Linear Algebra HW4 Invertible Matrices Ryzeson Maravich

1

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} -2 & 3 \\ 3 & 1 \end{bmatrix}$.

Claim: The set of 4-tuples (a, b, c, d) such that $AB = BA$ is $(-c+d, c, c, d)$ for $c, d \in \mathbb{R}$.
Proof: $AB = \begin{bmatrix} -2a+3b & 3a+b \\ -2c+3d & 3c+d \end{bmatrix}$, and $BA = \begin{bmatrix} -2a+3c & -2b+3d \\ 3a+c & 3b+d \end{bmatrix}$, so $AB - BA = \begin{bmatrix} 3b-3c & 3a+3b-3d \\ -3a-3c+3d & -3b+3c \end{bmatrix}$.
For $AB = BA$, $AB - BA = 0$, so each entry in the previous matrix must equal 0. We can then construct the augmented matrix $\left[\begin{array}{cccc|c} 0 & 3 & -3 & 0 & 0 \\ 3 & 3 & 0 & -3 & 0 \\ -3 & 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ and solve to get the 4-tuple $(-c+d, c, c, d)$.

2

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(x_1, x_2) = (5x_1 + 3x_2, 2x_2)$.

a

Claim: T is invertible.
Proof: Let A be the standard matrix of T. $A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Because A is an $n \times n$ matrix and has a pivot in every row, it is invertible by the Invertible Matrix Theorem, and therefore T is also invertible.

b

Claim: $T^{-1}(x_1, x_2) = (\frac{1}{5}x_1 - \frac{3}{10}x_2, \frac{1}{2}x_2)$.
Proof: The linear transformation T^{-1} should yield the inverse of the standard matrix of T. The standard matrix of T, $A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$, so the inverse matrix $A^{-1} = \frac{1}{10} \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{-3}{10} \\ 0 & \frac{1}{2} \end{bmatrix}$. For T^{-1} to have this as its standard matrix, $T^{-1}(x_1, x_2) = (\frac{1}{5}x_1 - \frac{3}{10}x_2, \frac{1}{2}x_2)$.

3

Let X be an $n \times m$ matrix and let Y be an $m \times p$ matrix.

Claim: If the columns of X are linearly independent and the columns of Y are linearly independent, then the columns of XY are linearly independent.
Proof: In order for the columns of XY to be linearly independent, the equation $XY(x_3) = 0$ must have only the trivial solution. This can be rewritten as $X(Yx_3) = 0$, and because the columns of X are linearly independent, we know that this equation has only the trivial solution, therefore $Yx_3 = 0$. Similarly, because the columns of Y are linearly independent, we know this equation also has only the trivial solution, therefore $x_3 = 0$. This shows that $XY(x_3) = 0$ only has the trivial solution, and therefore the columns of XY are linearly independent.

4

Suppose $AD = I_m$.

a

Claim: $\forall b \in \mathbb{R}^m$, $Ax = b$.
Proof: Let $x = Db$

$$\begin{aligned} & x = Db \\ \Rightarrow \quad & Ax = ADb \\ \Rightarrow \quad & Ax = I_m b \\ \Rightarrow \quad & Ax = b \end{aligned}$$

and thus $\forall b \in \mathbb{R}^m$, $Ax = b$, where $x = Db$.

b

Claim: A cannot have more rows than columns.
Proof: The only way $\forall b \in \mathbb{R}^m$, $Ax = b$ is if there is a pivot in every row of A, as this guarantees that there will never be a row $[0 \cdots 0|b]$ where $b \neq 0$. If A had more rows than columns, then there would not be a pivot in every row.

5

Claim: If A and B are $n \times n$ matrices and their product AB is invertible, then A must also be invertible.
Proof: Suppose AB is invertible, then there is some matrix C such that $ABC = CAB = I$.

$$\begin{aligned} & CAB = I \\ \Rightarrow \quad & ABCAB = ABI \\ \Rightarrow \quad & A(BCA)B = AB \end{aligned}$$

therefore $BCA = I$ and because we already know $ABC = I$, A is invertible.

6

A matrix A is idempotent if $A = A^2$.

a

Claim: $I - A$ is idempotent.
Proof:

$$\begin{aligned} (I - A)(I - A) &= I^2 - A - A + A^2 \\ &= I - A - A + A \\ &= I - A \end{aligned}$$

b

Claim: $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$.
Proof: $I + A$ is nonsingular if there is some matrix C such that $C(I + A) = (I + A)C = I$.

$$\begin{aligned}(I - \frac{1}{2}A)(I + A) &= I^2 + IA - \frac{1}{2}AI - \frac{1}{2}A^2 \\ &= I + A - \frac{1}{2}A - \frac{1}{2}A \\ &= I\end{aligned}$$

and

$$\begin{aligned}(I + A)(I - \frac{1}{2}A) &= I^2 + I(-\frac{1}{2}A) + AI + A(-\frac{1}{2}A) \\ &= I - \frac{1}{2}A + A - \frac{1}{2}A \\ &= I\end{aligned}$$

therefore $A + I$ is nonsingular and $(A + I)^{-1} = I - \frac{1}{2}A$.

c

Claim: $\begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix}$ is a 2×2 idempotent matrix.
Proof: $\begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix}$.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.