(\mathbf{com})$ . Also, we say that  $\mathcal{A}$  is a *test-visibly* RPDA if there exists a function  $\mathbf{vis}_t : \Sigma \rightarrow \mathbf{Tst}_k$  such that every rule  $(q, a, \mathbf{tst}) \rightarrow (q', \mathbf{asgn}, \mathbf{com})$  of  $\mathcal{A}$  satisfies  $\mathbf{vis}_t(a) = \mathbf{tst}$ . If  $\mathcal{A}$  is stack-visibly and test-visibly RPDA, we call  $\mathcal{A}$  a visibly RPDA. Let **DRPDA<sub>v</sub>** be the union of visibly  $k$ -DRPDA for all  $k \in \mathbb{N}_0$ , respectively.

#### 5.5 PDA simulating RPDA

Let  $\sigma : ((\Sigma \cup \{\tau\}) \times D) \rightarrow (\Sigma \cup \{\tau\})$  be the function defined as  $\sigma((a, d)) = a$  for any  $a \in \Sigma \cup \{\tau\}$  and  $d \in D$ . In this subsection, we show that we can construct an NRPDA  $\mathcal{A}'$  from a given  $k$ -NRPDA  $\mathcal{A}$  over  $\Sigma_{\mathfrak{i}}, \Sigma_{\mathfrak{o}}, D$  such that  $\sigma(L(\mathcal{A})) = L(\mathcal{A}')$ .

Let  $\Phi_k$  be the set of *equivalence relations* over the set of  $2k + 1$  symbols  $X_k = \{x_1, x_2, \dots, x_k, x'_1, x'_2, \dots, x'_k, x_{\mathbf{top}}\}$ . We write  $a \equiv_{\phi} b$  and  $a \not\equiv_{\phi} b$  to mean  $(a, b) \in \phi$  and  $(a, b) \notin \phi$ , respectively, for  $a, b \in X_k$  and  $\phi \in \Phi_k$ . Intuitively,

each  $\phi \in \Phi_k$  represents the equality and inequality among the data values in the registers and the stack top, as well as the transition of the values in the registers between two assignments. Two assignments  $\theta, \theta'$  and a value  $d$  at the stack top satisfy  $\phi$ , denoted as  $\theta, d, \theta' \models \phi$ , if and only if for  $i, j \in [k]$ ,

$$\begin{aligned} \mathbf{x}_i \equiv_{\phi} \mathbf{x}_j &\Leftrightarrow \theta(i) = \theta(j), & \mathbf{x}_i \equiv_{\phi} \mathbf{x}_{\text{top}} &\Leftrightarrow \theta(i) = d, \\ \mathbf{x}_i \equiv_{\phi} \mathbf{x}'_j &\Leftrightarrow \theta(i) = \theta'(j), & \mathbf{x}'_j \equiv_{\phi} \mathbf{x}_{\text{top}} &\Leftrightarrow \theta'(j) = d, \\ \mathbf{x}'_i \equiv_{\phi} \mathbf{x}'_j &\Leftrightarrow \theta'(i) = \theta'(j). \end{aligned}$$

Let  $\phi_{\perp} \in \Phi_k$  be the equivalence relation satisfying  $a \equiv_{\phi_{\perp}} b$  for any  $a, b \in X_k$ .

For  $\mathbf{tst} \subseteq [k] \cup \{\text{top}\}$  and  $\mathbf{asgn} \subseteq [k]$ , define a subset  $\Phi_k^{\mathbf{tst}, \mathbf{asgn}}$  of  $\Phi_k$  as:

$$\begin{aligned} \Phi_k^{\mathbf{tst}, \mathbf{asgn}} = \{ \phi \in \Phi_k \mid & (\forall i \in \mathbf{tst} : \forall j \in [k] \cup \{\text{top}\} : j \in \mathbf{tst} \Leftrightarrow \mathbf{x}_i \equiv_{\phi} \mathbf{x}_j), \\ & (\forall i \in \mathbf{asgn} : \forall j \in [k] \cup \{\text{top}\} : j \in \mathbf{tst} \Leftrightarrow \mathbf{x}_j \equiv_{\phi} \mathbf{x}'_i), \\ & (\forall i, j \in \mathbf{asgn} : \mathbf{x}'_i \equiv_{\phi} \mathbf{x}'_j), \\ & (\forall i \in [k] \setminus \mathbf{asgn} : \mathbf{x}_i \equiv_{\phi} \mathbf{x}'_i) \}. \end{aligned}$$

For  $j \in [k]$ , define  $\Phi_k^{\perp, j} = \{ \phi \in \Phi_k \mid \mathbf{x}_{\text{top}} \equiv_{\phi} \mathbf{x}_j, \forall i \in [k] : \mathbf{x}_i \equiv_{\phi} \mathbf{x}'_i \}$ . By definition,  $\theta, e, \theta' \models \phi$  for  $\phi \in \Phi_k^{\mathbf{tst}, \mathbf{asgn}}$  iff  $\theta, d, e \models \mathbf{tst}$  and  $\theta' = \theta[\mathbf{asgn} \leftarrow d]$  for some  $d \in D$ . Similarly,  $\theta, e, \theta' \models \phi$  for  $\phi \in \Phi_k^{\perp, j}$  iff  $\theta' = \theta$  and  $\theta(j) = e$ .

Let  $\odot$  and  $\odot_{\text{T}}$  be binary relations over  $\Phi_k$  defined as:

$$\begin{aligned} \phi_1 \odot \phi_2 &:\Leftrightarrow \mathbf{x}'_i \equiv_{\phi_1} \mathbf{x}'_j \Leftrightarrow \mathbf{x}_i \equiv_{\phi_2} \mathbf{x}_j \text{ for } i, j \in [k]. \\ \phi_1 \odot_{\text{T}} \phi_2 &:\Leftrightarrow \phi_1 \odot \phi_2 \text{ and } \mathbf{x}'_i \equiv_{\phi_1} \mathbf{x}_{\text{top}} \Leftrightarrow \mathbf{x}_i \equiv_{\phi_2} \mathbf{x}_{\text{top}} \text{ for } i \in [k]. \end{aligned}$$

Below we will define the *composition* of two equivalence relations, and  $\phi_1 \odot \phi_2$  means that  $\phi_1$  and  $\phi_2$  are composable. For  $\phi \in \Phi_k$  and  $\Phi' \subseteq \Phi_k$ , let  $\phi \odot \Phi' = \{ \phi' \in \Phi' \mid \phi \odot \phi' \}$  and  $\phi \odot_{\text{T}} \Phi' = \{ \phi' \in \Phi' \mid \phi \odot_{\text{T}} \phi' \}$ . By definition,  $\phi \odot_{\text{T}} \Phi_k^{\mathbf{tst}, \mathbf{asgn}}$  consists of at most one equivalence relation for any  $\phi \in \Phi_k$ ,  $\mathbf{tst} \subseteq [k] \cup \{\text{top}\}$ , and  $\mathbf{asgn} \subseteq [k]$ . Similarly,  $\phi \odot \Phi_k^{\perp, j}$  consists of exactly one equivalence relation for any  $\phi \in \Phi_k$  and  $j \in [k]$ .

For  $\phi_1, \phi_2 \in \Phi_k$  with  $\phi_1 \odot \phi_2$ , the *composition*  $\phi_1 \circ \phi_2$  of them is the equivalence relation in  $\Phi_k$  that satisfies the followings:

$$\begin{aligned} \mathbf{x}_i \equiv_{\phi_1} \mathbf{x}_j &\Leftrightarrow \mathbf{x}_i \equiv_{\phi_1 \circ \phi_2} \mathbf{x}_j \quad \text{for } i, j \in [k] \cup \{\text{top}\}, \\ \mathbf{x}'_i \equiv_{\phi_2} \mathbf{x}'_j &\Leftrightarrow \mathbf{x}'_i \equiv_{\phi_1 \circ \phi_2} \mathbf{x}'_j \quad \text{for } i, j \in [k], \\ (\exists l \in [k] : \mathbf{x}_i \equiv_{\phi_1} \mathbf{x}'_l \wedge \mathbf{x}_l \equiv_{\phi_2} \mathbf{x}'_j) &\Leftrightarrow \mathbf{x}_i \equiv_{\phi_1 \circ \phi_2} \mathbf{x}'_j \quad \text{for } i \in [k] \cup \{\text{top}\}, j \in [k]. \end{aligned}$$

By definition, if  $\theta_1, d_1, \theta_2 \models \phi_1$  and  $\theta_2, d_2, \theta_3 \models \phi_2$ , then  $\theta_1, d_1, \theta_3 \models \phi_1 \circ \phi_2$ . By definition,  $\circ$  is associative.

Let  $\mathcal{A} = (Q, Q_{\text{I}}, Q_{\text{O}}, q_0, \delta, c)$  be a  $k$ -NRPDA over  $\Sigma_{\text{I}}, \Sigma_{\text{O}}$ , and  $D$ . We construct a PDA  $\mathcal{A}' = (Q', Q'_{\text{I}}, Q'_{\text{O}}, q'_0, \phi_{\perp}, \delta', c')$  over  $\Sigma_{\text{I}}, \Sigma_{\text{O}}$ , and  $\Phi_k$ , where  $Q' = Q \times \Phi_k$ ,  $Q'_{\text{I}} = Q_{\text{I}} \times \Phi_k$ ,  $Q'_{\text{O}} = Q_{\text{O}} \times \Phi_k$ ,  $q'_0 = (q_0, \phi_{\perp})$ ,  $c'((q, \phi)) = c(q)$  for any  $q \in Q$  and

$\phi \in \Phi_k$ , and for any  $(q, \phi_2) \in Q'$ ,  $a \in \Sigma \cup \{\tau\}$ , and  $\phi_1 \in \Phi_k$ ,  $\delta'((q, \phi_2), a, \phi_1)$  is the smallest set satisfying the following inference rules:

$$\frac{\delta(q, a, \mathbf{tst}) \ni (q', \mathbf{asgn}, \mathbf{skip}), \phi_1 \odot \phi_2, \phi_3 \in \phi_2 \odot_{\mathbf{T}} \Phi_k^{\mathbf{tst}, \mathbf{asgn}}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_2 \circ \phi_3), \mathbf{skip})} \quad (1)$$

$$\frac{\delta(q, a, \mathbf{tst}) \ni (q', \mathbf{asgn}, \mathbf{pop}), \phi_1 \odot \phi_2, \phi_3 \in \phi_2 \odot_{\mathbf{T}} \Phi_k^{\mathbf{tst}, \mathbf{asgn}}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_1 \circ \phi_2 \circ \phi_3), \mathbf{pop})} \quad (2)$$

$$\frac{\delta(q, a, \mathbf{tst}) \ni (q', \mathbf{asgn}, \mathbf{push}(j)), \phi_1 \odot \phi_2, \phi_3 \in \phi_2 \odot_{\mathbf{T}} \Phi_k^{\mathbf{tst}, \mathbf{asgn}}, \phi_4 \in \phi_3 \odot \Phi_k^{-, j}}{\delta'((q, \phi_2), a, \phi_1) \ni ((q', \phi_4), \mathbf{push}(\phi_2 \circ \phi_3))} \quad (3)$$

Let  $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, \theta_{\perp}, \perp), \Sigma \times D, \{\tau\} \times D, \vdash_{\mathcal{A}}, c_{\mathcal{A}})$  and  $\mathcal{S}_{\mathcal{A}'} = (ID_{\mathcal{A}'}, (q'_0, \phi_{\perp}), \Sigma, \{\tau\}, \vdash_{\mathcal{A}'}, c_{\mathcal{A}'})$  be the TSs that represent the semantics of  $\mathcal{A}$  and  $\mathcal{A}'$ , respectively. Below we show that  $\mathcal{S}_{\mathcal{A}}$  is  $\sigma$ -bisimilar to  $\mathcal{S}_{\mathcal{A}'}$ . We define an intermediate TS  $\mathcal{S}_{\mathcal{A}}^{\text{aug}} = (ID_{\mathcal{A}}^{\text{aug}}, (q_0, \theta_{\perp}, (\perp, \theta_{\perp})), \Sigma \times D, \{\tau\} \times D, \vdash_{\mathcal{A}^{\text{aug}}}, c'_{\mathcal{A}})$  where  $ID_{\mathcal{A}}^{\text{aug}} = Q \times \Theta_k \times (D \times \Theta_k)^*$ ,  $c'_{\mathcal{A}}((q, \theta, v)) = c_{\mathcal{A}}((q, \theta, \text{fst}(v)))$  for any  $(q, \theta, v) \in ID_{\mathcal{A}}^{\text{aug}}$ , and  $\vdash_{\mathcal{A}^{\text{aug}}}$  is defined as follows:  $(q, \theta, v) \vdash_{\mathcal{A}^{\text{aug}}}^{(a, d)} (q', \theta', v')$  if and only if  $\delta(q, a, \mathbf{tst}) \ni (q', \mathbf{asgn}, \mathbf{com})$ ,  $d, \text{fst}(v(0)), \theta \models \mathbf{tst}$ ,  $\theta' = \theta[\mathbf{asgn} \leftarrow d]$ , and  $v' = v(1:)$ ,  $v$ , or  $(\theta'(j'), \theta')v$  if  $\mathbf{com} = \mathbf{pop}$ ,  $\mathbf{skip}$ , or  $\mathbf{push}(j')$ , respectively. Obviously,  $\mathcal{S}_{\mathcal{A}}$  and  $\mathcal{S}_{\mathcal{A}'}$  are bisimilar; the smallest relation  $R \subseteq ID_{\mathcal{A}} \times ID_{\mathcal{A}}^{\text{aug}}$  satisfying  $((q, \theta, \text{fst}(v)), (q, \theta, v)) \in R$  for every  $(q, \theta, v) \in ID_{\mathcal{A}}^{\text{aug}}$  is a bisimulation relation between  $\mathcal{S}_{\mathcal{A}}$  and  $\mathcal{S}_{\mathcal{A}}^{\text{aug}}$ . We can show the following lemma.

**Lemma 19.**  $\mathcal{S}_{\mathcal{A}}^{\text{aug}}$  is  $\sigma$ -bisimilar to  $\mathcal{S}_{\mathcal{A}'}$ .

*Proof.* Let  $R \subseteq ID_{\mathcal{A}}^{\text{aug}} \times ID_{\mathcal{A}'}$  be the smallest relation that satisfies for every  $q \in Q$ ,  $\theta_0, \dots, \theta_n \in \Theta_k$ ,  $d_0, \dots, d_{n-1} \in D$ , and  $\phi_0, \dots, \phi_n \in \Phi_k$ ,  $((q, \theta_n, (d_{n-1}, \theta_{n-1}) \dots (d_1, \theta_1)(d_0, \theta_0)), ((q, \phi_n), \phi_{n-1} \dots \phi_1 \phi_0)) \in R$  if  $\forall i \in [n] : \theta_{i-1}, d_{i-1}, \theta_i \models \phi_i$  and  $\theta_{\perp}, \perp, \theta_0 \models \phi_0$ . We can show that  $R$  is a  $\sigma$ -bisimulation relation from  $\mathcal{S}_{\mathcal{A}}^{\text{aug}}$  to  $\mathcal{S}_{\mathcal{A}'}$ .

Assume that  $((q, \theta_n, v), ((q, \phi_n), u)) \in R$  for  $v = (d_{n-1}, \theta_{n-1}) \dots (d_0, \theta_0)$  and  $u = \phi_{n-1} \dots \phi_0$ . Because  $\forall i \in [n] : \theta_{i-1}, d_{i-1}, \theta_i \models \phi_i$  and  $\theta_{\perp}, \perp, \theta_0 \models \phi_0$ ,  $\forall i \in [n] : \phi_{i-1} \odot \phi_i$ . If  $(q, \theta_n, v) \vdash_{\mathcal{A}^{\text{aug}}}^{(a, d)} (q', \theta', v')$ , then there exist  $\mathbf{tst}$ ,  $\mathbf{asgn}$ ,  $\mathbf{com}$ , and  $d$  such that  $\delta(q, a, \mathbf{tst}) \ni (q', \mathbf{asgn}, \mathbf{com})$ ,  $\theta_n, d, d_{n-1} \models \mathbf{tst}$ ,  $\theta' = \theta[\mathbf{asgn} \leftarrow d]$ , and  $v' = v(1:)$ ,  $v$ , or  $(\theta'(j'), \theta')v$  if  $\mathbf{com} = \mathbf{pop}$ ,  $\mathbf{skip}$ , or  $\mathbf{push}(j')$ , respectively. There exists  $\phi' \in \phi_n \odot_{\mathbf{T}} \Phi_k^{\mathbf{tst}, \mathbf{asgn}}$  because  $\theta_{n-1}, d_{n-1}, \theta_n \models \phi_n$  and  $\theta_n, d, d_{n-1} \models \mathbf{tst}$ . Thus, by definition,  $\delta'((q, \phi_n), a, \phi_{n-1}) \ni ((q', \phi''), \mathbf{com}')$  where  $\mathbf{com}' = \mathbf{pop}$  and  $\phi'' = \phi_{n-1} \circ \phi_n \circ \phi'$  if  $\mathbf{com} = \mathbf{pop}$ ,  $\mathbf{com}' = \mathbf{skip}$  and  $\phi'' = \phi_n \circ \phi'$  if  $\mathbf{com} = \mathbf{skip}$ , and  $\mathbf{com}' = \mathbf{push}(\phi_n \circ \phi')$  and  $\phi'' \in \phi' \odot \Phi_k^{-, j'}$  if  $\mathbf{com} = \mathbf{push}(j')$ . Therefore,  $((q, \phi_n), u) \vdash_{\mathcal{A}'}^a ((q', \phi''), u')$  where  $u' = u(1:)$ ,  $u$ , or  $(\phi_n \circ \phi')u$  for  $u = \phi_{n-1} \dots \phi_0$  if  $\mathbf{com}' = \mathbf{pop}$ ,  $\mathbf{skip}$ , or  $\mathbf{push}(\phi_n \circ \phi')$ , respectively. Because  $\theta_{n-1}, d_{n-1}, \theta_n \models \phi_n$  and  $\theta_n, d_{n-1}, \theta' \models \phi'$ ,  $\theta_{n-1}, d_{n-1}, \theta' \models \phi_n \circ \phi'$ . In the case of  $\mathbf{com} = \mathbf{pop}$ ,  $\theta_{n-2}, d_{n-2}, \theta' \models \phi_{n-1} \circ \phi_n \circ \phi'$  because  $\theta_{n-2}, d_{n-2}, \theta_{n-1} \models \phi_{n-1}$  and  $\theta_{n-1}, d_{n-1}, \theta' \models \phi_n \circ \phi'$ . In the case of  $\mathbf{com} = \mathbf{push}(j')$ ,  $\theta', \theta'(j'), \theta' \models \phi''$  because  $\phi'' \in \Phi_k^{-, j'}$ . Hence,  $((q', \theta', v'), ((q', \phi''), u')) \in R$  in any case.

On the other hand, if  $((q, \phi_n), u) \vdash_{\mathcal{A}'}^a ((q', \phi''), u')$ , then  $\delta'((q, \phi_n), a, \phi_{n-1}) \ni ((q', \phi''), \text{com}')$  where  $\text{com}' = \text{pop}, \text{skip}, \text{or } \text{push}(\phi''')$  if  $u' = u(1:), u$ , or  $\phi'''u$ , respectively. By definition, there exist  $\text{tst}, \text{asgn}, \text{com}$ , and  $\phi' \in \phi_n \circ \Phi_k^{\text{tst}, \text{asgn}}$  such that  $\delta(q, a, \text{tst}) \ni (q', \text{asgn}, \text{com})$ , and  $\text{com} = \text{pop}$  and  $\phi'' = \phi_{n-1} \circ \phi_n \circ \phi'$  if  $\text{com}' = \text{pop}$ ,  $\text{com} = \text{skip}$  and  $\phi'' = \phi_n \circ \phi'$  if  $\text{com}' = \text{skip}$ , or  $\text{com} = \text{push}(j')$ ,  $\phi'' \in \phi' \odot \Phi_j^{\text{push}, j'}$  and  $\phi''' = \phi_n \circ \phi'$  if  $\text{com}' = \text{push}(\phi''')$ . Because  $\theta_{n-1}, d_{n-1}, \theta_n \models \phi_n$  and  $\phi' \in \phi_n \odot_{\text{T}} \Phi_k^{\text{tst}, \text{asgn}}$ , there exists  $d \in D$  satisfying  $\theta_n, d, d_{n-1} \models \text{tst}$  and  $\theta_n, d_{n-1}, \theta' \models \phi'$  for  $\theta' = \theta_n[\text{asgn} \leftarrow d]$ . Therefore,  $(q, \theta_n, v) \vdash_{\mathcal{A}'^{\text{aug}}}^{(a, d)} (q', \theta', v')$  where  $v' = v(1:), v$ , or  $(\theta'(j'), \theta')v$  if  $\text{com} = \text{pop}, \text{skip}, \text{or } \text{push}(j')$ , respectively. We can show that  $((q', \theta', v'), ((q', \phi''), u')) \in R$  in the same way as the last paragraph.

By Lemmas 2 and 19, we obtain the following theorem.

**Theorem 20.** *For a given  $(m, k)$ -NRPDA  $\mathcal{A}$ , we can construct an NPDA  $m\text{-}\mathcal{A}'$  such that  $\sigma(L(\mathcal{A})) = L(\mathcal{A}')$ .*

## 6 Realizability problems for RPDA and RPDT

### 6.1 Finite actions

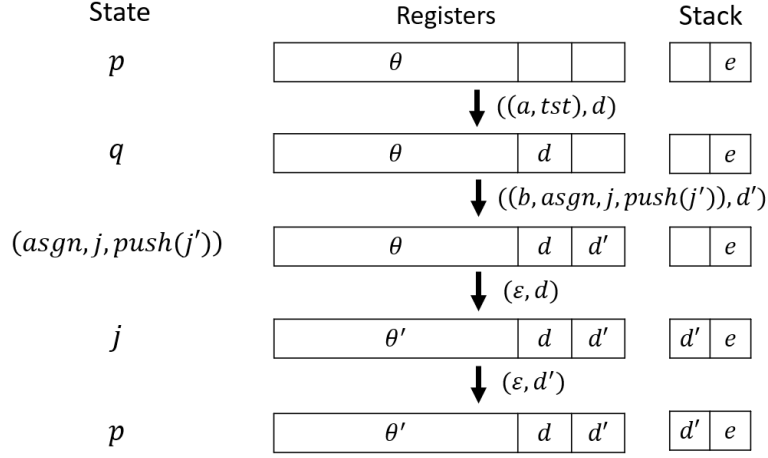
In [19], the abstraction of the behavior of  $k$ -register transducer ( $k$ -RT), called finite actions, was introduced to reduce the realizability problem of register automata (RA) and RT to the problem of finite alphabets. We extend the idea of [19] and define finite actions of  $k$ -RPDT.

For  $k \in \mathbb{N}_0$ , we define the set of finite input actions as  $A_k^{\text{i}} = \Sigma_{\text{i}} \times \text{Tst}_k$  and the set of finite output actions as  $A_k^{\text{o}} = \Sigma_{\text{o}} \times \text{Asgn}_k \times [k] \times \text{Com}([k])$ . A sequence  $w = (a_0^{\text{i}}, d_0^{\text{o}})(a_0^{\text{o}}, d_0^{\text{o}}) \cdots \in \text{DW}(\Sigma_{\text{i}} \times \text{Com}_v, \Sigma_{\text{o}} \times \text{Com}_v, D)$  is *compatible* with a sequence  $\bar{a} = (a_0^{\text{i}}, \text{tst}_0)(a_0^{\text{o}}, \text{asgn}_0, j_0, \text{com}_0) \cdots \in (A_k^{\text{i}} \cdot A_k^{\text{o}})^{\omega}$  iff there exists a sequence  $(\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^{\omega}$ , called a *witness*, such that  $\theta_0 = \theta_{\perp}^k$ ,  $u_0 = \perp$ ,  $(\theta_i, d_i^{\text{o}}, u_i(0)) \models \text{tst}_i$ ,  $\theta_{i+1} = \theta_i[\text{asgn}_i \leftarrow d_i^{\text{o}}]$ ,  $\theta_{i+1}(j_i) = d_i^{\text{o}}$  and  $u_{i+1} = \text{upds}(u_i, \theta_{i+1}, \text{com}_i)$ . Let  $\text{Comp}(\bar{a}) = \{w \in \text{DW}(\Sigma_{\text{i}}, \Sigma_{\text{o}}, D) \mid w \text{ is compatible with } \bar{a}\}$ . For a specification  $S \subseteq \text{DW}(\Sigma_{\text{i}}, \Sigma_{\text{o}}, D)$ , we define  $W_{S,k} = \{\bar{a} \mid \text{Comp}(\bar{a}) \subseteq S\}$ .

For a data word  $w \in \text{DW}(\Sigma_{\text{i}}, \Sigma_{\text{o}}, D)$  and a sequence  $\bar{a} \in (A_k^{\text{i}} \cdot A_k^{\text{o}})^{\omega}$  such that for each  $i \geq 0$ , there exists  $a \in \Sigma$  and we can write  $w(i) = (a, d)$  and  $\bar{a}(i) = (a, \text{tst})$  if  $i$  is even and  $w(i) = (a, d)$  and  $\bar{a}(i) = (a, \text{asgn}, j, \text{com})$  if  $i$  is odd, we define  $w \otimes \bar{a} \in \text{DW}(A_k^{\text{i}}, A_k^{\text{o}}, D)$  as  $w \otimes \bar{a}(i) = (\bar{a}(i), d)$  where  $w(i) = (a, d)$ .

### 6.2 Decidability and undecidability of realizability problems

**Lemma 21.**  $L_k = \{w \otimes \bar{a} \mid w \in \text{Comp}(\bar{a})\}$  is definable as the language of a  $(2, k+2)$ -DRPDA.



**Fig. 2.** An example of transitions of  $\mathcal{A}_k$ .

**Proof sketch.** Let  $(2, k+2)$ -DRPDA  $\mathcal{A}_1 = (Q_1, Q_1^{\mathfrak{f}}, Q_1^{\circ}, q_1^0, \delta_1, c_1)$  over  $A_k^{\mathfrak{f}}, A_k^{\circ}$  and  $D$  where  $Q_1 = \{p, q\} \cup (\text{Asgn}_k \times [k] \times \text{Com}([k])) \cup [k]$ ,  $Q_1^{\mathfrak{f}} = \{p\}$ ,  $Q_1^{\circ} = Q_1 \setminus Q_1^{\mathfrak{f}}$ ,  $q_1^0 = p$ ,  $c_1(s) = 2$  for every  $s \in Q$  and  $\delta_1$  consists of all the rules of the form

$$(p, (a_{\mathfrak{f}}, \text{tst}), \text{tst} \cup \text{tst}') \rightarrow (q, \{k+1\}, \text{skip}) \quad (4)$$

$$(q, (a_{\circ}, \text{asgn}, j, \text{com}), \text{tst}'') \rightarrow ((\text{asgn}, j, \text{com}), \{k+2\}, \text{skip}) \quad (5)$$

$$((\text{asgn}, j, \text{com}), \tau, \{k+1\} \cup \text{tst}'') \rightarrow (j, \text{asgn}, \text{com}) \quad (6)$$

$$(j, \tau, \{j, k+2\} \cup \text{tst}'') \rightarrow (p, \emptyset, \text{skip}) \quad (7)$$

for all  $(a_{\mathfrak{f}}, \text{tst}) \in A_k^{\mathfrak{f}}$ ,  $(a_{\circ}, \text{asgn}, j, \text{com}) \in A_k^{\circ}$ ,  $\text{tst}' \subseteq \{k+1, k+2\}$  and  $\text{tst}'' \in \text{Tst}_{k+2}$ . As in Fig. 2,  $\mathcal{A}_1$  checks whether an input sequence satisfies the conditions of compatibility by generating the witness of the compatibility step by step. We can show  $L(\mathcal{A}_k) = L_k$  by checking  $w \otimes \bar{a} \in L(\mathcal{A}_k) \Leftrightarrow w \in \text{Comp}(\bar{a})$  by the induction on the length of  $w \otimes \bar{a}$ .

**Lemma 22.** For a specification  $\mathcal{S}$  defined by some visibly  $k'$ -DRPDA,  $L_{\bar{\mathcal{S}}, k} = \{w \otimes \bar{a} \mid w \in \text{Comp}(\bar{a}) \cap \bar{\mathcal{S}}\}$  is definable as the language of a  $(4, k+k'+4)$ -DRPDA.

**Proof sketch.** Let  $L_{\bar{\mathcal{S}}} = \{w \otimes \bar{a} \mid w \in \bar{\mathcal{S}}, \bar{a} \in (A_k^{\mathfrak{f}} \cdot A_k^{\circ})^{\omega}\}$ . Because the class of languages defined by visibly DRPDA is closed under the complement, we can construct a visibly  $k'$ -DRPDA  $\mathcal{A}_2 = (Q_2, Q_2^{\mathfrak{f}}, Q_2^{\circ}, q_2^0, \delta_2, c_2)$  over  $A_k^{\mathfrak{f}}, A_k^{\circ}$  and  $D$  such that  $L(\mathcal{A}_2) = L_{\bar{\mathcal{S}}}$ . Let  $\mathcal{A}_1$  be the  $(2, k+2)$ -DRPDA such that  $L(\mathcal{A}_1) = L_k$ , which is given in Lemma 21. Because  $L_{\bar{\mathcal{S}}, k} = L_{\bar{\mathcal{S}}} \cap L_k$ , it is enough to show that we can construct a  $(4, k+k'+4)$ -DRPDA  $\mathcal{A}$  over  $A_k^{\mathfrak{f}}, A_k^{\circ}$  and  $D$  such that  $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ .

To construct  $\mathcal{A}$ , we use the facts that  $c_1(q)$  is even for every  $q \in Q_1$  and  $\delta_1$  consists of several groups of three consecutive rules having the following forms:

$$(q_1, a, \mathbf{tst}_1) \rightarrow (q_2, \mathbf{asgn}_1, \mathbf{skip}) \quad (5')$$

$$(q_2, \tau, \mathbf{tst}_2) \rightarrow (q_3, \mathbf{asgn}_2, \mathbf{com}_1) \quad (6')$$

$$(q_3, \tau, \mathbf{tst}_3) \rightarrow (q_4, \mathbf{asgn}_3, \mathbf{skip}). \quad (7')$$

Note that  $\mathbf{vis}(a) = v(\mathbf{com}_1)$  always holds for such three consecutive rules. (5'), (6') and (7') correspond to (5), (6) and (7), respectively, and (4) is also converted to three consecutive rules like (5')-(7') by adding dummy  $\tau$  rules.

We let  $k_1 = k + 2$  and  $k_2 = k'$ . We construct  $(4, k_1 + k_2 + 2)$ -DRPDA  $\mathcal{A} = (Q_{\mathfrak{i}} \cup Q_{\mathfrak{o}} \cup \{q_0\}, Q_{\mathfrak{i}} \cup \{q_0\}, Q_{\mathfrak{o}}, q_0, \delta, c)$  where  $Q_{\mathfrak{i}} = Q_1^{\mathfrak{i}} \times Q_2^{\mathfrak{i}} \times [5]$ ,  $Q_{\mathfrak{o}} = Q_1^{\mathfrak{o}} \times Q_2^{\mathfrak{o}} \times [5]$ .  $c$  is defined as  $c(q_0) = 1$  and  $c((q_1, q_2, i)) = c_2(q_2)$  for all  $(q_1, q_2, i) \in Q$ .  $\delta$  has a  $\tau$  rule  $(q_0, \tau, [k] \cup \{\mathbf{top}\}) \rightarrow ((q_0^1, q_0^2, 1), \mathbf{push}(1))$ . For all rules

$$(q_1, a, \mathbf{tst}_1) \rightarrow (q_2, \mathbf{asgn}_1, \mathbf{skip}) \in \delta_1 \quad (8)$$

$$(q_2, \tau, \mathbf{tst}_2) \rightarrow (q_3, \mathbf{asgn}_2, \mathbf{com}_1) \in \delta_1 \quad (9)$$

$$(q_3, \tau, \mathbf{tst}_3) \rightarrow (q_4, \mathbf{asgn}_3, \mathbf{skip}) \in \delta_1 \quad (10)$$

$$(q, a, \mathbf{tst}) \rightarrow (q', \mathbf{asgn}, \mathbf{com}) \in \delta_2 \quad (11)$$

such that  $v(\mathbf{com}_1) = v(\mathbf{com}) (= \mathbf{vis}(a))$  for  $a \in A_k^{\mathfrak{i}} \cup A_k^{\mathfrak{o}}$ , let  $\mathbf{tst}^{+k_1} = \{i + k_1 \mid i \in \mathbf{tst}\} \cup \{\mathbf{top} \mid \mathbf{top} \in \mathbf{tst} \setminus [k_1]\}$ ,  $\mathbf{asgn}^{+k_1} = \{i + k_1 \mid i \in \mathbf{asgn}\}$  and  $\mathbf{com}^{+k_1} = \mathbf{push}(j + k_1)$  if  $\mathbf{com} = \mathbf{push}(j)$  and  $\mathbf{com}^{+k_1} = \mathbf{com}$  otherwise, then  $\delta$  consists of the rules

$$((q_1, q, 1), \tau, T_1^{k_1+k_2+2} \cup \{\mathbf{top}\}) \rightarrow ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathbf{pop}) \quad (12)$$

$$((q_1, q, 2), \tau, T_1^{k_1+k_2+2} \cup \{\mathbf{top}\}) \rightarrow ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathbf{push}(k_1 + k_2 + 1)) \quad (13)$$

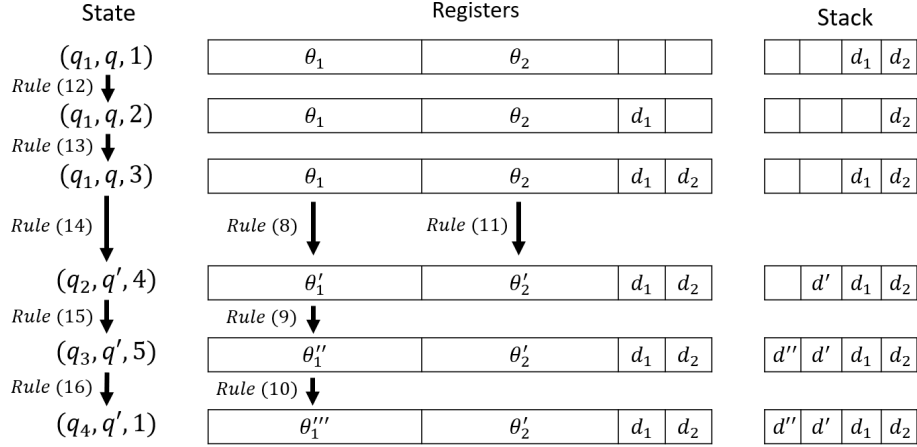
$$\begin{aligned} ((q_1, q, 3), a, \mathbf{tst}_1 \cup ((\mathbf{tst}^{+k_1} \setminus \{\mathbf{top}\}) \cup \mathbf{Top})) \\ \rightarrow ((q_2, q', 4), \mathbf{asgn}_1 \cup \mathbf{asgn}^{+k_1}, \mathbf{com}^{+k_1}) \end{aligned} \quad (14)$$

$$\begin{aligned} ((q_2, q', 4), \tau, ((\mathbf{tst}_2 \setminus \{\mathbf{top}\}) \cup \mathbf{Top}') \cup T_{k_1+1}^{k_1+k_2+1}) \\ \rightarrow ((q_3, q', 5), \mathbf{asgn}_2, \mathbf{com}_1) \end{aligned} \quad (15)$$

$$((q_3, q', 5), \tau, \mathbf{tst}_3 \cup T_{k_1+1}^{k_1+k_2+2}) \rightarrow ((q_4, q', 1), \mathbf{asgn}_3, \mathbf{skip}) \quad (16)$$

for all  $T_i^j \in \mathbf{Tst}_j \setminus \mathbf{Tst}_{i-1}$  (we assume  $\mathbf{Tst}_0 = \emptyset$ ), and  $\mathbf{Top} = \{k_1 + k_2 + 2\}$  ( $\mathbf{Top}' = \{k_1 + k_2 + 1\}$ ) if  $\mathbf{top} \in \mathbf{tst}$  ( $\mathbf{top} \in \mathbf{tst}_2$ ) and  $\mathbf{Top} = \emptyset$  ( $\mathbf{Top}' = \emptyset$ ) otherwise.

Fig. 3 illustrates an example of transitions of  $\mathcal{A}$  from  $(q_1, q, 1)$  to  $(q_4, q', 1)$  with updating contents of its registers and stack. The first to  $k_1$ -th registers simulate the registers of  $\mathcal{A}_1$ ,  $(k_1 + 1)$ -th to  $(k_1 + k_2)$ -th registers simulate the registers of  $\mathcal{A}_2$  and  $(k_1 + k_2 + 1)$ -th and  $(k_1 + k_2 + 2)$ -th registers are for keeping the first and second stack top contents, respectively. The stack contents of  $\mathcal{A}$  simulates that of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  by restoring the contents of stacks of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  alternately.



**Fig. 3.** An example of transitions of  $\mathcal{A}$  with  $\text{vis}(a) = \text{push}$ .

The transition rules (12) and (13) are for moving the two stack data values from the top to  $(k_1 + k_2 + 1)$ -th and  $(k_1 + k_2 + 2)$ -th registers. The transition rule (14) is for updating states, registers and stacks by simulating the rules (8) and (11). Because the first to  $k_2$ -th registers of  $\mathcal{A}_2$  are simulated by  $(k_1 + 1)$ -th to  $(k_1 + k_2)$ -th registers of  $\mathcal{A}$ , we use  $\text{tst}^{+k_1}$ ,  $\text{asgn}^{+k_1}$  and  $\text{com}^{+k_1}$  instead of  $\text{tst}$ ,  $\text{asgn}$  and  $\text{com}$ , and replace  $\text{top}$  in  $\text{tst}^{+k_1}$  to  $k_1 + k_2 + 2$ . The transition rules (15) and (16) simulate the rules (9) and (10), respectively.

We can show  $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$  by checking  $w \in L(\mathcal{A})$  iff  $w \in L(\mathcal{A}_1)$  and  $w \in L(\mathcal{A}_2)$  by the induction on the length of  $w$ .

**Lemma 23.**  $W_{S,k} = \overline{\text{Lab}(L_{\bar{S},k})}$ .

**Proof.** For every  $\bar{a} \in (A_k^\sharp A_k^\circ)^\omega$ ,  $\bar{a} \notin W_{S,k} \Leftrightarrow \text{Comp}(\bar{a}) \not\subseteq S \Leftrightarrow \exists w. w \in \text{Comp}(\bar{a}) \cap \bar{S} \Leftrightarrow \exists w. w \otimes \bar{a} \in L_{\bar{S},k} \Leftrightarrow \bar{a} \in \text{Lab}(L_{\bar{S},k})$ . Thus,  $W_{S,k} = \overline{\text{Lab}(L_{\bar{S},k})}$  holds.

**Theorem 24.** For all  $k \geq 0$ ,  $\text{REAL}(\text{DRPDAv}, \text{RPDT}[k])$  is decidable.

**Proof.** By Lemma 22 and Theorem 20,  $W_{S,k}$  is definable by 4-DPDA  $\mathcal{A}_f$ . By Lemma 7, we can construct 0-DPDA  $\mathcal{A}'_f$  where  $L(\mathcal{A}'_f) = L(\mathcal{A}_f)$ . By the construction of  $\mathcal{A}$  in Lemma 22, the stack height of current ID increases two during the transitions (12) to (16) if  $\text{vis}(a) = \text{push}$ , does not change if  $\text{vis}(a) = \text{skip}$  and decrease two if  $\text{vis}(a) = \text{pop}$ . Thus,  $\mathcal{A}'_f$  is visibly DPDA, that is, every transition rule  $(p, a, z) \rightarrow (q, \text{com})$  of  $\mathcal{A}'_f$  satisfies  $\text{vis}(a) = v(\text{com})$ . We show the following two conditions are equivalent.

- There exists a  $k$ -RPDT  $\mathcal{T}$  such that  $L(\mathcal{T}) \subseteq S$ .
- There exists a PDT  $\mathcal{T}'$  such that  $L(\mathcal{T}') \subseteq W_{S,k}$ .

Assume a  $k$ -RPDT  $\mathcal{T}$  over  $\Sigma_\sharp, \Sigma_\circ$  and  $\Gamma$  satisfies  $L(\mathcal{T}) \subseteq S$ . Then, consider the PDT  $\mathcal{T}'$  over  $A_k^\sharp, A_k^\circ$  and  $\Gamma$  such that  $(q, (a, \text{tst}), z) \rightarrow (q', (b, \text{asgn}, j, \text{com}), \text{com}')$



is a rule of  $\mathcal{T}'$  iff  $(q, a, \text{tst}, z) \rightarrow (q', b, \text{asgn}, j, \text{com})$  is a rule of  $\mathcal{T}$  where  $\text{com}' = \text{com}$  if  $v(\text{com}) = \text{pop}, \text{skip}$  and  $\text{com}' = \text{push}(z')$  if  $\text{com} = \text{push}(z', j')$  for some  $j' \in [k]$ . For  $\bar{a} \in L(\mathcal{T}')$ , every  $w \in \text{Comp}(\bar{a})$  has a witness and thus  $w \in L(\mathcal{T})$  holds. By the assumption  $L(\mathcal{T}) \subseteq S$ ,  $\text{Comp}(\bar{a}) \subseteq S$  holds and thus  $\bar{a} \in W_{S,k}$ . Hence, we obtain  $L(\mathcal{T}') \subseteq W_{S,k}$ .

Conversely, assume there exists an PDT  $\mathcal{T}'$  over  $A_k^{\mathfrak{I}}, A_k^{\mathfrak{O}}$  and  $\Gamma$  satisfies  $L(\mathcal{T}') \subseteq W_{S,k}$ . By the assumption that there exists an PDT  $\mathcal{T}'$  over  $A_k^{\mathfrak{I}}, A_k^{\mathfrak{O}}$  and  $\Gamma$  satisfies  $L(\mathcal{T}') \subseteq W_{S,k}$ , we can construct an PDT  $\mathcal{T}''$  such that every rule  $(q, (a, \text{tst}), z) \rightarrow (q', (b, \text{asgn}, j, \text{com}), \text{com}')$  satisfies  $\text{vis}(b) = v(\text{com}')$  (note that  $\text{vis}(a) = \text{skip}$  always hold) by the construction algorithm in [36], and thus  $v(\text{com}) = v(\text{com}')$  holds because  $\mathcal{A}'_f$  is visible and  $\text{vis}(b) = v(\text{com})$  holds. Consider the  $k$ -RPDT  $\mathcal{T}$  over  $\Sigma_{\mathfrak{I}}, \Sigma_{\mathfrak{O}}$  and  $\Gamma$  such that  $(q, a, \text{tst}, z) \rightarrow (q', b, \text{asgn}, j, \text{com}'')$  is a rule of  $\mathcal{T}$  iff  $(q, (a, \text{tst}), z) \rightarrow (q', (b, \text{asgn}, j, \text{com}), \text{com}')$  is a rule of  $\mathcal{T}''$  where  $\text{com}'' = \text{com}'$  if  $\text{com}' = \text{pop}, \text{push}$  and  $\text{com}'' = \text{push}(z', j')$  if  $\text{com}' = \text{push}(z')$  and  $\text{com} = \text{push}(j')$ . Further, assume  $w \in L(\mathcal{T})$ , and let  $\bar{a} \in (A_k^{\mathfrak{I}} \cdot A_k^{\mathfrak{O}})^{\omega}$  be the sequence that  $w$  is compatible with. Then, by the definition of  $\mathcal{T}$ ,  $\bar{a} \in L(\mathcal{T}'')$ . By the assumption  $L(\mathcal{T}'') \subseteq W_{S,k}$ , every  $\bar{a} \in L(\mathcal{T}'')$  satisfies  $\text{Comp}(\bar{a}) \subseteq S$ , and thus  $w \in S$  holds. Hence, we obtain  $L(\mathcal{T}) \subseteq S$ .

By the equivalence, we can check  $\text{REAL}(\text{DPDA}, \text{PDT})$  for  $\mathcal{A}'_f$ , which is shown to be decidable in Theorem 14, instead of checking  $\text{REAL}(\text{DRPDA}, \text{RPDT}[k])$ .

**Theorem 25.** *For all  $k \geq 0$ ,  $\text{REAL}(\text{NRPDA}, \text{RPDT}[k])$  is undecidable.*

**Proof.** We can easily reduce the  $\text{REAL}(\text{NPDA}, \text{PDT})$ , whose undecidability is proved in Theorem 15, to this problem.

## 7 Conclusion

We have discussed the realizability problem whose specification and implementation are DPDA (NPDA) and PDT in Section 4. To use the result in [36], we show  $\text{REAL}(\text{DPDA}, \text{PDT})$  is decidable. We also show the undecidability of  $\text{REAL}(\text{NPDA}, \text{PDT})$  by the reduction from the universality problem of NPDA. In Section 5, we have defined RPDT, RPDA and show the way to simulate the label of the language of RPDA by PDA. We have introduced the notions of stack-visibly [1] and test-visibly to discuss the decidability of realizability for RPDA and RPDT. We show that the behavior of registers and stack of RPDT can be simulated by the finite alphabels defined as finite actions, and prove that  $\text{REAL}(\text{DRPDA}, \text{RPDT}[k])$  can be reduced to  $\text{REAL}(\text{DPDA}, \text{PDT})$ .

It is still unknown whether the realizability problem is decidable when the specification is given by an universal PDA, DRPDA has no restriction of visibility and the number of registers of RPDT is not given. Investigating these cases are future work.

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## A Appendix

### A.1 A proof of Lemma 7

**Lemma 7.** *For a given  $m$ -DPDA  $\mathcal{A}$ , we can construct a 0-DPDA  $\mathcal{A}'$  such that  $L(\mathcal{A}) = L(\mathcal{A}')$*

**Proof.** For a given  $m$ -DPDA  $\mathcal{A}$ , we can construct an  $2m$ -DPDA  $\mathcal{A}'$  such that  $L(\mathcal{A}) = L(\mathcal{A}')$  and  $\mathcal{A}'$  has no skip rule by replacing every skip rule  $(q, a, z) \rightarrow (q', \text{skip})$  of  $\mathcal{A}$  to a pair of push and pop rules  $(q, a, z) \rightarrow (q'', \text{push}(z')), (q, \tau, z') \rightarrow (q', \text{pop})$  of  $\mathcal{A}'$  for  $a \in \Sigma \cup \{\tau\}$ . Thus, we show the lemma for  $m$ -DPDA  $\mathcal{A}$  that has no skip rule by the induction on  $m$ . The case  $m = 0$  is obvious. For an arbitrary  $m$ ,  $m$ -DPDA  $\mathcal{A} = (Q, Q_i, Q_o, q_0, z_0, \delta, c)$  over  $\Sigma_i, \Sigma_o$  and  $\Gamma$  can be converted to an  $(m-1)$ -DPDA  $\mathcal{A}'$  over  $\Sigma_i, \Sigma_o$  and  $\Gamma^2$  such that  $L(\mathcal{A}) = L(\mathcal{A}')$ . Let  $\mathcal{A}' = (Q \cup (Q \times \Gamma), Q_i \cup (Q_i \times \Gamma), Q_o \cup (Q_o \times \Gamma), (q_0, z_0), (z_0, z_0), \delta', c')$  such that  $c'(q) = c(q), c'((q, a)) = c(q)$  for all  $q \in Q, a \in \Sigma$  and

- $(q, a, z_1) \rightarrow (q', \text{pop}) \in \delta$  iff  $(q, a, (z_1, z_2)) \rightarrow ((q', z_2), \text{pop}), ((q, z_1), a, (z_c, z'_c)) \rightarrow (q', \text{skip}) \in \delta'$  for all  $z_c, z'_c \in \Gamma$ .
- $(q, a, z_1) \rightarrow (q', \text{skip}) \in \delta$  iff  $(q, a, (z_1, z_2)) \rightarrow (q', \text{skip}), ((q, z_1), a, (z_c, z'_c)) \rightarrow ((q', z_1), \text{skip}) \in \delta'$  for all  $z_c, z'_c \in \Gamma$ .
- $(q, a, z_1) \rightarrow (q', \text{push}(z')) \in \delta$  iff  $(q, a, (z_1, z_2)) \rightarrow ((q', z'), \text{skip}), ((q, z_1), a, (z_c, z'_c)) \rightarrow (q', \text{push}((z', z_1))) \in \delta'$  for all  $z_c, z'_c \in \Gamma$ .

for  $a \in \Sigma$ , and

- $(q, a, z_1) \rightarrow (q', \text{pop}), (q', b, z_2) \rightarrow (q'', \text{pop}) \in \delta$  iff  $(q, x, (z_1, z_2)) \rightarrow (q'', \text{pop}), ((q, z_1), x, (z_2, z_c)) \rightarrow ((q'', z_c), \text{pop}) \in \delta'$  for all  $z_c \in \Gamma$ .
- $(q, a, z_1) \rightarrow (q', \text{push}(z')) \in \delta', (q', b, z') \rightarrow (q'', \text{pop}) \in \delta$  iff  $(q, x, (z_1, z_c)) \rightarrow (q'', \text{skip}), ((q, z_1), x, (z_c, z'_c)) \rightarrow ((q'', z_1), \text{skip}) \in \delta'$  for all  $z_c, z'_c \in \Gamma$ .
- $(q, a, z_1) \rightarrow (q', \text{push}(z')), (q', b, z') \rightarrow (q'', \text{push}(z'')) \in \delta$  iff  $(q, x, (z_1, z_c)) \rightarrow (q'', \text{push}((z'', z'))), ((q, z_1), x, (z_c, z'_c)) \rightarrow ((q'', z''), \text{push}(z', z_1)) \in \delta'$  for all  $z_c, z'_c \in \Gamma$ .

where  $a, b \in \Sigma \cup \{\tau\}$ ,  $x = a$  if  $a \in \Sigma$  and  $x = b$  otherwise. As the definition of  $\mathcal{A}'$ , the ID  $(q, z_1 z_2 z_3 \cdots z_n)$  of  $\mathcal{A}$  corresponds to an ID  $(q, (z_1, z_2) \cdots (z_{n-1}, z_n))$  of  $\mathcal{A}'$  if  $n$  is even and an ID  $((q, z_1), (z_2, z_3) \cdots (z_{n-1}, z_n))$  if  $n$  is odd. We can check  $L(\mathcal{A}) = L(\mathcal{A}')$  by the induction on the length of a sequence  $w \in L(\mathcal{A})$ .

### A.2 A full proof of Lemma 21

**Lemma 21.**  $L_k = \{w \otimes \bar{a} \mid w \in \text{Comp}(\bar{a})\}$  is definable as the language of a  $(2, k+2)$ -DRPDA.

**Proof.** Let  $(2, k+2)$ -DRPDA  $\mathcal{A}_1 = (Q_1, Q_1^{\mathfrak{I}}, Q_1^{\circ}, q_1^0, \delta_1, c_1)$  over  $A_k^{\mathfrak{I}}, A_k^{\circ}$  and  $D$  where  $Q_1 = \{p, q\} \cup (\text{Asgn}_k \times [k] \times \text{Com}([k])) \cup [k]$ ,  $Q_1^{\mathfrak{I}} = \{p\}$ ,  $Q_1^{\circ} = Q_1 \setminus Q_1^{\mathfrak{I}}$ ,  $q_1^0 = p$ ,  $c_1(s) = 2$  for every  $s \in Q$  and  $\delta_1$  consists of rules of the form

$$(p, (a_{\mathfrak{I}}, \text{tst}), \text{tst} \cup \text{tst}') \rightarrow (q, \{k+1\}, \text{skip}) \quad (4)$$

$$(q, (a_{\circ}, \text{asgn}, j, \text{com}), \text{tst}'') \rightarrow ((\text{asgn}, j, \text{com}), \{k+2\}, \text{skip}) \quad (5)$$

$$((\text{asgn}, j, \text{com}), \tau, \{k+1\} \cup \text{tst}'') \rightarrow (j, \text{asgn}, \text{com}) \quad (6)$$

$$(j, \tau, \{j, k+2\} \cup \text{tst}'') \rightarrow (p, \emptyset, \text{skip}) \quad (7)$$

for all  $(a_{\mathfrak{I}}, \text{tst}) \in A_k^{\mathfrak{I}}$ ,  $(a_{\circ}, \text{asgn}, j, \text{com}) \in A_k^{\circ}$ ,  $\text{tst}' \subseteq \{k+1, k+2\}$  and  $\text{tst}'' \in \text{Tst}_{k+2}$ .

We show  $L(\mathcal{A}_k) = L_k$ . For this proof, we redefine compatibility for finite sequences  $w \in ((\Sigma_{\mathfrak{I}} \times D) \cdot (\Sigma_{\circ} \times D))^*$  and  $\bar{a} \in (A_k^{\mathfrak{I}} \cdot A_k^{\circ})^*$ . We show the following claim.

*Claim.* Let  $n \in \mathbb{N}_0$  and  $w \otimes \bar{a} = ((a_0^{\mathfrak{I}}, \text{skip}), d_0^{\mathfrak{I}})((a_0^{\circ}, \text{asgn}_0, j_0, \text{com}_0), d_0^{\circ}) \cdots \in ((A_k^{\mathfrak{I}} \times D) \cdot (A_k^{\circ} \times D))^*$  whose length is  $2n$  and  $\rho = (\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^*$  whose length is  $n+1$  and  $(\theta_0, u_0) = (\theta_{\perp}^k, \perp)$ . Then,  $\rho$  is a witness of the compatibility between  $w$  and  $\bar{a}$  iff  $(p, \theta'_0, u_0) \vdash^{w \otimes \bar{a}(0:1)(\tau, d_0^{\mathfrak{I}})(\tau, d_0^{\circ})} (\theta'_1, u_1) \vdash^{w \otimes \bar{a}(2:3)(\tau, d_1^{\mathfrak{I}})(\tau, d_1^{\circ})} \dots \vdash^{w \otimes \bar{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathfrak{I}})(\tau, d_{n-1}^{\circ})} (p, \theta'_n, u_n)$  where  $\theta'_i \in \Theta_{k+2}$  ( $i \in [n]$ ) satisfies  $\theta'_i(j) = \theta_i(j)$  for  $j \in [k]$ .

(Proof of the claim) We show the claim by induction on  $n$ . The case of  $n = 0$  is obvious. We show the claim for arbitrary  $n > 0$  with the induction hypothesis.

We first show left to right. By the induction hypothesis,  $(p, \theta'_0, u_0) \vdash^{w \otimes \bar{a}(0:1)(\tau, d_0^{\mathfrak{I}})(\tau, d_0^{\circ}) \cdots w \otimes \bar{a}(2n-4:2n-3)(\tau, d_{n-2}^{\mathfrak{I}})(\tau, d_{n-2}^{\circ})} (p, \theta'_{n-1}, u_{n-1})$  holds. By the assumption, because  $\rho$  is the witness, (a)  $\theta_{n-1}, d_{n-1}^{\mathfrak{I}}, u_{n-1}(0) \models \text{tst}_{n-1}$ , (b)  $\theta_n = \theta_{n-1}[\text{asgn}_{n-1} \leftarrow d_{n-1}^{\mathfrak{I}}]$ , (c)  $\theta_n(j_{n-1}) = d_{n-1}^{\circ}$  and (d)  $u_n = \text{upds}(u_{n-1}, \theta_n, \text{com}_{n-1})$  hold. By the condition (a),  $\mathcal{A}_k$  can do a transition  $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w \otimes \bar{a}(2n-2)} (q, \theta_{n-1}^1, u_{n-1})$  for unique  $\theta_{n-1}^1 \in \Theta_{k+2}$  by the rule  $(p, (a_{n-1}^{\mathfrak{I}}, \text{tst}_{n-1}), \text{tst}_{n-1} \cup \text{tst}') \rightarrow (q, \{k+1\}, \text{skip})$  of the form (4). We can also say  $(q, \theta_{n-1}^1, u_{n-1}) \vdash^{w \otimes \bar{a}(2n-1)} ((\text{asgn}_{n-1}, j_{n-1}, \text{com}_{n-1}), \theta_{n-1}^2, u_{n-1})$  by the rule of the form (5). Note that  $\theta_{n-1}^2(j) = \theta_{n-1}(j)$  if  $j \in [k]$ ,  $\theta_{n-1}^2(k+1) = d_{n-1}^{\mathfrak{I}}$  and  $\theta_{n-1}^2(k+2) = d_{n-1}^{\circ}$ .  $((\text{asgn}_{n-1}, j_{n-1}, \text{com}_{n-1}), \theta_{n-1}^2, u_{n-1}) \vdash^{(\tau, d_{n-1}^{\mathfrak{I}})} (j_{n-1}, \theta_{n-1}^3, u_n)$  is also valid transition of  $\mathcal{A}_k$  of the form (6) by the conditions (b) and (d) where  $\theta_{n-1}^3(j) = \theta_n(j)$  for  $j \in [k]$  and  $\theta_{n-1}^3(k+2) = d_{n-1}^{\circ}$ . By the condition (c),  $\theta_{n-1}^3(j_{n-1}) = \theta_{n-1}^3(k+1) = d_{n-1}^{\circ}$  holds. Thus, a transition  $(j_{n-1}, \theta_{n-1}^3, u_n) \vdash^{(\tau, d_{n-1}^{\circ})} (p, \theta'_n, u_n)$  is valid with the rule of the form (7). In conclusion,  $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w \otimes \bar{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathfrak{I}})(\tau, d_{n-1}^{\circ})} (p, \theta'_n, u_n)$  holds, and with the induction hypothesis, we obtain the left to right of the claim.

Next, we prove right to left. By the assumption,  $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w \otimes \bar{a}(2n-2:2n-1)(\tau, d_{n-1}^{\mathfrak{I}})(\tau, d_{n-1}^{\circ})} (p, \theta'_n, u_n)$  holds. By checking four transition rules that realize the above transition relation, we can obtain that  $\rho(n-1), \rho(n), w \otimes \bar{a}(2n-2)$  and  $w \otimes \bar{a}(2n-1)$  satisfies the conditions (a) to (d) described in the previous paragraph. Thus, by the induction hypothesis, we obtain  $\rho$  is a witness of the compatibility between  $w$  and  $\bar{a}$ .

(end of the proof of the claim)

By the claim,  $w \otimes \bar{a} \in L_k \Leftrightarrow$  there exists a witness  $(\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^\omega$  of  $w$  and  $\bar{a} \Leftrightarrow$  there exists a run  $(p, \theta'_0, u_0) \vdash^{w \otimes \bar{a}(0:1)(\tau, d_0^\sharp)(\tau, d_0^\circ)} (\theta'_1, u_1) \vdash^{w \otimes \bar{a}(2:3)(\tau, d_1^\sharp)(\tau, d_1^\circ)} \dots$  of  $\mathcal{A} \Leftrightarrow w \otimes \bar{a} \in L(\mathcal{A}_k)$  holds for all  $w \otimes \bar{a} \in \text{DW}(A_k^\sharp, A_k^\circ, D)$ .

### A.3 A full proof of Lemma 22

**Lemma 22.** *For a specification  $\mathcal{S}$  defined by some visibly  $k'$ -DRPDA,  $L_{\bar{\mathcal{S}},k} = \{w \otimes \bar{a} \mid w \in \text{Comp}(\bar{a}) \cap \bar{\mathcal{S}}\}$  is definable as the language of a  $(4, k+k'+4)$ -DRPDA.*

**Proof.** Let  $L_{\bar{\mathcal{S}}} = \{w \otimes \bar{a} \mid w \in \bar{\mathcal{S}}, \bar{a} \in (A_k^\sharp \cdot A_k^\circ)^\omega\}$ . We can construct a visibly  $k'$ -DRPDA  $\mathcal{A}_2 = (Q_2, Q_2^\sharp, Q_2^\circ, q_2^0, \delta_2, c_2)$  over  $A_k^\sharp, A_k^\circ$  and  $D$  such that  $L(\mathcal{A}_2) = L_{\bar{\mathcal{S}}}$ . Let  $\mathcal{A}_1$  be the  $(2, k+2)$ -DRPDA such that  $L(\mathcal{A}_1) = L_k$ , which is given in Lemma 21. Because  $L_{\bar{\mathcal{S}},k} = L_{\bar{\mathcal{S}}} \cap L_k$ , it is enough to show that we can construct a  $(4, k+k'+4)$ -DRPDA  $\mathcal{A}$  over  $A_k^\sharp, A_k^\circ$  and  $D$  such that  $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ .

To construct  $\mathcal{A}$ , we use the facts that  $c_1(q)$  is even for every  $q \in Q_1$  and  $\delta_1$  consists of several groups of three consecutive rules having the following forms:

$$(q_1, a, \text{tst}_1) \rightarrow (q_2, \text{asgn}_1, \text{skip}) \quad (5')$$

$$(q_2, \tau, \text{tst}_2) \rightarrow (q_3, \text{asgn}_2, \text{com}_1) \quad (6')$$

$$(q_3, \tau, \text{tst}_3) \rightarrow (q_4, \text{asgn}_3, \text{skip}). \quad (7')$$

Note that  $\text{vis}(a) = v(\text{com}_1)$  always holds for every consecutive rules. (5'), (6') and (7') correspond to (5), (6) and (7), respectively, and (4) is also converted to three consecutive rules like (5')-(7') by adding dummy  $\tau$  rules.

We let  $k_1 = k+2$  and  $k_2 = k'$ . We construct  $(4, k_1 + k_2 + 2)$ -DRPDA  $\mathcal{A} = (Q_\sharp \cup Q_\circ \cup \{q_0\}, Q_\sharp \cup \{q_0\}, Q_\circ, q_0, \delta, c)$  where  $Q_\sharp = Q_1^\sharp \times Q_2^\sharp \times [5]$ ,  $Q_\circ = Q_1^\circ \times Q_2^\circ \times [5]$ .  $c$  is defined as  $c(q_0) = 1$  and  $c((q_1, q_2, i)) = c_2(q_2)$  for all  $(q_1, q_2, i) \in Q$ .  $\delta$  has first  $\tau$  rule  $(q_0, \tau, [k] \cup \{\text{top}\}) \rightarrow ((q_0^1, q_0^2, 1), \text{push}(1))$ . For all rules

$$(q_1, a, \text{tst}_1) \rightarrow (q_2, \text{asgn}_1, \text{skip}) \in \delta_1 \quad (8)$$

$$(q_2, \tau, \text{tst}_2) \rightarrow (q_3, \text{asgn}_2, \text{com}_1) \in \delta_1 \quad (9)$$

$$(q_3, \tau, \text{tst}_3) \rightarrow (q_4, \text{asgn}_3, \text{skip}) \in \delta_1 \quad (10)$$

$$(q, a, \text{tst}) \rightarrow (q', \text{asgn}, \text{com}) \in \delta_2 \quad (11)$$

(note that  $v(\text{com}_1) = \text{vis}(a) = v(\text{com})$  always hold) for  $a \in A_k^\sharp \cup A_k^\circ$ , let  $\text{tst}^{+k_1} = \{i + k_1 \mid i \in \text{tst}\} \cup \{\text{top} \mid \text{top} \in \text{tst} \setminus [k_1]\}$ ,  $\text{asgn}^{+k_1} = \{i + k_1 \mid i \in \text{asgn}\}$  and  $\text{com}^{+k_1} = \text{push}(j + k_1)$  if  $\text{com} = \text{push}(j)$  and  $\text{com}^{+k_1} = \text{com}$  otherwise, then  $\delta$

consists of the rules

$$((q_1, q, 1), \tau, \mathbf{tst}[0, k_1 + k_2 + 2] \cup \{\mathbf{top}\}) \rightarrow ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathbf{pop}) \quad (12)$$

$$((q_1, q, 2), \tau, \mathbf{tst}[0, k_1 + k_2 + 2] \cup \{\mathbf{top}\}) \rightarrow ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathbf{push}(k_1 + k_2 + 1)) \quad (13)$$

$$((q_1, q, 3), a, \mathbf{tst}_1 \cup ((\mathbf{tst}^{+k_1} \setminus \{\mathbf{top}\}) \cup \mathbf{Top})) \rightarrow ((q_2, q', 4), \mathbf{asgn}_1 \cup \mathbf{asgn}^{+k_1}, \mathbf{com}^{+k_1}) \quad (14)$$

$$((q_2, q, 4), \tau, ((\mathbf{tst}_2 \setminus \{\mathbf{top}\}) \cup \mathbf{Top}') \cup \mathbf{tst}[k_1, k_1 + k_2 + 1]) \rightarrow ((q_3, q', 5), \mathbf{asgn}_2, \mathbf{com}_1) \quad (15)$$

$$((q_3, q', 5), \tau, \mathbf{tst}_3 \cup \mathbf{tst}[k_1, k_1 + k_2 + 2]) \rightarrow ((q_4, q', 1), \mathbf{asgn}_3, \mathbf{skip}) \quad (16)$$

for all  $\mathbf{tst}[i, j] \in \mathbf{Tst}_j \setminus \mathbf{Tst}_i$  (we assume  $\mathbf{Tst}_0 = \emptyset$ ) where  $\mathbf{Top} = \{k_1 + k_2 + 2\}$  ( $\mathbf{Top}' = \{k_1 + k_2 + 1\}$ ) if  $\mathbf{top} \in \mathbf{tst}$  ( $\mathbf{top} \in \mathbf{tst}_2$ ) and  $\mathbf{Top} = \emptyset$  ( $\mathbf{Top}' = \emptyset$ ) otherwise.

Fig. 3 indicates an example of transitions from  $(q_1, q, 1)$  to  $(q_4, q', 1)$  with updating contents of registers and stack. The first to  $k_1$ -th registers simulate the registers of  $\mathcal{A}_1$ ,  $(k_1 + 1)$ -th to  $(k_1 + k_2)$ -th register simulate the registers of  $\mathcal{A}_2$  and  $(k_1 + k_2 + 1)$ -th and  $(k_1 + k_2 + 2)$ -th registers is for keeping the first and second stack top contents, respectively. The stack contents of  $\mathcal{A}$  simulates that of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  by restoring contents of stacks of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  alternately.

The transition rules (12) and (13) is for restoring two stack data values from the top to  $(k_1 + k_2 + 1)$ -th and  $(k_1 + k_2 + 2)$ -th registers. The transition rule (14) is for updating states, registers and stacks by simulating the rules (8) and (11). Because the first to  $k_2$ -th registers of  $\mathcal{A}_2$  is simulated by  $(k_1 + 1)$ -th to  $(k_1 + k_2)$ -th registers of  $\mathcal{A}$ , we use  $\mathbf{tst}^{+k_1}$ ,  $\mathbf{asgn}^{+k_1}$  and  $\mathbf{com}^{+k_1}$  instead of  $\mathbf{tst}$ ,  $\mathbf{asgn}$  and  $\mathbf{com}$ , and replace  $\mathbf{top}$  in  $\mathbf{tst}^{+k_1}$  to  $k_1 + k_2 + 2$ . The transition rules (15) and (16) simulates the rules (9) and (10), respectively.

For two assignments  $\theta_1 \in \Theta_{k_1}$  and  $\theta_2 \in \Theta_{k_2}$ , let  $[\theta_1, \theta_2, d, d'] \in \Theta_{k_1+k_2+2}$  be the assignment such that  $[\theta_1, \theta_2, d, d'](i) = \theta_1(i)$  if  $i \in [k_1]$ ,  $[\theta_1, \theta_2, d, d'](i) = \theta_2(i)$  if  $k_1 + 1 \leq i \leq k_2$ ,  $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 1) = d$  and  $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 2) = d'$ . To prove  $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ , we show the following claim.

*Claim.* For all  $n \in \mathbb{N}_0$  and  $w \in ((A_k^i \cup A_k^o) \times D)^n$ , there exists sequences of transitions  $(q_0^1, \theta_0^1, u_0^1) \vdash_{\mathcal{A}_1}^{w(0)(\tau, d_0)(\tau, d'_0)} (q_1^1, \theta_1^1, u_1^1) \vdash_{\mathcal{A}_1}^{w(1)(\tau, d_1)(\tau, d'_1)} \dots \vdash_{\mathcal{A}_1}^{w(n-1)(\tau, d_{n-1})(\tau, d'_{n-1})} (q_n^1, \theta_n^1, u_n^1)$  and  $(q_0^2, \theta_0^2, u_0^2) \vdash_{\mathcal{A}_2}^{w(0)} (q_1^2, \theta_1^2, u_1^2) \vdash_{\mathcal{A}_2}^{w(1)} \dots \vdash_{\mathcal{A}_2}^{w(n-1)} (q_n^2, \theta_n^2, u_n^2)$  iff  $(q_0, \theta_{\perp}^{k_1+k_2+2}, \perp) \vdash_{\mathcal{A}}^{(\tau, \perp)} ((q_0^1, q_0^2, 1), [\theta_0^1, \theta_0^2, d_0^1, d_0^2], \langle u_0^1, u_0^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_0^1(0)(\tau, u_0^2(0))w(0)(\tau, d_0)(\tau, d'_0))} ((q_1^1, q_1^2, 1), [\theta_1^1, \theta_1^2, d_1^1, d_1^2], \langle u_1^1, u_1^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_1^1(0)(\tau, u_1^2(0))w(1)(\tau, d_1)(\tau, d'_1))} \dots \vdash_{\mathcal{A}}^{(\tau, u_{n-1}^1(0)(\tau, u_{n-1}^2(0))w(n-1)(\tau, d_{n-1})(\tau, d'_{n-1}))} ((q_n^1, q_n^2, 1), [\theta_n^1, \theta_n^2, d_n^1, d_n^2], \langle u_n^1, u_n^2 \rangle)$  holds where  $b \in \{1, 2\}$ ,  $i \in [n]$ ,  $\theta_0^b = \theta_{\perp}^{k_b}$ ,  $u_0^b = \perp$ ,  $q_i^b \in Q_b$ ,  $\theta_i^b \in \Theta_{k_b}$ ,  $u_i^b \in D^*$  and  $d_{i-1}, d'_{i-1} \in D$ .

(Proof of the claim) We show the claim by the induction on  $n$ . The case  $n = 0$  is obvious.

We first show left to right. By induction hypothesis,  $(q_0, \theta_\perp^A, \perp) \vdash_{\mathcal{A}}^{(\tau, \perp)}$   
 $((q_0^1, q_0^2, 1), [\theta_0^1, \theta_0^2, d_0^1, d_0^2], \langle u_0^1, u_0^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_0^1(0))(\tau, u_0^2(0))w(0)(\tau, d_0)(\tau, d'_0)} \dots$   
 $\vdash_{\mathcal{A}}^{(\tau, u_{n-2}^1(0))(\tau, u_{n-2}^2(0))w(n-2)(\tau, d_{n-2})(\tau, d'_{n-2})} ((q_{n-1}^1, q_{n-1}^2, 1), [\theta_{n-1}^1, \theta_{n-1}^2, d_{n-1}^1, d_{n-1}^2],$   
 $\langle u_{n-1}^1, u_{n-1}^2 \rangle)$  holds. Also, by the assumption,  $(q_{n-1}^1, \theta_{n-1}^1, u_{n-1}^1) \vdash_{\mathcal{A}_1}^{w(n-1)}$   
 $(q', \theta', u_{n-1}^1) \vdash_{\mathcal{A}_1}^{(\tau, d)} (q'', \theta'', u_n^1) \vdash_{\mathcal{A}_1}^{(\tau, d')} (q_n^1, \theta_n^1, u_n^1)$  and  $(q_{n-1}^2, \theta_{n-1}^2, u_{n-1}^2) \vdash_{\mathcal{A}_2}^{w(n-1)}$   
 $(q_n^2, \theta_n^2, u_n^2)$  for some  $q', q'' \in Q_1, \theta', \theta'' \in \Theta_{k_1}$  and  $d, d' \in D$ , and let the following  
be the rules used in these transitions.

$$(q_{n-1}, (a, v(\text{com}_1)), \text{tst}_1) \rightarrow (q', \text{asgn}_1, \text{skip}) \in \delta_1 \quad (8')$$

$$(q', \tau, \text{tst}_2) \rightarrow (q'', \text{asgn}_2, \text{com}_1) \in \delta_1 \quad (9')$$

$$(q'', \tau, \text{tst}_3) \rightarrow (q_n, \text{asgn}_3, \text{skip}) \in \delta_1 \quad (10')$$

$$(q_{n-1}, (a, v(\text{com})), \text{tst}) \rightarrow (q_n, \text{asgn}, \text{com}) \in \delta_2 \quad (11')$$

We can check the follows.

$$\begin{aligned} & ((q_{n-1}^1, q_{n-1}^2, 1), [\theta_{n-1}^1, \theta_{n-1}^2, d_{n-1}^1, d_{n-1}^2], \langle u_{n-1}^1, u_{n-1}^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_{n-1}^1(0))(\tau, u_{n-1}^2(0))} \\ & ((q_{n-1}^1, q_{n-1}^2, 3), [\theta_{n-1}^1, \theta_{n-1}^2, u_{n-1}^1(0), u_{n-1}^1(0)], \langle u_{n-1}^1, u_{n-1}^2 \rangle) \vdash_{\mathcal{A}}^{w(n-1)} \\ & ((q_{n-1}^1, q_{n-1}^2, 4), [\theta_n^1, \theta', u_{n-1}^1(0), u_{n-1}^1(0)], \langle u_{n-1}^1, u_n^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, d_{n-1})} \\ & ((q_{n-1}^1, q_{n-1}^2, 5), [\theta_n^1, \theta'', u_{n-1}^1(0), u_{n-1}^1(0)], \langle u_n^1, u_n^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, d'_{n-1})} \\ & ((q_n^1, q_n^2, 1), [\theta_n^1, \theta_n^2, u_{n-1}^1(0), u_{n-1}^1(0)], \langle u_n^1, u_n^2 \rangle) \end{aligned}$$

Thus, the right side of the claim holds. In a similar way, we can also show right to left.

(end of the proof of the claim)

By the claim,  $w \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2) \Leftrightarrow$  there exists runs  $(q_0^1, \theta_0^1, u_0^1) \vdash_{\mathcal{A}_1}^{w(0)(\tau, d_0)(\tau, d'_0)}$   
 $(q_1^1, \theta_1^1, u_1^1) \vdash_{\mathcal{A}_1}^{w(1)(\tau, d_1)(\tau, d'_1)} \dots$  and  $(q_0^2, \theta_0^2, u_0^2) \vdash_{\mathcal{A}_2}^{w(0)} (q_1^2, \theta_1^2, u_1^2) \vdash_{\mathcal{A}_2}^{w(1)}$   
 $\dots$  that satisfies the minimum number appearing in the sequence  
 $q_0^1, q_1^1, \dots$  infinitely is even.  $\Leftrightarrow$  there exists a run  $(q_0, \theta_\perp^A, \perp) \vdash_{\mathcal{A}}^{(\tau, \perp)}$   
 $((q_0^1, q_0^2, 1), [\theta_0^1, \theta_0^2, d_0^1, d_0^2], \langle u_0^1, u_0^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_0^1(0))(\tau, u_0^2(0))w(0)(\tau, d_0)(\tau, d'_0)} ((q_1^1, q_1^2, 1),$   
 $[\theta_1^1, \theta_1^2, d_1^1, d_1^2], \langle u_1^1, u_1^2 \rangle) \vdash_{\mathcal{A}}^{(\tau, u_1^1(0))(\tau, u_1^2(0))w(1)(\tau, d_1)(\tau, d'_1)} \dots$  that satisfies the min-  
imum number appearing in the sequence  $(q_0^1, q_0^2, 1), (q_0^1, q_0^2, 2) \dots, (q_1^1, q_1^2, 1), \dots$   
infinitely is even.  $\Leftrightarrow w \in L(\mathcal{A})$  holds.