Exercise sheet 1

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Problem 1 Computing the minimum

1. Let $X_i \in \{0, 1\}$ be the random variable that indicates whether line (*) is executed in the i-th iteration of the for-loop. Show that $E[X] = \sum_{i=2}^n E[X_i]$.

   Since $X = \sum_{i=2} nX_i$, it holds that

   $$E[X] = E\left[\sum_{i=2}^n X_i\right] \stackrel{(*)}{=} \sum_{i=2}^n E[X_i] \tag{1}$$

   where (*) holds by linearity of the expected value.

2. Find $E[X_i]$.

   Since the numbers are pairwise distinct, the probability that at some fixed position k $A[k]$ is minimal, is given by $Pr[A[k] \text{ minimal}] = \frac{1}{i}$, where i is the number of distinct numbers. It follows that

   $$E[X_i] \stackrel{\text{Def.}}{=} \sum_{a \in \{0,1\}} a \cdot Pr[X_i = a]$$
   $$= Pr[X_i = 1] = Pr[A[i] \text{ minimal in } i \text{ elements}] = \frac{1}{i} \tag{2}$$

3. Conclude that $E[X] = O(\log n)$.

   $$E[X] \stackrel{(1)}{=} \sum_{i=2}^n E[X_i] = \sum_{i=2}^n \frac{1}{i}$$
   $$\leq \sum_{i=1}^n \frac{1}{i} = H_n = O(\log n) \tag{3}$$

Problem 2 Induction

Problem 3 O-Notation

1. $\log(n!) = \Theta(n \log(n))$
   (1) $\log(n!) = O(n \log(n))$

   $$\log(n!) = \sum_{i=1}^n \log i \leq n \cdot \log n \tag{4}$$

   (2) $\log(n!) = \Omega(n \log(n))$

   $$TBD \tag{5}$$

2. $\log(mn) = O(\log(n + m))$ holds since

$$
\begin{aligned}
\log(mn) &= \log(m) + \log(n) \le \log(m + n) + \log(n + m) \\
&= 2\log(n + m) = O(\log(n + m))
\end{aligned}
\tag{6}
$$

3. Let $f, g \ge 2$ and $f(n) = O(g(n))$

   (a) $\sqrt{f(n)} = O(\sqrt{g(n)})$

$$
\begin{aligned}
\sqrt{f(n)} &= (f(n))^{\frac{1}{2}} \le (c \cdot g(n))^{\frac{1}{2}} \\
&= \sqrt{c} \cdot \sqrt{g(n)} = O(\sqrt{g(n)})
\end{aligned}
\tag{7}
$$

   (b) $2^{f}(n) = O(2^{g}(n))$

$$
TBD \tag{8}
$$