Exercise sheet 1

Max Wisniewski, Alexander Steen

Problem 1 Computing the minimum

1. Let $X_i \in \{0,1\}$ be the random variable that indicates whether line (*) is executed in the i-th iteration of the for-loop. Show that $E[X] = \sum_{i=2}^{n} E[X_i]$.

   Since $X = \sum_{i=2} nX_i$, it holds that

   $$E[X] = E\left[\sum_{i=2}^{n} X_i\right] \overset{(*)}{=} \sum_{i=2}^{n} E[X_i] \tag{1}$$

   where (*) holds by linearity of the expected value.
2. Find $E[X_i]$.

   Since the numbers are pairwise distinct, the probability that at some fixed position k $A[k]$ is minimal, is given by $Pr[A[k] \text{ minimal}] = \frac{1}{i}$, where i is the number of distinct numbers. It follows that

   $$\begin{aligned} E[X_i] &\overset{\text{Def.}}{=} \sum_{a \in \{0,1\}} a \cdot Pr[X_i = a] \\ &= Pr[X_i = 1] = Pr[A[i] \text{ minimal in } i \text{ elements}] = \frac{1}{i} \end{aligned} \tag{2}$$
3. Conclude that $E[X] = O(\log n)$.

   $$\begin{aligned} E[X] &\overset{(1)}{=} \sum_{i=2}^{n} E[X_i] = \sum_{i=2}^{n} \frac{1}{i} \\ &\le \sum_{i=1}^{n} \frac{1}{i} = H_n = O(\log n) \end{aligned} \tag{3}$$

Problem 2 Induction

Problem 3 O-Notation

1. $\log(n!) = \Theta(n \log(n))$ holds since
   (1) $\log(n!) = O(n \log(n))$

   $$\log(n!) = \sum_{i=1}^{n} \log i \le n \cdot \log n \tag{4}$$

   (2) $\log(n!) = \Omega(n \log(n))$

   $$\begin{aligned} \log(n!) &= \sum_{i=1}^{n} \log i \ge \sum_{i=\frac{n}{2}}^{n} \log i \\ &\ge \frac{n}{2} \log(\frac{n}{2}) = \Omega(n \log(n)) \end{aligned} \tag{5}$$

2. $\log(mn) = O(\log(n+m))$ holds since

$$\begin{aligned}\log(mn) &= \log(m) + \log(n) \leq \log(m+n) + \log(n+m) \\ &= 2\log(n+m) = O(\log(n+m))\end{aligned} \tag{6}$$

3. Let $f, g \geq 2$ and $f(n) = O(g(n))$

   (a) $\sqrt{f(n)} = O(\sqrt{g(n)})$ holds since

$$\begin{aligned}\sqrt{f(n)} &= (f(n))^{\frac{1}{2}} \leq (c \cdot g(n))^{\frac{1}{2}} \\ &= \sqrt{c} \cdot \sqrt{g(n)} = O(\sqrt{g(n)})\end{aligned} \tag{7}$$

   (b) $2^{f(n)} = O(2^{g(n)})$ does not hold, e.g. choose $f := 2 \cdot g$, then

$$2^{f(n)} = 2^{2 \cdot g(n)} = 2^{2^g(n)} \neq O(2^g(n)) \tag{8}$$