1. Assignment on

Computational Complexity Theory — WiSe 2012/13

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Due on 23. October 2012 in the tutorial session

Aufgabe 1 Collecting Stickers *10 points*

The final installment of the Twilight series will soon hit the movie theaters. Thus, every bag of Bircher-Müesli contains a sticker with the likeness of a Twilight-character. We assume that there are n different stickers, and that each bag contains a sticker that is chosen uniformly at random. We would like to know how many bags of Müesli we need to buy until our collection is complete.

Let X be the random variable that represents the required number of bags. We would like to find the expected value $\mathrm{E}[X]$.

(a) We subdivide the process into *rounds*. A round ends as soon as we obtain a sticker that we have not seen before. Thus, the first round ends with the first sticker, the second round ends as soon as we have two distinct stickers, and so on. The *length* of a round is defined as the number of Müesli bags that we need to buy during that round.

Let X_i be the random variable that represents the length of round i. Show that $\mathrm{E}[X] = \sum_{i=1}^{n} \mathrm{E}[X_i]$.

(b) Find $\mathrm{E}[X_i]$.

Hint: X_i follows a geometric distribution.

(c) Conclude that $\mathrm{E}[X] = O(n \log n)$.

Hint: $\sum_{i=1}^{n} (1/i) = O(\log n)$.

Aufgabe 2 Knapsack *10 points*

In the knapsack problem, we are given n items. Each item has a *weight* g_i and a *value* w_i. Furthermore, we have a maximum weight G. All inputs are positive integers.

We would like to find a set $I \subseteq \{1, \dots, n\}$ of items, such that the total value $\sum_{i \in I} w_i$ is maximum, subject to the constraint that the total weight is at most G, i.e., $\sum_{i \in I} g_i \leq G$.

(a) Define an appropriate decision version for the knapsack problem and show that it is NP-complete.

Hint: Reduce from SUBSET-SUM.

(b) Let $W := \sum_{i=1}^{n} w_i$. Show that the knapsack problem can be solved in $O(nW)$ time. Why does this not contradict (a)?

(c) Let $\varepsilon \geq 0$. Show that we can find in $\text{poly}(n, 1/\varepsilon)$ time a solution for the knapsack problem with value at least $(1 - \varepsilon)\text{OPT}$, where OPT is the value of an optimal solution.

Hint: Round appropriately and use (b).

Aufgabe 3 NP-completeness *10 points*

True or false? For every $\varepsilon > 0$, there exists an NP-complete problem that can be solved deterministically in $O(2^{n^\varepsilon})$ steps. Explain your answer.