Discretization of N-S and Energy Equation with the ADI Scheme by Using Finite Differences

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1 Equations

A vorticity-stream function formulation is used in the N-S equation to cancel out the pressure gradient term. For this purpose a curl is taken on both sides of the N-S equation.According to vorticity stream function formulation:

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega, \tag{1}$$

$$\frac{\partial (U\Omega)}{\partial X} + \frac{\partial (V\Omega)}{\partial Y} = \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}} \left(\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} \right) + \frac{(\rho\beta)_{nf}}{\rho_{nf}\beta_f} RaPr \frac{\partial T}{\partial X}, \tag{2}$$

$$\frac{\partial (UT)}{\partial X} + \frac{\partial (VT)}{\partial Y} = \frac{\alpha_{nf}}{\alpha_f} \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right). \tag{3}$$

All three of the above equations are non-dimensional. Here, Ψ, Ω, and T are stream function, vorticity and temperature respectively. In order to obtain the values of dependent variables(Ψ, Ω, and T) equations (1), (2), and (3) are solved numerically. Velocity components for (2) and (3) are calculated from their typical mathematical relation with stream function. Initially, (1), (2), and (3) are discretized by the Alternation Direction Implicit (ADI) scheme. The convective terms are discretized by Upwind Implicit method for stability reasons and center differences are used for the diffusion terms. Afterwards, the set of linear equations obtained for each of the dependent variables are solved by Tri-Diagonal Matrix Algorithm (TDMA).

2 Discretization

Equation (1),

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega,$$

$$\Rightarrow \frac{\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}}{\Delta x^2} + \frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{\Delta y^2} = -\Omega_{i,j},$$

$$\Rightarrow \Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j} + \beta^2\Psi_{i,j+1} - 2\beta^2\Psi_{i,j} + \beta^2\Psi_{i,j-1} = \Delta x^2 \Omega_{i,j}. \tag{4}$$

Where, Δ x, Δ y are uniform difference between two nodes in x and y direction respectively. Also, β is the ratio of these two differences. Two of the velocity components are calculated from the related we can obtain from the definition of stream function. After solving equation (4) for the values of Ψ, equations (5) and (6) are used to calculate U and V respectively for (2) and (3).

$$U_{i,j} = \frac{\partial \Psi}{\partial Y} \quad = \quad \frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{2\Delta Y} \tag{5}$$

$$V_{i,j} = -\frac{\partial \Psi}{\partial X} \quad = \quad -\frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2\Delta x} \tag{6}$$

Equation (2),

$$\frac{\partial (U\Omega)}{\partial X} + \frac{\partial (V\Omega)}{\partial Y} = \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}} \left(\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} \right) + \frac{(\rho\beta)_{nf}}{\rho_{nf}\beta_f} RaPr \frac{\partial T}{\partial X},$$

$$\Rightarrow a_e\Omega_{i+1,j} + a_w\Omega_{i-1,j} + a_p\Omega_{i,j} + a_n\Omega_{i,j+1} + a_s\Omega_{i,j-1} = \frac{RaPr(\rho\beta)_{nf}}{2\rho_{nf}\beta_f\Delta x}(T_{i+1,j} - T_{i-1,j}). \tag{7}$$

where,
$a_e = \frac{(1-e_u)}{2\Delta x}U_{i+1,j} - \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}\Delta x^2}$
$a_w = -\frac{(1+e_u)}{2\Delta x}U_{i-1,j} - \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}\Delta x^2}$
$a_p = \frac{e_u}{\Delta x}U_{i,j} + \frac{e_v}{\Delta y}V_{i,j} + \frac{2\mu_{nf}}{\rho_{nf}\alpha_{nf}\Delta x^2} + \frac{\mu_{nf}}{2\rho_{nf}\alpha_{nf}\Delta y^2}$
$a_n = \frac{(1-e_v)}{2\Delta y}V_{i,j+1} - \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}\Delta y^2}$
$a_s = -\frac{(1+e_v)}{2\Delta y}V_{i,j-1} - \frac{\mu_{nf}}{\rho_{nf}\alpha_{nf}\Delta y^2}$
Equation (3),

$$\frac{\partial(UT)}{\partial X} + \frac{\partial(VT)}{\partial Y} = \frac{\alpha_{nf}}{\alpha_f}\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right).$$
$$\Rightarrow a_e T_{i+1,j} + a_w T_{i-1,j} + a_p T_{i,j} + a_n T_{i,j+1} + a_s T_{i,j-1} = 0. \tag{8}$$

where,
$a_e = \frac{(1-e_u)}{2\Delta x}U_{i+1,j} - \frac{\alpha_{nf}}{\alpha_f\Delta x^2}$

$a_w = -\frac{(1+e_u)}{2\Delta x}U_{i-1,j} - \frac{\alpha_{nf}}{\alpha_f\Delta x^2}$

$a_p = \frac{e_u}{\Delta x}U_{i,j} + \frac{e_v}{\Delta y}V_{i,j} + \frac{2\alpha_{nf}}{\alpha_f\Delta x^2} + \frac{\alpha_{nf}}{\alpha_f\Delta y^2}$

$a_n = \frac{(1-e_v)}{2\Delta y}V_{i,j+1} - \frac{2\alpha_{nf}}{\alpha_f\Delta y^2}$

$a_s = -\frac{(1+e_v)}{2\Delta y}V_{i,j-1} - \frac{\alpha_{nf}}{\alpha_f\Delta y^2}$

In equations (7) and (8) the convective terms are approximated by implicit upwind scheme as equation (9).In (7) and (8), e_u and e_v are 1 when the velocity components are in positive direction and -1 when they are in negative direction.

$$\frac{\partial(U\Gamma)}{\partial X} = \frac{(1-e_u)(U\Gamma)_{i+1,j} + 2e_u(U\Gamma)_{i,j} - (1+e_u)(U\Gamma)_{i-1,j}}{2\Delta x}. \tag{9}$$

3 Alternating Direction Implicit Scheme

For equation (4), horizontal sweep:

$$\Psi^k_{i-1,j} - 2(1+\beta^2)\Psi^k_{i,j} + \Psi^k_{i+1,j} = -\left[\beta^2\left(\Psi^{k-1}_{i,j-1} + \Psi^{k-1}_{i,j+1}\right) + \Delta x^2\Omega^{k-1}_{i,j}\right]. \tag{10}$$

And the vertical sweep:

$$\beta^2\Psi^k_{i,j-1} - 2(1+\beta^2)\Psi^k_{i,j} + \beta^2\Psi^k_{i,j+1} = -\left(\Psi^k_{i-1,j} + \Psi^k_{i+1,j} + \Delta x^2\Omega^{k-1}_{i,j}\right). \tag{11}$$

In the same way for equation (7), horizontal sweep:

$$a_w\Omega^k_{i-1,j} + a_p\Omega^k_{i,j} + a_e\Omega^k_{i+1,j} = -\left(a_s\Omega^{k-1}_{i,j-1} + a_n\Omega^{k-1}_{i,j+1}\right) + \frac{RaPr(\rho\beta)_{nf}}{2\rho_{nf}\beta_f\Delta x}(T^{k-1}_{i+1,j} - T^{k-1}_{i-1,j}). \tag{12}$$

And the vertical sweep:

$$a_s\Omega^k_{i,j-1} + a_p\Omega^k_{i,j} + a_n\Omega^k_{i,j+1} = -\left(a_w\Omega^k_{i-1,j} + a_e\Omega^k_{i+1,j}\right) + \frac{RaPr(\rho\beta)_{nf}}{2\rho_{nf}\beta_f\Delta x}(T^{k-1}_{i+1,j} - T^{k-1}_{i-1,j}). \tag{13}$$

For equation (8), horizontal sweep:

$$a_w T^k_{i-1,j} + a_p T^k_{i,j} + a_e T^k_{i+1,j} = -\left(a_s T^{k-1}_{i,j-1} + a_n T^{k-1}_{i,j+1}\right). \tag{14}$$

And the vertical sweep:

$$a_s T^k_{i,j-1} + a_p T^k_{i,j} + a_n T^k_{i,j+1} = -\left(a_w T^k_{i-1,j} + a_e T^k_{i+1,j}\right). \tag{15}$$

Values on the right hand side of the equations (10) to (15) are obtained from the initial guess or previous iterations. The superscript k denotes a certain iteration. For the vertical sweeps k indicates the value of the independent variable obtained from the previous horizontal sweep. The error will be calculated from the differences of vertical and horizontal sweep.

4 Tri-Diagonal Matrix Algoritm

Each of the equations from (10) to (15) can be written in a generalized form as below,

$$A\Gamma_{n-1}^{k} + B\Gamma_{n}^{k} + C\Gamma_{n+1}^{k} = D^{k-1}. \tag{16}$$

Here, k denotes the node of interest(on a certain number of i or j) and n signifies the current iteration number.
If the domain is discretized in n×m nodes, then for each horizontal or vertical sweep there will be a n×n or m×m tri-diagonal coefficient matrix respectively. Each of those matrices will be solved in each iteration by TDMA algorithm which takes less time [$O(N)$] for solving a matrix ($N \times N$) compared to Gauss Elimination method [$O(N^3)$] or any iterative methods [$O(N^2)$].