Linear Algebra HW5 Determinants and Vector Spaces Ryzeson Maravich

1

Let A and B be 4×4 matrices with det A = 2 and det B = 6.

a

Claim: det $AB = 12$
Proof:

$$\begin{aligned} \det AB &= (\det A)(\det B) && (1) \\ &= 12 && (2) \end{aligned}$$

(1) Theorem 6, Pg. 175

b

Claim: det $A^5 = 32$
Proof:

$$\begin{aligned} \det A^5 &= \det(AAAAA) && (1) \\ &= (\det A)(\det A)(\det A)(\det A)(\det A) && (2) \\ &= 32 && (3) \end{aligned}$$

(2) Theorem 6, Pg. 175

c

Claim: det $3B = 486$
Proof: $3B$ is equivalent to B with each row multiplied by 3. When multiplying a row by some scalar multiple, the resulting determinant must also be multiplied by this scalar multiple (Theorem 3, Pg. 171). There are 4 rows, so det B must be multiplied by 3^4, which equals 486.

d

Claim: det $B^{-1}AB = 2$
Proof: We will first show that det $B^{-1} = \frac{1}{\det B}$. Because det $B \neq 0$, we know that B is invertible, and so $BB^{-1} = I_n$.

$$\begin{aligned} & BB^{-1} = I_n \\ \implies & \det(BB^{-1}) = \det I_n \\ \implies & (\det B)(\det B^{-1}) = 1 \\ \implies & \det B^{-1} = \frac{1}{\det B} \end{aligned}$$

This gives us

$$\begin{aligned} \det B^{-1}AB &= (\det B^{-1})(\det A)(\det B) \\ &= \frac{1}{6}(2)(6) \\ &= 2. \end{aligned}$$

2

A is an n×n matrix and r is a real number.
Claim: det $(rA) = r^n$ det A.
Proof: rA is equivalent to A with each row multiplied by r. When multiplying a row by some scalar multiple, the resulting determinant must also be multiplied by this scalar multiple (Theorem 3, Pg. 171). Because A has n rows, the scalar r is multiplied through n times, so det A must be also be multiplied by r^n.

3

Collaborators: Sam Kateman

Let S be the parallelogram determined by the vectors $b_1 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $A = \begin{bmatrix} 11 & 7 \\ 1 & 1 \end{bmatrix}$.
Claim: The area of the image of S under the mapping $x \mapsto Ax$ is 20.
Proof: S is the image of the unit square under the linear transformation with standard matrix $B = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$.
By calculating det B, we can find that the area of S is 5. S undergoes another linear transformation, this time with standard matrix A. The area of the image of S under this mapping is 5(det A), which is 20.

4

a

Let S be the set of all ordered pairs of real numbers.
Define addition ($\oplus$) on S by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

and scalar multiplication ($\star$) on S by

$$\alpha \star (x_1, x_2) = (\alpha x_1, \alpha x_2),$$

where $x_1, x_2, y_1, y_2, \alpha$ are real numbers.
Claim: S is not a vector space.
Proof: There is no identity element of addition (zero vector) in S. If $u \in S$ and $u = (u_1, u_2)$ where $u_2 \neq 0$, no matter what element you add to u, the second number in the ordered pair will always be 0, so you will never get u.

There is no identity element of addition, so there can be no inverse elements of addition.

There is no distributivity of scalar multiplication with respect to addition, that is $(c + d)v \neq cv + dv$ for some $v \in S$ where c, d are real numbers. Let $v = u$, as defined above. The left side of the equation will have an ordered pair where the second number will be nonzero (assuming $c + d \neq 0$). The right side however will always give an order pair where the second number is zero.

(The rest of the axioms hold.)

b

Let $\mathbb{R}$ denote the set of real numbers. Define scalar multiplication by $\alpha x = \alpha \cdot x$ and define addition, denoted $\oplus$, by $x \oplus y = \max(x, y)$.
Claim: $\mathbb{R}$ is not a vector space.
Proof: There is no identity element of addition in $\mathbb{R}$. Suppose there is an identity element, $0 \in \mathbb{R}$ and let $v \in \mathbb{R}$. No matter what 0 is, there is always some other element (0 - 1 using regularly defined operations in $\mathbb{R}$) such that $v + 0 = 0$, contradicting the axiom that $v + 0 = v$. Therefore there can be no identity element.

It then follows that there are no inverse elements of addition.

5

Let A be an invertible 2×2 matrix and let B be a 2×1 matrix.
Claim: The inverse of $\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & A^{-1}(-B) \\ 0 & 1 \end{bmatrix}$.
Proof: Let the inverse be denoted by $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, then $\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} I_2 & \vec{0} \\ 0 & 1 \end{bmatrix}$. Multiplying the two matrices on the right and setting equal to the left gives us the four equations

$$AX_{11} + BX_{21} = I_2 \tag{1}$$
$$AX_{12} + BX_{22} = 0 \tag{2}$$
$$X_{21} = 0 \tag{3}$$
$$X_{22} = 1. \tag{4}$$

Equation (3) can used to find $X_{11} = A^{-1}$ from equation (1). Equation (4) can be used to find $X_{12} = A^{-1}(-B)$ from equation (2). Substituting these values into X gives us the inverse matrix in terms of A and B.

6

Claim: The set of vectors $\left\{ \begin{bmatrix} h \\ -2 \\ \sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 57 \\ 13 \\ \sqrt{3} \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ h \\ \sqrt{3} \\ 0 \end{bmatrix} \right\}$ is linearly dependent for $h = \pm 4$.
Proof: Let us create the matrix $A = \begin{bmatrix} h & 57 & 0 & -8 \\ -2 & 13 & 0 & h \\ \sqrt{3} & \sqrt{3} & -1 & \sqrt{3} \\ 0 & 3 & 0 & 0 \end{bmatrix}$. The set of matrices is linearly dependent if the determinant of the matrix they comprise is zero.

$$\begin{aligned} 0 &= \det A \\ &= -1(-1)^{3+3}\left(\det \begin{vmatrix} h & 57 & -8 \\ -2 & 13 & h \\ 0 & 3 & 0 \end{vmatrix}\right) \\ &= -1\left(3(-1)^{3+2}\left(\det \begin{vmatrix} h & -8 \\ -2 & h \end{vmatrix}\right)\right) \\ &= 3(h^2 - 16) \\ 16 &= h^2 \\ \pm 4 &= h \end{aligned}$$

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7

Let A and B be $n \times n$ matrices where B has n pivot positions.
Claim: If det $(A^3B^2) = 0$ then A must be singular.
Proof: det $(A^3B^2) =$ (det A)(det A)(det A)(det B)(det B) $= 0$. Becuase B is an $n \times n$ matrix and has n pivots, it is invertible by the Invertible Matrix Theorem, and so det $B \neq 0$. Therefore to make the above equation true, det $A = 0$, therefore also making it singular.

8

Let $\mathbb{R}^+$ denote the set of positive real numbers. Define a sum operation $\oplus$ and a scalar multiplication operation $\star$ on $\mathbb{R}^+$ using the following rules: $x \oplus y = xy$ and $r \star x = x^r$ where $x, y \in \mathbb{R}^+$ and r is a real scalar.
Claim: $\mathbb{R}^+$ is a vector space under these operations.
Proof:

1. For any $x, y \in \mathbb{R}^+, x \oplus y = xy \in \mathbb{R}^+$ because the product of two positive real numbers is a positive real number.
2. For any $x, y \in \mathbb{R}^+, x \oplus y = xy = yx = y \oplus x$.
3. For any $x, y, z \in \mathbb{R}^+, (x \oplus y) \oplus z = (xy) \oplus z = (xy)z = x(yz) = x \oplus (yz) = x \oplus (y \oplus z)$.
4. For any $x \in \mathbb{R}^+, x \oplus 1 = x$, so 1 is the zero vector.
5. For every $x \in \mathbb{R}^+, x \oplus \frac{1}{x} = 1$, so $\frac{1}{x}$ is the inverse element of addition.
6. For any $c \in \mathbb{R}, c \star u = u^c \in \mathbb{R}^+$ because a positive real number to any real power is a positive real number.
7. For any $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^+, c \star (x \oplus y) = c \star (xy) = (xy)^c = x^c y^c = (x^c) \oplus (y^c) = (c \star x) \oplus (c \star y)$.
8. For any $c, d \in \mathbb{R}$ and $x \in \mathbb{R}^+, (c + d) \star x = x^{c+d} = x^c x^d = x^c \oplus x^d = (c \star x) \oplus (d \star x)$.
9. For any $c, d \in \mathbb{R}$ and $x \in \mathbb{R}^+, c \star (d \star x) = (d \star x)^c = (x^d)^c = x^{cd} = (cd) \star x$.
10. For any $x \in \mathbb{R}^+, 1 \star x = x^1 = x$, so 1 is the multiplicative identity element.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.