Linear Algebra HW8 Eigenvalues and Eigenvectors Ryzeson Maravich

1

Claim: There is no linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the range and kernel of T are the same subspace of $\mathbb{R}^3$.
Proof: For some $m \times n$ matrix A, Col $A = \{b|b = Ax \text{ for some } x \in \mathbb{R}^m\}$ (Pg 203), so Col A is equal to the range of A. Therefore the range of T is a subspace of $\mathbb{R}^3$ that is equal to dim Col A, which is in turn is equal to rank A. The kernel is the null space (Pg 206), so the kernel of T is a subspace of $\mathbb{R}^3$ that is equal to dim Nul A.
By the Rank Theorem, rank A + dim Nul $A = n$, where n is the number of columns of A. The standard matrix of T must be a 3×3 matrix, so $n = 3$. Because rank A and dim Nul A must be integer values, rank $A \neq$ dim Nul A, and therefore the range and kernel of T can not be the same subspace of $\mathbb{R}^3$.

2

Let $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$, and let E_λ be the eigenspace of A corresponding to λ.

a

Claim: $\lambda = 0$ is an eigenvalue of A.
Proof: λ is eigenvalue if the equation $Ax = \lambda x$ has a nontrivial solution. $Ax = 0x \Rightarrow Ax = 0$. This can be written as the augmented matrix $\left[\begin{array}{ccc|c} 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. x_2 and x_3 are free variables, so the equation has nontrivial solutions, and thus $\lambda = 0$ is an eigenvalue of A.

b

Claim: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors in E_0.

Proof: We can write the reduced augmented matrix from part (a) in parametric vector from as $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors in E_0, and are linearly independent because neither is a scalar multiple of the other. This can be verified by multiplying $\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, both of which equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the corresponding eigenvalue.

3

Claim: For the eigenspace of the matrix $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for $\lambda = 5$ to be two dimensional, $h = 6$.
Proof: The eigenspace of A is the set of all solutions to the equation $(A - \lambda I)x = 0$.

$A - 5I = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ - $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. To solve the equation $(A - 5I)x = 0$, create the augmented matrix $\left[\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right]$, which row reduces to $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 6-h & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. For an eigenspace to be two dimensional, its basis needs to contain 2 linearly independent vectors. This occurs when there are 2 free variables, which can only happen if $h = 6$

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.