2 (10 points)

Claim: There are 3168 four digit numbers that contain a one.
Proof: To figure out how many four digit numbers do contain the letter one, we can calculate the total number of four digit numbers, and subtract all of the ones that do not contain a 1. There are 9000 total 4 digit numbers. You can get this by figuring out how many combinations of 4 numbers you could make. You would have 9 options (1-9) for the thousands place digit, because if it were a 0, then it would not create a four digit number. You would then have 10 options for each of the other three place values. Due to the Fundamental Theorem of Counting, you can multiply $9 * 10 * 10 * 10$ to get 9000 total four digit numbers. You can figure out how many of these do not contain a 1 in a similar way. You have 8 options for the first digit, because you need to exclude both 0 and 1. You have 9 options for each of the other three places, as you only need to exclude 1. This totals $8 * 9 * 9 * 9 = 5832$. When you subtract this from the 9000 total four digit numbers, you get 3168.

5 (10 points)

Claim: 7,350 arrangements of the letters in MISSISSIPPI have no 2 I's adjacent.
Proof: First we need to figure out how many ways we could place the I's, which is essentially $\binom{8}{4}$. If each I were to be placed at either the beginning of the word, the end of the word, or between each of the letters of MSSSSPP, this would ensure that no two of them would be adjacent. The first I would have all 8 options to be placed, the second I would only have 7, the third I 6, and the last only 5. This gives us 1,680 possibilities by the Fundamental Theorem of Counting. These four I's, however, could be permuted in 4! different ways. For example, placing the I's in positions 1, 2, 3, and 4, is identical to placing them in positions 4, 3, 2, and 1. When we divide out these repeated arrangements from out initial over count, we are left with $1680/4! = 70$.

We then have to figure out how many different ways we can arrange the letters that are not I. There would be 7! ways to arrange these 7 letters, but because some of these would be identical due to the duplicate letters, we need to reduce this number. There would be 4! ways to arrange all the S's, and 2! ways to arrange the P's. These arrangements are independent of each other, so we can multiply them together, and divide them by our original count. $\frac{7!}{4!2!} = 105$.

To get the total arrangements of all letters so that no two I's are adjacent, we multiply the total ways we can place the I's and the total ways we can uniquely arrange the remaining letters to get $70 * 105 = 7,350$.

6 (20 points)

(a)
Claim: There are $\binom{10}{3}$ non-negative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 7$.
Proof: This problem is the mathematical equivalent of asking how many ways there are to put 7 items into 4 boxes, where each item is a 1, and each of the variables is a box. We can think of each of the four boxes as being separated by three dividers, just as the four variable are divided by three plus signs. The 7 total items, in addition to the 3 dividers, gives you 10 total items that you must arrange. There are 10! different arrangements for these items. This total counts each of the way the 3 dividers could be permuted as a separate arrangement, so we must divide by 3!. This is true for the 7 identical items too, so we must divide by 7!. This gives us $\frac{10!}{3!7!} = \frac{10!}{3!(10-3)!} = \binom{10}{3}$. (This comes out to 120 different solutions.)

(b)
Claim: There are $\binom{n+k-1}{k}$ non-negative integer solutions to the equation $x_1 + x_2 + \ldots + x_n = k$.
Proof: There will always be k items and $n-1$ dividers, which results in $(k+n-1)!$ arrangements total. There are $k!$ ways to permute the k objects, and $(n-1)!$ different ways to permute the $n-1$ dividers, so

you must divide these from the originally calculated arrangements. This gives us $\frac{(k+n-1)!}{k!(n-1)!} = \frac{(k+n-1)!}{k!(n+k-1-k)!} = \binom{n+k-1}{k}$.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code.