レーザー物理学 レポート No.6

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問 11

(2.30) 式の左辺は

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{d}{dt}\left(|\psi\rangle\langle\psi|\right) \\ &= \left(\frac{d}{dt}|\psi\rangle\right)\langle\psi| + |\psi\rangle\frac{d}{dt}\langle\psi|\end{aligned} \tag{11.1}$$

ここで Schrödinger 方程式から

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle \tag{11.2}$$

$$\frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi|H \tag{11.3}$$

を用いて (11.1) は

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{i}{\hbar}H|\psi\rangle\langle\psi| + \frac{i}{\hbar}|\psi\rangle\langle\psi|H \\ &= -\frac{i}{\hbar}H\rho + \frac{i}{\hbar}\rho H \\ &= \frac{i}{\hbar}[\rho, H]\end{aligned} \tag{11.4}$$

となる.

問 12

$\mu_{12}E_0^*/\hbar = \Omega$ とすると

$$\begin{pmatrix} \dot{\rho_{11}} \\ \dot{\rho_{22}} \\ \dot{\rho_{21}} \\ \dot{\rho_{12}} \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 0 & 0 & \Omega e^{i\delta t} & -\Omega^* e^{-i\delta t} \\ 0 & 0 & -\Omega e^{i\delta t} & \Omega^* e^{-i\delta t} \\ \Omega^* e^{-i\delta t} & -\Omega^* e^{-i\delta t} & 0 & 0 \\ -\Omega e^{i\delta t} & \Omega e^{i\delta t} & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{21} \\ \rho_{12} \end{pmatrix} \tag{11.5}$$

この係数行列について, 固有値を求める

$$\begin{aligned} \begin{vmatrix} -\lambda & 0 & \Omega e^{i\delta t} & -\Omega^* e^{-i\delta t} \\ 0 & -\lambda & -\Omega e^{i\delta t} & \Omega^* e^{-i\delta t} \\ \Omega^* e^{-i\delta t} & -\Omega^* e^{-i\delta t} & -\lambda & 0 \\ -\Omega e^{i\delta t} & \Omega e^{i\delta t} & 0 & -\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} -\lambda & -\Omega e^{i\delta t} & \Omega^* e^{-i\delta t} \\ -\Omega^* e^{-i\delta t} & -\lambda & 0 \\ \Omega e^{i\delta t} & 0 & -\lambda \end{vmatrix} \\ &+ \Omega^* e^{-i\delta t} \begin{vmatrix} 0 & \Omega e^{i\delta t} & -\Omega^* e^{-i\delta t} \\ -\lambda & -\Omega e^{i\delta t} & \Omega^* e^{-i\delta t} \\ \Omega e^{i\delta t} & 0 & -\lambda \end{vmatrix} \\ &+ \Omega e^{i\delta t} \begin{vmatrix} 0 & \Omega e^{i\delta t} & -\Omega^* e^{-i\delta t} \\ -\lambda & -\Omega e^{i\delta t} & \Omega^* e^{-i\delta t} \\ -\Omega^* e^{-i\delta t} & -\lambda & 0 \end{vmatrix} \\ &= \lambda^2(\lambda^2 - 4|\Omega|^2) \end{aligned} \tag{11.6}$$

よって固有値は $\lambda = 0, \pm 2|\Omega|$ である.

問 13

$[\rho, H]$ を計算する.

$$
\begin{aligned}
[\rho, H] &= \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} - \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \\
&= \begin{pmatrix} \rho_{12}H_{21} - \rho_{21}H_{12} & \rho_{11}H_{12} + \rho_{12}H_{22} - H_{11}\rho_{12} - H_{12}\rho_{22} \\ \rho_{21}H_{11} + \rho_{22}H_{21} - H_{21}\rho_{11} - H_{22}\rho_{21} & \rho_{21}H_{12} - H_{21}\rho_{12} \end{pmatrix}
\end{aligned} \tag{13.1}
$$

したがって Schrödinger 方程式は

$$
\begin{pmatrix} \dot{\rho_{11}} & \dot{\rho_{12}} \\ \dot{\rho_{21}} & \dot{\rho_{22}} \end{pmatrix} = \frac{i}{\hbar} \begin{pmatrix} \rho_{12}H_{21} - \rho_{21}H_{12} & \rho_{11}H_{12} + \rho_{12}H_{22} - H_{11}\rho_{12} - H_{12}\rho_{22} \\ \rho_{21}H_{11} + \rho_{22}H_{21} - H_{21}\rho_{11} - H_{22}\rho_{21} & \rho_{21}H_{12} - H_{21}\rho_{12} \end{pmatrix} \tag{13.2}
$$

この行列の 22 成分から 11 成分を引くと

$$
\dot{\rho_{22}} - \dot{\rho_{11}} = \frac{i}{\hbar}(2\rho_{21}H_{12} - 2H_{21}\rho_{12}) \tag{13.3}
$$

ここで左辺は $\dot{\rho_{22}} - \dot{\rho_{11}} = \dot{r_3}$ である. また

$$
\begin{aligned}
(\boldsymbol{F} \times \boldsymbol{r})_3 &= F_1 r_2 - F_2 r_1 \\
&= \frac{i}{\hbar}(H_{12} + H_{21})(\rho_{21} - \rho_{12}) + \frac{i}{\hbar}(H_{12} - H_{21})(\rho_{12} + \rho_{21}) \\
&= \frac{i}{\hbar}(2H_{12}\rho_{21} - 2H_{21}\rho_{12})
\end{aligned} \tag{13.4}
$$

である. (13.3), (13.4) から

$$
\dot{r_3} = (\boldsymbol{F} \times \boldsymbol{r})_3 \tag{13.5}
$$

である. 次に (13.2) の 12 成分と 21 成分を足すと

$$
\begin{aligned}
\dot{\rho_{12}} + \dot{\rho_{21}} =& \frac{i}{\hbar}\,(\rho_{11}H_{12} - \rho_{11}H_{21} + \rho_{12}H_{22} - \rho_{12}H_{11} \\
&+ \rho_{21}H_{11} - \rho_{21}H_{22} + \rho_{22}H_{21} - \rho_{22}H_{12}) \\
=& \frac{i}{\hbar}(H_{21} - H_{12})(\rho_{22} - \rho_{11}) + \frac{i}{\hbar}(H_{22} - H_{11})(\rho_{12} - \rho_{21}) \\
=& (F_2 r_3 - F_3 r_2) = (\boldsymbol{F} \times \boldsymbol{r})_1
\end{aligned} \tag{13.6}
$$

また $\dot{\rho_{12}} + \dot{\rho_{21}} = \dot{r_1}$ より

$$
\dot{r_1} = (\boldsymbol{F} \times \boldsymbol{r})_1 \tag{13.7}
$$

である. 次に (13.2) の 12 成分と 21 成分を引くと

$$\begin{aligned}\dot{\rho_{12}} - \dot{\rho_{21}} =& \frac{i}{\hbar}\,(\rho_{11}H_{12} + \rho_{11}H_{21} + \rho_{12}H_{22} - \rho_{12}H_{11} \\ &- \rho_{21}H_{11} + \rho_{21}H_{22} - \rho_{22}H_{21} - \rho_{22}H_{12}) \\ =& \frac{i}{\hbar}(H_{22} - H_{11})(\rho_{12} + \rho_{21}) - \frac{i}{\hbar}(H_{21} + H_{12})(\rho_{22} - \rho_{11}) \\ =& i(F_3 r_1 - F_1 r_3) = i(\boldsymbol{F} \times \boldsymbol{r})_2\end{aligned} \tag{13.8}$$

また $(\dot{\rho_{12}} - \dot{\rho_{21}})/i = \dot{r_2}$ より

$$\dot{r_2} = (\boldsymbol{F} \times \boldsymbol{r})_2 \tag{13.9}$$

以上から

$$\frac{d}{dt}\boldsymbol{r} = \boldsymbol{F} \times \boldsymbol{r} \tag{13.10}$$

が示された.