Combinatorics HW6 Stirling Numbers and Permutations Ryzeson Maravich

1 (10 points)

A group of ten children want to play cards. They split into three groups, one of these groups has four children in it, the other two have three each. Then each group sits around a table. Two seatings are considered the same if everyone's left neighbor is the same.

(a)
Claim: This can be done in 100,800 ways if the tables are identical.
Proof: First let's count the ways we can make the groups. We can pick 3 children from the group of 10, and then 3 more from this remaining group of 7, which then leaves us with the third group that contains 4 children. This is represented by $\binom{10}{3}\binom{7}{3} = 4,200$.

Then we need to figure out how many ways we can place each group at a table. There are $(n-1)!$ ways to place n people around a table. (You can just fix a person, and then there are $(n-1)!$ ways to linearly order the rest.) This gives us $(4-1)!(3-1)!(3-1)! = 24$. By multiplying the number of ways we can pick the groups, by the number of ways we can arrange them at the tables, we get 100,800 different ways.

(b)
Claim: This can be done in 604,800 ways if the tables are distinct.
Proof: If the tables are distinct, you also need to account for the different ways you could place the three groups at the tables. This is a linear order of three things, so $100,800 \cdot 3! = 604,800$

3 (10 points)

Collaborators: Connor Berson

Claim: $5x^4 - 10x^3$ as a linear representation of polynomials is $-5(x)_1 + 5(x)_2 + 20(x)_3 + 5(x)_4$
Proof: Using the equation $(x)_n = x(x-1)(x-2)\dots(x-n+1)$, we get that $(x)_4 = x^4 - 6x^3 + 11x^2 - 6x$. Because the coefficient of x^4 is a 5, we need $5(x)_4$, which is $5x^4 - 30x^3 + 55x^2 - 30x$. To then get a coefficient of -10 on x^3, we need $20(x)_3$. This process continues so that the remaining coefficient on each term is zero.

This leaves us with $-5(x)_1 + 5(x)_2 + 20(x)_3 + 5(x)_4$.

4 (10 points)

Collaborators: Matt Torrence

Claim: For all integers $n > 1$, $\sum_{k=0}^{n} s(n,k) = 0$.
Proof: The claim holds for $n = 2$. $\sum_{k=0}^{2} = s(2,0) + s(2,1) + s(2,2) = 0 + -1 + 1 = 0$. To prove the inductive step, we need to show $\sum_{k=0}^{n} s(n,k) = 0 \Rightarrow \sum_{k=0}^{n+1} s(n+1,k) = 0$, that is $\sum_{k=0}^{n} s(n,k) = \sum_{k=0}^{n+1} s(n+1,k)$.

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\begin{aligned}
\sum_{k=0}^{n} s(n,k) &= \sum_{k=0}^{n} s(n,k) - 1 + 1 && (1)\\
&= \sum_{k=-1}^{n} s(n,k) + 1 && (2)\\
&= \sum_{k=0}^{n} s(n,k-1) + 1 && (3)\\
&= \sum_{k=0}^{n} s(n,k-1) + -(n)\sum_{k=0}^{n} s(n,k) + 1 && (4)\\
&= \left[\sum_{k=0}^{n} s(n,k-1) + -(n)s(n,k)\right] + 1 && (5)\\
&= \sum_{k=0}^{n} s(n+1,k) + 1 && (6)\\
&= \sum_{k=0}^{n} s(n+1,k) + s(n+1,n+1) && (7)\\
&= \sum_{k=0}^{n+1} s(n+1,k) && (8)
\end{aligned}
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(4) $-(n)\sum_{k=0}^{n} s(n,k)$ is zero by our inductive assumption
(6) Recursive definition for the Stirling numbers of the first kind
(7) $s(n+1,n+1)$ is positive one because n and k have the same parity

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.