

Phép biến đổi tích phân

March 17, 2025

1 Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$[-\pi, \pi]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (n = 1, 2, \dots)$$

$$[0, 2\pi] \rightarrow \int_0^{2\pi}$$

1.1 Khoảng bất kỳ

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$[-p, p]$$

$$a_0 = \frac{1}{2p} \int_{-p}^p f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

1.2 Hàm chẵn lẻ

| HÀM | TÍNH CHẤT | Điều kiện |
|------|-----------------|-------------|
| Chẵn | $f(-x) = f(x)$ | $\forall x$ |
| Lẻ | $f(-x) = -f(x)$ | $\forall x$ |

- Chẵn \times Chẵn = Chẵn
- Chẵn \times Lẻ = Lẻ
- Lẻ \times Lẻ = Chẵn
- $\cos \rightarrow$ Chẵn.
- $\sin \rightarrow$ Lẻ.

1.2.1 Hàm chẵn

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx$$

Và

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

Trong đó

$$a_0 = \frac{1}{p} \int_0^p f(x) dx$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

1.2.2 Hàm lẻ

$$\int_{-p}^p f(x) dx = 0$$

Và

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

Trong đó

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx \quad (n = 1, 2, \dots)$$

1.3 Parseval

$$\frac{1}{2p} \int_{-p}^p f(x)^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

2 Fourier Transform

2.1 Biểu diễn

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega \quad (-\infty < x < \infty), \forall \omega \geq 0$$

Trong đó

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$$

Không liên tục tại x

$$\frac{f(x+) + f(x-)}{2}$$

2.2 Phép biến đổi

$$\hat{f}(\omega) = \mathcal{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad (-\infty < \omega < \infty)$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{f}(x) dx$$

2.2.1 Ngược

$$\check{f}(x) = f(x) = \mathcal{F}^{-1}(\hat{f}(\omega))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega) d\omega \quad (-\infty < x < \infty)$$

2.3 Tính chất

2.3.1 AXTT

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

2.3.2 Đạo hàm

$$\mathcal{F}(f^{(n)}) = (i\omega)^n \mathcal{F}(f)$$

$$\mathcal{F}(xf(x))(\omega) = i \left[\hat{f} \right]'(\omega) = i \frac{d}{d\omega} \mathcal{F}(f)(\omega)$$

$$\mathcal{F}(x^n f(x))(\omega) = i^n \left[\hat{f} \right]^{(n)}(\omega)$$

2.3.3 Tích chập

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t) dt$$

$$g * f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)f(t) dt$$

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(f * f)(\omega) = \hat{f}(\omega)^2$$

$$\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}^2(f) = f(-x)$$

Tìm $\mathcal{F}(g) \rightarrow$ Chọn $f(x)$ sao cho

$$\begin{aligned}\mathcal{F}(f(x)) &= g(\omega) \\ \mathcal{F}(\mathcal{F}(f)) &= \mathcal{F}(g(\omega)) \\ f(-x) &= \mathcal{F}(g)\end{aligned}$$

2.3.4 Gaussian Transform

$$a > 0$$

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}$$

$$a > 0 \quad (a = 2a)$$

$$\mathcal{F}\left(e^{-ax^2}\right)(\omega) = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

$$a = 1$$

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\omega) = e^{-\frac{\omega^2}{2}}$$

3 Basics

3.1 Euler formula

$$e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$$

$$e^{-i\omega x} = \cos(\omega x) - i \sin(\omega x)$$

$$|e^{-i\omega x}| = |\cos(\omega x) - i \sin(\omega x)| = \sqrt{\cos^2(\omega x) + \sin^2(\omega x)} = 1$$

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$$

$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu})$$

3.2 Differentiation

$$(c)' = 0$$

$$(kx)' = k, (ku)' = ku'$$

$$(x^n)' = nx^{n-1}, (u^n)' = u'nu^{n-1}$$

$$(uv)' = u'v + v'u$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}, \left(\frac{1}{u}\right)' = \frac{u'}{u^2}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}, (\sqrt{u})' = \frac{u'}{2\sqrt{u}}$$

$$(\sin x)' = \cos x, (\sin u)' = u' \cos u$$

$$(\cos x)' = -\sin x, (\cos u)' = -u' \sin u$$

$$(e^x)' = e^x, (e^u)' = u'e^u$$

$$(a^x)' = a^x \ln a, (a^u)' = u'a^u \ln a$$

$$(\ln x)' = \frac{1}{x}, (\ln u)' = \frac{u'}{u}$$

$$(\log_a x)' = \frac{1}{x \ln a}, (\log_a u)' = \frac{u'}{u \ln a}$$

$$(u \pm v)' = u' \pm v'$$

$$(u_1 \pm \dots \pm u_n)' = u_1' \pm \dots \pm u_n'$$

$$y = y[u(x)] \Rightarrow y'(x) = y'(u)u'(x)$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cx+d)^2}$$

3.3 Integral

$$\int dx = x, \int du = u$$
$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}, \int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1}$$

$$\int a^x dx = \frac{a^x}{\ln a}, \int a^u du = \frac{a^u}{\ln a}$$
$$\int \frac{dx}{x} = \ln|x|, \int \frac{du}{u} = \ln|u|$$

$$\int e^x dx = e^x, \int e^u du = e^u$$
$$\int e^{kx} dx = \frac{e^{kx}}{k}$$

$$\int \cos x dx = \sin x, \int \cos u du = \sin u$$
$$\int \sin x dx = -\cos x, \int \sin u du = -\cos u$$
$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b)$$
$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b)$$

$$\int (ax+b)^\alpha = \frac{1}{a} \frac{(ax+b)^{\alpha+1}}{\alpha+1}$$

3.3.1 Từng phần

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$
$$I = \int_a^b P(x)Q(x) dx$$
$$u = P(x)$$
$$dv = Q(x)dx$$

Chọn $P(x)$ đặt u : Nhất log - Nhì đa - Tam lượng - Tứ mũ.

Múa cột

VD1: Tính nguyên hàm : $I = \int (2x^2 - 3).e^x dx$

| (đạo hàm) $u = 2x^2 - 3$ | dấu | (nguyên hàm) $dv = e^x dx$ |
|-----------------------------|-----|-------------------------------|
| $4x$ | + | e^x |
| 4 | - | e^x |
| 0 | + | e^x |

$$\begin{aligned} \Rightarrow I &= e^x(2x^2 - 3) - 4x.e^x + 4e^x + C \\ &= e^x(2x^2 - 4x + 1) + C \end{aligned}$$

3.4 Trigonometry

$$\sin^2 a + \cos^2 a = 1$$

$$\tan a = \frac{\sin a}{\cos a}$$

$$\cot a = \frac{\cos a}{\sin a}$$

$$1 + \tan^2 a = \frac{1}{\cos^2 a}$$

$$1 + \cot^2 a = \frac{1}{\sin^2 a}$$

$$\tan a \cot a = 1$$

3.4.1 Cộng - Trừ

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

3.4.2 Nhân

$$\sin 2a = 2 \sin a \cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\cot 2a = \frac{\cot^2 a - 1}{2 \cot a}$$

$$\sin 3a = 3 \sin a - 4\sin^3 a$$

$$\cos 3a = 4\cos^3 a - 3 \cos a$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3\tan^2 a}$$

$$\cot 3a = \frac{3\cot^2 a - 1}{\cot^3 a - 3 \cot a}$$

3.4.3 Hạ bậc

$$\cos^2 a = \frac{1 + \cos 2a}{2}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2}$$

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a}$$

$$\sin^2 a \cos^2 a = \frac{1 - \cos 4a}{8}$$

$$\cos^3 a = \frac{3 \cos a + \cos 3a}{4}$$

$$\sin^3 a = \frac{3 \sin a - \sin 3a}{4}$$

$$\sin^4 a = \frac{\cos 4a - 4 \cos 2a + 3}{8}$$

$$\cos^4 a = \frac{\cos 4a + 4 \cos 2a + 3}{8}$$

3.4.4 Tích tổng

$$\cos a \cos b = \frac{1}{2} [\cos(a-b) + \cos(a+b)]$$

$$\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$$

$$\sin a \cos b = \frac{1}{2} [\sin(a+b) + \sin(a-b)]$$

3.4.5 Tổng tích

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\cos a + \sin a = \sqrt{2} \cos \left(\frac{\pi}{4} - a \right) = \sqrt{2} \sin \left(\frac{\pi}{4} + a \right)$$

$$\cos a - \sin a = \sqrt{2} \cos \left(\frac{\pi}{4} + a \right) = \sqrt{2} \sin \left(\frac{\pi}{4} - a \right)$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cos b}$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a \cos b}$$

$$\cot a + \cot b = \frac{\sin(a+b)}{\sin a \sin b}$$

$$\cot a - \cot b = \frac{\sin(b-a)}{\sin a \sin b}$$

$$\cot a + \tan a = \frac{2}{\sin 2a}$$

$$\cot a - \tan a = 2 \cot 2a$$

3.5 Cấp số nhân

$$S_n = u_1 + u_2 + \dots + u_n = \frac{u_1(1 - q^n)}{1 - q}$$

$$u_n = u_{n-1} \cdot q$$

$$u_n = u_1 \cdot q^{n-1}$$