5 (20 points)

Collaborators: Connor Berson

Let n be a composite number and p be the smallest prime dividing n.

(a)
Claim: $p \leq \sqrt{n}$
Proof: n has two factors, a and b such that $ab = n$ and $1 < a \leq b < n$. Assume $a > \sqrt{n}$. $ab > \sqrt{n}^2 \Rightarrow ab > n$. This results in a contradiction because $ab = n$. Therefor $a \leq \sqrt{n}$. By the Fundamental Theorem of Arithmetic, $\forall x, x > 1$ have a prime factorization, that is $\exists p \in \mathbb{N}, p|x$. Therefore $p|a \wedge a|n \Rightarrow p|n$ by the transitive property of multiplication.

(b)
Claim: There are 35 positive primes less than 150.
Proof: We can find the number of primes less than 150 by subtracting from 150 the number of positive composite numbers between 1 and 150. By part (a) we know that each number $1 < n < 150$ has a prime factor p such that $p \leq \sqrt{n}$. Therefore each number less than 150 will also have a prime less than $\sqrt{150}$. The positive composite numbers less than 150 will be those that are divisible by one of the prime divisors less than $\sqrt{150}$, which are 2, 3, 5, 7, and 11.

Let $C_{a,b,\ldots z}$ be the set of numbers such that $a, b, \ldots z$ all divide 150. By applying the Sieve Formula, we can find the number of positive primes less than 150 by calculating $150-(c_2+c_3+c_5+c_7+c_{11})+(c_{2,3}+c_{2,5}+c_{2,7}+c_{2,11}+c_{3,5}+c_{3,7}+c_{3,11}+c_{5,7}+c_{5,11}+c_{7,11})-(c_{2,3,5}+c_{2,3,7}+c_{2,3,11}+c_{2,5,7}+c_{2,5,11}+c_{2,7,11}+c_{3,5,7}+c_{3,5,11}+c_{3,7,11}+c_{5,7,11})+(c_{2,3,5,7}+c_{2,3,5,11}+c_{2,3,7,11}+c_{2,5,7,11}+c_{3,5,7,11})-(c_{2,3,5,7,11}) = 150-189+84-14+0-0 = 31$.

We must add to this total the five primes we removed. They were initially removed because they are divisible by themselves. We must also subtract 1 from this total because it is not a prime, but it was not removed by the Sieve Formula because it is not composite either. This leaves us with a final answer of $31 + 5 - 1 = 35$.

6 (20 points)

Collaborators: Kayl Murdough

(a)
i.
Claim: You can distribute 20 identical objects to 10 distinct recipients so that each recipient receives at most 5 objects in 2,930,455 different ways.
Proof: This is the same as asking how many solutions there are to the equation $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} = 20$ such that $0 \leq x_i \leq 5$. We can calculate this by subtracting the number of solutions where $x_i > 5$ from the total number of integer solutions. The total number of solutions is the number of weak compositions, which is $\binom{29}{20}$.

One possibility where $x_i > 5$ is when just one of the variables satisfies this condition. To ensure this condition, WLOG substitute $x_1' + 6$ for x_1. This results in $(x_1'+6)+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} = 20 \Rightarrow x_1'+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} = 14$, which has $\binom{23}{14}$ solutions. The variable that can be chosen to satisfy this condition can be chosen in $\binom{10}{1}$ ways. Similarly, $x_i > 5$ can happen with two variables and three variables, which result in $\binom{17}{8}\binom{10}{2}$ and $\binom{11}{2}\binom{10}{3}$ ways respectively. It is impossible for four variables to satisfy $x_i > 5$ as this would be more than 20.

By applying the Sieve Formula, we get $\binom{29}{20} - \binom{23}{14}\binom{10}{1} + \binom{17}{8}\binom{10}{2} - \binom{11}{2}\binom{10}{3} = 2,930,455$.

ii.
Claim: You can distribute 20 identical objects to 10 distinct recipients so that each recipient receives at least 1 but at most 5 objects in 90,221 different ways.
Proof: Again, this is the same as asking how many solutions there are to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 20$ such that $1 \leq x_i \leq 5$. First we can give each recipient one item, that is substitute $x_i' + 1$ for x_i, resulting in $x_1' + x_2' + x_3' + x_4' + x_5' + x_6' + x_7' + x_8' + x_9' + x_{10}' = 10$. Because each recipient already has one item, the new condition becomes $0 \leq x_i \leq 4$. (Now you can't add more than four objects to a recipient without it having more than 5 objects total.) You can then use the same process and formula as part (a) to get $\binom{19}{10} - \binom{14}{5}\binom{10}{1} + \binom{9}{0}\binom{10}{2} = 90,221$ ways.

(b)
Claim: You can distribute n identical objects to k distinct recipients so that each recipient receives at most r objects with the formula $\sum_{i=0}^{k}(-1)^i\binom{k+n-1-i(r+1)}{k-1}\binom{k}{i}$
Proof: $(-1)^i$ accounts for the proper sign of the Sieve Formula. The first term should be positive, and it alternates every term. $\binom{k+n-1-i(r+1)}{k-1}$ counts the number of solutions such that the condition $0 \leq x_a \leq r$ is satisfied for i recipients. $\binom{k}{i}$ counts the way you can choose the recipients that satisfy the condition. The summation goes from 0 to k, because the minimum number of recipients the condition must be applied to is 0, and the maximum is all k recipients. If it is not possible to apply the condition to i number of recipients, then the term will become zero, which does not affect the total.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.