1

Let A= $\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

a

Claim: $\mathbf{w}$ is in Col A.
Proof: $\mathbf{w}$ is in Col A if it is spanned by the the columns of A, or if the equation $Ax = \mathbf{w}$ is consistent for some x. By creating and row reducing the augmented matrix $\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This augmented matrix is consistent, so $\mathbf{w}$ is in Col A.

b

Claim: $\mathbf{w}$ is in Nul A.
Proof: $\mathbf{w}$ is in Nul A if $A\mathbf{w} = \mathbf{0}$. This is true, because

$$\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2

For any real number k, define S_k to be the set of all polynomials p in $\mathbb{P}_n$ such that $p(0) = k$.

a

Claim: S_0 is a subspace of $\mathbb{P}_n$.
Proof: The zero vector is in $\mathbb{P}_n$, which is just 0. S_0 is closed under addition, because any polynomial in S_0 will not have a constant, so added the sum will also not have a constant, so will be in S_0. Similarly, S_0 is closed under multiplication, because every polynomial in S_0 has no constant, so no matter which scalar it is multiplied by, the product will have no constant, and will also be in S_0.

b

Claim: S_1 is not a subspace of $\mathbb{P}_n$?.
Proof The zero vector, 0, is not in S_1.

3

Let F be a fixed $n \times n$ matrix and define a function $T : M_{n\times n} \longrightarrow M_{n\times n}$ by $T(X) = FX - XF$.

a

Claim: T is a linear transformation.
Proof: T is linear transformation if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \text{for all } \vec{u}, \vec{v} \in M_{n\times n}, \text{ and} \tag{1}$$
$$T(c\vec{u}) = cT(\vec{u}) \quad \text{for all } \vec{u} \in M_{n\times n} \text{ and all scalars } c. \tag{2}$$

Showing (1),

$$\begin{aligned} T(A+B) &= F(A+B)-(A+B)F \\ &= FA+FB-AF-BF \\ &= FA-AF+FB-BF \\ &= T(A)+T(B) \end{aligned}$$

Showing (2),

$$\begin{aligned} T(cA) &= F(cA)-(cA)F \\ &= cFA-cAF \\ &= c(FA-AF) \\ &= cT(A) \end{aligned}$$

b

Claim: Kernel $T = \{X | X \text{ commutes with } F\}$.
Proof: The kernel of $T = \{\vec{u} \in M_{n\times n} | T(\vec{u}) = 0\}$. To find this, set $T(X) = 0 = FX - XF$. Adding XF to both sides results in the equation $FX = XF$, so X must commute with F.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the honor code.