CSCC73 A7

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Q1 Max Flow True/False

a) True. This follows from the Max Flow Min Cut Theorem. If there is a min cut then the flow on it must be saturated because if it was not then it is possible to increase the flow so that it matches the capacity of the min cut. Furthermore, if we have multiple min cuts the flow on all of them must be saturated because if for any min cut if the flow was not saturated this implies that the cut itself is not minimum because it is able to support greater capacity.

b) False. Consider a graph with 5 nodes s, t, u, v, w. Let $E = \{(s, u), (s, v), (v, w), (u, w), (w, t)\}$ and the capacity of all edges is 1. The capacity of the maximum flow is 1. Clearly there is only one possible min cut which is $S = \{s, u, v, w\}$ and $T = \{t\}$. However I can construct the flow using this min cut to be the edges $(s, u) \rightarrow (u, w) \rightarrow (w, t)$ where the flow though all of these edges is 1. Clearly all of these edges are saturated but only one actually crosses the min cut which is (w, t).

c) False. Consider a graph with 5 nodes s, t, u, v, w. Let $E = \{(s, t), (u, v), (v, w), (w, u)\}$ and the capacity of all edges is 1. Let the flow in the edges $(u, v), (v, w), (w, u)$ be 1. Then in this case the overall value of the flow is 0 since there is no flow leaving s however the flow on all edges is not 0.

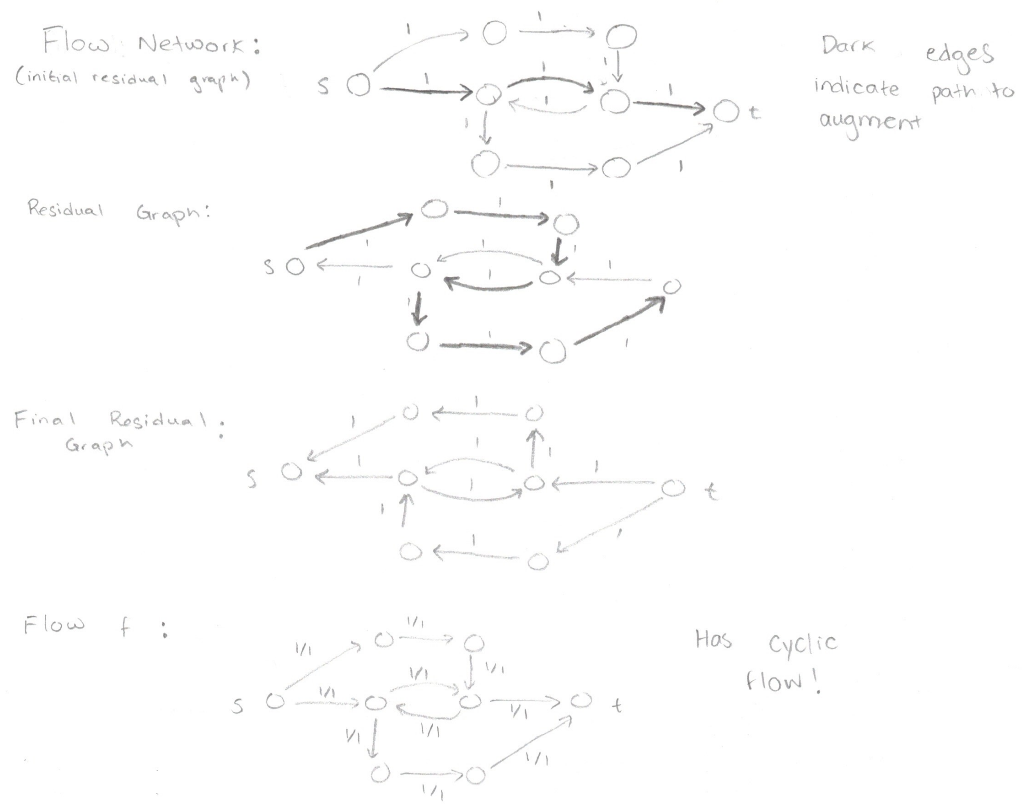
d) False. Consider a graph with 6 nodes s, t, u, v, w, x. Let $E = \{(s, u), (s, v), (s, w), (u, x), (v, x), (w, x), (x, t)\}$. Furthermore the capacity of all edges is 1 except the edge (x, t) which has a capacity of 3. Clearly, the max flow is 3. The are three possible min cuts:

1. $S = \{s\}$, $T = \{t, u, v, w, x\}$
2. $S = \{s, u, v, w\}$, $T = \{t, x\}$
3. $S = \{s, u, v, w, x\}$, $T = \{t\}$

However if I increase all capacities by 1, there is only one min cut which is $S = \{s, u, v, w, x\}$, $T = \{t\}$ and the max flow is 4. As a result neither 1 or 2 remain min cuts.

Q2 Cyclic Flows

PART A: Consider the following example,



The figure above outlines the Ford-Fulkerson algorithm performed on the graph above. As seen by picking the shortest augmentation paths the algorithm is fored to create a cyclic flow.

PART B:

If we have a flow f then there are two cases:

CASE 1. If f is already an acyclic flow in $\mathcal{F}$ then it is trivial that there is an acyclic flow in $\mathcal{F}$ with the same value as f since it is simply f.

CASE 2. If f is a cyclic flow, consider the following procedure:

1. Let C denote a cycle with flow in $\mathcal{F}$. Find the edge with the smallest flow in C, let x be the flow in that edge
2. For all edges in C decrease the flow of each edge by x, call the new flow f'
3. If there are multiple cycles repeat steps 1 and 2 until no cycle is left

f' is a valid flow. This is because C is a cycle so each edge in C has both x less flow going in and x less flow going out. Thus both the capacity and conservation constraints still hold.

Furthermore, the value of f' is the same as the value for f. This is because the flow leaving s is the same in f as in f'. This is because s cannot have an incoming edges by definition. Thus the flow leaving s has not been manipulated by this procedure. As a result the value of f and f' must be equal.

Thus we have constructed an acyclic flow f' with the same value as the cyclic flow f.

$\square$

Q3 Max Flow Modification

PART A:

Proof for $\mathcal{F}^+$:

The proof will be done in two cases, either e is part of some min cut in $\mathcal{F}$ or e is not part of a in min cut in $\mathcal{F}$. Note that we say e is part of a min cut if given a cut (S, T), e is an edge (u, v) where $u \in S$ and $v \in T$.

CASE 1. If e is part of a min cut in $\mathcal{F}$ then the capacity of that cut is $V(f)$ by Max Flow Min Cut Theorem. There are two subcases: either $\mathcal{F}$ has only one min cut or it has multiple min cuts. If $\mathcal{F}$ has only one min cut then by increasing the capacity of e by one unit, the min cut of $\mathcal{F}^+$ has a capacity of $V(f)+1$. Note that non-integral capacities are irrelevant since the capacities of $\mathcal{F}$ are all integral. Thus the new min cut has capacity $V(f)+1$ and by Max Flow Min Cut Theorem the value of the max flow in $\mathcal{F}^+$ is $V(f)+1$. In subcase two if there are multiple min cuts then the capacity of the min cut in $\mathcal{F}^+$ is still $V(f)$ this is because we have only increased the capacity of a single min cut so the remaining min cuts will not have their capacities' changed. Thus in $\mathcal{F}^+$ the capacity of the min cut is still $V(f)$ so by Max Flow Min Cut Theorem the value of the max flow is $V(f)$.

CASE 2. If e is not part of a min cut in $\mathcal{F}$ then the capacity of any cut that includes e is at minimum 1 greater than the capacity of a min cut. This is because $\mathcal{F}$ is a flow network with integral capacities. Note that the capacity of the min cut is $V(f)$. Thus if we increase the capacity of e by one unit then the capacity of any cut in $\mathcal{F}^+$ which includes e must be at minimum 2 greater than the capacity of a min cut. By Max Flow Min Cut Theorem the value of the max flow is equivalent to the capacity of the min cut, so in this case the value of max flow is $V(f)$.

Therefore the value of the max flow in $\mathcal{F}^+$ is $V(f)$ or $V(f)+1$.

$\square$

Proof for $\mathcal{F}^-$:

The proof will be done in two cases, either e is part of some min cut in $\mathcal{F}$ or e is not part of a in min cut in $\mathcal{F}$. Note that we say e is part of a min cut if given a cut (S, T), e is an edge (u, v) where $u \in S$ and $v \in T$.

CASE 1. If e is part of a min cut in $\mathcal{F}$ then the capacity of that cut is $V(f)$ by Max Flow Min Cut Theorem. By decreasing the capacity of e by one unit, the min cut of $\mathcal{F}^-$ has a capacity of $V(f)-1$. Note that non-integral capacities are irrelevant since the capacities of $\mathcal{F}$ are all integral. Thus the new min cut has capacity $V(f)11$ and by Max Flow Min Cut Theorem the value of the max flow in $\mathcal{F}^-$ is $V(f)-1$.

CASE 2. If e is not part of a min cut in $\mathcal{F}$ then the capacity of any cut that includes e is at minimum 1 greater than the capacity of a min cut. This is because $\mathcal{F}$ is a flow network with integral capacities. Note that the capacity of the min cut is $V(f)$. Thus if we decrease the capacity of e by one unit then the capacity of any cut in $\mathcal{F}^-$ which includes e must be greater than or equal to the capacity of a min cut. By Max Flow Min Cut Theorem the value of the max flow is equivalent to the capacity of the min cut, so in this case the value of max flow is $V(f)$.

Therefore the value of the max flow in $\mathcal{F}^-$ is $V(f)$ or $V(f)-1$.

$\square$

PART B:

Consider the following algorithm,

$\mathcal{F}^+$MaxFlow($\mathcal{F}$, f, e)

Increase the capacity of e by one unit and construct the residual graph $\mathcal{F}_f$

$p =$ some simple $s \to t$ path in $\mathcal{F}_f$ or None if no such path exists

if $p \neq$ None then $f =$ augment(f, p)

return f

CORRECTNESS: As proved in part a, a maximum flow in $\mathcal{F}^+$ will have the value $V(f)$ or $V(f)+1$. This depends on whether e is part of the minimum cut of $\mathcal{F}^+$. There are two cases:

CASE 1. If e is not part of the min cut then the value of the max flow in $\mathcal{F}$ is the same as the value of the max flow in $\mathcal{F}^+$. In this case there is no $s \to t$ path in the residual graph since the flow is already maximum. Thus in the algorithm above p will be None and f is returned without being modified. f is a feasible flow for $\mathcal{F}^+$ since $\mathcal{F}$ and $\mathcal{F}^+$ are the exact same expect for the differing edge e. Thus the flow returned is both feasible and maximum as wanted.

CASE 2. If e is part of a min cut then by increasing the capacity of the edge e it may be possible to return a flow with value $V(f) + 1$ provided no other min cuts exist. In this case if there are multiple $s \to t$ paths then p will still be None and the logic follows from case 1. However, if there is only a single min cut then the residual graph will have a $s \to t$ path meaning that p is not None. Thus the algorithm will augment the flow f. Since the augmentation step is guaranteed to augment the flow by at least 1 and we know that the flow can only be 1 greater the flow returned is a max flow. Thus the flow returned is a max flow.

RUNNING TIME: The complexity to construct the residual graph is $O(m + n)$ where m is the number of edges and n the number of nodes. Additionally we can use BFS to find a simple path in $\mathcal{F}_f$ so finding p can be done in $O(m + n)$ time. Lastly the augmentation step will run in $O(n)$ as discussed in lecture. Thus the total complexity is $O(m + n)$ which is linear with respect to the number of nodes and edges.

<u>PART C:</u>

Note that we assume that $e = (u, v)$. Furthermore, assume that there is no cycle when finding p_1 and p_2. We can do this since we proved that any cyclic flow has an equivalent acyclic flow. Consider the following algorithm,

$\mathcal{F}^-$MaxFlow($\mathcal{F}$, f, e)

Decrease the capacity of e by one unit

If the flow through $e \leq$ capacity of e then return f

p_1 = some simple $s \to u$ path using edges in f

p_2 = some simple $v \to t$ path using edges in f

Decrease the flow in f by one unit in all the edges in $\{e\} \cup p_1 \cup p_2$

Construct the residual graph $\mathcal{F}_f$

p = some simple $s \to t$ path in $\mathcal{F}_f$ or None if no such path exists

if $p \neq$ None then f = augment(f, p)

return f

CORRECTNESS: As proved in part a, a maximum flow in $\mathcal{F}^-$ will have the value $V(f)$ or $V(f) - 1$. There are two cases in the algorithm either the edge e is saturated or it is not saturated. There are two cases:

CASE 1. If e is not saturated then the *value of the flow through* $e + 1 \leq$ *capacity of* e. Thus by decreasing the capacity of e by one unit we can still maintain the current max flow. This is because the current flow has value $V(f)$ which as proved earlier is the greatest value a max flow in $\mathcal{F}^-$ can have. Thus the algorithm will return a feasible flow f with value $V(f)$ as expected.

CASE 2. If e is saturated then by decreasing the flow in e by one unit the *flow through* $e >$ *capacity of* e which is not allowed in a feasible flow. Thus the current flow f cannot be returned and must be modified. First the flow must be adjusted so that the flow through e is reduced by 1. We can do this by finding a simple path from $s \to u$ and $v \to t$ and decreasing the flow through all the edges and e by one. Note that we can assume there are no cycles here since we proved that any cyclic flow has an equivalent acyclic flow. The resulting flow has value $V(f) - 1$ however it may be possible to generate a flow of value $V(f)$ thus we must construct the residual graph to check for an $s \to t$ path. If a path exists then the flow is augmented and we return a flow of value $V(f)$. Otherwise we return a flow of value $V(f) - 1$ as wanted.

RUNNING TIME: The complexity to reduce the flow through e and check the capacity of e can be done in $O(1)$ time. Using BFS finding p_1 and p_2 can be done in $O(m + n)$ time, similarly decreasing the flow in p_1 and p_2 is done in $O(m + n)$ time. Note m is the number of edges and n the number of nodes. Construct the residual graph is also $O(m + n)$. Additionally we can use BFS to find a simple path in $\mathcal{F}_f$ so finding p can be done in $O(m+n)$ time. Lastly the augmentation step will run in $O(n)$ as discussed in lecture. Thus the total complexity is $O(m+n)$ which is linear with respect to the number of nodes and edges.