Linear Algebra HW1 Systems of Linear Equations Ryzeson Maravich

1

Let

$$\begin{aligned} m_1x_1 + x_2 &= b_1 \\ m_2x_1 + x_2 &= b_2 \end{aligned}$$

where m_1, m_2, b_1, b_2 are constants.

a

Claim: If $m_1 = m_2$ then the system is consistent only if $b_1 = b_2$.
Proof: Let $m_1 = m_2$.

$$\begin{bmatrix} -m_1 & 1 & b_1 \\ -m_1 & 1 & b_2 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} -m_1 & 1 & b_1 \\ 0 & 0 & b_2 - b_1 \end{bmatrix}$$

The system will only be consistent if $b_2 - b_1 = 0$, therefore $b_1 = b_2$.

b

Provide a geometric interpretation for part (a).
If $m_1 = m_2$, then the two equations would have the same coefficients in front of the variables, and therefore have the same slope. Because they will never intersect, the only way the system would ever have a solution is if the two equations represented the same line. The only way this is possible is if they had the same y-intercepts, that is if $b_2 = b_1$.

2

a

Claim: The statement "Elementary row operations on an augmented matrix never change the solution set of the associated linear system." is true.
Proof: Elementary row operations on a matrix give you a new matrix that is row equivalent (Pg 6), and if two linear systems are row equivalent, then the two systems have the same solution set (Pg 7).

b

Claim: The statement "Two matrices are row equivalent if they have the same number of rows." is false.
Proof: Take for example the two matrices A and B below:

$$A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right], B = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Matrices are row equivalent if there is a sequence of elementary row operations that transforms one matrix into another (Pg 6). A and B have the same number of rows, but there is no way to transform matrix A into B.

c

Claim: The statement "Two linear systems are equivalent if they have the same solution set." is true.
Proof: Defined verbatim (Pg 3).

d

Claim: The statement “A system of equations where there are more equations than the number of unknowns (known as an overdetermined system) cannot have a unique solution.” is false.
Proof: Consider the matrix

$$\left[\begin{array}{c|c} 1 & 3 \\ 3 & 9 \end{array}\right].$$

There are more equations than variables, but there is only one unique solution, $x_1 = 3$. (This is because both equations describe the same line.)

3

Suppose the coefficient matrix of a linear system of n equations in n variables has a pivot in every row.

Claim: The linear system has a unique solution.
Proof: A linear system can either have a unique solution, infinitely many solutions, or no solutions.

For there to be infinitely many solutions, there must be a row in the augmented matrix that takes the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 0 \end{array}\right]$, and the corresponding row of the coefficient matrix would take the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$, which is not possible if there a pivot in every row.

Similarly, for there to be no solution, there must be a row in the augmented matrix that takes the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$, and the corresponding row of the coefficient matrix would take the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$, which is not possible if there a pivot in every row.

Therefore the linear system can only have a unique solution.

4

Choose h and k such that the system

$$\begin{aligned} 2x_1 + 5x_2 &= h + 2k \\ 6x_1 + 3hx_2 &= k \end{aligned}$$

has (a) no solution, (b) a unique solution, (c) infinitely many solutions.

a

Claim: For the system to have no solution, $h = 5, k \neq -3$.
Proof: The corresponding augmented matrix can be written as $\left[\begin{array}{cc|c} 2 & 5 & h+2k \\ 6 & 3h & k \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 2 & 5 & h+2k \\ 0 & -15+3h & -3h-5k \end{array}\right]$.

For the linear system to have no solution, the last two columns of the final row must take the form $\left[\begin{array}{c|c} \cdots 0 & b \end{array}\right]$, where $b \neq 0$. For this to be true, we set $-15 + 3h = 0 \Rightarrow h = 5$. Then we can set $-3(h) - 5k \neq 0$ and after substituting $h = 5$, get $k \neq -3$.

b

Claim: For the system to have a unique solution, $h \neq 5$.
Proof: For the linear system to have a unique solution, the second to last column of the final row must be

nonzero. Using the matrix from (a), we can solve the equations to get $h \neq 5$.

c

Claim: For the system to have infinitely many solutions, $h = 5, k = -3$.
Proof: For the linear system to have infinitely many solutions, the last two columns of the final row must take the form $\left[\cdots 0 \mid 0\right]$. Using the matrix from (a), we can solve the equations to get $h = 5, k = -3$.

5

Claim: $\begin{cases} x_1 = 2x_3 + 10x_1 + 17 \\ x_2 = 2x_3 + 6x_4 + 5 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$ is the general solution to the matrix $\left[\begin{array}{ccccc|c} 1 & -3 & 4 & 8 & 0 & 2 \\ 0 & 1 & -2 & -6 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Proof: $\left[\begin{array}{ccccc|c} 1 & -3 & 4 & 8 & 0 & 2 \\ 0 & 1 & -2 & -6 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \xrightarrow{3R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & -10 & 0 & -17 \\ 0 & 1 & -2 & -6 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The solution set can be found by solving the equations for the respective variable, and writing them in terms of free variables if necessary.

6

Claim: There are 7 possible forms of a 2x3 matrix, as listed below.
Proof: There can not be more pivots than rows, so there are between 0 and 2 possible pivots.

If there are no pivots, the matrix looks like $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

If there is only one pivot, then it must on the top row, as all nonzero rows are above any rows of all zeroes. There are no restrictions to what column it could be placed in, so the possible arrangements are $\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc|c} 0 & \blacksquare & * \\ 0 & 0 & 0 \end{array}\right]$, and $\left[\begin{array}{cc|c} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{array}\right]$.

If there are two pivots, then for each of the previous 3 forms, a pivot can be added in the 2nd row, so long as it is placed to the right of the leading entry above it. This gives us $\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \end{array}\right]$, $\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{array}\right]$, and $\left[\begin{array}{cc|c} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{array}\right]$.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.