Phép biến đổi tích phân

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1 Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$[-\pi, \pi]$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, ...)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad (n = 1, 2, ...)$$

$[0, 2\pi] \to \int_0^{2\pi}$

1.1 Khoảng bất kỳ

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$[-p, p]$

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx \quad (n = 1, 2, ...)$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx \quad (n = 1, 2, ...)$$

1.2 Hàm chẳn lẻ

HÀM	TÍNH CHẤT	Điều kiện
Chẳn	$f(-x) = f(x)$	$\forall x$
Lẻ	$f(-x) = -f(x)$	$\forall x$

- Chẳn $\times$ Chẳn $=$ Chẳn
- Chẳn $\times$ Lẻ $=$ Lẻ
- Lẻ $\times$ Lẻ $=$ Chẳn
- cos $\rightarrow$ Chẳn.
- sin $\rightarrow$ Lẻ.

1.2.1 Hàm chẳn

$$\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)\,dx$$

Và

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

Trong đó

$$a_0 = \frac{1}{p}\int_{0}^{p} f(x)\,dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, ...)$$

1.2.2 Hàm lẻ

$$\int_{-p}^{p} f(x)\,dx = 0$$

Và

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

Trong đó

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, ...)$$

1.3 Parseval

$$\frac{1}{2p}\int_{-p}^{p} f(x)^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

2 Fourier Transform

2.1 Biểu diễn

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \quad (-\infty < x < \infty),\ \forall\omega \geq 0$$

Trong đó

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

Không liên tục tại x

$$\frac{f(x+) + f(x-)}{2}$$

2.2 Phép biến đổi

$$\hat{f}(\omega) = \mathcal{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \quad (-\infty < \omega < \infty)$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \check{f}(x)\,dx$$

2.2.1 Ngược

$$\check{f}(x) = f(x) = \mathcal{F}^{-1}(\hat{f}(\omega))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)\,d\omega \quad (-\infty < x < \infty)$$

2.3 Operational properties

2.3.1 AXTT

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

2.3.2 Đạo hàm

$$\mathcal{F}(f^{(n)}) = (i\omega)^n \mathcal{F}(f)$$

$$\mathcal{F}(xf(x))(\omega) = i\left[\hat{f}\right]'(\omega) = i\frac{d}{d\omega}\mathcal{F}(f)(\omega)$$

$$\mathcal{F}(x^n f(x))(\omega) = i^n\left[\hat{f}\right]^{(n)}(\omega)$$

2.3.3 Tích chập

$$f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

$$g * f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)f(t)\,dt$$

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(f * f)(\omega) = \hat{f}(\omega)^2$$

$$\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}^2(f) = f(-x)$$

Tìm $\mathcal{F}(g) \rightarrow$ Chọn $f(x)$ sao cho

$$\begin{aligned}\mathcal{F}(f(x)) &= g(\omega)\\ \mathcal{F}(\mathcal{F}(f)) &= \mathcal{F}(g(\omega))\\ f(-x) &= \mathcal{F}(g)\end{aligned}$$

2.3.4 Gaussian Transform

$a > 0$

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}$$

$a > 0 \quad (a = 2a)$

$$\mathcal{F}\left(e^{-ax^2}\right)(\omega) = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

$a = 1$

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\omega) = e^{-\frac{\omega^2}{2}}$$

3 Partial Derivatives Transform

3.1 Operational properties

$$\mathcal{F}\left(\frac{\partial^n}{\partial t^n}u(x,t)\right)(\omega) = \frac{d^n}{dt^n}\hat{u}(\omega,t), \quad n = \overline{1,n}$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial x^n}u(x,t)\right)(\omega) = (i\omega)^n\hat{u}(\omega,t), \quad n = \overline{1,n}$$

3.2 Method (Boundary Value Problem)

- **Step 1**: *FT* the given *BVP* (Hint: *Operational properties*) $\rightarrow$ *ODE*.
- **Step 2**: Sovle *ODE* (Hint: Integrated the given *Boundary value*) $\rightarrow$ $\hat{u}(\omega,t)$.
- **Step 3**: *Inverse FT* $\hat{u}(\omega,t) \rightarrow u(x,t)$.

4 ODE

4.1 PTVP CẤP 1

4.1.1 Tách biến

$$f(x)dx = g(y)dy$$

phuong phap

$$\int f(x)dx = \int g(y)dy$$
$$F(x) = G(y) + C$$

4.1.2 Tuyến tính cấp 1

Thuần nhất

$$y' + p(x)y = 0$$

phuong phap

$$\begin{aligned}
\frac{dy}{dx} &= -p(x)y \\
\frac{1}{y}dy &= -p(x)dx \\
\int \frac{1}{y}dy &= -\int p(x)dx \\
\ln|y| &= -\int p(x)dx + \ln|c| \\
\ln|y| - \ln|c| &= -\int p(x)dx \\
\ln|\frac{y}{c}| &= -\int p(x)dx \\
\frac{y}{c} &= e^{-\int p(x)dx} \\
y &= c.e^{-\int p(x)dx}
\end{aligned}$$

Không thuần nhất

$$y' + p(x)y = q(x)$$

phuong phap (biến thiên hằng số)

$$y' + p(x)y = 0 \implies y = C.e^{-\int p(x)dx}$$

$$C = C(x)$$

$$\begin{cases} y = C(x).e^{-\int p(x)dx} \\ y' = C'(x).e^{-\int p(x)dx} - p(x)C(x).e^{-\int p(x)dx} \end{cases}$$

$$\begin{aligned}
&\implies y' + p(x)y = q(x) \\
C'(x).e^{-\int p(x)dx} - p(x)C(x).e^{-\int p(x)dx} &+ p(x)C(x).e^{-\int p(x)dx} = q(x) \\
C'(x).e^{-\int p(x)dx} &= q(x) \\
C'(x) &= q(x).e^{\int p(x)dx} \\
C(x) &= \int q(x).e^{\int p(x)dx}dx + K
\end{aligned}$$

$$\begin{aligned}
\implies y &= C(x).e^{-\int p(x)dx} \\
y &= \left(\int q(x).e^{\int p(x)dx}dx + K \right) e^{-\int p(x)dx}
\end{aligned}$$

4.1.3 Bernoulli

$$y' + p(x)y = q(x).y^{\alpha}$$

phuong phap

$$y^{-\alpha}.y' + p(x)y^{1-\alpha} = q(x)$$

$$\begin{aligned} z &= y^{1-\alpha} \\ z' &= (1-\alpha).y^{-\alpha}.y' \\ y^{-\alpha}.y' &= \frac{1}{1-\alpha}z' \end{aligned}$$

$$\begin{aligned} \implies y^{-\alpha}.y' + p(x)y^{1-\alpha} &= q(x) \\ \frac{1}{1-\alpha}z' + p(x).z &= q(x) \\ z' + (1-\alpha)p(x).z &= q(x)(1-\alpha) \end{aligned}$$

4.1.4 Toàn phần

$$P(x,y)dx + Q(x,y)dy = 0$$

$$P'_y = Q'_x$$

$$\begin{aligned} u(x,y) &= \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy + C \\ &\text{or} \\ u(x,y) &= \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy + C \end{aligned}$$

4.2 PTVP CẤP 2

4.2.1 Thuần nhất

$$y'' + py' + qy = 0$$

phuong phap

$$\lambda^2 + p\lambda + q = 0 \implies \text{ giai ptrinh } \implies \lambda_1, \lambda_2$$

- $\lambda_1 \neq \lambda_2 \implies y = C_1.e^{\lambda_1 x} + C_2.e^{\lambda_2 x}$
- $\lambda_1 = \lambda_2 = \lambda \implies y = (C_1 + x.C_2)e^{\lambda x}$
- $\lambda_{1,2} = \alpha \pm i\beta \implies y = e^{\alpha x}(C_1.\cos\beta x + C_2.\sin\beta x)$

4.2.2 Không thuần nhất

$$y'' + py' + qy = f(x)$$

1) $f(x) = e^{ax}Q_n(x)$ phuong phap

$$y'' + py' + qy = 0 \implies \lambda_1, \lambda_2 \implies y_{tn}$$

$$(P_n(x) \text{ cung bac voi } Q_n(x), \text{ vi du: } P_n(x) = Ax + B)$$

- $a \neq \lambda_1 \ \wedge \ a \neq \lambda_2 \implies y_r = e^{ax}P_n(x)$
- $a = \lambda_1 \vee a = \lambda_2 \implies y_r = xe^{ax}P_n(x)$
- $a = \lambda_1 = \lambda_2 \implies y_r = x^2e^{ax}P_n(x)$

$$\begin{aligned} &\implies y'' + py' + qy = f(x) \\ &\quad y_r'' + py_r' + qy_r = f(x) \\ \implies &\text{ He so } A,\ B, \ldots \implies y_r \implies y = y_{tn} + y_r \end{aligned}$$

2) $f(x) = P_n(x)\sin\beta x + Q_m(x)\cos\beta x$ phuong phap

$$y'' + py' + qy = 0 \implies \lambda_1, \lambda_2 \implies y_{tn}$$

$$l = \max\{m, n\} R_l(x) = Ax + B \quad H_l(x) = Cx + D$$

- $\pm i\beta \neq \lambda_{1,2} \implies y_r = R_l(x)\sin\beta x + H_l(x)\cos\beta x$
- $\pm i\beta = \lambda_{1,2} \implies y_r = x\big(R_l(x)\sin\beta x + H_l(x)\cos\beta x\big)$

$$\begin{aligned} &\implies y'' + py' + qy = f(x) \\ &\quad y_r'' + py_r' + qy_r = f(x) \\ \implies &\text{ He so } A,\ B, C, D,\ \ldots \implies y_r \implies y = y_{tn} + y_r \end{aligned}$$

3) Biến thiên hằng số

$$y'' + py' + qy = 0 \implies \lambda_2, \lambda_2 \implies y_{tn} = C_1.y_1(x) + C_2.y_2(x)$$

$$C_1 = C_1(x) C_2 = C_2(x)$$

$$\text{HPT} \implies \begin{cases} C_1'y_1 + C_2'y_2 = 0 \\ C_1'y_1' + C_2'y_2' = f(x) \end{cases} \implies C_1', C_2' \implies C_1, C_2$$

$$\implies y = C_1.y_1(x) + C_2.y_2(x)$$

5 Basics

5.1 Euler formula

$$e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$$

$$e^{-i\omega x} = \cos(\omega x) - i\sin(\omega x)$$

$$|e^{-i\omega x}| = |\cos(\omega x) - i\sin(\omega x)| = \sqrt{\cos^2(\omega x) + \sin^2(\omega x)} = 1$$

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$$

$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu})$$

5.2 Differentiation

$$(c)' = 0$$

$$(kx)' = k \, , \; (ku)' = ku'$$

$$(x^n)' = nx^{n-1} \, , \; (u^n)' = u' n u^{n-1}$$

$$(uv)' = u'v + v'u$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2} \, , \; \left(\frac{1}{u}\right)' = \frac{u'}{u^2}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}} \, , \; (\sqrt{u})' = \frac{u'}{2\sqrt{u}}$$

$$(\sin x)' = \cos x \, , \; (\sin u)' = u' \cos u$$

$$(\cos x)' = -\sin x \, , \; (\cos u)' = -u' \sin u$$

$$(e^x)' = e^x \, , \; (e^u)' = u' e^u$$

$$(a^x)' = a^x \ln a \, , \; (a^u)' = u' a^u \ln a$$

$$(\ln x)' = \frac{1}{x} \, , \; (\ln u)' = \frac{u'}{u}$$

$$(\log_a x)' = \frac{1}{x \ln a} \, , \; (\log_a u)' = \frac{u'}{u \ln a}$$

$$(u \pm v)' = u' \pm v'$$

$$(u_1 \pm ... \pm u_n)' = u_1' \pm ... \pm u_n'$$

$$y = y[u(x)] \Rightarrow y'(x) = y'(u)u'(x)$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cx+d)^2}$$

5.3 Integral

$$\int dx = x \; , \; \int du = u$$
$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} \; , \; \int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1}$$

$$\int a^x dx = \frac{a^x}{\ln a} \; , \; \int a^u dx = \frac{a^u}{\ln a}$$
$$\int \frac{dx}{x} = \ln|x| \; , \; \int \frac{du}{u} = \ln|u|$$

$$\int e^x dx = e^x \; , \; \int e^u du = e^u$$
$$\int e^{kx} dx = \frac{e^k x}{k} \; , \; \int e^{ax+b} dx = \frac{1}{a} e^{ax+b}$$

$$\int \cos x dc = \sin x \; , \; \int \cos u du = \sin u$$
$$\int \sin x dx = -\cos x \; , \; \int \sin u du = -\cos u$$
$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b)$$
$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b)$$

$$\int (ax+b)^{\alpha} = \frac{1}{a} \frac{(ax+b)^{\alpha+1}}{\alpha+1}$$

5.3.1 Từng phần

$$\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du$$
$$I = \int_a^b P(x)Q(x) \, dx$$
$$u = P(x)$$
$$dv = Q(x)dx$$

Chọn $P(x)$ đặt u: Nhất log - Nhì đa - Tam lượng - Tứ mũ.

Múa cột

VD1: Tính nguyên hàm : $I = \int (2x^2 - 3).e^x dx$

(đạo hàm) $u = 2x^2 - 3$	dấu	(nguyên hàm) $dv = e^x dx$
$4x$	+	e^x
4	-	e^x
0	+	e^x

$$\Rightarrow I = e^x(2x^2 - 3) - 4x.e^x + 4e^x + C$$

$$= e^x(2x^2 - 4x + 1) + C$$

5.4 Trigonometry

$$\sin^2 a + \cos^2 a = 1$$

$$\tan a = \frac{\sin a}{\cos a}$$

$$\cot a = \frac{\cos a}{\sin a}$$

$$1 + \tan^2 a = \frac{1}{\cos^2 a}$$

$$1 + \cot^2 a = \frac{1}{\sin^2 a}$$

$$\tan a \cot a = 1$$

5.4.1 Cộng - Trừ

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

5.4.2 Nhân

$$\sin 2a = 2\sin a\cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$

$$\tan 2a = \frac{2\tan a}{1-\tan^2 a}$$

$$\cot 2a = \frac{\cot^2 a - 1}{2\cot a}$$

$$\sin 3a = 3\sin a - 4\sin^3 a$$

$$\cos 3a = 4\cos^3 a - 3\cos a$$

$$\tan 3a = \frac{3\tan a - \tan^3 a}{1-3\tan^2 a}$$

$$\cot 3a = \frac{3\cot^2 a - 1}{\cot^3 a - 3\cot a}$$

5.4.3 Hạ bậc

$$\cos^2 a = \frac{1 + \cos 2a}{2}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2}$$

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a}$$

$$\sin^2 a \cos^2 a = \frac{1 - \cos 4a}{8}$$

$$\cos^3 a = \frac{3\cos a + \cos 3a}{4}$$

$$\sin^3 a = \frac{3\sin a - \sin 3a}{4}$$

$$\sin^4 a = \frac{\cos 4a - 4\cos 2a + 3}{8}$$

$$\cos^4 a = \frac{\cos 4a + 4\cos 2a + 3}{8}$$

5.4.4 Tích tổng

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

5.4.5 Tổng tích

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a–b}{2}$$

$$\cos a– \cos b = –2\sin\frac{a+b}{2}\sin\frac{a–b}{2}$$

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a–b}{2}$$

$$\sin a– \sin b = 2\cos\frac{a+b}{2}\sin\frac{a–b}{2}$$

$$\cos a + \sin a = \sqrt{2}\cos\left(\frac{\pi}{4}–a\right) = \sqrt{2}\sin\left(\frac{\pi}{4}+a\right)$$

$$\cos a– \sin a = \sqrt{2}\cos\left(\frac{\pi}{4}+a\right) = \sqrt{2}\sin\left(\frac{\pi}{4}–a\right)$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a\cos b}$$

$$\tan a– \tan b = \frac{\sin(a–b)}{\cos a\cos b}$$

$$\cot a + \cot b = \frac{\sin(a+b)}{\sin a\sin b}$$

$$\cot a– \cot b = \frac{\sin(b–a)}{\sin a\sin b}$$

$$\cot a + \tan a = \frac{2}{\sin 2a}$$

$$\cot a– \tan a = 2\cot 2a$$

5.5 Cấp số nhân

$$S_n = u_1 + u_2 + ... + u_n = \frac{u_1(1-q^n)}{1-q}$$

$$u_n = u_{n-1}.q$$

$$u_n = u_1.q^{n-1}$$