Cálculo I — LEI

Teste 2A — 13 de Dezembro de 2006

1. Sejam $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ funções com derivadas contínuas. Sabendo que $P(x) = 2x+1$ coincide simultaneamente com o polinómio de Taylor de primeira ordem da função f em torno do ponto $a = 0$ e com o polinómio de Taylor de segunda ordem da função g em torno do ponto $b = 1$, determine $f(0)$, $f'(0)$, $g(1)$, $g'(1)$ e $g''(1)$.

   $f(0) = P(0) = 1,\ f'(0) = P'(0) = 2.$

   $g(1) = P(1) = 3,\ g'(1) = P'(1) = 2,\ g''(1) = P''(1) = 0.$

2. Calcule apenas duas das seguintes primitivas:

   (a) $\displaystyle\int \frac{1}{x^2}\cos\left(\frac{2}{x}\right) dx$; (b) $\displaystyle\int x \operatorname{arctg}(x^2)\, dx$;

   (c) $\displaystyle\int \frac{x+1}{x(x-1)^2}\, dx$; (d) $\displaystyle\int x\sqrt[4]{1+x}\, dx$.

(a) $$\int \frac{1}{x^2}\cos\left(\frac{2}{x}\right) dx = -\frac{1}{2}\int \left(-\frac{2}{x^2}\right)\cos\left(\frac{2}{x}\right) dx = -\frac{1}{2}\operatorname{sen}\left(\frac{2}{x}\right) + C.$$

(b) $$\int x \operatorname{arctg}(x^2)\, dx = \frac{x^2}{2}\operatorname{arctg}(x^2) - \int \frac{x^2}{2}\frac{2x}{1+x^4}\, dx = \frac{x^2}{2}\operatorname{arctg}(x^2) - \frac{1}{4}\int \frac{4x^3}{1+x^4}\, dx$$
$$= \frac{x^2}{2}\operatorname{arctg}(x^2) - \frac{1}{4}\ln(1+x^4) + C.$$

(c) $$\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$
$$\iff x+1 = A(x-1)^2 + Bx + Cx(x-1)$$
$$\iff x+1 = (A+C)x^2 + (-2A+B-C)x + A$$
$$\iff A+C = 0 \wedge -2A+B-C = 1 \wedge A = 1 \iff A = 1 \wedge B = 2 \wedge C = -1$$

$$\int \frac{x+1}{x(x-1)^2}\, dx = \int \frac{1}{x}\, dx + \int \frac{2}{(x-1)^2}\, dx - \int \frac{1}{x-1}\, dx$$
$$= \ln|x| - \frac{2}{x-1} - \ln|x-1| + C.$$

(d) Substituição $1+x = t^4$, $t \geq 0$. Então $g(t) = t^4 - 1$ e $g'(t) = 4t^3$.

Calculamos
$$\int 4t^3(t^4-1)\, t\, dt = 4\int (t^8 - t^4)\, dt = \frac{4}{9}t^9 - \frac{4}{5}t^5 + C$$
e resulta
$$\int x\sqrt[4]{1+x}\, dx = \frac{4}{9}(1+x)^{9/4} - \frac{4}{5}(1+x)^{5/4} + C.$$

1. Seja $P(x) = 2x^3 + 3x^2 + 1$ o polinómio de Taylor de terceira ordem em torno do ponto $a = 1$ duma função $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ possuindo derivadas contínuas. Determine o correspondente polinómio de Taylor de segunda ordem.

$f(1) = P(1) = 6,\ f'(1) = P'(1) = 12,\ f''(1) = P''(1) = 18.$
$P_{2,1}(x) = 6 + 12(x-1) + \frac{18}{2}(x-1)^2 = 9x^2 - 6x + 3.$

2. Calcule apenas duas das seguintes primitivas:

(a) $\displaystyle\int \frac{1}{x^3}\, e^{1/x^2}\, dx$; (b) $\displaystyle\int \operatorname{arctg}(2x)\, dx$;

(c) $\displaystyle\int \frac{2x-1}{(x^2-1)(x-2)}\, dx$; (d) $\displaystyle\int x\sqrt{x-1}\, dx$.

(a) $\displaystyle\int \frac{1}{x^3}\, e^{1/x^2}\, dx = -\frac{1}{2} e^{1/x^2} + C.$

(b) $\displaystyle\int \operatorname{arctg}(2x)\, dx = x \operatorname{arctg}(2x) - \frac{1}{4}\ln(1+4x^2) + C.$

(c) $\displaystyle\frac{2x-1}{(x^2-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} \iff A = -\frac{1}{2} \wedge B = -\frac{1}{2} \wedge C = 1$

$$\int \frac{2x-1}{(x^2-1)(x-2)}\, dx = -\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \ln|x-2| + C.$$

(d) Substituição $x - 1 = t^2,\ t \geq 0$. Então $g(t) = t^2 + 1$ e $g'(t) = 2t$.
Calculamos
$$\int 2t(t^2+1)\, t\, dt = \frac{2}{5}\, t^5 + \frac{2}{3}\, t^3 + C$$
e resulta
$$\int x\sqrt{x-1}\, dx = \frac{2}{5}\,(x-1)^{5/2} + \frac{2}{3}\,(x-1)^{3/2} + C.$$