Combinatorics HW9 Exponential Generating Functions Ryzeson Maravich

1 (20 points)

Collaborators: Kayl Murdough

(a)
Claim: There are 739,792 ways to distribute 10 different toys to 4 different children if the first child receives at least one toy and the last child receives at least two toys.
Proof: The generating function for this problem is $F(x) = (\frac{x}{1!} + \frac{x^2}{2!} + \cdots)(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots)^2(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)$, representing each of the four children, respectively. $F(x) = (e^x - 1)e^{2x}(e^x - 1 - x) = e^{4x} - 2e^{3x} + e^{2x} - xe^{3x} + xe^{2x} \Rightarrow a_n = 4^n - 2(3^n) + 2^n - n3^{n-1} + n2^{n-1}$, therefore $a_{10} = 739,792$.

(b)
Claim: There are $\frac{4^n+2^n-3^n-1}{2}$. ways to pile red, white, blue and orange poker chips in a stack of height n such that the stack contains an even number of blue chips and at least one orange chip.
Proof: $F(x) = (1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots)^2((1+\frac{x^2}{2!}+\frac{x^4}{4!}\cdots)(\frac{x}{1!}+\frac{x^2}{2!}+\cdots) = e^{2x}(\frac{e^x+e^{-x}}{2})(e^x-1) = \frac{e^{4x}+e^{2x}-e^{3x}-e^x}{2} \Rightarrow a_n = \frac{4^n+2^n-3^n-1}{2}$.

2 (10 points)

$a_0 = 2$ and $a_n = na_{n-1} - n!$ for $n \geq 2$.

Claim: A closed formula for a_n defined above is $(2-n)n!$.
Proof: Let $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ be the generating function of the sequence $\{f_n\}_{n\geq 0}$. Let us first re-index the equation to $a_{n+1} = (n+1)a_n - (n+1)!$, then multiply both sides by $\frac{x^{n+1}}{(n+1)!}$ and sum over all natural numbers n to get $\sum_{n\geq 0} a_{n+1}\frac{x^{n+1}}{(n+1)!} = \sum_{n\geq 0}(n+1)a_n\frac{x^{n+1}}{(n+1)!} - \sum_{n\geq 0}(n+1)!\frac{x^{n+1}}{(n+1)!}$. By writing our equation in terms of our generating function, we get

$$\begin{aligned}
&F(x) - a_0 = xF(x) - \frac{x}{1-x} \\
\Longrightarrow\ &F(x)(1-x) = \frac{2-3x}{1-x} \\
\Longrightarrow\ &F(x) = \frac{3}{1-x} - \frac{1}{(1-x)^2} \\
\Longrightarrow\ &F(x) = \sum_{n\geq 0} 3x^n - \sum_{n\geq 0}\left(\!\!\binom{2}{n}\!\!\right)x^n \\
\Longrightarrow\ &a_n = \left[3 - \left(\!\!\binom{2}{n}\!\!\right)\right]n! \\
\Longrightarrow\ &a_n = (2-n)n!.
\end{aligned}$$

4 (20 points)

For a fixed integer $m \geq 1$, let a_n be the number of surjective functions $f : [n] \to [m]$.

(a)
Claim: The EGF for a_n is $(e^x - 1)^m$.
Proof: We can form a bijection from the number of surjective functions, to the number of n-letter words, such that each letter in an m-length alphabet is used once. The generating function would be represented

as $(\frac{x}{1!} + \frac{x^2}{2!} + \cdots)^m = (e^x - 1)^m$.

(b)
Claim: $a_n = \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k}k^n$.
Proof: $F(x) = (e^x - 1)^m \Rightarrow F(x) = \sum_{k=0}^{m} \binom{m}{k} e^{kx}(-1)^{m-k}$ by the Binomial Theorem. After substituting in the power series for e^{kx} and rearranging, we get $F(x) = \sum_{n\geq 0}\sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k}k^n\frac{x^n}{n!}$, therefore $a_n = \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k}k^n$.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.