Linear Algebra HW7 Bases and Coordinate Vectors Ryzeson Maravich

Let $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$. Then A is row equivalent to $B = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a

Claim: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for Nul A.

Proof: In this case, Nul $A = \{x | x \in \mathbb{R}^5 \text{ and } Ax = 0\}$. We can find these by creating the augmented matrix corresponding to the equation $Ax = 0$, and rewriting it in terms of free variables.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \frac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$$

The set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$ spans Nul A, and because it is linearly independent, is also a basis for Nul A.

b

Claim: The dimension of Nul A is 2.
Proof: The dimension of Nul A is equivalent to the number of free variables in B, which is 2.

c

Claim: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$ is a basis for Col A.
Proof: A basis for Col A can be constructed from the columns of A such that the corresponding columns of B are pivot columns.

d

Claim: The dimension of Col A is 3.
Proof: The dimension of Col A is equivalent to the number of pivot columns in B, which is 3.

1

The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$.

Claim: The coordinate vector of $p(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$ is $\begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$.

Proof: The coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ satisfies the equation $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$, where $P_{\mathcal{B}}$ is the change-of-coordinate matrix, and x is the vector we wish to transform. We can construct the columns of $P_{\mathcal{B}}$ with the elements in $\mathcal{B}$. We can do the same for $\mathbf{x}$ by using $p(t)$. This gives us the equation

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}.$$

Solving this equation gives us $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$.

2

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$.

Claim: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for $\mathbb{R}^3$.

Proof: A basis for $\mathbb{R}^3$ must be a set of linearly independent vectors that span $\mathbb{R}^3$. The matrix formed the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ row reduces as follows.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & 3 & -2 \end{bmatrix} \tilde{} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Because it does not row reduce to the 3×3 identity matrix, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbb{R}^3$ by the Invertible Matrix Theorem, so it is not a basis for $\mathbb{R}^3$.

Additionally, $4\mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent either.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.