Question-1

$$y = \beta_0 + \beta_1 x + u, \quad E(u) = \alpha_0$$

Lets add and substract α_0 to the existing model.

$$y = \beta_0 + \beta_1 x + u + \alpha_0 - \alpha_0$$

$$y = \beta_0 + \alpha_0 + \beta_1 x + u - \alpha_0$$

In more clear form

$$y = \underbrace{(\beta_0 + \alpha_0)}_{\text{new intercept}} + \underbrace{\beta_1}_{\text{slope}} x + \underbrace{(u - \alpha_0)}_{\text{new error.}}$$

Thus the model can be rewritten with same slope and new intercept and new error.

Now $E(\text{new error}) = E(u - \alpha_0)$

$= E(u) - \alpha_0$ Given $E(u) = \alpha_0$

$= 0$

Thus the new error has zero expected value.

Question -2

Show that β_0's estimator is also unbiased

Answer We know $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$$\bar{y} = \beta_0 + \beta_1 \bar{x} + \bar{u}$$

$$\hat{\beta}_0 = \beta_0 + \beta_1 \bar{x} + \bar{u} - \hat{\beta}_1 \bar{x}$$

$$= \beta_0 + \beta_1 (\bar{x}) - \bar{u} - \hat{\beta}_1 \bar{x}$$

$$E(\hat{\beta}_0) = E(\beta_0) + E(\bar{x}(\beta_1 - \hat{\beta}_1)) - E(\bar{u})$$

$$= \beta_0 + \bar{x}\beta_1 - \bar{x} E(\hat{\beta}_1) - E(\bar{u})$$

⇓

we know $\hat{\beta}_1$ is unbiased

$$E(\hat{\beta}_1) = \beta_1$$

$$= \beta_0 + \bar{x}\beta_1 - \bar{x}\beta_1 - E(\bar{u})$$

⇓

zero assumption

$$= \beta_0$$

$$\boxed{E(\hat{\beta}_0) = \beta_0}$$

Thus OLS estimator for β_0 is unbiased.

Question 3

Given Sample size N

$\bar{x} = 0 \quad \bar{y} = 2$

$$\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}) = 1$$

$$\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x}) = 2$$

$$\frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})(y_i - \bar{y}) = 3$$

(a) Regression of Y on X

$$Y_i = b_1 + b_2 X_i + \epsilon_i$$

We know $b_1 = \bar{y} - b_2 \bar{x}$

$$b_2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$b_2 = \frac{(N-1)}{2(N-1)} \quad \left.\right\} \text{Given}$$

Substitute

$b_2 = \frac{1}{2}$

$b_1 = \bar{y} - \frac{1}{2}\bar{x}$

$b_1 = 2 - \frac{1}{2} \times 0$

$b_1 = 2$

(b) Regression of x on Y

$$x_i = a_1 + a_2 Y_i + \eta_i$$

we know $a_2 = \dfrac{\sum_{i=1}^{N} (x_i - \bar{x})(Y_i - \bar{Y})}{\sum_{i=1}^{N} (Y_i - \bar{Y})^2}$ $\quad a_1 = \bar{x} - a_2 \bar{Y}$

$$a_2 = \frac{N-1}{3(N-1)} \Big\} \text{ given}$$

$$\boxed{a_2 = \frac{1}{3}}$$

Substitute this in a_1

$$a_1 = \bar{x} - a_2 \bar{Y}$$

$$= 0 - \frac{1}{3} \times 2$$

$$\boxed{a_1 = \frac{-2}{3}}$$

(c) Regression of Z on X

$$z_i = c_1 + c_2 x_i + \xi_i$$

we know $\boxed{c_1 = \bar{z} - c_2 \bar{x}}$

$$\boxed{c_2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}}$$

Given $c_1 = 2$ and $c_2 = 1$

$2 = \bar{z} - 1 \cdot \bar{x}$ ⟸ Given $\bar{x} = 0$

$\boxed{\bar{z} = 2}$ → Mean.

Now

$$1 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$1 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})}{2 \times (N-1)}$$

$$\boxed{2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})}$$

Covariance of $x, z = 2$ → Covariance.

Mean of $z = 2$