1 Randomized Rounding

1.1 Set Cover

Given is $E = \{e_1, ..., en\}$ set of elements, $S_1, .., S_m$ with $S_i \subset E \forall 1 \leq i \leq m$ and weights $w_j \geq 0 \forall j \in [m]$.
We want to find $I \subset \{1, ..., m\}$ such that $\sum_{j \in I} w_j$ is minimized subject ot $underset j \in I \bigcup S_j = E$.

Integer Program

$$
\begin{array}{lcl}
\min & \sum_{j=1}^{m} w_j x_j & \\
s.t. & \underset{j:e_i \in S_j}{x}_j & \geq \quad 1, \quad i = 1, .., n \\
x_k \in \{0, 1\}, \quad j = 1, .., n & &
\end{array}
$$

Solve the relaxation of the IP you get the optimal solution X^* for the LP. Next we want to round every component $x_j \mapsto \begin{cases} 1 \\ 0 \end{cases}$ To gain an integer solution.

We know from the relaxed condition, that $0 \leq x_j^* \leq 1$ so we will set $P[S_j \text{ in the solution}] = x_j^*$.

Now $X_j = \begin{cases} 1 & S_j \text{ in the solution} \\ 0 & \text{otherwise} \end{cases}$ is the random variable that indicates, whether S_j is in the solution.

With this we can calculate the expected cost of the integer program.

$$
\begin{aligned}
E[\sum_{j=1}^{m} w_j x_j &= \sum_{j=1}^{m} w_j P[x_j = 1] \\
&= \sum_{j=1}^{m} w_j x_j^* \\
&= Z_{LP}^* \leq Z_{IP}^* = OPT
\end{aligned}
$$

This looks to good to be true and our suspissions are fullfilled, because with this simple rounding we might not have a feasable solution.

We now look at the probaility with which an elemnt is not covered.

$$
\begin{array}{rcl}
\forall e_i \ : \ P[e_i \text{ not covered}] & = & \prod\limits_{j:e_i \in S_j} (1 - x_j^*) \\
& \leq & \prod\limits_{j:e_i \in S_j} e^{-x_j^*} \\
& = & e^{-\sum\limits_{j:e_i \in S_j 0} x_j^*} \\
& \overset{LPconstraint}{\geq} & e^{-1}
\end{array}
$$

With this error we will now construct an algorithm, that satisfies the following conditions

- The Failiureprobability has to be less than n^{-c} with c constant.
- We will run the algorithm many times, until it is satisfied.
- The constant has to be great enough, that the expected times of iterations is small.

Solution Idea:
Toss a coin for $S_j \quad \forall j$ more than once.

To be precise, we will toss the coin $c \cdot \log n$ times. The probabilities will be the same es in the first rounding.

Put S_j in the solution if the coin comes up heads AT LEAST once $P[X_j = 1] = (1 - x_j)^{c \cdot \ln n}$ so the probability, that an edge was not covered is

$$
\begin{array}{rcl}
P[e_i \text{ not covered}] & = & \prod\limits_{j:e_i \in S_j} (1 - x_j)^{c \cdot \log n} \\
& \leq & \prod\limits_{j:e_i \in S_j} e^{-x_j^*(c \cdot \log n)} \\
& = & e^{-(c \cdot \log n) \sum x_j^*} \\
& \leq & \frac{1}{n^c}
\end{array}
$$

Now we have to merge all the probabilites, but keep in mind, that they are not indipended

$$
\begin{array}{rcl}
P[\exists e_t \text{ not covered}] & \leq & \sum\limits_{i=1}^{n} P[e_i \text{ not covered}] \\
& \leq & n \cdot \frac{1}{n^c} \\
& \leq & \frac{1}{n^{c-1}}
\end{array}
$$

Theorem 1. *$\mathcal{A}$ is a (randmized) $O(\log n)$-approximation algorithm and produces a set-cover with high probability.*

Proof 1:

To proof this, we will first show a slitly stronger claim on the approximation factor:

If $\mathcal{A}$ returnes a set-cover, that the approximation-factor is $O(\log\, n)$.

$x_j^* \in [0,1],\ c\log\, n \geq 1$
$p_j'(x_j) = (c\log\, n)(1-x_j^*)^{c\log\, n-1} \leq c\log n$
$pj(0) = 0 \Rightarrow p_j(x_j^*) \leq (c\log\, n)x_j^*$

$$\begin{aligned}
E[\sum_{j=1}^{m} w_j x_j] &= \sum_{j=1}^{m} w_j P[X_j = 1] \\
&\leq \sum_{j=1}^{m} w_j (c\log\, n) x_j* \\
&= c\log\, n \cdot z_{LP}^* \\
&\leq OPT
\end{aligned}$$

Next we have to proof the more general case.

$A_1, ..., A_n$ disjoined events that form a partition fo the sample space:

$$\begin{aligned}
E[X] &= \sum_{i}^{n} P(A_i) \cdot E[X \mid A_i] \\
&= \sum_{x} xPX|A(x) \\
&= \frac{P(\{X=x\}\cap A)}{P(A)}
\end{aligned}$$

Let F be events, when the solution is feasale and $\overline{F}$ the complement of F in sample space.
And we conclude the probability $P[F] \geq 1 - \frac{1}{n^{c-1}}$

$$E[\sum_{i=1}^{m} w_j X_j] = E[\sum_{j=1}^{m} w_j X_j|F] \cdot P[F] + E[\sum_{j=1}^{m} w_j X_j|\overline{F}] \cdot P[\overline{F}]$$

$$\begin{aligned}
E[\sum |F] &\geq \frac{1}{P[F]}\left(E[\sum] - E[\sum|\overline{F}]P[\overline{F}]\right) \\
&\leq \frac{1}{P[F]} E[\sum w_j x_j] \\
&\leq \frac{c\log\, n z_{LP}^*}{1-\frac{1}{n^{c-1}}} \\
&\leq 2c(\log\, n)Z_{LP}^* \leq 2c(\log\, n)OPT
\end{aligned}$$

$\square$