The Birth of Analytic Philosophy

Mathematics has always served as the cornerstone of rationality and exactitude in the academic community. Scientific laws are modeled by equations, experimental data is gathered through quantifiable measurements, and even visual and auditory art adhere to mathematical patterns and principles to enhance their aesthetic value. When there is intellectual doubt in one's field, one turns to mathematics to provide answers. This is because math is precise, objective, and infallible. When a conclusion is reached using math, it can be believed with absolute certainty. But how do we know this? What justification do we have to know that the logic used by mathematicians can be trusted, and more importantly, how do we know the premises (axioms) deduced from are true? Questions like these were presupposed as true until the development of Non-Euclidean geometry and transfinite numbers in the late 19th century. These discoveries, which seemed to contradict the foundations of mathematics, led many to question which theory to believe, and subsequently question the validity of math entirely.

When truth itself is under question, philosophers take it upon themselves to find answers. As a result, philosophers and mathematicians alike had to solve the problem of the axioms by reevaluating and justifying the fundamental propositions of mathematics, as well as solve the problem of proof, that is, provide an unobjectionable defense for the rules of the system.

Rigor in mathematics begins with the Greek mathematician Euclid's work *Elements*, in which he is able to axiomatize geometry, beginning with only twenty-three definitions, five postulates, and five common notions. Of course there were important discoveries before this, and math was employed by the masses in daily activities, but this is the first documented place where every new finding is strictly deduced from a given axiom, or from a previous proven proposition. The proofs in the *Elements* were so airtight that there was no possible doubt about the legitimacy of the claims. With this standard in place, mathematicians attempted to emulate this rigor in all future endeavors.

The fifth of Euclid's postulates, nicknamed the "Parallel Postulate" because it indirectly gives a definition for parallel lines, was slightly different from the other four. It seemed obvious, but its lengthy formulation compared to the others caused suspicion in academics for centuries, including al-Khwarizmi, Descartes, and even Newton (Greenberg). Eventually, in the early 19th century, Hungarian János Bolyai and Russian Nikolai Lobachevsky were able to (independently) formulate a new system of geometry, with equally sound logic as Euclid's, but without the Parallel Postulate. They realized that seemingly parallel lines could eventually cross or bend away from each other if the plane they were on was thought of as not flat, but curved. With this new system introduced, nobody was sure which was true. Further complicating matters, Bernhard Reimann confirmed shortly after an infinitude of such Non-Euclidean geometries. With infinite valid interpretations of geometry, there in turn seemed to be infinite truths, a serious epistemological problem.

Several decades later, Georg Cantor unveiled another counter-intuitive, yet groundbreaking discovery. With his very clever diagonal argument, he showed that there is no bijection between the real numbers, itself is an infinite set, and the natural numbers, also an infinite set. In other words, he showed that there are different degrees of infinity, and some are "larger" than others. Much like Bolyai and Lobachevsky's work, this made mathematicians question even their most fundamental assumptions.

Hilbert's speech in 1900 outlined 23 essential problems that he urged mathematicians to solve within the next century, one of which challenged mathematicians "To prove that they [the axioms of mathematics] are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results” (Hilbert 414). Axiomatizing mathematics and ensuring it's consistency was an extraordinary task to undertake, but with the uncertainty surrounding Non-Euclidean geometry and Cantor’s theory of transfinite numbers, mathematicians were desperate to find an answer to Hilbert’s problem and save mathematics from the emerging crisis. They needed reasons to believe the fundamental truths they were working from were in fact true.

The problem of the axioms, as opposed to the problem of the proof, seems to be the more difficult one to provide a convincing answer for. As evidenced by Euclid, axioms are the basis for any mathematical system, so how can you justify something that is self-evidently or definitionally true? There are four different approaches to this answer: logicism, intuitionism, conventionalism, and formalism.

Proponents of logicism (such as Russell, Whitehead, and Frege, all of whom are essential in helping to solve the problem of the proofs) believe that the entirety of mathematics can be derived from basic logic. Intuitionists think that math is in some sense pre-programmed into the human mind, and so math is not really about some real, objective reality. They claim that there is only truth in the mind, and we can use math to express this to others. Conventionalists, such as Henri Poincaré, argue that math is just a language, and as such, it does not matter how you say a certain thing, it still is representing one thing, which is truth. To a conventionalist, to say that Euclidean geometry is true while Non-Euclidean geometry is not, is equivalent to saying that a sentence spoken in German is true while the same sentence spoken in French is not. They are in fact both true, because they represent the same idea, just with a syntactical difference. Formalists, like Hilbert himself, advocate that truth is only relative to the axiomatic set, so the axioms one uses are arbitrary. If you want to use the rules defined by Euclidean geometry you can, and if you want to use the rules defined by Non-Euclidean geometry you can. In both cases you would be correct, because there is no one absolute "truth", just many governed by their respective axiomatic system.

Certainly, there are advantages and disadvantages to each position. For logicists, the infallible nature of logic is reassuring, but using it to define axioms themselves leads to an infinite regress. Intuitionism, while intuitively sensical, is difficult to prove, and lacks overall rigor. Conventionalists and formalists alike account for the coexistence of multiple systems, but because of this, never admit to any universal truth. To this day there is no definitive consensus on which solution is correct, and there are several prominent mathematicians and philosophers that have represented each school of thought.

After the proposal of several solutions to the problem of the axioms, mathematicians had to figure out the problem of the proofs and determine how to logically proceed step by step from the axioms to ensure that the results were just as valid. Without rigorous justification for such a method, any derived theorems and proofs could be questioned, and the truth of mathematics as a whole could be doubted. This process would take considerably more work than the problem of the axioms, with the majority of the groundwork needing to be developed from scratch, and with meticulous care at each moment.

In the late 19th century, Gottlieb Frege, whom many consider to be the father of analytic philosophy, published the *Begriffsschrift*, in which he set forth his project of logicism. By building off of pre-existing logic with the invention of quantifier notation, Frege was able to construct a rigorous system for representing and calculating logical expressions. Although his style and diagrams seem esoteric and obtuse by today's standards, much of modern logic is based on his terminology and methodology. With a logical system ready, the only thing left to do was develop a way to translate language into logical statements that could be dealt with by such a system.

Frege takes this burden upon himself as well in *On Sense and Reference*. Everyday language is slippery, with the meaning of words being hidden, and the core content of what is being said remaining elusive in many cases. The example Frege uses to illustrate this point is our use of the terms "morning star" and "evening star", both of which refer to the planet Venus (Frege 27). The sentence "The morning star is the evening star" conveys new information when said. If we were to replace "morning star" with its synonym "evening star", we get the vacuous tautology "The evening star is the evening star." This causes problems for logic, because substituting y for x when $x = y$ in such a system should not yield different results. Similarly, if the respective definitions for such "synonymous" words are contradictory, such as the case for the evening star and the morning star (Frege 33), these contradictions suggest an even bigger logical problem. Obviously, substituting two identical things that have the same meaning should not affect the overall meaning of the proposition, so our meaning of the term meaning must be faulty.

To get to the underlying cause of this problem, Frege provides three different definitions of how we use the word "meaning." The first he calls the image, which is the subjective idea of the word that appears in one's mind. This can be thought of as the word's connotation, except more personal and emotional, and Frege describes this image as "saturated with feeling" (Frege 29). This almost always varies from person to person, so it is much too subjective for a logician like Frege to use in his theory of language. He instead focuses on a word's sense and reference. The referent of a word is the set of all objects that exist in the world that the word is supposed to signify. The sense of a word consists of the properties and qualities that determine the referent. The sense of a shoe, for example, would be "a covering of the foot with a sturdy sole." This abstract set of conditions then leads us to the referent of shoe, that is the set of all things that match this description.

With this distinction between sense and reference as two different concepts of "meaning," Frege shows that the sense of the word is the essential definition of meaning. Two words can only be truly synonymous if they have the same sense. The morning star/evening star complication vanishes because the two have different senses. The former is the first star seen before the sunrise in the east, while the latter is the last star before sundown in the west.

Knowing how words function and how we are able to use them is important, but the only way that logic can be employed is with propositions, so it is also necessary to determine what these words mean as a whole. "Every declarative sentence … and its referent, if it exists, is either the true or the false" (Frege 34). By this, Frege is making the obvious point that sentences do not have a reference in the physical world, but can only be true or false. It is through this realization that we can conclude that sense yields meaning, whereas the reference yields truth value.

At this point, Frege has created an extensive logical system for dealing with nearly every kind of proposition, as well as a precise method for converting language into these propositions that can be defined in logical terms. The stage is set for building mathematics from the ground up, but even then there remains a question: What is a number? If there is a definition for a given word, then it is possible to work with it within Frege's logical system. If a word has no sense then this can not be translated. (Or rather in this case, the sense which we conceive matches poorly with what the sense of number actually is, so the translation into formal logic does us no good.)

Frege in his *Foundations of Arithmetic*, insists that "Number is not abstracted from things in the way that colour, weight and hardness are… Number is not anything physical" (Frege 98). When a number is used adjectively to a group of things, it does not describe each individual thing the way any other adjective would. If there are two sturdy shoes, then each shoe is comfy, but it is meaningless to say that each shoe is two. Two refers to the aggregate of items. To reconcile this, Frege claims that the concept of a number is directly related to the existence of an object. There exists a shoe, and there exists a (different) shoe, implying that there are two shoes. "In this respect existence is similar to number," Frege remarks (Frege 103). For Frege, it is the instantiation itself, through the use of quantifiers he devised in the *Begriffsschrift*, that creates the number.

But as Frege said before, a "number" itself is not a physical entity. For example, the two mentioned earlier describes the quantity (repetitions of instantiation) of shoes, but this two is not the same as the two in the equation 1 + 1 = 2. The first describes the quality of two-ness, or having the collective property of encompassing two, whereas the second use is the number two itself. Two is not a thing in the real world to Frege; it "has no location" (Frege 109). To define number, we simply make a bijection to the set of all sets that contain that many members. For example, 0 is equivalent to the set of all sets for which there is a bijection into the empty set. Similarly, 1 is the set of all sets for which there is a bijection into the set containing only the empty set. This lets Frege define numbers in a concrete way that is constructed purely from logic. And with all of the other parts in place, Frege had singlehandedly laid the foundation for totally rigorous and uncompromising true mathematics. That is until Bertrand Russell noticed a single flaw.

While the second volume of Frege's *Foundations on Arithmetic* was being published, he received a letter from Russell. It contained what is now known as Russell's Paradox, which calls for the creation of “the class of all those classes which are not members of themselves.” Frege's central axioms should theoretically allow for such a set to be created, but if the set does contain itself, then it should not contain itself, but if it does not, then it should. At first he is unsure whether this portends any major implications to his theories, but as he continues to think about it, he realizes that this simple paradox is devastating to his logicism project, and nullifies the majority of his work on the matter. Despite this, Russell recounts, "I realize there is nothing in my knowledge to compare with Frege’s dedication to truth, … and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure…" (Irvine and Deutsch). Thankfully, Frege was not insulted or depressed by the matter, and with his remarkable action of intellectual humility, perfectly embodied the principles of the analytic tradition that he helped create.

In "What There Is" and "'Characteristics of Mental Phenomenon", Russel explicates his position that he deems neutral monism**.** Russell believes there to be only one kind of substance that comprises the world, and the only thing that changes is our way of viewing it. When we look at the world through the lens of psychology, we characterize the world as consisting of mental elements, but when viewing through the lens of physics, we characterize the world as consisting of physical elements. Russell compares this to organizing Londoners alphabetically or geographically (Ammerman 32). The categorizations do not create two different groups of people, it just creates two different interpretations of the same people. This is obviously different from the mind-body dualists, but also differs from the materialists, who reject the mind entirely, and the idealists, who reject matter entirely. This metaphysical insight is important for understanding his method of translating reality into language, which he details in one of his most famous works, "On Denoting."

"On Denoting" is Russell's attempt to fix the flaws of Frege's theory of translation, as well as contemporary philosopher Alexus Meinong's ontology. In Frege's case, Russell took issue with his commitment to the Principle of Compositionality and his failure to confront the Law of Excluded Middle. The Principle of Compositionality claims that the meaning of the sentence is the sum of the meaning of its words, and is the reason Frege was so worried about the problems of substituting "synonymous" words in *On Sense and Reference*. The Law of Excluded Middle is a logical rule that simply states that either the given proposition is true, or the negation of that proposition is true. Russell demonstrates that Frege's theory fails on both ends.

Russell provides the sentence "The present King of France is bald" (Russell 490), which he uses to raise some objections to Frege. (To immediately dispel any uncertainty, there is in fact no King of France.) For the Principle of Compositionality to hold, every word must contain meaning in isolation. So what does "the" mean? It is a definite article, and so by itself is meaningless. Perhaps a more convincing example is the segment "the present King." This could possibly refer to many different people, and then refers to nothing when used in the full context. It changes its meaning when in the context of other words, and therefore does not carry its original, isolated meaning.

"The present King of France is bald" also seems to defy the Law of Excluded Middle, at least when following Frege's rules. "The present King of France is bald" is false, but "The present King of France is not bald" is also false. Russell carefully deconstructs this to show how normal grammar deceives us by hiding parts of the complete meaning. For Russell, this sentence actually asserts three different things: There is a thing *x* such that *x* is the present King of France, there is only one such *x*, and *x* is bald (Russell 482). This is false because the first part is false. For the negation, it would logically read "There is no such thing *x* such that *x* is the present King of France, there is only one such *x*, and *x* is bald." This is true, because there is no such thing. By breaking the proposition into its fundamental pieces and negating it in its entirety, rather than just the last part, Russell saves sentences of this form from contradicting the Law of Excluded Middle.

Finally, Russell addresses the problem he sees with Meinong's theory, which contains an important distinction between existence and subsistence. If a thing exists, it belongs in the set of all things that are in reality. It is the typical meaning that “exists” possesses. A thing subsists if an idea can be formulated of it or it can be talked about in a sentence. It does not necessarily have to be a physical object in the real world. This categorization lets Meinong talk about things that do not actually exist, such as unicorns or numbers, but this also causes some problems. According to Meinong, sentences like "The round square is round" are true, because the round square subsists and can therefore be talked about, and a round thing being round is an a priori truth. This statement should be false, however; there is no such thing as a round square. This is why Meinong's theory becomes too metaphysically expensive, as it claims false things are true. Conveniently, Russell's solution to the Law of Excluded Middle solves this too. By dissecting "The round square is round" it becomes clear that the sentence is false from the initial assertion that "there exists an *x* such that *x* is a round square" (Russell 491). With that, Russell provides a preferable method of translation that eliminates the flaws of past philosophers' theories.

There is still one slight problem with Russell's theory at this point. With the way that he sets up his theory in "On Denoting", it is still susceptible to his very own paradox: the set of all sets that do not contain themselves. Russell explores the nature of this paradox and its solution in his "Mathematical Logic as based on the Theory of Types." He begins by analyzing similar paradoxes such as "I am lying" and the clever "the least integer not nameable in fewer than nineteen syllables" (Russell 223) and comes to the realization that the root issue of each paradox is that they are all self-referential. By self-referencing, they deceitfully abuse the vagueness of language to appear paradoxical. Russell therefore proposes the rule that “no totality can contain members defined in terms of itself” (237). To avoid this from occurring, different levels or types of propositions must be created such that “the *n* + 1th logical type will consist of propositions of order *n*, which will be such as contain propositions of order *n* - 1, but of no higher order, as apparent variables” (238). First order propositions are those about the world, second order propositions are those about first order propositions, third order are those about the second, ad infinitum. With the stipulation that propositions can only talk about those with a lower type, such “paradoxes” become meaningless, and so Russell's system is now immune to such contradictions.

In 1910, Russell and his Cambridge teacher turned colleague, Alfred North Whitehead, published the momunemental *Principia Mathematica*. With the first volume alone totalling 510 pages, they expounded a system of mathematical logic in the same vein as Frege did with the *Begrsiffsschrift*, avoiding his pitfalls and accounting for Russell's new theory of translation. The proof of 1 + 1 = 2 appearing 200 pages in gives an idea of how painstakingly exact and exhausting this project was. But with his own paradox solved, a superior theory of language, and a system of rigorous logic detailed, Russell finally finished the mission that Frege had set out to accomplish. With both the problem of the axioms and the problem of proof solved, mathematics was rectified as the logical and dependable subject it is known to be.

Works Cited

Ammerman, Robert R. Classics of Analytic Philosophy. Hackett Publishing, 1990.

Frege, Gottlob. *The Foundations of Arithmetic: A Logical-mathematical Investigation into the Concept of Number 188*4. New York, Pearson Education, 2007.

Frege, Gottlob. “Sense and Reference.” *The Philosophical Review*, vol. 57, no. 3, 1948, pp. 209–230., www.jstor.org/stable/2181485. Accessed 20 Mar. 2020.

Greenberg, M. J. Euclidean and Non-Euclidean Geometries: Development and History, 3rd ed. San Francisco, CA: W. H. Freeman, 1994.

Hilbert, David. "Mathematical Problems." *Bulletin of the American Mathematical Society*, ns, vol. 37, no. 4, 26 June 2000, pp. 407-36. *American Mathematical Society*, www.ams.org/journals/bull/2000-37-04/S0273-0979-00-00881-8/S0273-0979-00-00881-8.pdf.

Irvine, Andrew David and Deutsch, Harry, "Russell's Paradox", *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2016/entries/russell-paradox/>.

Russell, Bertrand. "Mathematical Logic as Based on the Theory of Types." *American Journal of Mathematics*, vol. 30, 1 July 1908, pp. 222-62. *JSTOR*, archive.org/details/jstor-2369948/page/n1/mode/2up.

Russell, Bertrand. “On Denoting.” *Mind*, vol. 14, no. 56, 1905, pp. 479–493. *JSTOR*, www.jstor.org/stable/2248381. Accessed 21 Mar. 2020.