ON GENERALIZATIONS OF A CONJECTURE OF KANG AND PARK

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ABSTRACT. Let $\Delta_d^{(a,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-)}(n)$ where $q_d^{(a)}(n)$ counts the number of partitions of n into parts with difference at least d and size at least a, and $Q_d^{(a,-)}(n)$ counts the number of partitions into parts $\equiv \pm a \pmod{d+3}$ excluding the $d+3-a$ part. Motivated by generalizing a conjecture of Kang and Park, Duncan, Khunger, Swisher, and the second author conjectured that $\Delta_d^{(3,-)}(n) \geq 0$ for all $d \geq 1$ and $n \geq 1$ and were able to prove this when $d \geq 31$ is divisible by 3. They were also able to conjecture an analog for higher values of a that the modified difference function $\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0$ where $Q_d^{(a,-,-)}(n)$ counts the number of partitions into parts $\equiv \pm a \pmod{d+3}$ excluding the a and $d+3-a$ parts and proved it for infinitely many classes of n and d.

We prove that $\Delta_d^{(3,-)}(n) \geq 0$ for all but finitely many d. We also provide a proof of the generalized conjecture for all but finitely many d for fixed a and strengthen the results of Duncan et.al. We provide a conditional proof of a linear lower bound on d for the generalized conjecture, which improves our unconditional result based on a conjectural modification of a recently proven conjecture of Alder. Using this modification, we obtain a strengthening of this generalization of Kang and Park's conjecture which remarkably allows a as a part. Additionally, we provide asymptotic evidence that this strengthened conjecture holds.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, that sum to n. Let $p(n \mid \text{condition})$ be the number of partitions of n satisfying a certain condition. Euler famously proved that the number of partitions of a positive integer n into odd parts equals the number of partitions of n into distinct parts. Two other famous partition identities are those of Rogers and Ramanujan. The first Rogers-Ramanujan identity states that the number of partitions of n with parts having difference at least 2 is equal to the number of partitions with parts congruent $\pm 1 \pmod 5$ and the second Rogers-Ramanujan identity states the number of partitions of n with parts at least 2 and difference at least 2 is equal to the number of partitions with parts congruent to $\pm 2 \pmod 5$. These identities are encapsulated by the $q-$series relations

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

where $(a;q)_0 = 1$, and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, where $n = \infty$ is allowed.

Date: June 25, 2022.

2010 *Mathematics Subject Classification.* 05A17, 11P82, 11P84, 11F37.

Key words and phrases. partitions, Rogers-Ramanujan identities, Alder's conjecture.

Motivated in generalizing the Rogers-Ramanujuan identities, Schur discovered the number of partitions with parts having difference at least 3 which have no consecutive multiples of 3 as parts is equal to the number of partitions with parts congruent to $\pm 1 \pmod 6$.

Remarkably, there are no other such partition identities that exist as shown by Alder [2] and Lehmer [8]. In 1956, Alder [1] conjectured a generalization of a related family of partition identities. Alder's conjecture states the number of partitions with difference of at least d is greater than or equal to the number of partitions with parts congruent to ± 1 $\pmod{d+3}$. Note that this conjecture generalizes the Euler, Rogers-Ramanujan, and Schur identities. Alder's conjecture was proven by Andrews [4] for $n \geq 1$ and $d = 2^r - 1, r \geq 4$ in 1971. In 2004 and 2008, Yee [10, 11] proved the conjecture for $n \geq 1$, $d \geq 32$ and $d = 7$. In 2011, the remaining cases of Alder's conjecture were proven by Alfes, Jameson, and Lemke Oliver [3] by using the asymptotic methods of Meinardus [9].

In 2020, Kang and Park [7] investigated how to construct an analog of Alder's conjecture for the second Rogers-Ramanujan identity. Kang and Park compared the partition functions

$$\begin{aligned} q_d^{(a)}(n) &:= p(n| \text{ parts} \geq a \text{ and parts differ by at least } d), \\ Q_d^{(a)}(n) &:= p(n| \text{ parts} \equiv \pm a \pmod{d+3}), \end{aligned}$$

by utilizing the difference function

$$\Delta_d^{(a)}(n) := q_d^{(a)}(n) - Q_d^{(a,-)}(n)\,.$$

In their attempts to create an analog of Alder's conjecture for the second Rogers-Ramanujan identity, Kang and Park found that

$$\Delta_d^{(2)}(n) < 0 \text{ for some choices of } d, n \geq 1.$$

However, by employing a minor modification of $Q_d^{(a)}(n)$ by defining for $d, a, n \geq 1$,

$$\begin{aligned} Q_d^{(a,-)}(n) &:= p(n \,|\, \text{parts} \equiv \pm a \pmod{d+3}\,,\ \text{excluding the part } d+3-a), \\ \Delta_d^{(a,-)}(n) &:= q_d^{(a)}(n) - Q_d^{(a,-)}(n)\,, \end{aligned}$$

they presented the following conjecture.

Conjecture 1.1 (Kang, Park [7], 2020). *For all d, $n \geq 1$,*

$$\Delta_d^{(2,-)}(n) \geq 0.$$

Kang and Park's conjecture was proven for all but finitely many d by Duncan, et al. by employing a modification of Alder's conjecture and the results of Yee [11].

Theorem 1.2 (Duncan, et al. [6], 2021). *For all $d \geq 62$ and $n \geq 1$,*

$$\Delta_d^{(2,-)}(n) \geq 0.$$

Motivated by Kang and Park's conjecture, Duncan et al. attempted to find a generalization for higher values of a. Remarkably, they found that the removal of the $d+3-a$ as a part was sufficient when $a = 3$.

Conjecture 1.3 (Duncan, et al. [6], 2021). *For all $d, n \geq 1$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

By employing their methods in Duncan, et al. [6], we prove Conjecture 1.3 for all but finitely many d.

Theorem 1.4. *For $n \geq 1$ and $d = 1, 2, 91, 92, 93$ or $d \geq 187$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

When $a \geq 4$, it is not the case that $\Delta_d^{(a,-)}(n)$ is non-negative for all $d, n \geq 1$. By considering the functions

$$Q_d^{(a,-,-)}(n) := p(n \mid \text{parts} \equiv \pm a \pmod{d+3}, \text{excluding the parts } a \text{ and } d+3-a),$$
$$\Delta_d^{(a,-,-)}(n) := q_d^{(a)}(n) - Q_d^{(a,-,-)}(n),$$

Duncan, et al. were able to conjecture the following analog of Kang and Park's conjecture.

Conjecture 1.5 (Duncan, et al. [6], 2021). *For $d, a \geq 1$ with $1 \leq a \leq d+2$ and $n \geq 1$,*

$$\Delta_d^{(a,-,-)}(n) \geq 0.$$

They were able to prove infinitely many cases of Conjecture 1.5 by employing the methods of Yee [11] and Andrews [4].

Throughout this paper, we let $h_d^{(a)}$ and $h_n^{(a)}$ denote the least non-negative residues of $-d$ and $-n$ modulo a respectively. Note that $\frac{d+h_d^{(a)}}{a} = \lceil \frac{d}{a} \rceil$ and $\frac{n+h_n^{(a)}}{a} = \lceil \frac{n}{a} \rceil$.

We prove a strengthening of [6, Theorem 1.6] by employing our modification of Alder's conjecture.

Theorem 1.6. *For $a \geq 4$ and $\lceil \frac{d}{a} \rceil = 31$ or $\lceil \frac{d}{a} \rceil \geq 63$ such that $h_d^{(a)} \leq 3$ and $n \geq 1$,*

$$\Delta_d^{(a,-)}(n) \geq 0.$$

In particular, for $a = 4$, $121 \leq d \leq 124$ or $d \geq 249$, and $n \geq 1$,

$$\Delta_d^{(4,-)}(n) = q_d^{(4)}(n) - Q_d^{(4,-)}(n) \geq 0$$

Remark 1.7. *This is a strengthening of* [6, Theorem 1.6] *since*

$$\Delta_d^{(a,-,-)}(n) \geq \Delta_d^{(a,-)}(n) \text{ for all } d, a, n \geq 1.$$

By developing a new method in comparing these partition functions, we also prove Conjecture 1.5 for all but finitely many d for fixed a.

Theorem 1.8. *For $d, a, n \geq 1$ such that $\lceil \frac{d}{a} \rceil \geq 2^{a+3} - 1$,*

$$\Delta_d^{(a,-,-)}(n) \geq 0.$$

Remark 1.9. *The cases of Conjecture 1.5 for $1 \leq a \leq 4$ have already been proven for all but finitely many d by Alder's conjecture, Theorem 1.2, and Theorem 1.4. Hence, we'll focus on proving Theorem 1.8 when $a \geq 5$.*

It is a natural question of determining for fixed $a \geq 1$ which $d, n \geq 1$ are sufficient to allow

$$\Delta_d^{(a,-)}(n) \geq 0.$$

Towards answering this question, we present the following conjectural modification of Alder's conjecture and our strengthening of Conjecture 1.5.

Conjecture 1.10. *For $k, m \geq 12$ such that $m \geq k+2$,*

$$q_k^{(1)}(m) - Q_{k-4}^{(1,-)}(m) \geq 0.$$

If Conjecture 1.10 holds, we are able to prove a strengthening of Conjecture 1.5 which allows a as a part. This conditional result remarkably reduces the exponential lower bound of d in Theorem 1.8 to a linear one.

Theorem 1.11. *Suppose that Conjecture 1.10 holds for the prescribed bounds. Then for $a \geq 1$, $\lceil \frac{d}{a} \rceil \geq 12$, and $n \geq 1$,*

$$\Delta_d^{(a,-)}(n) \geq 0.$$

Moreover, suppose the bounds on a and d are as above, and $1 \leq \lceil \frac{n}{a} \rceil \leq 5\lceil \frac{d}{a} \rceil$. Unconditionally, we have

$$\Delta_d^{(a,-)}(n) \geq 0.$$

Remark 1.12. *The choice of the upper bound $\lceil \frac{n}{a} \rceil \leq 5\lceil \frac{d}{a} \rceil$ for the unconditional component of Theorem 1.11 comes from applying the function $\mathcal{G}_d^{(1)}(n)$ as defined by Yee* [11].

Finally by employing the methods of Duncan, et al. [6], Alfes, et al. [3], and Andrews [5], we obtain the following asymptotic result.

Theorem 1.13. *For $a, n \geq 1$ and $d \geq 4$ such that $a < \frac{d+3}{2}$ and $gcd(a, d+3) = 1$,*

$$\lim_{n\to\infty} \Delta_d^{(a,-)}(n) = +\infty.$$

We now outline the rest of this paper. In Section 2, we present a modification of [4, Theorem 1.3] and establish some other important lemmas. In Section 3, we prove a modification of Alder's conjecture, which forms the critical component of the proofs of Theorems 1.4 and 1.6. In Section 4, we prove Theorems 1.4 and 1.6 by reducing to our unconditional modification of Alder's conjecture. In Section 5, we prove Theorem 1.8 by using the generating functions from Yee [11]. In Section 6, we conditionally prove Theorem 1.11 by reducing to the case of $a = 1$ and employing Conjecture 1.10. We also prove Conjecture 1.10 for small m to derive our unconditional result in Theorem 1.11. Finally, in Section 7, we employ the methods of Duncan, et al. [6], Alfes, et al. [3], and Andrews [5] to prove Theorem 1.13. We conclude with potential avenues for resolving more small d cases of Conjectures 1.3, 1.5 and extending Theorem 1.6.

2. Preliminaries

In this section, we first establish generating functions for our partition counting functions. We then introduce several lemmas which our results are based on.

We find by employing combinatorial methods that the generating function for $q_d^{(a)}(n)$ is

$$\sum_{n=0}^{\infty} q_d^{(a)}(n)q^n = \sum_{k=0}^{\infty} \frac{q^{d\binom{k}{2}+ka}}{(q;q)_k}.$$

One can see from the work of Alfes, Jameson, and Lemke Oliver [3] that $Q_d^{(a)}(n)$ is

$$\sum_{n=0}^{\infty} Q_d^{(a)}(n)q^n = \frac{1}{(q^{d+3-a};q^{d+3})_\infty (q^a;q^{d+3})_\infty}.$$

For positive integers d, a, n, we find that the generating function for $Q_d^{(a,-)}(n)$ is

$$\sum_{n=0}^{\infty} Q_d^{(a,-)}(n)q^n = \begin{cases} \frac{1}{(q^{2d+6-a};q^{d+3})_\infty} & \text{for } a = \frac{d+3}{2}, \\ \frac{1}{(q^{2d+6-a};q^{d+3})_\infty (q^a;q^{d+3})_\infty} & \text{otherwise.} \end{cases}$$

by accounting for the deletion of the term $d+3-a$ from the expanded product.

Similarly, we find that the generating function of $Q_d^{(a,-,-)}(n)$ to be

$$\sum_{n=0}^{\infty} Q_d^{(a,-,-)}(n)q^n = \begin{cases} \frac{1}{(q^{2d+6-a};q^{d+3})_\infty} & \text{for } a = \frac{d+3}{2}, \\ \frac{1}{(q^{2d+6-a};q^{d+3})_\infty (q^{d+3+a};q^{d+3})_\infty} & \text{otherwise.} \end{cases}$$

We employ the following notation: for a set of positive integers R, we define

$$\rho(R;n) := p(n \mid \text{parts from the set } R).$$

For fixed positive integers d and r, we use the partition function $\rho(T_{r,d};n)$ from Andrews [4] with set of parts

$$T_{r,d} = \{x \in \mathbb{N} | x \equiv 1, d+2, \cdots, d+2^{r-1} \pmod{2d}\}.$$

From the definition of $\rho(T_{r,d};n)$, we find that the associated generating function to be

$$\sum_{n=0}^{\infty} \rho(T_{r,d};n)q^n = \frac{1}{(q^1;q^{2d})_\infty (q^{d+2};q^{2d})_\infty \cdots (q^{d+2^{r-1}};q^{2d})_\infty}.$$

Throughout this paper, we denote r_d to be the largest positive integer r for fixed d such that $d \geq 2^r - 1$.

We also use the partition function $\mathcal{G}_d^{(1)}(n)$ considered by Yee [11], which is defined by

$$\sum_{n=0}^{\infty} \mathcal{G}_d^{(1)}(n)q^n := \frac{(q^{d+2^{r_d-1}};q^{2d})_\infty}{(q^1;q^{2d})_\infty (q^{d+2};q^{2d})_\infty \cdots (q^{d+2^{r_d-2}};q^{2d})_\infty}. \tag{2.1}$$

From (2.1), we find that $\mathcal{G}_d^{(1)}(n)$ counts partitions with distinct parts from the set $\{x \in \mathbb{N} | x \equiv d + 2^{r_d-1} \pmod{2d}\}$ and unrestricted parts from the set

$$T_{r_d-1,d} = \{y \in \mathbb{N} | y \equiv 1, d+2, \cdots, d+2^{r_d-2} \pmod{2d}\}.$$

Using these auxillary partition counting functions, we outline the following lemmas that will allow us to prove inequality chains involving partition functions.

We first present a comparison theorem of Andrews [4].

Theorem 2.1 (Andrews [4], 1971)**.** *Let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $y_1 = 1$ and $x_i \geq y_i$ for all i. Then for all $n \geq 1$,*

$$\rho(T;n) \geq \rho(S;n).$$

We present the following modification of Theorem 2.1, which will be extensively employed throughout the rest of this paper.

Lemma 2.2. *Let $S = \{x_i\}$ and $T = \{y_i\}$ be two strictly increasing sequences of positive integers such that $y_1 = a$, a divides each y_i, and $x_i \geq y_i$. Then for all $n \geq 1$,*

$$\rho(T; n + h_n^{(a)}) \geq \rho(S;n),$$

Proof. Suppose $\lambda \vdash n$, $\lambda = (x_{i_j})_{j=1}^m \in X$ with X defined to be the set of partitions of n counted by $\rho(S;n)$. We construct an injection $\varphi : X \to Y$ with Y being the set of partitions of $n + h_n^{(a)}$ counted by $\rho(T; n + h_n^{(a)})$.

Let $n = an' - m$ with $n' \geq 0, m \in \{0, 1, ..., a-1\}$. We define the sum of differences

$$\alpha := \sum_{j=1}^{m} (x_{i_j} - y_{i_j}).$$

We observe that α is non-negative since $x_{i_j} \geq y_{i_j}$ for all i_j. Note

$$n = \sum_{j=1}^{m} y_{i_j} + \alpha \equiv -m \pmod{a},$$

thus $\alpha \equiv -m \pmod{a}$ due to $a|y_{i_j}$ for all j. Hence, let $\alpha = a\beta - m$ with $\beta \geq 0$. We define $\varphi(\lambda)$ to be the partition with corresponding parts y_{i_j} with β additional appearances of a. We notice that $\varphi(\lambda) \vdash n + h_n^{(a)}$ since

$$\sum_{j=1}^{m} y_{i_j} + a\beta = \sum_{j=1}^{m} x_{i_j} + m = an' = n + h_n^{(a)}.$$

Injectivity of the function is immediate when the image of φ is viewed as

$$q_i = \begin{cases} \rho_1 + \beta, i = 1 \\ \rho_i, i \geq 2 \end{cases}$$

where ρ_i is the multiplicity number of x_i occurring as a part of λ and q_i is the multiplicity number of y_i of $\varphi(\lambda)$.

□

We also employ the following lemmas of Duncan et al. [6] which allows us to use our modification of Alder's conjecture.

Lemma 2.3 (Duncan et al. [6], 2021)**.** *For all $d, a \geq 1$ and $n \geq d + 2a$,*

$$q_d^{(a)}(n) \geq q^{(1)}_{\lceil \frac{d}{a} \rceil} \left(\left\lceil \frac{n}{a} \right\rceil \right).$$

Lemma 2.4 (Duncan et al. [6], 2021)**.** *For $d, a, n \geq 1$ such that a divides $d + 3$,*

$$Q_d^{(a,-)}(an) = Q^{(1,-)}_{\frac{d+3}{a}-3}(n),$$

$$Q_d^{(a,-,-)}(an) = Q^{(1,-,-)}_{\frac{d+3}{a}-3}(n).$$

We now present a combined result of Andrews [4], which will be employed in the proofs of Theorems 1.4 and 1.6.

Theorem 2.5 (Andrews [4], 1971)**.** *For $d = 2^r - 1, r \geq 1$ and $n \geq 1$,*

$$q_d^{(1)}(n) \geq \rho(T_{r,d}; n).$$

Proof. The result follows by combining the proofs of [4, Theorems 1 and 4]. □

We conclude this section with a result from Yee [11] which will be used in our modification of Alder's conjecture.

Lemma 2.6 (Yee [11], 2008)**.** *For $k \geq 31, s \geq 1, k \neq 2^s - 1,$ and $m \geq 4k + 2^{r_k}$,*

$$q_k^{(1)}(m) \geq \mathcal{G}_k^{(1)}(m).$$

Proof. Combine both [11, Lemmas 2.2 and 2.7] to obtain the result. □

3. A Modification of Alder's Conjecture

In this section, we use the work of Andrews [4] and Yee [11] to prove a modification of Alder's conjecture. We use this modification to give simple proofs of Theorems 1.4 and 1.6.

Proposition 3.1. *For $k = 31$ or $k \geq 63$ and $m \geq k+2$,*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m).$$

We prove Proposition 3.1 in three cases based on the form and size of k and m. We will use throughout the paper the notation λ^x to denote that λ appears as a part x times in a partition λ.

Lemma 3.2. *For $k+2 \leq m \leq 5k$ and $k \geq 31$,*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m).$$

Proof. We define

$$S_k = \{x \in \mathbb{N} | x \equiv \pm 1 \pmod{k}\} \setminus \{k-1\}$$

so that $\rho(S_k; m) = Q_{k-3}^{(1,-)}(m)$. We prove Lemma 3.2 based on relating the size of the parts in S_k and the size of m.

We note that $q_k^{(1)}(m)$ is a weakly increasing function since for any partition of m counted by $q_k^{(1)}(m)$, we can add 1 to its largest part to create a partition of $m+1$ counted by $q_k^{(1)}(m+1)$. In a similar fashion, $Q_{k-3}^{(1,-)}(m)$ is weakly increasing since for any partition of m counted by $Q_{k-3}^{(1,-)}(m)$, we can adjoin 1 as a part to create a partition of $m+1$ counted by $Q_{k-3}^{(1,-)}(m+1)$.

Notice that for $k+2 \leq m \leq 2k-2$ that we have $q_k^{(1)}(k+2) = 2$ with partitions $(k+1, 1), (k+2)$. Note $Q_{k-3}^{(1,-)}(2k-2) = 2$ with partitions $(k+1, 1^{k-3}), (1^{2k-2})$, hence the inequality holds.

We now consider the interval $2k-1 \leq m \leq 4k-1$. We notice that $q_k^{(1)}(2k-1) \geq 16$ since the partitions $(2k-1), (2k-1-i, i)$ with $1 \leq i \leq 15$ are counted by $q_k^{(1)}(2k-1)$ due to $k \geq 31$. Note that at $m = 4k-1$, we have $Q_{k-3}^{(1,-)}(4k-1) = 12$ partitions since there is one partition with largest part for each element in $\{4k-1, 3k+1, 3k-1\}$, two with largest part $2k+1$, three with largest part for each element in $\{2k-1, k+1\}$, and one with largest part 1. Hence for all $2k-1 \leq m \leq 4k-1$ the inequality holds.

We now verify for all $4k \leq m \leq 5k$ that the inequality holds. Notice that we have the lower bound $q_k^{(1)}(4k) \geq 47$ since the partitions $(4k), (4k-i, i)$ with $1 \leq i \leq 46$ are counted by $q_k^{(1)}(4k)$ due to $k \geq 31$. We observe that $Q_{k-3}^{(1,-)}(5k) = 26$ since there is one partition with largest part for each element in $\{5k-1, 4k+1\}$, two partitions with largest part $4k-1$, three partitions with largest part $3k+1$, four partitions with largest part $3k-1$, five partitions with largest part for each element in $\{2k+1, 2k-1\}$, four partitions with largest part $k+1$, and one partition with largest part 1. Hence, we obtain $q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m)$ for $4k \leq m \leq 5k$. ◻

Lemma 3.3. *For any $k \geq 63$ such that $k \neq 2^s - 1$, $s \geq 1$ and $m \geq 4k + 2^{r_k}$,*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m).$$

Proof. We prove Lemma 3.3 by showing the following inequalities,

$$q_k^{(1)}(m) \geq \mathcal{G}_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m).$$

Recall that from Lemma 2.6 that $q_k^{(1)}(m) \geq \mathcal{G}_k^{(1)}(m)$. It is also clear $\mathcal{G}_k^{(1)}(m) \geq \rho(T_{r_k-1,k}; m)$, thus we reduce to showing $\rho(T_{r_k-1,k}; m) \geq Q_{k-3}^{(1,-)}(m)$. Since $r_k \geq 6$, we have $\rho(T_{r_k-1,k}; m) \geq \rho(T_{5,k}; m)$. Hence, it suffices to show $\rho(T_{5,k}; m) \geq Q_{k-3}^{(1,-)}(m)$.

Let S and T denote the sets of partitions counted by $Q_{k-3}^{(1,-)}(m)$ and $\rho(T_{5,k}; m)$ respectively. We set $x_i \in S_k$ and $y_i \in T_{5,k}$ to denote the associated ith smallest element of S_k and $T_{5,k}$. Using Table 1, note that the only i where $x_i < y_i$ is when $i = 2$.

i	x_i	y_i
1	1	1
$5\alpha+1$: (i is even)	$\left(\frac{5\alpha+1}{2}\right)k+1$	$2k\alpha+1$
$5\alpha+1$: (i is odd and $i \neq 1$)	$\left(\frac{5\alpha+2}{2}\right)k-1$	$2k\alpha+1$
$5\alpha+\bar{i}$: ($2 \leq \bar{i} \leq 4$ and i is even)	$\left(\frac{5\alpha+\bar{i}}{2}\right)k+1$	$2k\alpha+k+2^{\bar{i}-1}$
$5\alpha+\bar{i}$: ($2 \leq \bar{i} \leq 4$ and i is odd)	$\left(\frac{5\alpha+\bar{i}+1}{2}\right)k-1$	$2k\alpha+k+2^{\bar{i}-1}$
5α : (i is even and positive)	$\left(\frac{5\alpha}{2}\right)k+1$	$2k\alpha-k+16$
5α : (i is odd and positive)	$\left(\frac{5\alpha+1}{2}\right)k-1$	$2k\alpha-k+16$

TABLE 1. Values of $x_i \in S_k, y_i \in T_{5,k}$ for $i = 5\alpha + \bar{i}$ where $\bar{i}$ and α are respectively the remainder and quotient from Euclidean division of i by 5.

We will construct an injection $\varphi : S \to T$. Let $\lambda \vdash m$ be an element in S and ρ_i and q_i denote the number of times x_i (wrt y_i) occurs as a part of λ (wrt $\varphi(\lambda)$). Set

$$\alpha := \sum_{i \neq 2} (x_i - y_i)\rho_i$$

to be the difference sum.

Let S_1 denote the subset of S where the partitions satisfy the constraint $\rho_1 + \alpha \geq \rho_2$. We define the function $\varphi_1 : S_1 \to T$ as follows:

I: $p_1 + \alpha \geq p_2$. We set

$$q_i = \begin{cases} -p_2 + p_1 + \alpha, i = 1 \\ p_i, i \geq 2 \end{cases}$$

Note this encapsulates the case when $\rho_2 = 1$ since we presuppose that $m \geq k + 2$.

Let S_2 denote the set of partitions of S with $\rho_2 > \rho_1 + \alpha$ and ρ_2 even. We define $S_{(2,\beta)} \subset S_2$ to be the set of partitions which additionally satisfy $\rho_1 + \rho_6 = \beta(k-2) + \bar{\rho}$ with $\bar{\rho} \in \{0, \cdots, k-3\}$ and $\beta \in \mathbb{Z}_{\geq 0}$. Observe that $\bigcup_{\beta \in \mathbb{Z}_{\geq 0}} S_{(2,\beta)} = S_2$. For each β, we define the function $\varphi_{(2,\beta)} : S_{(2,\beta)} \to T$ as follows:

II: $\rho_1 + \alpha < \rho_2, \rho_2 \in 2\mathbb{Z}$ and $\lambda \in S_{(2,\beta)}$. We set

$$q_i = \begin{cases} \frac{\rho_2 - 2\beta}{2} + \alpha + \rho_1 - 2\beta, i = 1 \\ 2\beta, i = 2 \\ \rho_i, 3 \leq i \leq 5 \\ \frac{\rho_2 - 2\beta}{2} + \rho_6, i = 6 \\ \rho_i, i \geq 7. \end{cases}$$

Note $\rho_2 - 2\beta \geq 0$ since $\rho_2 - \frac{2\rho_2}{k-2} \geq 0$ if $k \geq 4$. Additionally, $\alpha + \rho_1 - 2\beta \geq 0$ if $k \geq 4$, thus $q_1, q_6 \geq 0$. Hence, we obtain that $\varphi_{(2,\beta)}$ is well defined.

Let $S_3 \subset S$ denote the set of partitions with ρ_2 odd and $\rho_2 > \alpha + \rho_1$. We define $S_{(3,\beta)} \subset S_3$ to be the set of partitions which additionally satisfy $\rho_1 + \rho_6 = \beta(k-2) + \bar{\rho}$ with $\bar{\rho} \in \{0, \cdots, k-3\}$ and $\beta \geq 0$. Observe that $\bigcup_{\beta \in \mathbb{Z}_{\geq 0}} S_{(3,\beta)} = S_3$. For each β, we define the function $\varphi_{(3,\beta)} : S_{(3,\beta)} \to T$ as follows:

III: $\rho_1 + \alpha < \rho_2, \rho_2 \in 2\mathbb{Z}+1$ and $\lambda \in S_{(3,\beta)}$. We set

$$q_i = \begin{cases} \frac{\rho_2 - 3 - 2\beta}{2} + \alpha + \rho_1 - 2\beta, i = 1 \\ 2\beta + 1, i = 2 \\ \rho_i, 3 \leq i \leq 5 \\ \frac{\rho_2 - 1 - 2\beta}{2} + \rho_6, i = 6 \\ \rho_i, i \geq 7 \end{cases}$$

Note if $\rho_2 = 3$, we must have $\beta = 0$ if $k \geq 5$ since otherwise $\lambda \in S_1$. For $\rho_2 \geq 5$, note $\rho_2 > 3 + \frac{2\rho_2}{k-2}$ if $k \geq 7$, implying $\rho_2 - 3 - 2\beta \geq 0$. Observe from the definition of β that $\alpha + \rho_1 - 2\beta \geq 0$ for $k \geq 4$. Hence $q_1, q_6 \geq 0$. Thus, $\varphi_{(3,\beta)}$ is well-defined.

We define $\varphi : S \to T$ to be the function defined piecewise from $\varphi_1, \varphi_{(2,\beta)}, \varphi_{(3,\beta)}$ as above. In order to show that φ is injective, it suffices to show that $\varphi_1, \varphi_{(2,\beta)}, \varphi_{(3,\beta)}$ are injective and that the images of distinct cases are disjoint.

It is clear that φ_1 is injective from construction, hence we focus on showing for fixed β that $\varphi_{(2,\beta)}$ is injective on its domain $S_{(2,\beta)}$. Suppose that $\lambda, \lambda' \in S_{(2,\beta)}$ are distinct partitions such that $\varphi_{(2,\beta)}(\lambda) = \varphi_{(2,\beta)}(\lambda')$. Let ρ_i and ρ_i' denote the multiplicity numbers of the x_i occurring as parts of λ and λ' respectively. Similarly, let $\bar{\rho}, \bar{\rho}'$ denote the remainders when $\rho_1 + \rho_6, \rho_1' + \rho_6'$ are divided by $k-2$ respectively.

Observe from the definition of $\varphi_{(2,\beta)}$ that we may assume for all $i \neq 1, 2, 6$ that $\rho_i = \rho_i' = 0$. We obtain from the definition of $\varphi_{(2,\beta)}$ and using that β is fixed the following system of equations

$$\frac{\rho_2}{2} + k\rho_6 + \rho_1 = \frac{\rho_2'}{2} + k\rho_6' + \rho_1'$$
$$\frac{\rho_2}{2} + \rho_6 = \frac{\rho_2'}{2} + \rho_6'$$
$$\rho_1 + (k-1)\rho_6 = \rho_1' + (k-1)\rho_6'.$$

From these three equations, we observe that $\rho_2 \neq \rho_2', \rho_1 \neq \rho_1', \rho_6 \neq \rho_6'$, otherwise $\lambda = \lambda'$. Hence, assume without loss of generality that $\rho_1 > \rho_1'$. Observe that since $\lambda, \lambda' \in S_{(2,\beta)}$, we have $(\rho_1 - \rho_1') + (\rho_6 - \rho_6') = \bar{\rho} - \bar{\rho}' < k-2$. We note that this yields $|\bar{\rho} - \bar{\rho}'| < k-2$ since $0 \leq \bar{\rho}, \bar{\rho}' < k-2$. Using this and the third equation above, we have

$$(\rho_1 - \rho_1') = (\bar{\rho} - \bar{\rho}') + (\rho_6' - \rho_6) = (k-1)(\rho_6' - \rho_6).$$

Note that $\bar{\rho} \neq \bar{\rho}'$ since $(k-2) \nmid (\bar{\rho} - \bar{\rho}')$. If $\bar{\rho} = \bar{\rho}'$, then $k-1 = 1$, which implies that $k = 2$, which is a contradiction. Hence in both cases, we obtain a contradiction. Thus $\lambda = \lambda'$.

The verification of $\varphi_{(3,\beta)}$ is injective on $S_{(3,\beta)}$ yields the same exact system of equations and thus is injective via the same analysis.

We observe from construction that if $\beta \neq \beta'$ that $\mathrm{im}\varphi_{(j,\beta)} \cap \mathrm{im}\varphi_{(j,\beta')} = \varnothing$ for $j = 2, 3$. It also follows from construction that $\mathrm{im}\varphi_{(2,\beta)} \cap \mathrm{im}\varphi_{(3,\beta')} = \varnothing$ for any $\beta, \beta' \in \mathbb{Z}_{\geq 0}$ since q_2 is even in the first case and odd in the second.

We now verify that $\mathrm{im}\varphi_1 \cap \mathrm{im}\varphi_{(2,\beta)} = \varnothing$ for all $\beta \in \mathbb{Z}_{\geq 0}$. Again we can assume from the constructions of the functions that $\rho_i = \rho'_i = 0$ for all $i \neq 1, 2, 6$. Let $\lambda \in S_1$, $\lambda' \in S_{(2,\beta)}$, and suppose that $\varphi_1(\lambda) = \varphi_{(2,\beta)}(\lambda')$.

We obtain the following system of equations

$$\begin{aligned}\rho_1 + k\rho_6 - \rho_2 &= \rho'_1 + k\rho'_6 + \frac{\rho'_2 - 2\beta}{2} - 2\beta \\ \rho_2 &= 2\beta \\ \rho_6 &= \rho'_6 + \frac{\rho'_2 - 2\beta}{2}.\end{aligned}$$

Using these three equations and $\rho'_1 < \rho'_2$, we obtain

$$\rho_1 = \rho'_1 - \frac{k}{2}(\rho'_2 - 2\beta) + (\frac{\rho'_2 - 2\beta}{2}) < \rho'_2 + \frac{(-k+1)}{2}(\rho'_2 - 2\beta).$$

In order to show that $\rho_1 < 0$, it suffices to show that $\rho'_2 + (k-1)\beta \leq \frac{k-1}{2}\rho'_2$. Observe from the definition of β that we have $\beta \leq \frac{(\rho'_1+\rho'_6)}{k-2} < \frac{\rho'_2}{k-2}$ since $\lambda' \in S_{(2,\beta)}$. This yields $\rho'_2 + (k-1)\beta \leq \rho'_2 + \frac{k-1}{k-2}\rho'_2$. Hence, it suffices to show that

$$(\frac{2k-3}{k-2})\rho'_2 = \rho'_2 + (\frac{k-1}{k-2})\rho'_2 \leq \frac{k-1}{2}(\rho'_2).$$

Note that this reduces to showing that the inequality

$$0 \leq k^2 - 7k + 8$$

holds, which is true for $k \geq 6$. Hence, we obtain that $\rho_1 < 0$, which is a contradiction. This yields that $\mathrm{im}\varphi_1 \cap \mathrm{im}\varphi_{(2,\beta)} = \varnothing$.

We now verify that $\mathrm{im}\varphi_1 \cap \mathrm{im}\varphi_{(3,\beta)} = \varnothing$. Suppose that $\lambda \in S_1$ and $\lambda' \in S_{(3,\beta)}$ such that $\varphi(\lambda) = \varphi_{(3,\beta)}(\lambda')$. We observe that we obtain the system of equations

$$\begin{aligned}\rho_1 + k\rho_6 - \rho_2 &= \rho'_1 + k\rho'_6 + \frac{\rho'_2 - 2\beta - 3}{2} - 2\beta \\ \rho_2 &= 2\beta + 1 \\ \rho_6 &= \rho'_6 + \frac{\rho'_2 - 2\beta - 1}{2}.\end{aligned}$$

Using these three equations and $\rho'_1 < \rho'_2$, we obtain

$$\rho_1 = -k(\rho_6 - \rho'_6) + \rho'_1 + \frac{\rho'_2 - 2\beta - 3}{2} + 1 < \frac{-k+1}{2}(\rho'_2 - 2\beta - 1) + \rho'_2.$$

Note from this representation of ρ_1 that it suffices to show

$$\frac{-k+1}{2}(\rho'_2 - 2\beta - 1) + \rho'_2 \leq 0.$$

From this inequality and $(k-1)\beta \leq \frac{k-1}{k-2}\rho'_2$, it suffices to show that $\frac{k-1}{2}(\rho'_2 - 1) - \frac{k-1}{k-2}\rho'_2 \geq \rho'_2$. We notice that $\rho'_2 - 1 \geq \frac{\rho'_2}{2}$ since $\rho'_2 \geq 3$. Hence it suffices to show that

$$\frac{k-1}{4} \geq 1 + \frac{k-1}{k-2},$$

which is true for $k \geq 10$. This yields that $\rho_1 < 0$ which is a contradiction. $\square$

Lemma 3.4. *For $k = 2^s - 1$ with $s \geq 5$ and $m \geq 4k + 2^s$,*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m).$$

Proof. We prove Lemma 3.4 by showing the following inequality holds,

$$q_k^{(1)}(m) \geq \rho(T_{s,k}; m) \geq Q_{k-3}^{(1,-)}(m).$$

Note that the first inequality holds by Theorem 2.5, hence we reduce to showing $\rho(T_{s,k}; m) \geq Q_{k-3}^{(1,-)}(m)$. However, the inequality follows by adapting the injection in the proof of Lemma 3.3. □

Proof of Proposition 3.1. Combine Lemmas 3.2, 3.3, and 3.4 to obtain the result. □

4. Proofs of Theorems 1.4 and 1.6

In this section, we provide proofs of Theorems 1.4 and 1.6 using the lemmas established in Section 2 as well as Proposition 3.1.

4.1. **Proof of Theorem 1.4.** We first prove Theorem 1.4 for $d = 1$ by employing the Glaishier bijection.

Proposition 4.1. *For $n \geq 1$ and $d = 1$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

Proof. Let S and T denote the sets of partitions counted by $Q_1^{(3,-)}(n)$ and $q_1^{(3)}(n)$ respectively. Observe that the parts of partitions in S are congruent to $\pm 3 \pmod 4 \setminus \{1\}$, hence each part is odd and is greater than or equal to 3. Let $\lambda_i = 2\mu_i - 1$ with $\mu_i \geq 2$ be the *ith* part arranged in increasing order allowed for partitions in S. Suppose $\lambda \vdash n$ is an element of S. We let ρ_i denote the multiplicity of λ_i as a part of λ. We write $\rho_i = 2^{a_1(i)} + \cdots + 2^{a_j(i)}$ (with $a_1(i) < a_2(i) < \cdots < a_j(i)$) in its binary expansion. Consider the Glaisher mapping defined by

$$\varphi(\lambda) = \varphi(\sum_{i=1}^{r}(2\mu_i - 1)\rho_i) = \sum_{i=1}^{r}(2\mu_i - 1)2^{a_1(i)} + \cdots + (2\mu_i - 1)2^{a_j(i)}.$$

Note that each part of $\varphi(\lambda)$ is greater than or equal to 3 since $\mu_i \geq 2$. Observe that φ is injective and has desired codomain by the same reasoning in [**?**]. □

We now provide a proof of the $d = 2$ case for Theorem 1.4.

Proposition 4.2. *For $n \geq 1$ and $d = 2$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

Proof. Recall that the second Rogers-Ramanujan identity states for all $n \geq 1$,

$$\sum_{n=0}^{\infty} q_2^2(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \sum_{n=0}^{\infty} Q_2^{(2)}(n)q^n. \tag{4.1}$$

Multiplying by the factor $(1 - q^3)$ on both sides of (4.1) yields

$$(1 - q^3)\sum_{n=0}^{\infty} q_2^{(2)}(n)q^n = \frac{(1 - q^3)}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \sum_{n=0}^{\infty} Q_2^{(2,-)}(n)\, q^n.$$

Note that when $n = 1, 2$ it's clear that $q_2^{(3)}(n) = Q_2^{(3,-)}(n) = 0$. By setting $m = n + 3$, it suffices to show for $m \geq 3$ the inequality,

$$q_2^{(2)}(m) - q_2^{(2)}(m-3) \leq q_2^{(3)}(m).$$

Let $q_2^{(2)}(m)^*$ denote the set of partitions of m counted by $q_2^{(2)}(m)$ with the additional property that their smallest part is 2. Note that $q_2^{(2)}(m) - q_2^{(2)}(m)^* = q_2^{(3)}(m)$ by construction. Thus it suffices to show that $q_2^{(2)}(m)^* \leq q_2^{(2)}(m-3)$.

Let X and Y denote the set of partitions counted by $q_2^{(2)}(m)^*$ and $q_2^{(2)}(m-3)$ respectively. We also let $\lambda = (\lambda_1, \cdots, \lambda_{i-1}, 2) \vdash m$ be a partition counted by $q_2^{(2)}(m)^*$. Assuming that X is non-empty, we define the function $\varphi : X \to Y$,

$$\varphi(\lambda) = (\lambda_1, \cdots, \lambda_{i-1} - 1).$$

It is clear from construction that $\varphi(\lambda) \in Y$. We now show that φ is injective. Note that if $\lambda, \lambda' \in X$ have different lengths that $\varphi(\lambda) \neq \varphi(\lambda')$ since φ subtracts the length of the partitions by 1. Hence we may assume that λ, λ' have the same length. Since φ only removes the last part of partitions of X, we immediately must have $\lambda = \lambda'$ if $\varphi(\lambda) = \varphi(\lambda')$. This yields $q_2^{(2)}(m)^* \leq q_2^{(2)}(m-3)$, which completes the proof. ☐

Proof of Theorem 1.4 for $91 \leq d \leq 93$ *and* $d \geq 187$. We observe for $d \geq 1$ and $3 \leq n \leq d+5$ that

$$\Delta_d^{(3,-)}(n) \geq 0$$

since 3 is the only possible part of partition counted by $Q_d^{(3,-)}(n)$ in the interval. Thus $Q_d^{(3,-)}(n) \leq 1 \leq q_d^{(3)}(n)$. In the case when $n = d+6$, we have $q_d^{(3)}(d+6) \geq 2 = Q_d^{(3,-)}(d+6)$. Now let $n \geq d+7$. To prove Theorem 1.4 for this range of d and n, it suffices to show the inequality chain

(4.2)
$$q_d^{(3)}(n) \geq q^{(1)}_{\frac{d+h_d^{(3)}}{3}}\left(\frac{n+h_n^{(3)}}{3}\right) \geq Q^{(1,-)}_{\frac{d+h_d^{(3)}}{3}-3}\left(\frac{n+h_n^{(3)}}{3}\right) = Q^{(3,-)}_{d+h_d^{(3)}-3}\left(n+h_n^{(3)}\right) \geq Q_d^{(3,-)}(n).$$

The first inequality in (4.2) is justified by Lemma (2.3) for $n \geq d+6$. For d such that $d+h_d^{(3)} = 93$ or $d + h_d^{(3)} \geq 189$, the second inequality follows from Proposition 3.1 and multiplication by 3 on the bounds of $k = \frac{d+h_d^{(3)}}{3}$. The equality follows from applying Lemma 2.4. The final inequality follows by applying Lemma 2.2 with $\rho(T; n + h_n^{(3)}) = Q^{(3,-)}_{d+h_d^{(3)}-3}\left(n+h_n^{(3)}\right)$ and $\rho(S; n) = Q_d^{(3,-)}(n)$. Note that the application of Lemma 2.2 is justified since $h_d^{(3)} \leq 3$. ☐

4.2. **The Proof of Theorem 1.6.**

Proof of Theorem 1.6. The case when $1 \leq n \leq a-1$ is trivial since $q_d^{(a)}(n) = Q_d^{(a,-)}(n) = 0$. We observe for $d + h_d^{(a)} \geq 31a$ and $a \leq n \leq d + (a+2)$ that a is the only possible part of a partition counted by $Q_d^{(a,-)}(n)$, hence $Q_d^{(a,-)}(n) \leq 1 \leq q_d^{(a)}(n)$. Note for $d + (3+a) \leq n \leq d + 2a$ that the only potential partitions counted by $Q_d^{(a,-)}(n)$ are $(a^{\lambda_a}), (d+3+a)$ with λ_a

such that $d+(3+a) \leq \lambda_a a \leq d+2a$. Hence $Q_d^{(a,-)}(n) \leq 2 \leq q_d^{(a)}(n)$ since $(n), (n-a,a)$ are counted by $q_d^{(a)}(n)$. Thus for $d+h_d^{(a)} \geq 31a$ and $1 \leq n \leq d+2a$,

$$\Delta_d^{(a)}(n) \geq 0.$$

Now let $n > d+2a$. In order to prove Theorem 1.6, we derive the following inequality chain (4.3)

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq Q^{(1,-)}_{\frac{d+h_d^{(a)}}{a}-3}\left(\frac{n+h_n^{(a)}}{a}\right) = Q^{(a,-)}_{d+h_d^{(a)}-3}\left(n+h_n^{(a)}\right) \geq Q_d^{(a,-)}(n).$$

We utilize the same argument present in our proof of Theorem 1.4. The first inequality in (4.3) is justified by Lemma 2.3 for $n \geq d+2a$. For d such that $d+h_d^{(a)} = 31a$ or $d+h_d^{(a)} \geq 63a$, the second inequality follows from Proposition 3.1 and multiplication by a on the bounds of $k = \frac{d+h_d^{(a)}}{a}$. The equality is a result of applying Lemma 2.4. The final inequality follows by applying Lemma 2.2 with $\rho(T; n+h_n^{(a)}) = Q^{(a,-)}_{d+h_d^{(a)}-3}(n+h_n^{(a)})$ and $\rho(S;n) = Q_d^{(a,-)}(n)$.

We end by remarking that $h_d^{(4)}$ is always less than or equal to 3. Thus, for $n \geq 1$ and $\lceil \frac{d}{4} \rceil = 31$ or $\lceil \frac{d}{4} \rceil \geq 63$,

$$q_d^{(4)}(n) \geq Q_d^{(4,-)}(n).$$

□

5. On the generalized Kang-Park conjecture

In this section, we provide a proof of Theorem 1.8. This allows us to give an extension of [6, Theorem 1.6].

We now present the partition functions we'll use throughout this section. We define $r_{d,a}$ to be the largest non-negative integer r such that $2^{r_{d,a}} - 1 \leq \frac{d+h_d^{(a)}}{a}$. Recall that $\mathcal{G}^{(1)}_{\frac{d+h_d^{(a)}}{a}}(\frac{n+h_n^{(a)}}{a})$ counts the number partitions of $\frac{n+h_n^{(a)}}{a}$ with the set of parts

$$\{\lambda_i \equiv 1, \frac{d+h_d^{(a)}}{a}+2, \cdots, \frac{d+h_d^{(a)}}{a}+2^{r_{d,a}-2}, \frac{d+h_d^{(a)}}{a}+2^{r_{d,a}-1} \pmod{2(\frac{d+h_d^{(a)}}{a})}\},$$

where parts congruent to $\frac{d+h_d^{(a)}}{a}+2^{r_{d,a}-1}$ are distinct.

We define $\rho_{(k,1)}(T;m)$ for $k, m \geq 1$ to be

$$\rho_{(k,1)}(T;m) := \rho(\{\lambda_i \equiv 1, k+2, \cdots, k+2^{a-1} \pmod{2k}\}; m).$$

We also define for $k, m \geq 1$,

$$\rho_{(ka,a)}(T^a; ma) := \rho(\{\lambda_i \equiv a, (k+2)a, \cdots, (k+2^{a-1})a \pmod{2ka}\}; am).$$

We will prove Theorem 1.8 in three cases based on the form and size of d and n.

Lemma 5.1. *Let d, a, and n be positive integers such that $a \geq 5$, $\frac{n+h_n^{(a)}}{a} \leq \frac{4(d+h_d^{(a)})}{a} + 2^{r_{d,a}}$, and $d+h_d^{(a)} \geq a2^{a+3} - a$. Then $q_d^{(a)}(n) \geq Q_d^{(a,-,-)}(n)$.*

Lemma 5.2. *Let d, a, and n be positive integers such that $a \geq 5$, $\frac{n+h_n^{(a)}}{a} \geq \frac{4(d+h_d^{(a)})}{a} + 2^{r_{d,a}}$, $d + h_d^{(a)} \geq a2^{a+3} - a$, and $\frac{d+h_d^{(a)}}{a} \neq 2^s - 1$ for $s \geq 1$. Then the chain of inequalities*

$$\begin{aligned} q_d^{(a)}(n) &\geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq \mathcal{G}^{(1)}_{\frac{d+h_d^{(a)}}{a}}(\frac{n+h_n^{(a)}}{a}) \\ &\geq \rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a}) = \rho_{(d+h_d^{(a)},a)}(T^a; n+h_n^{(a)}) \geq Q_d^{(a,-,-)}(n), \end{aligned}$$

holds.

Lemma 5.3. *Let d, a, and n be positive integers such that $a \geq 5$, $\frac{n+h_n^{(a)}}{a} \geq \frac{4(d+h_d^{(a)})}{a} + 2^{r_{d,a}}$, $d + h_d^{(a)} \geq a2^{a+3} - a$, and $\frac{d+h_d^{(a)}}{a} = 2^{r_{d,a}} - 1$. Then the chain of inequalities*

$$\begin{aligned} q_d^{(a)}(n) &\geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq \rho(T;\frac{n+h_n^{(a)}}{a}) \\ &\geq \rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a}) = \rho_{(d+h_d^{(a)},a)}(T^a; n+h_n^{(a)}) \geq Q_d^{(a,-,-)}(n), \end{aligned}$$

holds.

In proving all three of the lemmas, we use the following result.

Lemma 5.4. *Let d, a, and n be positive integers such that $a \geq 5$, $d + h_d^{(a)} \geq 2^{a+3}a - a$. Then,*

$$\rho_{(d+h_d^{(a)},a)}(T^a; n+h_n^{(a)}) \geq Q_d^{(a,-,-)}(n).$$

i	x_i	y_i
$a\alpha:$ (α is positive)	$(\frac{a}{2}\alpha+1)(d+3)-a$	$2(d+h_d^{(a)})(\alpha-1)+(d+h_d^{(a)})+2^{a-1}a$
$a\alpha+1$	$\left(\frac{a}{2}\alpha+1\right)(d+3)+a$	$2(d+h_d^{(a)})\alpha+a$
$a\alpha+\bar{i}:$ ($\bar{i}\geq 2$ and $\bar{i}$ is even)	$\left(\frac{a}{2}\alpha+\frac{\bar{i}+2}{2}\right)(d+3)-a$	$2(d+h_d^{(a)})\alpha+(d+h_d^{(a)})+2^{\bar{i}-1}a$
$a\alpha+\bar{i}:$ ($\bar{i}\geq 2$ and $\bar{i}$ is odd)	$\left(\frac{a}{2}\alpha+\frac{\bar{i}+1}{2}\right)(d+3)+a$	$2(d+h_d^{(a)})\alpha+(d+h_d^{(a)})+2^{\bar{i}-1}a$

TABLE 2. Values of x_i, y_i over $\bar{i}$ for even $a \geq 5$ where $\bar{i}$ and α are respectively the remainder and quotient from Euclidean division of i by a.

i	x_i	y_i
$a\alpha:$ (α is even and positive)	$\left(\frac{a}{2}\alpha+1\right)(d+3)-a$	$2(d+h_d^{(a)})(\alpha-1)+(d+h_d^{(a)})+2^{a-1}a$
$a\alpha:$ (α is odd and positive)	$\left(\frac{a}{2}\alpha+\frac{1}{2}\right)(d+3)+a$	$2(d+h_d^{(a)})(\alpha-1)+(d+h_d^{(a)})+2^{a-1}a$
$a\alpha+1:$ (α is even)	$\left(\frac{a}{2}\alpha+1\right)(d+3)+a$	$2(d+h_d^{(a)})\alpha+a$
$a\alpha+1:$ (α is odd)	$\left(\frac{a}{2}\alpha+\frac{3}{2}\right)(d+3)-a$	$2(d+h_d^{(a)})\alpha+a$
$a\alpha+\bar{i}:$ ($\bar{i}\geq 2$ and $\alpha \pmod 2 \not\equiv \bar{i}$)	$\left(\frac{a}{2}\alpha+\frac{\bar{i}+1}{2}\right)(d+3)+a$	$2(d+h_d^{(a)})\alpha+(d+h_d^{(a)})+2^{\bar{i}-1}a$
$a\alpha+\bar{i}:$ ($\bar{i}\geq 2$ and $\alpha \pmod 2 \equiv \bar{i}$)	$\left(\frac{a}{2}\alpha+\frac{\bar{i}+2}{2}\right)(d+3)-a$	$2(d+h_d^{(a)})\alpha+(d+h_d^{(a)})+2^{\bar{i}-1}a$

TABLE 3. Values of x_i, y_i over i for odd $a\geq 5$ where $\bar{i}$ and α are respectively the remainder and quotient from Euclidean division of i by a.

Proof of Lemma 5.4. Let S_d and T^a denote the set of parts of partitions counted by the functions $Q_d^{(a,-,-)}(n)$ and $\rho_{(d+h_d^{(a)},a)}(T^a;n+h_n^{(a)})$ respectively. We set $x_i\in S_d$ and $y_i\in T^a$ to denote the *ith* part of their respective sets with respective to increasing order. To compare the values of x_i, y_i, we use that $d+h_d^{(a)}\leq d+a$, $d+h_d^{(a)}\geq a2^{a+3}-a$, and $a\leq \frac{d}{15}$. Note that by considering various *is* and using Tables 2 and 3, we find that a sufficient bound for d to ensure $x_i\geq y_i$ is

$$d\geq 2^{a-1}a+2a-3. \tag{5.1}$$

By our assumption on d, inequality (5.1) is satisfied, hence $x_i\geq y_i$. Since $x_i\geq y_i$ for all positive i, we can apply Lemma 2.2 with $\rho(T;n+h_n^{(a)})=\rho_{(d+h_d^{(a)},a)}(T^a;n+h_n^{(a)})$ and $\rho(S;n)=Q_d^{(a,-,-)}(n)$ to obtain the result.

□

5.1. **Proof of Lemma 5.1.**

Proof of Lemma 5.1. In the case when $1\leq n\leq d+2a$, one may verify with direct calculation that the generalized Kang-Park conjecture holds for this range of n.

We now consider the case when $n>d+2a$. For these n we consider the following inequality chain:

$$q_d^{(a)}(n)\geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right)\geq \rho_{\left(\frac{d+h_d^{(a)}}{a},1\right)}\left(T;\frac{n+h_n^{(a)}}{a}\right)=\rho_{(d+h_d^{(a)},a)}(T^a;n+h_n^{(a)})\geq Q_d^{(a,-,-)}(n). \tag{5.2}$$

We note that the first inequality in (5.2) holds by Lemma 2.3. The last inequality holds by Lemma 5.4. The equality follows from the natural bijection of multiplying the parts by a.

Set $m=\frac{n+h_n^{(a)}}{a}$ and $k=\frac{d+h_d^{(a)}}{a}$. It suffices to show for $k\geq 1$ and $1\leq m\leq 5k+1$ the inequality

$$q_k^{(1)}(m)\geq \rho_{(k,1)}(T;m), \tag{5.3}$$

where $\rho_{(k,1)}(T;m)$ is the counting function whose parts satisfy the congruence properties

$$T=\{\lambda_i\equiv 1,k+2,\cdots,k+2^{a-1} \pmod{2k}\}.$$

We again use that $q_k^{(1)}(m)$ is a weakly increasing function. We also observe that $\rho_{(k,1)}(T;m)$ is weakly increasing since we can add 1 to be an additional part of a partition of m to obtain a partition for $m+1$.

We prove (5.3) for the interval $1 \leq m \leq 2k+6$. Note that both functions on the interval $1 \leq m \leq k+1$ return one, hence we suppose that $k+2 \leq m \leq 2k$. We notice that $\rho_{(k,1)}(T;k+2^i) = i+1$ for $1 \leq i \leq a-1$ and is constant on the intervals $2^i + k \leq m \leq k + 2^{i+1} - 1$. We note that $q_k^{(1)}(2^i + k) \geq i+1$ for $i \geq 1$ since the partitions $(k+2^i), (k+2^i-\alpha, \alpha)$ with $\alpha \leq 2^{i-1}$ are counted by $q_k^{(1)}(2^i+k)$.

Note that in the interval $k + 2^{a-1} \leq m \leq 2k$ that we have $\rho_{(k;1)}(T;m) = a$. Hence, we consider the value of the functions on the interval $2k+1 \leq m \leq 2k+6$. Since $k \geq 2^{a+3}-1$, we have that $k + 2^{a-1} < 2k+1$, yielding $q_k^{(1)}(2k+1) \geq a+1$. We have $\rho_{(k;1)}(T;2k+1) = a+1$. Note that $\rho_{(k;1)}(T;m)$ in this interval increases by 1 only at $2k+4$ and $2k+6$. However, each increase of 2 increases $q_k^{(1)}(m)$ by at least one yielding the desired inequality.

We now show (5.3) for the interval $2k+6 \leq m \leq 3k+1$. Since $q_k^{(1)}(m)$ weakly increasing, we will find a lower bound for $q_k^{(1)}(2k+6)$. The partitions of $2k+6$ of the form $(2k+6-i, i)$ where $i \in \{1, ..., \lfloor \frac{k}{2} + 3 \rfloor\}$ and the trivial partition $(2k+6)$ are counted by $q_k^{(1)}(2k+6)$. From this, we obtain the lower bound $q_k^{(1)}(2k+6) \geq \lfloor \frac{k}{2} + 3 \rfloor + 1$. By $k \geq 2^{a+3} - 1$, we find that

$$q_k^{(1)}(2k+6) \geq \lfloor \frac{2^{a+3}-1}{2} + 3 \rfloor + 1 = 2^{a+2} + 3.$$

Since $\rho_{(k,1)}(T;m)$ weakly increases, we will provide an upper bound for $\rho_{(k,1)}(T;m)(3k+1)$. The largest part that could be in a partition counted by $\rho_{(k,1)}(3k+1)$ is $2k+1$. We observe that $(2k+1, 1^k)$ is the only partition of $3k+1$ counted by $\rho(T;3k+1)$ that includes $2k+1$ since $3k+1-2k-1 = k$.

We now consider partitions counted by $\rho_{(k,1)}(T;3k+1)$ with largest part $k+2^\ell$ where $\ell \in \{1, 2, ..., a-1\}$. For each ℓ, we notice that $3k+1-k-2^\ell = 2k+1-2^\ell$. Therefore, any partition whose largest part is of the form $k+2^\ell$ can have at most one other element of the form $k+2^w$ with $w \in \{1, ..., \ell\}$. Thus, for each fixed $\ell \in \{1, ..., a-1\}$, there are at most $(\ell+1)$ partitions of $3k+1$ such that $k+2^\ell$ is the largest part. Finally, there is only one partition of $3k+1$ with largest part of 1. Therefore, we obtain that

$$\rho_{(k,1)}(T;3k+1) \leq \frac{(a+2)(a-1)}{2} + 1 + 1 = 1 + \frac{a(a+1)}{2}.$$

Note that since $2^{a+2} + 3 \geq \frac{a(a+1)}{2} + 1$ for $a \geq 1$, we have that (5.3) holds for this interval.

We now prove (5.3) for $3k+1 \leq m \leq 4k+1$. Since $q_k^{(1)}(m)$ weakly increasing, we will find a lower bound for $q_k^{(1)}(3k+1)$. The partitions of $3k+1$ of the form $(3k+1-i, i)$ for $i \in \{1, ..., \lfloor \frac{2k+1}{2} \rfloor\}$ and the trivial partition $(3k+1)$ are counted by $q_k^{(1)}(3k+1)$. Thus, we have that

$$q_k^{(1)}(3k+1) \geq \lfloor \frac{2k+1}{2} \rfloor + 1 \geq \lfloor \frac{2 \cdot (2^{a+3}-1)+1}{2} \rfloor + 1 = 2^{a+3}.$$

We now provide an upper bound for $\rho_{(k,1)}(T;m)$ in $3k+1 \leq m \leq 4k+1$ by obtaining an upper bound for $\rho(T;4k+1)$.

By repeating the same procedure by considering partitions with fixed largest part, there is one partition with largest part $4k+1$, there are at most $a-1$ partitions with largest part of the form $3k+2^\ell$ with $\ell \in \{1,2,...,a-1\}$, at most a partitions with largest part $2k+1$, and for each $h \in \{1,\cdots,a-1\}$ at most $(h+1)^2$ partitions with largest part $k+2^h$ by considering the partitions with one, two, and three parts that are not equal to 1. Finally, there is only one partition of $4k+1$ whose largest part is 1. Using this method, we have

$$\rho_{(k,1)}(T;4k+1) \leq 1+(a-1)+a+(2^2+3^2+...+a^2)+1 = 2a+\frac{a(a+1)(2a+1)}{6}.$$

Thus, we have that (5.3) holds for $3k+1 \leq m \leq 4k+1$ since $k \geq 2^{a+3}-1$.

Finally, we show that (5.3) holds for $4k+1 \leq m \leq 5k+1$. Since $q_k^{(1)}(m)$ weakly increasing, we will find a lower bound for $q_k^{(1)}(4k+1)$. The partitions of $4k+1$ of the form $(4k+1-i, i)$ where $i \in \{1, ..., \lfloor\frac{3k+1}{2}\rfloor\}$ and the trivial partition $(4k+1)$ are counted by $q_k^{(1)}(4k+1)$. This yields the lower bound,

$$q_k^{(1)}(4k+1) \geq \lfloor\frac{3k+1}{2}\rfloor+1 \geq \lfloor\frac{3\cdot(2^{a+3}-1)+1}{2}\rfloor+1 = 3\cdot 2^{a+2}.$$

We now provide an upper bound for $\rho_{(k,1)}(T;m)$ in $4k+1 \leq m \leq 5k+1$, which we'll do by obtaining an upper bound for $\rho(T;5k+1)$. The only partition with largest part $4k+1$ is $(4k+1,1^k)$. By repeating the same procedure used in the interval $2k+6 \leq m \leq 3k+1$, we find that there are at most $a(a-1)$ partitions with largest part of the form $3k+2^\ell$ with $\ell \in \{1,2,...,a-1\}$, at most $1+\frac{a(a+1)}{2}$ partitions with largest part $2k+1$ by considering various fixed second largest potential parts. We also find for each $h \in \{1,\cdots,a-1\}$ at most $(h+1)^3$ partitions whose largest part is $k+2^h$ by considering that there are at most four parts that are not equal to 1. Finally there is only one partition of $5k+1$ whose largest part is 1. We obtain by adding the upper bound

$$\rho_{(k,1)}(T;5k+1) \leq 1+a(a-1)+1+\frac{a(a+1)}{2}+\frac{a^2(a+1)^2}{4}.$$

Because $k \geq 2^{a+3}-1$, we have that (5.3) holds for this interval.

To summarize, we obtain bounds for $q_k^{(1)}(m)$ and $\rho_{(k,1)}(T;m)$ displayed in Table 4. Notice that for $a \geq 5$, this implies that $q_k^{(1)}(m) \leq \rho_{(k,1)}(T;m)$ for $2k+6 \leq m \leq 5k+1$.

m	$q_k^{(1)}(m)$	$\rho_{(k,1)}(T;m)$
$2k+6 \leq m \leq 3k+1$	$\geq 2^{a+2}+3$	$\leq 1+\frac{a(a+1)}{2}$
$3k+1 \leq m \leq 4k+1$	$\geq 2^{a+3}$	$\leq 2a+\frac{a(a+1)(2a+1)}{6}$
$4k+1 \leq m \leq 5k+1$	$\geq 3\cdot 2^{a+2}$	$\leq 2+a(a-1)+\frac{a(a+1)}{2}+\frac{a^2(a+1)^2}{4}$.

TABLE 4. Bounds on $q_k^{(1)}(m)$ and $\rho_{(k,1)}(T;m)$ for intervals of m.

$\square$

5.2. **Proof of Lemmas 5.2, 5.3, and Theorem 1.8.**

Proof of Lemma 5.2. Recall that we are proving the inequality

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq \mathcal{G}^{(1)}_{\frac{d+h_d^{(a)}}{a}}(\frac{n+h_n^{(a)}}{a})$$

$$\geq \rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a}) = \rho_{(d+h_d^{(a)},a)}(T^a;n+h_n^{(a)}) \geq Q_d^{(a,-,-)}(n).$$

The first and second inequalities follow by applying Lemmas 2.3 and 2.6. The third inequality follows from $\frac{d+h_d^{(a)}}{a} \geq 2^{a+3}-1$ and that there are more parts allowed for partitions counted by $\mathcal{G}^{(1)}_{\frac{d+h_d^{(a)}}{a},r_{d,a}}(\frac{n+h_n^{(a)}}{a})$ than $\rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a})$. The equality follows from multiplying corresponding parts by a. The last inequality holds by applying Lemma 5.4. □

Proof of Lemma 5.3. Recall that we are showing

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq \rho(T;\frac{n+h_n^{(a)}}{a})$$

$$\geq \rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a}) = \rho_{(d+h_d^{(a)},a)}(T^a;n+h_n^{(a)}) \geq Q_d^{(a,-,-)}(n),$$

where $\rho(T;\frac{n+h_n^{(a)}}{a})$ counts partitions with parts of the form

$$\{\lambda_i \equiv 1, \frac{d+h_d^{(a)}}{a}+2, \cdots, \frac{d+h_d^{(a)}}{a}+2^{r_{d,a}-1} \pmod{\frac{2(d+h_d^{(a)})}{a}}\}.$$

We observe that it suffices to prove Lemma 5.3 when $r_{d,a} = a+3$ since this yields the minimum number of congruence classes for $\rho(T;\frac{n+h_n^{(a)}}{a})$. The first inequality is justified by Lemma 2.3 for $n \geq d+2a$. Observe that the second inequality follows from Theorem 2.5. The third inequality follows from $\frac{d+h_d^{(a)}}{a} \geq 2^{a+3}-1$ and that there are more parts allowed for partitions counted by $\rho(T;\frac{n+h_n^{(a)}}{a})$ than $\rho_{(\frac{d+h_d^{(a)}}{a},1)}(T;\frac{n+h_n^{(a)}}{a})$. The equality follows from multiplying corresponding parts by a. The last inequality follows by applying Lemma 5.4. □

We finally finish the proof of Theorem 1.8.

Proof of Theorem 1.8. Employ Lemmas 5.1, 5.2, and 5.3 to obtain for positive integers d, a, n such that $a \geq 5$ and $d + h_d^{(a)} \geq 2^{a+3}a - a$,

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0.$$

By Remark 1.9, we have the desired result. □

6. A strengthening of the generalized Kang-Park conjecture

In this section, we provide a proof of Theorem 1.11 by using Conjecture 1.10. We also prove Conjecture 1.10 for $k+2 \leq m \leq 5k$ to obtain that Theorem 1.11 holds unconditionally for small n.

Proof of conditional part of Theorem 1.11. Via work done in the unconditional component of Theorem 1.11, we assume that $n \geq d+2a$. We employ the following inequality chain:

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq Q^{(1,-)}_{\frac{d+h_d^{(a)}}{a}-4}\left(\frac{n+h_n^{(a)}}{a}\right) = Q^{(a,-)}_{d+h_d^{(a)}-a-3}\left(n+h_n^{(a)}\right) \geq Q_d^{(a,-)}(n).$$

We note that the first inequality holds for $n \geq d+2a$ by Lemma 2.3. The second inequality follows by assuming Conjecture 1.10. The equality follows from the bijection by multiplying the parts of partitions counted by $Q^{(1,-)}_{\frac{d+h_d^{(a)}}{a}-4}\left(\frac{n+h_n^{(a)}}{a}\right)$ by a. We now show the last inequality.

Let X and Y be the set of partitions counted by $Q_d^{(a,-)}(n)$ and $Q^{(a,-)}_{d+h_d^{(a)}-3-a}\left(n+h_n^{(a)}\right)$ respectively. Let x_i and y_i denote the parts of partitions in X and Y with indexing with respect to increasing size. Note that for all $i \geq 1$ that we have $x_i \geq y_i$ since $h_d^{(a)} \leq a$. This allows us to apply Lemma 2.2, yielding $Q^{(a,-)}_{d+h_d^{(a)}-a-3}\left(n+h_n^{(a)}\right) \geq Q_d^{(a,-)}(n)$. □

We now proceed with the unconditional part of Theorem 1.11. We first prove Theorem 1.11 for $1 \leq n \leq d+2a$.

Lemma 6.1. *For $a \geq 1$, $\lceil \frac{d}{a} \rceil \geq 12$, and $1 \leq n \leq d+2a$,*

$$\Delta_d^{(a,-)}(n) \geq 0.$$

Proof. The case when $1 \leq n \leq a-1$ is trivial since $q_d^{(a)}(n) = Q_d^{(a,-)}(n) = 0$. Observe for $a \leq n \leq d+2+a$ that $q_d^{(a)}(n) \geq 1 \geq Q_d^{(a,-)}(n)$. Note for $d+3+a \leq n \leq d+2a$ that $q_d^{(a)}(n) \geq 2 \geq Q_d^{(a,-)}(n)$. □

We now give an unconditional proof of the second statement of Theorem 1.11 for the case where $\lceil \frac{d+2a}{a} \rceil \leq \lceil \frac{n}{a} \rceil \leq 5\lceil \frac{d}{a} \rceil$. For this we will need to use the following lemma.

Lemma 6.2. *For integers $k \geq 12$ and $k+2 \leq m \leq 5k$,*

$$q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m).$$

Proof of Lemma 6.2. Recall that $q_k^{(1)}(m)$ is weakly increasing since for any partition of positive integer m counted by the function $q_k^{(1)}(m)$, one can add 1 to the largest part of the partition to obtain a partition counted by $q_k^{(1)}(m+1)$. Similarly, $Q_{k-4}^{(1,-)}(m)$ is weakly increasing since for any partition of m counted by $Q_{k-4}^{(1,-)}(m)$, we can adjoin 1 as a part to create a partition of $m+1$ counted by $Q_{k-4}^{(1,-)}(m+1)$.

We begin by showing that $q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m)$ for $k+2 \leq m \leq 2k-4$. Note that $q_k^{(1)}(k+2) \geq 2$ since the partitions $(1, k+1)$ and $(k+2)$ are counted. Note that $Q_{k-4}^{(1,-)}(2k-4) = 2$ with associated partitions $(k, 1^{k-4}), (1^{2k-4})$. Thus within the interval $k+2 \leq m \leq 2k-4$ the inequality holds.

We now show $q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m)$ for $2k-3 \leq m \leq 3k-5$. We note that $Q_{k-4}^{(1,-)}(3k-5) = 5$ with associated partitions $((2k-1), 1^{k-4}), ((2k-3), 1^{k-2}), (k^2, 1^{k-5}), (k, 1^{2k-5}), (1^{3k-5})$, hence it suffices to show that $q_k^{(1)}(m) \geq 5$ in this interval. We notice that the partitions of

$2k-3$ in the set $\{(2k-3)\}\cup\{(2k-3-i,i) : i \in \{1,2,...,\lfloor\frac{k-3}{2}\rfloor\}\}$ are counted by $q_k^{(1)}(2k-3)$. Therefore, $q_k^{(1)}(2k-3) \geq 1+\lfloor\frac{k-3}{2}\rfloor$. Since $k \geq 12$, we have that $q_k^{(1)}(2k-3) \geq 5$ as desired.

We now show that $q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m)$ for $3k-4 \leq m \leq 4k-6$. We note that $Q_{k-4}^{(1,-)}(4k-6) = 11$ since there is one partition with largest part for each element in $\{3k-2, 3k-4\}$, two with largest part of $2k-1$, three with largest part for each element in $\{2k-3, k\}$, and one partition of largest part 1. Hence, it suffices to show $q_k^{(1)}(m) \geq 11$ in the interval. We notice that the partitions of $3k-4$ in the set $\{(3k-4)\}\cup\{(3k-4-i,i) : i \in \{1,2,...,k-2\}\}$ are counted by $q_k^{(1)}(3k-4)$. Therefore, $q_k^{(1)}(3k-4) \geq k-1 \geq 11$.

We now show that $q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m)$ for $4k-5 \leq m \leq 5k-8$. We find that $Q_{k-4}^{(1,-)}(5k-8) = 20$ since there is one partition with largest part for each element in $\{4k-3, 4k-5\}$, two with largest part for each element in $\{3k-2, 3k-4\}$, five with largest part $2k-1$, four with largest part for each element in $\{2k-3, k\}$, and one with largest part 1. Hence, it suffices to show $q_k^{(1)}(m) \geq 20$. We notice the partitions of $4k-5$ in the set $\{(4k-5)\}\cup\{(i, 4k-5-i) : i \in \{1,2,...,\lfloor\frac{3k-5}{2}\rfloor\}\}$ are counted by $q_k^{(1)}(4k-5)$. Note that $q_k^{(1)}(4k-5)$ also counts partitions of the form $(3k-6-j, j+k, 1)$ with $j \in \{1,2,...,\lfloor\frac{k-6}{2}\rfloor\}$ and partitions of form $(3k-8-\ell, \ell+1+k, 2)$ with $\ell \in \{1,2,...,\lfloor\frac{k-9}{2}\rfloor\}$. Therefore, $q_k^{(1)}(4k-5) \geq 1+\lfloor\frac{3k-5}{2}\rfloor+\lfloor\frac{k-6}{2}\rfloor+\lfloor\frac{k-9}{2}\rfloor$. Since $k \geq 12$, we have the desired inequality.

Finally, we prove that $q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m)$ for $5k-8 \leq m \leq 5k$. We find that $Q_{k-4}^{(1,-)}(5k) = 36$ since there is one partition with largest part for each element in $\{5k-4, 5k-6\}$, two with largest part for each element in $\{4k-3, 4k-5\}$, five with largest part for each element in $\{3k-2, 3k-4\}$, eight with largest part $2k-1$, six with largest part $2k-3$, five with largest part k, and one with largest part 1. Hence it suffices to show that $q_k^{(1)}(m) \geq 36$. We notice that the partitions of $5k-8$ in set $\{(5k-8)\}\cup\{(5k-8-i,i) : i \in \{1,2,...,2k-4\}\}$ are counted by $q_k^{(1)}(5k-8)$. In addition, $q_k^{(1)}(5k-8)$ counts the partitions of the form $(4k-9-j, k+j, 1)$ for $j \in \{1,\cdots,\lfloor\frac{2k-9}{2}\rfloor\}$. We also note $q_k^{(1)}(5k-8)$ counts the partitions of the form $(4k-11-\ell, k+1+\ell, 2)$ for $\ell \in \{1,\cdots,k-6\}$ and the partitions of the form $(4k-13-r, k+2+r, 3)$ for $r \in \{1,\cdots,\lfloor\frac{2k-15}{2}\rfloor\}$. Therefore,

$$q_k^{(1)}(5k-8) \geq 2k-4+\lfloor\frac{2k-9}{2}\rfloor+k-6+\lfloor\frac{2k-15}{2}\rfloor \geq 36$$

using the assumption that $k \geq 12$.

□

Proof of unconditional part of Theorem 1.11. We employ the same inequality chain as used in the conditional part of Theorem 1.11. Recall that,

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+h_d^{(a)}}{a}}\left(\frac{n+h_n^{(a)}}{a}\right) \geq Q^{(1,-)}_{\frac{d+h_d^{(a)}}{a}-4}\left(\frac{n+h_n^{(a)}}{a}\right) = Q^{(a,-)}_{d+h_d^{(a)}-a-3}\left(n+h_n^{(a)}\right) \geq Q_d^{(a,-)}(n).$$

All inequalities and equalities except for the second one are justified by our work in the conditional component of Theorem 1.11. Note that $d+h_d^{(a)}+2a \leq d+3a$. Hence, it suffices to show for $k \geq 12, k+2 \leq m \leq 5k$ that

$$q_k^{(1)}(m) \geq Q_{k-4}^{(1,-)}(m),$$

but this is resolved by Lemma 6.2.

□

7. Asymptotic Results

In this section, we provide a short proof of Theorem 1.13 by using the asymptotic methods developed in Duncan, et al. [6] and Alfes, et al. [3]. We also describe potential methods in proving more cases of Conjectures 1.3, 1.5 and extending the results of Theorem 1.6.

Proof of Theorem 1.13. One can repeat the arguments presented in the proof of [6, Theorem 1.9] to obtain,

$$\lim_{n\to\infty} \Delta_d^{(a)}(n) = \lim_{n\to\infty} q_d^{(a)}(n)\left(1 - \frac{Q_d^{(a)}(n)}{q_d^{(a)}(n)}\right) = +\infty.$$

Observe Remark 1.7 indicates that $\Delta_d^{(a,-)}(n) \geq \Delta_d^{(a)}(n)$, implying our desired result. □

7.1. **Potential methods and Future Directions.** We note from Theorem 1.4 that the remaining cases of Conjecture 1.3 are $3 \leq d \leq 90,\ 94 \leq d \leq 186$. The sub-case when $d \geq 31$ are divisible by 3 are addressed by [6, Proposition 4.1].

We now describe a potential method in resolving more cases of Conjecture 1.3. First, one should derive computationally effective asymptotic expressions for $q_k^{(1)}(m)$ and $Q_{k-3}^{(1,-)}(m)$ by using the work of Alfes, et.al [3]. One can then use these expressions to show $q_k^{(1)}(m) \geq Q_{k-3}^{(1,-)}(m) \geq 0$ for suitable k and use (4.2) to prove more small d cases of Conjecture 1.3.

From computational data of verifying for $k+2 \leq m \leq 100000$, it appears that this approach will be feasible for $k \geq 10$, hence the potential remaining cases of Conjecture 1.3 are $3 \leq d \leq 27$ by (4.2). Unfortunately, this method fails for those values of d since for $3 \leq k \leq 9$ computation suggests

$$\lim_{m\to\infty}\left(q_k^{(1)}(m) - Q_{k-3}^{(1,-)}(m)\right) = -\infty.$$

To address this problematic case, one can use the explicit asymptotic expressions in Duncan, et al. [6] for $q_d^{(3)}(n)$ and $Q_d^{(3,-)}(n)$ to find an $\Omega(d)$ such that $n > \Omega(d)$,

$$\Delta_d^{(3,-)}(n) \geq 0$$

to address the values of $d > 3$ such that $\gcd(d,3) = 1$. One can then employ a finite computation to show $\Delta_d^{(3)}(n) \geq 0$ for $n \leq \Omega(d)$ for these values of d. In a similar fashion as in the $a = 3$ case, one can extend Theorem 1.6 for $d + h_d^{(a)} \geq 10a$ for arbitrary a.

Unfortunately, due to the additional a parameter, it appears that reducing the status of Conjecture 1.5 to a finite computation is unlikely unless new combinatorial or analytic methods are developed. However, we believe that the reduction to an unconditional linear lower bound for d outlined in Theorem 1.11 is feasible by modifying the injection in Section 3.

Acknowledgements

We are immensely thankful for Holly Swisher in many useful conversations, comments, and deeply encouraging feedback regarding this paper. We also thank Robert Lemke Oliver for giving us his code which greatly aided our calculations of $q_k^{(1)}(m)$ and $Q_k^{(1)}(m)$ for large m.

References

[1] H. L. Alder. Research Problems. *Bulletin of the American Mathematical Society*, 62(1):76, January 1956.

[2] Henry L. Alder. The nonexistence of certain identities in the theory of partitions and compositions. *Bulletin of the American Mathematical Society*, 54(8):712–723, August 1948.

[3] Claudia Alfes, Marie Jameson, and Robert J. Lemke Oliver. Proof of the Alder-Andrews conjecture. *Proceedings of the American Mathematical Society*, 139(01):63–78, January 2011.

[4] George Andrews. On a partition problem of H. L. Alder. *Pacific Journal of Mathematics*, 36(2):279–284, February 1971.

[5] George E. Andrews. *The Theory of Partitions*, volume 2 of *Number Theory.* Addison-Wesley Publishing Company, Reading, Massachusetts, 1976.

[6] Adriana L. Duncan, Simran Khunger, Holly Swisher, and Ryan Tamura. Generalizations of Alder's conjecture via a conjecture of Kang and Park. *Res. Number Theory*, 7(1):Paper No. 11, 26, 2021.

[7] Soon-Yi Kang and Eun Young Park. An analogue of Alder–Andrews Conjecture generalizing the 2nd Rogers–Ramanujan identity. *Discrete Mathematics*, 343(7), July 2020.

[8] D. H. Lehmer. Two nonexistence theorems on partitions. *Bulletin of the American Mathematical Society*, 52(6):538–545, June 1946.

[9] Gunter Meinardus. Asymptotische Aussagen uber Partitionen. *Mathematische Zeitschrift*, 59:388–398, 1954.

[10] A. J. Yee. Partitions with difference conditions and Alder's conjecture. *Proceedings of the National Academy of Sciences*, 101(47):16417–16418, November 2004.

[11] Ae Ja Yee. Alder's conjecture. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2008(616), January 2008.

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