相対性理論 レポート No.11

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Q80.

共変微分は計量性から

$$\begin{aligned}\nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} + X^\alpha{}_{\lambda\mu} g_{\alpha\nu} + X^\beta{}_{\lambda\nu} g_{\mu\beta} \\ &= \partial_\lambda g_{\mu\nu} + X_{\nu\lambda\mu} + X_{\mu\lambda\nu} = 0 \\ \therefore\ \partial_\lambda g_{\mu\nu} &= -X_{\nu\lambda\mu} - X_{\mu\lambda\nu}\end{aligned} \tag{1}$$

同様に

$$\begin{aligned}\nabla_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} + X^\alpha{}_{\mu\nu} g_{\alpha\lambda} + X^\beta{}_{\mu\lambda} g_{\nu\beta} \\ &= \partial_\mu g_{\nu\lambda} + X_{\lambda\mu\nu} + X_{\nu\mu\lambda} = 0 \\ \therefore\ \partial_\mu g_{\nu\lambda} &= -X_{\lambda\mu\nu} - X_{\nu\mu\lambda}\end{aligned} \tag{2}$$

$$\begin{aligned}\nabla_\nu g_{\lambda\mu} &= \partial_\nu g_{\lambda\mu} + X^\alpha{}_{\nu\lambda} g_{\alpha\mu} + X^\beta{}_{\nu\mu} g_{\lambda\beta} \\ &= \partial_\nu g_{\lambda\mu} + X_{\mu\nu\lambda} + X_{\lambda\nu\mu} = 0 \\ \therefore\ \partial_\nu g_{\lambda\mu} &= -X_{\mu\nu\lambda} - X_{\lambda\nu\mu} = 0\end{aligned} \tag{3}$$

ここで $X_{\alpha\beta\gamma} = X_{\alpha\gamma\beta}$ を用いて $(2)+(3)-(1)$ を計算すると

$$\begin{aligned}&\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \\ =&X_{\nu\lambda\mu} + X_{\mu\lambda\nu} - X_{\lambda\mu\nu} - X_{\nu\mu\lambda} - X_{\mu\nu\lambda} - X_{\lambda\nu\mu} \\ =&-2X_{\lambda\mu\nu} \\ \therefore\ X_{\lambda\mu\nu} =& X^\alpha{}_{\mu\nu} g_{\alpha\lambda} = -\frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\end{aligned} \tag{4}$$

(4) の両辺に $g^{\lambda\rho}$ を掛けると $g_{\alpha\lambda} g^{\lambda\rho} = \delta^\rho{}_\alpha$ より

$$X^\rho{}_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \tag{5}$$

添字を $\rho \to \mu$, $\mu \to \rho$, $\nu \to \sigma$, $\lambda \to \nu$ と置き直せば

$$\begin{aligned}X^\mu{}_{\rho\sigma} &= -\frac{1}{2} g^{\nu\mu}(\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma}) \\ &= -\Gamma^\mu{}_{\rho\sigma}\end{aligned} \tag{6}$$

を得る.