Exercise sheet 2

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Problem 1 Convexity

1. Let $\{C_i\}_{i\in I}$ be a set of convex sets. Show that $\bigcap_{i\in I} C_i$ is convex.

   Proof: Since each C_i is convex, it holds that the line $\overline{pq}$, for $p, q \in C_i$, is completely contained in C_i. Since for all $p, q \in \bigcap_{i\in I} C_i$, the segment $\overline{pq}$ is contained in each C_i, it holds that $\overline{pq} \in \bigcap_{i\in I} C_i$.
   $\Rightarrow \bigcap_{i\in I} C_i$ is convex. □

   A similar property can be found for unions of convex sets:
   Claim: For any non-decreasing series of convex sets $(C_i)_{i\in I}$ (with respect to set inclusion), the set $\bigcup_{i\in I} C_i$ is convex.
   Proof: For all $p, q \in \bigcup_{i\in I} C_i$ there exists a $\tilde{i} \in I$ such that $p, q \in C_{\tilde{i}}$, hence $\overline{pq} \in C_{\tilde{i}}$. Since $(C_i)_{i\in I}$ is non-decreasing, $\overline{pq} \in \bigcup_{i\in I} C_i$. □
2. Let P be a finite point set in the plane. Show that the boundary of the convex hull $CH(P)$ of P is a convex polygon whose vertices are points of P.

   Proof:
   We first proove, that $CH(P)$ is a polygon and second, the points of the polygon are points of P.

   Let C be an arbitrary convex set containing P. Assume C is not a polygon. Fix a point x on that curve [1]. We now take $\varepsilon > 0$ steps on the curve to the point x_ε and look at the set C' where we substituted the line $\overline{xx_\varepsilon}$ for the curveseqment that connected them before.
   We know if ε is small enough there is no point of P in the cut part. If the curve was bend left, the resulting line will only do left turns at the end. Therefore C' is convex.
   Hence C could not be the convex hull.

   Next we fix a convex polygon CP that has points of P on the nodes. This can be found by intersecting all halfplanes that contain all points in P as in the Brute-Force algorithm.

   Because CP is a convex set containing P only $CH(P) \subset CP$ can hold.

   Assume there is a point on the polygon $CH(P)$ $y = p_k \notin P$. Then $p_{k-1}p_kp_{k+1}$ is a triangle pointing to the outside of the polygon. Otherwise we would have taken a left turn in cw order. But this is not possible due to the previous shown fact, that a polygon only of points of P is a convex set containing P.

[1] We assume complete sets. If the set has no boundary we consider a series of points converging to the boundary.

Hence p_k can not be on the hull.

$\square$

3. Show that the segment between two points $p, q \in P$ is an edge of $CH(P)$ if and only if all points of P lie on the same side of the line through p and q.

Proof:
"$\Rightarrow$": By contraposition. Let $\tilde{p}$ a point on the one side of the line through p and q and $\tilde{q}$ a point on the other side. Let $s \in \overline{pq}$. Then, one of $\overline{s\tilde{p}}$ and $\overline{s\tilde{q}}$ is not contained in P. Hence, $\overline{pq}$ cannot be an edge of $CH(P)$.
"$\Leftarrow$":
By the previous part, we know that the $CH(P)$ is a polygon of the points in P. Because $p, q \in P$ we know that $\overline{pq} \in CH(P)$.

Assume $\overline{pq}$ is not on the boundary. Then there exists points $u, v \in P$ on both sides of $\overline{pq}$ by the previous part. But this is not possible due to the premise.

$\square$

Problem 2 Computing the maximal tangent to a subconvex hull in Chens Algorithm in $O(logh^*)$.

Proof:

Let p_{k-1}, p_k be the last computed points on the convex hull. Let $q_1, ..., q_{h^*}$ be the points on the convex hull of the subconvex hull given in ccw or cw order. Then we will compute the point on the hull that maximizes the angle as follows. Assume we have the points given in an array `hull`.

```
pos ← 1
h'← h* / 2
while h' > 0 do
  l ← deg(p_{k-1},p_k,hull[pos-1 mod h^*]))
  c ← deg(p_{k-1},p_k,hull[pos mod h^*]))
  r ← deg(p_{k-1},p_k,hull[pos+1 mod h^*]))
  if l < c && c < r || l == r
    pos ← pos + h'␣mod␣h*
  else if l > c && c > r
    pos ← pos - h' mod h*
  h'← h' / 2
od
return pos
```

Next we have to verify the algorithm.

Claim 1. *The above algorithm maximizes the angle and can be computed in* $O(\log\ h^*)$ *time.*

⌟

Proof 1:
The running time of the algorithm is obviously $O(\log\ h^*)$. We start with $\frac{h^*}{2}$. In the while loop we only compute for constant time c. We decrease h' until we hit zero.

$$\Rightarrow T(n) = logh^*.$$

We know that $\sum\limits_{i=1}^{\infty} \frac{h^*}{2} = h^*$ and if our steps are discrete we can reach the point in $logh^*$ time (Binary Search). Let R_i be the set of reachable points on the hull in step i of the algorithm and q the optimal point. Then

$$\forall i \in \mathbb{N} : q \in R_i, |R_i| = \frac{h^*}{2^i}$$

Inducion on i.

I.A. $i = 1$
In the first step we can do at most $\sum_{i=1}^{\infty} \frac{h^*}{2^i} = h^*$ steps. Therefore all points are reachable espacially q.

I.S. $i \rightsquigarrow i + 1$
We now in $q \in R_i$. On the last step we were according to the algorithm at one endpoint and we were on the edge of R_i. In the algorithm we devide `h'` by two, therefore we are in the middle of the set according to the given order because it had size `h'` $= |R_i|$. We divide R_i into two subsets R^l_{i+1} and R^r_{i+1}. If the first case is true, we know $q \in R^l_{i+1}$ because the hull was convex therefore we can only decrease the angle again on the otherside of the optimum. We take $R_{i+1} = R^l_{i+1}$ as the algorithm says. The othercase is equivalent with R^r_{i+1}.
$|R_{i+1}| = |\frac{R_i}{2}|$.

□

Problem 3 Chan's algorithm and superexponential search

1.
2.