3

Exercício 3.1

f[x_, y_] = x^2 y;

$\partial_x f(0,0) = 0$

$\partial_x f(x_0, y_0) = 2\, x_0\, y_0$

$\partial_y f(1,2) = 1$

$\partial_y f(x_0, y_0) = x_0^2$

Exercício 3.2

f[x_, y_] = 3 x^2 + 2 y^2;

$\partial_x f = 6\, x$

$\partial_y f = 4\, y$

f[x_, y_] = Sin[x^2 - 3 x y];

$\partial_x f = (2\, x - 3\, y)\, \mathrm{Cos}\left[x^2 - 3\, x\, y\right]$

$\partial_y f = -3\, x\, \mathrm{Cos}\left[x^2 - 3\, x\, y\right]$

f[x_, y_] = x^2 y^2 Exp[2 x y];

$\partial_x f = 2\, e^{2\,x\,y}\, x\, y^2 + 2\, e^{2\,x\,y}\, x^2\, y^3$

$\partial_y f = 2\, e^{2\,x\,y}\, x^2\, y + 2\, e^{2\,x\,y}\, x^3\, y^2$

f[x_, y_] = Exp[x] Log[x y];

$\partial_x f = \frac{e^x}{x} + e^x\, \mathrm{Log}[x\, y]$

$\partial_y f = \frac{e^x}{y}$

f[x_, y_] = Exp[Sin[x √y]];

$\partial_x f = e^{\mathrm{Sin}\left[x\sqrt{y}\right]}\, \sqrt{y}\, \mathrm{Cos}\left[x\, \sqrt{y}\right]$

$\partial_y f = \frac{e^{\mathrm{Sin}\left[x\sqrt{y}\right]}\, x\, \mathrm{Cos}\left[x\, \sqrt{y}\right]}{2\, \sqrt{y}}$

$f[x\_,\ y\_] = \frac{x^2 + y^2}{x^2 - y^2};$

$$\partial_x f = -\frac{4\,x\,y^2}{\left(x^2 - y^2\right)^2}$$
$$\partial_y f = \frac{4\,x^2\,y}{\left(x^2 - y^2\right)^2}$$

```
f[x_, y_] = x Cos[x] Cos[y];
```

$\partial_x f =$ Cos[y] (Cos[x]-x Sin[x])
$\partial_y f =$ -x Cos[x] Sin[y]

```
f[x_, y_] = ArcTan[x^2 y^3];
```

$$\partial_x f = \frac{2\,x\,y^3}{1 + x^4\,y^6}$$
$$\partial_y f = \frac{3\,x^2\,y^2}{1 + x^4\,y^6}$$

```
f[x_, y_] = x + x y^2 + Log[Sin[x^2 + y]];
```

$$\partial_x f = 1 + y^2 + 2\,x\,\mathrm{Cot}\left[x^2 + y\right]$$
$$\partial_y f = 2\,x\,y + \mathrm{Cot}\left[x^2 + y\right]$$

```
f[x_, y_, z_] = z Exp[x^2 + y^2];
```

$$\partial_x f = 2\,e^{x^2+y^2}\,x\,z$$
$$\partial_y f = 2\,e^{x^2+y^2}\,y\,z$$
$$\partial_z f = e^{x^2+y^2}$$

```
f[x_, y_, z_] = Log[Exp[x] + z^y];
```

$$\partial_x f = \frac{e^x}{e^x + z^y}$$
$$\partial_y f = \frac{z^y\,\mathrm{Log}[z]}{e^x + z^y}$$
$$\partial_z f = \frac{y\,z^{-1+y}}{e^x + z^y}$$

$$f[x\_, y\_, z\_] = \frac{x\,y^3 + \mathrm{Exp}[z]}{x^3\,y - \mathrm{Exp}[z]};$$

$$\partial_x f = -\frac{y\left(2\,x^3\,y^3 + e^z\left(3\,x^2 + y^2\right)\right)}{\left(e^z - x^3\,y\right)^2}$$
$$\partial_y f = \frac{2\,x^4\,y^3 - e^z\,x\left(x^2 + 3\,y^2\right)}{\left(e^z - x^3\,y\right)^2}$$
$$\partial_z f = \frac{e^z\,x\,y\left(x^2 + y^2\right)}{\left(e^z - x^3\,y\right)^2}$$

Exercício 3.3

a)

$\partial_x f(0,0) = 0$
$\partial_y f(0,0) = 0$

b)

$\partial_x f(0,0) = 0$
$\partial_y f(0,0) = 0$

Exercício 3.4

a)

```
f[x_, y_] = Exp[x y];
```

$x\partial_x f = e^{x y}\, x\, y$
$y\partial_y f = e^{x y}\, x\, y$

b)

```
f[x_, y_] = Log[x^2 + y^2 + x y];
```

$$x\partial_x f = \frac{x\,(2\,x + y)}{x^2 + x\,y + y^2}$$
$$y\partial_y f = \frac{y\,(x + 2\,y)}{x^2 + x\,y + y^2}$$

$x\partial_x f \ + \ y\partial_y f \ = \ 2$

c)

$$f[x\_,\ y\_,\ z\_] = x + \frac{x - y}{y - z};$$

$$\partial_x f = 1 + \frac{1}{y - z}$$
$$\partial_y f = \frac{-x + z}{(y - z)^2}$$
$$\partial_z f = \frac{x - y}{(y - z)^2}$$

$\partial_x f \ + \ \partial_y f \ + \ \partial_z f \ = 1$

Exercício 3.5

$$\text{Ddirecao}[f\_,\ P\_,\ u\_] := \text{Limit}\left[\frac{f\ @@\ (P + h\,u)\ -\ f\ @@\ P}{h},\ h \to 0\right];$$

a)

```
f[x_, y_] = x^2 y + x; P = {1, 0}; u = {1, 1};
```

```
Ddirecao[f, P, Normalize[u]]
```

$\sqrt{2}$

b)

$$f[x\_,\ y\_] = x^2\ \text{Sin}[2\,y];\ P = \left\{1,\ \frac{\pi}{2}\right\};\ u = \{3,\ -4\};$$

```
Ddirecao[f, P, Normalize[u]]
```

$\frac{8}{5}$

c)

```
f[x_, y_, z_] = x^2 + y^2 + z^2; P = {1, 2, 3}; u = {1, 1, 1};

Ddirecao[f, P, Normalize[u]]
```

$4\sqrt{3}$

Exercício 3.6

a)

```
Grad[x Exp[-x + y], {x, y}]
```

$\{e^{-x+y} - e^{-x+y} x, e^{-x+y} x\}$

b)

```
Grad[x Exp[-x^2 - y^2 - z^2], {x, y, z}]
```

$\left\{e^{-x^2-y^2-z^2} - 2 e^{-x^2-y^2-z^2} x^2, -2 e^{-x^2-y^2-z^2} x y, -2 e^{-x^2-y^2-z^2} x z\right\}$

c)

```
Grad[x y z / (x^2 + y^2 + z^2 + 1), {x, y, z}] // Simplify
```

$\left\{\frac{y z \left(1 - x^2 + y^2 + z^2\right)}{\left(1 + x^2 + y^2 + z^2\right)^2}, \frac{x z \left(1 + x^2 - y^2 + z^2\right)}{\left(1 + x^2 + y^2 + z^2\right)^2}, \frac{x y \left(1 + x^2 + y^2 - z^2\right)}{\left(1 + x^2 + y^2 + z^2\right)^2}\right\}$

d)

```
Grad[z^2 Exp[x] Cos[y], {x, y, z}] // Simplify
```

$\{e^x z^2 \mathrm{Cos}[y], -e^x z^2 \mathrm{Sin}[y], 2 e^x z \mathrm{Cos}[y]\}$

Exercício 3.7

```
planoTangente[f_, a_, b_] := f[a, b] + Derivative[1, 0][f][a, b] (x - a) + Derivative[0, 1][f][a, b] (y - b);

f[x_, y_] = x^2 + y^3;
```

$z = -11 + 6 x + 3 y$

Exercício 3.8

```
f[x_, y_] = x^2 + y^2;
```

$z = 0$

```
g[x_, y_] = -x^2 - y^2 + x y^3;
```

```
z = 0
```

Exercício 3.9

```
f[x_, y_] = Exp[x - y];
```

```
z = 1 + x - y
```

```
Solve[z == planoTangente[f, 1, 1] && x == 0 && y == 0, {x, y, z}]
```

```
{{x → 0, y → 0, z → 1}}
```

Exercício 3.10

Ver diapositivos

Exercício 3.11

Ver diapositivos

Exercício 3.12

a)

$$f[x\_, y\_] = \frac{x^2 y}{x^2 + y^2};$$

```
f[0, 0] = 0;
```

```
Simplify[Ddirecao[f, {0, 0}, {u1, u2}], Element[u1 | u2, Reals]]
```

$$\frac{u1^2\, u2}{u1^2 + u2^2}$$

```
Simplify[Ddirecao[f, {x, y}, {u1, u2}], Element[u1 | u2, Reals]]
```

$$\frac{x\left(2\, u1\, y^3 + u2\left(x^3 - x\, y^2\right)\right)}{\left(x^2 + y^2\right)^2}$$

b)

f não é diferenciável na origem

Exercício 3.13

a)

$$f[x\_, y\_] = \left(x^2 + y^2\right) \mathrm{Sin}\left[\frac{1}{\sqrt{x^2 + y^2}}\right];$$

```
f[0, 0] = 0;
```

$\partial_x f(0,0) = 0$

$\partial_y f(0,0) = 0$

b)

$$\partial_x f(x,y) = -\frac{x \operatorname{Cos}\left[\frac{1}{\sqrt{x^2+y^2}}\right]}{\sqrt{x^2+y^2}} + 2\,x \operatorname{Sin}\left[\frac{1}{\sqrt{x^2+y^2}}\right]$$

$$\partial_y f(x,y) = -\frac{y \operatorname{Cos}\left[\frac{1}{\sqrt{x^2+y^2}}\right]}{\sqrt{x^2+y^2}} + 2\,y \operatorname{Sin}\left[\frac{1}{\sqrt{x^2+y^2}}\right]$$