Exercise sheet 1

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Problem 1 Computing the minimum

1. Let $X_i \in \{0,1\}$ be the random variable that indicates whether line (*) is executed in the i-th iteration of the for-loop. Show that $E[X] = \sum_{i=2}^{n} E[X_i]$.

   Since $X = \sum_{i=2} nX_i$, it holds that

$$E[X] = E\left[\sum_{i=2}^{n} X_i\right] \overset{(*)}{=} \sum_{i=2}^{n} E[X_i] \tag{1}$$

   where (*) holds by linearity of the expected value.

2. Find $E[X_i]$.

   Since the numbers are pairwise distinct, the probability that at some fixed position k $A[k]$ is minimal, is given by $Pr[A[k] \text{ minimal}] = \frac{1}{i}$, where i is the number of distinct numbers. It follows that

$$\begin{aligned} E[X_i] &\overset{\text{Def.}}{=} \sum_{a\in\{0,1\}} a \cdot Pr[X_i = a] \\ &= Pr[X_i = 1] = Pr[A[i] \text{ minimal in } i \text{ elements}] = \frac{1}{i} \end{aligned} \tag{2}$$

3. Conclude that $E[X] = O(\log n)$.

$$\begin{aligned} E[X] &\overset{(1)}{=} \sum_{i=2}^{n} E[X_i] = \sum_{i=2}^{n} \frac{1}{i} \\ &\leq \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n) \end{aligned} \tag{3}$$

Problem 2 Induction

Let L be a set of n lines in the plane. We would like to assign a color to each face such that no two adjacent faces have the same color. Show that two colors are always sufficient.

Proof by induction:

Base step : $n = 1$

Since L consists of only one line, the plane is divided into two halfs. We can color the one black, and the other one white in order to gain a valid coloring.

Induction step : $n \rightsquigarrow n + 1$

Now consider the set L of $n + 1$ lines. By removing one line, say l, we gain a set of n lines which can, by induction assumption, be colored properly. If we add the $n + 1$-st line l again, it divides the plane into two halfs.

Let $L_1, \ldots, L_a$ be the faces that are entirely on the 'left side' of this plane and let $R_1, \ldots, R_b$ be the faces that are entirely on the 'right side' of this plane. Let $M_1, \ldots, M_c$ be the faces that are divided by l, where M_i^l denotes the left part of the divided face M_i and M_i^r the right side.

The coloring of the set L containing $n+1$ lines is achieved by the following rules:

L_i keep their coloring, R_i swap their coloring, M_i^l takes the color of M_i and M_i^r takes the opposite color of M_i^l.

Claim: The coloring gained by the above rules is valid.

Proof: The coloring of L_i and M_i^l are valid by induction assumption, and since the coloring of R_i and M_i^r were valid by induction assumption, they are still valid with swapped colors. The borders of the faces along the $n+1$st line are valid since the other side has the opposite color. If we assume general position no two faces R_i and L_j can share an edge, because then two lines have to equal. □

Problem 3 O-Notation

1. $\log(n!) = \Theta(n\log(n))$ holds since
   (1) $\log(n!) = O(n\log(n))$

$$\log(n!) = \sum_{i=1}^{n} \log i \leq n \cdot \log n \tag{4}$$

   (2) $\log(n!) = \Omega(n\log(n))$

$$\begin{aligned} \log(n!) &= \sum_{i=1}^{n} \log i \geq \sum_{i=\frac{n}{2}}^{n} \log i \\ &\geq \frac{n}{2}\log(\frac{n}{2}) = \Omega(n\log(n)) \end{aligned} \tag{5}$$

2. $\log(mn) = O(\log(n+m))$ holds since

$$\begin{aligned} \log(mn) &= \log(m) + \log(n) \leq \log(m+n) + \log(n+m) \\ &= 2\log(n+m) = O(\log(n+m)) \end{aligned} \tag{6}$$

3. Let $f, g \geq 2$ and $f(n) = O(g(n))$
   (a) $\sqrt{f(n)} = O(\sqrt{g(n)})$ holds since

$$\begin{aligned} \sqrt{f(n)} &\leq \sqrt{c \cdot g(n)} \\ &= \sqrt{c} \cdot \sqrt{g(n)} = O(\sqrt{g(n)}) \end{aligned} \tag{7}$$

   (b) $2^{f(n)} = O(2^{g(n)})$ does not hold, e.g. choose $f := 2 \cdot g$, then

$$2^{f(n)} = 2^{2 \cdot g(n)} = 2^{2^{g(n)}} \neq O(2^{g(n)}) \tag{8}$$