4.

Exercício 4.1

```
MatrizJacobiana[funcao_List?VectorQ, variaveis_List] :=
 D[funcao, {variaveis, 1}] /. MapThread[Rule, {variaveis, {##}}] &

Derivada[funcao_List?VectorQ, variaveis_List, ponto_List] :=
 Dot[MatrizJacobiana[funcao, variaveis][Sequence @@ ponto], {##}] &
```

a)

```
f[x_, y_] = {x, y};
```

$$Jf(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b)

```
f[x_, y_] = {x Exp[y] + Cos[y], x, x + Exp[y]};
```

$$Jf(x,y) = \begin{pmatrix} e^{y} & e^{y}\, x - \mathrm{Sin}[y] \\ 1 & 0 \\ 1 & e^{y} \end{pmatrix}$$

c)

```
f[x_, y_] = {x y Exp[x y], x Sin[y], 5 x y^2};
```

$$Jf(x,y) = \begin{pmatrix} e^{x\,y}\, y + e^{x\,y}\, x\, y^{2} & e^{x\,y}\, x + e^{x\,y}\, x^{2}\, y \\ \mathrm{Sin}[y] & x\, \mathrm{Cos}[y] \\ 5\, y^{2} & 10\, x\, y \end{pmatrix}$$

d)

```
f[x_, y_, z_] = {x - y, y + z};
```

$$Jf(x,y,z) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

e)

```
f[x_, y_, z_] = {x + y + Exp[z], x^2 y};
```

$$Jf(x,y,z) = \begin{pmatrix} 1 & 1 & e^{z} \\ 2\, x\, y & x^{2} & 0 \end{pmatrix}$$

Exercício 4.2

```
f[x_, y_, z_] = {x - y + z, x^2 y z, x y z};
g[x_, y_, z_] = {x y, y z, 2 x, x y z};
```

a)

```
Df((-1,0,-1);(2,3,-1)) = {-2, -3, 3}

Dg((-1,0,-1);(2,3,-1)) = {-3, -3, 4, 3}
```

b)

```
Df(-1,0,-1)(x,y,z) = {x - y + z, -y, y}

Dg(-1,0,-1)(x,y,z) = {-y, -y, 2 x, y}
```

Exercício 4.3

```
f[x_, y_] = {3 x, x + 2 y};
```

a)

$$\mathrm{Jf}(x,y) = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

b)

A função f é de classe C^1.

c)

```
Print["Df(1,2)(x,y) = ", MatrizJacobiana[f[x, y], {x, y}][1, 2].{x, y}]

Df(1,2)(x,y) = {3 x, x + 2 y}
```

d)

$$\mathrm{Df}(x_0,y_0)(x,y) = \{3x,\ x + 2y\}$$

Exercício 4.4

```
f[x_, y_] = {2 x^2, 3 y, 2 x y};
```

a)

$$\mathrm{Jf}(x,y) = \begin{pmatrix} 4x & 0 \\ 0 & 3 \\ 2y & 2x \end{pmatrix}$$

b)

A função f é de classe $\mathcal{C}^1$.

```
Df(1,1)(x,y) = {4 x, 3 y, 2 x + 2 y}
```

c)

```
Df(1,1)(2,3) = {8, 9, 10}
```

Exercício 4.5

a)

```
f[x_, y_] = 2 x y / (x^2 + y^2)^2;
```

$$\text{Hess } f(x,y) = \begin{pmatrix} \frac{24\, y\, (x^3 - x\, y^2)}{(x^2+y^2)^4} & -\frac{6\,(x^4 - 6\, x^2\, y^2 + y^4)}{(x^2+y^2)^4} \\ -\frac{6\,(x^4 - 6\, x^2\, y^2 + y^4)}{(x^2+y^2)^4} & \frac{24\, x\, y\, (-x^2 + y^2)}{(x^2+y^2)^4} \end{pmatrix}$$

b)

```
f[x_, y_] = Cos[x y^2];
```

$$\text{Hess } f(x,y) = \begin{pmatrix} -y^4 \operatorname{Cos}[x\, y^2] & -2\, y\, (x\, y^2 \operatorname{Cos}[x\, y^2] + \operatorname{Sin}[x\, y^2]) \\ -2\, y\, (x\, y^2 \operatorname{Cos}[x\, y^2] + \operatorname{Sin}[x\, y^2]) & -2\, x\, (2\, x\, y^2 \operatorname{Cos}[x\, y^2] + \operatorname{Sin}[x\, y^2]) \end{pmatrix}$$

c)

```
f[x_, y_] = Exp[-x y^2] + y^3 x^4;
```

$$\text{Hess } f(x,y) = \begin{pmatrix} y^3\, (12\, x^2 + e^{-x\, y^2}\, y) & 2\, e^{-x\, y^2}\, y\, (-1 + 6\, e^{x\, y^2}\, x^3\, y + x\, y^2) \\ 2\, e^{-x\, y^2}\, y\, (-1 + 6\, e^{x\, y^2}\, x^3\, y + x\, y^2) & 2\, e^{-x\, y^2}\, x\, (-1 + 3\, e^{x\, y^2}\, x^3\, y + 2\, x\, y^2) \end{pmatrix}$$

d)

```
f[x_, y_] = 1 / (Cos[x]^2 + Exp[-y]);
```

$$\text{Hess } f(x,y) = \begin{pmatrix} \frac{2\, e^{2\, y}\, (e^y \operatorname{Cos}[x]^4 - \operatorname{Sin}[x]^2 + \operatorname{Cos}[x]^2\, (1 + 3\, e^y \operatorname{Sin}[x]^2))}{(1 + e^y \operatorname{Cos}[x]^2)^3} & \frac{4\, e^{2\, y} \operatorname{Cos}[x] \operatorname{Sin}[x]}{(1 + e^y \operatorname{Cos}[x]^2)^3} \\ \frac{4\, e^{2\, y} \operatorname{Cos}[x] \operatorname{Sin}[x]}{(1 + e^y \operatorname{Cos}[x]^2)^3} & -\frac{e^y\, (-1 + e^y \operatorname{Cos}[x]^2)}{(1 + e^y \operatorname{Cos}[x]^2)^3} \end{pmatrix}$$

Exercício 4.6

```
g[x_, t_] = 2 + Exp[-t] Sin[x];
```

$$\frac{\partial g}{\partial t}(x,t) = -e^{-t} \operatorname{Sin}[x]$$

$$\frac{\partial^2 g}{\partial x^2}(x,t) = -e^{-t}\,\mathrm{Sin}[x]$$

Exercício 4.7

```
f[x, y, z, w] = Exp[x y z] Sin[x w];
```

$$f_{xzw} = e^{x\,y\,z}\,x\,y\,((2+x\,y\,z)\,\mathrm{Cos}[w\,x] - w\,x\,\mathrm{Sin}[w\,x])$$

$$f_{zwx} = e^{x\,y\,z}\,x\,y\,((2+x\,y\,z)\,\mathrm{Cos}[w\,x] - w\,x\,\mathrm{Sin}[w\,x])$$

Exercício 4.8

a)

```
D[2 x^3, y]
```

0

```
D[x^2 y + x, x]
```

$1 + 2\,x\,y$

A função f seria de classe C^2 mas $f_{xy} \neq f_{yx}$

b)

```
D[x Sin[y], y]
```

x Cos[y]

```
D[y Sin[x], x]
```

y Cos[x]

A função f seria de classe C^2 mas $f_{xy} \neq f_{yx}$

Exercício 4.9

```
f[0, 0] = 0;
f[x_, y_] = x y^3 / (x^2 + y^2);
```

a)

$$f_x[0,0] = 0$$

$$f_x[x,y] = \frac{y^3\,(-x^2+y^2)}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0)$$

$$f_y[0,0] = 0$$

$$f_y[x,y] = \frac{x\left(3x^2y^2+y^4\right)}{\left(x^2+y^2\right)^2}, \quad (x,y)\neq(0,0)$$

b)

$$f_{xy}[0,0] = 1$$

$$f_{yx}[0,0] = 0$$

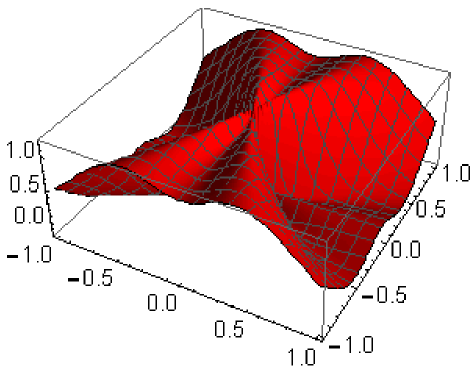
c)

$$f_{xy}[x,y] = \frac{-3x^4y^2+6x^2y^4+y^6}{\left(x^2+y^2\right)^3}, \quad (x,y)\neq(0,0)$$

```
Limit[fxy[x, x], x → 0]
```

$$\frac{1}{2}$$

Conclui-se assim que f não é de classe C^2.



Exercício 4.10

```
u[x_, y_] = x y;
v[x_, y_] = Sin[x y];
w[x_, y_] = Exp[x];
f[x_, y_, z_] = x^2 y + y^2 z;
h[x_, y_] = f[u[x, y], v[x, y], w[x, y]]

Grad[h[x, y], {x, y}] // Simplify
```

$$\{x^2y^3\,\mathrm{Cos}[x\,y] + \mathrm{Sin}[x\,y]\left(2xy^2 + 2e^x y\,\mathrm{Cos}[x\,y] + e^x\,\mathrm{Sin}[x\,y]\right),$$
$$x\left(x^2y^2\,\mathrm{Cos}[x\,y] + 2xy\,\mathrm{Sin}[x\,y] + e^x\,\mathrm{Sin}[2\,x\,y]\right)\}$$

Exercício 4.11

```
f[x_, y_, z_] = x^2 y - x z;
h[t_] = {a t^2, a t, t^3};
```

a)

```
Df(1,0,0)(1,2,2) = 0
```

b)

$$g'(t) = 5\,a\,(-1 + a^2)\,t^4$$

$$g'(t)=0 \Leftrightarrow a \in \{-1,\ 0,\ 1\}$$

Exercício 4.13

a)

```
u[x_, y_] = Log[Sin[x/y]]; x[t_] = 3 t^2; y[t_] = Sqrt[1 + t^2];
```

$$\frac{du}{dt} = \left(-\frac{3\,t^3}{(1+t^2)^{3/2}} + \frac{6\,t}{\sqrt{1+t^2}}\right) \mathrm{Cot}\left[\frac{3\,t^2}{\sqrt{1+t^2}}\right]$$

b)

```
w[r_, s_] = r^2 + s^2; r[p_, q_] = p q^2; s[p_, q_] = p^2 Sin[q];
```

$$\frac{\partial w}{\partial p} = 2\,p\,\left(q^4 + 2\,p^2\,\mathrm{Sin}[q]^2\right)$$

$$\frac{\partial w}{\partial q} = 4\,p^2\,q^3 + p^4\,\mathrm{Sin}[2\,q]$$

c)

```
z[x_, y_] = x^2 Sin[y]; x[s_, t_] = s^2 + t^2; y[s_, t_] = 2 s t;
```

$$\frac{\partial z}{\partial s} = 2\,(s^2+t^2)\,\left(t\,(s^2+t^2)\,\mathrm{Cos}[2\,s\,t] + 2\,s\,\mathrm{Sin}[2\,s\,t]\right)$$

$$\frac{\partial z}{\partial t} = 2\,(s^2+t^2)\,\left(s\,(s^2+t^2)\,\mathrm{Cos}[2\,s\,t] + 2\,t\,\mathrm{Sin}[2\,s\,t]\right)$$