Linear Algebra HW3 Transformations & Dependence Ryzeson Maravich

1

Let $A = [\vec{v_1} \quad \vec{v_2} \quad \vec{v_3} \quad \vec{v_4} \quad \vec{v_5} \quad \vec{v_6}] = \begin{bmatrix} 6 & 3 & 0 & 11 & 9 & 2 \\ 2 & 1 & 0 & 2 & 13 & -1 \\ -4 & 0 & -8 & -1 & 2 & -1 \\ 8 & 5 & -4 & 7 & 4 & -8 \\ 1 & -4 & 18 & 15 & 6 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Claim: $B = [\vec{v_1} \quad \vec{v_2} \quad \vec{v_4} \quad \vec{v_5}]$ is a solution to the equation $B\vec{x} = \vec{0}$ so that it has only the trivial solution, and so it has the maximum amount of columns from A.

Proof: $B\vec{x} = \vec{0}$ has only the trivial solution if and only if B has no free variable, which is the case for $B = [\vec{v_1} \quad \vec{v_2} \quad \vec{v_4} \quad \vec{v_5}]$.

This is the maximum amount of columns from A that B can have, because for every case where B has five or six columns, that is $\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, there is at least one free variable.

2

Consider each of the following linear transformations of the plane $\mathbb{R}^2$.
(i) Find $T(\vec{e_1})$ and $T(\vec{e_2})$.
(ii) Find the standard matrix for T.

a

Vertical shear that sends $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\vec{e_1} - 3\vec{e_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, leaving $\vec{e_2}$ unchanged.

i

$T(\vec{e_1}) = \vec{e_1} - 3\vec{e_2}$ and $T(\vec{e_2}) = \vec{e_2}$

ii

$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b

Rotation counterclockwise about the origin by $\frac{3\pi}{4}$ radians.

i

$T(\vec{e_1}) = -\frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2}$ and $T(\vec{e_2}) = -\frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2}$

ii

$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

c

Reflection through the line $x_2 = x_1$.

i

$T(\vec{e_1}) = \vec{e_2}$ and $T(\vec{e_2}) = \vec{e_1}$

ii

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d

The composite linear transformation T_d that first performs (a), then (b), then (c)

i

$$\begin{aligned}
T_d(\vec{e_1}) &= T_c(T_b(T_a(\vec{e_1}))) \\
&= T_c(T_b(\vec{e_1} - 3\vec{e_2})) \\
&= T_c(T_b(\vec{e_1}) - 3T_b(\vec{e_2})) \\
&= T_c(-\frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2} - 3(-\frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2})) \\
&= T_c(\sqrt{2}\vec{e_1} + 2\sqrt{2}\vec{e_2}) \\
&= \sqrt{2}T_c(\vec{e_1}) + 2\sqrt{2}T_c(\vec{e_2}) \\
&= 2\sqrt{2}\vec{e_1} + \sqrt{2}\vec{e_2}
\end{aligned}$$

ii

$\begin{bmatrix} 2\sqrt{2} & -\frac{\sqrt{2}}{2} \\ \sqrt{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

3

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{\vec{v_1} \cdots \vec{v_p}\}$ be a set of linearly independent vectors in $\mathbb{R}^n$.
Claim: If $\{T(\vec{v_1}) \cdots T(\vec{v_p})\}$ is linearly dependent then T is not bijective.
Proof: $\{T(\vec{v_1}) \cdots T(\vec{v_p})\}$ is linearly dependent, so it can be written as $c_1T(\vec{v_1}) + c_2T(\vec{v_2}) + \cdots + c_pT(\vec{v_p}) = 0$ for some $c_i \neq 0$, which can be rewritten as $T(c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_p\vec{v_p}) = 0$. Because some $c_i \neq 0$, $T(x) = 0$ has more than the trivial solution, and is therefore not bijective.

4

Claim: $h = -\frac{28}{3}$ for the vectors $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ to be linearly dependent.

Proof: We can construct the augmented matrix $\left[\begin{array}{ccc|c} -2 & -6 & 8 & 0 \\ 4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{28}{3}+h & 0 \end{array}\right]$. For the vectors to be linearly dependent, they need to have more than the trivial solution to $A\vec{x} = \vec{0}$, so they need to have at least one free variable, which is the case when $h = -\frac{28}{3}$.

5

Claim: The statement "If $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are vectors in $\mathbb{R}^4$ such that no vector $\vec{v_i}$ is a scalar multiple of one of the other three vectors, then the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is linearly independent." is false.

Proof: Let $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$. No vector is a scalar multiple of another vector, but the augmented matrix row reduces to $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, which has more than just the trivial solution, therefore making the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ linearly dependent.

The linear dependency of a set of vectors is determined by whether at least one of the vectors can be written as a linear combination of the others, not by whether some are scalar multiples of another.

6

Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is an onto linear transformation with standard matrix A.

a

Claim: A must have 3 pivot positions.
Proof: For $T : \mathbb{R}^4 \to \mathbb{R}^3$ the standard matrix A must be a 3 x 4 matrix. If it is onto, then for every $\vec{b}$, $A\vec{x} = \vec{b}$ must have a solution. In order to ensure every $\vec{b}$ satisfies this equation, there needs to be a pivot in each row, so 3 pivots.

b

Claim: There are four possible echelon forms of A.
Proof: $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.