CS F214 Homework 2

1. We have m, a nullary function (a constant), f, a unary function, S and B, two binary predicates.

- **i.** Since S is a binary predicate, it can take two terms. f is a function that returns one term. m is a constant term. Hence, we can say that $S(m, x)$ is a **valid formula in predicate logic.**
- **ii.** Since B is a binary predicate, it takes two terms. f is a function that returns one term. m is a constant term. $f(m)$ will also return a constant term. Hence, $B(m, f(m))$ is actually B receiving two constants terms as inputs and therefore a **valid formula in predicate logic**.
- **iii.** Since f is a function, it will output a term and not a truth value. Hence $f(m)$ is **an invalid formula in predicate logic.**
- **iv.** $B(m, x)$ is a predicate. B can only accept terms. It cannot accept predicates. In $B(B(m, x), y)$, B is trying to accept $B(m, x)$, a predicate, which is not allowed. Hence $B(B(m, x), y)$ is an **invalid formula in predicate logic.**
- **v.** Since B is a binary predicate, it has to take in two terms. Hence $B(m)$ is not allowed. Hence $S(B(m), z)$ is an **invalid formula in predicate logic.**
- **vi.** In $B(x, y) \rightarrow \exists z\ (\ S(y, z)\ )$, all the rules of predicate logic are being followed. Hence $B(x, y) \rightarrow \exists z\ (\ S(y, z)\ )$ is a **valid formula in predicate logic.**
- **vii.** Since f is a function, it will output a term. Hence $f(f(x))$ will also output a term. Hence $S(x, y) \rightarrow S(y, f(f(x)))$ is a **valid formula in predicate logic.**
- **viii.** Since B is a binary predicate, it has to take in two terms. Hence $B(x)$ is not allowed. Hence $B(x) \rightarrow B(B(x))$ is an **invalid formula in predicate logic.**

2. We are given
$W(x, y)$: x wrote y
$L(x, y)$: x is longer than y
$N(x)$: x is a novel
h: Hardy
a: Austen
j: Jude the Obscure
p: Pride and Predjudice

Looking at the first option, we have $\forall x\ (W(h, x) \rightarrow L(x, a)\ )$. This translates to "All of what Hardy has written, is longer than Austen". This clearly does not make sense in English as saying that a written item is longer than a human is nonsensical. Hence the first option is incorrect.
Looking at the second option, we have $\forall x\ \exists y\ (L(x, y) \rightarrow (W(h, y) \wedge W(a, x\ )))$. This translates to "For all x and some y, if x is longer than y, then Hardy has written y and Austen has written x". While this may seem correct at first glance, note that we have not defined what x and y are. They could be anything. Hence, the second option is incorrect.
Looking at the third option, we have $\forall x\ \forall y\ (W(h, x) \wedge W(a, y) \rightarrow L(x, y)))$. This translates to "For all x and all y, if Hardy has written x and Austen has written y, then x is longer than y". This appears to be the converse of the second option, the wrong quantifier for y has been used and again, there is no mention of what x and y are. Hence, the third option is incorrect.

Looking at the fifth option, we have $\exists x\ \forall y\ (W(h,x)\ \rightarrow\ (W(a,x\ ) \wedge L(x,y)))$. This translates to "For some x and all y, if Hardy has written x , then Austen has written y and x is longer than y". This is not the meaning we intend. Hence, the fifth option is incorrect.
Looking at the fourth option, we have $\exists x\ (N(x)\ \wedge\ W(h,x)\ \wedge\ \forall y\ (N(y)\ \wedge\ W(a,y))\ \rightarrow\ L(x,y))$. This translates to, "For some x, if x is a novel, written by Hardy, and for every single y that is a novel and written by Austen, then x is longer than y. This is exactly the meaning that we want, and it makes perfect sense according to the grammatical rules of English.
Hence, the sentence "Hardy wrote a novel which is longer than any of Austen's" can be represented in predicate logic as $\exists \boldsymbol{x}\ (\boldsymbol{N(x)}\ \wedge\ \boldsymbol{W(h,x)}\ \wedge\ \forall \boldsymbol{y}\ (\boldsymbol{N(y)}\ \wedge\ \boldsymbol{W(a,y)})\ \rightarrow\ \boldsymbol{L(x,y)})$.
Option 4 is correct.

3. We are required to prove that the following sequent is valid.
$\forall x\ (P(x)\ \rightarrow\ Q(x))\ \vdash\ \forall x\ (\neg Q(x))\ \rightarrow\ \forall x\ (\neg P(x))$

1 $\forall x\ \left(P(x) \rightarrow Q(x)\right)$ premise
2 $\forall x\ \neg Q(x)$ *assumption*
x_0
3 $\neg Q(x_0)$ $\forall x\ e$ 2
4 $P(x_0) \rightarrow Q(x_0)$ $\forall x\ e$ 1
5 $\neg P(x_0)$ MT 4, 3
6 $\forall x\ \neg P(x)$ $\forall x\ i$ 3-5
7 $(\forall x\ \neg Q(x)) \rightarrow (\forall x\ \neg P(x))$ $\rightarrow i$ 2-6

4. We are required to prove that the following sequent is valid.
$\forall x\ (P(x)\ \rightarrow\ \neg Q(x))\ \vdash\ \neg(\exists x\ (P(x) \wedge Q(x))$

1 $\forall x\ (P(x) \rightarrow\ \neg Q(x))$ premise
2 $\exists x\ \left(P(x) \wedge Q(x)\right)$ *assumption*
x_0
3 $P(x_0) \wedge Q(x_0)$
4 $P(x_0)$ $\wedge e_1$ 3
5 $Q(x_0)$ $\wedge e_2$ 3
6 $P(x_0) \rightarrow Q(x_0)$ $\forall x\ e$ 1
7 $\neg\left(\neg Q(x_0)\right)$ $\neg\neg i$ 5
8 $\neg P(x_0)$ MT 6, 7
9 $\perp$ $\neg e$ 4, 8
10 $\neg\left(\exists x\ \left(P(x) \wedge Q(x)\right)\right)$ $\neg i$ 2-9

5. We are required to prove that the following sequent is valid.
$\forall x\, \forall y\, \forall z\, ((S(x,y) \wedge S(y,z)) \rightarrow S(x,z)), \forall x\, \neg(S(x,x)) \vdash \forall x\, \forall y\, (S(x,y) \rightarrow \neg(S(y,x)))$

1 $\forall x\, \forall y\, \forall z\, \big((S(x,\, y) \wedge S(y,\, z)) \rightarrow S(x,\, z)\big)$ premise
2 $\forall x\, \neg(S(x,\, x))$

x_0
- 3 $\forall y\, \forall z\, \big((S(x_0,\, y) \wedge S(y,\, z)) \rightarrow S(x_0,\, z)\big)$ $\forall x\, e$ 1
- 4 $\neg(S(x_0, x_0))$ $\forall x\, e$ 2
- y_0
  - 5 $\forall z\, \big((S(x_0,\, y_0) \wedge S(y_0,\, z)) \rightarrow S(x_0,\, z)\big)$ $\forall y\, e$ 3
  - 6 $(S(x_0,\, y_0) \wedge S(y_0,\, x_0)) \rightarrow S(x_0,\, x_0)$ $\forall z\, e$ 5
    - 7 $S(x_0, y_0)$ *assumption*
      - 8 $S(y_0, x_0)$ *assumption*
      - 9 $S(x_0, y_0) \wedge S(y_0, x_0)$ $\wedge i$ 7, 8
      - 10 $S(x_0, x_0)$ $\rightarrow e$ 9, 6
      - 11 $\neg(S(x_0, x_0))$ *copy* 4
      - 12 $\perp$ $\neg e$ 10, 11
    - 13 $\neg S(y_0, x_0)$ $\neg i$ 8-12
  - 14 $S(x_0, y_0) \rightarrow \neg(S(y_0, x_0))$ $\rightarrow i$ 7-13
- 15 $\forall y\, S(x_0, y) \rightarrow \neg(S(y, x_0))$ $\forall y$ 5-14

16 $\forall x\, \forall y\, \big(S(x,y) \rightarrow \neg(S(y,\, x))\big)$ $\forall x$ 3-15

6. We have $\phi \stackrel{\text{def}}{=} \forall x\, \exists y\, \exists z\, (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x))$

(a) The model $\mathcal{M}$ consists of the set of natural numbers with $P^{\mathcal{M}} \stackrel{\text{def}}{=} \{(m,n\} \mid m < n\}$.

$P(x,y) \equiv x < y$
$P(z,y) \equiv z < y$
$\big(P(x,z) \rightarrow P(z,x)\big) \equiv \big((x < z) \rightarrow (z < x)\big)$
We can choose three natural numbers x, y and z such that $z < x < y$. Hence $P(x,y)$ will hold *true*, $P(z,y)$ will hold *true* and $\big(P(x,z) \rightarrow P(z,x)\big)$ will also hold *true* as $P(x,z)$ is *false*, and $P(z,x)$ is *true*, hence we get $F \rightarrow T$, which is *true*. Hence $(P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x))$ evaluates to $T \wedge T \wedge T$, which is *true*. Also, $z < x < y$ fit the quantifiers $\forall x\, \exists y\, \exists z$. Hence, we can say that **the model satisfies ϕ**.

(b) The model $\mathcal{M}'$ consists of the set of natural numbers with $P^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(m, 2 * m\} \mid m\ natural\ number\}$.

$P(x,y) \equiv (y = 2x)$
$P(z,y) \equiv (y = 2z)$
$\big(P(x,z) \rightarrow P(z,x)\big) \equiv \big((z = 2x) \rightarrow (x = 2z)\big)$

If we choose $x = k$, $y = 2k$ and $z = k$, then $y = 2x$. Also, $y = 2z$. Hence $P(x,y)$ will hold *true*, $P(z,y)$ will hold *true* and $\big(P(x,z) \rightarrow P(z,x)\big)$ will also hold *true* as $P(x,z)$ is

$false$, and $P(z,x)$ is $false$, hence we get $F \rightarrow F$, which is $true$ ($\big(P(x,z) \rightarrow P(z,x)\big) \equiv \big((z = 2x) \rightarrow (x = 2z)\big)$).
$(P(x,y)\ \land P(z,y) \land (P(x,z) \rightarrow P(z,x))$ evaluates to $T \land T \land T$, which is $true$. Also, $x = k$, $y = 2k$ and $z = k$, fit the quantifiers $\forall x\ \exists y\ \exists z$ as we can choose any arbitrary value of k (which will be the value of x), then decide values of y and z accordingly. Hence, we can say that **the model satisfies $\boldsymbol{\phi}$**.

(c) The model $\mathcal{M}''$ consists of the set of natural numbers with $P^{\mathcal{M}''} \stackrel{\text{def}}{=} \{(m,n\} \mid m < n + 1\}$.
$(x,y) \equiv x < y + 1$
$P(z,y) \equiv z < y + 1$
$\big(P(x,z) \rightarrow P(z,x)\big) \equiv \big((x < z + 1) \rightarrow (z < x + 1)\big)$

We can choose three natural numbers x, y and z such that $z\ <\ x\ + 1 <\ y + 2$. Hence $P(x,y)$ will hold $true$, $P(z,y)$ will hold $true$ and $\big(P(x,z) \rightarrow P(z,x)\big)$ will also hold $true$ as $P(x,z)$ is $false$, and $P(z,x)$ is $true$, hence we get $F \rightarrow T$, which is $true$. Hence $(P(x,y)\ \land P(z,y) \land (P(x,z) \rightarrow P(z,x))$ evaluates to $T \land T \land T$, which is $true$. Also, $z\ <\ x\ <\ y$ fit the quantifiers $\forall x\ \exists y\ \exists z$. Hence, we can say that **the model satisfies $\boldsymbol{\phi}$**.