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A Unified Approach to Construct Generalized B-Splines for Isogeometric Applications*

XU Gang · SUN Ningning · XU Jinlan · HUI Kin-Chuen · WANG Guozhao

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Abstract Generalized B-splines have been employed as geometric modeling and numerical simulation tools for isogeometric analysis (IGA for short). However, the previous models used in IGA, such as trigonometric generalized B-splines or hyperbolic generalized B-splines, are not the unified mathematical representation of conics and polynomial parametric curves/surfaces. In this paper, a unified approach to construct the generalized non-uniform B-splines over the space spanned by $\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$ is proposed, and the corresponding isogeometric analysis framework for PDE solving is also studied. Compared with the NURBS-IGA method, the proposed frameworks have several advantages such as high accuracy, easy-to-compute derivatives and integrals due to the non-rational form. Furthermore, with the proposed spline models, isogeometric analysis can be performed on the computational domain bounded by transcendental curves/surfaces, such as the involute of circle, the helix/helicoid, the catenary/catenoid and the cycloid. Several numerical examples for isogeometric heat conduction problems are presented to show the effectiveness of the proposed methods.

Keywords Generalized B-splines, heat conduction, isogeometric analysis, unified construction.

1 Introduction

Recently seamless integration of computer aided design (CAD for short) and computer aided engineering (CAE for short) is one of the main challenge in the field of advanced manufacturing.

XU Gang (Corresponding author) · SUN Ningning · XU Jinlan
Department of Computer Science, Hangzhou Dianzi University, Hangzhou 310018*, China; Key Laboratory of Complex Systems Modeling and Simulation, Ministry of Education, Hangzhou* 310018*, China.*
Email: gxu@hdu.edu.cn; snm-sun@163.com; jlxu@hdu.edu.cn.
HUI Kin-chuen
Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Hong Kong, China. Email: kchui@mae.cuhk.edu.hk.
WANG Guozhao
Department of Mathematics, Zhejiang University, Hangzhou 310027*, China.* Email: wanggz@zju.edu.cn.
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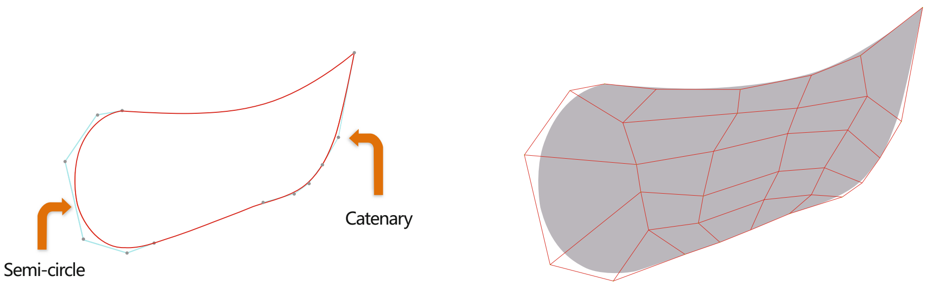
In order to realize the unified data representation in CAD and CAE, Hughes, et al.[1] proposed isogeometric analysis (IGA for short) approach to overcome the gap between CAD and finite element analysis by using the same mathematical language for all design and analysis tasks. The main advantages of IGA can be described as follows:

1) IGA avoids data exchanges between the design and analysis phases

2) IGA achieves more accurate solution by high-order approximation;

3) IGA achieves smooth solution field with high continuity;

4) IGA allows to perform refinement operations by knot insertion and degree elevation without changing the geometry.



(a) Given boundary curves

(b) Parameterization with generalized B-spline

Figure 1 Given boundary curves and the corresponding parameterization of computational domain

As a basic mathematic language of IGA, NURBS have become de facto standards in CAD field. However, since NURBS is a rational model , it is difficult to evaluate its derivatives and integrals. Moreover, the NURBS model cannot represent exactly transcendental curves such as the helix, the cycloid, the catenary and the exponential curve, which play an important role in mechanical engineering. In order to overcome the described disadvantages of NURBS models, several generalized B-splines in non-polynomial space, such as trigonometric generalized B-splines in the space span$\{\sin t, \cos t, 1, t, \cdots, t^{n-2}\}$ and hyperbolic generalized B-splines in the space span$\{\sinh t, \cosh t, 1, t, \cdots, t^{n-2}\}$, are proposed in geometric modeling and isogeometric analysis. In contrast with NURBS, they can achieve arc-length parameterization of conic sections such that equally spaced points in the parameter domain correspond to equally spaced points on the described curve. However, they are not the unified mathematical representation of conics and polynomial parametric curves/surfaces. As an example shown in Figure 1(a), given a computational domain Ω bounded by a semi-circle arc and a catenary in the same parametric direction, it is impossible to parameterize Ω only by trigonometric generalized B-splines or hyperbolic generalized B-splines. In order to parameterize Ω exactly, generalized B-splines in the space spanned by $\{\sin t, \cos t, \sinh t, \cosh t, 1, t, \cdots, t^{n-4}\}$, have to be considered (Figure 1(b)). Furthermore, if other special curves are involved, such as cardioid or involute of circle, $\sin(\omega t)$ and $t \sin t$ has to be introduced into the spline space. Hence, how to construct the generalized non-uniform B-spline models in these spaces, is an important and challenging problem. In this paper, we investigated this problem and apply the new spline models in isogeometric analysis. Our main contributions can be described as follows:

1) We propose a unified approach to construct the generalized non-uniform B-splines in the space $\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$. All the cases of knot vectors can be included by using matrix determinant form.

2) Isogeometric analysis framework with the proposed generalized B-spline models is proposed for PDE solving. The proposed frameworks have several advantages such as high accuracy, easy-to-compute derivatives and integrals due to the non-rational form, and some remarkable transcendental curves such as the helix, the cycloid and the catenary can be involved in the unified framework.

3) Several numerical examples have been tested on isogeometric heat conduction problems to illustrate the efficiency of proposed IGA methods with generalized B-spline models.

The rest of the paper is organized as follows. Section 2 reviews the related work in isogeometric analysis and generalized B-splines. In Section 3, a unified method to construct the generalized non-uniform B-splines over the space span$\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$ is presented. The isogeometric analysis framework based on the proposed splines is explored in Section 4. Several numerical examples are given and analyzed for two-dimensional heat conduction problem in Section 5. Section 6 concludes the paper with a summary and some future research topics.

2 Related Work

In this section, some related works in IGA and generalized B-splines will be reviewed.

Construction of generalized B-splines Several new kinds of bases are proposed in new space for geometric modeling in CAGD. Zhang[2] investigated C-curves in the space span$\{1, t, \cos t, \sin t\}$; Sânchez-Reyes[3] proposed a basis of the space spanned by $\{1, \sin t, \cos t, \cdots, \sin mt, \cos mt\}$. Some normalized B-bases for the spaces spanned by $\{1, t, \cos t, \sin t, \cos 2t, \sin 2t\}$, $\{1, t, t^2, \cos t, \sin t\}$, and $\{1, t, \cos t, \sin t, t\cos t, t\sin t\}$ are constructed in [4]. Lü, et al.[5] proposed the uniform hyperbolic polynomial B-splines curves in the space span$\{\sinh t, \cosh t, 1, t, \cdots, t^{n-2}\}$. Chen and Wang[6], Wang, et al.[7] constructed C-Bézier basis and the non-uniform algebraic trigonometric B-spline basis of the space spanned by $\{\sin t, \cos t, 1, t, \cdots, t^{n-2}\}$. Costantini, et al.[8] studied the approximation power, the existence of a normalized B-basis and the structure of a degree-raising process for the spline space spanned by $\{u(t), v(t), 1, t, \cdots, t^{n-2}\}$. As we know, there is few work about the unified construction of non-uniform generalized B-splines in the space span$\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$.

IGA with generalized B-splines Various spline tools have been introduced into the isogeometric analysis framework, such as NURBS[1], T-splines[9, 10], analysis-suitable T-splines[11], PHT-splines[12], LR-splines[13] and Powell-Sabin splines[14]. Manni, et al.[15] firstly used generalized B-splines in isogeometric analysis, and quasi-interpolation method can be suitably used to set Dirichlet boundary conditions[16]. Tension spline approximation method for advection dominated advection diffusion problem was studied in [17]. The multilevel representations in terms of a hierarchy of tensor-product generalized B-splines and their application in IGA was investigated in [18]. Bracco, et al.[19] extended the T-spline approach to trigonometric generalized B-splines, and studied the generalized spline spaces over T-meshes[20]. The corresponding dimension formula and the definition of locally refined generalized B-splines are also given[20]. Recently, isogeometric collocation methods based on generalized B-splines is proposed and their performance through numerical examples are also studied[21].

In this paper, a unified approach to construct the generalized non-uniform B-splines in the space spanned by $\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$ will be presented, and the corresponding isogeometric analysis framework for PDE solving will be investigated.

3 Unified Approach for Constructing Generalized B-Splines

In this section, a unified approach to construct generalized B-splines in the space span$\{\alpha(t)$, $\beta(t)$, $\xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$ of order k will be given, which is the geometric foundation for isogeometric analysis framework.

3.1 The Construction of Generalized B-Spline Basis

3.1.1 Definition of Kernel Function

In the integration definition of generalized B-splines, kernel function is a key element for different spaces span$\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-5}\}$. Suppose that the kernel function $\kappa(t)$ is a linear combination of $\alpha(t), \beta(t), \xi(t)$, and $\eta(t)$, that is,

$$\kappa(t) = a\alpha(t) + b\beta(t) + c\xi(t) + d\eta(t). \tag{1}$$

The coefficients a, b, c, d will be determined from the conditions $\kappa(0) = \kappa'(0) = \kappa''(0) = 0$, and $\kappa'''(0) = 1$. That is,

$$\begin{cases} a\alpha(0) + b\beta(0) + c\xi(0) + d\eta(0) = 0, \\ a\alpha'(0) + b\beta'(0) + c\xi'(0) + d\eta'(0) = 0, \\ a\alpha''(0) + b\beta''(0) + c\xi''(0) + d\eta''(0) = 0, \\ a\alpha'''(0) + b\beta'''(0) + c\xi'''(0) + d\eta'''(0) = 1. \end{cases} \tag{2}$$

Then we have,

$$a = -\frac{\begin{vmatrix} \beta(0) & \xi(0) & \eta(0) \\ \beta'(0) & \xi'(0) & \eta'(0) \\ \beta''(0) & \xi''(0) & \eta''(0) \end{vmatrix}}{D}, \qquad b = \frac{\begin{vmatrix} \alpha(0) & \xi(0) & \eta(0) \\ \alpha'(0) & \xi'(0) & \eta'(0) \\ \alpha''(0) & \xi''(0) & \eta''(0) \end{vmatrix}}{D}, \tag{3}$$

$$c = -\frac{\begin{vmatrix} \alpha(0) & \beta(0) & \eta(0) \\ \alpha'(0) & \beta'(0) & \eta'(0) \\ \alpha''(0) & \beta''(0) & \eta''(0) \end{vmatrix}}{D}, \qquad d = \frac{\begin{vmatrix} \alpha(0) & \beta(0) & \xi(0) \\ \alpha'(0) & \beta'(0) & \xi'(0) \\ \alpha''(0) & \beta''(0) & \xi''(0) \end{vmatrix}}{D}, \tag{4}$$

in which

$$D = \begin{vmatrix} \alpha(0) & \beta(0) & \xi(0) & \eta(0) \\ \alpha'(0) & \beta'(0) & \xi'(0) & \eta'(0) \\ \alpha''(0) & \beta''(0) & \xi''(0) & \eta''(0) \\ \alpha'''(0) & \beta'''(0) & \xi'''(0) & \eta'''(0) \end{vmatrix}. \tag{5}$$

3.1.2 Basis Construction

Suppose that T is the knot vector $\{t_i\}_{i=-\infty}^{+\infty}$, and $\Omega_k[T]$ is the space span$\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-5}\}$ of order k.

We first define l_{i+j}, s_{i+j}, $j = 0, 1, 2, 3, 4$, such that

$$t_{i+j} = \cdots = t_{i+j+l_{i+j}-1} < t_{i+j+l_{i+j}}; \quad t_{i+j-s_{i+j}} < t_{i+j-s_{i+j}+1} = \cdots = t_{i+j}. \tag{6}$$

Then $f(t)$ is defined as follows

$$f(t) = \begin{cases} \kappa(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{7}$$

in which $\kappa(t)$ is the kernel function defined in Subsection 2.1. It is easy to check that $f(t)$ satisfies

$$f(0) = f'(0) = f''(0) = 0, \quad f'''(0) = 1. \tag{8}$$

That is, $f^{(i)}(t)$ has a zero of multiplicity $(3 - i)$ at 0, $i = 0, 1, 2, 3$.

Define

$$f_{i+j}(t) = \begin{cases} f^{(l_{i+j}-1)}(t - t_{i+j}), & 1 \leq j + l_{i+j} \leq 5, \\ f^{(4-j)}(t - t_{i+j}), & j + l_{i+j} > 5, \end{cases} \quad j = 0, 1, 2, 3, \tag{9}$$

where l_{i+j} is defined as in (6). Obviously, $f_{i+j}(t)$ has a zero of multiplicity $(4 - l_{i+j})$ at t_{i+j}.

We firstly define $N(t)$ as follows:

$$N(t) = \begin{vmatrix} f_i(t_{i+4}) & f_{i+1}(t_{i+4}) & f_{i+2}(t_{i+4}) & f_{i+3}(t_{i+4}) \\ f_i'(t_{i+4}) & f_{i+1}'(t_{i+4}) & f_{i+2}'(t_{i+4}) & f_{i+3}'(t_{i+4}) \\ f_i''(t_{i+4}) & f_{i+1}''(t_{i+4}) & f_{i+2}''(t_{i+4}) & f_{i+3}''(t_{i+4}) \\ f_i(t) & f_{i+1}(t) & f_{i+2}(t) & f_{i+3}(t) \end{vmatrix}. \tag{10}$$

Then we define

$$N_{i,4}(t) = \begin{cases} (-1)^{1+s_{i+4}} N(t), & t_i \leq t \leq t_{i+4}, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where s_{i+4} is defined as in (6).

Thus, a set of desired initial functions $N_{i,4}(t)$ is obtained with the continuity property as described in Theorem 1.

Theorem 1 *$N_{i,4}(t)$ is a piecewise polynomial over T and it has a zero of multiplicity $(4-l_i)$ at t_i, $1 \le l_i \le 4$, it is $(3-l_{i+j})$ times continuously differential at knots $t_{i+j}, 1 \le l_{i+j} \le 4, j = 1,2,3$, and it has a zero of multiplicity $(4-s_{i+4})$ at $t_{i+4}, 1 \le l_{i+j} \le 4$.*

Proof Obviously, $N_{i,4}(t)$ is a piecewise polynomial, and it has a zero of multiplicity $(4-l_i)$ at t_i, $1 \le l_i \le 4$, it is $(3-l_{i+j})$ times continuously differential at knots t_{i+j}, $1 \le l_{i+j} \le 4$, $j = 1,2,3$. We only need to prove that it has a zero of multiplicity $(4-s_{i+4})$ at t_{i+4}, $1 \le s_{i+4} \le 4$.

(i) $s_{i+4} = 1$, $N(t)$ has the same expression as in (10). From the property of determinant, we can obtain

$$N_{i,4}(t_{i+4}) = N'_{i,4}(t_{i+4}) = N''_{i,4}(t_{i+4}) = 0.$$

(ii) $s_{i+4} = 2$, that is, $t_i < t_{i+1} < t_{i+2} < t_{i+3} = t_{i+4}$, so $f_{i+3}(t)$=$f'(t-t_{i+3})$,when $t_i \le t < t_{i+4}(t_{i+3})$. Substituting them into (10), and expanding $N(t)$ according to the forth column, we can obtain

$$N(t) = -\begin{vmatrix} f(t_{i+4}-t_i) & f(t_{i+4}-t_{i+1}) & f(t_{i+4}-t_{i+2}) \\ f'(t_{i+4}-t_i) & f'(t_{i+4}-t_{i+1}) & f'(t_{i+4}-t_{i+2}) \\ f(t-t_i) & f(t-t_{i+1}) & f(t-t_{i+2}) \end{vmatrix}.$$

Hence, $N_{i,4}(t_{i+4}) = N'_{i,4}(t_{i+4}) = 0$.

(iii) $s_{i+4} = 3$, that is, $t_i < t_{i+1} < t_{i+2} = t_{i+3} = t_{i+4}$. Using the method listed in (ii), we have

$$N(t) = \begin{vmatrix} f_i(t_{i+4}) & f_{i+1}(t_{i+4}) \\ f_i(t) & f_{i+1}(t) \end{vmatrix}.$$

Hence, $N_{i,4}(t_{i+4}) = 0$.

(iv) $s_{i+4} = 4$, that is, $t_i < t_{i+1} = t_{i+2} = t_{i+3} = t_{i+4}$. Using the method listed in (ii), we have $N(t) = -f_i(t)$, $N_{i,4}(t) = f_i(t)$.

(v) $s_{i+4} = 5$, that is, $t_i = t_{i+1} = t_{i+2} = t_{i+3} = t_{i+4}$, obviously, $N_{i,4}(t) \equiv 0$. ∎

Hence, from Theorem 1, $N_{i,4}(t)$ satisfies all the requirements for basis construction and it contains all the 16 cases of knots listed in Table 1.

For $k \ge 5$, $N_{i,k}(t)$ is defined recursively by

$$N_{i,k}(t) = \int_{-\infty}^{t} (\delta_{i,k-1}N_{i,k-1}(s) - \delta_{i+1,k-1}N_{i+1,k-1}(s))\, ds, \quad k \ge 5, \quad i = 0, \pm 1, \cdots, \tag{12}$$

where

$$\delta_{i,k} := \left(\int_{-\infty}^{+\infty} N_{i,k}(t) dt \right)^{-1}.$$

$N_{i,k}$, $k \ge 5$, $i = 0, \pm 1, \cdots$, constitute a non-uniform quasi-B-spline basis of $\Omega_k[T]$, for $k \ge 5$, which is called generalized B-splines of order k.

Table 1 The 16 cases of knot vector, N is the number of knot intervals between t_i and t_{i+4}

N	cases of knot	
0	$t_i = t_{i+1} = t_{i+2} = t_{i+3} = t_{i+4}$	
1	$t_i < t_{i+1} = t_{i+2} = t_{i+3} = t_{i+4}$	$t_i = t_{i+1} < t_{i+2} = t_{i+3} = t_{i+4}$
	$t_i = t_{i+1} = t_{i+2} < t_{i+3} = t_{i+4}$	$t_i = t_{i+1} = t_{i+2} = t_{i+3} < t_{i+4}$
2	$t_i < t_{i+1} < t_{i+2} = t_{i+3} = t_{i+4}$	$t_i < t_{i+1} = t_{i+2} < t_{i+3} = t_{i+4}$
	$t_i < t_{i+1} = t_{i+2} = t_{i+3} < t_{i+4}$	$t_i = t_{i+1} < t_{i+2} < t_{i+3} = t_{i+4}$
	$t_i = t_{i+1} < t_{i+2} = t_{i+3} < t_{i+4}$	$t_i = t_{i+1} = t_{i+2} < t_{i+3} < t_{i+4}$
3	$t_i < t_{i+1} < t_{i+2} < t_{i+3} = t_{i+4}$	$t_i < t_{i+1} < t_{i+2} = t_{i+3} < t_{i+4}$
	$t_i < t_{i+1} = t_{i+2} < t_{i+3} < t_{i+4}$	$t_i = t_{i+1} < t_{i+2} < t_{i+3} < t_{i+4}$
4	$t_i < t_{i+1} < t_{i+2} < t_{i+3} < t_{i+4}$	

3.1.3 Examples of Generalized B-Splines

In this subsection, we will construct several generalized B-spline models by the proposed approach.

GB-spline I If we have $\alpha(t) = \sin t, \beta(t) = \cos t, \xi(t) = \sinh t, \eta(t) = \cosh t$, then the kernel function should be:

$$\kappa(t) = \frac{1}{2}(\sinh t - \sin t). \tag{13}$$

The corresponding spline model is denoted as GB-spline I, and it is a unified mathematic representation of conic and polynomial curves, which can be considered as an alternative to NURBS models. Some remarkable transcendental curves/surfaces such as the helix/helcoid and the catenary/catenoid/cycloid can be represented exactly by this model.

GB-spline II If $\alpha(t) = \sin t, \beta(t) = \cos t, \xi(t) = \sin(\omega t), \eta(t) = \cos(\omega t), \omega \neq 1$, then the kernel function should be

$$\kappa(t) = \frac{1}{\omega^2 - 1}\left(\sin t - \frac{1}{\omega}\sin(\omega t)\right). \tag{14}$$

The corresponding spline model is denoted as GB-spline II, and can be employed to represent the epicycloid/cardioid/astroid curves, which has the following parametric form:

$$\text{Epicycloid:} \quad \begin{cases} x(t) = (R+r)\cos t - r\cos\left(\dfrac{R+r}{r}t\right), \\ y(t) = (R+r)\sin t - r\sin\left(\dfrac{R+r}{r}t\right), \end{cases} \tag{15}$$

$$\text{Astroid:} \quad \begin{cases} x(t) = R\cos^3 t, \\ y(t) = R\sin^3 t. \end{cases} \tag{16}$$

GB-spline III If we set $\alpha(t) = \sin t, \beta(t) = \cos t, \xi(t) = t\sin t, \eta(t) = t\cos t$, then the kernel function should be:

$$\kappa(t) = \frac{1}{2}(\sin t - t\cos t). \tag{17}$$

This kind of spline model is denoted as GB-spline III, and it can exactly represent some special curves such as the involute of circle and the spiral of archimedes, which has the following parametric form:

$$\text{Spiral of archimedes:} \quad \begin{cases} x(t) = t\cos t, \\ y(t) = t\sin t, \end{cases} \tag{18}$$

$$\text{Involute of circle:} \quad \begin{cases} x(t) = r(\cos t + t\sin t), \\ y(t) = r(\sin t - t\cos t). \end{cases} \tag{19}$$

It should be mentioned that circle involute is employed in most gear-tooth process. The offset of a circle involute is another circle involute and,consequently, this spline model contains the simplest families of curves that are closed with respect to the offset operation: Straight lines, circles and circle involutes. Hence, GB-spline III has important applications in gear-tooth engineering.

3.2 Properties of the Basis

Similarly with the classical B-splines, the proposed generalized B-splines basis $N_{i,k}$ has the following properties:

1) Differential: $N_{i,k}$ is $(k - r_j - 1)$ times continuously differential at the knot t_j with r_j the number of times t_j appears in the knot sequence $(t_j)_i^{i+k}$.

2) Positivity and Local support:

$$N_{i,k}(t) \begin{cases} > 0, & t_i < t < t_{i+k}, \\ = 0, & \text{otherwise}. \end{cases} \tag{20}$$

3) Zero function: $N_{i,k}(t) \equiv 0$ if and only if $t_i = t_{i+1} = \cdots = t_{i+k}$.

4) Partition of unity: $\sum_i N_{i,k}(t) \equiv 1, t \in [t_i, t_{i+1})$, $k \geq 5$.

5) Derivative: $N'_{i,k}(t) = \delta_{i,k-1} N_{i,k-1}(t) - \delta_{i+1,k-1} N_{i+1,k-1}(t)$.

3.3 *h*-Refinement by Knot Insertion

For the proposed spline model, a unified knot insertion formula is proposed as follows.

Theorem 2 *Let $T := \{t_i\}_{i=-\infty}^{+\infty}$ be a knot sequence, where $t_i \leq t_{i+1}, i = 0, \pm 1, \pm 2, \cdots$. Inserting a new knot u into T, $t_i \leq u < t_{i+1}$, we can obtain a new knot sequence T^1 : $\{t_i^1\}_{i=-\infty}^{+\infty}$. $N_{j,k}(t)$ and $N_{j,k}^1(t)$ are defined as in* (12) *for the knot sequence T and T^1, respectively. Then we have for all $j, k \geq 4$,*

$$N_{j,k}(t) = \alpha_{j,k} N_{j,k}^1(t) + \beta_{j+1,k} N_{j+1,k}^1(t), \tag{21}$$

where for $0 \leq r < k$,

$$\alpha_{j,k} = \begin{cases} 1, & j \leq i-k, \\ \dfrac{N_{j,k}^{(k-l_j)}(t_j)}{N_{j,k}^{(k-l_j)}(t_j)}, & i-k < j < i-r+1, \\ 0, & j \geq i-r+1, \end{cases}$$
$$\beta_{j,k} = \begin{cases} 0, & j \leq i-k+1, \\ \dfrac{N_{j-1,k}^{(k-s_{j+k-1})}(t_{j+k-1})}{N_{j,k}^{(k-s_{j+k-1})}(t_{j+k-1})}, & i-k+1 < j < i-r+2, \\ 1, & j \geq i-r+2, \end{cases} \tag{22}$$

and for $r \geq k$,

$$\alpha_{j,k} = \begin{cases} 1, & j \leq i-k+1, \\ 0, & j > i-k+1, \end{cases} \quad \beta_{j,k} = \begin{cases} 0, & j \leq i-k+1, \\ 1, & j > i-k+1, \end{cases} \tag{23}$$

where l_j *and* s_{j+k-1} *are defined as follows*:

$$t_j = \cdots = t_{j+l_{i+j}-1} < t_{j+l_{i+j}} t_{j+k-1-s_{j+k-1}} < t_{j+k-s_{j+k-1}} = \cdots = t_{j+k-1},$$

and r *is the multiplicity of the knot* u *in* T. *If* $t_i < u < t_{i+1}$, *then* $r = 0$.

3.4 Generalized B-Spline Curves / Surfaces / Volumes

Based on the proposed generalized B-spline model, a kind of spline curve in $\Omega_k\,[t_k, t_{n+1}]$ can be defined by

$$\boldsymbol{p}(t) = \sum_{i=1}^{n} N_{i,k}(t)\boldsymbol{P}_i, \tag{24}$$

where $\boldsymbol{P}_i$ are the control points.

In a tensor-product way, the corresponding spline surface can be defined as

$$\boldsymbol{p}(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_i(u)N_j(v)\boldsymbol{P}_{i,j},$$

in which $\boldsymbol{P}_{i,j}$ are the control points.

Similarly, we can define the corresponding generalized B-spline volume as

$$\boldsymbol{p}(u,v,w) = \sum_{i=1}^{l}\sum_{j=m}^{m}\sum_{k=1}^{n} N_i(u)N_j(v)N_k(v)\boldsymbol{P}_{i,j,k},$$

where $\boldsymbol{P} = [\boldsymbol{P}_{i,j,k}]$ is the control lattice.

4 Isogeometric Analysis with Generalized B-Spline Models

Isogeometric analysis method is proposed by Hughes, et al.[1] to achieve the seamless integration between numerical simulation and geometric design. The main idea is to use the unified mathematical representation both for the physical field to be solved and for the input CAD models, which avoids the costly forth and back data exchange. In this section, we will study the application of the proposed generalized B-spline models in isogeometric analysis, that is, the proposed generalized B-spline are employed both for the computational domain and for the physical field.

4.1 Model Problem and Isogeometric Framework

In this paper, we will consider the following two-dimensional heat conduction problem as an illustrative model problem:

$$-\Delta U(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \text{in } \Omega, \quad U(\boldsymbol{x}) = 0, \quad \text{on } \partial\Omega_D, \tag{25}$$

where Ω is the computation domain parameterized by the proposed generalized B-spline models, $U(\boldsymbol{x})$ is the unknown heat field to be solved, $\boldsymbol{x}$ are the Cartesian coordinates, $g(\boldsymbol{x})$ is the source function.

From the variational approach, a solution $U \in H^1(\Omega)$ will be found, such as $U(\boldsymbol{x}) = \mathbf{0}$ on $\partial\Omega_D$ and:

$$-\int_\Omega \Delta U(\boldsymbol{x})\, \psi(\boldsymbol{x})\, d\Omega = \int_\Omega g(\boldsymbol{x})\, \psi(\boldsymbol{x})\, d\Omega, \quad \forall \psi \in H^1_{\partial\Omega_D}(\Omega),$$

in which $\psi(\boldsymbol{x})$ are test functions.

From the boundary conditions, we have

$$-\int_\Omega \boldsymbol{\nabla} U(\boldsymbol{x})\, \boldsymbol{\nabla}\psi(\boldsymbol{x})\, d\Omega = \int_\Omega g(\boldsymbol{x})\, \psi(\boldsymbol{x})\, d\Omega. \tag{26}$$

With the key idea of isogeometric analysis, we will represent the solution field U using the same generalized B-spline models as for the computational domain Ω, that is,

$$U(\xi,\eta) = \sum_{i=1}^{n_i}\sum_{j=1}^{n_j} \widehat{N}_i^{p_i}(\xi)\, \widehat{N}_j^{p_j}(\eta)\, u_{ij}, \tag{27}$$

where $\widehat{N}_i$ functions are generalized B-Spline basis functions and $\boldsymbol{u} = (\xi,\eta) \in \mathcal{P}$ are domain parameters. After defining the test functions $\psi(\boldsymbol{x})$ in the physical domain such as:

$$\psi(\boldsymbol{x}) = \widehat{N}_{ij}(\xi,\eta) = \widehat{N}_i^{p_i}(\xi)\, \widehat{N}_j^{p_j}(\eta). \tag{28}$$

Substituting Equations (27) and (28) into Equation (26), we obtain

$$\sum_{k=1}^{n_k}\sum_{l=1}^{n_l} u_{kl} \int_\Omega \boldsymbol{\nabla} N_{kl}(\boldsymbol{x})\, \boldsymbol{\nabla} N_{ij}(\boldsymbol{x})\, d\Omega = \int_\Omega g(\boldsymbol{x})\, N_{ij}(\boldsymbol{x})\, d\Omega. \tag{29}$$

Then a linear system can be obtained from Equation (29),

$$\boldsymbol{M}\boldsymbol{u} = \boldsymbol{S}, \tag{30}$$

in which $\boldsymbol{u}$ are unknown variables, the entries in stiffness matrix $\boldsymbol{M} = [M_{ij,kl}]$ and right-hand side $\boldsymbol{S} = [S_{ij}]$ can be computed as follows:

$$\begin{aligned} M_{ij,kl} &= \int_{\Omega} \boldsymbol{\nabla} N_{kl}(\boldsymbol{x})\ \boldsymbol{\nabla} N_{ij}(\boldsymbol{x})\ d\Omega \\ &= \int_{\mathcal{P}} \boldsymbol{\nabla}_{\boldsymbol{u}} \widetilde{N}_{kl}(\boldsymbol{u}) B(\boldsymbol{u})^T B(\boldsymbol{u})\ \boldsymbol{\nabla}_{\boldsymbol{u}} \widetilde{N}_{kl}(\boldsymbol{u}) J(\boldsymbol{u})\ d\mathcal{P}, \\ S_{ij} &= \int_{\Omega} g(\boldsymbol{x})\ N_{ij}(\boldsymbol{x})\ d\Omega \\ &= \int_{\mathcal{P}} g(T(\boldsymbol{u}))\ \widetilde{N}_{kl}(\boldsymbol{u}) J(\boldsymbol{u})\ d\mathcal{P}, \end{aligned}$$

where $J(\boldsymbol{u})$ is the Jacobian of the transformation,

$$J(\boldsymbol{u}) = \begin{vmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{vmatrix}, \tag{31}$$

$B(\boldsymbol{u})$ is the transposed of the inverse of the Jacobian matrix.

4.2 Isogeometric Analysis Framework with Generalized B-Splines

The isogeometric analysis framework with generalized B-splines can be described as follows:

Step 1 Construct the analysis-suitable parameterization of the computational domain with generalized B-splines using the similar method as in [22];

Step 2 Computation of the stiffness matrix coefficients: For each knot span

1) Evaluation of the generalized B-spline function and gradient values at the Gauss points;

2) Evaluation of the transformation matrix and Jacobian at the Gauss points;

3) Calculation of the matrix coefficients;

4) Assembly of the stiffness matrix.

Step 3 Computation of the right-hand side terms similarly as Step 1;

Step 4 Imposition of boundary conditions;

Step 5 Solve the linear system;

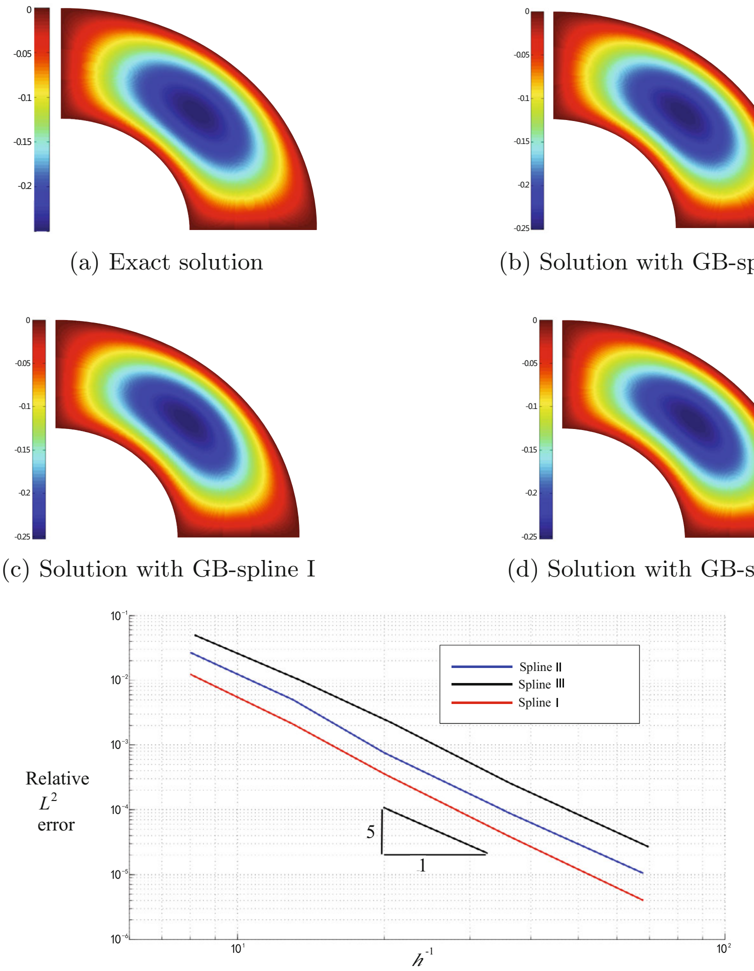
Step 6 Perform h-refinement to achieve more accurate simulation result;

Step 7 Visualization of simulation results.

5 Numerical Examples

Starting from a planar generalized B-spline surface as computational domain, the isogeometric analysis framework described in Section 4 has been implemented as a plugin in the AXEL[†] platform, achieving a solution field with generalzied B-spline representation.

[†] http://axel.inria.fr/.



(a) Exact solution

(b) Solution with GB-spline I

(c) Solution with GB-spline I

(d) Solution with GB-spline I

(e) Comparison of the convergence results during h-refinement

Figure 2 The benchmark example

5.1 A Benchmark Example

The first example is an annulus region bounded by two concentric circles as shown in Figure 2(a). Here the source function is

$$g(\boldsymbol{x}) = \frac{8 - 9\sqrt{x^2 + y^2}}{x^2 + y^2} \sin(2 \arctan(y/x)),$$

and the corresponding exact solution for heat conduction problem (25) is

$$U(\boldsymbol{x}) = (x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2) \sin(2 \arctan(y/x))$$

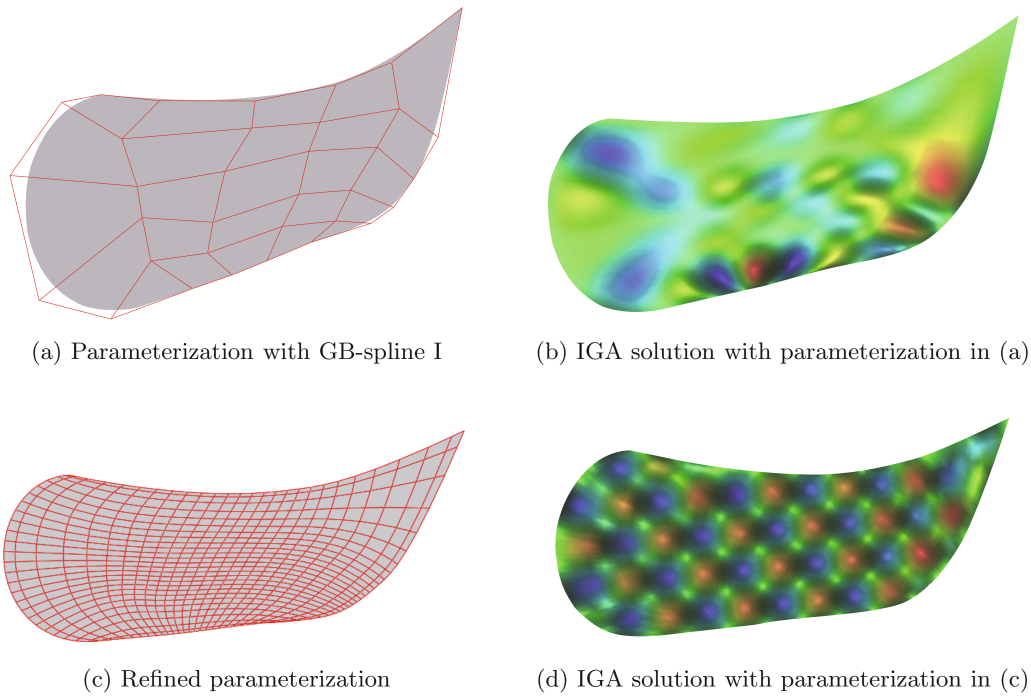
as shown in Figure 2(a). We solve this problem by IGA method with the generalized B-spline models of degree five respectively: GB-spline I, GB-spline II, and GB-spline III. Firstly, we represent the computational domain by these GB-spline models of degree five exactly. After isogeometric solving, we can obtain different solution field for different GB-spline model shown in Figure 2(b), (c), (d). Figure 2(e) shows the comparison of the convergence results between the IGA method using different generalized B-spline models during h-refinement. We can find that GB-spline I has a better convergence for this benchmark example.

5.2 More Examples

More examples will be given for the heat conduction problem with homogeneous boundary condition and the following source function

$$f(\boldsymbol{x}) = -2\pi^2 \sin(\pi x) \sin(\pi y).$$

As shown in Figure 1, by using the GB-spline I of degree five, we can construct the analysis-suitable parameterization for the given computational domain bounded by semi-circle and catenary. The corresponding IGA solution of the heat conduction problem is shown in Figure 3(b). The parameterization with h-refinements and the corresponding IGA solution are illustrated in Figure 3(c) and Figure 3(d).



(a) Parameterization with GB-spline I

(b) IGA solution with parameterization in (a)

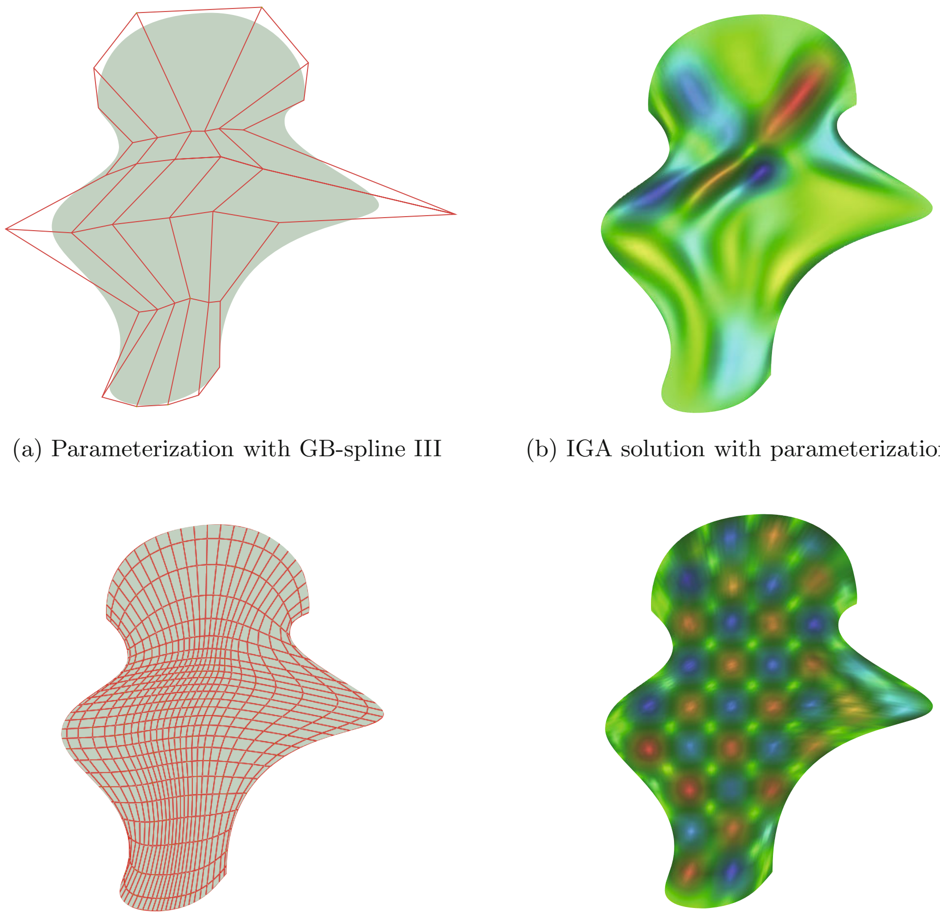
(c) Refined parameterization

(d) IGA solution with parameterization in (c)

Figure 3 IGA example using GB-spline I with semi-circle and catenary as boundary

Figure 4 (a) shows a computational domain bounded by semi-circe (top) and the spiral of archimedes (bottom), which is parameterized by using the GB-spline III of degree five. The corresponding IGA solution of the heat conduction problem is presented in Figure 4(b). Figure 4(c) and Figure 4(d) illustrate the parameterization with h-refinements and the corresponding IGA solution respectively.

From the above two examples, the proposed generalized B-spline models can be used to parameterize the computational domain bounded by some special parametric curves, which can not be represented exactly by the conventional NURBS model and previous generalized B-splines.



(a) Parameterization with GB-spline III

(b) IGA solution with parameterization in (a)

(c) Refined parameterization

(d) IGA solution with parameterization in (c)

Figure 4 IGA example using GB-spline III with semi-circle and the spiral of archimedes as boundary

6 Conclusion

In this paper, a unified approach to construct the generalized non-uniform B-splines over the space span$\{\alpha(t), \beta(t), \xi(t), \eta(t), 1, t, \cdots, t^{n-4}\}$ is proposed , and the corresponding isogeometric analysis framework for PDE solving is also studied. Compared with the NURBS-IGA method, the proposed frameworks have several advantages such as high accuracy, easy-to-compute derivatives and integrals due to the non-rational form. Furthermore, with the proposed spline models, isogeometric analysis can be performed on the computational domain bounded by transcendental curves/surfaces, such as the helix/helicoid, the catenary/catenoid and the cycloid. Several comparison examples for isogeometric heat conduction problems are presented to show the effectiveness of the proposed methods.

In the future, the Bézier extraction approach for the proposed generalized B-splines will be studied, such that the proposed IGA framework can be implemented in a similar way as finite element method.

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