Lecture 31

We will continue with more rules for the integrals.

Integrals of Exponential functions

1. $\int e^x dx = e^x + C$

2. $\int e^{kx} dx = \frac{e^{kx}}{k} + C$

3. $\int a^x dx = \frac{a^x}{\ln a} + C$

4. $\int a^{kx} dx = \frac{a^{kx}}{k \cdot \ln(a)} + C$

Integration of some Trigonometric functions

1. $\int \sin x \, dx = -\cos x + C$

2. $\int \sin(kx) dx = -\frac{\cos(kx)}{k} + C$

3. $\int \cos x \, dx = \sin x + C$

4. $\int \cos(kx)\,dx = \dfrac{\sin(kx)}{k} + C$

5. $\int \sec^2 x\,dx = \tan x + C$

6. $\int \sec^2(kx)\,dx = \dfrac{\tan(kx)}{k} + C$

7. $\int \operatorname{cosec}^2(x)\,dx = -\cot x + C$

8. $\int \operatorname{cosec}^2(kx)\,dx = \dfrac{-\cot(kx)}{k} + C$

We'll find integrals of other trig functions later.

<u>Ques.</u> Find the following integrals.

1. $\int \cos(2x) + \frac{1}{x} - e^{3x}\,dx$

$$= \int \cos(2x)\,dx + \int \frac{1}{x}\,dx - \int e^{3x}\,dx$$

$$= \frac{\sin(2x)}{2} + \ln|x| - \frac{e^{3x}}{3} + C$$

2. $\int 10^x + \sin x - 2^{3x}\,dx = \dfrac{10^x}{\ln 10} - \cos x - \dfrac{2^{3x}}{3\ln 2} + C$

Ques. Find $f(x)$ if $f'(x) = x - \sec^2 x$ and $f(\pi) = 0$.

Sol. We know that

$$f(x) = \int f'(x)\,dx = \int (x - \sec^2 x)\,dx$$

$$= \frac{x^2}{2} - \tan x + C$$

now $f(\pi) = 0 \Rightarrow \frac{\pi^2}{2} - \tan\pi + C = 0$

$$\Rightarrow \frac{\pi^2}{2} - 0 + C = 0 \Rightarrow C = -\frac{\pi^2}{2}$$

$$\therefore \boxed{f(x) = \frac{x^2}{2} - \tan x - \frac{\pi^2}{2}}$$

Integration By Substitution

Let's see the substitution rule which is another method to find more complicated integrals.

We know by chain rule that

$$\left((x^2+1)^2\right)' = 2(x^2+1)\cdot 2x = 4x\,(x^2+1)$$

$$\Rightarrow \int 4x\,(x^2+1)\,dx = (x^2+1)^2 + C$$

Can we find this using some other method.

Let's substitute $u = x^2+1$ in $\int 4x(x^2+1)\,dx$

then $du = 2x\,dx$

$\Rightarrow \dfrac{du}{2x} = dx$

$$\therefore \int 4x\,(x^2+1)\,dx = \int \cancel{4x}^{\,2} \cdot u \cdot \frac{du}{\cancel{2x}} = \int 2u\,du$$

$$= 2 \cdot \int u\,du$$

$$= 2 \cdot \frac{u^2}{2} + C = u^2 + C$$

$$= (x^2+1)^2 + C \quad \left[\text{put back } u = (x^2+1)\right]$$

So we substituted a new variable u for x^2+1 in our original integral and the process became easier.

This, precisely, is the substitution rule.

e.g. $\int x^2\sqrt{x^3+1}\,dx$.

We cannot find this integral by the rules described till now.

The problematic term is x^3+1 so we let

$$u = x^3+1$$

$$\Rightarrow \quad du = 3x^2\,dx \quad \Rightarrow \quad \frac{du}{3} = x^2\,dx$$

$$\therefore \int x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\;x^2.dx = \int \frac{\sqrt{u}\,du}{3}$$

$$= \frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{1}{3}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{9}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}} + C$$

— Answer

General Strategy

- Assume u = some function
- find $du = \ldots dx$
- Replace all x and dx by expressions of u and du.
- integrate and put back the expression of u.

Common Substitution Rules

1. u = function inside ugly power
2. u = function inside sin, cos etc.
3. u = function inside exponential
4. u = function and its derivative are both present in integral.