Phép biến đổi tích phân

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1 Phép Biến Đổi Tích Phân

1.1 Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$[-\pi, \pi]$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, ...)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad (n = 1, 2, ...)$$

$[0, 2\pi] \to \int_0^{2\pi}$

1.1.1 Khoảng bất kỳ

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$[-p, p]$

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx \quad (n = 1, 2, ...)$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx \quad (n = 1, 2, ...)$$

1.1.2 Hàm chẵn lẻ

HÀM	TÍNH CHẤT	Điều kiện
Chẵn	$f(-x) = f(x)$	$\forall x$
Lẻ	$f(-x) = -f(x)$	$\forall x$

- Chẵn $\times$ Chẵn $=$ Chẵn
- Chẵn $\times$ Lẻ $=$ Lẻ
- Lẻ $\times$ Lẻ $=$ Chẵn
- $\cos \to$ Chẵn.
- $\sin \to$ Lẻ.

Hàm chẵn

$$\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)\,dx$$

Và

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

Trong đó

$$a_0 = \frac{1}{p}\int_{0}^{p} f(x)\,dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \quad (n = 1, 2, ...)$$

Hàm lẻ

$$\int_{-p}^{p} f(x)\,dx = 0$$

Và

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

Trong đó

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \quad (n = 1, 2, ...)$$

1.1.3 Parseval

$$\frac{1}{2p}\int_{-p}^{p} f(x)^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

1.2 Fourier Transform

1.2.1 Biểu diễn

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \quad (-\infty < x < \infty),\ \forall\omega \geq 0$$

Trong đó

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

Không liên tục tại x

$$\frac{f(x+) + f(x-)}{2}$$

1.2.2 Phép biến đổi

$$\hat{f}(\omega) = \mathcal{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \quad (-\infty < \omega < \infty)$$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \check{f}(x)\,dx$$

Ngược

$$\check{f}(x) = f(x) = \mathcal{F}^{-1}(\hat{f}(\omega))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)\,d\omega \quad (-\infty < x < \infty)$$

1.2.3 Tính chất

AXTT

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Đạo hàm

$$\mathcal{F}(f^{(n)}) = (i\omega)^n \mathcal{F}(f)$$

$$\mathcal{F}(xf(x))(\omega) = i\left[\hat{f}\right]'(\omega) = i\frac{d}{d\omega}\mathcal{F}(f)(\omega)$$

$$\mathcal{F}(x^n f(x))(\omega) = i^n \left[\hat{f}\right]^{(n)}(\omega)$$

Tích chập

$$f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

$$g * f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)f(t)\,dt$$

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(f * f)(\omega) = \hat{f}(\omega)^2$$

$$\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}^2(f) = f(-x)$$

Tìm $\mathcal{F}(g) \to$ Chọn $f(x)$ sao cho

$$\begin{aligned} \mathcal{F}(f(x)) &= g(\omega) \\ \mathcal{F}(\mathcal{F}(f)) &= \mathcal{F}(g(\omega)) \\ f(-x) &= \mathcal{F}(g) \end{aligned}$$

Gaussian Transform $\quad a > 0$

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}$$

$a > 0 \quad (a = 2a)$

$$\mathcal{F}\left(e^{-ax^2}\right)(\omega) = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

$a = 1$

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\omega) = e^{-\frac{\omega^2}{2}}$$

1.3 Basics

1.3.1 Euler formula

$$e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$$

$$e^{-i\omega x} = \cos(\omega x) - i\sin(\omega x)$$

$$|e^{-i\omega x}| = |\cos(\omega x) - i\sin(\omega x)| = \sqrt{\cos^2(\omega x) + \sin^2(\omega x)} = 1$$

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$$

$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu})$$

1.3.2 Differentiation

$$(c)^{'} = 0$$
$$(kx)^{'} = k\ ,\ (ku)^{'} = ku^{'}$$
$$(x^n)^{'} = nx^{n-1}\ ,\ (u^n)^{'} = u^{'}nu^{n-1}$$

$$(uv)^{'} = u^{'}v + v^{'}u$$
$$\left(\frac{u}{v}\right)^{'} = \frac{u^{'}v - v^{'}u}{v^2}$$
$$\left(\frac{1}{x}\right)^{'} = -\frac{1}{x^2}\ ,\ \left(\frac{1}{u}\right)^{'} = \frac{u^{'}}{u^2}$$

$$(\sqrt{x})^{'} = \frac{1}{2\sqrt{x}}\ ,\ (\sqrt{u})^{'} = \frac{u^{'}}{2\sqrt{u}}$$

$$(\sin x)^{'} = \cos x\ ,\ (\sin u)^{'} = u^{'}\cos u$$
$$(\cos x)^{'} = -\sin x\ ,\ (\cos u)^{'} = -u^{'}\sin u$$

$$(e^x)^{'} = e^x\ ,\ (e^u)^{'} = u^{'}e^u$$
$$(a^x)^{'} = a^x\ln a\ ,\ (a^u)^{'} = u^{'}a^u\ln a$$
$$(\ln x)^{'} = \frac{1}{x}\ ,\ (\ln u)^{'} = \frac{u^{'}}{u}$$
$$(\log_a x)^{'} = \frac{1}{x\ln a}\ ,\ (\log_a u)^{'} = \frac{u^{'}}{u\ln a}$$

$$(u \pm v)^{'} = u^{'} \pm v^{'}$$
$$(u_1 \pm ... \pm u_n)^{'} = u_1^{'} \pm ... \pm u_n^{'}$$
$$y = y[u(x)] \Rightarrow y^{'}(x) = y^{'}(u)u^{'}(x)$$
$$\left(\frac{ax+b}{cx+d}\right)^{'} = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cx+d)^2}$$

1.3.3 Integral

$$\int dx = x \; , \; \int du = u$$
$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} \; , \; \int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1}$$

$$\int a^{x} dx = \frac{a^{x}}{\ln a} \; , \; \int a^{u} dx = \frac{a^{u}}{\ln a}$$
$$\int \frac{dx}{x} = \ln |x| \; , \; \int \frac{du}{u} = \ln |u|$$

$$\int e^{x} dx = e^{x} \; , \; \int e^{u} du = e^{u}$$
$$\int e^{kx} dx = \frac{e^{k} x}{k}$$

$$\int \cos x dc = \sin x \; , \; \int \cos u du = \sin u$$
$$\int \sin x dx = -\cos x \; , \; \int \sin u du = -\cos u$$
$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b)$$
$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b)$$

$$\int (ax+b)^{\alpha} = \frac{1}{a} \frac{(ax+b)^{\alpha+1}}{\alpha+1}$$

Từng phần

$$\int_{a}^{b} u \, dv = uv|_{a}^{b} - \int_{a}^{b} v \, du$$
$$I = \int_{a}^{b} P(x) Q(x) \, dx$$
$$u = P(x)$$
$$dv = Q(x) dx$$

Chọn $P(x)$ đặt u: Nhất log - Nhì đa - Tam lượng - Tứ mũ.

Múa cột

VD1: Tính nguyên hàm : $I=\int(2x^2-3).e^x dx$

(đạo hàm) $u=2x^2-3$	dấu	(nguyên hàm) $dv=e^x dx$
$4x$	+	e^x
4	-	e^x
0	+	e^x

$$\Rightarrow I=e^x(2x^2-3)-4x.e^x+4e^x+C$$

$$=e^x(2x^2-4x+1)+C$$

1.3.4 Trigonometry

$$\sin^2 a+\cos^2 a=1$$

$$\tan a=\frac{\sin a}{\cos a}$$

$$\cot a=\frac{\cos a}{\sin a}$$

$$1+\tan^2 a=\frac{1}{\cos^2 a}$$

$$1+\cot^2 a=\frac{1}{\sin^2 a}$$

$$\tan a\cot a=1$$

Cộng - Trừ

$$\cos(a-b)=\cos a\cos b+\sin a\sin b$$

$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

$$\sin(a-b)=\sin a\cos b-\cos a\sin b$$

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}$$

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}$$

Nhân

$$\sin 2a = 2\sin a\cos a$$

$$\cos 2a = \cos^2 a – \sin^2 a = 2\cos^2 a – 1 = 1 – 2\sin^2 a$$

$$\tan 2a = \frac{2\tan a}{1 – \tan^2 a}$$

$$\cot 2a = \frac{\cot^2 a – 1}{2\cot a}$$

$$\sin 3a = 3\sin a – 4\sin^3 a$$

$$\cos 3a = 4\cos^3 a – 3\cos a$$

$$\tan 3a = \frac{3\tan a – \tan^3 a}{1 – 3\tan^2 a}$$

$$\cot 3a = \frac{3\cot^2 a – 1}{\cot^3 a – 3\cot a}$$

Hạ bậc

$$\cos^2 a = \frac{1 + \cos 2a}{2}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2}$$

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a}$$

$$\sin^2 a \cos^2 a = \frac{1 - \cos 4a}{8}$$

$$\cos^3 a = \frac{3\cos a + \cos 3a}{4}$$

$$\sin^3 a = \frac{3\sin a - \sin 3a}{4}$$

$$\sin^4 a = \frac{\cos 4a - 4\cos 2a + 3}{8}$$

$$\cos^4 a = \frac{\cos 4a + 4\cos 2a + 3}{8}$$

Tích tổng

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

Tổng tích

$$\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a–b}{2}$$

$$\cos a–\cos b=–2\sin\frac{a+b}{2}\sin\frac{a–b}{2}$$

$$\sin a+\sin b=2\sin\frac{a+b}{2}\cos\frac{a–b}{2}$$

$$\sin a–\sin b=2\cos\frac{a+b}{2}\sin\frac{a–b}{2}$$

$$\cos a+\sin a=\sqrt{2}\cos\left(\frac{\pi}{4}–a\right)=\sqrt{2}\sin\left(\frac{\pi}{4}+a\right)$$

$$\cos a–\sin a=\sqrt{2}\cos\left(\frac{\pi}{4}+a\right)=\sqrt{2}\sin\left(\frac{\pi}{4}–a\right)$$

$$\tan a+\tan b=\frac{\sin(a+b)}{\cos a\cos b}$$

$$\tan a–\tan b=\frac{\sin(a–b)}{\cos a\cos b}$$

$$\cot a+\cot b=\frac{\sin(a+b)}{\sin a\sin b}$$

$$\cot a–\cot b=\frac{\sin(b–a)}{\sin a\sin b}$$

$$\cot a+\tan a=\frac{2}{\sin 2a}$$

$$\cot a–\tan a=2\cot 2a$$

1.3.5 Cấp số nhân

$$S_n=u_1+u_2+...+u_n=\frac{u_1(1-q^n)}{1-q}$$

$$u_n=u_{n-1}.q$$

$$u_n=u_1.q^{n-1}$$