2 (10 points)

Let $a_0 = 0, a_1 = 1$ and let $a_{n+2} = 6a_{n+1} - 9a_n$ for $n \geq 0$

Claim: $a_n = n \cdot 3^{n-1}$ for n ≥ 0.
Proof: First, we need to check to see if the statement holds true for the base cases of $n = 0$ and $n = 1$. For $n = 0, 0 = 0 \cdot 3^{0-1} \Rightarrow 0 = 0$, which is true. For $n = 1, 1 = 1 \cdot 3^{1-1} \Rightarrow 1 = 1$, which is true.

Assume for both $n = k$ and $n = k - 1$ that $a_k = k \cdot 3^{k-1}$ is true. Next, we need to prove the inductive step, that is, proving $a_{k+1} = (k+1) \cdot 3^k$, given our assumption. We know from $a_{n+2} = 6a_{n+1} - 9a_n$, that $a_{k+1} = 6a_k - 9a_{k-1}$. By substituting both of our inductive assumptions into this equation, we get $a_{k+1} = 6(k \cdot 3^{k-1}) - 9[(k-1) \cdot 3^{k-2}]$. If we can show that $(k+1) \cdot 3^k = 6(k \cdot 3^{k-1}) - 9[(k-1) \cdot 3^{k-2}]$, then this will prove our inductive step.

$$\begin{aligned}
(k+1) \cdot 3^k &= 6(k \cdot 3^{k-1}) - 9[(k-1) \cdot 3^{k-2}] \\
(k+1) \cdot 3^k &= 6(k \cdot 3^{k-1}) - 3^2[(k-1) \cdot 3^{k-2}] \\
(k+1) \cdot 3^k &= 2(k \cdot 3^k) - [(k-1) \cdot 3^k] \\
(k+1) \cdot 3^k &= 2(k \cdot 3^k) - k \cdot 3^k + 3^k \\
(k+1) &= 2k - k + 1 \\
k+1 &= k+1
\end{aligned}$$

3 (10 points)

Claim: For any positive integer n, if $x_1, \ldots, x_n \in R$, then $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$
Proof: First we need to prove our base case of $n = 1$. $|x_1| \leq |x_1|$, so the base case is true.

Assume $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$ is true. Next we need to prove our inductive step, that is, proving $|x_1 + \cdots + x_{n+1}| \leq |x_1| + \cdots + |x_{n+1}|$, given our assumption. If we add $|x_{n+1}|$ to both sides of our assumption, we get $|x_1 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + \cdots + |x_n| + |x_{n+1}|$. If we can show that $|x_1 + \cdots + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}|$, then we can prove the inductive step through the transitive property. Because $|x_1 + \cdots + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}|$ fits the form $|x + y| \leq |x| + |y|$, we know it is true, as it is the triangle inequality.

4 (10 points)

Let $x \neq 1$ be a real number.

Claim: For all $n \geq 0$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

Proof: First we need to check to see if the statement holds true for the base case, $n = 0$. This is true because $x^0 = \frac{1-x^{0+1}}{1-x} \Rightarrow 1 = 1$.

Assume for $n = k$, $\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$. Next we need to prove the inductive step, that is, proving $\sum_{i=0}^{k+1} x^i = \frac{1-x^{k+2}}{1-x}$, given our assumption. If we add x^{k+1} to both sides of our assumption, we get $\sum_{i=0}^{k} x^i + x^{k+1} = \frac{1-x^{k+1}}{1-x} + x^{k+1}$. This can simplify to $\sum_{i=0}^{k+1} x^i = \frac{1-x^{k+1}}{1-x} + x^{k+1}$. If we can show that

$\frac{1-x^{k+2}}{1-x} = \frac{1-x^{k+1}}{1-x} + x^{k+1}$, then this will prove our inductive step.

$$\begin{aligned}
\frac{1-x^{k+2}}{1-x} &= \frac{1-x^{k+1}}{1-x} + x^{k+1} \\
&= \frac{1-x^{k+1}+(1-x)x^{k+1}}{1-x} \\
&= \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x} \\
\frac{1-x^{k+2}}{1-x} &= \frac{1-x^{k+2}}{1-x}
\end{aligned}$$

7 (20 points)

Collaborators: Matt Torrence

Claim: Every positive integer n is the sum of one or more distinct Fibonacci numbers.
Proof: The claim is true for $n = 1(F_1), n = 2(F_2), n = 3(F_3)$. Assume that the claim is true for all $1 < n \leq k$. We need to prove that this assumption implies that our claim is true for $k+1$.

If $k+1$ is a Fibonacci number, then the inductive step is proven. If $k+1$ is not a Fibonacci number, then $F_m < k+1 < F_{m+1}$. Let $d = k+1-F_m \Rightarrow k+1 = F_m + d$. We know that F_m is (trivially) the sum of distinct positive Fibonacci numbers. If we can show that $d < F_m \wedge d \leq k$ then we know from our inductive assumption that our claim is true for $k+1$.

$$k+1 < F_{m+1}$$
$$\Rightarrow d = k+1-F_m < F_{m+1} - F_m = F_{m-1} < F_m$$
$$\Rightarrow d < F_m.$$

$$F_m \geq 1$$
$$-F_m \leq -1$$
$$1 - F_m \leq 0$$
$$k+1-F_m \leq k$$
$$d \leq k$$

Because $d \leq k$, we know by our inductive assumption that d is a sum of distinct Fibonacci numbers, and because $d < F_m$, none of these addends will be F_m, thus ensuring that $k+1$ will be the sum of distinct Fibonacci numbers, thereby completing the inductive step.

I affirm that I have upheld the highest standards of honesty and integrity in my academic work and have not witnessed a violation of the honor code. https://brilliant.org/wiki/strong-induction/ helped me with understanding strong induction.