Linear Algebra HW9 Diagonalization Ryzeson Maravich

1

One of the eigenvalues of the matrix $F = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$ is $\lambda = 5$, and $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector of F.

Claim: F is diagonalizable.

Proof: F is diagonalizable if $F = PDP^{-1}$ for some invertible matrix P and some diaongal matrix D. In order to construct P, we need 3 linearly independent eigenvectors of F. Let us find the eigenvectors when $\lambda = 5$, which satisfy the equation $F - 5I = 0$. This can be represented as

$$\left[\begin{array}{ccc|c} -12 & -16 & 4 & 0 \\ 6 & 8 & -2 & 0 \\ 12 & -16 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \alpha \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}.$$

These two egienvectors, along with the one provided, give 3 linearly independent eigenvectors needed to construct $P = \begin{bmatrix} -\frac{4}{3} & \frac{1}{3} & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Note that P row reduces to the 3×3 identity matrix, so is invertible by the Invertible Matrix Theorem.

To construct D, we need the eigenvectors' corresponding eigenvalues. $\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$ correspond to $\lambda = 5$, so we just need to find the eigenvalue for the provided vector. We can do this by multiplying

$$\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Therefore, because $F\vec{v} = -3\vec{v}$, we know $\lambda = 3$ is an eigenvector of F. We can then construct the matrix $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, and thus the diagonalization is complete.

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Claim: The matrix $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ is diagonalizable.

Proof: A is diagonalizable if $A = PDP^{-1}$ for some invertible matrix P and some diaongal matrix D. Because A is a lower triangular matrix, the eigenvalues of A are the entries of the main diagonal, so $\lambda = 4$, 4, 2, 2. To construct P, we need to find 4 linearly independent eigenvectors of A, which are solutions to the equation $A - \lambda I = 0$. When $\lambda = 4$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and when $\lambda = 2$,

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. Note that P row reduces to the 4×4 identity matrix, so is invertible by the Invertible Matrix Theorem. Using the corresponding eigenvalues, we can then construct $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, and thus A is diagonizable.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.