Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 607934, 14 pages
http://dx.doi.org/10.1155/2013/607934

Research Article

Fuzzy Z-Continuous Posets

S. P. Rao[1,2] and Q. G. Li[1]

[1] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*
[2] *Faculty of Sciences, Nanchang Institute of Technology, Nanchang 330099, China*

Correspondence should be addressed to Q. G. Li; liqingguoli@yahoo.com.cn

Received 27 April 2013; Revised 7 July 2013; Accepted 8 July 2013

Academic Editor: Stefan Siegmund

The aim of this paper is to generalize fuzzy continuous posets. The concept of fuzzy subset system on fuzzy posets is introduced; some elementary definitions such as fuzzy Z-continuous posets and fuzzy Z-algebraic posets are given. Furthermore, we try to find some natural classes of fuzzy Z-continuous maps under which the images of such fuzzy algebraic structures can be preserved; we also think about fuzzy Z-continuous closure operators in alternative ways. An extension theorem is presented for extending a fuzzy monotone map defined on the Z-compact elements to a fuzzy Z-continuous map defined on the whole set.

1. Introduction

The concept of continuous lattice was initiated by Scott in [1, 2] in a topological manner as a mathematical tool in computer sciences (domain theory). Although it has appeared in other fields of mathematics as well, such as general topology, real analysis, algebra, category theory, logic, this concept was defined later in purely order theoretical terms and now has become the one used in almost all references.

To introduce higher type variables into recursion equations, Wright et al. [3] introduced the notion of subset systems Z in the seventies, replacing the system of all directed subsets by other types of subsets enjoying a certain stability property under the monotone maps. Here, the authors devoted their study to the so-called Z-inductive posets, which is really a generalization of Z version of algebraic posets. At the end of the paper, the authors suggested an attempt to study the generalized counterpart of continuous poset (lattice) obtained by replacing directed subsets by Z-subsets, where Z is an arbitrary subset system. That undertaking was begun by Bandelt and Erné [4, 5] and independently by Novak [6]; there was subsequent research concerned by other authors [7–9].

On the other side, quantitative domain theory has been developed to supply models for concurrent systems. Now it forms a new focus on domain theory and has undergone active research. Rutten's generalized (ultra)metric spaces [10], Flagg's continuity spaces [11], and Wager's Ω-categories [12] are good examples, which consist of basic frameworks of quantitative domain theory (cf. [13]).

In [13], Zhang and Fan investigated quantitative domains based on frames. From the work go, they defined a fuzzy partial order which is really a degree function on a nonempty set; afterwards, they defined and studied fuzzy dcpos and fuzzy domains. Yao and Shi [14, 15] studied fuzzy dcpos and their continuity over complete residuated lattices; Su and Li [16] discussed algebraic fuzzy dcpos and exploited their relationship with fuzzy domains. Furthermore, from the viewpoint of category, Hofmann and Waszkiewicz [17–19] dealt with quantitative domains; Stubbe [20, 21] made a study of quantitative completely distributive lattices; Lai and Zhang [22] presented a systematic investigation of completeness and directed completeness of Ω-categories.

In view of the increasing interest in quantitative domain theory and Z-continuous posets. Therefore, it is natural to give a presentation of these matters in a more general framework. For this purpose, we are motivated to introduce the notion of fuzzy subset systems as a structure to study quantitative domain theory. We try to extend the theory of quantitative domain theory to a more general fuzzy order structure.

This paper is arranged as follows. In Section 2, we recall some basic materials related to fuzzy posets and fuzzy Galois connections. In Section 3, we give the definition of fuzzy subset systems, then present the notions of fuzzy

Z-continuous posets and fuzzy strongly Z-continuous posets, and study the relationship between such algebraic structures. We also discuss the fuzzy Z-continuous section-retraction pair between fuzzy Z-continuous posets. In Section 4, we introduce the concept of fuzzy Z-complete closure systems and associate a fuzzy Z-continuous closure operator with a fuzzy Z-complete closure system. We prove that each fuzzy Z-complete closure system of a fuzzy Z-continuous poset is fuzzy Z-continuous. In Section 5, the notion of fuzzy Z-algebraic posets is given, then some algebraic properties of such a structure are studied. An extension theorem based on Z-compact elements is obtained. In the last section, conclusions are made.

2. Preliminaries

In this paper, we will use a complete residuated lattice as the structures of truth values. Such an algebraic structure is significant in fuzzy logic in a narrow sense [23, 24]. If no other conditions are imposed, in the sequel, L always denotes a complete residuated lattice.

Definition 1. A complete residuated lattice is an algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ such that

(1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1;

(2) $(L, *, 1)$ is a commutative monoid; that is, $*$ is commutative, associative, and $a * 1 = a$ holds for all $a \in L$;

(3) $*$ and $\rightarrow$ form an adjoint pair, that is, for any $a, b, c \in L$, $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$.

Proposition 2. *For a complete residuated lattic L, one has*

(1) $0 * a = 0$ *and* $1 \rightarrow a = a$,

(2) $a \leq b \Leftrightarrow a \rightarrow b = 1$,

(3) $(a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$,

(4) $a \rightarrow \bigwedge_{j\in J} a_j = \bigwedge_{j\in J}(a \rightarrow a_j)$, *and hence* $a \rightarrow b \leq a \rightarrow c$ *whenever* $b \leq c$,

(5) $(\bigvee_{j\in J} a_j) \rightarrow c = \bigwedge_{j\in J}(a_j \rightarrow c)$, *and hence* $a \rightarrow c \geq b \rightarrow c$ *whenever* $a \leq b$,

(6) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = a * b \rightarrow c$,

(7) $a * (a \rightarrow b) \leq b$,

(8) $(a \rightarrow b) * (c \rightarrow d) \leq a * c \rightarrow b * d$,

(9) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$.

More properties about complete residuated lattices can be found in [24].

Let X be a nonempty set. An L-subset on X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X. All algebraic operations on L can be extended pointwisely to the power set L^X. That is, for any $A, B \in L^X$, $x \in X$, we have $A \leq B \Leftrightarrow A(x) \leq B(x)$, $(A \rightarrow B)(x) = A(x) \rightarrow B(x)$ and $(A * B)(x) = A(x) * B(x)$.

Definition 3 (see [13, 25]). A fuzzy poset is a pair (X, e) such that X is a non-empty set and $e : X \times X \rightarrow L$ is a map, called a fuzzy order, that satisfies for any $x, y, z \in X$,

(1) $e(x, x) = 1$;

(2) $e(x, y) * e(y, z) \leq e(x, z)$;

(3) $e(x, y) = e(y, x) = 1 \Rightarrow x = y$.

If e is an L-partial order on X, then (X, e) is called an L-partial ordered set (simply, a fuzzy poset). To study fuzzy relational systems, Belohlávek [23] defined and studied an L-order over complete residuated lattices. It is shown in [14] that the previous notion is equivalent to Belohlávek's one.

Example 4. (1) For a non-empty set X, the subsethood degree map $\mathrm{sub}(-,-) : L^X \times L^X \rightarrow L$ is defined by for each pair $(A, B) \in L^X \times L^X$, $\mathrm{sub}(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$, then $\mathrm{sub}(-,-)$ is a fuzzy partial order on L^X and (L^X, sub) is a fuzzy poset. Especially, when $A \leq B$, we have $\mathrm{sub}(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x)) = 1$.

(2) If $M \subseteq X$, then (M, e_M) is also a fuzzy poset (relative to the induced order from (X, e)), where e_M is the restriction of e to $M \times M$.

Based on the introduction of fuzzy posets, the basic notions, such as join, meet, fuzzy closure operator and fuzzy Galois connection, can be established as an approach to generalizing the classic order theory. Here, we only recall some fundamental notions and give some basic properties needed in this paper. One can refer to [13–16, 23, 26–29] for further details.

Definition 5. In a fuzzy poset (X, e), an element $x_0 \in X$ is called a join (or meet) of a fuzzy subset A, in symbols $x_0 = \sqcup A$ (or $x_0 = \sqcap A$) if

(1) for any $x \in X$, $A(x) \leq e(x, x_0)$ (or $A(x) \leq e(x_0, x)$),

(2) for any $y \in X$, $\bigwedge_{x\in X}(A(x) \rightarrow e(x, y)) \leq e(x_0, y)$ (or $\bigwedge_{x\in X}(A(x) \rightarrow e(y, x)) \leq e(y, x_0)$).

For any $x \in X$, $\downarrow x \in L^X$ (or $\uparrow x \in L^X$) is defined as for any $y \in X$, $\downarrow x(y) = e(y, x)$ (or $\uparrow x(y) = e(x, y)$). For $A \in L^X$, $\downarrow A$ is defined as $\downarrow A(x) = \bigvee_{d\in X} A(d) * e(x, d)$. $A \in L^X$ is called a fuzzy upper set (or a fuzzy lower set) if for any $x, y \in X$, $A(x) * e(x, y) \leq A(y)$ (or $A(x) * e(y, x) \leq A(y)$).

Proposition 6. *Let (X, e) be a fuzzy poset. Then*

(1) $x_0 = \sqcup A$ *if and only if for any* $y \in X$, $e(x_0, y) = \bigwedge_{z\in X}(A(z) \rightarrow e(z, y))$,

(2) $x_0 = \sqcap A$ *if and only if for any* $y \in X$, $e(y, x_0) = \bigwedge_{z\in X}(A(z) \rightarrow e(y, z))$,

particularly, when $A = \downarrow x$, *one has*

$$\begin{aligned} e(x, y) &= \bigwedge_{z\in X} (e(z, x) \longrightarrow e(z, y)) \\ &= \bigwedge_{z\in X} (e(y, z) \longrightarrow e(x, z)). \end{aligned} \tag{1}$$

$x \in X$ is called the maximal (or minimal) element of $A \in L^X$, in symbols $x = \max A$ (or $x = \min A$) if $A(x) = 1$ and for all $y \in X$, $A(y) \le e(y, x)$ (or $A(y) \le e(x, y)$). It is easy to check that if A has a maximal (or minimal) element, then it is unique.

Definition 7. Given two fuzzy posets (X, e_X) and (Y, e_Y), a map $f : X \to Y$ is said to be fuzzy monotone if for any $x, y \in X$, $e_X(x, y) \le e_Y(f(x), f(y))$. Furthermore, a fuzzy monotone map $f : X \to X$ is called a projection on X if for all $x \in X$, $f \circ f(x) = f(x)$. We say a projection is a fuzzy closure operator if $e(x, c(x)) = 1$ for all $x \in X$.

Definition 8. Let (X, e_X) and (Y, e_Y) be two fuzzy posets, $f : (X, e_X) \to (Y, e_Y)$ and $g : (Y, e_Y) \to (X, e_X)$ two fuzzy monotone maps. The pair (f, g) is called a fuzzy Galois connection between (X, e_X) and (Y, e_Y) provided that

$$\text{for any } x \in X,\ y \in Y, \quad e_Y(y, f(x)) = e_X(g(y), x), \tag{2}$$

where f is called the upper adjoint of g and dually g is the lower adjoint of f.

Obviously, a fuzzy Galois connection is an extension of a crisp Galois connection. The crisp Galois connection is defined as follows: $y \le f(x) \Leftrightarrow g(y) \le x$ for any $x \in X$, $y \in Y$, and its relative properties can be found in [30].

Example 9. Let $f : X \to Y$ be a map. For each $B \in L^Y$, let $f_L^{\leftarrow}(B) = B \circ f$. Then we obtain a powerset operator: $f_L^{\leftarrow} : L^Y \to L^X$. Conversely, define a powerset operator $f_L^{\rightarrow} : L^X \to L^Y$ by

$$\text{for any } y \in Y,\ A \in L^X, \quad f_L^{\rightarrow}(A)(y) = \bigvee_{x \in X} A(x) * e_Y(y, f(x)). \tag{3}$$

Then $(f_L^{\rightarrow}, f_L^{\leftarrow})$ is a fuzzy Galois connection between (L^X, sub) and (L^Y, Low).

Proposition 10. *Let (X, e_X) and (Y, e_Y) be two fuzzy posets, $f : (X, e_X) \to (Y, e_Y)$ and $g : (Y, e_Y) \to (X, e_X)$ two maps. Then the following are equivalent:*

(1) *(f, g) is a fuzzy Galois connection,*

(2) *f is fuzzy monotone and $g(y) = \min f_L^{\leftarrow}(\uparrow y)$ for all $y \in Y$,*

(3) *g is fuzzy monotone and $f(x) = \max f_L^{\leftarrow}(\downarrow x)$ for all $x \in X$.*

Proposition 11. *Let (X, e_X) and (Y, e_Y) be two fuzzy posets, $f : (X, e_X) \to (Y, e_Y)$ and $g : (Y, e_Y) \to (X, e_X)$ two maps.*

(1) *If f is a fuzzy monotone map and has a lower adjoint, then for any $A \in L^X$ such that $\sqcap A$ exists, $f(\sqcap A) = \sqcap f_L^{\rightarrow}(A)$.*

(2) *If g is a fuzzy monotone map and has an upper adjoint, then for any $D \in L^Y$ such that $\sqcup D$ exists, $g(\sqcup D) = \sqcup g_L^{\rightarrow}(D)$.*

For any map $f : X \to Y$, we denote the corestriction to the image as $f^{\circ} : X \to f(X)$ and the inclusion of the image into Y accordingly as $f_{\circ} : f(X) \to Y$. Thus, each f has the decomposition $f = f_{\circ} f^{\circ}$. If $X = Y$, then $f^{\circ} f_{\circ}$ is the restriction and corestriction $f|_{f(X)} : f(X) \to f(X)$.

Proposition 12. *Let (X, e) be a fuzzy poset and $f : X \to X$ a fuzzy monotone map. Then the following are equivalent:*

(1) *f is a fuzzy closure operator,*

(2) *$(f_{\circ}, f^{\circ})$ is a fuzzy Galois connection between $(f(X), e_{f(X)})$ and (X, e_X),*

(3) *There is a fuzzy Galois connection (g, d) between some fuzzy posets (S, e_S) and (X, e_X) such that $f = gd$.*

3. Fuzzy Subset Systems and Fuzzy Z-Continuous Posets

In this section, we present the definition of fuzzy subset systems, and in such a framework we propose the notions of fuzzy Z-continuous posets and fuzzy strongly Z-continuous posets and study the relationship between them. We finally discuss the fuzzy Z-continuous section-retraction pair between fuzzy Z-continuous posets and get the similar results in [29].

Let FPO denote the category of all fuzzy posets with fuzzy monotone maps as morphisms and let FSET denote the category of fuzzy subsets with fuzzy maps as morphisms.

Definition 13. A fuzzy subset system on FPO is a functor Z_L : FPO $\to$ FSET satisfying the following conditions.

(1) For any fuzzy poset (X, e_X), $Z_L(X) \subseteq L^X$.

(2) If (X, e_X) and (Y, e_Y) are two fuzzy posets and $f : (X, e_X) \to (Y, e_Y)$ a fuzzy monotone map, then for any $A \in Z_L(X)$, $Z_L(f)(A) = f_L^{\rightarrow}(A) \in Z_L(Y)$.

(3) For any $x \in X$, $\downarrow x \in Z_L(X)$.

It is clear that in order to define a fuzzy subset system Z_L, it suffices to define its object maps satisfying the previous conditions. Obviously, it is exactly the formal generalization of the classic definition of a Z-subset system in [3].

Let (X, e) be a fuzzy poset. The following are some examples of fuzzy subset systems.

(1) $\mathscr{D}_L(X)$ is the family of all fuzzy direct subsets of X.

(2) $\mathscr{P}_L(X)$ is the family of all fuzzy arbitrary subsets of X.

(3) $\mathscr{L}_L(X)$ is the family of all fuzzy lower subsets of X.

Based on a commutative, unital quantale Ω, the authors [22, 31, 32] introduced the concept of a class of weights and studied the complete properties of such a structure from the viewpoint of category, where $Z_L(X)$ is a fuzzy lower set and $Z_L(f) = f_L^{\rightarrow}$, but $Z_L(f)(A) = f_L^{\rightarrow}(A) \in Z_L(Y)$ is not required.

Definition 14. A fuzzy poset (X, e) is said to be fuzzy Z-complete if for any $A \in Z_L(X)$, $\sqcup_X A$ exists. A fuzzy subset I of a fuzzy poset (X, e) is called a fuzzy Z-ideal of X if it is a fuzzy lower set generated by some $A \in Z_L(X)$; that is, for any fuzzy Z-ideal I, there exists $A \in Z_L(X)$ with $I = \downarrow A$. The sets of all fuzzy Z-subsets and all fuzzy Z-ideals on X are denoted by $Z_L(X)$ and $ZI_L(X)$, respectively.

Definition 15. If (X, e) is a fuzzy Z-complete poset, then for any $x, y \in X$, define $\Downarrow_Z x \in L^X$ by

$$\Downarrow_Z x(y) = \bigwedge_{I \in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)). \tag{4}$$

$\Downarrow_Z : X \rightarrow L^X$ is called the fuzzy Z-way-below relation. For $x \in X$, if $\Downarrow_Z x(x) = 1$, then we call x a compact element in X, and all compact elements in X are denoted by $K(X)$.

Next we give an equivalent definition of the fuzzy Z-way-below relation in terms of fuzzy Z-subsets.

Proposition 16. *Let (X, e) be a fuzzy Z-complete poset. Then for any $x, y \in X$,*

$$\bigwedge_{I \in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) = \bigwedge_{S \in Z_L(X)} \left(e(x, \sqcup S) \longrightarrow \left(\bigvee_{d \in X} S(d) * e(y, d)\right)\right). \tag{5}$$

That is, $\Downarrow_Z x(y) = \bigwedge_{S \in Z_L(X)} (e(x, \sqcup S) \rightarrow (\bigvee_{d \in X} S(d) * e(y, d)))$.

Some basic properties of the fuzzy relation are listed in the following.

Proposition 17. *Let (X, e) be a fuzzy Z-complete poset. For any $x, y, u, v \in X$, then*

(1) $\Downarrow_Z x \le \downarrow x$;

(2) $e(u, x) * \Downarrow_Z y(x) * e(y, v) \le \Downarrow_Z v(u)$.

Proof. (1) By Definition 13(3), for any $x \in X$, $\downarrow x \in ZI_L(X)$, then

$$\begin{aligned}\Downarrow_Z x(y) &= \bigwedge_{I \in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) \\ &\le e(x, \sqcup \downarrow x) \longrightarrow \downarrow x(y) \\ &= 1 \longrightarrow \downarrow x(y) \\ &= \downarrow x(y).\end{aligned} \tag{6}$$

(2) Is straightforward. □

Definition 18. A fuzzy Z-complete poset (X, e) is called a fuzzy Z-continuous poset if for any $x \in X$, $\Downarrow_Z x \in Z_L(X)$ (or $\Downarrow_Z x \in ZI_L(X)$) and $x = \sqcup \Downarrow_Z x$. A fuzzy Z-continuous poset is called strongly fuzzy Z-continuous if the fuzzy Z-way-below relation has the interpolation property: for any $x, y \in X$, $\Downarrow_Z x(y) = \bigvee_{d \in X} \Downarrow_Z x(d) * \Downarrow_Z d(y)$.

Fuzzy $\mathscr{D}$-continuous posets are known in the literature as fuzzy domains. See [11–15, 18, 19, 25, 29] for further details. Fuzzy $\mathscr{P}$-continuous posets are known as fuzzy completely distributive lattices which have been widely studied by [20–22] from the viewpoint of category.

Definition 19. A fuzzy subset system Z_L is said to be fuzzy union-complete if for any $\Phi \in Z_L(Z_L(X))$, $\sqcup\Phi$ exists and $\sqcup\Phi \in Z_L(X)$.

Remark 20. The fuzzy subset systems $\mathscr{D}_L$, $\mathscr{P}_L$, and $\mathscr{L}_L$ are fuzzy union-complete.

Proof. It is trivial that the statement holds for $\mathscr{P}_L$ and $\mathscr{L}_L$. We now give the proof in terms of $\mathscr{D}_L$. Recall that a fuzzy subset D is a fuzzy directed subset if $\bigvee_{x \in X} D(x) = 1$, and for any $a, b \in X$, $D(a) * D(b) \le \bigvee_{d \in X} D(d) * e(a, d) * e(b, d)$.

For any $\Phi \in \mathscr{D}_L(\mathscr{D}_L(X))$, put $\phi = \bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi) * \varphi$. We now show that $\sqcup\Phi = \phi$ and $\phi \in \mathscr{D}_L(X)$. For each $\psi \in \mathscr{D}_L(X)$,

$$\begin{aligned}\operatorname{sub}(\phi, \psi) &= \bigwedge_{x \in X} (\phi(x) \longrightarrow \psi(x)) \\ &= \bigwedge_{x \in X} \left(\bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi) * \varphi(x) \longrightarrow \psi(x)\right) \\ &= \bigwedge_{x \in X} \bigwedge_{\varphi \in \mathscr{D}_L(X)} (\Phi(\varphi) \longrightarrow (\varphi(x) \longrightarrow \psi(x))) \\ &= \bigwedge_{\varphi \in \mathscr{D}_L(X)} \left(\Phi(\varphi) \longrightarrow \bigwedge_{x \in X} (\varphi(x) \longrightarrow \psi(x))\right) \\ &= \bigwedge_{\varphi \in \mathscr{D}_L(X)} (\Phi(\varphi) \longrightarrow \operatorname{sub}(\varphi, \psi)), \\ \bigvee_{x \in X} \phi(x) &= \bigvee_{x \in X} \bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi) * \varphi(x) = \left(\bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi)\right) \\ &\quad * \left(\bigvee_{x \in X} \varphi(x)\right) = 1,\end{aligned} \tag{7}$$

and for any $a, b \in X$,

$$\begin{aligned}\phi(a) * \phi(b) &= \bigvee_{\varphi_1, \varphi_2 \in \mathscr{D}_L(X)} \Phi(\varphi_1) * \varphi_1(a) * \Phi(\varphi_2) * \varphi_2(b) \\ &\le \bigvee_{\varphi_1, \varphi_2 \in \mathscr{D}_L(X)} \bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi) * \operatorname{sub}(\varphi_1, \varphi) \\ &\quad * \operatorname{sub}(\varphi_2, \varphi) * \varphi_1(a) * \varphi_2(b) \\ &\le \bigvee_{\varphi \in \mathscr{D}_L(X)} \Phi(\varphi) * \varphi(a) * \varphi(b)\end{aligned}$$

$$\leq \bigvee_{d\in X}\bigvee_{\varphi\in\mathscr{D}_L(X)}\Phi(\varphi)*\varphi(d)*e(a,d)*e(b,d)$$
$$=\bigvee_{d\in X}\phi(d)*e(a,d)*e(b,d). \tag{8}$$

□

Lemma 21. *In a fuzzy union-complete subset system Z_L, if (X,e) is a fuzzy Z-continuous poset, then for any $A\in Z_L(X)$, we have $\bigvee_{x\in X}A(x)*\Downarrow_Z x\in Z_L(X)$.*

Proof. Since (X,e) is fuzzy Z-continuous, then $\Downarrow_Z x\in Z_L(X)$ for all $x\in X$. Hence the map $f=\Downarrow_Z : X\to Z_L(X)$ is welldefined. It is easy to verify that $\Downarrow_Z$ is fuzzy monotone, then by Definition 13(2), for any $A\in Z_L(X)$, we have $f_L^{\rightarrow}(A)=\bigvee_{x\in X}A(x)*\mathrm{sub}(-,\Downarrow_Z x)\in Z_L(Z_L(X))$. Since Z_L is fuzzy union-complete, to show $\bigvee_{x\in X}A(x)*\Downarrow_Z x\in Z_L(X)$, it suffices to show that $\sqcup(\bigvee_{x\in X}A(x)*\mathrm{sub}(-,\Downarrow_Z x))=\bigvee_{x\in X}A(x)*\Downarrow_Z x$. To this end, for any $\psi\in Z_L(X)$,

$$\begin{aligned}
&\mathrm{sub}\left(\bigvee_{x\in X}A(x)*\Downarrow_Z x,\psi\right)\\
&=\bigwedge_{y\in X}\left(\bigvee_{x\in X}A(x)*\Downarrow_Z x(y)\longrightarrow\psi(y)\right)\\
&=\bigwedge_{x\in X}\left(A(x)\longrightarrow\bigwedge_{y\in X}(\Downarrow_Z x(y)\longrightarrow\psi(y))\right)\\
&=\bigwedge_{x\in X}(A(x)\longrightarrow\mathrm{sub}(\Downarrow_Z x,\psi))\\
&=\bigwedge_{x\in X}\left(A(x)\longrightarrow\bigwedge_{\varphi\in Z_L(X)}(\mathrm{sub}(\varphi,\Downarrow_Z x)\longrightarrow\mathrm{sub}(\varphi,\psi))\right)\\
&=\bigwedge_{\varphi\in Z_L(X)}\left(\bigvee_{x\in X}A(x)*\mathrm{sub}(\varphi,\Downarrow_Z x)\longrightarrow\mathrm{sub}(\varphi,\psi)\right)\\
&=\bigwedge_{\varphi\in Z_L X}(f_L^{\rightarrow}(A)(\varphi)\longrightarrow\mathrm{sub}(\varphi,\psi)),
\end{aligned}\tag{9}$$

which indicates that $\sqcup f_L^{\rightarrow}(A)=\bigvee_{x\in X}A(x)*\Downarrow_Z x$. □

Lemma 22. *In a fuzzy union-complete subset system Z_L, if (X,e) is a fuzzy Z-continuous poset, and for any $A\in Z_L(X)$, set $a=\sqcup A$, then $\Downarrow_Z a=\bigvee_{d\in X}A(d)*\Downarrow_Z d$ holds.*

Proof. We first show that $\sqcup(\bigvee_{d\in X}A(d)*\Downarrow_Z d)=\sqcup A$. For any $y\in X$,

$$\begin{aligned}
&\bigwedge_{x\in X}\left(\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d(x)\right)\longrightarrow e(x,y)\right)\\
&\quad=\bigwedge_{x\in X}\bigwedge_{d\in X}(A(d)\longrightarrow(\Downarrow_Z d(x)\longrightarrow e(x,y)))\\
&\quad=\bigwedge_{d\in X}\left(A(d)\longrightarrow\bigwedge_{x\in X}(\Downarrow_Z d(x)\longrightarrow e(x,y))\right)\\
&\quad=\bigwedge_{d\in X}(A(d)\longrightarrow e(\sqcup\Downarrow_Z d,y))\\
&\quad=\bigwedge_{d\in X}(A(d)\longrightarrow e(d,y))\\
&\quad=e(\sqcup A,y).
\end{aligned}\tag{10}$$

On the one hand, for any $z\in X$, by Proposition 17(2),

$$\begin{aligned}
\bigvee_{d\in X}A(d)*\Downarrow_Z d(z)&\leq\bigvee_{d\in X}e(d,\sqcup A)*\Downarrow_Z d(z)\\
&\leq\bigvee_{d\in X}\Downarrow_Z a(z)=\Downarrow_Z a(z).
\end{aligned}\tag{11}$$

On the other hand, since the fuzzy subset system is fuzzy union-complete, by Lemma 21, we have

$$\begin{aligned}
&\Downarrow_Z a(z)\\
&=\bigwedge_{I\in ZI_L(X)}(e(a,\sqcup I)\longrightarrow I(z))\\
&\leq e\left(a,\sqcup\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d\right)\right)\longrightarrow\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d(z)\right)\\
&=e(a,\sqcup A)\longrightarrow\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d(z)\right)\\
&=\bigvee_{d\in X}A(d)*\Downarrow_Z d(z).
\end{aligned}\tag{12}$$

Therefore, $\Downarrow_Z a=\bigvee_{d\in X}A(d)*\Downarrow_Z d$. □

Proposition 23. *Let a fuzzy subset system Z_L be fuzzy union-complete. If (X,e) is a fuzzy Z-continuous poset, then for any $x,y\in X$,*

$$\Downarrow_Z x(y)=\bigwedge_{A\in Z_L(X)}\left(e(x,\sqcup A)\longrightarrow\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d(y)\right)\right). \tag{13}$$

Proof. For any $A\in Z_L(X)$, $\bigvee_{d\in X}A(d)*\Downarrow_Z d\in ZI_L(X)$, then by Lemma 22,

$$\begin{aligned}
&\Downarrow_Z x(y)*e(x,\sqcup A)\\
&=\Downarrow_Z x(y)*e\left(x,\sqcup\left(\bigvee_{d\in X}A(d)*\Downarrow_Z d\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= e\left(x, \sqcup\left(\bigvee_{d\in X} A(d) * \Downarrow_Z d\right)\right) * \bigwedge_{I\in ZI_L(X)} (e(x,\sqcup I) \longrightarrow I(y)) \\
&\le \left(\bigvee_{d\in X} A(d) * \Downarrow_Z d\right)(y) \\
&= \bigvee_{d\in X} A(d) * \Downarrow_Z d(y).
\end{aligned} \tag{14}
$$

This implies that $\Downarrow_Z x(y) \le e(x,\sqcup A) \rightarrow (\bigvee_{d\in X} A(d) * \Downarrow_Z d(y))$.

By the arbitrariness of A, we have

$$
\Downarrow_Z x(y) \le \bigwedge_{A\in Z_L(X)} \left(e(x,\sqcup A) \longrightarrow \left(\bigvee_{d\in X} A(d) * \Downarrow_Z d(y)\right)\right). \tag{15}
$$

Conversely, by Proposition 17(1),

$$
\begin{aligned}
e(x,\sqcup A) \longrightarrow \left(\bigvee_{d\in X} A(d) * \Downarrow_Z d(y)\right) &\le e(x,\sqcup A) \\
&\longrightarrow \left(\bigvee_{d\in X} A(d) * \downarrow d(y)\right).
\end{aligned} \tag{16}
$$

Thus

$$
\bigwedge_{A\in Z_L(X)} \left(e(x,\sqcup A) \longrightarrow \left(\bigvee_{d\in X} A(d) * \Downarrow_Z d(y)\right)\right) \le \Downarrow_Z x(y). \tag{17}
$$

Hence, $\Downarrow_Z x(y) = \bigwedge_{A\in Z_L(X)} (e(x,\sqcup A) \longrightarrow (\bigvee_{d\in X} A(d) * \Downarrow_Z d(y)))$. □

Theorem 24. *In a fuzzy union-complete subset system Z_L, a fuzzy Z-continuous poset (X,e) is exactly a strongly fuzzy Z-continuous poset.*

Proof. By Definition 18, it suffices to show $\Downarrow_Z x(y) = \bigvee_{d\in X} \Downarrow_Z x(d) * \Downarrow_Z d(y)$ for all $x, y \in X$. Proposition 17 implies that the left side of the equation is larger than or equal to the right side. We only need to show the other side. By Proposition 23,

$$
\begin{aligned}
&\Downarrow_Z x(y) \\
&\quad = \bigwedge_{A\in Z_L(X)} \left(e(x,\sqcup A) \longrightarrow \bigvee_{d\in X} A(d) * \Downarrow_Z d(y)\right) \\
&\quad \le e(x,\sqcup \Downarrow_Z x) \longrightarrow \bigvee_{d\in X} \Downarrow_Z x(d) * \Downarrow_Z d(y) \\
&\quad = 1 \longrightarrow \bigvee_{d\in X} \Downarrow_Z x(d) * \Downarrow_Z d(y) \\
&\quad = \bigvee_{d\in X} \Downarrow_Z x(d) * \Downarrow_Z d(y).
\end{aligned} \tag{18}
$$

□

Stubbe [20] presented an elegant characterization of the fuzzy completely distributive lattices by fuzzy Galois connection; Albert and Kelly [31] and Lai and Zhang [22] gave an equivalent characterization of Φ-cocomplete Ω-category. In the next theorem, we will give a more explicit one and find construction methods which allow us to obtain a multitude of fuzzy Z-continuous posets by using the given ones as building blocks.

Theorem 25. *Let (X,e) be a fuzzy Z-complete poset. Then the following statements are equivalent:*

(1) *(X,e) is a fuzzy Z-continuous poset;*

(2) *for each $x \in X$, $\Downarrow_Z x$ is the smallest fuzzy Z-ideal I with $e(x,\sqcup I) = 1$;*

(3) *for each $x \in X$, there exists a smallest fuzzy Z-ideal I with $e(x,\sqcup I) = 1$;*

(4) *the sup map $r = (I \rightarrow \sqcup I) : ZI_L(X) \rightarrow X$ has a lower adjoint. These conditions imply*

(5) *the sup map $r = (I \rightarrow \sqcup I) : ZI_L(X) \rightarrow X$ preserves all existing inf.*

Proof. (1) ⇒ (2): Condition (1) holds if and only if for each $x \in X$, $\Downarrow_Z x \in ZI_L(X)$ and $x = \sqcup \Downarrow_Z x$. Hence $\Downarrow_Z x$ is a fuzzy Z-ideal with $e(x,\sqcup \Downarrow_Z x) = e(x,x) = 1$.

Moreover, for all $y \in X$, $I' \in ZI_L(X)$ with $e(x,\sqcup I') = 1$, we have

$$
\begin{aligned}
\Downarrow_Z x(y) &= \bigwedge_{I\in ZI_L(X)} (e(x,\sqcup I) \longrightarrow I(y)) \\
&\le e\left(x,\sqcup I'\right) \longrightarrow I'(y) \\
&= 1 \longrightarrow I'(y) = I'(y).
\end{aligned} \tag{19}
$$

This establishes (2).

Condition (2) trivially implies (3).

(3) ⇔ (4): It is easy to check that r is a fuzzy monotone map. By Proposition 10, r has a lower adjoint if and only if $\min r_L^{\leftarrow}(\uparrow x)$ exists for all $x \in X$, where $\min r_L^{\leftarrow}(\uparrow x) = \min\{I \in ZI_L(X), e(x,\sqcup I) = 1\}$. This means $\min r_L^{\leftarrow}(\uparrow x)$ is precisely the smallest fuzzy Z-ideal I with $e(x,\sqcup I) = 1$.

(4) ⇒ (1): Let $s : X \rightarrow ZI_L(X)$ be the lower adjoint of r. To show that (X,e) is a fuzzy Z-continuous poset, it suffices to prove $s(x) = \Downarrow_Z x$ for all $x \in X$.

On the one hand, for all $y \in X$, since $s(x) \in ZI_L(X)$, then

$$
\begin{aligned}
\Downarrow_Z x(y) &= \bigwedge_{I\in ZI_L(X)} (e(x,\sqcup I) \longrightarrow I(y)) \\
&\le e(x,\sqcup s(x)) \longrightarrow s(x)(y) \\
&= s(x)(y).
\end{aligned} \tag{20}
$$

On the other hand, $s(x)(y) * \mathrm{sub}(s(x), I) \leq I(y)$,

$$\begin{aligned} s(x)(y) &\leq \bigwedge_{I \in ZI_L(X)} (\mathrm{sub}(s(x), I) \longrightarrow I(y)) \\ &= \bigwedge_{I \in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) \\ &= \Downarrow_Z x(y). \end{aligned} \tag{21}$$

(4) $\Rightarrow$ (5): It is immediate from Proposition 11. □

In [29], the authors studied section-retraction pair of fuzzy domains in a systematic way; here, we further give an application in fuzzy Z-continuous posets.

Definition 26 (see [29]). Let (X, e_X) and (Y, e_Y) be two fuzzy posets, $s : (X, e_X) \rightarrow (Y, e_Y)$ and $r : (Y, e_Y) \rightarrow (X, e_X)$ two fuzzy monotone maps. The pair (s, r) is called a fuzzy monotone section-retraction pair if $r \circ s = id_X$. In this situation, we call (X, e_X) a fuzzy monotone retraction of (Y, e_Y).

It is clear that r is surjective and s is injective in a fuzzy monotone section-retraction pair (s, r).

Definition 27. Given two fuzzy Z-complete posets (X, e_X) and (Y, e_Y), a fuzzy monotone map $f : (X, e_X) \rightarrow (Y, e_Y)$ is said to be fuzzy Z-continuous if for any $A \in Z_L(X)$, $f(\sqcup_X A) = \sqcup_Y f_L^{\rightarrow}(A)$.

Definition 28. A fuzzy monotone section-retraction pair (s, r) is called a fuzzy Z-continuous section-retraction pair if (X, e_X) and (Y, e_Y) are fuzzy Z-complete posets and s, r are both fuzzy Z-continuous, then we speak of (X, e_X) as a fuzzy Z-continuous retraction of (Y, e_Y).

Lemma 29. *Let the pair (s, r) be a fuzzy Z-continuous section-retraction-pair between (X, e_X) and (Y, e_Y). Then for any $x \in X$, $y \in Y$, $\Downarrow_Z s(x)(y) \leq \Downarrow_Z x(r(y))$.*

Proof. Note that for any $I' \in ZI_L(X)$, $s_L^{\rightarrow}(I') \in ZI_L(Y)$, then

$$\begin{aligned} &\Downarrow_Z s(x)(y) \\ &= \bigwedge_{I \in ZI_L(Y)} (e_Y(s(x), \sqcup I) \longrightarrow I(y)) \\ &\leq \bigwedge_{I' \in ZI_L(X)} \left(e_Y\left(s(x), \sqcup s_L^{\rightarrow}(I')\right) \longrightarrow s_L^{\rightarrow}(I')(y)\right) \\ &= \bigwedge_{I' \in ZI_L(X)} \left(e_Y\left(s(x), s(\sqcup I')\right) \longrightarrow s_L^{\rightarrow}(I')(y)\right) \\ &\leq \bigwedge_{I' \in ZI_L(X)} \left(e_X\left(x, \sqcup I'\right) \longrightarrow s_L^{\rightarrow}(I')(y)\right) \\ &= \bigwedge_{I' \in ZI_L(X)} \left(e_X\left(x, \sqcup I'\right) \longrightarrow \left(\bigvee_{a \in X} I'(a) * e_Y(y, s(a))\right)\right) \\ &\leq \bigwedge_{I' \in ZI_L(X)} \left(e_X\left(x, \sqcup I'\right) \longrightarrow \left(\bigvee_{a \in X} I'(a) * e_X(r(y), r(s(a)))\right)\right) \\ &= \bigwedge_{I' \in ZI_L(X)} \left(e_X\left(x, \sqcup I'\right) \longrightarrow \left(\bigvee_{a \in X} I'(a) * e_X(r(y), a)\right)\right) \\ &= \bigwedge_{I' \in ZI_L(X)} \left(e_X\left(x, \sqcup I'\right) \longrightarrow I'(r(y))\right) \\ &= \Downarrow_Z x(r(y)). \end{aligned} \tag{22}$$

□

Theorem 30. *A fuzzy Z-continuous retraction of a fuzzy Z-continuous poset is also a fuzzy Z-continuous poset.*

Proof. Assume that (s, r) is a fuzzy Z-continuous section-retraction-pair between (X, e_X) and (Y, e_Y), where (Y, e_Y) is a fuzzy Z-continuous poset, we need to show that (X, e_X) is also a fuzzy Z-continuous poset.

Since $r \circ s = id_X$, then for any $x \in X$, $x = r(s(x))$. Note that r is fuzzy Z-continuous and (Y, e_Y) is a fuzzy Z-continuous poset, then $r_L^{\rightarrow}(\Downarrow_Z s(x)) \in ZI_L(X)$ and

$$\sqcup r_L^{\rightarrow}(\Downarrow_Z s(x)) = r(\sqcup \Downarrow_Z s(x)) = r(s(x)) = x. \tag{23}$$

Therefore, to prove (X, e_X) is fuzzy Z-continuous, it suffices to show that $r_L^{\rightarrow}(\Downarrow_Z s(x)) = \Downarrow_Z x$. On the one hand, for any $y \in X$ and by Lemma 29,

$$\begin{aligned} r_L^{\rightarrow}(\Downarrow_Z s(x))(y) &= \bigvee_{z \in Y} \Downarrow_Z s(x)(z) * e_X(y, r(z)) \\ &\leq \bigvee_{z \in Y} \Downarrow_Z x(r(z)) * e_X(y, r(z)) \\ &\leq \bigvee_{z \in Y} \Downarrow_Z x(y) = \Downarrow_Z x(y). \end{aligned} \tag{24}$$

On the other hand,

$$\begin{aligned} \Downarrow_Z x(y) &= \bigwedge_{I \in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) \\ &\leq e(x, \sqcup(r_L^{\rightarrow}(\Downarrow_Z s(x)))) \longrightarrow r_L^{\rightarrow}(\Downarrow_Z s(x))(y) \\ &= r_L^{\rightarrow}(\Downarrow_Z s(x))(y). \end{aligned} \tag{25}$$

Therefore, $r_L^{\rightarrow}(\Downarrow_Z s(x)) = \Downarrow_Z x$. □

4. Fuzzy Z-Complete Closure Systems

In [33], the authors studied fuzzy closure systems on L-order sets, where the L-order sets are really fuzzy posets, and discussed their relationship with fuzzy closure operators.

In this section, we further introduce the concept of fuzzy Z-complete closure systems and associate a fuzzy Z-continuous closure operator with a fuzzy Z-complete closure system. We prove that each fuzzy Z-complete closure system of a fuzzy Z-continuous poset is fuzzy Z-continuous.

Definition 31 (see [33]). If (X, e) is a fuzzy poset, a fuzzy closure system on X is a subset M of X such that for each $x \in X$, $\min\{\uparrow x * 1_M\}$ exists, where 1_M is the constant fuzzy set with the value 1 on M.

Theorem 32 (see [33]). *Let (X, e) be a fuzzy poset, $c : X \to X$ a fuzzy monotone map, and $c^\circ : X \to c(X)$ the corestriction to the image. Then one has the following.*

(1) *If M is a fuzzy closure system on X and for each $x \in X$, $c^\circ(x) = \min\{\uparrow x * 1_M\}$, then the map $c : X \to X$ is a fuzzy closure operator.*

(2) *If $c : X \to X$ is a fuzzy closure operator, then $c(X)$ is a fuzzy closure system on X.*

(3) *The map defined by (1) from the set of fuzzy closure systems on X to the set of fuzzy closure operators on X is bijective, and its converse is the map defined by (2).*

Lemma 33. *Let (X, e) be a fuzzy Z-complete poset and $M \subseteq X$. Then for any $A \in Z_L(M)$, $\sqcup_X A = \sqcup_X i_{ML}^{\to}(A)$. Here, i_M is an inclusion map from M to X. Moreover, if $\sqcup_X A \in M$, then $\sqcup_X A = \sqcup_M A$.*

Proof. It is trivial that i_M is fuzzy monotone, then for any $A \in Z_L(M)$, $i_{ML}^{\to}(A) \in Z_L(X)$ and $\sqcup_X i_{ML}^{\to}(A)$ exists. For any $z \in X$,

$$\begin{aligned}
e_X\left(\bigsqcup_X i_{ML}^{\to}(A), z\right) &= \bigwedge_{x\in X}\left(i_{ML}^{\to}(A)(x) \longrightarrow e_X(x,z)\right)\\
&= \bigwedge_{x\in X}\left(\bigvee_{y\in M} A(y) * e_X(x, i_M(y)) \longrightarrow e_X(x,z)\right)\\
&= \bigwedge_{y\in M}\left(A(y) \longrightarrow \bigwedge_{x\in X}\left(e_X(x,y) \longrightarrow e_X(x,z)\right)\right)\\
&= \bigwedge_{y\in M}\left(A(y) \longrightarrow e_X(y,z)\right).
\end{aligned} \tag{26}$$

Thus $\sqcup_X A = \sqcup_X i_{ML}^{\to}(A)$.

Furthermore, if $\sqcup_X A \in M$, then for any $z' \in M$, we have

$$\begin{aligned}
e_X\left(\bigsqcup_X i_{ML}^{\to}(A), z'\right) &= e_M\left(\bigsqcup_X i_{ML}^{\to}(A), z'\right)\\
&= \bigwedge_{y\in M}\left(A(y) \longrightarrow e_M(y,z')\right).
\end{aligned} \tag{27}$$

This shows that $\sqcup_M A = \sqcup_X i_{ML}^{\to}(A)$. Hence $\sqcup_M A = \sqcup_X A$. □

Lemma 34. *Let (X, e) be a fuzzy poset, $f : X \to X$ a projection, and $i_{f(X)} : f(X) \to X$ an inclusion map. Then (1) for any $A \in Z_L(f(X))$, $f_L^{\to}(i_{f(X)L}^{\to}(A)) = f_L^{\to}(A)$; (2) for any $B \in Z_L(X)$, $f_L^{\to}(f_L^{\to}(B)) = f_L^{\to}(B)$.*

Proof. (1) For any $y \in f(X)$, note that $y = f(y)$,

$$\begin{aligned}
f_L^{\to}\left(i_{f(X)L}^{\to}(A)\right)(y) &= \bigvee_{z\in X} i_{f(X)L}^{\to}(A)(z) * e_{f(X)}(y, f(z))\\
&= \bigvee_{z\in X}\bigvee_{x\in f(X)} A(x) * e_X(z,x) * e_{f(X)}(y, f(z))\\
&\le \bigvee_{z\in X}\bigvee_{x\in f(X)} A(x) * e_{f(X)}(f(z), f(x))\\
&\quad * e_{f(X)}(y, f(z))\\
&\le \bigvee_{z\in X}\bigvee_{x\in f(X)} A(x) * e_{f(X)}(y, f(x))\\
&= \bigvee_{x\in f(X)} A(x) * e_{f(X)}(y, f(x))\\
&= f_L^{\to}(A)(y).
\end{aligned} \tag{28}$$

For the converse,

$$\begin{aligned}
f_L^{\to}\left(i_{f(X)L}^{\to}(A)\right)(y) \bigvee_{z\in X}\bigvee_{x\in f(X)} &= A(x) * e_X(z,x)\\
&\quad * e_{f(X)}(y, f(z))\\
&\ge \bigvee_{x\in f(X)} A(x) * e_{f(X)}(y, f(x))\\
&= f_L^{\to}(A)(y).
\end{aligned} \tag{29}$$

Similarly, we can prove (2). □

Proposition 35. *Let (X, e) be a fuzzy Z-complete poset and $f : X \to X$ a fuzzy Z-continuous projection. Then $(f(X), e_{f(X)})$ is a fuzzy Z-complete poset.*

Proof. By Lemma 33, it suffices to show that $\sqcup_X A \in f(X)$ for each $A \in Z_L(f(X))$. Actually, assume that $a = \sqcup_X A$, which means for all $z \in X$, $e_X(a,z) = \bigwedge_{x\in f(X)}(A(x) \to e_X(x,z))$. Then

$$\begin{aligned}
&\bigwedge_{x\in f(X)}\left(f_L^{\to}(A)(x) \longrightarrow e_X(x,z)\right)\\
&= \bigwedge_{x\in f(X)}\left(\bigvee_{y\in f(X)} A(y) * e_{f(X)}(x, f(y)) \longrightarrow e_X(x,z)\right)
\end{aligned}$$

$$= \bigwedge_{y\in f(X)} \left(A(y) \longrightarrow \bigwedge_{x\in f(X)} \left(e_{f(X)}(x, f(y)) \longrightarrow e_X(x,z) \right) \right)$$
$$= \bigwedge_{y\in f(X)} \left(A(y) \longrightarrow e_X(y,z) \right), \tag{30}$$

which implies that $e_X(a,z) = \bigwedge_{x\in f(X)}(f_L^{\rightarrow}(A)(x) \rightarrow e_X(x,z))$.

Since f is fuzzy Z-continuous and by Lemmas 33, 34, we have

$$a = \underset{X}{\sqcup} f_L^{\rightarrow}(A) = \underset{X}{\sqcup} f_L^{\rightarrow}\left(i_{f(X)L}^{\rightarrow}(A)\right) = f\left(\underset{X}{\sqcup} i_{f(X)L}^{\rightarrow}(A)\right)$$
$$= f\left(\underset{X}{\sqcup} A\right) = f(a). \tag{31}$$

□

Remark 36. The preceding proposition states that for any fuzzy Z-continuous projection $f : X \rightarrow X$, the inclusion map from $f(X)$ to X is fuzzy Z-continuous.

Proof. Since for any $A \in Z_L(f(X))$, $i_{f(X)}(\sqcup_{f(X)}A) = \sqcup_{f(X)}A$ and $\sqcup_X i_{f(X)L}^{\rightarrow}(A) = \sqcup_X A$, then by Lemma 33 and Proposition 35, we have $i_{f(X)}(\sqcup_{f(X)}A) = \sqcup_X i_{f(X)L}^{\rightarrow}(A)$; that is, the inclusion map from $f(X)$ to X is fuzzy Z-continuous. □

Proposition 37. *Let (X, e) be a fuzzy Z-complete poset, $f : X \rightarrow X$ a fuzzy Z-continuous projection and $Y = f(X)$. Then for all $A \in Z_L(X)$, $f(\sqcup_X A) = \sqcup_Y f_L^{\rightarrow}(A)$ (in other words, the corestriction of f to Y is fuzzy Z-continuous).*

Proof. Since f is fuzzy Z-continuous, then for any $A \in Z_L(X)$, $f(\sqcup_X A) = \sqcup_X f_L^{\rightarrow}(A)$. Moreover, f is fuzzy monotone, which indicates that $f_L^{\rightarrow}(A) \in Z_L(Y)$. By Lemma 33 and Proposition 35, $\sqcup_X f_L^{\rightarrow}(A) = \sqcup_Y f_L^{\rightarrow}(A)$. Thus $f(\sqcup_X A) = \sqcup_Y f_L^{\rightarrow}(A)$. □

Theorem 38. *Let (X, e) be a fuzzy Z-continuous poset and $f : X \rightarrow X$ a fuzzy Z-continuous projection. Then $(f(X), e_{f(X)})$ is fuzzy Z-continuous (relative to the induced fuzzy order).*

Proof. By Proposition 35, $(f(X), e_{f(X)})$ is a fuzzy Z-complete lattice. For any $x \in X$, $f(x) = f \circ f(x) = f(i_{f(X)}(f(x))) = (f \circ i_{f(X)})(f(x))$, which implies $f \circ i_{f(X)} = id_{f(X)}$. By Remark 36 and Proposition 37, $(i_{f(X)}, f)$ is a fuzzy Z-continuous section-retraction pair between $(f(X), e_{f(X)})$ and (X, e). Since (X, e) is a fuzzy Z-continuous poset, then $(f(X), e_{f(X)})$ is a fuzzy Z-continuous poset which follows from Theorem 30. □

Definition 39. A fuzzy closure system on X is fuzzy Z-complete if for each $A \in Z_L(M)$, such that $\sqcup_X A$ exists, we have $\sqcup_X A \in M$ (in other words, M is closed in X under the formation of sups of fuzzy Z-subsets of M).

Theorem 40. *In a fuzzy Z-complete poset (X, e), the one-to-one correspondence established by Theorem 32 induces a one-to-one correspondence between fuzzy Z-complete closure systems on X and fuzzy Z-continuous closure operators on X.*

Proof. Suppose that a subset M is a fuzzy Z-complete closure system on X; by Theorem 32, it suffices to show that $c : X \rightarrow X$ is fuzzy Z-continuous. Since c is a fuzzy monotone map, then for any $A \in Z_L(X)$, $1 = e_X(\sqcup_X c_L^{\rightarrow}(A), c(\sqcup_X A))$. It remains to show $1 = e_X(c(\sqcup_X A), \sqcup_X c_L^{\rightarrow}(A))$, to this end, for any $y \in X$,

$$e_X\left(\underset{X}{\sqcup} c_L^{\rightarrow}(A), y\right)$$
$$= \bigwedge_{x\in M} \left(c_L^{\rightarrow}(A)(x) \longrightarrow e_X(x,y) \right)$$
$$= \bigwedge_{x\in M} \left(\bigvee_{x'\in X} A(x') * e_M(x, c(x')) \longrightarrow e_X(x,y) \right)$$
$$= \bigwedge_{x'\in X} \left(A(x') \longrightarrow \bigwedge_{x\in M} \left(e_M(x, c(x')) \longrightarrow e_X(x,y) \right) \right)$$
$$= \bigwedge_{x'\in X} \left(A(x') \longrightarrow e_X(x', c(x')) * e_X(c(x'), y) \right)$$
$$\leq \bigwedge_{x'\in X} \left(A(x') \longrightarrow e_X(x', y) \right)$$
$$= e_X\left(\underset{X}{\sqcup} A, y\right). \tag{32}$$

Thus for any $y \in X$, $e_X(\sqcup_X c_L^{\rightarrow}(A), y) \leq e_X(\sqcup_X A, y)$, especially, set $y = \sqcup_X c_L^{\rightarrow}(A)$ and note that $\sqcup_X c_L^{\rightarrow}(A) = \sqcup_M c_L^{\rightarrow}(A) = c(\sqcup_X c_L^{\rightarrow}(A))$; then

$$1 = e_X\left(\underset{X}{\sqcup} c_L^{\rightarrow}(A), \underset{X}{\sqcup} c_L^{\rightarrow}(A)\right) \leq e_X\left(\underset{X}{\sqcup} A, \underset{X}{\sqcup} c_L^{\rightarrow}(A)\right)$$
$$\leq e_X\left(c\left(\underset{X}{\sqcup} A\right), \underset{X}{\sqcup} c_L^{\rightarrow}(A)\right). \tag{33}$$

Assume that $c : X \rightarrow X$ is a fuzzy Z-continuous closure operator, then $c(X)$ is a fuzzy Z-complete closure system which follows from Proposition 35 and Theorem 38. □

By Theorems 32(2), 38, and 40, we can deduce the following result.

Theorem 41. *Any fuzzy Z-complete closure system of a fuzzy Z-continuous poset is fuzzy Z-continuous.*

5. Fuzzy Z-Algebraic Posets

In this section, the notion of fuzzy Z-algebraic posets is given, then we investigate some algebraic properties of such a structure. Moreover, an extension theorem of a fuzzy Z-algebraic poset is obtained.

In universal algebra, algebraic lattice has become familiar objects as lattices of congruences and lattices of subalgebras of an algebra. Yao [14] gave the definition of fuzzy (directed) algebraic posets, and Stubbe [21] presented a systematic investigation of fuzzy (directed) algebraic posets. Here, we study such an algebraic structure from the viewpoint of the fuzzy subset systems.

Definition 42. Let (X, e) be a fuzzy Z-complete poset and $x \in X$. Define a map $k_x : X \to L$ by $k_x = \downarrow x|_{K(X)}$, k_x restricted on $K(X)$, that is, $k_x(y) = e(y, x)$ if $y \in K(X)$ and otherwise 0. The fuzzy Z-complete poset (X, e) is said to be a fuzzy Z-algebraic poset if and only if $k_x \in Z_L(X)$ and $x = \sqcup k_x$.

Next we give an example of fuzzy Z-algebraic posets.

Example 43. In a fuzzy union-complete subset system Z_L, $(ZI_L(X), \mathrm{sub})$ is a fuzzy Z-algebraic poset.

Proof. Let $\Phi \in ZI_L(ZI_L(X))$. It is easy to verify that $\sqcup\Phi = \bigvee_{I \in ZI_L(X)} \Phi(I) * I$. Since Z_L is fuzzy union-complete, then $\sqcup\Phi \in Z_L(X)$. Moreover, $\bigvee_{I \in ZI_L(X)} \Phi(I) * I$ is a fuzzy lower set; hence $(ZI_L(X), \mathrm{sub})$ is a fuzzy Z-complete poset.

Next we show $\downarrow y \in K(ZI_L(X))$ for any $y \in X$. To this end, it suffices to show $\Downarrow I(I) = \bigvee_{x \in X} I(x) * \mathrm{sub}(I, \downarrow x)$ for any $I \in ZI_L(X)$.

On the one hand, note that $(\sqcup\Phi)(x) = \Phi(\downarrow x)$; then

$$\begin{aligned} I(x) * \mathrm{sub}(I, \downarrow x) * \mathrm{sub}(I, \sqcup\Phi) &\leq \mathrm{sub}(I, \downarrow x) * (\sqcup\Phi)(x) \\ &= \mathrm{sub}(I, \downarrow x) * \Phi(\downarrow x) \\ &\leq \Phi(I). \end{aligned} \tag{34}$$

Thus $I(x) * \mathrm{sub}(I, \downarrow x) \leq \mathrm{sub}(I, \sqcup\Phi) \to \Phi(I)$, which implies that $\bigvee_{x \in X} I(x) * \mathrm{sub}(I, \downarrow x) \leq \Downarrow I(I)$.

On the other hand, since $f = \downarrow : X \to ZI_L(X)$ is a fuzzy monotone map, then for any $I \in ZI_L(X)$, we have $f_L^{\to}(I) = \bigvee_{x \in X} I(x) * \mathrm{sub}(-, f(x)) = \bigvee_{x \in X} I(x) * \mathrm{sub}(-, \downarrow x) \in ZI_L(ZI_L(X))$ and $\sqcup f_L^{\to}(I) = I$. Then

$$\begin{aligned} \Downarrow I(I) \bigvee_{\Phi \in ZI_L(ZI_L(X))} &= (\mathrm{sub}(I, \sqcup\Phi) \longrightarrow \Phi(I)) \\ &\leq \mathrm{sub}\left(I, \sqcup f_L^{\to}(I)\right) \longrightarrow f_L^{\to}(I)(I). \\ &= \bigvee_{x \in X} I(x) * \mathrm{sub}(I, \downarrow x). \end{aligned} \tag{35}$$

Hence $\Downarrow I(I) = \bigvee_{x \in X} I(x) * \mathrm{sub}(I, \downarrow x)$. This indicates that $\downarrow y \in K(ZI_L(X))$.

At last, we show for any $I \in ZI_L(X)$, k_I satisfies Definition 42. It is easy to check that $k_I \in ZI_L(X)$, and it remains to show $\sqcup k_I = I$. Appealing to the previous proof, we obtain

$$\begin{aligned} \sqcup k_I &= \bigvee_{I' \in ZI_L(X)} k_I\left(I'\right) * I' \geq \bigvee_{x \in X} \mathrm{sub}(\downarrow x, I) * \downarrow x = I, \\ \sqcup k_I &= \bigvee_{I' \in ZI_L(X)} k_I\left(I'\right) * I' = \bigvee_{I' \in ZI_L(X)} \mathrm{sub}\left(I', I\right) * I' \leq I. \end{aligned} \tag{36}$$

These complete the proof. □

Definition 44. In a fuzzy Z-complete poset, a fuzzy subset system Z_L is said to be consistent if for any $x \in X$, there exists a fuzzy subset $A \in Z_L(X)$ satisfying that $A \leq \Downarrow_Z x$ and $x = \sqcup A$, then $\Downarrow_Z x \in Z_L(X)$ and $x = \sqcup \Downarrow_Z x$.

Remark 45. Fuzzy $\mathscr{D}$-complete posets, fuzzy $\mathscr{P}$-complete posets, and fuzzy $\mathscr{L}$-complete posets are consistent.

Proof. Obviously, fuzzy $\mathscr{P}$-complete posets and fuzzy $\mathscr{L}$-complete posets are consistent. Next we give the proof in terms of fuzzy $\mathscr{D}$-complete posets.

Let (X, e) be a fuzzy dcpo. For any $x \in X$, if there exists a fuzzy directed subset A such that $A \leq \Downarrow x$ and $x = \sqcup A$, we should show that $\Downarrow x \in \mathscr{D}_L(X)$ and $x = \sqcup \Downarrow x$. For any $y \in X$, we first show $\Downarrow x(y) \leq \bigvee_{d \in X} A(d) * e(y, d)$. Indeed,

$$\begin{aligned} \Downarrow x(y) &= \bigwedge_{I \in \mathscr{I}_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) \\ &\leq e(x, \sqcup \downarrow A) \longrightarrow \left(\bigvee_{d \in X} A(d) * e(y, d)\right) \\ &= 1 \longrightarrow \left(\bigvee_{d \in X} A(d) * e(y, d)\right) \\ &= \bigvee_{d \in X} A(d) * e(y, d). \end{aligned} \tag{37}$$

Thus, for any $a, b \in X$, we have

$$\begin{aligned} \Downarrow x(a) * \Downarrow x(b) &\leq \bigvee_{d_1, d_2 \in X} A(d_1) * A(d_2) * e(a, d_1) * e(b, d_2) \\ &\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} A(d) * e(d_1, d) * e(d_2, d) * e(a, d_1) * e(b, d_2) \\ &\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} A(d) * e(a, d) * e(b, d) \\ &= \bigvee_{d \in X} A(d) * e(a, d) * e(b, d) \\ &\leq \bigvee_{d \in X} \Downarrow x(d) * e(a, d) * e(b, d). \end{aligned} \tag{38}$$

Moreover, $\bigvee_{y \in X} \Downarrow x(y) = 1$ follows from $1 = \bigvee_{y \in X} A(y) \leq \bigvee_{y \in X} \Downarrow x(y)$. Hence, $\Downarrow x$ is fuzzy directed.

Since $A \leq \Downarrow x$ and $\sqcup$ is fuzzy monotone, then $1 = \mathrm{sub}(A, \Downarrow x) \leq e(\sqcup A, \sqcup \Downarrow x) = e(x, \sqcup \Downarrow x)$. Meanwhile, $1 = \mathrm{sub}(\Downarrow x, \downarrow x) \leq e(\sqcup \Downarrow x, \sqcup \downarrow x) = e(\sqcup \Downarrow x, x)$. Therefore, $x = \sqcup \Downarrow x$. □

Lemma 46. *In a fuzzy union-complete subset system Z_L, if (X, e) is a fuzzy Z-algebraic poset, then for any $x \in X$, we have $\bigvee_{z \in K(X)} \downarrow x(z) * \downarrow z \in Z_L(X)$.*

Proof. By Definition 13(3), $\downarrow x \in Z_L(X)$ for any $x \in X$. Therefore, the map $f = \downarrow: X \to Z_L(X)$ is well defined. Assume that (X, e) is a fuzzy Z-algebraic poset, then $k_x \in Z_L(X)$. Since the map $f = \downarrow$ is fuzzy monotone, by Definition 13(2), $f_L^{\to}(k_x) = \bigvee_{z\in K(X)} k_x(z) * \mathrm{sub}(-, \downarrow z) \in Z_L(Z_L(X))$. Since Z_L is fuzzy union-complete, to show $\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z \in Z_L(X)$, it suffices to show that $\sqcup(\bigvee_{z\in K(X)} k_x(z) * \mathrm{sub}(-, \downarrow z)) = \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z$. For any $\psi \in Z_L(X)$,

$$\begin{aligned}
&\mathrm{sub}\left(\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z, \psi\right) \\
&= \bigwedge_{y\in X}\left(\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y) \longrightarrow \psi(y)\right) \\
&= \bigwedge_{z\in K(X)} (\downarrow x(z) \longrightarrow \mathrm{sub}(\downarrow z, \psi)) \\
&= \bigwedge_{\varphi\in Z_L(X)}\left(\bigvee_{z\in K(X)} \downarrow x(z) * \mathrm{sub}(\varphi, \downarrow z) \longrightarrow \mathrm{sub}(\varphi, \psi)\right) \\
&= \bigwedge_{\varphi\in Z_L(X)} (f_L^{\to}(k_x)(\varphi) \longrightarrow \mathrm{sub}(\varphi, \psi)).
\end{aligned} \tag{39}$$

Thus $\sqcup f_L^{\to}(k_x) = \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z$. □

Proposition 47. *In a fuzzy union-complete and consistent subset system Z_L, if (X, e) is a fuzzy Z-algebraic poset, then (X, e) is a fuzzy Z-continuous poset and for any $x, y \in X$,*

$$\Downarrow_Z x(y) = \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y). \tag{40}$$

Proof. The Z-continuity of (X, e) is immediate from Definition 44; it remains to show that for any $x, y \in X$, $\Downarrow_Z x(y) = \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y)$.

On the one hand, by Proposition 17(2),

$$\begin{aligned}
\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y) &= \bigvee_{z\in K(X)} \Downarrow_Z z(z) * \downarrow x(z) * \downarrow z(y) \\
&\le \Downarrow_Z x(y).
\end{aligned} \tag{41}$$

On the other hand, since Z_L is fuzzy union-complete and $\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z$ is a fuzzy lower set, by Lemma 46, $\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z \in ZI_L(X)$. It is easy to check that $x = \sqcup(\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z)$, then

$$\begin{aligned}
&\Downarrow_Z x(y) \\
&= \bigwedge_{I\in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(y)) \\
&\le e\left(x, \sqcup\left(\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z\right)\right) \longrightarrow \left(\bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y)\right) \\
&= \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y).
\end{aligned} \tag{42}$$

Therefore, $\Downarrow_Z x(y) = \bigvee_{z\in K(X)} \downarrow x(z) * \downarrow z(y)$. □

Proposition 48. *Let (X, e) be a fuzzy Z-complete poset and $c : (X, e) \to (X, e)$ a fuzzy Z-continuous closure operator. Then $c(K(X)) \subseteq K(c(X))$ and the equality holds if (X, e) is a fuzzy Z-algebraic poset.*

Proof. For any $x \in K(X)$, $e(x, c(x)) = 1$ and for any $I' \in ZI_L(c(X))$, by Proposition 35, we have $\sqcup I' \in c(X)$. Note that $i_{c(X)L}^{\to}(I') \in ZI_L(X)$; then

$$\begin{aligned}
&\Downarrow_Z x(x) \\
&= \bigwedge_{I\in ZI_L(X)} (e(x, \sqcup I) \longrightarrow I(x)) \\
&\le \bigwedge_{I'\in ZI_L(c(X))} \left(e_X\left(x, \sqcup i_{c(X)L}^{\to}(I')\right) \longrightarrow i_{c(X)L}^{\to}(I')(x)\right) \\
&= \bigwedge_{I'\in ZI_L(c(X))} \left(e_X(x, \sqcup I') \longrightarrow \bigvee_{d'\in c(X)} I'(d') * e_X(x, d')\right) \\
&\le \bigwedge_{I'\in ZI_L(c(X))} \left(e_{c(X)}(c(x), \sqcup I') * e_X(x, c(x)) \longrightarrow \bigvee_{d'\in c(X)} I'(d') * e_{c(X)}(c(x), d')\right) \\
&= \bigwedge_{I'\in ZI_L(c(X))} (e_{c(X)}(c(x), \sqcup I') \longrightarrow I'(c(x))) \\
&= \Downarrow_Z c(x)(c(x)).
\end{aligned} \tag{43}$$

Further, if (X, e) is a fuzzy Z-algebraic poset, then for any $y \in K(c(X))$, there exists $k_y \in Z_L(X)$ such that $y = \sqcup k_y$. Since c is fuzzy monotone, then $c_L^{\to}(k_y) \in ZI_L(c(X))$. Consider

$$\begin{aligned}
1 = \Downarrow_{c(X)} y(y) &= \bigwedge_{I\in ZI_L(c(X))} (e_{c(X)}(y, \sqcup I) \longrightarrow I(y)) \\
&\le e_{c(X)}\left(y, \sqcup c_L^{\to}(k_y)\right) \longrightarrow c_L^{\to}(k_y)(y) = c_L^{\to}(k_y)(y) \\
&= \bigvee_{x\in K(X)} e(x, y) * e(y, c(x)) \\
&\le \bigvee_{x\in K(X)} e(c(x), y) * e(y, c(x)),
\end{aligned} \tag{44}$$

which indicates that there exists $x' \in K(X)$ such that $1 = e(c(x'), y) * e(y, c(x'))$, hence $y = c(x')$. Therefore, $y \in c(K(X))$. □

Proposition 49. *Let (X, e) be a fuzzy Z-algebraic poset and $c : (X, e) \to (X, e)$ a fuzzy Z-continuous closure operator. Then $(c(X), e_{c(X)})$ is a fuzzy Z-algebraic poset (relative to the induced fuzzy order).*

Proof. For any $y \in c(X) \subseteq X$, there exists $k_y \in Z_L(X)$ such that $y = \sqcup k_y$. Note that $c : (X, e) \to (X, e)$ is a fuzzy Z-continuous closure operator; then

$$y = c(y) = c(\sqcup k_y) = \sqcup c_L^{\to}(k_y). \tag{45}$$

To show $(c(X), e_{c(X)})$ is a fuzzy Z-algebraic poset, it suffices to show that $c_L^{\to}(k_y) = k_y$. On the one hand, for any $z \in K(c(X))$,

$$\begin{aligned} c_L^{\to}(k_y)(z) &= \bigvee_{x \in K(X)} k_y(x) * e(z, c(x)) \\ &\leq \bigvee_{x \in K(X)} e(x, \sqcup k_y) * e(z, c(x)) \\ &\leq \bigvee_{x \in K(X)} e(c(x), y) * e(z, c(x)) \\ &\leq e(z, y) = k_y(z). \end{aligned} \tag{46}$$

On the other hand, for $z \in K(c(X))$, by Proposition 48, there exists $z' \in K(X)$ with $c(z') = z$. Since c is a fuzzy closure operator, by Proposition 12, $(i_{c(X)}, c)$ is a fuzzy Galois connection between $c(X)$ and X. Thus $k_y(z) = e(z, y) = e(c(z'), y) = e(z', i_{c(X)}(y)) = e(z', y)$, and

$$\begin{aligned} c_L^{\to}(k_y)(z) &= \bigvee_{x \in K(X)} e(x, y) * e(z, c(x)) \\ &\geq e(z', y) * e(z, c(z')) = e(z', y). \end{aligned} \tag{47}$$

□

By Theorem 32 and Proposition 49, we get the following.

Theorem 50. *Any Z-complete closure system of a fuzzy Z-algebraic poset is a fuzzy Z-algebraic poset.*

Lemma 51. *Let (X, e) be a fuzzy Z-complete poset and $A \in Z_L(X)$. Then $K_A = k_{\sqcup A}$, where $K_A : X \to L$ defined by $K_A = \bigvee_{x' \in X} A(x') * \downarrow x'|_{K(X)}$ for any $A \in Z_L(X)$, K_A restricted on $K(X)$, that is, $K_A(y) = \bigvee_{x' \in X} A(x') * e(y, x')$ if $y \in K(X)$ and otherwise* 0.

Proof. For any $x \in K(X)$,

$$\begin{aligned} K_A(x) &= \bigvee_{y \in X} A(y) * e(x, y) \leq \bigvee_{y \in X} e(y, \sqcup A) * e(x, y) \\ &= e(x, \sqcup A) = k_{\sqcup A}(x). \end{aligned} \tag{48}$$

For the converse,

$$\begin{aligned} k_{\sqcup A}(x) &= \Downarrow_Z(\sqcup A)(x) = \bigwedge_{I \in ZI_L(X)} (e(\sqcup A, \sqcup I) \longrightarrow I(x)) \\ &\leq \bigvee_{d \in X} A(d) * e(x, d) = K_A(x). \end{aligned} \tag{49}$$

□

Lai and Zhang [22] pointed out that for any dcpo (Y, e) and fuzzy monotone map $f : (X, e) \to (Y, e)$, there is a unique fuzzy Scott continuous map $\overline{f} : (\mathcal{I}(X), e) \to (Y, e)$ such that $f = \overline{f} \circ \mathbf{y}$, where $\mathbf{y}$ is a yoneda embedding. The following is the similar conclusion of that. It says that any fuzzy monotone maps defined on the compact elements of fuzzy Z-algebraic posets extend uniquely to fuzzy Z-continuous maps on the whole fuzzy poset.

Theorem 52. *Let (X, e) be a fuzzy Z-algebraic poset, (Y, e) a fuzzy Z-complete poset, and $f : K(X) \to Y$ a fuzzy monotone map. Then there exists a unique fuzzy Z-continuous $\overline{f} : X \to Y$, which extends f.*

Proof. For any $x \in X$, there exists $k_x \in Z_L(X)$ such that $x = \sqcup k_x$. Since f is fuzzy monotone and (Y, e) a fuzzy Z-complete poset, then $\sqcup f_L^{\to}(k_x)$ exists in Y. Now for $\overline{f}$ to extend f to be fuzzy Z-continuous, we must have $\overline{f}(\sqcup A) = \sqcup f_L^{\to}(A)$ for any $A \in Z_L(X)$. We need to show that $\overline{f}$ defined in this way is indeed well defined, fuzzy Z-continuous.

For any $x \in K(X)$, $\overline{f}(x) = \overline{f}(\sqcup k_x) = \sqcup f_L^{\to}(k_x)$. Next we show that $\sqcup f_L^{\to}(k_x) = f(x)$. For any $y \in Y$,

$$\begin{aligned} f_L^{\to}(k_x)(y) &= \bigvee_{z \in K(X)} k_x(z) * e(y, f(z)) \\ &\leq \bigvee_{z \in K(X)} e(f(z), f(x)) * e(y, f(z)) \\ &\leq e(y, f(x)), \end{aligned} \tag{50}$$

and for any $z \in Y$,

$$\begin{aligned} &\bigwedge_{y \in Y} (f_L^{\to}(k_x)(y) \longrightarrow e(y, z)) \\ &\quad = \bigwedge_{y \in Y} \left(\bigvee_{z' \in K(X)} k_x(z') * e(y, f(z')) \longrightarrow e(y, z) \right) \\ &\quad = \bigwedge_{z' \in K(X)} (k_x(z') \longrightarrow e(f(z'), z)) \\ &\quad \leq k_x(x) \longrightarrow e(f(x), z) \\ &\quad = e(f(x), z). \end{aligned} \tag{51}$$

Hence by Definition 5, $f(x) = \sqcup f_L^{\to}(k_x)$; that is, $\overline{f}(x) = f(x)$ holds for all $x \in K(X)$. Moreover, for $x \in X$, $\overline{f}(x) = \sqcup f_L^{\to}(k_x)$, and these indicate that $\overline{f}$ is well defined.

Next we show that $\overline{f}$ is fuzzy Z-continuous. Since f is a fuzzy monotone map, then for any $x, y \in X$,

$$\begin{aligned} e(x,y) &\leq \operatorname{sub}\left(k_x, k_y\right) \leq \operatorname{sub}\left(f_L^{\rightarrow}(k_x), f_L^{\rightarrow}\left(k_y\right)\right) \\ &\leq \operatorname{sub}\left(\sqcup f_L^{\rightarrow}(k_x), \sqcup f_L^{\rightarrow}\left(k_y\right)\right) \\ &= e\left(\overline{f}(x), \overline{f}(y)\right), \end{aligned} \tag{52}$$

which implies that $\overline{f}$ is fuzzy monotone; hence we have $1 = e(\sqcup \overline{f}_L^{\rightarrow}(A), \overline{f}(\sqcup A))$, and it remains to show the other side. To this end, by Lemma 51, $\overline{f}(\sqcup A) = \sqcup f_L^{\rightarrow}(k_{\sqcup A}) = \sqcup f_L^{\rightarrow}(K_A)$, and note that $\overline{f}(x) = f(x)$ for all $x \in K(X)$, it suffices to show that $f_L^{\rightarrow}(K_A) \leq \overline{f}_L^{\rightarrow}(A)$. For any $y \in Y$,

$$\begin{aligned} f_L^{\rightarrow}(K_A)(y) &= \bigvee_{x\in K(X)} K_A(x) * e(y, f(x)) \\ &= \bigvee_{x\in K(X)} \bigvee_{x'\in X} A\left(x'\right) * e\left(x, x'\right) * e(y, f(x)) \\ &\leq \bigvee_{x\in K(X)} \bigvee_{x'\in X} A\left(x'\right) * e\left(\overline{f}(x), \overline{f}\left(x'\right)\right) \\ &\quad * e\left(y, \overline{f}(x)\right) \\ &\leq \bigvee_{x'\in X} A\left(x'\right) * e\left(y, \overline{f}\left(x'\right)\right) \\ &= \overline{f}_L^{\rightarrow}(A)(y). \end{aligned} \tag{53}$$

These indicate that $\overline{f}$ is fuzzy Z-continuous and the uniqueness of f is obvious.

Thus, we complete the proof. □

6. Conclusions

Through introducing the concept of fuzzy subset systems, we study fuzzy Z-continuous posets, strongly fuzzy Z-continuous posets, and fuzzy Z-algebraic posets. Because of the outstanding properties of complete residuated lattices, the fuzzy Z-continuous posets and fuzzy Z-algebraic posets inherit many good properties from Z-continuous posets and Z-algebraic posets, such as the fuzzy Z-continuous section-retraction pair between fuzzy Z-continuous posets, and we also give the definition of fuzzy Z-complete closure systems and associate a fuzzy Z-continuous closure operator with a fuzzy Z-complete closure system; it is shown that each fuzzy Z-complete closure system of a fuzzy Z-continuous poset is fuzzy Z-continuous; we also present an extension theorem of fuzzy Z-algebraic posets.

Acknowledgment

This work was supported by the National Natural Science Foundation of China, Grant no. 11071061.

References

[1] D. S. Scott, "Outline of a mathematical theory of computation," in *Proceedings of the 4th Annual Princeton Conference on Information Sciences and Systems*, pp. 169–176, Princeton University Press, Princeton, NJ, USA, 1970.

[2] D. S. Scott, "Continuous lattices," in *Toposes, Algebraic Geometry and Logic*, vol. 274 of *Lecture Notes in Mathematics*, pp. 97–136, Springer, Berlin, Germany, 1972.

[3] J. B. Wright, E. G. Wagner, and J. W. Thatcher, "A uniform approach to inductive posets and inductive closure," *Theoretical Computer Science*, vol. 7, no. 1, pp. 57–77, 1978.

[4] H. Bandelt and M. Erné, "The category of Z-continuous posets," *Journal of Pure and Applied Algebra*, vol. 30, no. 3, pp. 219–226, 1983.

[5] H. J. Bandelt and M. Erne, "Representations and embeddings of M-distributive lattices," *Journal of Pure and Applied Algebra*, vol. 10, no. 3, pp. 315–324, 1984.

[6] D. Novak, "Generalization of continuous posets," *Transactions of the American Mathematical Society*, vol. 272, no. 2, pp. 645–667, 1982.

[7] M. Erné, "Z-continuous posets and their topological manifestation," *Applied Categorical Structures*, vol. 7, no. 1-2, pp. 31–70, 1999.

[8] M. Ernè and D. S. Zhao, "Z-join spectra of Z-supercompactly generated lattices," *Applied Categorical Structures*, vol. 9, no. 1, pp. 41–63, 2001.

[9] P. Venugopalan, "Z-continuous posets," *Houston Journal of Mathematics*, vol. 12, no. 2, pp. 275–294, 1986.

[10] J. J. M. M. Rutten, "Elements of generalized ultrametric domain theory," *Theoretical Computer Science*, vol. 170, no. 1-2, pp. 349–381, 1996.

[11] B. Flagg, P. Sunderhauf, and K. Wagner, *A Logical Approach to Quantitative Domain Theory*, Elsevier, New York, NY, USA, 1996, http://at.yorku.ca/e/a/p/p/23.htm.

[12] K. R. Wager, *Solving recursive domain equations with enriched categories [Ph.D. thesis]*, School of Computer Science, Carnegie-Mellon University, Pittsburgh, Pa, USA, 1994.

[13] Q. Y. Zhang and L. Fan, "Continuity in quantitative domains," *Fuzzy Sets and Systems*, vol. 154, no. 1, pp. 118–131, 2005.

[14] W. Yao, "Quantitative domains via fuzzy sets—part I: continuity of fuzzy directed complete posets," *Fuzzy Sets and Systems*, vol. 161, no. 7, pp. 973–987, 2010.

[15] W. Yao and F. G. Shi, "Quantitative domains via fuzzy sets—part II: fuzzy Scott topology on fuzzy directed-complete posets," *Fuzzy Sets and Systems*, vol. 173, pp. 60–80, 2011.

[16] S. H. Su and Q. G. Li, "Algebraic fuzzy directed-complete posets," *Neural Computing and Applications*, vol. 21, no. 1, supplement, pp. 255–265, 2012.

[17] D. Hofmann and P. Waszkiewicz, "Approximation in quantale-enriched categories," *Topology and Its Applications*, vol. 158, no. 8, pp. 963–977, 2011.

[18] D. Hofmann and P. Waszkiewicz, "A duality of quantale-enriched categories," *Journal of Pure and Applied Algebra*, vol. 216, no. 8-9, pp. 1866–1878, 2012.

[19] P. Waszkiewicz, "On domain theory over Girard quantales," *Fundamenta Informaticae*, vol. 92, no. 1-2, pp. 169–192, 2009.

[20] I. Stubbe, "Categorical structures enriched in a quantaloid: categories, distributors and functors," *Theory and Applications of Categories*, vol. 14, no. 1, pp. 1–45, 2005.

[21] I. Stubbe, "Towards "dynamic domains": totally continuous cocomplete Q-categories," *Theoretical Computer Science*, vol. 373, no. 1-2, pp. 142–160, 2007.

[22] H. Lai and D. X. Zhang, "Complete and directed complete Ω-categories," *Theoretical Computer Science*, vol. 388, no. 1–3, pp. 1–25, 2007.

[23] R. Belohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic, New York, NY, USA, 2002.

[24] P. Hajek, *Mathematics of Fuzzy Logic*, Kluwer Academic, Dordrecht, The Netherlands, 1998.

[25] L. Fan, "A new approach to quantitative domain theory," *Electronic Notes in Theoretical Computer Science*, vol. 45, pp. 77–87, 2001.

[26] R. Bělohlávek, "Fuzzy Galois connections," *Mathematical Logic Quarterly*, vol. 45, no. 4, pp. 497–504, 1999.

[27] R. Bělohlávek, "Fuzzy closure operators," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 473–489, 2001.

[28] W. Yao and L. X. Lu, "Fuzzy Galois connections on fuzzy posets," *Mathematical Logic Quarterly*, vol. 55, no. 1, pp. 105–112, 2009.

[29] Q. Y. Zhang and W. X. Xie, "Section-retraction-pairs between fuzzy domains," *Fuzzy Sets and Systems*, vol. 158, no. 1, pp. 99–114, 2007.

[30] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, vol. 93 of *Encyclopedia of Mathematics and its Applications*, Combridge University Press, Cambridge, UK, 2003.

[31] M. H. Albert and G. M. Kelly, "The closure of a class of colimits," *Journal of Pure and Applied Algebra*, vol. 51, no. 1-2, pp. 1–17, 1988.

[32] G. M. Kelly and V. Schmitt, "Notes on enriched categories with colimits of some class," *Theory and Applications of Categories*, vol. 14, no. 17, pp. 399–423, 2005.

[33] L. Guo, G. Q. Zhang, and Q. G. Li, "Fuzzy closure systems on L-ordered sets," *Mathematical Logic Quarterly*, vol. 57, no. 3, pp. 281–291, 2011.