Folhas

Cálculo I
LEI

5.1

a) $\cos^2 x = \dfrac{\cos 2x + 1}{2}$; $x \in \mathbb{R}$

Sabemos que

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1 - \cos^2 x) \\ &= 2\cos^2 x - 1 \end{aligned}$$

fórmula fundamental da trigonometria

Logo: $2\cos^2 x = \cos 2x + 1$

$\Leftrightarrow \cos^2 x = \dfrac{\cos 2x + 1}{2}$

b) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$; $x \in \mathbb{R}$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

fórmula fundamental da trigonometria

Logo $-2\sin^2 x = \cos 2x - 1$

$\Leftrightarrow \sin^2 x = \dfrac{1 - \cos 2x}{2}$

5.2

a) $\operatorname{sen}(\operatorname{arcsen}(-1/2)) = -1/2$

Em geral $\operatorname{sen}(\operatorname{arcsen}(x)) = x$, $\forall x \in [-1, 1]$

$$[-1, 1] \xrightarrow{\text{arcsen}} [-\pi/2, \pi/2] \xrightarrow{\text{sen}} [-1, 1]$$

Mas $\operatorname{arcsen}(\operatorname{sen}(y)) = y$ apenas quando $y \in [-\pi/2, \pi/2]$

b) $\operatorname{arcsen}\left(\operatorname{sen}\left(\frac{7\pi}{6}\right)\right)$

$\operatorname{sen}\left(\frac{7\pi}{6}\right) = \operatorname{sen}\left(\pi + \frac{\pi}{6}\right) = -\operatorname{sen}\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$\operatorname{arcsen}\left(-\frac{1}{2}\right) = y \iff \operatorname{sen}(y) = -\frac{1}{2}$ e $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\iff y = -\frac{\pi}{6}$

$\operatorname{arcsen}\left(\operatorname{sen}\left(\frac{7\pi}{6}\right)\right) = -\frac{\pi}{6}$

Note que não podemos simplesmente escrever

$\operatorname{arcsen}\left(\operatorname{sen}\left(\frac{7\pi}{6}\right)\right) = \frac{7\pi}{6}$ ✗

pois, por convenção, $\operatorname{arcsen}: [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

c) $\cos\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{\sqrt{3}}{2}$

d) $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right)$

$\arccos: [-1, 1] \longrightarrow [0, \pi]$

Temos $\underbrace{\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \arccos\left(\cos\left(\frac{\pi}{3}\right)\right)}_{\text{a função cos é par}} \underbrace{=}_{\frac{\pi}{3} \in [0,\pi]} \frac{\pi}{3}$

e) $\operatorname{arctg}\left(\operatorname{tg}\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

Note que $\operatorname{arctg}: \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ e $-\frac{\pi}{4} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

f) $\operatorname{tg}(\operatorname{arctg}(-1)) = -1.$

53

a) $\operatorname{sen}(\arccos x) = \sqrt{1-x^2}$

Usando a formula fundamental da trigonometria

$\cos^2(\arccos x) + \operatorname{sen}^2(\arccos x) = 1$

$\Rightarrow \quad x^2 + \operatorname{sen}^2(\arccos x) = 1$

$\Rightarrow \operatorname{sen}^2(\arccos x) = 1 - x^2$

$\Rightarrow \operatorname{sen}(\arccos x) = \pm\sqrt{1-x^2}$

Ora:

$$[-1,1] \xrightarrow{\arccos} [0,\pi] \xrightarrow{\operatorname{sen}} [-1,1]$$

No intervalo $[0,\pi]$ a função seno é positiva

logo, $\operatorname{sen}(\arccos x) = \sqrt{1-x^2}$

Domínio $= [-1,1]$

b) $\operatorname{tg}(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$

$$\operatorname{tg}(\arccos x) = \frac{\operatorname{sen}(\arccos x)}{\cos(\arccos x)} = \frac{\sqrt{1-x^2}}{x}$$

Domínio $= [-1,1] \setminus \{0\}$

c) $\cos(\operatorname{arcsen} x) = \sqrt{1-x^2}$

Análogo à alínea a)

d) $\operatorname{tg}(\operatorname{arcsen} x) = \dfrac{x}{\sqrt{1-x^2}}$

$$\operatorname{tg}(\operatorname{arcsen} x) = \frac{\operatorname{sen}(\operatorname{arcsen} x)}{\cos(\operatorname{arcsen} x)} = \frac{x}{\sqrt{1-x^2}}$$

Domínio $=]-1,1[$

e) $\operatorname{sen}(\operatorname{arctg} x)$

Vamos usar a formula

$$1 + \operatorname{tg}^2\alpha = \frac{1}{\cos^2\alpha}$$

Fazendo $\alpha = \operatorname{arctg} x$ vem

$$1 + \operatorname{tg}^2(\operatorname{arctg} x) = \frac{1}{\cos^2(\operatorname{arctg} x)}$$

$\Rightarrow 1+x^2 = \frac{1}{\cos^2(\text{arctg}\,x)} \Rightarrow \cos^2(\text{arctg}\,x) = \frac{1}{1+x^2}$

$\Rightarrow \cos(\text{arctg}\,x) = \pm \frac{1}{\sqrt{1+x^2}}$

ora $\mathbb{R} \xrightarrow{\text{arctg}}]-\pi/2, \pi/2[\xrightarrow{\cos} [-1,1]$

No intervalo $]-\pi/2, \pi/2[$ a função cosseno é positiva

logo, $\cos(\text{arctg}\,x) = \frac{1}{\sqrt{1+x^2}}$

Como $\text{tg}(\alpha) = \frac{\text{sen}\,\alpha}{\cos\alpha}$ logo $\text{sen}\,\alpha = \text{tg}(\alpha)\cos(\alpha)$.

Assim, $\text{sen}(\text{arctg}\,x) = \text{tg}(\text{arctg}\,x)\cos(\text{arctg}\,x)$

$= \frac{x}{\sqrt{1+x^2}}$

Domínio $= \mathbb{R}$

f) $\cos(\text{arctg}\,x) = \frac{1}{\sqrt{1+x^2}}$

Foi feito na alínea (e).

5.4

a) $e^x = e^{1-x}$

$e^x = e^{1-x} \Leftrightarrow x = 1-x$ (pois a função exponencial é injectiva)

$\Leftrightarrow 2x = 1$

$\Leftrightarrow x = 1/2$

Conjunto solução $= \{1/2\}$

b) $e^{2x} + 2e^x - 3 = 0$

Seja $y = e^x$. Então

$e^{2x} + 2e^x - 3 = 0 \Leftrightarrow y^2 + 2y - 3 = 0 \Leftrightarrow$

$\Leftrightarrow y = 1 \vee y = -3$

$\Leftrightarrow e^x = 1 \vee \underbrace{e^x = -3}_{\text{impossível}}$

$\Leftrightarrow x = 0$

Conjunto solução: $\{0\}$

c) $e^{3x} - 2e^{-x} = 0$

$$e^{3x} - 2e^{-x} = 0 \Leftrightarrow e^{-x}\left(e^{4x} - 2\right) = 0$$

$$\Leftrightarrow \underbrace{e^{-x} = 0}_{\text{impossível}} \vee e^{4x} - 2 = 0$$

$$\Leftrightarrow e^{4x} = 2 \Leftrightarrow 4x = \ln 2 \Leftrightarrow x = \frac{\ln 2}{4}$$

Conjunto solução: $\left\{\frac{\ln 2}{4}\right\}$

d) $\ln(x^2-1) + 2\ln 2 = \ln(4x-1)$

$$\ln(x^2-1) + 2\ln 2 = \ln(4x-1) \Leftrightarrow \ln(x^2-1) + \ln(2^2) = \ln(4x-1)$$

$$\Leftrightarrow \ln(x^2-1) + \ln(4) = \ln(4x-1)$$

$$\Leftrightarrow \ln\left(4(x^2-1)\right) = \ln(4x-1)$$

(Note que ln é injectiva) $\Rightarrow 4x^2 - 4 = 4x - 1$

$$\Leftrightarrow 4x^2 - 4x - 3 = 0$$

$$\Leftrightarrow x = \frac{4 \pm \sqrt{16 + 4\times 4\times 3}}{2\times 4}$$

$$\Leftrightarrow x = \frac{4 \pm 4\times 2}{8} \Leftrightarrow x = \frac{3}{2} \vee x = \frac{-1}{2}$$

Note agora que $-1/2$ não é solução da equação dada pois a expressão não está definida nesse ponto

Conjunto solução: $\{3/2\}$

5.5

a) $f(x) = ch(3x)$

$f'(x) = 3\, sh(3x)$

b) $f(x) = sh(x^2+1)$

$f'(x) = 2x\, ch(x^2+1)$

c) $f(x) = x^2\, sh^3 x$

$f'(x) = 2x\, sh^3 x + 3x^2\, ch x\, sh^2 x$

d) $f(x) = \ln\left(ch(x+1)\right)$

$$f'(x) = \frac{sh(x+1)}{ch(x+1)} = tgh(x+1)$$

5.6

a) $ch^2 x - sh^2 x = 1$

$$ch^2 x - sh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} =$$

$$= \frac{4}{4} = 1$$

b) $ch\,x + sh\,x = e^x$

$$ch\,x + sh\,x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

c) $sh(-x) = -sh\,x$

$$sh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -sh(x)$$

d) $ch(-x) = ch(x)$

$$ch(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = ch(x)$$

e) $sh(x+y) = sh\,x\,ch\,y + ch\,x\,sh\,y$

$$sh(x)\,ch(y) + ch(x)\,sh(y) = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{e^x e^y + \cancel{e^x e^{-y}} - \cancel{e^{-x} e^y} - e^{-x} e^{-y}}{4} + \frac{e^x e^y - \cancel{e^x e^{-y}} + \cancel{e^{-x} e^y} - e^{-x} e^{-y}}{4}$$

$$= \frac{e^x e^y - e^{-x} e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = sh(x+y)$$

f) $ch(x+y) = ch\,x\,ch\,y + sh\,x\,sh\,y$

Análogo à alínea (e).

g) $th^2 x + \frac{1}{ch^2 x} = 1$

De $ch^2 x - sh^2 x = 1$ vem

$$1 - th^2 x = \frac{1}{ch^2 x} \iff th^2 x + \frac{1}{ch^2 x} = 1$$

h) $\coth^2 x - \dfrac{1}{sh^2 x} = 1 \qquad x \neq 0.$

Análogo à alínea (g).

5.7 Este exercício está repetido – é o mesmo que 5.5.

5.8

a) $\operatorname{argsh} x = \ln\left(x + \sqrt{x^2+1}\right)$; $x \in \mathbb{R}$

argsh é a inversa de sh

$$\frac{e^x - e^{-x}}{2} = y \Leftrightarrow e^x - e^{-x} = 2y$$

Fazendo $z = e^x$ vem $z - \dfrac{1}{z} = 2y$

$$\Leftrightarrow z^2 - 1 = 2yz$$

$$\Leftrightarrow z^2 - 2yz - 1 = 0$$

$$\Leftrightarrow z = \frac{2y \pm \sqrt{4y^2+4}}{2}$$

$$\Leftrightarrow z = y \pm \sqrt{y^2+1}$$

Portanto $e^x = y \pm \sqrt{y^2+1}$ e como $e^x > 0 \ \forall x$

$$e^x = y + \sqrt{y^2+1} \Leftrightarrow x = \ln\left(y + \sqrt{y^2+1}\right)$$

Então $\operatorname{argsh} x = sh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

b) $\operatorname{argch} x = \ln\left(x + \sqrt{x^2-1}\right)$; $x \in [1, +\infty[$

Análogo à alínea (a).

c) $\operatorname{argth} x = \ln\left(\sqrt{\dfrac{1+x}{1-x}}\right)$; $x \in]-1, 1[$

argth é a inversa de th

$$th\, x = \frac{sh\, x}{ch\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$th\, x = y \Leftrightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} = y \Leftrightarrow$$

$$\Leftrightarrow e^x - e^{-x} = y\left(e^x + e^{-x}\right)$$

$$\Leftrightarrow (1-y)e^{x} - (1+y)e^{-x} = 0$$

$$\Leftrightarrow e^{-x}\left((1-y)e^{2x} - (1+y)\right) = 0$$

$$\Leftrightarrow (1-y)e^{2x} - (1+y) = 0 \qquad \text{pois } e^{-x} > 0,\ \forall x$$

$$\Leftrightarrow e^{2x} = \frac{1+y}{1-y}$$

$$\Leftrightarrow 2x = \ln\left(\frac{1+y}{1-y}\right) \Leftrightarrow x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

$$\Leftrightarrow x = \ln\left(\left(\frac{1+y}{1-y}\right)^{1/2}\right) \Leftrightarrow x = \ln\left(\sqrt{\frac{1+y}{1-x}}\right)$$

Portanto $\operatorname{argth}(x) = th^{-1}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$

d) $\operatorname{argcoth} x = \ln\left(\sqrt{\frac{x+1}{x-1}}\right)$, $x \in \mathbb{R} \setminus]-1,1[$

Análogo à alínea (c)

<u>5.9</u>

a) $\operatorname{argsh}' x = \dfrac{1}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$

De $\operatorname{argsh} x = \ln\left(x + \sqrt{x^2+1}\right)$ vem

$$\operatorname{argsh}' x = \frac{\left(x+\sqrt{x^2+1}\right)'}{x+\sqrt{x^2+1}} = \frac{1 + \frac{1}{2}\,2x\,\frac{1}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$= \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}\left(x+\sqrt{x^2+1}\right)} = \frac{1}{\sqrt{x^2+1}}$$

b) $\operatorname{argch}' x = \dfrac{1}{\sqrt{x^2-1}}$, $x \in]1,+\infty[$

Análogo à alínea (a).

c) $\operatorname{argth}' x = \dfrac{1}{1-x^2}, \quad |x|<1$

$$\operatorname{argth} x = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) =$$

$$= \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right)$$

$$\operatorname{argth}' x = \frac{1}{2}\left(\frac{1}{1+x} - \frac{-1}{1-x}\right) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) =$$

$$= \frac{1}{2}\left(\frac{1-x+1+x}{(1+x)(1-x)}\right) = \frac{1}{1-x^2}$$

d) $\operatorname{argcoth}' x = \dfrac{1}{1-x^2}, \quad |x|>1.$

Análogo à alínea (c).