Máximos e mínimos condicionados

Sejam $\mathcal{U}$ um aberto de $\mathbb{R}^n$, $f: \mathcal{U} \to \mathbb{R}$ e $g: \mathcal{U} \to \mathbb{R}$ funções de classe C^1. Pretende-se determinar

$$\max f|_{\Sigma_c} \quad \text{e} \quad \min f|_{\Sigma_c}$$

em que $\Sigma_c = g^{-1}(c) = \{ x \in \mathcal{U} : g(x) = c \}$ é a linha ou superfície de nível (consoante $n=2$ ou $n=3$) c da função g.

Os maximizantes ou minimizantes locais da função f restrita a Σ_c estão entre as soluções dos sistemas

$$(I) \begin{cases} x \in \Sigma_c \\ \nabla g(x) = 0 \end{cases} \iff \begin{cases} g(x) = c \\ \nabla g(x) = 0 \end{cases}$$

(pontos singulares de Σ_c)

$$II \begin{cases} x \in \Sigma_c \\ \nabla f(x) = \lambda \nabla g(x) \end{cases} \iff \begin{cases} g(x) = c \\ \nabla f(x) = \lambda \nabla g(x) \end{cases}$$

(pontos de Σ_c onde ∇f e ∇g são colineares)

Determinadas as soluções destes sistemas, o maior valor que a função toma nesses pontos será o $\max f|_{\Sigma_c}$ e o menor valor o $\min f|_{\Sigma_c}$.

Exemplo

Determine $\min f|_K$, sendo $f(x,y,z) = xy+z$ e $K = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$.

Resolução: Seja $g(x,y,z) = x^2+y^2+z^2$. Então $K = g^{-1}(1) = \Sigma_1$

(I) Pontos singulares de Σ_1: $\begin{cases} x \in K \\ \nabla g(x) = 0 \end{cases}$

$$\begin{cases} x^2+y^2+z^2=1 \\ (2x,2y,2z)=(0,0,0) \end{cases} \quad \begin{cases} x^2+y^2+z^2=1 \\ x=y=z=0 \end{cases} \text{ Impossível}$$

Então K não tem pontos singulares.

II $\begin{cases} X \in K \\ \nabla f(X) = \lambda \nabla g(X) \end{cases} \quad \begin{cases} x^2+y^2+z^2=1 \\ (y,x,1)=\lambda(2x,2y,2z) \end{cases}$

$$\begin{cases} x^2+y^2+z^2=1 \\ y=2\lambda x \\ x=2\lambda y \\ 1=2\lambda z \end{cases} \quad \begin{cases} y=2\lambda\cdot 2\lambda y \\ - \end{cases} \quad \begin{cases} y(1-4\lambda^2)=0 \\ - \end{cases} \quad \begin{cases} y=0 \vee \lambda^2=1/4 \end{cases}$$

- $\boxed{y=0}$

$$\begin{cases} x^2+z^2=1 \\ 0=2\lambda x \\ x=0 \\ 1=2\lambda z \end{cases} \quad \begin{cases} z^2=1 \\ - \\ x=0 \\ \lambda=1/2z \end{cases} \quad \begin{cases} z=1 \vee z=-1 \\ - \\ x=0 \\ - \end{cases}$$

Obtivemos os pontos $A=(0,0,1)$ e $B=(0,0,-1)$

$\boxed{\lambda^2=1/4} \iff \lambda=1/2 \vee \lambda=-1/2$

- $\boxed{\lambda=1/2}$

$$\begin{cases} x^2+y^2+z^2=1 \\ y=x \\ x=y \\ 1=z \end{cases} \quad \begin{cases} 2x^2+1=1 \\ y=x \\ y=x \\ z=1 \end{cases} \quad \begin{cases} x=0 \\ y=0 \\ z=1 \end{cases}$$

Obtivemos de novo o ponto A

- $\boxed{\lambda=-1/2}$

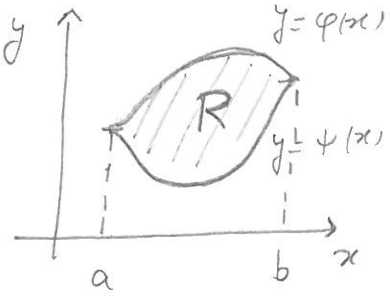
$$\begin{cases} x^2+y^2+z^2=1 \\ y=-x \\ x=-y \\ 1=-z \end{cases} \quad \begin{cases} 2x^2+1=1 \\ y=-x \\ y=-x \\ z=-1 \end{cases} \quad \begin{cases} x=0 \\ y=0 \\ z=-1 \end{cases}$$

Obtivemos de novo o ponto B

$f(A) = 1$, $f(B) = -1$

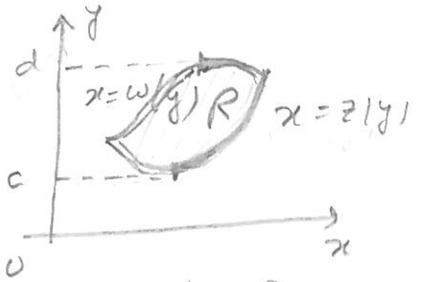
Então $\max f|_K = 1$ e $\min f|_K = -1$

Integrais duplos



$$\iint_R f(x,y)\,d(x,y) = \int_a^b \int_{\psi(x)}^{\varphi(x)} f(x,y)\,dy\,dx$$

$$= \int_c^d \int_{\omega(y)}^{z(y)} f(x,y)\,dx\,dy$$



Interpretação geométrica do integral

Se $f : R \longrightarrow \mathbb{R}$ é positiva

$$\iint_R f(x,y)\,d(x,y) = \text{Volume do sólido de base } R \text{ e altura } f(x,y) \text{ em cada ponto } (x,y) \in R$$

$$\iint_R 1\,d(x,y) = \text{volume do cilindro de base } R \text{ e altura } 1.$$

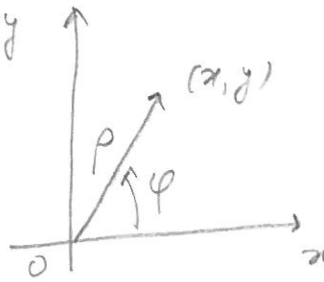
Fórmula de mudança de variável

- $\Phi : D \longrightarrow R$, D, R abertos de $\mathbb{R}^2$
- Φ função bijetiva (então $D = \Phi^{-1}(R)$) de classe C^1.

NOTA: $\Phi(u,v) = (x,y)$ ou, equivalentemente, $(u,v) = \Phi^{-1}(x,y)$.

$$\boxed{\iint_R f(x,y)\,d(x,y) = \iint_{\Phi^{-1}(R)} |\det J_{(u,v)}\Phi|\; f(\Phi(u,v))\,d(u,v)}$$

Coordenadas polares



$$\begin{cases} x = \rho\cos\varphi \\ y = \rho \operatorname{sen}\varphi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \operatorname{tg}\varphi = y/x \quad \text{se } x \neq 0 \end{cases}$$

$\rho \in \mathbb{R}^+$, $\varphi \in [0, 2\pi[$

$$\Phi : \mathbb{R}^+ \times [0,2\pi[\longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$
$$(\rho,\varphi) \longmapsto (\underbrace{\rho\cos\varphi}_{= x}, \underbrace{\rho\operatorname{sen}\varphi}_{= y})$$

$$|\det J_{(\rho,\varphi)}\Phi| = \rho$$

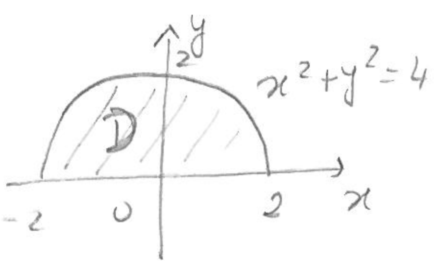
$$\boxed{\iint_R f(x,y)\, d(x,y) = \iint_{\Phi^{-1}(R)} \rho\, f(\rho\cos\varphi, \rho\operatorname{sen}\varphi)\, d(\rho,\varphi)}$$

Exemplo

$D = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4,\ y \geq 0 \}$

Em coordenadas cartesianas

$$\iint_D x\, d(x,y) = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} x\, dy\, dx$$



$$= \int_{-2}^{2} \Big[x y \Big]_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= -\frac{1}{2} \int_{-2}^{2} (-2x)(4-x^2)^{1/2}\, dx$$

$$= \left[-\frac{1}{2} \frac{(4-x^2)^{3/2}}{3/2} \right]_{-2}^{2} = \left[-3\sqrt{(4-x^2)^3} \right]_{-2}^{2}$$

$$= 0 - 0 = 0$$

Em coordenadas polares

$$\begin{cases} x^2+y^2\leq 4 \\ y\geq 0 \end{cases} \quad \begin{cases} \rho^2\leq 4 \\ \rho\,\text{sen}\varphi\geq 0 \end{cases} \quad \begin{cases} 0\leq\rho\leq 2 \\ \varphi\in[0,\pi] \end{cases}$$

$$\iint_D x\,d(x,y) = \int_0^{\pi}\int_0^2 \rho\cdot\rho\cos\varphi\,d\rho\,d\varphi = \int_0^{\pi}\left[\rho^3/_3 \cos\varphi\right]_{\rho=0}^{\rho=2} d\varphi$$

$$= \frac{8}{3}\left[-\text{sen}\varphi\right]_0^{\pi} = \frac{8}{3}(-0+0)=0$$

Integrais triplos

Dada uma função $f: V\to\mathbb{R}$ integrável, V aberto do $\mathbb{R}^3$, $f\geq 0$, a interpretação geométrica de $\iiint_V f(x,y,z)\,d(x,y,z)$ é o volume do "hipersólido" (em dimensão 4) com base V e altura, em (x,y,z), dada por $f(x,y,z)$.

Tal como com os integrais duplos, o valor do integral não depende da ordem que escolhemos para integrar. No caso dos integrais triplos há 6 escolhas possíveis, a saber: $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, $dz\,dy\,dx$.

Exemplo

$$V=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2\geq z^2,\ x^2+y^2\leq 1,\ z\geq 0\}$$

$$\iiint_V z\,d(x,y,z)$$

Variação total de z

$z\geq 0$, $z^2\leq x^2+y^2\leq 1 \Rightarrow z^2\leq 1$

Então $0\leq z\leq 1$

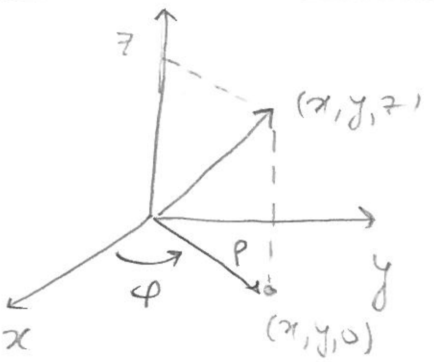
Mudança de variável

$\phi : \mathcal{U} \longrightarrow \mathcal{V}$ função bijetiva de classe C^1, $\mathcal{U}, \mathcal{V}$ abertos de $\mathbb{R}^3$

(então $\phi^{-1}(\mathcal{V}) = \mathcal{U}$). <u>NOTA</u>: $\phi(u,v,w) = (x,y,z)$, $(u,v,w) = \phi^{-1}(x,y,z)$

$$\boxed{\iiint_{\mathcal{V}} f(x,y,z)\, d(x,y,z) = \iiint_{\phi^{-1}(\mathcal{V})} |\det J_{(u,v,w)} \phi| \, f(\phi(u,v,w))\, d(u,v,w)}$$

Coordenadas cilíndricas



$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\,\mathrm{sen}\,\varphi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \mathrm{tg}\,\varphi = y/x \quad \text{se } x \neq 0 \\ z = z \end{cases}$$

$\rho \in \mathbb{R}^+$, $\varphi \in [0, 2\pi[$, $z \in \mathbb{R}$

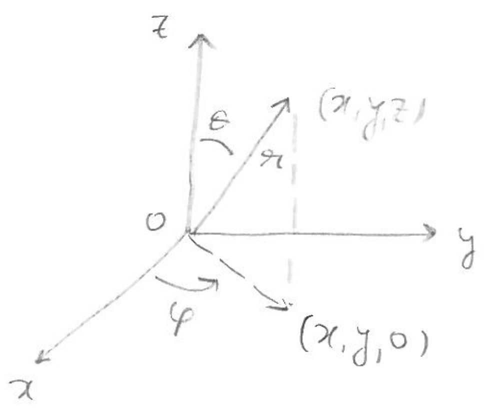
$$\phi : \mathbb{R}^+ \times [0,2\pi[\times \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$(\rho, \varphi, z) \longmapsto (\underbrace{\rho\cos\varphi}_{= x}, \underbrace{\rho\,\mathrm{sen}\,\varphi}_{= y}, \underbrace{z}_{= z})$$

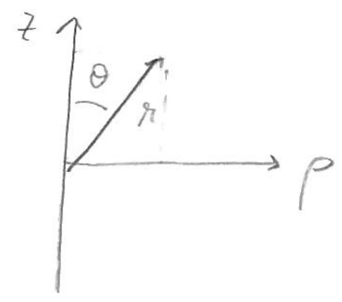
$$|\det J_{(\rho,\varphi,z)} \phi| = \rho$$

$$\boxed{\iiint_{D} f(x,y,z)\, d(x,y,z) = \iiint_{\phi^{-1}(D)} \rho\, f(\rho\cos\varphi, \rho\,\mathrm{sen}\,\varphi, z)\, d(\rho,\varphi,z)}$$

Coordenadas esféricas



No semi-plano φ = constante



$\rho = r\,\mathrm{sen}\,\theta \quad , \quad z = r\cos\theta$

Como $x = \rho\cos\varphi$ e $y = \rho\,\mathrm{sen}\,\varphi$, então

$$\begin{cases} x = r\cos\varphi\,\mathrm{sen}\,\theta \\ y = r\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \mathrm{tg}\,\varphi = y/x \quad \text{se } x \neq 0 \\ \cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

$r \in \mathbb{R}^+, \ \varphi \in [0, 2\pi[\ , \ \theta \in [0, \pi]$

$$\begin{aligned} \Phi : \mathbb{R}^+ \times [0, 2\pi[\times [0, \pi] &\longrightarrow \mathbb{R}^3 \\ (r, \varphi, \theta) &\longmapsto (r\cos\varphi\,\mathrm{sen}\,\theta, r\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta, r\cos\theta) \end{aligned}$$

$$|\det J_{(r,\varphi,\theta)}\Phi| = r^2\,\mathrm{sen}\,\theta$$

$$\iiint_D f(x,y,z)\,d(x,y,z) = \iiint_{\Phi^{-1}(D)} r^2\,\mathrm{sen}\,\theta\, f(r\cos\varphi\,\mathrm{sen}\,\theta, r\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta, r\cos\theta)\,d(r,\varphi,\theta)$$

Exemplo

$$V = \{ (x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ 0 \leq z \leq 1,\ z^2 \geq x^2 + y^2 \}$$

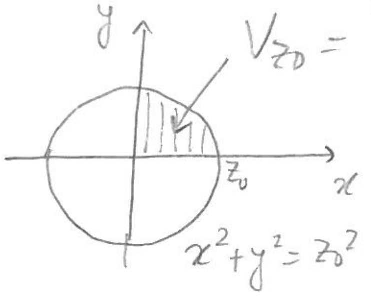
$$\iiint_V x \, d(x,y,z)$$

Coordenadas cartesianas

Variação total de z: $0 \leq z \leq 1$

Corte por $z = z_0$ $(0 \leq z_0 \leq 1)$

$$V_{z_0} = V \cap \{ (x,y,z) \in \mathbb{R}^3 : z = z_0 \}$$



$$\iiint_V x \, d(x,y,z) = \int_0^1 \left(\iint_{V_z} x \, d(x,y) \right) dz$$

$$= \int_0^1 \int_0^z \int_0^{\sqrt{z^2-x^2}} x \, dy \, dx \, dz = \int_0^1 \int_0^z \Big[xy \Big]_{y=0}^{y=\sqrt{z^2-x^2}} dx \, dz$$

$$= -\frac{1}{2} \int_0^1 \int_0^z (-2)x \, (z^2 - x^2)^{1/2} \, dx \, dz = -\frac{1}{2} \int_0^1 \left[\frac{(z^2-x^2)^{3/2}}{3/2} \right]_{x=0}^{x=z} dz$$

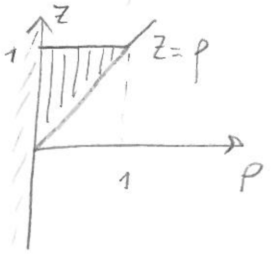
$$= -\frac{1}{2} \int_0^1 \left(0 - \frac{2}{3} (z^2)^{3/2} \right) dz = \frac{1}{3} \int_0^1 z^3 \, dz = \left[\frac{1}{3} \frac{z^4}{4} \right]_0^1 = \frac{1}{12}$$

Coordenadas cilíndricas

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 0 \leq z \leq 1 \\ z^2 \geq x^2+y^2 \end{cases} \quad \begin{cases} \rho\cos\varphi \geq 0 \\ \rho\,\text{sen}\varphi \geq 0 \\ 0 \leq z \leq 1 \\ z^2 \geq \rho^2 \end{cases} \quad \begin{cases} \cos\varphi \geq 0 \\ \text{sen}\varphi \geq 0 \end{cases} \quad \Big\} \; 0 \leq \varphi \leq \pi/2$$

$$\begin{cases} 0 \leq z \leq 1 \\ z \geq \rho \end{cases}$$

Corte por $\varphi =$ constante



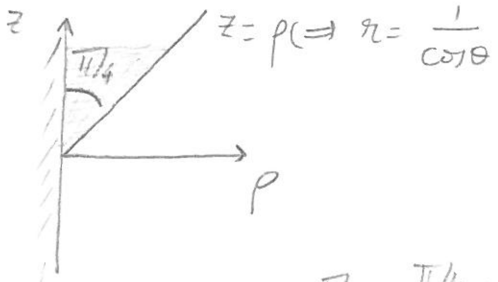
$$\iiint_V x \, d(x,y,z) = \int_0^{\pi/2}\int_0^1\int_0^z \rho \cdot \rho\cos\varphi \, d\rho \, dz \, d\varphi$$

$$= \int_0^{\pi/2}\int_0^1 \left[\cos\varphi \, \frac{\rho^3}{3}\right]_{\rho=0}^{\rho=z} dz\,d\varphi = \int_0^{\pi/2}\int_0^1 \cos\varphi \, \frac{z^3}{3} \, dz\,d\varphi$$

$$= \int_0^{\pi/2}\left[\cos\varphi \, \frac{z^4}{12}\right]_{z=0}^{z=1} d\varphi = \frac{1}{12}\left[\text{sen}\varphi\right]_0^{\pi/2} = \frac{1}{12}$$

Coordenadas esféricas

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 0 \leq z \leq 1 \\ z^2 \geq x^2+y^2 \end{cases} \quad \begin{cases} 0 \leq \varphi \leq \pi/2 \\ 0 \leq z \leq 1 \\ \rho \leq z \end{cases} \quad \begin{cases} 0 \leq \varphi \leq \pi/2 \\ 0 \leq r\cos\theta \leq 1 \iff r \leq 1/\cos\theta \\ \cancel{r}\,\text{sen}\theta \leq \cancel{r}\cos\theta \iff 0 \leq \theta \leq \pi/4 \end{cases}$$



$$\iiint_V x \, d(x,y,z) = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{1/\cos\theta} r^2\,\text{sen}\theta \cdot r\cos\varphi\,\text{sen}\theta \, dr \, d\theta \, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\left[\text{sen}^2\theta\cos\varphi \, \frac{r^4}{4}\right]_{r=0}^{r=\frac{1}{\cos\theta}} d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{4}\,\frac{\text{sen}^2\theta}{\cos\theta}\cos\varphi \, d\theta\,d\varphi$$

$= \ldots$

(não acabei os cálculos porque dão trabalho, não vale a pena).