

## 相対性理論 レポート No.9

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Q70.(3)

等価原理から局所 Lorentz 系での質点の運動方程式は

$$\frac{d^2 X^\mu}{d\tau^2} = 0 \quad (1)$$

であった. 連鎖律から

$$\frac{d}{d\tau} X^\mu = \frac{dx^\rho}{d\tau} \frac{\partial X^\mu}{\partial x^\rho} \quad (2)$$

同様に

$$\begin{aligned} 0 &= \frac{d^2 X^\mu}{d\tau^2} = \frac{d}{d\tau} \left( \frac{dx^\rho}{d\tau} \frac{\partial X^\mu}{\partial x^\rho} \right) \\ &= \frac{d^2 x^\rho}{d\tau^2} \frac{\partial X^\mu}{\partial x^\rho} + \frac{dx^\nu}{d\tau} \left( \frac{d}{d\tau} \frac{\partial X^\mu}{\partial x^\nu} \right) \end{aligned} \quad (3)$$

ここで連鎖律から

$$\frac{d}{d\tau} = \frac{dx^\sigma}{d\tau} \frac{\partial}{\partial x^\sigma} \quad (4)$$

より

$$\begin{aligned} 0 &= \frac{d^2 X^\mu}{d\tau^2} = \frac{d^2 x^\rho}{d\tau^2} \frac{\partial X^\mu}{\partial x^\rho} + \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} \frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \\ &= \ddot{x}^\rho \frac{\partial X^\mu}{\partial x^\rho} + \dot{x}^\nu \dot{x}^\sigma \frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \end{aligned} \quad (5)$$

ここで両辺に  $\partial x^\rho / \partial X^\mu$  を掛けると

$$0 = \ddot{x}^\rho + \frac{\partial x^\rho}{\partial X^\mu} \dot{x}^\nu \dot{x}^\sigma \frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \quad (6)$$

ここで  $\rho \rightarrow \mu, \mu \rightarrow \nu, \nu \rightarrow \rho$  と置き換えれば

$$0 = \ddot{x}^\mu + \frac{\partial x^\mu}{\partial X^\nu} \frac{\partial X^\nu}{\partial x^\rho \partial x^\sigma} \dot{x}^\rho \dot{x}^\sigma \quad (7)$$

を得る.

Q71.(2)

$\partial_\sigma g_{\nu\rho}$  は

$$\begin{aligned}\frac{\partial}{\partial x_\sigma} g_{\nu\rho} &= \frac{\partial}{\partial x_\sigma} \eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho} \\ &= \eta_{\alpha\beta} \left( \frac{\partial X^\alpha}{\partial x^\sigma \partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho} + \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \right)\end{aligned}\tag{8}$$

より

$$\begin{aligned}\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} &= \eta_{\alpha\beta} \left( \cancel{\frac{\partial X^\alpha}{\partial x^\sigma \partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho}} + \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \right) \\ &\quad + \eta_{\alpha\beta} \left( \frac{\partial X^\alpha}{\partial x^\rho \partial x^\sigma} \frac{\partial X^\beta}{\partial x^\nu} + \cancel{\frac{\partial X^\alpha}{\partial x^\sigma} \frac{\partial X^\beta}{\partial x^\rho \partial x^\nu}} \right) \\ &\quad + \eta_{\alpha\beta} \left( \cancel{\frac{\partial X^\alpha}{\partial x^\nu \partial x^\rho} \frac{\partial X^\beta}{\partial x^\sigma}} + \cancel{\frac{\partial X^\alpha}{\partial x^\rho} \frac{\partial X^\beta}{\partial x^\nu \partial x^\sigma}} \right) \\ &= \eta_{\alpha\beta} \left( \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} + \frac{\partial X^\alpha}{\partial x^\rho \partial x^\sigma} \frac{\partial X^\beta}{\partial x^\nu} \right) \\ &= 2\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\ &= 2\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\ &= 2g_{\nu\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho}\end{aligned}\tag{9}$$

したがって

$$\begin{aligned}\Gamma^\mu_{\rho\sigma} &= \frac{1}{2} g^{\mu\nu} (\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma}) \\ &= g^{\mu\nu} g_{\nu\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\ &= \delta^\mu_\gamma \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\ &= \frac{\partial x^\mu}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho}\end{aligned}\tag{10}$$

となる.