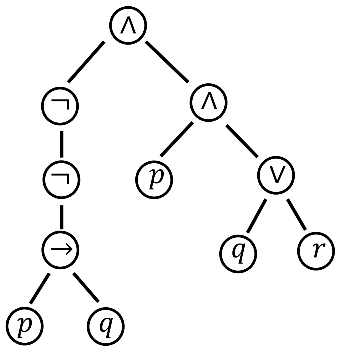
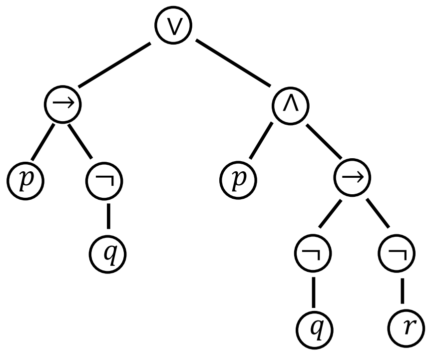
CS F214 Homework 1

1. (a)



(b)



2. (a) $(p_1 \vee p_3 \vee p_5 \vee p_6 \vee \neg p_3) \wedge (p_2 \vee p_4 \vee p_5 \vee p_6 \vee \neg p_2 \vee p_8) \wedge (p_1 \vee p_2 \vee p_6 \vee p_7 \vee \neg p_6)$
In the first term, i.e. $(p_1 \vee p_3 \vee p_5 \vee p_6 \vee \neg p_3)$, there is a p_3 and a $\neg p_3$ in disjunction. For any valuation, the entire term will result to true because of LEM $(p_3 \vee \neg p_3)$.
In the second term, i.e. $(p_2 \vee p_4 \vee p_5 \vee p_6 \vee \neg p_2 \vee p_8)$, there is a p_2 and a $\neg p_2$ in disjunction. For any valuation, the entire term will result to true because of LEM $(p_2 \vee \neg p_2)$.
In the third term, i.e. $(p_1 \vee p_2 \vee p_6 \vee p_7 \vee \neg p_6)$, there is a p_6 and a $\neg p_6$ in disjunction. For any valuation, the entire term will result to true because of LEM $(p_6 \vee \neg p_6)$.
Since the whole expression is a conjunction of the above three terms, and since we have shown that each term valuates to true, no matter what valuation we take, we arrive at $T \wedge T \wedge T$ which results to T.
This shows that the propositional formula
$(\boldsymbol{p_1} \vee \boldsymbol{p_3} \vee \boldsymbol{p_5} \vee \boldsymbol{p_6} \vee \neg \boldsymbol{p_3}) \wedge (\boldsymbol{p_2} \vee \boldsymbol{p_4} \vee \boldsymbol{p_5} \vee \boldsymbol{p_6} \vee \neg \boldsymbol{p_2} \vee \boldsymbol{p_8}) \wedge (\boldsymbol{p_1} \vee \boldsymbol{p_2} \vee \boldsymbol{p_6} \vee \boldsymbol{p_7} \vee \neg \boldsymbol{p_6})$
is valid.

(b) $(p_1 \vee p_2 \vee p_3 \vee p_6 \vee \neg p_3) \wedge (p_1 \vee p_3 \vee p_5 \vee p_6 \vee \neg p_2 \vee p_8) \wedge (p_1 \vee p_2 \vee p_6 \vee p_7 \vee p_8)$
If we can find a valuation such that atleast one of the terms in conjunction is F, then the entire propositional logic formula will be F because $F \wedge \phi$ results to F.
Take the term $(p_1 \vee p_2 \vee p_6 \vee p_7 \vee p_8)$. If we take p_1 as F, p_2 as F, p_6 as F, p_7 as F and p_8 as F, we will get $F \vee F \vee \mathrm{F} \vee \mathrm{F} \vee \mathrm{F}$, which results to F. Hence, we have atleast one valuation where the entire propositional formula valuates to false.
This shows that the propositional formula
$(\boldsymbol{p_1} \vee \boldsymbol{p_2} \vee \boldsymbol{p_3} \vee \boldsymbol{p_6} \vee \neg \boldsymbol{p_3}) \wedge (\boldsymbol{p_1} \vee \boldsymbol{p_3} \vee \boldsymbol{p_5} \vee \boldsymbol{p_6} \vee \neg \boldsymbol{p_2} \vee \boldsymbol{p_8}) \wedge (\boldsymbol{p_1} \vee \boldsymbol{p_2} \vee \boldsymbol{p_6} \vee \boldsymbol{p_7} \vee \boldsymbol{p_8})$
is not valid.

3. (a)

p	q	r	ϕ
1	1	1	**1**
1	1	0	**1**
1	0	1	**1**
1	0	0	**0**
0	1	1	**1**
0	1	0	**0**
0	0	1	**1**
0	0	0	**0**

Looking at the truth table given above, there are three valuations for which ϕ evaluates to F.
These are :
$p = 1, q = 0, r = 0$
$p = 0, q = 1, r = 0$
$p = 0, q = 0, r = 0$
We can represent the above valuations in propositional logic as given below respectively :
$\neg p \vee q \vee r$
$p \vee \neg q \vee r$
$p \vee q \vee r$
CNF of $\boldsymbol{\phi}$ will hence be $(\neg \boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r}) \wedge (\boldsymbol{p} \vee \neg \boldsymbol{q} \vee \boldsymbol{r}) \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r})$

PTO

(b)

p	q	r	s	ψ
1	1	1	1	**0**
1	1	1	0	**1**
1	1	0	1	**1**
1	1	0	0	**1**
1	0	1	1	**1**
1	0	1	0	**1**
1	0	0	1	**0**
1	0	0	0	**1**
0	1	1	1	**1**
0	1	1	0	**1**
0	1	0	1	**1**
0	1	0	0	**1**
0	0	1	1	**0**
0	0	1	0	**1**
0	0	0	1	**1**
0	0	0	0	**1**

Looking at the truth table given above, there are three valuations for which ϕ evaluates to F.
These are :
$p = 1, q = 1, r = 1, s = 1$
$p = 1, q = 0, r = 0, s = 1$
$p = 0, q = 0, r = 1, s = 1$
We can represent the above valuations in propositional logic as given below respectively :
$\neg p \vee \neg q \vee \neg r \vee \neg s$
$\neg p \vee q \vee r \vee \neg s$
$p \vee q \vee \neg r \vee \neg s$
CNF of $\boldsymbol{\phi}$ will hence be $(\neg \boldsymbol{p} \vee \neg \boldsymbol{q} \vee \neg \boldsymbol{r} \vee \neg \boldsymbol{s}) \wedge (\neg \boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r} \vee \neg \boldsymbol{s}) \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \neg \boldsymbol{r} \vee \neg \boldsymbol{s})$

4. We have $F_{(n+1)} = F_n + F_{(n-1)}$, where $F_1 = 1, F_2 = 1$. We wish to prove F_{3n} is even for $n \geq 1$ using mathematical induction. In any proof by mathematical induction, we first have a base case to show it is true for $n = 1$, then an induction hypothesis to assume it is true for $n = k$, then the induction step to show it is true for $n = k + 1$ if it is true for $n = k$.
Let us assume $M(n)$ is the statement that "F_{3n} is even"

BASE CASE : $F_3 = F_1 + F_2$

$$= 1 + 1 = 2 \text{ (is even)}$$

Hence we have $M(1)$ is true.

INDUCTION HYPOTHESIS : Assume $M(k)$ is true. This means that we assume F_{3k} is even.

INDUCTIVE STEP : $F_{3(k+1)} = F_{3k+3}$

$$\begin{aligned} &= F_{3k+3-1} + F_{3k+3-2} \\ &= F_{3k-1} + F_{3k-2} \\ &= F_{3k-1} + F_{3k-1} + F_{3k} \\ &= 2 \cdot F_{3k-1} + F_{3k} \end{aligned}$$

$2 \cdot F_{3k-1}$ will be even because any number multiplied with an even number will be an even number. From our induction hypothesis, we have F_{3k} is even. The sum of two even numbers will be even.

PROOF : Let x and y be two even numbers such that $x = 2a$ and $y = 2b$ $(a, b \in \mathbb{N})$.
$x + y = 2a + 2b$
$= 2(a + b)$ (is even)

Hence, $M(k + 1)$ is true, i.e. $F_{3(k+1)}$ will be even.

Hence, we have used mathematical induction to prove that F_{3n} is even for $n \geq 1$.

5. We wish to prove that the semantic entailment $\vDash p \wedge q \rightarrow p$ holds. The semantic entailment $\vDash p \wedge q \rightarrow p$ holds, hence the sequent $\vdash p \wedge q \rightarrow p$ must be valid, by the completeness theorem. We have two propositional atoms in the above sequent, p and q. Hence, we need $2^2 = 4$ different valuations where we need to show that $p \wedge q \rightarrow p$ results to T. We go about by using the Law of Excluded Middle (LEM) to arrive at each and every possible valuation, since LEM implies a tautology.

1 $p \vee \neg p$ LEM
2 p $assumption$
3 $q \vee \neg q$ LEM
4 q $assumption$
5 $p \wedge q$ $assumption$
6 p $copy\ 2$
7 $p \wedge q \rightarrow p$ $\rightarrow i\ 5,6$
8 $\neg q$ $assumption$
9 $p \wedge q$ $assumption$
10 p $copy\ 2$
11 $p \wedge q \rightarrow p$ $\rightarrow i\ 9,10$
12 $p \wedge q \rightarrow p$ $\vee e\ 3$, 4-7, 8-11
13 $\neg p$ $assumption$
14 $q \vee \neg q$ LEM
15 q $assumption$
16 $p \wedge q$ $assumption$
17 p $\wedge e_1 16$
18 $p \wedge q \rightarrow p$ $\rightarrow i\ 5,6$
19 $\neg q$ $assumption$
20 $p \wedge q$ $assumption$
21 p $\wedge e_1 20$
22 $p \wedge q \rightarrow p$ $\rightarrow i\ 9,10$
23 $p \wedge q \rightarrow p$ $\vee e\ 14$, 15-18, 19-22
24 $p \wedge q \rightarrow p$ $\vee e\ 1$, 2-12, 13-23

Hence, we have showed that the sequent $\vdash p \wedge q \rightarrow p$ is valid.
The semantic entailment $\vDash p \wedge q \rightarrow p$ holds.