Honors Stat Mech
Project 4

Physics 423 Spring 2017

Due: Thurs, Mar. 30

Self-Organized Criticality (SOC) and the Abelian Sandpile Model. Sandpiles exhibit interesting behavior. When you add sand a little bit at a time, it might make the pile taller, or it might trigger a small avalanche, or it might trigger a large avalanche. It turns out that sandpiles and various other systems in nature seek out *critical states*, where the response to small perturbations can have essentially any size: there is no "characteristic size" for the amplitude of the response to the perturbation. Operationally, critical behavior is defined by *power law* dependences of observables on scales. In the sandpile case, the distribution in the size of avalanches $D(s) \sim s^{-\alpha}$ for some *critical exponent* α.

Many systems that we study in physics are *not* critical, and observables depend on sizes, distances, and times exponentially instead of in a power law. For example, in a magnetic system at finite temperature, we could measure the correlation between two spins separated by a distance r. Such correlations typically fall off exponentially at long distances: $C(r) \sim r^n e^{-r/\xi}$, where ξ is called the *correlation length* and n is a critical exponent. ξ sets the typical or the "maximum" range of the correlations. However, by tuning parameters, it is sometimes possible to make ξ diverge, so that instead the behavior is governed for all r by the power law r^n. In this case it is sometimes said that "fluctuations occur over all length scales," meaning that the state of one degree of freedom is not exponentially insensitive to the state of another far-away degree of freedom. (Or, connecting to the avalanche model, the state of one degree of freedom is not exponentially insensitive to perturbations made to distant degrees of freedom.) Such points in parameter space are called critical points, and the terminology is closely connected to critical points in the context of phase transitions, as we will see later. In magnets, critical behavior can sometimes be achieved by tuning the temperature of the system to a special value T_C. If the correlation length behaves as $\xi \sim |T - T_C|^{-\nu}$, then long-range correlations are obtained as T approaches T_C. (The exponent becomes small, so that $e^{-r/\xi} \approx 1$.)

When criticality requires parameter tuning, it is not particularly natural, in the sense that a random choice of the parameter will not be near the critical point. It is therefore remarkable that some systems approach criticality naturally: criticality is said to be "self-organized." In this project, you will simulate a simple toy model for avalanches in sandpiles, with no continuous free parameters, and show that the model nevertheless tends toward critical behavior: after some time has passed, the distribution of avalanche sizes follows a power law, just as in real sandpiles.

An abelian sandpile model is a model for defined by three types of rules: an adding rule (in which a "grain" is added to a square on a grid), a toppling rule (how avalanches start and evolve when there is too much sand on a square), and a boundary rule (what to do with avalanches occurring near the edge of the grid.) Take the following rules, as in Bak 1998:

- Choose a random square on a finite square grid. Increment the amount of sand $Z(x, y)$ on the square by one unit, $Z(x, y) \to Z(x, y) + 1$.
- If $Z(x, y) = 4$, topple the square. Set $Z(x, y) \to 0$ and redistribute the four grains, one grain at each of the nearest neighbor sites. In the process, some of the neighbors might reach four grains. Apply this toppling rule to each site on the grid repeatedly until it is finished (no squares have four grains).
- If toppling occurs on the boundary, remove from the simulation any sand that would be redistributed off the edge.
- Repeat from the top, adding another grain of sand to a random point.

Implement these rules into a simulation and run it for a long time (5000 steps or more on a 50x50 grid if you can). Each time you add a grain, count how many topplings it initiates. This is your data, representing the "size of avalanches" s. Show that the distribution of sizes $D(s)$ follows a power law with exponent approximately -1, $D(s) \sim s^{-1}$. If you start with no sand on the board, it takes quite awhile for the system to approach the critical state, so you might drop the first 1000 steps or so before fitting the distribution to the power law. **OPTIONAL:** Add a plotting feature so that you can view the state of the system during or after an avalanche. (For example, make a density plot.) Then modify your simulation to add sand only at a central point of the grid, rather than randomly. Repeat a bunch of avalanches and see how the system looks after each one. Try to animate the frames of an avalanche.