1 (10 points)

Collaborators: Kayl

Burnside's Lemma states $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Claim: You can color five white squares of a 3x3 grid black in 34 different ways, under rotation.
Proof: Under rotation, the group $G = \{R_0, R_1, R_2, R_3\}$ where each element g in G represents a rotation of the 3x3 grid by 0, 90, 180, and 270 degrees, respectively. We must also determine the number of fixed points in X, that is the number of colorings of our 3x3 grid that remain identical after a single rotation, for each g. If we label our grid as such

1	2	3
4	5	6
7	8	9

we know that the elements in G can be defined as $R_0 = (1)(2)(3)(4)(5)(6)(7)(8)(9), R_1 = (5)(8426)(9713), R_2 = (5)(64)(73)(82)(91), R_3 = (5)(8624)(9317)$. The fixed points occur when each square in the same cycle is the same color, so when selecting only five squares, the number of fixed points of R_0 is $\binom{9}{5}$. For R_1, the two fixed points occur when 5 and one other cycle is colored, which can occur in $\binom{2}{1}$ ways. For R_2, the fixed points would occur when 5 and two of the other cycles are also colored, which can occur in $\binom{4}{2}$ ways. For R_3, the two fixed points would occur when 5 and one other cycle is colored, which can occur in $\binom{2}{1}$ ways. Using Burnside's Lemma, we have that the number of orbitals, which is what we wish to count, is equal to $\frac{1}{4}\left(\binom{9}{5}+\binom{2}{1}+\binom{4}{2}+\binom{2}{1}\right) = 34$.

2 (10 points)

Claim: The beads of a five bead necklace can be colored in 39 different ways using only three colors, under symmetry.
Proof: Let us order the beads 1, 2, 3, 4, and 5, and arrange them as a regular pentagon. Under symmetry, the group $G = \{R_0, R_1, R_2, R_3, R_4, F_1, F_2, F_3, F_4, F_5\}$ where the first five elements represent rotations of the beads by 0, 1, 2, 3, and 4 places, and the last five elements represent flips among the lines of symmetry that pass through bead 1, 2, 3, 4, and 5, respectively. The elements can be defined as $R_0 = (1)(2)(3)(4)(5), R_1 = (51234), R_2 = (52413), R_3 = (53142), R_4 = (54321), F_1 = (1)(43)(52), F_2 = (2)(31)(54), F_3 = (21)(3)(54), F_4 = (21)(4)(53), F_5 = (32)(41)(5)$. The fixed points occur when each bead in the same cycle is the same color, and because there are three possible colorings for each bead, we know that there are 3^k fixed points for each element in G, where k is the number of cycles. By applying Burnside's Lemma, the number of distinct colorings can happen in $\frac{1}{10}\left(3^5+3+3+3+3+3^3+3^3+3^3+3^3+3^3\right) = 39$ ways.

3 (10 points)

Claim: The beads of a six bead necklace can be colored in 130 different ways using only three colors, under rotation.
Proof: Let us order the beads 1, 2, 3, 4, 5, and 6, and arrange them as a regular hexagon. Under rotation, the group $G = \{R_0, R_1, R_2, R_3, R_4, R_5\}$ where the elements represent rotations of the beads by 0, 1, 2, 3, 4, and 5 places, respectively. The elements can be defined as $R_0 = (1)(2)(3)(4)(5)(6), R_1 = (612345), R_2 =$

$(531)(624), R_3 = (41)(52)(63), R_4 = (531)(642), R_5 = (654321)$. By applying Burnside's Lemma, the number of distinct colorings can happen in $\frac{1}{6}\left(3^6 + 3 + 3^2 + 3^3 + 3^2 + 3\right) = 130$ ways.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.