1 (20 points)

Collaborators: Dr. Conceicao, Connor Berson, Kayl Murdough

(a)
Claim: There are 336 ways someone can order a dozen of donuts where there are six different varieties available and they order between one and four (inclusive) of each variety.
Proof: The generating function for this problem is $G(x) = (x + x^2 + x^3 + x^4)^6 = x^6(1 + x + x^2 + x^3)^6 = x^6(\frac{1-x^4}{1-x})^6 = x^6(1 - x^4)^6(\frac{1}{1-x})^6$. The formal power series of $G(x)$ gives us the ordinary generating function $x^6 \sum_{k\geq 0} \binom{6}{k}(-x^4)^k \sum_{j\geq 0} \left(\!\binom{6}{j}\!\right) x^j$. The number of ways we can place an order for a dozen donuts is $\left[x^6 \sum_{k\geq 0} \binom{6}{k}(-x^4)^k \sum_{j\geq 0} \left(\!\binom{6}{j}\!\right) x^j\right]_{x^{12}} = \left[\sum_{k\geq 0} \binom{6}{k}(-x^4)^k \sum_{j\geq 0} \left(\!\binom{6}{j}\!\right) x^j\right]_{x^6}$. The only solution sets (k, j) that satisfy this are (0,6) and (1,2). This results in $[\binom{6}{0} \left(\!\binom{6}{6}\!\right) - \binom{6}{1} \left(\!\binom{6}{2}\!\right)]x^6$, with the coefficient computing to 336 different ways.

(b)
Claim: There are 91 10-combinations of the letters A, B, C, D, and E, if A can be used any number of times, B must be used at least once, C can be used at most once, D is used exactly twice or not at all, and E is used an even number of times.
Proof: $G(x) = (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})(1+x)(1+x^2)(1+x^2+x^4+x^6+x^8+x^{10})$, representing the number of A's, B's, C's, D's, and E's, respectively. As the quantities representing the number of A's, B's, and E's can be represented as infinite sums, $G(x) = (\frac{1}{1-x})(\frac{x}{1-x})(1+x)(1+x^2)(\frac{1}{(1-x)^2}) = x(\frac{1}{1-x})^3(1+x^2)$. The number of ways to get 10-combinations with the specified restrictions is $\left[x(\frac{1}{1-x})^3(1+x^2)\right]_{x^{10}} = \left[(\frac{1}{1-x})^3(1+x^2)\right]_{x^9} = \left[(1+x^2)\sum_{k\geq 0} \left(\!\binom{3}{k}\!\right) x^k\right]_{x^9}$. The coefficient is $\left(\!\binom{3}{9}\!\right)$ when you choose 1, and $\left(\!\binom{3}{7}\!\right)$ when you choose x^2, for a total of 91 total ways.

4 (20 points)

Collaborators: Matt Torrence

$a_0 = 1, a_1 = 4$, and for $n \geq 2$, $a_n = 4a_{n-1} + 4a_{n-2}$

Claim: A closed formula for the sequence defined above is $2^n(n + 1)$.
Proof: Let $G(x)$ be the generating function of the sequence $\{a_n\}_{n\geq 0}$. We will first re-index the equation to $a_{n+2} = 4a_{n+1} + 4a_n$, then multiply both sides by x^{n+2} and sum over all natural numbers n to get $\sum_{n\geq 0} a_{n+2}x^{n+2} = 4\sum_{n\geq 0} a_{n+1}x^{n+2} - 4\sum_{n\geq 0} a_n x^{n+2} \Rightarrow \sum_{n\geq 0} a_{n+2}x^{n+2} = 4x\sum_{n\geq 0} a_{n+1}x^{n+1} - 4x^2\sum_{n\geq 0} a_n x^n$. By writing this equation in terms of our generating function $G(x)$, we get $G(x) - a_0 - a_1x = 4x(G(x) - a_0) - 4x^2G(x)$. After substituting initial conditions and rearranging, we get $G(x) = \frac{1}{(1-2x)^2}$, which is equivalent to $\sum_{n\geq 0} 2^{n-1}nx^{n-1}$. Because a_n is the coefficient on x^n, we must substitute $n+1$ for n, to get $a_n = 2^n(n + 1)$.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.