Teste 1 - novembro 2012

1) $\frac{dy}{dx} - 6xy = 2xy^2$

$u = (y)^{-1}$

(a) $u = \frac{1}{y}$

$\frac{dy}{dx} - 6xy = 2xy^2$] · multiplicação p/ y^{-2}

$y \cdot y^{-2} = y^{1-2} = y^{-1} = \frac{1}{y}$

$y^{-2}\frac{dy}{dx} - 6x\frac{1}{y} = 2x$] m. variável $u = \frac{1}{y}$

$\frac{du}{dx} = -1 \cdot \frac{dy}{dx} \cdot y^{-2}$

$-\frac{du}{dx} = y^{-2}\frac{dy}{dx}$

$\Rightarrow -\frac{du}{dx} - 6xu = 2x$

$\Leftrightarrow \frac{du}{dx} + 6xu = -2x$ c.q.p.

(b) $\frac{du}{dx} + \underbrace{6x}_{P(x)}u = \underbrace{-2x}_{Q(x)}$

$u(x) = e^{-\int 6x\,dx}\left[\int e^{\int 6x\,dx}(-2x)\,dx + C\right]$

$= e^{-6\frac{x^2}{2}}\left[\int e^{6\frac{x^2}{2}}(-2x)\,dx + C\right] = e^{-3x^2}\left[\int e^{3x^2}(-2x)\,dx + C\right]$

$= e^{-3x^2}\left[-\frac{1}{6}\int e^{3x^2}(6x)\,dx + C\right] = e^{-3x^2}\left(-\frac{1}{6}e^{3x^2}\right) + C$

$= -\frac{1}{6}e^{0} + C = -\frac{1}{6}\times 1 + C = -1/6 + C$

$1 : \frac{1}{6}$
$= 1\times 6$

(2) $u(x) = -\frac{1}{6} + C$

Fizemos a mudança de variável $u = \frac{1}{y}$, logo:

(1) : $\frac{1}{y} = -\frac{1}{6} + C \Leftrightarrow y\left(-\frac{1}{6} + C\right) = 1 \Leftrightarrow y = \frac{1}{-\frac{1}{6} + C}$

$\Leftrightarrow y = 6 + C$

$y(x) = 6 + C$

2) a) $\begin{cases} y' = 2y \\ y(1) = 2 \end{cases}$

$\frac{dy}{dt} = \underbrace{2y}_{f(y)} \Rightarrow \int \frac{1}{2y}\,dy = \int dt$

$g(t) = 1$

$\Rightarrow \frac{1}{2}\int \frac{1}{y}\,dy = t + C \Rightarrow \frac{1}{2}\ln|y| = t + C$

$\Rightarrow \ln|y| = 2t + C \Rightarrow y = e^{2t+C}$, $C (\in \mathbb{R})$ cte arbitrária

$y(t) = e^{2t+C}$

$y(1) = 2 \Leftrightarrow e^{2\times 1 + C} = 2 \Leftrightarrow e^{2+C} = 2 \Leftrightarrow \ln 2 = 2 + C$

$\Leftrightarrow C = \ln 2 - 2$

$y(t) = e^{2t + \ln(2) - 2} = e^{2t} \cdot e^{\ln 2} \cdot e^{-2} = e^{2t-2} \cdot 2 = 2e^{2t-2}$

A solução é a função $y(t) = 2e^{2t-2}$

(b) $y(t) = \cos(t^2)$

$y' + aty = 0$

$y'(t) = -2t \,\text{sen}\,(t^2)$

$-2t\,\text{sen}(t^2) + at\cos(t^2) = 0$

$\Leftrightarrow -2\cancel{t}\,\text{sen}\,(t^2) = -a\cancel{t}\cos(t^2)$?

$\Leftrightarrow t(-2\,\text{sen}(t^2) + a\cos(t^2)) = 0$

$\Leftrightarrow t = 0 \quad \vee \quad \cancel{+}2\,\text{sen}\,(t^2) = \cancel{+}a\cos(t^2)$

$\Leftrightarrow t = 0 \quad \vee \quad \frac{a}{2} = tg\,(t^2) \quad \Leftrightarrow t = 0 \quad \vee \quad a = 2\,tg\,(t^2)$

(c) $y' = \dfrac{\cos(t^2y) - 2ty}{t^2} \Rightarrow y' \cdot t^2 = \cos(t^2y) - 2ty$

$u = t^2 y$

$\dfrac{du}{dt} = 2ty + t^2\dfrac{dy}{dt}$

$\Rightarrow t^2\dfrac{dy}{dt} + 2ty = \cos(t^2y)$

$\Rightarrow \dfrac{du}{dt} = \cos(u)$

$\Rightarrow u' = \cos u$ (2ª opção)

(d) $y'' + 16y = e^t$

$\underbrace{y'' + 16y = 0}$ → EDO homogénea

• consideremos soluções da forma $y(t) = e^{mt}$

$y'(t) = m e^{mt}$

$y''(t) = m^2 e^{mt}$

· Substituindo na EDO: $m^2 e^{mt} + 16 e^{mt} = 0 \Leftrightarrow e^{mt}[m^2 + 16] = 0 \Leftrightarrow \underbrace{e^{mt} = 0}_{i(x)} \vee m^2 + 16 = 0$

$\Leftrightarrow m^2 = -16 \Rightarrow m = \pm 4i$

$y_H(t) = e^{0t}[c_1\cos(4t) + c_2\,\text{sen}\,(4t)] = c_1\cos(4t) + c_2\,\text{sen}(4t)$, com c_1, c_2 con^{tes} arbitrárias ($\in \mathbb{R}$)

$y_p(t) = ?$ Conjunto C.I. $= \{e^t\}$

$y_p(t) = Ae^t$

$y'_p(t) = Ae^t$

$y''_p(t) = Ae^t$

Substituindo: $Ae^t + 16Ae^t = e^t$

$\Leftrightarrow e^t(16A + A) = e^t \Leftrightarrow e^t(17A) = e^t$

Donde vem: $17A = 1 \Leftrightarrow \underline{A = 1/17}$

$y_p(t) = \frac{1}{17}e^t$

Solução geral:

$y(t) = c_1\cos(4t) + c_2\,\text{sen}(4t) + \frac{1}{17}e^t$, com c_1, c_2 c^{tes} arbitrárias ($\in \mathbb{R}$)

(e) $y_1(t) = \ln t$
$y_2(t) = 2t + \ln t$ > soluções de uma EDO linear homogénea de ordem 2.

$$\begin{cases} y(1) = 2 \\ y'(1) = 2 \end{cases}$$

$2t c_2 + c_2 \ln(t)$
$2c_2 + \frac{1}{t} c_2$

$y(t) = c_1 \ln t + c_2 (2t + \ln t)$, c_1, c_2 c^{tes} arbitrárias $(\in \mathbb{R})$

$y'(t) = c_1 \cdot \frac{1}{t} + c_2 \left(2 + \frac{1}{t}\right)$

- $y(1) = 2 \Rightarrow c_1 \ln(1) + c_2 (2 + \ln 1) = 2$
$\Leftrightarrow \cancel{c_1 \times 0} + c_2 (2 + 0) = 2 \Leftrightarrow 2c_2 = 2 \Leftrightarrow \underline{c_2 = 1}$

- $y'(1) = 2 \Rightarrow c_1 \cdot \frac{1}{1} + 1 (2 + 1) = 2$
$\Rightarrow c_1 + 3 = 2 \Rightarrow \underline{c_1 = -1}$

Fica então: $y(t) = -\ln t + (2t + \ln t)$
$= \cancel{-\ln t} + 2t + \cancel{\ln t} = 2t$

$y(t) = 2t$ (2ª opção)

4) Dados:
- temperatura (T)
- tempo (t)
- temp. $c^{te} \rightarrow T_0$

$$\frac{dT}{dt} + KT = KT_0 \quad , \; c/ \; K \in \mathbb{R} \; (\text{constante})$$

| horas a que foi desligado ? |

$T = 70°C$
$T_0 = 20°C$

temperatura às 18 h $\rightarrow 50°C$
" às 19h $\rightarrow 40°C$

$$\frac{dT}{dt} + \underbrace{K}_{P(t)} T = \underbrace{KT_0}_{Q(t)} \Leftrightarrow T(t) = e^{-\int K\,dt} \left[\int e^{\int K\,dt} KT_0 \, dt + C \right]$$

$= e^{-Kt} \left[\int e^{Kt} KT_0 \, dt + C \right] = e^{-Kt} \left[T_0 \int e^{Kt} K \, dt + C \right]$

$= e^{-Kt} \left[T_0 e^{Kt} \right] = T_0 e^0 + C = T_0 + C$

$T(t) = T_0 + C$

$\frac{dT}{dt} = KT_0 - KT \Leftrightarrow \frac{dT}{dt} = K(T_0 - T)$

$\Leftrightarrow \frac{dT}{dt} = K (20 - 70)$

$\Leftrightarrow \frac{dT}{dt} = -50K \Rightarrow \int \frac{1}{-50K} dT = \int dt$

$\Rightarrow -\frac{1}{50K} T = t \Leftrightarrow T = -50Kt$

$T(t) = \frac{50}{18} t = \frac{25}{9} t$

$T(t) = -50Kt$ $\quad T(18) = 50 \Leftrightarrow 50 = -50K \cdot 18$
$\Leftrightarrow K = -\frac{1}{18}$

5)

$$\frac{d^2x}{dt^2} + w^2 x = \cos(wt)$$

(a) $\frac{d^2x}{dt^2} + w^2 x = 0$

Soluções da forma $x(t) = e^{mt}$

$x' = m e^{mt}$

$x'' = m^2 e^{mt}$

Substituindo: $m^2 + w^2 = 0$

$\Leftrightarrow m^2 = \underbrace{-w^2}_{w>0 \Rightarrow w^2>0 \Rightarrow -w^2<0}$ $\quad \boxed{\sqrt{-w^2} = \sqrt{w^2}\cdot i}$

$\Leftrightarrow m = \pm wi$

$x_H(t) = c_1 \cos(wt) + c_2 \operatorname{sen}(wt)$, com c_1, c_2 c^{tes} arbitrárias.

(b) Conjunto CI. $= \{\operatorname{sen}(wt); \cos(wt)\}$

$x_p(t) = At\operatorname{sen}(wt)$

$x'_p = A\operatorname{sen}(wt) + Atw\cos(wt)$

$x''_p = A\cos(wt) + Aw\cos(wt) - Atw^2\operatorname{sen}(wt)$

Substituindo:

$A\cos(wt) + Aw\cos(wt) - \cancel{Atw^2\operatorname{sen}(wt)} + \cancel{w^2 At\operatorname{sen}(wt)} = \cos(wt)$

$\Leftrightarrow \cos(wt)(A + Aw) = \cos(wt)$

$\Leftrightarrow A + Aw = 1$

$\Leftrightarrow A(1+w) = 1 \Leftrightarrow A = \frac{1}{1+w}$

$x_p(t) = \frac{1}{1+w} t \operatorname{sen}(wt)$

(c)

$x_p(t) = At\operatorname{sen}(wt) + Bt\cos(wt)$

$x'_p = A\operatorname{sen}(wt) + Atw\cos(wt) + B\cos(wt) - Btw\operatorname{sen}(wt)$

$x''_p = Aw\cos(wt) + Aw\cos(wt) - Atw^2\operatorname{sen}(wt) - B\operatorname{sen}(wt) - (Bw\operatorname{sen}(wt) + Btw^2\cos(wt))$

$= 2Aw\cos(wt) - Atw^2\operatorname{sen}(wt) - B\operatorname{sen}(wt) - Bw\operatorname{sen}(wt) - Btw^2\cos(wt)$

Substituindo

$2Aw\cos(wt) - \cancel{Atw^2\operatorname{sen}(wt)} - B\operatorname{sen}(wt) - Bw\operatorname{sen}(wt) - \cancel{Btw^2\cos(wt)} + \cancel{w^2 At\operatorname{sen}(wt)} + \cancel{w^2 Bt\cos(wt)} = \cos(wt)$

$2Aw\cos(wt) - B\operatorname{sen}(wt) - Bw\operatorname{sen}(wt) = \cos(wt)$

$\Leftrightarrow 2Aw\cos(wt) + \operatorname{sen}(wt)(-B - Bw) = \cos(wt)$

$\Leftrightarrow \cos(wt)(2Aw)$

$\begin{cases} 2Aw = 1 \\ -B - Bw = 0 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2}w \\ B(-1-w) = 0 \end{cases} \Leftrightarrow \begin{cases} A = w/2 \\ B = 0 \lor w = -1 \end{cases} \Leftrightarrow \begin{cases} A = -1/2 \\ B = 0 \end{cases}$

$x(t) = c_1\cos(-t) + c_2\operatorname{sen}(-t) - \frac{1}{2} t \operatorname{sen}(-t)$, c/ $c_1, c_2 \in \mathbb{R}$ (c^{tes})

d) $\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \left(c_1 \cos(-t) + c_2 \operatorname{sen}(-t) - \frac{1}{2} t \operatorname{sen}(-t) \right)$

$= \cos(-\infty) + \operatorname{sen}(-\infty) - (-\infty) \operatorname{sen}(-\infty)$

$=$ $\underline{\underline{\text{Não existe}}}$

e) $x(0) = 0 \Rightarrow c_1 \cos(0) + \cancel{c_2 \operatorname{sen}(0)} - \cancel{\frac{1}{2} \times 0} \operatorname{sen}(0) = 0$

$\Rightarrow c_1 = 0$

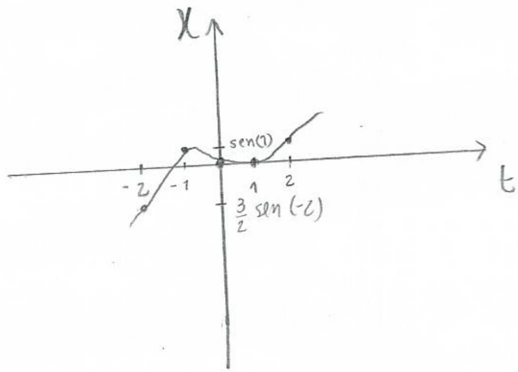
$x'(t) = + c_1 \operatorname{sen}(-t) - c_2 \cos(-t) - \frac{1}{2} \operatorname{sen}(-t) + \frac{1}{2} \cos(-t)$

$x'(0) = 0 \Leftrightarrow - c_2 \cos(0) - \cancel{\frac{1}{2} \operatorname{sen} 0} + \frac{1}{2} \cos 0 = 0$

$\Leftrightarrow - c_2 + \frac{1}{2} = 0 \Leftrightarrow c_2 = 1/2$

$x(t) = \frac{1}{2} \operatorname{sen}(-t) - \frac{1}{2} t \operatorname{sen}(-t)$

$= \frac{1}{2} \operatorname{sen}(-t) (1 - t)$



t	$x(t)$
0	0
1	0
-1	$\frac{1}{2} \operatorname{sen}(1) \cancel{2} = \operatorname{sen} 1 \simeq 0{,}017$
-2	$\frac{1}{2} \operatorname{sen}(-2)(3) = \frac{3}{2} \operatorname{sen}(-2) \simeq -0{,}052$
2	$\frac{1}{2} \operatorname{sen}(-2)(-1) = -\frac{1}{2} \operatorname{sen}(-2) \simeq 0{,}017$