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Ryoma Senda[1], Yoshiaki Takata[2], and Hiroyuki Seki[1]

[1] Graduate School of Informatics, Nagoya University,
Furo-cho, Chikusa, Nagoya 464-8601, Japan
ryoma.private@sqlab.jp, seki@i.nagoya-u.ac.jp
[2] Graduate School of Engineering, Kochi University of Technology,
Tosayamada, Kami City, Kochi 782-8502, Japan
takata.yoshiaki@kochi-tech.ac.jp

Abstract.

1 Introduction

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $[n] = \{1, \cdots, n\}$ for $n \in \mathbb{N}$. For a set A, let $\mathscr{P}(A)$ be the power set of A, let A^* and A^ω be the sets of finite and infinite words over A, respectively. We denote $A^+ = A^* \setminus \{\varepsilon\}$ and $A^\infty = A^* \cup A^\omega$. For a word $\alpha \in A^\infty$ over a set A, let $\alpha(i) \in A$ be the i-th element of α $(i \geq 0)$, $\alpha(i : j) = \alpha(i)\alpha(i+1)\cdots\alpha(j-1)\alpha(j)$ for $i \geq j$ and $\alpha(i :) = \alpha(i)\cdots$ for $i \geq 0$. Let $\langle u, w\rangle = u(0)w(0)u(1)w(1)\cdots \in A^\infty$ for words $u, w \in A^\infty$ and $\langle B, C\rangle = \{\langle u, w\rangle \mid u \in B, w \in C\}$ for sets $B, C \subseteq A^\infty$. By $|\beta|$, we mean the cardinality of β if β is a set and the length of β if β is a finite sequence.

In this paper, disjoint sets $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ denote a (finite) input alphabet, an output alphabet and a stack alphabet, respectively, and $\Sigma = \Sigma_{\mathbb{i}} \cup \Sigma_{\mathbb{o}}$. For a set Γ, let $Com(\Gamma) = \{pop, skip\} \cup \{push(z) \mid z \in \Gamma\}$ be the set of stack commands over Γ.

2.1 Transition Systems

Definition 1. *A transition system (TS) is $\mathcal{S} = (S, s_0, A, E, \to_{\mathcal{S}}, c)$ where*

- *S is a (finite or infinite) set of states,*
- *$s_0 \in S$ is the initial state,*
- *A, E is (finite or infinite) alphabets such that $A \cap E = \emptyset$,*
- *$\to_{\mathcal{S}} \subseteq S \times (A \cup E) \times S$ is a set of transition relation, written as $s \to^a s'$ if $(s, a, s') \in \to_{\mathcal{S}}$ and*
- *$c : S \to [n]$ is a coloring function where $n \in \mathbb{N}$.*

An element of A is an observable label and an element of E is an internal label. A *run* of TS $\mathcal{S} = (S, s_0, A, E, \to_{\mathcal{S}}, c)$ is a pair $(\rho, w) \in S^\omega \times (A \cup E)^\omega$ that satisfies $\rho(0) = s_0$ and $\rho(i) \to^{w(i)} \rho(i+1)$ for $i \geq 0$. Let $\min_{\text{inf}} : S^\omega \to [n]$ be a minimal coloring function such that $\min_{\inf}(\rho) = \min\{m \mid$ there exist an infinite number of $i \geq 0$ such that $c(\rho(i)) = m\}$. We call $\mathcal{S}$ deterministic if $s \to^a s_1$ and $s \to^a s_2$ implies $s_1 = s_2$ for all $s, s_1, s_2 \in S$ and $a \in A \cup E$.

For $w \in (A \cup E)^\omega$, let $ef(w) = a_0 a_1 \cdots \in A^\infty$ be the sequence obtained from w by removing all symbols belonging to E. Note that $ef(w)$ is not always an infinite sequence even if w is an infinite sequence. We define the *language* of $\mathcal{S}$ as $L(\mathcal{S}) = \{ef(w) \in A^\omega \mid$ there exists a run (ρ, w) such that $\min_{\inf}(\rho)$ is even$\}$. For $m \in \mathbb{N}_0$, we call an $\mathcal{S}$ m-TS if for every run (ρ, w) of $\mathcal{S}$, w contains no consecutive subsequence $w' \in E^{m+1}$.

3 Pushdown Transducers, Automata and Games

3.1 Pushdown Transducers

Definition 2. *A pushdown transducer (PDT) over finite alphabets Σ_{i}, Σ_{o} and Γ is $\mathcal{T} = (P, p_0, z_0, \Delta)$ where P is a finite set of states, $p_0 \in P$ is the initial state, $z_0 \in \Gamma$ is the initial stack symbol and $\Delta : P \times \Sigma_{\mathrm{i}} \times \Gamma \to P \times \Sigma_{\mathrm{o}} \times Com(\Gamma)$ is a finite set of deterministic transition rules having one of the following forms:*

- $(p, a, z) \to (q, b, pop)$ *(pop rule)*
- $(p, a, z) \to (q, b, skip)$ *(skip rule)*
- $(p, a, z) \to (q, b, push(z))$ *(push rule)*

where $p, q \in P$, $a \in \Sigma_{\mathrm{i}}$, $b \in \Sigma_{\mathrm{o}}$ and $z \in \Gamma$.

For a state $p \in P$ and a finite sequence representing stack contents $u \in \Gamma^*$, (p, u) is called a *configuration* or *instantaneous description (abbreviated as ID)* of PDT $\mathcal{T}$. Let $ID_{\mathcal{T}}$ denote the set of all IDs of $\mathcal{T}$. Let $\Rightarrow_{\mathcal{T}} \subseteq ID_{\mathcal{T}} \times \Sigma_{\mathrm{i}} \cdot \Sigma_{\mathrm{o}} \times ID_{\mathcal{T}}$ be the transition relation of $\mathcal{T}$ that satisfies the following conditions. For $u \in \Gamma^+$ and $com \in Com(\Gamma)$, let us define $upds(u, com)$ as $upds(u, com) = u(1:)$, $upds(u, skip) = u$ and $upds(u, push(z')) = z'u$.

For two IDs $(p, u), (q, u') \in ID_{\mathcal{T}}$, $a \in \Sigma_{\mathrm{i}}$ and $b \in \Sigma_{\mathrm{o}}$, $((p, u), ab, (q, u')) \in \Rightarrow_{\mathcal{T}}$, written as $(p, u) \Rightarrow^{ab}_{\mathcal{T}} (q, u')$, if there exist a rule $(p, a, z) \to (q, b, com) \in \Delta$ such that $z = u(0)$ and $u' = upds(u, com)$. If $\mathcal{T}$ is clear from the context, we abbreviate $\Rightarrow^{ab}_{\mathcal{T}}$ as $\Rightarrow^{ab}$. By definition, any ID $(p, \varepsilon) \in ID_{\mathcal{T}}$ has no successor. That is, there is no transition from an ID with empty stack. We define a run and the language $L(\mathcal{T}) \subseteq (\Sigma_{\mathrm{i}} \cdot \Sigma_{\mathrm{o}})^\omega$ of PDT $\mathcal{T}$ as those of deterministic 0-TS $(ID_{\mathcal{T}}, (q_0, z_0), \Sigma_{\mathrm{i}} \cdot \Sigma_{\mathrm{o}}, \emptyset, \Rightarrow_{\mathcal{T}}, c)$ where $c(s) = 2$ for all $s \in ID_{\mathcal{T}}$. Let **PDT** be the class of PDT.

Example 3. Let us consider PDT $\mathcal{T} = (\{p\}, p, z_0, \Delta)$ over $\{a, b\}, \{a, b\}$ and $\{z_0, z\}$ where $\Delta = \{(p, a, z_0) \to (p, b, push(z))(p, b, z_0) \to (p, a, push(z))(p, a, z) \to (p, a, push(z))(p, b, z) \to (p, b, pop)\}$ Let $(\rho, w) \in ID^\omega_{\mathcal{T}} \times \{a, b\}^\omega$ be a pair of sequences where $\rho = (p, z_0)(p, zz_0)(p, zzz_0)(p, zzzz_0) \cdots$ and $w = aba^\omega$, then (ρ, w) is a run of $\mathcal{T}$.

Let $\#_a(w), \#_b(w) \in \mathbb{N}_0$ denote the number of a, b appearing in $w \in \{a, b\}^*$, respectively. $L(\mathcal{T})$ is the set of the sequence $w \in \{a, b\}^\omega$ that satisfies one of

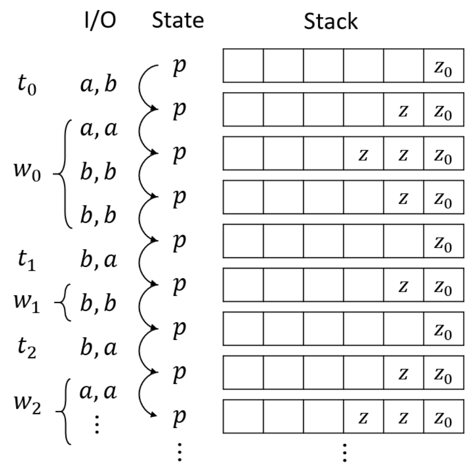

Fig. 1. An example of run for a sequence $w = t_0 w_0 t_1 w_1 \cdots$.

the following conditions: (i) $w = t_0 w_0 t_1 w_1 \cdots \in (\{ab, ba\} \times \{aa, bb\}^*)^\omega$ where for all $i \geq 0$, $t_i \in \{ab, ba\}$ and $w_i \in \{aa, bb\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$. (ii) $w = t_0 w_0 t_1 w_1 \cdots t_n w_n \in (\{ab, ba\} \times \{aa, bb\}^*)^* \cdot (\{ab, ba\} \times \{aa, bb\}^\omega)$ where for all $0 \leq i \leq n$, $t_i \in \{ab, ba\}$ and $w_i \in \{aa, bb\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$ for $0 \leq i < n$ and $w_n \in \{aa, bb\}^\omega$ such that $\#_a(w') - \#_b(w') \geq 0$ for all subsequence $w' = w(0 : m)$ of w for all $m \in \mathbb{N}_0$.

3.2 Pushdown Automata

Definition 4. *A nondeterministic pushdown automata (NPDA) over finite alphabets $\Sigma_{\mathbb{i}}$, $\Sigma_{\mathbb{o}}$ and Γ is $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, z_0, \delta, c)$ where Q, $Q_{\mathbb{i}}, Q_{\mathbb{o}}$ are finite sets of states that satisfy $Q = Q_{\mathbb{i}} \cup Q_{\mathbb{o}}$ and $Q_{\mathbb{i}} \cap Q_{\mathbb{o}} = \emptyset$, $q_0 \in Q_{\mathbb{i}}$ is the initial state, $z_0 \in \Gamma$ is the initial stack symbol, $c : Q \to [n]$ is a coloring function where $n \in \mathbb{N}$ is the number of priorities and $\delta : Q \times \Sigma \times \Gamma \to \mathscr{P}(Q \times Com(\Gamma))$ is a finite set of transition rules, having one of the following forms:*

- $(q_{\mathbb{x}}, a_{\mathbb{x}}, z) \to (q_{\overline{\mathbb{x}}}, com)$ *(input/output rules)*
- $(q_{\mathbb{x}}, \tau, z) \to (q'_{\mathbb{x}}, com)$ *(τ rules)*

where $(\mathbb{x}, \overline{\mathbb{x}}) \in \{(\mathbb{i}, \mathbb{o}), (\mathbb{o}, \mathbb{i})\}$, $q_{\mathbb{x}}, q'_{\mathbb{x}} \in Q_{\mathbb{x}}, q_{\overline{\mathbb{x}}} \in Q_{\overline{\mathbb{x}}}, a_{\mathbb{x}} \in \Sigma_{\mathbb{x}}$, $z \in \Gamma$ and $com \in Com(\Gamma)$.

We define $ID_{\mathcal{A}} = Q \times \Gamma^*$ and a transition relation $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times (\Sigma \cup \{\tau\}) \times ID_{\mathcal{A}}$ as $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$ iff there exist a rule $(p, a, z) \to (q, com) \in \delta$ and a sequence $u \in \Gamma^*$ such that $z = u(0)$ and $u' = upds(u, com)$. We write $(q, u) \vdash^a_{\mathcal{A}} (q', u')$ iff $((q, u), a, (q', u')) \in \vdash_{\mathcal{A}}$. We write $\vdash^a_{\mathcal{A}}$ as $\vdash^a$ if $\mathcal{A}$ is clear from comtext. We define a run and the language $L(\mathcal{A})$ of $\mathcal{A}$ as those of TS $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, z_0), \Sigma, \{\tau\}, \vdash_{\mathcal{A}}, c')$ where $c'((q, u)) = c(q)$ for every $(q, u) \in ID_{\mathcal{A}}$. We call a PDA $\mathcal{A}$ deterministic if $\mathcal{S}_{\mathcal{A}}$ is deterministic. We call $\mathcal{A}$ an m-NPDA (or

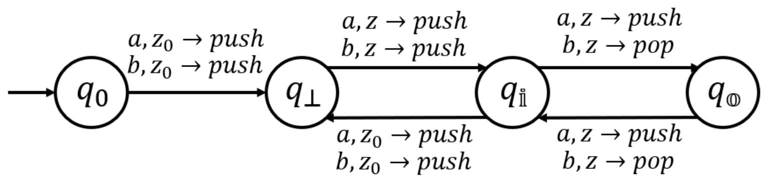

Fig. 2. States and transitions of $\mathcal{A}$. (A label $a, b \to c$ from q to q' means $(q, a, b) \to (q', c) \in \delta$.)

m-DPDA when $\mathcal{A}$ is deterministic) if $\mathcal{S}_\mathcal{A}$ is an m-TS. We abbreviate 0-NPDA (0-DPDA) as NPDA (DPDA). Let **DPDA** and **NPDA** be the classes of DPDA and NPDA, respectively.

Example 5. Let us consider DPDA $\mathcal{A} = (\{q_0, q_\perp, q_{\mathbb{I}}, q_{\mathbb{O}}\}, \{q_0, q_{\mathbb{I}}\}, \{q_\perp, q_{\mathbb{O}}\}, q_0, z_0, \delta, c)$ over $\{a, b\}, \{a, b\}$ and $\{z_0, z\}$ where $c(q_\perp) = 3$, $c(p) = 2$ for $p = q_0, q_{\mathbb{I}}, q_{\mathbb{O}}$ and δ is defined as Fig. 2, in detail, $\delta = \{(p, x, z_0) \to (q_\perp, push(z)) \mid p \in \{q_0, q_{\mathbb{I}}\}, x \in \{a, b\}\} \cup \{(q_\perp, x, z) \to (q_{\mathbb{I}}, push(z)) \mid x \in \{a, b\}\} \cup \{(p, a, z) \to (p', push(z)), (p, b, z) \to (p', pop) \mid (p, p') \in \{(q_{\mathbb{I}}, q_{\mathbb{O}}), (q_{\mathbb{O}}, q_{\mathbb{I}})\}\}$. For example, we can check the sequence $(\rho, w) \in ID_\mathcal{T}^\omega \times \{a, b\}^\omega$ where $\rho = (q_0, z_0)(q_\perp, zz_0)(q_{\mathbb{I}}, zzz_0)(q_{\mathbb{O}}, zzzz_0)(q_{\mathbb{I}}, zzzzz_0)(q_{\mathbb{O}}, zzzzzz_0)\cdots$ and $w = aba^\omega$ is a run of $\mathcal{A}$. Because $q_\perp$ appears only one times in ρ, $w \in L(\mathcal{A})$ holds.

$L(\mathcal{A})$ is the set of the sequence $w \in \{a, b\}^\omega$ that satisfies one of the following conditions: (i) $w = t_0 w_0 t_1 w_1 \cdots \in (\{ab, ba\} \times \{a, b\}^*)^\omega$ where for all $i \geq 0$, $t_i \in \{ab, ba\}$ and $w_i \in \{a, b\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$. (ii) $w = t_0 w_0 t_1 w_1 \cdots t_n w_n \in (\{ab, ba\} \times \{a, b\}^*)^* \cdot (\{ab, ba\} \times \{a, b\}^\omega)$ where for all $0 \leq i \leq n$, $t_i \in \{ab, ba\}$ and $w_i \in \{a, b\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$ for $0 \leq i < n$ and $w_n \in \{a, b\}^\omega$ such that $\#_a(w') - \#_b(w') \geq 0$ for all subsequence $w' = w(0 : m)$ of w for all $m \in \mathbb{N}_0$. To compare with PDT $\mathcal{T}$ defined in Example 16, we can check $L(\mathcal{T}) \subseteq L(\mathcal{A})$.

Lemma 6. *For a given m-DPDA $\mathcal{A}$, we can construct a 0-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$*

3.3 Pushdown Games

Definition 7. *A pushdown game of DPDA $\mathcal{A} = (Q, Q_{\mathbb{I}}, Q_{\mathbb{O}}, q_0, z_0, \delta, c)$ over $\Sigma_{\mathbb{I}}, \Sigma_{\mathbb{O}}$ and Γ is $\mathcal{G}_\mathcal{A} = (V, V_{\mathbb{I}}, V_{\mathbb{O}}, E, C)$ where $V = Q \times \Gamma^*$ is the set of vertices with $V_{\mathbb{I}} = Q_{\mathbb{I}} \times \Gamma^*, V_{\mathbb{O}} = Q_{\mathbb{O}} \times \Gamma^*$, $E \subseteq V \times V$ is the set of edges defined as $E = \{(v, v') \mid v \vdash^a v'$for some $a \in \Sigma_{\mathbb{I}} \cup \Sigma_{\mathbb{O}}\}$ and $C : V \to [n]$ is the coloring function such that $C((q, u)) = c(q)$ for all $(q, u) \in V$.*

The game starts with some $(q_0, z_0) \in V_{\mathbb{I}}$. When the current vertex is $v \in V_{\mathbb{I}}$, Player I chooses a successor $v' \in V_{\mathbb{O}}$ of v as the next vertex. When the current vertex is $v \in V_{\mathbb{O}}$, Player II chooses a successor $v' \in V_{\mathbb{I}}$ of v. Formally, a finite or infinite sequence $\rho \in V^\infty$ is *valid* if $\rho(0) = (q_0, z_0)$ and $(\rho(i-1), \rho(i)) \in E$ for every $i \geq 1$. A *play* of $\mathcal{G}_\mathcal{A}$ is an infinite and valid sequence $\rho \in V^\omega$. Let PL be the set of plays. A play $\rho \in PL$ is *winning* for Player I iff $\min\{m \in [n] \mid$ there exist an infinite number of $i \geq 0$ such that $c(\rho(i)) = m\}$ is even.

By the definition of $\mathcal{G}_\mathcal{A}$, the following lemma holds.

Lemma 8. *Let $f_1 : PL^\omega \to (Q \times Com(\Gamma))^\omega$ and $f_2 : PL^\omega \to \Sigma^\omega$ be the functions defined as follows. For every play $\rho = (q_0, u_0)(q_1, u_1)\cdots \in PL$ of $\mathcal{G}_\mathcal{A}$,*

- *$f_1(\rho) = (q_0, com_0)(q_1, com_1)\cdots \in (Q \times Com)^\omega$ where $u_{i+1} = upds(u_i, com_i)$ for all $i \geq 0$ and*
- *$f_2(\rho) = w$ where $\rho(i) \vdash^{w(i)} \rho(i+1)$ for all $i \geq 0$.*

Then, f_1 and f_2 are well-defined and both of f_1 and f_2 are injections.

[Walukiewucz, 2001] proved that we can construct a PDT $\mathcal{T}$ that gives a winning strategy of $\mathcal{G}_\mathcal{A}$, that is, $L(\mathcal{T}) = \{f_1(\rho) \mid \rho$ is winning for Player I $\}$.

Theorem 9. *[Walukiewucz, 2001] If player I has a winning strategy of $\mathcal{G}_\mathcal{A}$, we can construct a PDT $\mathcal{T}$ over $Q_{\mathbb{i}} \times Com(\Gamma), Q_{\mathbb{o}} \times Com(\Gamma)$ and a stack alphabet Γ' that gives a winning strategy of $\mathcal{G}_\mathcal{A}$. That is, $\rho \in PL$ is winning for Player I iff $f_1(\rho) \in L(T)$.*

By Lemma 8, a winning strategy can be also given as the set of sequences $w \in \Sigma^\omega$ such that the play $f_2^{-1}(w)$ is winning. Thus, we can obtain the following lemma in a similar way to Theorem 9.

Lemma 10. *If player I has a winning strategy of $\mathcal{G}_\mathcal{A}$, we can construct a PDT $\mathcal{T}$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ' that gives a winning strategy of $\mathcal{G}_\mathcal{A}$. That is, $\rho \in PL$ is winning for Player I iff $f_2(\rho) \in L(T)$.*

4 Realizability problems for PDA and PDT

For a specification S and an implementation I, we write $I \models S$ if $L(I) \subseteq L(S)$.

Definition 11. *Realizability problem* $\textsc{Real}(\mathcal{S}, \mathcal{I})$ *for a class of specifications $\mathcal{S}$ and of implementations $\mathcal{I}$: For a specification $S \in \mathcal{S}$, is there an implementation $I \in \mathcal{I}$ such that $I \models S$?*

Example 12. By Examples 16 and 5, $L(\mathcal{T}) \subseteq L(\mathcal{A})$ holds for PDT $\mathcal{T}$ and DPDA $\mathcal{A}$ defined in the examples. Thus, $\mathcal{T} \models \mathcal{A}$ holds. Let us consider $\mathcal{A}' = (\{q_0, q_\perp, q_{\mathbb{i}}, q_{\mathbb{o}}\}, \{q_0, q_{\mathbb{i}}\}, \{q_\perp, q_{\mathbb{o}}\}, q_0, z_0, \delta, c')$ which is obtained from $\mathcal{A}$ but the coloring function c' is defined as $c'(q_\perp) = 1$ and $c'(p) = 2$ for $p = q_0, q_{\mathbb{i}}, q_{\mathbb{o}}$. For example, the pair of the sequences $(\rho, w) \in ID_\mathcal{A}^\omega \times \Sigma^\omega$ in Example 5 is a run of $\mathcal{A}'$ but $w \notin L(\mathcal{A})$ because ρ has only one $q_\perp$ in its states. We can check that all sequence $w' \in (\{a\} \cdot \Sigma)^\omega$ is not in $L(\mathcal{A}')$ because the run (ρ', w') of w' never visit $q_\perp$ over one times. Then, we can check there are no PDT $\mathcal{T}$ such that $\mathcal{T} \models \mathcal{A}$ because every language $L(\mathcal{T})$ of PDT $\mathcal{T}$ always includes a word $w \in (\{a\} \cdot \Sigma)^\omega$, and thus there exists a word $w \in L(\mathcal{T}) \cap \overline{L(\mathcal{A})}$, it means $L(\mathcal{T}) \not\subseteq L(\mathcal{A})$.

Theorem 13. $\textsc{Real}(\mathbf{DPDA}, \mathbf{PDT})$ *is decidable.*

Proof. Let $\mathcal{A}$ be a given DPDA. By Lemma 10, we can construct a PDT $\mathcal{T}$ such that $w \in L(T)$ iff $f_2^{-1}(w)$ is winning play of $\mathcal{G}_\mathcal{A}$. By the definition of f_2, $\rho(i) \vdash^{w(i)} \rho(i+1)$ holds for all $i \geq 0$ where $\rho = f_2^{-1}(w)$. Then, $w \in L(\mathcal{A})$ holds, and thus $\mathcal{T} \models \mathcal{A}$. Hence, we can say $\mathcal{T} \models \mathcal{A}$ iff player I has a winning strategy for the game $\mathcal{G}_\mathcal{A}$. Because there is an algorithm for constructing $\mathcal{T}$ in [Walukiewucz, 2001], $\textsc{Real}(\mathbf{DPDA}, \mathbf{PDT})$ is decidable.

Theorem 14. $\textsc{Real}(\mathbf{NPDA}, \mathbf{PDT})$ *is undecidable.*

Proof. We prove the theorem by a reduction from the universality problem of NPDA, which is undecidable. For a given NPDA $\mathcal{A} = (Q, Q_\mathrm{i}, Q_\mathrm{o}, q_0, z_0, \delta, c)$ over $\Sigma_\mathrm{i}, \Sigma_\mathrm{o}$ and Γ, we can construct an NPDA $\mathcal{A}' = (Q \times [2], Q \times \{1\}, Q \times \{2\}, q_0, z_0, \delta', c')$ over $\Sigma'_\mathrm{i}, \Sigma'_\mathrm{o}$ and Γ where $\Sigma'_\mathrm{i} = \Sigma_\mathrm{i} \cup \Sigma_\mathrm{o}$, Σ'_o is an arbitrary (nonempty) alphabet, $c'((q,1)) = c'((q,2)) = c(q)$ for all $q \in Q$ and δ' satisfies that $((q,1),a,z) \to ((q',2), com) \in \delta$ iff $(q,a,z) \to (q', com) \in \delta'$, and $((q',2),b,z) \to ((q',1), skip) \in \delta'$ for all $b \in \Sigma'_\mathrm{o}$ and $z \in \Gamma$.

We show $L(\mathcal{A}) = (\Sigma'_\mathrm{i})^\omega$ iff there exists $\mathcal{T}$ such that $\mathcal{T} \models \mathcal{A}$. By the construction of $\mathcal{A}'$, $L(\mathcal{A}') = \langle L(\mathcal{A}), (\Sigma'_\mathrm{o})^\omega \rangle$ holds. If $L(\mathcal{A}) = (\Sigma'_\mathrm{i})^\omega$, then $L(\mathcal{A}') = \langle (\Sigma'_\mathrm{i})^\omega, (\Sigma'_\mathrm{o})^\omega \rangle$ and thus $\mathcal{T} \models \mathcal{A}$ holds for every $\mathcal{T}$. Assume that $L(\mathcal{A}) \neq (\Sigma'_\mathrm{i})^\omega$. Then, there exists a word $w \in (\Sigma'_\mathrm{i})^\omega$ such that $w \notin L(\mathcal{A})$. For any PDT $\mathcal{T}$ and any $u \in (\Sigma'_\mathrm{i})^\omega$, there is $v \in (\Sigma'_\mathrm{o})^\omega$ such that $\langle u, v \rangle \in L(\mathcal{A}')$. On the other hand, $\langle w, v \rangle \notin L(\mathcal{A}')$ holds for any $v \in (\Sigma'_\mathrm{o})^\omega$. Hence, $\mathcal{T} \not\models \mathcal{A}'$ holds for any PDT $\mathcal{T}$. This completes the reduction and the realizability problem for PDT and NPDA is undecidable.

5 Register Pushdown Transducers and Register Pushdown Automata

5.1 Data words and registers

We assume a countable set D of *data values*. For finite alphabets $\Sigma_\mathrm{i}, \Sigma_\mathrm{o}$, an infinite sequence $(a_1^\mathrm{i}, d_1)(a^\mathrm{o}, d'_1) \cdots \in ((\Sigma_\mathrm{i} \times D) \cdot (\Sigma_\mathrm{o} \times D))^\omega$ is called a *data word*. We let $DW(\Sigma_\mathrm{i}, \Sigma_\mathrm{o}, D) = ((\Sigma_\mathrm{i} \times D) \cdot (\Sigma_\mathrm{o} \times D))^\omega$.

For $k \in \mathbb{N}_0$, a mapping $\theta : [k] \to D$ is called an *assignment* (of data values to k registers). Let Θ_k denote the collection of assignments to k registers. We assume $\perp \in D$ as the initial data value and let $\theta_\perp^k \in \Theta_k$ be the initial assignment such that $\theta_\perp^k(i) = \perp$ for all $i \in [k]$.

We denote $Tst_k = \mathscr{P}([k] \cup \{top\})$ and $Asgn_k = \mathscr{P}([k])$ where $top \notin \mathbb{N}$ is a unique symbol that represents a stack top value. Tst_k is the set of guard conditions. For $tst \in Tst_k$, $\theta \in \Theta_k$ and $d, e \in D$, we denote $(\theta, d, e) \models tst$ if $(\theta(i) = d \Leftrightarrow i \in tst)$ and $(e = d \Leftrightarrow top \in tst)$ hold. In the definitions of register pushdown transducer and automaton in the next section, the data values d and e correspond to an input data value and a stack top data value, respectively. $Asgn_k$ is the set of assignment conditions. For $asgn \in Asgn_k$, $\theta, \theta' \in \Theta_k$ and $d \in D$, let $\theta[asgn \leftarrow d]$ be the assignment θ' such that $\theta'(i) = d$ for $i \in asgn$ and $\theta'(i) = \theta(i)$ for $i \notin asgn$.

5.2 Register pushdown transducers

Definition 15. *A k-register pushdown transducer (k-RPDT) over finite alphabets $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and an infinite set D of data values is $\mathcal{T} = (P, p_0, \Delta)$ where P is a finite set of states, $p_0 \in P$ is the initial state, $\Delta : P \times \Sigma_{\mathbb{i}} \times Tst_k \to P \times \Sigma_{\mathbb{o}} \times Asgn_k \times [k] \times Com([k])$ is a finite set of deterministic transition rules.*

D is used as a stack alphabet. For $u \in D^+$, $\theta' \in \Theta_k$ and $com \in Com([k])$, let us define $upds(u, \theta', com)$ as $upds(u, \theta', pop) = u(1:)$, $upds(u, \theta', skip) = u$ and $upds(u, \theta', push(j')) = \theta'(j')u$. Let $ID_{\mathcal{T}} = P \times \Theta_k \times D^*$ and $\Rightarrow_{\mathcal{T}} \subseteq ID_{\mathcal{T}} \times ((\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D)) \times ID_{\mathcal{T}}$ be the transition relation of $\mathcal{T}$ such that $((p, \theta, u), (a, d^{\mathbb{i}})(b, d^{\mathbb{o}}), (q, \theta', u')) \in \Rightarrow_{\mathcal{T}}$ iff there exists a rule $(p, a, tst) \to (q, b, asgn, j, com) \in \Delta$ that satisfies the following conditions: $(d^{\mathbb{i}}, u(0), \theta) \models tst$, $\theta' = \theta[asgn \leftarrow d^{\mathbb{i}}]$, $\theta'(j) = d^{\mathbb{o}}$ and $u' = upds(u, \theta', com)$, and we write $(p, \theta, u) \Rightarrow_{\mathcal{T}}^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})} (q, \theta', u')$. If $\mathcal{T}$ is clear from the context, we abbreviate $\Rightarrow_{\mathcal{T}}^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})}$ as $\Rightarrow^{(a,d^{\mathbb{i}})(b,d^{\mathbb{o}})}$.

A run and the language $L(\mathcal{T})$ of $\mathcal{T}$ are those of deterministic 0-TS $(ID_{\mathcal{T}}, (q_0, \theta_{\perp}^k, \perp), (\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D), \emptyset, \Rightarrow_{\mathcal{T}}, c)$ where $c(s) = 2$ for all $s \in ID_{\mathcal{T}}$. Let $\mathbf{RPDT}[k]$ be the class of k-RPDT and $\mathbf{RPDT} = \bigcup_{k \in \mathbb{N}_0} \mathbf{RPDT}[k]$.

Example 16. Let us consider 1-RPDT $\mathcal{T} = (\{p, p'\}, p, z_0, \Delta)$ over $\{a\}, \{b\}$ and $\{z_0, z\}$ where $\Delta = \{(p, a, tst) \to (p', b, \{1\}, 1, push(1))(p', a, \{\}) \to (p, b, \{1\}, 1, push(1))(p', a, \{1\}) \to (p, b, \{1\}, 1, push(1))(p, a, \{top\}) \to (p', b, \{1\}, 1, pop)(p, a, \{1, top\}) \to (p', b, \{1\}, 1, pop)\}$ Let $(\rho, w) \in ID_{\mathcal{T}}^{\omega} \times \{a, b\}^{\omega}$ be a pair of sequences where $\rho = (p, z_0)(p, zz_0)(p, zzz_0)(p, zzzz_0)\cdots$ and $w = aba^{\omega}$, then (ρ, w) is a run of $\mathcal{T}$.

Let $\#_a(w), \#_b(w) \in \mathbb{N}_0$ denote the number of a, b appearing in $w \in \{a, b\}^*$, respectively. $L(\mathcal{T})$ is the set of the sequence $w \in \{a, b\}^{\omega}$ that satisfies one of the following conditions: (i) $w = t_0 w_0 t_1 w_1 \cdots \in (\{ab, ba\} \times \{aa, bb\}^*)^{\omega}$ where for all $i \geq 0$, $t_i \in \{ab, ba\}$ and $w_i \in \{aa, bb\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$. (ii) $w = t_0 w_0 t_1 w_1 \cdots t_n w_n \in (\{ab, ba\} \times \{aa, bb\}^*)^* \cdot (\{ab, ba\} \times \{aa, bb\}^{\omega})$ where for all $0 \leq i \leq n$, $t_i \in \{ab, ba\}$ and $w_i \in \{aa, bb\}^*$ such that $\#_b(w_i) - \#_a(w_i) = 2$ for $0 \leq i < n$ and $w_n \in \{aa, bb\}^{\omega}$ such that $\#_a(w') - \#_b(w') \geq 0$ for all subsequence $w' = w(0:m)$ of w for all $m \in \mathbb{N}_0$.

5.3 Register pushdown automata

Definition 17. *A nondeterministic k-register pushdown automaton (k-NRPDA) over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and D is $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, \delta, c)$, where*

- *Q is a finite set of states,*
- *$Q_{\mathbb{i}} \cup Q_{\mathbb{o}} = Q, Q_{\mathbb{i}} \cap Q_{\mathbb{o}} = \emptyset$,*
- *$q_0 \in Q$ is the initial state, and*
- *$\delta : Q \times (\Sigma \cup \{\tau\}) \times Tst_k \to \mathscr{P}(Q \times Asgn_k \times Com([k]))$ is a transition function having one of the forms:*
  - *$(q_{\mathbb{x}}, a_{\mathbb{x}}, tst) \to (q_{\overline{\mathbb{x}}}, asgn, com)$ (input/output rule)*

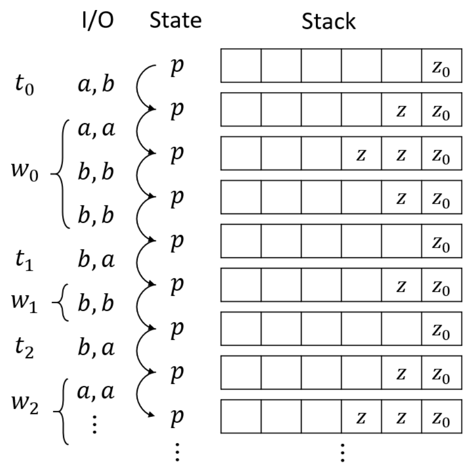

Fig. 3. An example of run for a sequence $w = t_0 w_0 t_1 w_1 \cdots$.

- $(q_{\mathbb{x}}, \tau, tst) \to (q'_{\mathbb{x}}, asgn, com)$ *(τ rule)*
  where $(\mathbb{x}, \overline{\mathbb{x}}) \in \{(\mathbb{i}, \mathbb{o}), (\mathbb{o}, \mathbb{i})\}$, $q_{\mathbb{x}}, q'_{\mathbb{x}} \in Q_{\mathbb{x}}, q_{\overline{\mathbb{x}}} \in Q_{\overline{\mathbb{x}}}, a_{\mathbb{x}} \in \Sigma_{\mathbb{x}}$, $tst \in Tst_k$, $asgn \in Asgn_k$ *and* $com \in Com([k])$.

– $c : Q \to [n]$ *where* $n \in \mathbb{N}$ *is the number of priorities.*

Let $ID_{\mathcal{A}} = Q \times \Theta_k \times D^*$. We define the transition relation $\vdash_{\mathcal{A}} \subseteq ID_{\mathcal{A}} \times ((\Sigma \cup \{\tau\}) \times D) \times ID_{\mathcal{A}}$ as $((q, \theta, u), (a, d), (q', \theta', u')) \in \vdash_{\mathcal{A}}$, written as $(q, \theta, u) \vdash^{(a,d)} (q', \theta', u')$, iff there exists a rule $(p, a, tst) \to (q, asgn, com) \in \delta$ such that $(d, u(0), \theta) \models tst$, $\theta' = \theta[asgn \leftarrow d]$ and $u' = upds(u, \theta', com)$. We write $\vdash_{\mathcal{A}}^{(a,d)}$ as $\vdash^{(a,d)}$ if $\mathcal{A}$ is clear from the context. For $s, s' \in ID_{\mathcal{A}}$ and $w \in ((\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D))^m$, we write $s \vdash^w s'$ if there exists $\rho \in ID_{\mathcal{A}}^{m+1}$ such that $\rho(0) = s, \rho(m) = s'$, and $\rho(0) \vdash^{w(0)} \cdots \vdash^{w(m-1)} \rho(m)$.

A run and the language $L(\mathcal{A})$ of k-DRPDA $\mathcal{A}$ are those of TS $\mathcal{S}_{\mathcal{A}} = (ID_{\mathcal{A}}, (q_0, \theta_{\perp}^k, \perp), \Sigma \times D, \{\tau\} \times D, \Rightarrow_{\mathcal{A}}, c')$ where $c'((q, \theta, u)) = c(q)$ for all $(q, \theta, u) \in ID_{\mathcal{A}}$. We call an $\mathcal{A}$ deterministic, or k-DRPDA, if $\mathcal{S}_{\mathcal{A}}$ is deterministic. We call an $\mathcal{A}$ (m, k)-NRPDA (or an (m, k)-DRPDA when $\mathcal{A}$ is deterministic) if $\mathcal{S}_{\mathcal{A}}$ is an m-TS. We abbreviate $(0, k)$-NRPDA ($(0, k)$-DPDA) as k-NRPDA (k-DRPDA).

5.4 Visibly RPDA

Let **DRPDA** and **NRPDA** be the unions of k-DRPDA and k-NRPDA for all $k \in \mathbb{N}_0$, respectively. Let $Com_v = \{pop, skip, push\}$ and $v : Com([k]) \to Com_v$ be the function such that $v(push(j)) = push$ for $j \in [k]$ and $v(com) = com$ otherwise. We say that an (m, k)-RPDA $\mathcal{A}$ visibly manipulates its stack (or a *visibly* RPDA) if its input and output alphabets are $\Sigma_{\mathbb{i}} \times Com_v$ and $\Sigma_{\mathbb{o}} \times Com_v$, respectively, and every rule of $\mathcal{A}$ has the form $(q, (a, v(com)), tst) \to (q', asgn, com)$. Let **DRPDAv** be the union of visibly k-DRPDA for all $k \in \mathbb{N}_0$, respectively.

6 Realizability problems for RPDA and RPDT

6.1 Finite actions

For $k \in \mathbb{N}_0$, we define the set of finite input actions as $A_k^{\mathbb{i}} = \Sigma_{\mathbb{i}} \times Tst_k$ and the set of finite output actions as $A_k^{\mathbb{o}} = \Sigma_{\mathbb{o}} \times Asgn_k \times [k] \times Com([k])$. A sequence $w = ((a_0^{\mathbb{i}}, skip), d_0^{\mathbb{i}})((a_0^{\mathbb{o}}, v(com_0)), d_0^{\mathbb{o}}) \cdots \in DW(\Sigma_{\mathbb{i}} \times Com_v, \Sigma_{\mathbb{o}} \times Com_v, D)$ is *compatible* with a sequence $\overline{a} = (a_0^{\mathbb{i}}, tst_0)(a_0^{\mathbb{o}}, asgn_0, j_0, com_0) \cdots \in (A_k^{\mathbb{i}} \cdot A_k^{\mathbb{o}})^\omega$ iff there exists a sequence $(\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^\omega$, called a *witness*, such that $\theta_0 = \theta_\perp^k$, $u_0 = \perp$, $(\theta_i, d_i^{\mathbb{i}}, u_i(0)) \models tst_i, \theta_{i+1} = \theta_i[asgn_i \leftarrow d_i^{\mathbb{i}}], \theta_{i+1}(j_i) = d_i^{\mathbb{o}}$ and $u_{i+1} = upds(u_i, \theta_{i+1}, com_i)$. Let $Comp(\overline{a}) = \{w \in DW(\Sigma_{\mathbb{i}} \times Com_v, \Sigma_{\mathbb{o}} \times Com_v, D) \mid w \text{ is compatible with } \overline{a}\ \}$. For a specification $S \subseteq DW(\Sigma_{\mathbb{i}} \times Com_v, \Sigma_{\mathbb{o}} \times Com_v, D)$, we define $W_{S,k} = \{\overline{a} \mid Comp(\overline{a}) \subseteq S\}$.

Theorem 18. *For a specification $S \subseteq DW(\Sigma_{\mathbb{i}} \times Com_v, \Sigma_{\mathbb{o}} \times Com_v, D)$, the following statements are equivalent.*

- *There exists a k-RPDT $\mathcal{T}$ such that $L(\mathcal{T}) \subseteq S$.*
- *There exists a PDT $\mathcal{T}'$ such that $L(\mathcal{T}') \subseteq W_{S,k}$.*

For a data word $w \in DW(\Sigma_{\mathbb{i}} \times Com_v, \Sigma_{\mathbb{o}} \times Com_v, D)$ and a sequence $\overline{a} \in (A_k^{\mathbb{i}} \cdot A_k^{\mathbb{o}})^\omega$ such that for each $i \geq 0$, there exists $(a, comv) \in \Sigma \times Com_v$ and we can write $w(i) = ((a, skip), d)$ and $\overline{a}(i) = (a, tst)$ if i is even and $w(i) = ((a, v(com)), d)$ and $\overline{a}(i) = (a, asgn, j, com)$ if i is odd, we define $w \otimes \overline{a} \in DW(A_k^{\mathbb{i}} \times Com_v, A_k^{\mathbb{o}} \times Com_v, D)$ as $w \otimes \overline{a}(i) = ((\overline{a}(i), comv), d)$ where $w(i) = ((a, comv), d)$.

6.2 Decidability and undecidability of realizability problems

Lemma 19. *$L_k = \{w \otimes \overline{a} \mid w \in Comp(\overline{a})\}$ is definable as the language of a $(2, k+2)$-DRPDA.*

Proof sketch. Let $(2, k+2)$-DRPDA $\mathcal{A}_k = (Q, Q_{\mathbb{i}}, Q \setminus Q_{\mathbb{i}}, p, \delta_k, c_k)$ over $A_k^{\mathbb{i}}, A_k^{\mathbb{o}}$ and D where $Q = \{p, q\} \cup (Asgn_k \times [k] \times Com([k])) \cup [k]$, $Q_{\mathbb{i}} = \{p\}$ and $c_k(s) = 2$ for every $s \in Q$ and δ_k consists of rules of the form

$$(p, ((a_{\mathbb{i}}, tst), skip), tst \cup tst') \to (q, \{k+1\}, skip) \tag{1}$$

$$(q, ((a_{\mathbb{o}}, asgn, j, com), v(com)), tst'') \to ((asgn, j, com), \{k+2\}, skip) \tag{2}$$

$$((asgn, j, com), \tau, \{k+1\} \cup tst'') \to (j, asgn, com) \tag{3}$$

$$(j, \tau, \{j, k+2\} \cup tst'') \to (p, \emptyset, skip) \tag{4}$$

for all $(a_{\mathbb{i}}, tst) \in A_k^{\mathbb{i}}$, $(a_{\mathbb{o}}, asgn, j, com) \in A_k^{\mathbb{o}}$, $tst' \subseteq \{k+1, k+2\}$ and $tst'' \in Tst_{k+2}$. We can show $L(\mathcal{A}_k) = L_k$.

Lemma 20. *For a specification $\mathcal{S}$ defined by some visibly k'-DRPDA, $L_{\overline{S},k} = \{w \otimes \overline{a} \mid w \in Comp(\overline{a}) \cap \overline{S}\}$ is definable as the language of a $(4, k+k'+4)$-DRPDA.*

Proof. Let $L_{\overline{S}} = \{w \otimes \overline{a} \mid w \in \overline{S}\}$. We can construct visibly $\mathcal{A}_{\overline{S}}$ be a k'-DRPDA such that $L(\mathcal{A}_{\overline{S}}) = L_{\overline{S}}$. Let $\mathcal{A}_k$ be the $(2, k+2)$-DRPDA such that $L(\mathcal{A}_k) = L_k$, which is given in Lemma 19. Because $L_{\overline{S},k} = L_k \cap L_{\overline{S}}$ and both L_k and $L_{\overline{S}}$ are visibly DRPDA, it is enough to show that we can construct a visibly $(4, k+k'+4)$-DRPDA $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{A}_{\overline{S}}) \cap L(\mathcal{A}_k)$.

We can convert $\mathcal{A}_k$ to a $(2, k_1)$-DRPDA $\mathcal{A}_1 = (Q_1, Q_1^{\mathbb{i}}, Q_1^{\mathbb{o}}, q_1^0, \delta_1, c_1)$ over $A^{\mathbb{i}}_{k_1+k_2+2} \times \mathit{Com}_v, A^{\mathbb{o}}_{k_1+k_2+2} \times \mathit{Com}_v$ and $\mathcal{A}_{\overline{S}}$ to a visibly k_2-DRPDA $\mathcal{A}_2 = (Q_2, Q_2^{\mathbb{i}}, Q_2^{\mathbb{o}}, q_2^0, \delta_2, c_2)$ over $A^{\mathbb{i}}_{k_1+k_2+2} \times \mathit{Com}_v, A^{\mathbb{o}}_{k_1+k_2+2} \times \mathit{Com}_v$ where $c_1(q)$ is even for every $q \in Q_1$ and every rule in δ_1 consists of several groups of three consecutive rules having the following forms:

$$(q_1, (a, v(\mathit{com}_1)), \mathit{tst}_1) \to (q_2, \mathit{asgn}_1, \mathit{skip}) \tag{15'}$$
$$(q_2, \tau, \mathit{tst}_2) \to (q_3, \mathit{asgn}_2, \mathit{com}_1) \tag{16'}$$
$$(q_3, \tau, \mathit{tst}_3) \to (q_4, \mathit{asgn}_3, \mathit{skip}) \quad . \tag{17'}$$

Note that $(15'), (16')$ and $(17')$ correspond to $(15), (16)$ and (17), respectively, and (14) can be treated as the triple sequencial rules by adding meaningless τ rules.

We construct $(4, k_1+k_2+2)$-DRPDA $\mathcal{A} = (Q_1 \times Q_2 \times [5], Q_1^{\mathbb{i}} \times Q_2^{\mathbb{i}} \times [5], Q_1^{\mathbb{o}} \times Q_2^{\mathbb{o}} \times [5], (q_0^1, q_0^2, 1), \delta, c)$ where $c((q_1, q_2, i)) = c_2(q_2)$ for all $(q_1, q_2, i) \in Q$. For all rules

$$(q_1, (a, v(\mathit{com}_1)), \mathit{tst}_1) \to (q_2, \mathit{asgn}_1, \mathit{skip}) \in \delta_1 \tag{5}$$
$$(q_2, \tau, \mathit{tst}_2) \to (q_3, \mathit{asgn}_2, \mathit{com}_1) \in \delta_1 \tag{6}$$
$$(q_3, \tau, \mathit{tst}_3) \to (q_4, \mathit{asgn}_3, \mathit{skip}) \in \delta_1 \tag{7}$$
$$(q, (a, v(\mathit{com})), \mathit{tst}) \to (q', \mathit{asgn}, \mathit{com}) \in \delta_2 \tag{8}$$

$(v(\mathit{com}_1) = v(\mathit{com}))$ for $a \in A^{\mathbb{i}}_{k_1+k_2+2} \cup A^{\mathbb{o}}_{k_1+k_2+2}$, let $\mathit{tst}^{+k_1} = \{i + k_1 \mid i \in \mathit{tst}\} \cup \{\mathit{top} \mid \mathit{top} \in \mathit{tst} \setminus [k_1]\}$, $\mathit{asgn}^{+k_1} = \{i + k_1 \mid i \in \mathit{asgn}\}$ and $\mathit{com}^{+k_1} = \mathit{push}(j + k_1)$ if $\mathit{com} = \mathit{push}(j)$ and $\mathit{com}^{+k_1} = \mathit{com}$ otherwise, then δ consists of the rules

$$((q_1, q, 1), \tau, \mathit{tst}' \cup \{\mathit{top}\}) \;\to\; ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathit{pop}) \tag{9}$$
$$((q_1, q, 2), \tau, \mathit{tst}' \cup \{\mathit{top}\}) \;\to\; ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathit{push}(k_1 + k_2 + 1)) \tag{10}$$
$$\begin{aligned}((q_1, q, 3), (a, v(\mathit{com}_1)),\ &\mathit{tst}_1 \cup ((\mathit{tst}^{+k_1} \setminus \{\mathit{top}\}) \cup \mathit{Top})) \\ &\to\; ((q_2, q', 4), \mathit{asgn}_1 \cup \mathit{asgn}^{+k_1}, \mathit{com}^{+k_1})\end{aligned} \tag{11}$$
$$((q_2, q, 4), \tau, \mathit{tst}_2 \cup \mathit{tst}') \;\to\; ((q_3, q', 5), \mathit{asgn}_2, \mathit{com}_1) \tag{12}$$
$$((q_3, q', 5), \tau, \mathit{tst}_3 \cup \mathit{tst}') \;\to\; ((q_4, q', 1), \mathit{asgn}_3, \mathit{skip}) \tag{13}$$

for all $\mathit{tst}' \in \mathit{Tst}_{k_1+k_2+2}$ where $\mathit{Top} = \{k_1 + k_2 + 2\}$ if $\mathit{top} \in \mathit{tst}$ and $\mathit{Top} = \emptyset$ otherwise. We can show $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.

Lemma 21. $W_{S,k} = \overline{\mathit{Lab}(L_{\overline{S},k})}$.

Proof. For every $\overline{a} \in (A^{\mathbb{i}}_k A^{\mathbb{o}}_k)^\omega$, $\overline{a} \notin W_{S,k} \Leftrightarrow \mathit{Comp}(\overline{a}) \not\subseteq S \Leftrightarrow \exists w.w \in \mathit{Comp}(\overline{a}) \cap \overline{S} \Leftrightarrow \exists w.w \otimes \overline{a} \in L_{\overline{S},k} \Leftrightarrow \overline{a} \in \mathit{Lab}(L_{\overline{S},k})$. Thus, $W_{S,k} = \overline{\mathit{Lab}(L_{\overline{S},k})}$ holds.

Theorem 22. *For all $k \geq 0$,* $\textsc{Real}(\mathbf{DRPDAv}, \mathbf{RPDT}[k])$ *is decidable.*

Proof. By Lemma 20, $L_{\overline{S},k}$ is definable by some visibly DRPDA. Because every language recognized by a visibly DRPDA can be converted to the language of visibly DPDA by taking the projection on its label, $W_{S,k}$ is definable by some visibly DPDA by Lemma 21. By Theorem 18, we can check $\textsc{Real}(\mathbf{DPDA}, \mathbf{PDT})$ for $W_{S,k}$, which is shown to be decidable in Theorem 13, instead of checking $\textsc{Real}(\mathbf{DRPDAv}, \mathbf{RPDT}[k])$.

Theorem 23. *For all $k \geq 0$,* $\textsc{Real}(\mathbf{NRPDA}, \mathbf{RPDT}[k])$ *is undecidable.*

Proof. We can easily reduce the problem from $\textsc{Real}(\mathbf{NPDA}, \mathbf{PDT})$, whose undecidability is proved in Theorem 14.

7 Conclusion

References

A Appendix

A.1 A full proof of Lemma 6

Lemma 6. *For a given m-DPDA $\mathcal{A}$, we can construct a 0-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$*

Proof. For a given m-DPDA $\mathcal{A}$, we can construct an $2m$-DPDA $\mathcal{A}'$ such that $L(\mathcal{A}) = L(\mathcal{A}')$ and $\mathcal{A}'$ has no skip rule by replacing every skip rule $(q, a, z) \to (q', \mathit{skip})$ of $\mathcal{A}$ to a pair of push and pop rules $(q, a, z) \to (q'', \mathit{push}(z')), (q, \tau, z') \to (q', \mathit{pop})$ of $\mathcal{A}'$ for $a \in \Sigma \cup \{\tau\}$. Thus, we show the lemma for m-DPDA $\mathcal{A}$ that has no skip rule by the induction on m. The case $m = 0$ is obvious. For an arbitrary m, m-DPDA $\mathcal{A} = (Q, Q_{\mathbb{i}}, Q_{\mathbb{o}}, q_0, z_0, \delta, c)$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and Γ can be converted to an $(m-1)$-DPDA $\mathcal{A}'$ over $\Sigma_{\mathbb{i}}, \Sigma_{\mathbb{o}}$ and $\Gamma \cup \Gamma^2$ such that $L(\mathcal{A}) = L(\mathcal{A}')$. Let $\mathcal{A}' = (Q \times (\Gamma \cup \{\bot\}), Q_{\mathbb{i}} \times (\Gamma \cup \{\bot\}), Q_{\mathbb{o}} \times (\Gamma \cup \{\bot\}), (q_0, \bot), (\bot, z_0), \delta', c')$ such that

- $(q, a, z_1) \to (q', \mathit{pop}), (q', \tau, z_2) \to (q'', \mathit{pop}) \in \delta$ iff $((q, \bot), a, (z_1, z_2)) \to ((q'', \bot), \mathit{pop})$, $((q, z_c), a, (z_c, z_1)) \to ((q'', z_2), \mathit{pop}) \in \delta'$ for all $z_c \in \Gamma$.
- $(q, a, z_1) \to (q', \mathit{push}(z')) \in \delta', (q', \tau, z') \to (q'', \mathit{pop}) \in \delta$ or $(q, \tau, z_1) \to (q', \mathit{push}(z')) \in \delta', (q', a, z') \to (q'', \mathit{pop}) \in \delta$ iff $((q, \bot), a, (z_1, z_2)) \to ((q'', \bot), \mathit{skip})$, $((q, z_c), a, (z_c, z_1)) \to ((q'', z_c), \mathit{skip}) \in \delta'$ for all $z_c, z_2 \in \Gamma$.
- $(q, \tau, z_1) \to (q', \mathit{push}(z')), (q', a, z') \to (q'', \mathit{push}(z'')) \in \delta$ iff $((q, \bot), a, (z_1, z_2)) \to ((q'', \bot), \mathit{push}((z'', z')))$, $((q, z_c), a, (z_c, z_1)) \to ((q'', z''), \mathit{pop}) \in \delta'$ for all $z_c \in \Gamma$.

A.2 A full proof of Lemma 19

Lemma 19. *$L_k = \{w \otimes \overline{a} \mid w \in \mathit{Comp}(\overline{a})\}$ is definable as the language of a $(2, k+2)$-DRPDA.*

Proof. Let $(2, k+2)$-DRPDA $\mathcal{A}_k = (Q, Q_{\mathbb{i}}, Q \setminus Q_{\mathbb{i}}, p, \delta_k, c_k)$ over $A_k^{\mathbb{i}}, A_k^{\mathbb{o}}$ and D where $Q = \{p, q\} \cup (\mathit{Asgn}_k \times [k] \times \mathit{Com}([k])) \cup [k]$, $Q_{\mathbb{i}} = \{p\}$ and $c_k(s) = 2$ for every $s \in Q$ and δ_k consists of rules of the form

$$(p, ((a_{\mathbb{i}}, \mathit{tst}), \mathit{skip}), \mathit{tst} \cup \mathit{tst}') \to (q, \{k+1\}, \mathit{skip}) \tag{14}$$

$$(q, ((a_{\mathbb{o}}, \mathit{asgn}, j, \mathit{com}), v(\mathit{com})), \mathit{tst}'') \to ((\mathit{asgn}, j, \mathit{com}), \{k+2\}, \mathit{skip}) \tag{15}$$

$$((\mathit{asgn}, j, \mathit{com}), \tau, \{k+1\} \cup \mathit{tst}'') \to (j, \mathit{asgn}, \mathit{com}) \tag{16}$$

$$(j, \tau, \{j, k+2\} \cup \mathit{tst}'') \to (p, \emptyset, \mathit{skip}) \tag{17}$$

for all $(a_{\mathbb{i}}, \mathit{tst}) \in A_k^{\mathbb{i}}$, $(a_{\mathbb{o}}, \mathit{asgn}, j, \mathit{com}) \in A_k^{\mathbb{o}}$, $\mathit{tst}' \subseteq \{k+1, k+2\}$ and $\mathit{tst}'' \in \mathit{Tst}_{k+2}$.

We show $L(\mathcal{A}_k) = L_k$. For this proof, we redefine compatibility for finite sequences $w \in ((\Sigma_{\mathbb{i}} \times D) \cdot (\Sigma_{\mathbb{o}} \times D))^*$ and $\overline{a} \in (A_k^{\mathbb{i}} \cdot A_k^{\mathbb{o}})^*$. We show the following claim.

Claim. Assume $n \in \mathbb{N}_0$ and let $w \otimes \overline{a} = (((a_0^{\mathbb{i}}, \mathit{tst}_0), \mathit{skip}), d_0^{\mathbb{i}})(((a_0^{\mathbb{o}}, \mathit{asgn}_0, j_0, \mathit{com}_0), v(\mathit{com}_0)), d_0^{\mathbb{o}}) \cdots \in ((A_k^{\mathbb{i}} \times D) \cdot$

$(A^{\mathsf{o}}_k \times D))^*$ whose length is $2n$ and $\rho = (\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^*$ whose length is $n+1$ and $(\theta_0, u_0) = (\theta^k_\perp, \perp)$. Then, ρ is a witness of the compatibility between w and $\overline{a}$ iff $(p, \theta'_0, u_0) \vdash^{w\otimes\overline{a}(0:1)(\tau,d^{\mathsf{i}}_0)(\tau,d^{\mathsf{o}}_0)} (\theta'_1, u_1) \vdash^{w\otimes\overline{a}(2:3)(\tau,d^{\mathsf{i}}_1)(\tau,d^{\mathsf{o}}_1)} \cdots \vdash^{w\otimes\overline{a}(2n-2:2n-1)(\tau,d^{\mathsf{i}}_{n-1})(\tau,d^{\mathsf{o}}_{n-1})} (p, \theta'_n, u_n)$ where $\theta'_i \in \Theta_{k+2}$ ($i \in [n]$) satisfies $\theta'_i(j) = \theta_i(j)$ for $j \in [k]$.

(Proof of the claim) We show the claim by induction on n. The case of $n = 0$ is obvious. We show the claim for arbitrary $n > 0$ with the induction hypothesis.

We first show left to right. By the induction hypothesis, $(p, \theta'_0, u_0) \vdash^{w\otimes\overline{a}(0:1)(\tau,d^{\mathsf{i}}_0)(\tau,d^{\mathsf{o}}_0)\cdots w\otimes\overline{a}(2n-4:2n-3)(\tau,d^{\mathsf{i}}_{n-2})(\tau,d^{\mathsf{o}}_{n-2})} (p, \theta'_{n-1}, u_{n-1})$ holds. By the assumption, because ρ is the witness, (a) $\theta_{n-1}, d^{\mathsf{i}}_{n-1}, u_{n-1}(0) \models tst_{n-1}$, (b) $\theta_n = \theta_{n-1}[asgn_{n-1} \leftarrow d^{\mathsf{i}}_{n-1}]$, (c) $\theta_n(j_{n-1}) = d^{\mathsf{o}}_{n-1}$ and (d) $u_n = upds(u_{n-1}, \theta_n, com_{n-1})$. By the condition (a), $\mathcal{A}_k$ can do a transition $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w\otimes\overline{a}(2n-2)} (q, \theta^1_{n-1}, u_{n-1})$ for unique $\theta^1_{n-1} \in \Theta_{k+2}$ by the rule $(p, ((a^{\mathsf{i}}_{n-1}, tst_{n-1}), skip), tst_{n-1} \cup tst') \to (q, \{k+1\}, skip)$ of the form (14). We can also say $(q, \theta^1_{n-1}, u_{n-1}) \vdash^{w\otimes\overline{a}(2n-1)} ((asgn_{n-1}, j_{n-1}, com_{n-1}), \theta^2_{n-1}, u_{n-1})$ by the rule of the form (15). Note that $\theta^2_{n-1}(j) = \theta_{n-1}(j)$ if $j \in [k]$, $\theta^2_{n-1}(k+1) = d^{\mathsf{i}}_{n-1}$ and $\theta^2_{n-1}(k+2) = d^{\mathsf{o}}_{n-1}$. $((asgn_{n-1}, j_{n-1}, com_{n-1}), \theta^2_{n-1}, u_{n-1}) \vdash^{(\tau,d^{\mathsf{i}}_{n-1})} (j_{n-1}, \theta^3_{n-1}, u_n)$ is also valid transition of $\mathcal{A}_k$ of the form (16) by the conditions (b) and (d) where $\theta^{n-1}_3(j) = \theta_n(j)$ for $j \in [k]$ and $\theta^3_{n-1}(k+2) = d^{\mathsf{o}}_{n-1}$. By the condition (c), $\theta^{n-1}_3(j_{n-1}) = \theta^{n-1}_3(k+1) = d^{\mathsf{o}}_{n-1}$ holds. Thus, a transition $(j_{n-1}, \theta^3_{n-1}, u_n) \vdash^{(\tau,d^{\mathsf{o}}_{n-1})} (p, \theta'_n, u_n)$ is valid with the rule of the form (17). In conclusion, $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w\otimes\overline{a}(2n-2:2n-1)(\tau,d^{\mathsf{i}}_{n-1})(\tau,d^{\mathsf{o}}_{n-1})} (p, \theta'_n, u_n)$ holds, and with the induction hypothesis, we obtain the left to right of the claim.

Next, we prove right to left. By the assumption, $(p, \theta'_{n-1}, u_{n-1}) \vdash^{w\otimes\overline{a}(2n-2:2n-1)(\tau,d^{\mathsf{i}}_{n-1})(\tau,d^{\mathsf{o}}_{n-1})} (p, \theta'_n, u_n)$ holds. By checking four transition rules that realize the above transition relation, we can obtain that $\rho(n-1), \rho(n), w \otimes \overline{a}(2n-2)$ and $w \otimes \overline{a}(2n-1)$ satisfies the conditions (a) to (d) described in the previous paragraph. Thus, by the induction hypothesis, we obtain ρ is a witness of the compatibility between w and $\overline{a}$.

(end of the proof of the claim)

By the claim, $w \otimes \overline{a} \in L_k \Leftrightarrow$ there exists a witness $(\theta_0, u_0)(\theta_1, u_1) \cdots \in (\Theta_k \times D^*)^\omega$ of w and $\overline{a}$ $\Leftrightarrow$ there exists a run $(p, \theta'_0, u_0) \vdash^{w\otimes\overline{a}(0:1)(\tau,d^{\mathsf{i}}_0)(\tau,d^{\mathsf{o}}_0)} (\theta'_1, u_1) \vdash^{w\otimes\overline{a}(2:3)(\tau,d^{\mathsf{i}}_1)(\tau,d^{\mathsf{o}}_1)} \cdots$ of $\mathcal{A}$ $\Leftrightarrow$ $w \otimes \overline{a} \in L(\mathcal{A}_k)$ holds for all $w \otimes \overline{a} \in DW(A^{\mathsf{i}}_k, A^{\mathsf{o}}_k, D)$.

A.3 A full proof of Lemma 20

Lemma 20. *For a specification $\mathcal{S}$ defined by some visibly k'-DRPDA, $L_{\overline{S},k} = \{w \otimes \overline{a} \mid w \in Comp(\overline{a}) \cap \overline{S}\}$ is definable as the language of a $(4, k+k'+4)$-DRPDA.*

Proof. Let $L_{\overline{S}} = \{w \otimes \overline{a} \mid w \in \overline{S}\}$ and let $\mathcal{A}_{\overline{S}}$ be a visibly k'-DRPDA such that $L(\mathcal{A}_{\overline{S}}) = L_{\overline{S}}$ and $\mathcal{A}_k$ be the $(2, k+2)$-DRPDA such that $L(\mathcal{A}_k) = L_k$, which is given in Lemma 19. Because $L_{\overline{S},k} = L_k \cap L_{\overline{S}}$ and both L_k and $L_{\overline{S}}$ are visibly DRPDA, it is enough to show that we can construct a visibly $(4, k+k'+4)$-DRPDA $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{A}_{\overline{S}}) \cap L(\mathcal{A}_k)$.

We can convert $\mathcal{A}_k$ to a $(2, k_1)$-DRPDA $\mathcal{A}_1 = (Q_1, Q_1^{\mathrm{i}}, Q_1^{\mathrm{c}}, q_1^0, \delta_1, c_1)$ over $A^{\mathrm{i}}_{k_1+k_2+2} \times \mathit{Com}_v, A^{\mathrm{c}}_{k_1+k_2+2} \times \mathit{Com}_v$ and $\mathcal{A}_{\overline{S}}$ to a visibly k_2-DRPDA $\mathcal{A}_2 = (Q_2, Q_2^{\mathrm{i}}, Q_2^{\mathrm{c}}, q_2^0, \delta_2, c_2)$ over $A^{\mathrm{i}}_{k_1+k_2+2} \times \mathit{Com}_v, A^{\mathrm{c}}_{k_1+k_2+2} \times \mathit{Com}_v$ where $c_1(q)$ is even for every $q \in Q_1$ and every rule in δ_1 consists of several groups of three consecutive rules having the following forms:

$$(q_1, (a, v(\mathit{com}_1)), \mathit{tst}_1) \to (q_2, \mathit{asgn}_1, \mathit{skip}) \tag{15'}$$
$$(q_2, \tau, \mathit{tst}_2) \to (q_3, \mathit{asgn}_2, \mathit{com}_1) \tag{16'}$$
$$(q_3, \tau, \mathit{tst}_3) \to (q_4, \mathit{asgn}_3, \mathit{skip}) \quad . \tag{17'}$$

Note that $(15'), (16')$ and $(17')$ correspond to $(15), (16)$ and (17), respectively, and (14) can be treated as the triple sequencial rules by adding meaningless τ rules.

We construct $(4, k_1+k_2+2)$-DRPDA $\mathcal{A} = (Q_1 \times Q_2 \times [5], Q_1^{\mathrm{i}} \times Q_2^{\mathrm{i}} \times [5], Q_1^{\mathrm{c}} \times Q_2^{\mathrm{c}} \times [5], (q_0^1, q_0^2, 1), \delta, c)$ where $c((q_1, q_2, i)) = c_2(q_2)$ for all $(q_1, q_2, i) \in Q$. For all rules

$$(q_1, (a, v(\mathit{com}_1)), \mathit{tst}_1) \to (q_2, \mathit{asgn}_1, \mathit{skip}) \in \delta_1 \tag{18}$$
$$(q_2, \tau, \mathit{tst}_2) \to (q_3, \mathit{asgn}_2, \mathit{com}_1) \in \delta_1 \tag{19}$$
$$(q_3, \tau, \mathit{tst}_3) \to (q_4, \mathit{asgn}_3, \mathit{skip}) \in \delta_1 \tag{20}$$
$$(q, (a, v(\mathit{com})), \mathit{tst}) \to (q', \mathit{asgn}, \mathit{com}) \in \delta_2 \tag{21}$$

$(v(\mathit{com}_1) = v(\mathit{com}))$ for $a \in A^{\mathrm{i}}_{k_1+k_2+2} \cup A^{\mathrm{c}}_{k_1+k_2+2}$, let $\mathit{tst}^{+k_1} = \{i + k_1 \mid i \in \mathit{tst}\} \cup \{\mathit{top} \mid \mathit{top} \in \mathit{tst} \setminus [k_1]\}$, $\mathit{asgn}^{+k_1} = \{i + k_1 \mid i \in \mathit{asgn}\}$ and $\mathit{com}^{+k_1} = \mathit{push}(j + k_1)$ if $\mathit{com} = \mathit{push}(j)$ and $\mathit{com}^{+k_1} = \mathit{com}$ otherwise, then δ consists of the rules

$$((q_1, q, 1), \tau, \mathit{tst}' \cup \{\mathit{top}\}) \;\to\; ((q_1, q, 2), \{k_1 + k_2 + 1\}, \mathit{pop}) \tag{22}$$
$$((q_1, q, 2), \tau, \mathit{tst}' \cup \{\mathit{top}\}) \;\to\; ((q_1, q, 3), \{k_1 + k_2 + 2\}, \mathit{push}(k_1 + k_2 + 1)) \tag{23}$$
$$\begin{aligned}((q_1, q, 3), (a, v(\mathit{com}_1)),\ &\mathit{tst}_1 \cup ((\mathit{tst}^{+k_1} \setminus \{\mathit{top}\}) \cup \mathit{Top})) \\ &\to\; ((q_2, q', 4), \mathit{asgn}_1 \cup \mathit{asgn}^{+k_1}, \mathit{com}^{+k_1})\end{aligned} \tag{24}$$
$$((q_2, q, 4), \tau, \mathit{tst}_2 \cup \mathit{tst}') \;\to\; ((q_3, q', 5), \mathit{asgn}_2, \mathit{com}_1) \tag{25}$$
$$((q_3, q', 5), \tau, \mathit{tst}_3 \cup \mathit{tst}') \;\to\; ((q_4, q', 1), \mathit{asgn}_3, \mathit{skip}) \tag{26}$$

for all $\mathit{tst}' \in \mathit{Tst}_{k_1+k_2+2}$ where $\mathit{Top} = \{k_1 + k_2 + 2\}$ if $\mathit{top} \in \mathit{tst}$ and $\mathit{Top} = \emptyset$ otherwise.

For two assignments $\theta_1 \in \Theta_{k_1}$ and $\theta_2 \in \Theta_{k_2}$, let $[\theta_1, \theta_2, d, d'] \in \Theta_{k_1+k_2+2}$ be the assignment such that $[\theta_1, \theta_2, d, d'](i) = \theta_1(i)$ if $i \in [k_1]$, $[\theta_1, \theta_2, d, d'](i) = \theta_2(i)$ if $k_1 + 1 \le 1 \le k_2$, $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 1) = d$ and $[\theta_1, \theta_2, d, d'](k_1 + k_2 + 2) = d'$. To prove $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$, we show the following claim.

Claim. For all $n \in \mathbb{N}_0$ and $w \in ((A^{\mathrm{i}}_{k_1} \cup A^{\mathrm{c}}_{k_1}) \times D)^n$, there exists sequences of transitions $(q_0^1, \theta_0^1, u_0^1) \vdash_{\mathcal{A}_1}^{w(0)(\tau, d_0)(\tau, d_0')}$

$(q_1^1,\theta_1^1,u_1^1) \vdash_{\mathcal{A}_1}^{w(1)(\tau,d_1)(\tau,d_1')} \cdots \vdash_{\mathcal{A}_1}^{w(n-1)(\tau,d_{n-1})(\tau,d_{n-1}')} (q_n^1,\theta_n^1,u_n^1)$ and $(q_0^2,\theta_0^2,u_0^2) \vdash_{\mathcal{A}_2}^{w(0)} (q_1^2,\theta_1^2,u_1^2) \vdash_{\mathcal{A}_2}^{w(1)} \cdots \vdash_{\mathcal{A}_2}^{w(n-1)} (q_n^2,\theta_n^2,u_n^2)$ iff $(q_0,\theta_\perp^{\mathcal{A}},\perp) \vdash_{\mathcal{A}}^{(\tau,\perp)} ((q_0^1,q_0^2,1),[\theta_0^1,\theta_0^2,d_0^1,d_0^2],\langle u_0^1,u_0^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_0^1(0))(\tau,u_0^2(0))w(0)(\tau,d_0)(\tau,d_0')} ((q_1^1,q_1^2,1),[\theta_1^1,\theta_1^2,d_1^1,d_1^2],\langle u_1^1,u_1^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_1^1(0))(\tau,u_1^2(0))w(1)(\tau,d_1)(\tau,d_1')} \cdots \vdash_{\mathcal{A}}^{(\tau,u_{n-1}^1(0))(\tau,u_{n-1}^2(0))w(n-1)(\tau,d_{n-1})(\tau,d_{n-1}')} ((q_n^1,q_n^2,1),[\theta_n^1,\theta_n^2,d_n^1,d_n^2],\langle u_n^1,u_n^2\rangle)$ holds where $b\in\{1,2\}$, $i\in[n]$, $\theta_0^b=\theta_\perp^{k_b}$, $u_0^b=\perp$, $q_i^b\in Q_b$, $\theta_i^b\in\Theta_{k_b}$, $u_i^b\in D^*$ and $d_{i-1},d_{i-1}'\in D$.

(Proof of the claim) We show the claim by the induction on n. The case n_0 is obvious.

We first show left to right. By induction hypothesis, $(q_0,\theta_\perp^{\mathcal{A}},\perp) \vdash_{\mathcal{A}}^{(\tau,\perp)} ((q_0^1,q_0^2,1),[\theta_0^1,\theta_0^2,d_0^1,d_0^2],\langle u_0^1,u_0^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_0^1(0))(\tau,u_0^2(0))w(0)(\tau,d_0)(\tau,d_0')} \cdots \vdash_{\mathcal{A}}^{(\tau,u_{n-2}^1(0))(\tau,u_{n-2}^2(0))w(n-2)(\tau,d_{n-2})(\tau,d_{n-2}')} ((q_{n-1}^1,q_{n-1}^2,1),[\theta_{n-1}^1,\theta_{n-1}^2,d_{n-1}^1,d_{n-1}^2],\langle u_{n-1}^1,u_{n-1}^2\rangle)$ holds. Also, by the assumption, $(q_{n-1}^1,\theta_{n-1}^1,u_{n-1}^1) \vdash_{\mathcal{A}_1}^{w(n-1)} (q',\theta',u_{n-1}^1) \vdash_{\mathcal{A}_1}^{(\tau,d)} (q'',\theta'',u_n^1) \vdash_{\mathcal{A}_1}^{(\tau,d')} (q_n^1,\theta_n^1,u_n^1)$ and $(q_{n-1}^2,\theta_{n-1}^2,u_{n-1}^2) \vdash_{\mathcal{A}_2}^{w(n-1)} (q_n^2,\theta_n^2,u_n^2)$ for some $q',q''\in Q_1,\theta',\theta''\in\Theta_{k_1}$ and $d,d'\in D$, and let the following be the rules used in these transitions.

$$(q_{n-1},(a,v(com_1)),tst_1) \to (q',asgn_1,skip)\in\delta_1 \tag{18'}$$
$$(q',\tau,tst_2) \to (q'',asgn_2,com_1)\in\delta_1 \tag{19'}$$
$$(q'',\tau,tst_3) \to (q_n,asgn_3,skip)\in\delta_1 \tag{20'}$$
$$(q_{n-1},(a,v(com)),tst) \to (q_n,asgn,com)\in\delta_2 \tag{21'}$$

We can check $((q_{n-1}^1,q_{n-1}^2,1),[\theta_{n-1}^1,\theta_{n-1}^2,d_{n-1}^1,d_{n-1}^2],\langle u_{n-1}^1,u_{n-1}^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_{n-1}^1(0))(\tau,u_{n-1}^2(0))} ((q_{n-1}^1,q_{n-1}^2,3),[\theta_{n-1}^1,\theta_{n-1}^2,u_{n-1}^1(0),u_{n-1}^1(0)],\langle u_{n-1}^1,u_{n-1}^2\rangle) \vdash_{\mathcal{A}}^{w(n-1)} ((q_{n-1}^1,q_{n-1}^2,4),[\theta_n^1,\theta',u_{n-1}^1(0),u_{n-1}^1(0)],\langle u_{n-1}^1,u_n^2\rangle) \vdash_{\mathcal{A}}^{(\tau,d_{n-1})} ((q_{n-1}^1,q_{n-1}^2,5),[\theta_n^1,\theta'',u_{n-1}^1(0),u_{n-1}^1(0)],\langle u_n^1,u_n^2\rangle) \vdash_{\mathcal{A}}^{(\tau,d_{n-1}')} ((q_n^1,q_n^2,1),[\theta_n^1,\theta_n^2,u_{n-1}^1(0),u_{n-1}^1(0)],\langle u_n^1,u_n^2\rangle)$, and thus the right side of the claim holds. In a similar way, we can also show right to left.

(end of the proof of the claim)

By the claim, $w\in L(\mathcal{A}_1)\cap L(\mathcal{A}_2) \Leftrightarrow$ there exists runs $(q_0^1,\theta_0^1,u_0^1) \vdash_{\mathcal{A}_1}^{w(0)(\tau,d_0)(\tau,d_0')} (q_1^1,\theta_1^1,u_1^1) \vdash_{\mathcal{A}_1}^{w(1)(\tau,d_1)(\tau,d_1')} \cdots$ and $(q_0^2,\theta_0^2,u_0^2) \vdash_{\mathcal{A}_2}^{w(0)} (q_1^2,\theta_1^2,u_1^2) \vdash_{\mathcal{A}_2}^{w(1)} \cdots$ that satisfies the minimum number appearing in the sequence $q_0^1,q_1^1,\cdots$ infinitely is even. $\Leftrightarrow$ there exists a run $(q_0,\theta_\perp^{\mathcal{A}},\perp) \vdash_{\mathcal{A}}^{(\tau,\perp)} ((q_0^1,q_0^2,1),[\theta_0^1,\theta_0^2,d_0^1,d_0^2],\langle u_0^1,u_0^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_0^1(0))(\tau,u_0^2(0))w(0)(\tau,d_0)(\tau,d_0')} ((q_1^1,q_1^2,1),[\theta_1^1,\theta_1^2,d_1^1,d_1^2],\langle u_1^1,u_1^2\rangle) \vdash_{\mathcal{A}}^{(\tau,u_1^1(0))(\tau,u_1^2(0))w(1)(\tau,d_1)(\tau,d_1')} \cdots$ that satisfies the minimum number appearing in the sequence

$(q_0^1, q_0^2, 1), (q_0^1, q_0^2, 2) \cdots, (q_1^1, q_1^2, 1), \cdots$ infinitely is even. $\Leftrightarrow w \in L(\mathcal{A})$ holds for all $w \in DW(A^{\mathrm{i}}_{k_1}, A^{\Phi}_{k_1}, D)$.