Lecture 30

We are going to start with integral calculus.

Suppose we know $f'(x)$. How can we find $f(x)$?
(Recall Prob. 2 from Assignment 3).

Definition A function $F(x)$ is called an anti-derivative of $f(x)$ if

$$F'(x) = f(x).$$

e.g. If $f(x) = 3x^2$ then $F(x) = x^3$ is an antiderivative of $f(x)$. as $F'(x) = (x^3)' = 3x^2 = f(x)$.

But, x^4 is not the only anti-derivative.
as $G(x) = x^4 + 2$, $H(x) = x^4 + \pi$ etc. are all anti-derivatives of $f(x)$. Thus,

Any two anti-derivatives of $f(x)$ differ by a constant.

$\therefore$ the general antiderivative for $f(x) = 3x^2$ is

$F(x) = x^3 + c$ where $c \in \mathbb{R}$ is a constant.

Thus we can define the indefinite integral.

$$\underbrace{\int}_{\text{integral of } f(x)} \underbrace{f(x)\, dx}_{\text{with respect to } x} = \underbrace{F(x)}_{\text{any anti-derivative of } f(x)} + \underbrace{C}_{\text{constant}} \quad \text{——} \ (1)$$

(1) defines the indefinite integral of $f(x)$. So, e.g.

$$\int 3x^2\, dx = x^3 + c$$

Just like the derivatives, let's see rules for finding integrals.

Power Rule

If $n \neq -1$ then $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

If $n = -1$ then $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$

Since ln only makes sense for positive real numbers, that's why we write $\ln|x|$.

Addition/Substraction and coefficient rule

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx \quad \text{where } k \in \mathbb{R}$$

is a constant.

Let's do some questions.

1. $$\int 5x^4\,dx \underset{\text{coefficient rule}}{=} 5\int x^4\,dx \underset{\text{power rule}}{=} 5\,\frac{x^5}{5} + C = x^5 + C$$

2. $$\int x^5 - x^4 + 3x + \pi \; dx = \int x^5\,dx - \int x^4\,dx + 3\int x\,dx + \int \pi\,dx$$

$$= \frac{x^6}{6} + C - \frac{x^5}{5} + C + \frac{3x^2}{2} + C \underbrace{+\pi x + C}_{\text{as } \pi = \pi x^0}$$

$$= \frac{x^6}{6} - \frac{x^5}{5} + \frac{3x^2}{2} + \pi x + C$$

where we combined C's as another constant which we still denote by C.

3. $\int (x^2+1)^2 dx = \int (x^4 + 1 + 2\cdot x^2) dx$ $\quad [(a+b)^2 \text{ formula}]$

$$= \frac{x^5}{5} + x + \frac{2}{3}x^3 + C$$

4. $\int \sqrt{x} + x^{\frac{1}{3}} dx = \int x^{\frac{1}{2}} dx + \int x^{\frac{1}{3}} dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{4}x^{\frac{4}{3}} + C$$

(Use power rule on both)

5. $\int \frac{x^3+x+2}{x^2} dx = \int \frac{x^3}{x^2} + \frac{x}{x^2} + \frac{2}{x^2} dx$

$$= \int x\, dx + \int \frac{1}{x} dx + \int 2x^{-2} dx$$

$$= \frac{x^2}{2} + \ln|x| + \frac{2x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} + \ln|x| - 2x^{-1} + C$$

If we have some extra information then we can explicitly find the constant C.

Ques find $f(x)$ if $f'(x) = x^2+1$ and $f(0)=1$

Sol We know that

$$f(x) = \int f'(x)\,dx = \int x^2+1\,dx$$

$$= \frac{x^3}{3} + x + C$$

now $f(0)=1 \Rightarrow 1 = \frac{0^3}{3} + 0 + C \Rightarrow C = 1$

$$\therefore \boxed{f(x) = \frac{x^3}{3} + x + 1}$$

Ans