Combinatorics HW5 Bionomial Theorem - Compositions Ryzeson Maravich

3 (15 points)

(a)
Claim: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2}$ answers the question "What is they number of ways you can choose a card from a $n-1$ card stack, plus the number of ways you can draw 2 cards from the stack after that?"
Proof: There are $\binom{n+1}{1}$ ways to choose a card from a $n+1$ card stack, and $\binom{n+1}{1} = \binom{n}{0} + \binom{n}{1}$. After this, the stack will have n cards left, so the number of ways of choosing 2 additional cards is simply $\binom{n}{2}$. The two possibilities summed together yields $\binom{n}{0} + \binom{n}{1} + \binom{n}{2}$.

(b)
Claim: $\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1}$ answers the question "How many ways can you arrange 5 black marbles and 5 white marbles in a line?"
Proof: You have a multi-set of 10 objects, 5 of each type. To compute the number of ways that you can linearly order them, use the equation $\frac{10!}{5!5!}$, which simplifies to $\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1}$.

(c)
Claim: $\binom{20}{10}\binom{10}{5}$ answers the question "How many ways can you make two groups of 5 from a group of 20 people?"
Proof: This counts the number of different ways to make a group of 10 people from 20, and then the number of ways to pick a group of 5 people from the remaining 10. This leaves you with two groups of 5 people, the last group that was picked, and the group of 5 people who were not picked.

4 (15 points)

(a)
Claim: $x_1 + x_2 + x_3 + x_4 = 12$ when $x_1, x_2 \geq 1$ and $x_3, x_4, \geq 2$ has 84 solutions.
Proof: Because x_1 and x_2 must be greater than 1, we can ensure this by setting $x_1 = x_1 + 1$ and $x_2 = x_2 + 1$. Likewise, we can set $x_3 = x_3 + 2$ and $x_4 = x_4 + 2$. When substituted in, we get $x_1 + 1 + x_2 + 1 + x_3 + 2 + x_4 + 2 = 12 \Rightarrow x_1 + x_2 + x_3 + x_4 = 6$. Because the additional stipulations are now already satisfied, we can continue by just finding the number of non negative integer solutions. This is the same as finding the number of weak compositions, which can be calculated by $\binom{9}{3} = 84$.

(b)
Claim: $x_1 + x_2 + x_3 + 3x_4 + 5x_5 = 7$ when $x_i \geq 0$ has 57 solutions.
Proof: x_4 could be either 0, 1, or 2, so we will compute the total solutions for each case. If $x_4 = 2$, then $x_5 = 0$, and this would leave us with $x_1 + x_2 + x_3 = 1$. There are $\binom{3}{2} = 3$ solutions to this equation. Similarly, if $x_4 = 1$, then $x_5 = 0$ and there would be $\binom{6}{2} = 15$ solutions.

If $x_4 = 0$, then x_5 could either be 0 or 1. If $x_5 = 0$, then $x_1 + x_2 + x_3 = 7$, giving us $\binom{9}{2} = 36$ solutions. If If $x_5 = 1$, then $x_1 + x_2 + x_3 = 2$, which has $\binom{3}{2} = 3$ solutions. By adding the totals for each possibility, we arrive at 57 solutions.

(c)
Claim: $x_1 + x_2 + x_3 + \frac{1}{2}x_4 = \frac{11}{2}$ when $x_i \geq 0$ has 56 solutions.
Proof: When x_4 is even, the $\frac{1}{2}x_4$ term becomes an integer, and because the other terms also must be integers, they will never add to an integer. Therefore x_4 must be odd. If $x_4 = 1$, then $x_1 + x_2 + x_3 = 5$, which has $\binom{7}{2} = 21$ solutions. Similarly, there are $\binom{6}{2} = 15$ solutions when $x_4 = 3$, $\binom{5}{2} = 10$ solutions when $x_4 = 5$, $\binom{4}{2} = 6$ solutions when $x_4 = 7$, $\binom{3}{2} = 3$ solutions when $x_4 = 9$ and $\binom{2}{2} = 1$ solution when $x_4 = 11$. This last case is when $x_1 = x_2 = x_3 = 0$, so there are no remaining non-negative integer solutions for larger values of x_4. The total number of solutions is 56.

5 (20 points)

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Claim: $F_{n+1} = \sum_{k=0}^{n} \binom{n-k+1}{k}$.
Proof: The statement holds true for $n = 0$. $F_1 = \sum_{k=0}^{0} \binom{0-0+1}{0} \Rightarrow 1 = 1$. This statement also holds true for $n = 1$. $F_{1+1} = \sum_{k=0}^{1} \binom{1-k+1}{k} \Rightarrow F_2 = \binom{2}{0} + \binom{1}{1} \Rightarrow 2 = 2$.

Because we already proved the initial conditions, if we can show that the right hand side satisfies the Fibonacci numbers recurrence formula, that is show $\sum_{k=0}^{n} \binom{n-k+1}{k} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n=2} \binom{n-k-1}{k}$, we know that it will hold for all n.

$$\sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$$

$$\sum_{k=0}^{n-1} \binom{n-k}{k} + \binom{0}{k-1} + \sum_{k=1}^{n-1} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$$

$$\sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} = \sum_{k=}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$$

$$\sum_{k=0}^{n-1} \binom{n-k}{k} + 0 + \sum_{k=1}^{n} \binom{n-k}{k-1} = \sum_{k=}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$$

$$\sum_{k=0}^{n-1} \binom{n-k}{k} + \binom{n}{-1} + \sum_{k=1}^{n} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$$

$$0 + \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n-2} \binom{n-k-1}{k}$$

$$\binom{0}{n} + \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n-2} \binom{n-k-1}{k}$$

$$\sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n} \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n-2} \binom{n-k-1}{k}$$

$$\sum_{k=0}^{n} \binom{n-k}{k} + \binom{n-k}{k-1} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n-2} \binom{n-k-1}{k}$$

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=0}^{n-2} \binom{n-k-1}{k}$$

*Note: Some trivial steps are shown to aid my understanding

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.