Linear Algebra HW2 Matrix Equation and Solution Sets Ryzeson Maravich

1

Let $A = \begin{bmatrix} 6 & 2 & 11 & 2 & 8 \\ -3 & -3 & -2 & 10 & 2 \\ -12 & 5 & -7 & 8 & 0 \\ 9 & 6 & 20 & 4 & 3 \end{bmatrix}$.

a

The reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & \frac{-8}{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

b

Claim: The columns of A span $\mathbb{R}^4$.
Proof: There is a pivot position in every row, therefore the columns of A span $\mathbb{R}^4$, by Theorem 4 in Section 1.4.

c

Claim: Any of the first four columns of A are the only ones that may be individually deleted to leave a 4x4 matrix whose columns span $\mathbb{R}^4$.
Proof: Removing any one of these columns still leaves a pivot in each row, so the columns of A will still span $\mathbb{R}^4$.

d

Claim: You can not remove more than one row so that the columns of A span $\mathbb{R}^4$.
Proof: If you remove more than one column, you have more rows than columns. Because you can only have one pivot per column, you would not have enough pivots for there to be a pivot in every row. Therefore the columns of A would not span $\mathbb{R}^4$ by Theorem 4 in Section 1.4.

2

Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

a

Claim: $A\vec{x} = \vec{b}$ has a solution only when $b_1 + 2b_2 + b_3 = 0$.
Proof: We can use A and $\vec{b}$ to make the augmented matrix $\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{array}\right] \xrightarrow{3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 5 & -1 & -8 & b_3 \end{array}\right]$

$\xrightarrow{-5R_1+R_3} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 14 & 12 & -5b_1+b_3 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{array}\right]$.

The only possible solution to this augmented matrix is $b_1 + 2b_2 + b_3 = 0$.

b

Claim: $A\vec{x} = \vec{b}$ does not have solution for all $\vec{b}$.
Proof: If $b_1 + 2b_2 + b_3 \neq 0$, then the last two columns of the final row of the matrix have the form $\begin{bmatrix} 0 & b \end{bmatrix}$ where $b \neq 0$, therefore there is no solution.

3

a

Claim: The statement "If A is a 3x4 matrix, then the homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution" is true.
Proof: There are 4 variables, but there can only be 3 pivots because there are 3 rows, and therefore there can be at most 3 basic variables. This guarantees at least one free variable, which guarantees a nontrivial solution by definition (Pg 44).

b

Claim: The statement "If the equation $A\vec{x} = \vec{b}$ is inconsistent, then $\vec{b}$ is not in the set spanned by the columns of A" is true.
Proof: If $A\vec{x} = \vec{b}$ is inconsistent, then the vector equation $x_1a_1 + x_2a_2 + \cdots x_pa_p = b$ does not have a solution (Pg 36), therefore $\vec{b}$ is not in the set spanned by $\{a_1, a_2, \ldots a_p\}$, the columns of A (Def Pg 30).

c

Claim: The statement "The equation $A\vec{x} = \vec{b}$ is homogeneous if the zero vector is a solution" is true.
Proof: If the zero vector is a solution, that is $\vec{x} = \vec{0}$, then for every matrix A, $A\vec{0} = \vec{0}$, so $\vec{b} = 0$, which fits the form of a homogeneous matrix equation (Pg 43).

d

Claim: The statement "If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $\vec{v}$ is in the span of $\{\vec{u}, \vec{u} - \vec{v}\}$" is true.
Proof: $\vec{v}$ can be written as a linear combination as such

$$\begin{aligned}\vec{v} &= 1\vec{u} - 1(\vec{u} - \vec{v}) \\ &= \vec{u} - \vec{u} + \vec{v} \\ &= \vec{v}\end{aligned}$$

therefore it is the span$\{\vec{u}, \vec{u} - \vec{v}\}$ (Pg 30).

4

Let $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$.
Claim: $h = \frac{-7}{2}$ for $\vec{y}$ to be in the plane spanned by $\vec{v_1}\vec{v_2}$.
Proof: $\vec{y}$ is in the plane spanned by $\vec{v_1}\vec{v_2}$ if it can be written as the vector equation $c_1\vec{v_1} + c_2\vec{v_2} = \vec{y}$.
This can be written as the augmented matrix $\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{cc|c} 1 & 3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{array}\right]$.
This will have a solution only when $2h + 7 = 0$, that is when $h = \frac{-7}{2}$.

5

a

Claim: The solution set for the single-equation linear system $3x_1 + x_2 - 4x_3 = 0$ is $x = x_2 \begin{bmatrix} \frac{-1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

Proof: $3x_1 + x_2 - 4x_3 = 0 \Rightarrow x_1 = \frac{-1}{3}x_2 + \frac{4}{3}x_3$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3}x_2 + \frac{4}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

b

Claim: The solution set for the single-equation linear system $3x_1 + x_2 - 4x_3 = 6$ is $x = x_2 \begin{bmatrix} \frac{-1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Proof: One solution would be $x_1 = 1, x_2 = 3, x_3 = 0$.

$3x_1 + x_2 - 4x_3 = 6 \Rightarrow x_1 = \frac{-1}{3}x_2 + \frac{4}{3}x_3 + 2$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3}x_2 + \frac{4}{3}x_3 + 2 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

c

The solution in part (a) is the plane in the span$\{\vec{u}, \vec{v}\}$. The solution in (b) is the plane in (a) translated by the vector $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, parallel to the plane in (a).

6

Suppose $A\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ for an unknown A and two unknown vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$. Let $\vec{w} = 2\vec{u} - 3\vec{v}$.

Claim: $A\vec{w} = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$.

Proof:

$$\begin{aligned} A\vec{w} &= A(2\vec{u} - 3\vec{v}) && (1) \\ &= A(2\vec{u}) + A(-3\vec{v}) && (2) \\ &= 2A\vec{u} + -3A\vec{v} && (3) \\ &= 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + -3 \begin{bmatrix} 1 \\ -3 \end{bmatrix} && (4) \\ &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} && (5) \\ &= \begin{bmatrix} -1 \\ 11 \end{bmatrix} && (6) \end{aligned}$$

(1) Given
(2) Theorem 5a, Pg 39
(3) Theorem 5b, Pg 39
(4) Given
(5) Scalar Multiplication, Pg 25
(6) Vector Addition, Pg 25

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.