Folha 1

1.1 $x<y$

a) $x^2<y^2$. Falsa. Contra-exemplo $x=-1$, $y=0$.

b) $x^3<y^3$. Verdadeira.

(i) Se $x<y$ com $x>0$ (e $y>0$) então
$x^2<xy<y^2$, pelo que
$x<y \Rightarrow x\cdot x^2<y\cdot x^2<yy^2$, que dá $x^3<y^3$.

(ii) se $x<y$ com $x=0$ (e $y>0$) então $x^2=0$ e $y^2>0$,
donde $x^3=0$ e $y^3>0$, pelo que $x^3<y^3$.

(iii) Se $x<y$ com $y=0$ (e $x<0$) então $x^2>0$ e $y^2>0$,
donde $x^3<0$ e $y^3=0$ e, portanto, $x^3<y^3$.

(iv) Se $x<y$ com $y<0$ (e $x<0$) então
$x^2>xy>y^2$, donde
$x<y \Rightarrow x^2x<x^2y<y^2y$, ou seja, $x^3<y^3$.

(v) se $x<y$ com $x<0$ e $y>0$ então $x^2>0$, $y^2>0$ e
$x^3=x^2\cdot x<0$, $y^3=y^2\cdot y>0$, donde $x^3<y^3$.

c) $\frac{1}{x}<\frac{1}{y}$ ($x\neq0$, $y\neq0$). Falsa. Contra-exemplo $x=1$, $y=2$.

d) $\frac{1}{x^3}>\frac{1}{y^3}$ ($x\neq0$, $y\neq0$). Falsa. Contra-exemplo $x=-2$, $y=1$.

1.2 - Escrever como intervalo ou reunião de intervalos.

a) $\{x \in \mathbb{R} : 1 - x \leq 2\} = [-1, +\infty[$

b) $\{x \in \mathbb{R} : 0 \leq 1 - 2x \leq 1\} = \{x \in \mathbb{R} : -1 \leq -2x \leq 0\}$

$= \{x \in \mathbb{R} : 0 \leq 2x \leq 1\} = [0, \frac{1}{2}]$

c) $\{x \in \mathbb{R} : x^2 > 5\} = \{x \in \mathbb{R} : x > \sqrt{5} \vee x < -\sqrt{5}\}$

$=]-\infty, -\sqrt{5}[\cup]5, +\infty[$

(*) d) $\{x \in \mathbb{R} : x^2(x^2 - 1) \geq 0\} = \{x \in \mathbb{R} : x = 0 \vee x^2 - 1 \geq 0\}$

$= \{x \in \mathbb{R} : x = 0 \vee x \geq 1 \vee x \leq -1\}$

$=]-\infty, -1] \cup [0, 0] \cup [1, +\infty[$

(*) e) $\{x \in \mathbb{R} : |5 - \frac{1}{x}| < 1\} = \{x \in \mathbb{R} : -1 < 5 - \frac{1}{x} < 1\}$

$= \{x \in \mathbb{R} : 4 < \frac{1}{x} < 6\} = \{x \in \mathbb{R}^+ : \frac{1}{6} < x < \frac{1}{4}\}$

$=]\frac{1}{6}, \frac{1}{4}[$

f) $\{x \in \mathbb{R} : |3 - x| \geq 2\} = \{x \in \mathbb{R} : 3 - x \geq 2 \vee 3 - x \leq -2\}$

$= \{x \in \mathbb{R} : x \leq 1 \vee x \geq 5\} =]-\infty, 1] \cup [5, +\infty[$

g) $\{x \in \mathbb{R} : |5x + 2| \leq 1\} = \{x \in \mathbb{R} : -1 \leq 5x + 2 \leq 1\}$

$= \{x \in \mathbb{R} : -\frac{3}{5} \leq x \leq -\frac{1}{5}\} = [-\frac{3}{5}, -\frac{1}{5}]$

(*) h) $\{x \in \mathbb{R} : \sqrt{3x+1} = 2x\}$

$= \{x \in \mathbb{R} : 3x+1 = 4x^2 \wedge x \geq 0\}$

$= \{x \in \mathbb{R} : (x = -\frac{1}{4} \vee x = 1) \wedge x \geq 0\}$

$= [1, 1]$

(*) i) $\{x \in \mathbb{R} : x^3 \geq 4x\}$

$= \{x \in \mathbb{R} : (x=0) \vee (x>0 \wedge x^2 \geq 4) \vee (x<0 \wedge x^2 \leq 4)\}$

$= \{x \in \mathbb{R} : x=0 \vee x \geq 2 \vee -2 \leq x < 0\}$

$= [-2, 0] \cup [2, +\infty[$

j) $\{x \in \mathbb{R} : 6x^2 - 5x \leq -1\} = \{x \in \mathbb{R} : 6x^2 - 5x + 1 \leq 0\}$

$\left(6x^2 - 5x + 1 = 0 \Leftrightarrow x = \frac{1}{2} \vee x = \frac{1}{3}\right)$

$= \left[\frac{1}{3}, \frac{1}{2}\right)$

k) $\{x \in \mathbb{R} : |3x-2| \leq 1\} = \{x \in \mathbb{R} : -1 \leq 3x - 2 \leq 1\}$

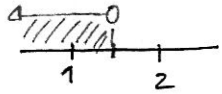
$= \{x \in \mathbb{R} : \frac{1}{3} \leq x \leq 1\} = \left[\frac{1}{3}, 1\right)$

l) $\{x \in \mathbb{R} : 2 < |x| < 3\} = \{x \in \mathbb{R} : 2 < x < 3 \vee -3 < x < -2\}$

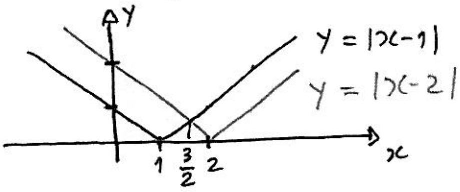
$=]-3, -2[\cup]2, 3[$

(*) m) $\{x \in \mathbb{R} : |x-1| < |x-2|\}$

→ $\{x \in \mathbb{R} : d(x,1) < d(x,2)\} =]-\infty, \frac{3}{2}[$



→ Graficamente



→ $|x-1| < |x-2| \Leftrightarrow (x-1 < x-2 \wedge x \geq 2) \vee (x-1 < -x+2 \wedge 1 \leq x < 2)$

$\vee (-x+1 < -x+2 \wedge x < 1)$ ...

m) $\{x\in\mathbb{R} : \frac{1-x}{2x+3} > 0\}$

$= \{x\in\mathbb{R} : (1-x>0 \wedge 2x+3>0) \vee (1-x<0 \wedge 2x+3<0)\}$

$= \{x\in\mathbb{R} : (x<1 \wedge x>-\frac{3}{2}) \vee (x>1 \wedge x<-\frac{3}{2})\}$

$= \{x\in\mathbb{R} : -\frac{3}{2} < x < 1\} =]-\frac{3}{2}, 1[$

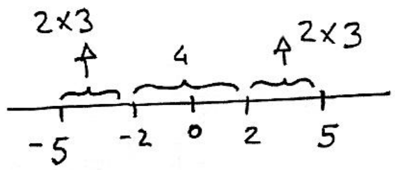
o) $\{x\in\mathbb{R} : |x|\,|x+3| = 4\} = \{x\in\mathbb{R} : |x^2+3x| = 4\}$

$= \{x\in\mathbb{R} : x^2+3x=4 \vee x^2+3x=-4\}$

$= \{x\in\mathbb{R} : x^2+3x-4=0 \vee \underbrace{x^2+3x+4=0}_{\text{imp.}}\}$

$= \{-4, 1\} = [-4,-4] \cup [1,1]$

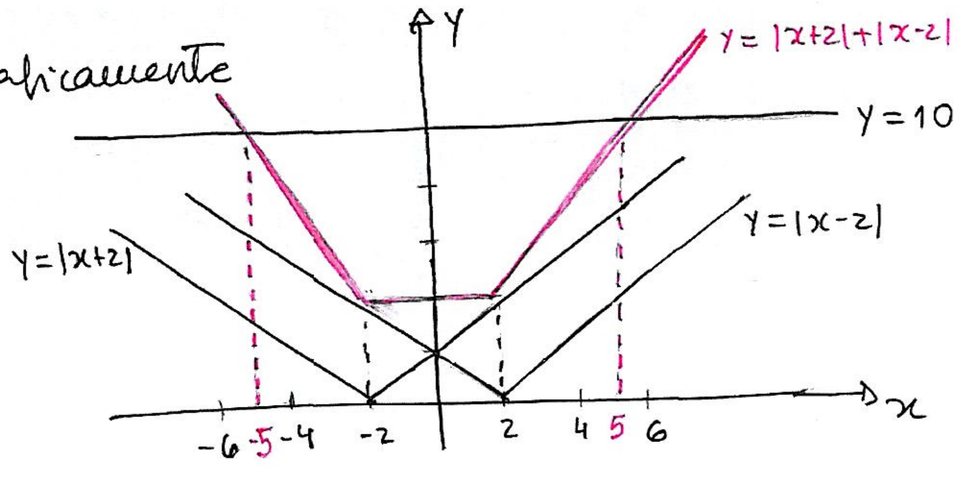
p) $\{x\in\mathbb{R} : |x+2| + |x-2| < 10\}$



→ $\{x\in\mathbb{R} : d(x-2) + d(x,2) < 10\}$

$=]-5, 5[$

→ Graficamente



→ Eliminar o módulo, estudando separadamente os

q) $\{x \in \mathbb{R} : |x^2-1| \leq 1\} = \{x \in \mathbb{R} : 0 \leq x^2 \leq 2\}$

$= [-\sqrt{2}, \sqrt{2}]$

r) $\{x \in \mathbb{R} : 2x^2 \leq 4\} = \{x \in \mathbb{R} : x^2 \leq 2\} = [-\sqrt{2}, \sqrt{2}]$

s) $\{x \in \mathbb{R} : 4 < x^2 < 9\} = \{x \in \mathbb{R} : -3 < x < -2 \vee 2 < x < 3\}$

$=]-3, -2[\cup]2, 3[$

t) $\{x \in \mathbb{R} : \dfrac{x}{x-2} \leq 0\} = \{x \in \mathbb{R} : (x \leq 0 \wedge x-2 > 0) \vee (x \geq 0 \wedge x < 2)\}$

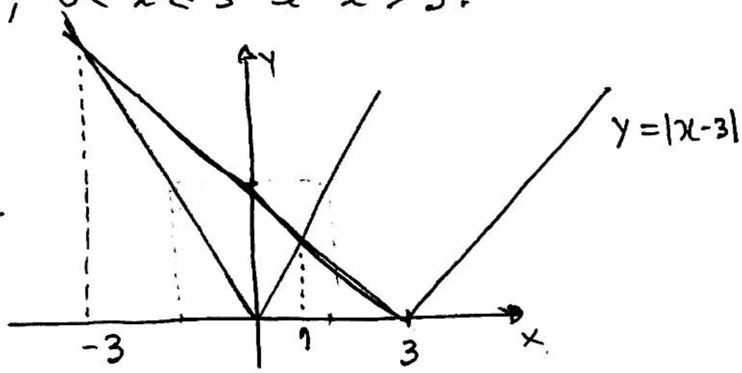
$= [0, 2[$

u) $\{x \in \mathbb{R} : |x-3| < 2|x|\}$

→ Eliminar os módulos, estudando separadamente os casos $x \leq 0$, $0 < x \leq 3$ e $x > 3$.

→ Graficamente

$]-\infty, -3[\cup]1, +\infty[$



v) $\{x \in \mathbb{R} : |x+1| > |x-3|\}$

Como em u). $]1, +\infty[$

1,3 Majorantes, minorantes, ...

a) $[-\sqrt{5},3] \cap \mathbb{Q} = A$

$\text{Maj } A = [3,+\infty[\,, \quad \text{Min } A =]-\infty, -\sqrt{5}]$

$\inf A = -\sqrt{5}, \quad \sup A = 3 = \max A$

A não possui mínimo.

b) $B = [0,\sqrt{3}] \cap \mathbb{R}\setminus\mathbb{Q}$

Semelhante, mas agora B ñ possui mínimo e possui máximo.

c) $C = \{x \in \mathbb{Q} : x^2 < 11\} =]-\sqrt{11}, \sqrt{11}[\, \cap \mathbb{Q}$

Semelhante aos anteriores mas C ñ possui min nem max.

d) $D = \{x \in \mathbb{R} : |x-5| < 3\} =]2,8[$

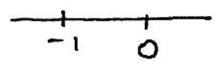
...

e) $\{x \in \mathbb{Z} : x^2 < \frac{25}{16}\} = \{x \in \mathbb{Z} : -\frac{5}{4} < x < \frac{5}{4}\} = D$

$D = \{-1,0,1\}$. $\text{Maj } D = [1,+\infty[\,, \text{Min } D =]-\infty,-1]$

$\sup D = 1 = \max D, \quad \inf D = -1 = \min D.$

f) $\{x \in \mathbb{R}\setminus\mathbb{Q} : \underline{x \leq 0 \wedge |x^2-1| < x+5}\} = F$



$(x \geq -5 \wedge x \leq -1 \wedge x^2-1 < x+5)$

$\vee (x \geq -5 \wedge -1 < x \leq 0 \wedge 1-x^2 < x+5)$

$\Leftrightarrow (-5 \leq x \leq -1 \wedge x^2-x-6 < 0) \vee (-1 < x \leq 0 \wedge \underbrace{x^2+x+4>0}_{\text{sempre}})$

$\Leftrightarrow \left(-5 \leq x \leq -1 \ \wedge\ -2 < x < 3\right) \vee \left(-1 < x \leq 0\right)$

$]-2,-1] \cup]-1,0] =]-2,0]$

Então $F =]-2,0] \cap \mathbb{R}\setminus\mathbb{Q}$

Maj $F = [0,+\infty[$, Min $F =]-\infty,-2]$

$\sup F = 0$, $\inf F = -2$

F não possui máx nem min.

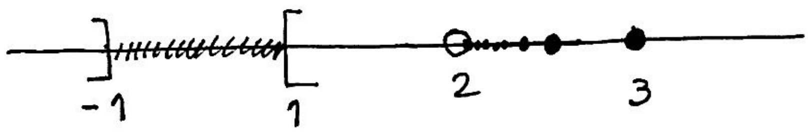
g) $G = \{x \in \mathbb{R} : 5 - x^2 < 1\} = \{x \in \mathbb{R} : x^2 > 4\}$

$=]-\infty,-2[\cup]2,+\infty[$

Maj $G = \emptyset =$ Min G e G ñ possui sup, inf, max, min

h) $H = \{2 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{x \in \mathbb{R} : x^2 < 1\}$

$= \{2 + \frac{1}{n} : n \in \mathbb{N}\} \cup]-1,1[$



Maj $H = [3,+\infty[$ Min $H =]-\infty,-1]$

$\sup H = 3$, $\inf H = -1$, $\max H = 3$

H ñ possui mínimo.

1.4 ...

1.5 a) $\forall x \in \mathbb{R}: \quad x > 7 \Rightarrow |x| > 7$ Verdadeira.

b) $\forall x \in \mathbb{R}: \quad |1+4x| < 1 \Rightarrow x \geq -\frac{1}{2}$ Verdadeira.

$-1 < 1+4x < 1$
$-\frac{1}{2} < x < 0$

c) $\forall x \in \mathbb{R}: \quad |x| \geq 1 \Rightarrow x \geq 1$ Falsa.

Podemos ter $|x| \geq 1$ com $x \leq -1$.

d) $\forall x \in \mathbb{R}: \quad |x-5| \leq 2 \Rightarrow 3 < x < 7$ Falsa.

$-2 \leq x - 5 \leq 2$
$3 \leq x \leq 7$

Podemos ter $|x-5| \leq 2$ com $x = 3$ (ou $x = 7$).

1.6 - Não é válida a lei do corte (3ª implicação), pq $a-b=0$.

1.7 - $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$: Falsa ($x=1, y=1$)

$\sqrt{xy} = \sqrt{x}\sqrt{y}$: Falsa ($x<0 \;\; y<0$)
Verdadeira se $x \geq 0 \wedge y \geq 0$.

supus $n \in \mathbb{N}$
$$\begin{cases} (x+y)^n = x^n + y^n : \text{Falsa } (n=2,\ x=1,\ y=1) \\ (xy)^n = x^n y^n : \text{Verdadeira} \end{cases}$$