Linear Algebra HW10 Orthogonality Ryzeson Maravich

1

Let $\vec{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

a

Claim: $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for R^3.
Proof: $\mathcal{B}$ is an orthogonal set if each element is orthogonal to every other element. This is true for $\mathcal{B}$ because $\vec{u}_1 \cdot \vec{u}_2 = 6$ - 6 - $0 = 0$, $\vec{u}_1 \cdot \vec{u}_3 = 3$ - 3 - $0 = 0$, and $\vec{u}_2 \cdot \vec{u}_3 = 2 + 2$ - $4 = 0$. Because $\mathcal{B}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^3$, they are linearly independent and hence $\mathcal{B}$ is a basis for $\mathbb{R}^3$ by Thm 4, Pg 340.

b

Claim: $[x]_{\mathcal{B}} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

Proof: To find $[x]_{\mathcal{B}}$, we need to solve the equation $\begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} [x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$. By creating and then row reducing the augmented matrix $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$, we see that $[x]_{\mathcal{B}} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

2

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $W = \text{Span } \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, and $y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

Claim: $y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, such that y is the sum of a vector in W and a vector orthogonal to W.

Proof: First find $\text{proj}_W y$ denoted $\hat{y}$. Because $\hat{y} \in W$, $\hat{y} = \frac{y \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{y \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 + \frac{y \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}\vec{u}_3$ by the Orthogonal Decomposition Theorem. Substituting values we get

$$\hat{y} = \frac{1}{3}\vec{u}_1 + \frac{14}{3}\vec{u}_2 + \frac{-5}{3}\vec{u}_3 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ -\frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{14}{3} \\ 0 \\ \frac{14}{3} \\ \frac{14}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{5}{3} \\ -\frac{5}{3} \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}.$$

To find the component of y orthogonal to W, subtract

$$y - \hat{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

To check if this resulting vector is orthogonal to W it suffices to check if it orthogonal to $\vec{u}_1, \vec{u}_2$, and $\vec{u}_3$. The dot product between each of these and our new vector is 0, therefore the new vector is orthogonal to W. Because $\begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = y$ the proof is complete.

3

Let $\vec{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$ and let $A = \begin{bmatrix} 2 & 5 \\ 0 & -2 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$

a

Let $\hat{z} = \text{proj}_{\text{Col}A}\vec{z}$.

Claim: $\hat{z} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$.

Proof: Denote the columns of A as a_1 and a_2. Because neither a_1 or a_2 can be written as a linear combination of the other, they are linearly independent. $a_1 \cdot a_2 = 0$, so they are also also orthogonal. Because $\{a_1, a_2\}$ are linearly independent, orthogonal, and span Col A, they form an orthogonal basis for Col A, and so $\hat{z} = \frac{z \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{z \cdot a_2}{a_2 \cdot a_2} a_2$ by the Orthogonal Decomposition Theorem. Substituting values, we get

$$\hat{z} = \frac{7}{14} a_1 + 0 a_2 = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}.$$

b

Claim: If $Ax = \hat{z}$ then $x = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$.

Proof:

$$Ax = \hat{z} \implies \begin{bmatrix} 2 & 5 \\ 0 & -2 \\ -1 & 4 \\ -3 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \implies \left[\begin{array}{cc|c} 2 & 5 & 1 \\ 0 & -2 & 0 \\ -1 & 4 & -\frac{1}{2} \\ -3 & 2 & -\frac{3}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies x = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

c

Claim: The distance from $\vec{z}$ to Col A is $\sqrt{17.5}$.
Proof: By the Best Approximation Theorem, the distance from $\vec{z}$ to Col A is $\|\vec{z} - \hat{z}\|$.

$$\vec{z} - \hat{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\|\vec{z} - \hat{z}\|^2 = 1^2 + 4^2 + \frac{1}{2}^2 + \frac{1}{2}^2 = 17.5$$

Therefore the distance from $\vec{z}$ to Col A is $\sqrt{17.5}$

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.