Recall :-

Integration by substitution

Strategy

① Let $u = \cdots$, find $du = \cdots$

② Replace dx and all x's by du and expressions in u.

③ Integrate

④ Substitute back the expression for u from ①.

Some useful substitutions

① $u =$ function inside a complicated power

$=$ argument of trig functions

$=$ exponent of e^x or a^x.

$=$ function whose derivative is also present.

However, there is <u>NO FIXED RULE</u> for substitution.

It varies from problem to problem.

Ques.

(1) $\int e^{x} \cos(e^{x})\, dx$

Suppose $u = e^{x}$

$\Rightarrow du = e^{x} dx$

$\therefore \int e^{x} \cos(e^{x})\, dx = \int \cos u \, du \quad (du = e^{x} dx)$

$= \sin u + C$

$= \sin(e^{x}) + C$

Answer

Integrate

(1) $\int x^2 e^{x^3} dx$

(11) $\int \frac{x}{(x+1)^6} dx.$

Sol. (1) In e^{x^3}, suppose $u = x^3$

$\Rightarrow du = 3x^2 dx$

$\therefore \int x^2 \cdot e^{x^3} dx = \int \cancel{x^2} \cdot e^u \frac{du}{3\cancel{x^2}}$

$= \int \frac{e^u}{3} du = \frac{1}{3} \int e^u du$

$= \frac{1}{3} \cdot e^u + C$

$= \frac{e^{x^3}}{3} + C$

Answer.

(11) $\int \frac{x}{(x+1)^6} dx.$

Suppose $u = x+1 \Rightarrow \boxed{x = u-1}$

$\Rightarrow du = dx.$

$\int \frac{x}{(x+1)^6} dx = \int \frac{x}{u^6} du = \int \frac{u-1}{u^6} du$

$= \int \frac{u}{u^6} du - \int \frac{1}{u^6} du \quad \left(\frac{u-1}{u^6} = \frac{u}{u^6} - \frac{1}{u^6} \right.$

$= \int u^{-5} du - \int u^{-6} du$

$$= \frac{u^{-4}}{-4} + \frac{u^{-5}}{5} + C$$

$$= \frac{(x+1)^{-4}}{-4} + \frac{(x+1)^{-5}}{5} + C$$

Answer

Example

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx.$$

Suppose $u = \cos x.$

$\Rightarrow \quad du = -\sin x \, dx$

$$\therefore \int \tan x \, dx = \int \frac{-du}{u}$$

$$= -\ln|u| + C$$

$$\Rightarrow -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C$$

$$= \ln|\sec x| + C$$

Aside

$u = \sin x$

$du = \cos x \, dx$

X

↳ why choosing $u = \sin x$ won't work.

Ques $\displaystyle\int \frac{1}{x\ln x}\,dx = \int \frac{dx}{x}\cdot\frac{1}{\ln x}$

Assume $u = \ln x$

$\Rightarrow du = \frac{1}{x}\,dx \quad \left(\frac{d}{dx}(\ln x) = \frac{1}{x}\right)$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\ln x| + C$$

final answer

Geometric Interpretation of integrals;

Fundamental Theorem of Calculus

If $F(x)$ is an anti-derivative of $f(x)$, i.e.

$F'(x) = f(x)$

$\Rightarrow \int f(x) = F(x) + C$ ← indefinite integral.

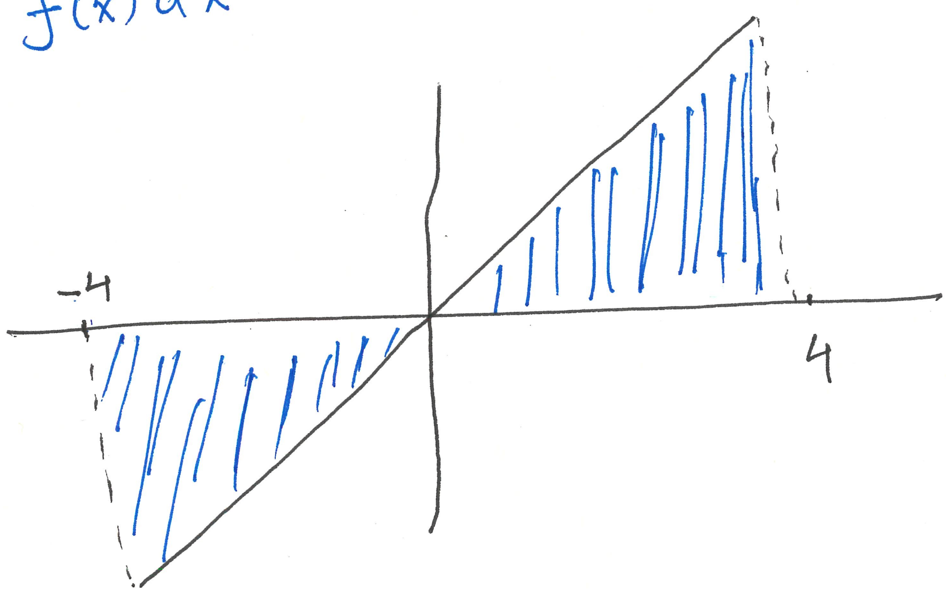
geometric interpretation for $f'(x)$ = slope of the tangent line to $f(x)$ at the point x.

Ques.

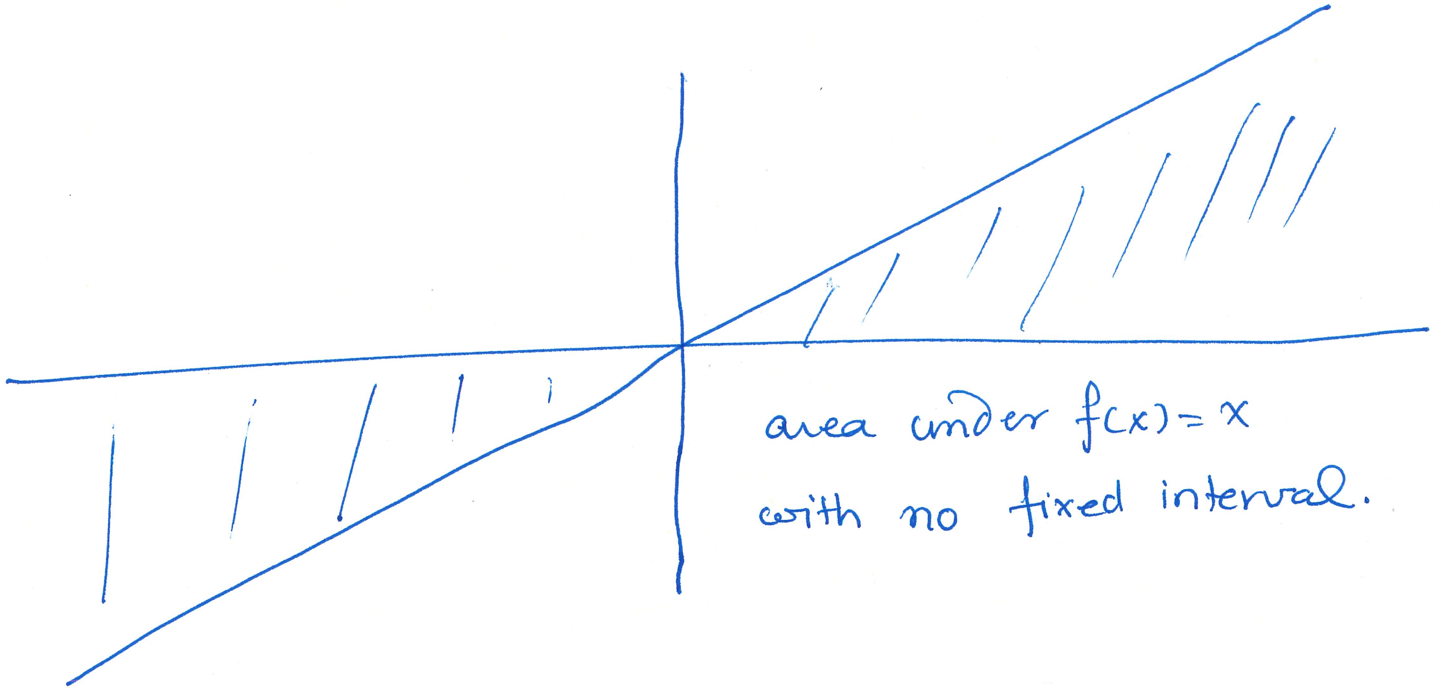
$\int f(x)\,dx$ ⤳ geometrically.

↳ area under the curve.

ex. $\int f(x)\,dx$ where $f(x) = x$. b/w $x=-4$ and $x=4$



Integration of a function is area under the curve of $f(x)$.



Definite Integral of $f(x)$ is just indefinite integral in a fixed interval.

$$\int_a^b f(x)\,dx$$ — notation for definite integral.

— integral of $f(x)$ between $x = a$ and $x = b$.

$$\int_{-4}^{4} x\,dx$$ = integral of $f(x) = x$ b/w $x = -4$ and $x = 4$

= area under the curve of $f(x) = x$ b/w $x = -4$ and 4.

Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$, i.e., $F'(x) = f(x)$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b}$$

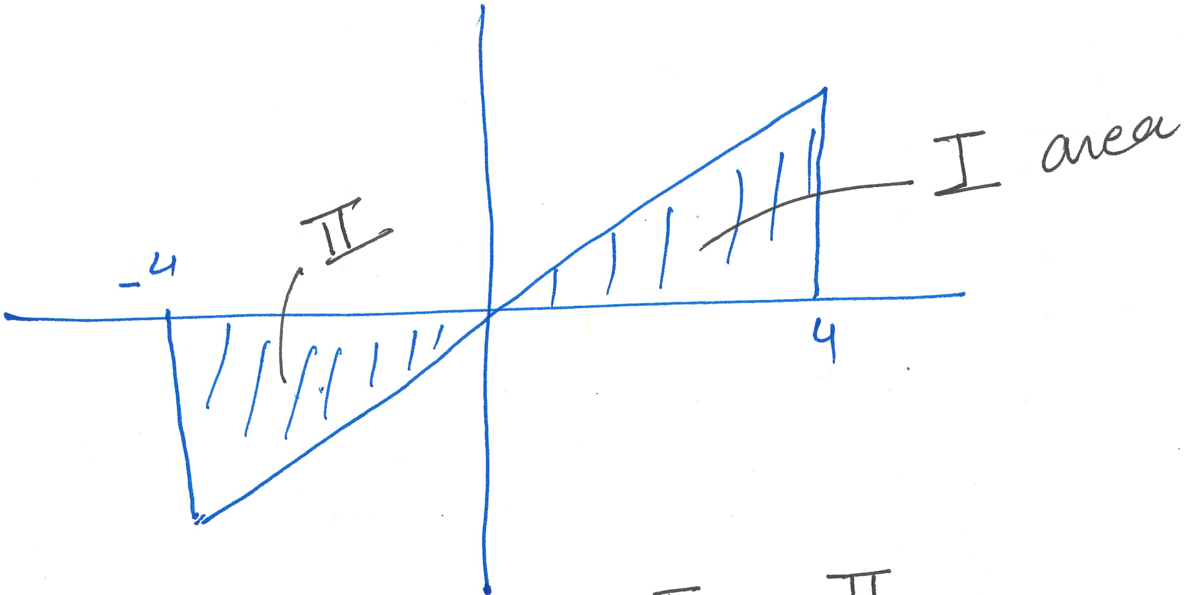
$F(b)$ = evaluate $F(x)$ at $x=b$.

$$\int_{-4}^{4} x\,dx = \left[\frac{x^2}{2} + c\right]_{-4}^{4}$$

$$= \left(\frac{4^2}{2} + c\right) - \left(\frac{(-4)^2}{2} + c\right)$$

$$= \frac{16}{2} + C - \left(\frac{16}{2} + c\right)$$

$$= \cancel{\frac{16}{2}} + \cancel{c} - \cancel{\frac{16}{2}} - \cancel{c} = 0$$

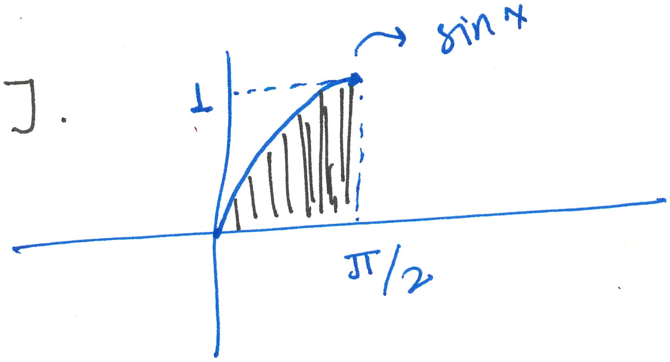


I = II

total area under the curve = 0

- Definite integral is "signed" area under the curve.

Ques: Find the area under $f(x) = \sin x$ between $[0, \frac{\pi}{2}]$.



Area under $f(x)$ b/w $x = a$ and $x = b$

$$= \int_a^b f(x)\,dx$$

$$= \int_0^{\pi/2} \sin x \, dx$$

$$= \left(-\cos x\right)_0^{\pi/2}.$$

→ we drop $+C$ because when $F(b) - F(a)$ then C will cancel.

$$= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos 0\right)$$

$$= -0 + 1 = \boxed{1}$$

Ques Find the integrals.

① $\int_0^1 x^3+x^2+2\,dx$ ② $\int_{+1}^{e} \frac{1}{x}\,dx$

Sol^n ① We have

$$\int_0^1 (x^3+x^2+2)\,dx = \left[\frac{x^4}{4}+\frac{x^3}{3}+2x\right]_0^1$$

$$= \left[\frac{1^4}{4}+\frac{1^3}{3}+2.1\right]-[0]$$

$$= \frac{1}{4}+\frac{1}{3}+2 = \frac{3+4+24}{12}$$

$$= \boxed{\frac{31}{12}}$$

② $\int_{+1}^{e} \frac{1}{x}\,dx = [\ln|x|]_1^e$

$$= [\ln|e|]-[\ln|1|]$$

$$= 1-0 = \boxed{1}$$

Remark :- We do not need $+C$ for definite integral as it will cancel in the end.