相対性理論 レポート No.9

佐々木良輔

Q70.(3)

等価原理から局所 Lorentz 系での質点の運動方程式は

$$\frac{d^2X^\mu}{d\tau^2} = 0 \tag{1}$$

であった. 連鎖律から

$$\frac{d}{d\tau}X^\mu = \frac{dx^\rho}{d\tau}\frac{\partial X^\mu}{\partial x^\rho} \tag{2}$$

同様に

$$\begin{aligned} 0 = \frac{d^2X^\mu}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{dx^\rho}{d\tau}\frac{\partial X^\mu}{\partial x^\rho}\right) \\ &= \frac{d^2x^\rho}{d\tau^2}\frac{\partial X^\mu}{\partial x^\rho} + \frac{dx^\nu}{d\tau}\left(\frac{d}{d\tau}\frac{\partial X^\mu}{\partial x^\nu}\right) \end{aligned} \tag{3}$$

ここで連鎖律から

$$\frac{d}{d\tau} = \frac{dx^\sigma}{d\tau}\frac{\partial}{\partial x^\sigma} \tag{4}$$

より

$$\begin{aligned} 0 = \frac{d^2X^\mu}{d\tau^2} &= \frac{d^2x^\rho}{d\tau^2}\frac{\partial X^\mu}{\partial x^\rho} + \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \\ &= \ddot{x}^\rho\frac{\partial X^\mu}{\partial x^\rho} + \dot{x}^\nu\dot{x}^\sigma\frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \end{aligned} \tag{5}$$

ここで両辺に $\partial x^\rho/\partial X^\mu$ を掛けると

$$0 = \ddot{x^\rho} + \frac{\partial x^\rho}{\partial X^\mu}\dot{x}^\nu\dot{x}^\sigma\frac{\partial X^\mu}{\partial x^\nu \partial x^\sigma} \tag{6}$$

ここで $\rho \to \mu$, $\mu \to \nu$, $\nu \to \rho$ と置き換えれば

$$0 = \ddot{x}^\mu + \frac{\partial x^\mu}{\partial X^\nu}\frac{\partial X^\nu}{\partial x^\rho \partial x^\sigma}\dot{x}^\rho\dot{x}^\sigma \tag{7}$$

を得る.

Q71.(2)

$\partial_\sigma g_{\nu\rho}$ は

$$
\begin{aligned}
\frac{\partial}{\partial x_\sigma} g_{\nu\rho} &= \frac{\partial}{\partial x_\sigma} \eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho} \\
&= \eta_{\alpha\beta} \left(\frac{\partial X^\alpha}{\partial x^\sigma \partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho} + \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \right)
\end{aligned}
\tag{8}
$$

より

$$
\begin{aligned}
\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma} &= \eta_{\alpha\beta} \left(\cancel{\frac{\partial X^\alpha}{\partial x^\sigma \partial x^\nu} \frac{\partial X^\beta}{\partial x^\rho}} + \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \right) \\
&\quad + \eta_{\alpha\beta} \left(\frac{\partial X^\alpha}{\partial x^\rho \partial x^\sigma} \frac{\partial X^\beta}{\partial x^\nu} + \cancel{\frac{\partial X^\alpha}{\partial x^\sigma} \frac{\partial X^\beta}{\partial x^\rho \partial x^\nu}} \right) \\
&\quad - \eta_{\alpha\beta} \left(\cancel{\frac{\partial X^\alpha}{\partial x^\nu \partial x^\rho} \frac{\partial X^\beta}{\partial x^\sigma}} + \cancel{\frac{\partial X^\alpha}{\partial x^\rho} \frac{\partial X^\beta}{\partial x^\nu \partial x^\sigma}} \right) \\
&= \eta_{\alpha\beta} \left(\frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} + \frac{\partial X^\alpha}{\partial x^\rho \partial x^\sigma} \frac{\partial X^\beta}{\partial x^\nu} \right) \\
&= 2\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\
&= 2\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\nu} \frac{\partial X^\beta}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\
&= 2 g_{\nu\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho}
\end{aligned}
\tag{9}
$$

したがって

$$
\begin{aligned}
\Gamma^\mu{}_{\rho\sigma} &= \frac{1}{2} g^{\mu\nu} \left(\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma} \right) \\
&= g^{\mu\nu} g_{\nu\gamma} \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\
&= \delta^\mu{}_\gamma \frac{\partial x^\gamma}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho} \\
&= \frac{\partial x^\mu}{\partial X^\beta} \frac{\partial X^\beta}{\partial x^\sigma \partial x^\rho}
\end{aligned}
\tag{10}
$$

となる.