

1 Rough Outline

Outlining the steps of LQR control design:

1. Derive the dynamic equations for the system
2. Express them in state-space form
3. Linearize them about a fixed point (equilibrium point), here it is the pendulum up position
4. Find the eigenvalues and poles of the linear system and check for its stability
- 5.

2 Cart-Inverted Pendulum dynamics

In this section, we intend to model the dynamics of the inverted pendulum + cart system using the Euler-Lagrange method, considering the following notations:

$$x_p = x + L \sin \theta$$

$$y_p = L \cos \theta$$

Kinetic energy of the cart is given as:

$$T_{cart} = \frac{1}{2} M \dot{x}^2$$

Kinetic energy of the pendulum is given as:

$$\begin{aligned} T_{pend} &= \frac{1}{2} m \dot{x}_p^2 + \frac{1}{2} m \dot{y}_p^2 \\ &= \frac{1}{2} m [\dot{x}^2 + 2L\dot{x}\dot{\theta} \cos \theta + L^2 \dot{\theta}^2] \end{aligned}$$

Hence, the total kinetic energy of the cart-inverted pendulum system is given as:

$$\begin{aligned} T_{net} &= T_{cart} + T_{pend} \\ T &= \frac{1}{2} (M+m) \dot{x}^2 + mL\dot{x}\dot{\theta} \cos \theta + \frac{1}{2} mL^2 \dot{\theta}^2 \end{aligned}$$

Considering the cart to be the reference, the system possesses potential energy only due to the bob-mass of the pendulum, which is given by:

$$V = mg y_p = mgL \cos \theta$$

With the above formulations for kinetic energy and potential energy, the Lagrangian for the system is written as:

$$\begin{aligned} L &= T - V \\ L &= \frac{1}{2} (M+m) \dot{x}^2 + mL\dot{x}\dot{\theta} \cos \theta + \frac{1}{2} mL^2 \dot{\theta}^2 - mgL \cos \theta \end{aligned}$$

For the co-ordinate x :

$$(M+m)\ddot{x} + mL\ddot{\theta} \cos \theta - mL\dot{\theta}^2 \sin \theta = F \quad (1)$$

For the co-ordinate θ :

$$L\ddot{\theta} + \ddot{x} \cos \theta - g \sin \theta = 0 \quad (2)$$

De-coupling the terms comprising $\ddot{\theta}$ and \ddot{x} using equations (1) and (2), we get the following non-linear, 2nd order ordinary differential equations (ODEs):

$$\ddot{x} = \frac{F + mL\dot{\theta}^2 \sin \theta - mg \sin \theta \cos \theta}{M + m \sin^2 \theta} \quad (3)$$

$$\ddot{\theta} = \frac{(M+m)g \sin \theta - F \cos \theta - mL\dot{\theta}^2 \sin \theta \cos \theta}{L(M + m \sin^2 \theta)} \quad (4)$$

The general state-space form of a system of non-linear equations is given as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

Since we have two 2nd order non-linear ODEs, we have to define 4 states as - $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, $x_4 = \dot{\theta}$ and an input $u = F$

From equations (3) and (4), we can write the non-linear state-space form of equations as shown below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{u + mLx_4^2 \sin x_3 - mg \sin x_3 \cos x_3}{M + m \sin^2 x_3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{(M+m)g \sin x_3 - u \cos x_3 - mLx_4^2 \sin x_3 \cos x_3}{L(M + m \sin^2 x_3)} \end{aligned}$$

3 Equilibrium Point Analysis

To find the equilibrium points of the state-space system, we set all derivatives equal to zero and solve for the state variables.

3.1 Equilibrium Conditions

At equilibrium: $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$

This gives us the following system of equations:

$$0 = x_2 \quad (5)$$

$$0 = \frac{u + mLx_4^2 \sin x_3 - mg \sin x_3 \cos x_3}{M + m \sin^2 x_3} \quad (6)$$

$$0 = x_4 \quad (7)$$

$$0 = \frac{(M+m)g \sin x_3 - u \cos x_3 - mLx_4^2 \sin x_3 \cos x_3}{L(M + m \sin^2 x_3)} \quad (8)$$

3.2 Solving for Equilibrium Points

From equations (5) and (7), we immediately obtain:

$$x_2^* = 0, \quad x_4^* = 0 \quad (9)$$

Substituting $x_4^* = 0$ into equation (6):

$$0 = u - mg \sin x_3 \cos x_3 \quad (10)$$

which yields:

$$u = mg \sin x_3 \cos x_3 \quad (11)$$

Substituting $x_4^* = 0$ into equation (8):

$$0 = (M + m)g \sin x_3 - u \cos x_3 \quad (12)$$

which gives:

$$u = \frac{(M + m)g \sin x_3}{\cos x_3} = (M + m)g \tan x_3 \quad (13)$$

3.3 Consistency Condition

For both conditions (11) and (13) to be satisfied simultaneously:

$$mg \sin x_3 \cos x_3 = (M + m)g \tan x_3 \quad (14)$$

Simplifying:

$$mg \sin x_3 \cos x_3 = (M + m)g \frac{\sin x_3}{\cos x_3} \quad (15)$$

Multiplying both sides by $\cos x_3$:

$$mg \sin x_3 \cos^2 x_3 = (M + m)g \sin x_3 \quad (16)$$

If $\sin x_3 \neq 0$, we can divide both sides by $g \sin x_3$:

$$m \cos^2 x_3 = M + m \quad (17)$$

This implies:

$$\cos^2 x_3 = \frac{M + m}{m} > 1 \quad (18)$$

This is **impossible** since $\cos^2 x_3 \leq 1$ for all real values of x_3 .

Therefore, we must have:

$$\sin x_3 = 0 \Rightarrow x_3^* = n\pi, \quad n \in \mathbb{Z} \quad (19)$$

When $\sin x_3 = 0$, from equation (11):

$$u^* = 0 \quad (20)$$

3.4 Equilibrium Points

The equilibrium points of the system are:

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (x_1^*, 0, n\pi, 0), \quad u^* = 0 \quad (21)$$

where:

- x_1^* can be any value (the cart position is arbitrary at equilibrium)
- $n = 0, \pm 1, \pm 2, \dots$

3.5 Physical Interpretation

The two primary equilibrium configurations are:

1. **Upright Position** ($x_3^* = 0$): The pendulum points upward. This is an *unstable equilibrium*.
2. **Downward Position** ($x_3^* = \pi$): The pendulum hangs downward. This is a *stable equilibrium*.

Both equilibrium positions require zero control input ($u^* = 0$) and zero velocities ($x_2^* = x_4^* = 0$).

4 Linearization About the Unstable Equilibrium Point

To design a linear controller for the inverted pendulum system, we linearize the nonlinear state-space equations about the upright (unstable) equilibrium point.

4.1 Equilibrium Point

The equilibrium point for the upright position is:

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{u} = 0 \quad (22)$$

where \bar{x}_1 is arbitrary (the cart can be at any position).

4.2 Perturbation Variables

We define small perturbations from the equilibrium:

$$\delta x_i = x_i - \bar{x}_i, \quad i = 1, 2, 3, 4 \quad (23)$$

$$\delta u = u - \bar{u} = u \quad (24)$$

4.3 Linearized System Form

The linearized system has the form:

$$\delta \dot{\mathbf{x}} = A\delta \mathbf{x} + B\delta u \quad (25)$$

where A is the state matrix and B is the input matrix, obtained by computing partial derivatives of the nonlinear system at the equilibrium point.

4.4 State Equations

The nonlinear state equations are:

$$f_1 = \dot{x}_1 = x_2 \quad (26)$$

$$f_2 = \dot{x}_2 = \frac{u + mLx_4^2 \sin x_3 - mg \sin x_3 \cos x_3}{M + m \sin^2 x_3} \quad (27)$$

$$f_3 = \dot{x}_3 = x_4 \quad (28)$$

$$f_4 = \dot{x}_4 = \frac{(M+m)g \sin x_3 - u \cos x_3 - mLx_4^2 \sin x_3 \cos x_3}{L(M + m \sin^2 x_3)} \quad (29)$$

4.5 Computing the State Matrix A

The state matrix is:

$$A = \left[\frac{\partial f_i}{\partial x_j} \right]_{\bar{x}, \bar{u}} \quad (30)$$

4.5.1 First Row of A

From $f_1 = x_2$:

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial x_3} = 0, \quad \frac{\partial f_1}{\partial x_4} = 0 \quad (31)$$

4.5.2 Second Row of A

At the equilibrium point where $x_3 = 0, x_4 = 0, u = 0$:

- $\sin(0) = 0, \cos(0) = 1$
- Denominator: $M + m \sin^2(0) = M$

The partial derivatives are:

$$\frac{\partial f_2}{\partial x_1} = 0, \quad \frac{\partial f_2}{\partial x_2} = 0 \quad (32)$$

For $\frac{\partial f_2}{\partial x_3}$, differentiating f_2 and evaluating at equilibrium:

$$\frac{\partial f_2}{\partial x_3} \Big|_{eq} = \frac{-mg}{M} \quad (33)$$

$$\frac{\partial f_2}{\partial x_4} = 0 \quad (\text{at equilibrium}) \quad (34)$$

4.5.3 Third Row of A

From $f_3 = x_4$:

$$\frac{\partial f_3}{\partial x_1} = 0, \quad \frac{\partial f_3}{\partial x_2} = 0, \quad \frac{\partial f_3}{\partial x_3} = 0, \quad \frac{\partial f_3}{\partial x_4} = 1 \quad (35)$$

4.5.4 Fourth Row of A

At the equilibrium point:

$$\frac{\partial f_4}{\partial x_1} = 0, \quad \frac{\partial f_4}{\partial x_2} = 0 \quad (36)$$

For $\frac{\partial f_4}{\partial x_3}$:

$$\frac{\partial f_4}{\partial x_3} \Big|_{eq} = \frac{(M+m)g}{LM} \quad (37)$$

$$\frac{\partial f_4}{\partial x_4} = 0 \quad (\text{at equilibrium}) \quad (38)$$

4.6 Computing the Input Matrix B

The input matrix is:

$$B = \left[\frac{\partial f_i}{\partial u} \right]_{\bar{x}, \bar{u}} \quad (39)$$

Computing each element:

$$\frac{\partial f_1}{\partial u} = 0 \quad (40)$$

$$\frac{\partial f_2}{\partial u} = \frac{1}{M} \quad (41)$$

$$\frac{\partial f_3}{\partial u} = 0 \quad (42)$$

$$\frac{\partial f_4}{\partial u} = \frac{-1}{LM} \quad (43)$$

4.7 Linearized State-Space Model

The complete linearized system about the upright equilibrium point is:

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \\ \delta \dot{x}_3 \\ \delta \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{LM} & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{1}{M} \\ \frac{-1}{LM} \end{bmatrix} \delta u \quad (44)$$

4.8 Physical Interpretation

This linearized model is valid for small deviations from the upright position. The key observations are:

- The system is **unstable** (positive term $\frac{(M+m)g}{LM}$ in A matrix indicates instability)
- The state x_1 (cart position) does not appear in the dynamics, indicating neutral stability in the horizontal direction
- The coupling between cart acceleration and pendulum angle ($\frac{-mg}{M}$) and between control input and angular acceleration ($\frac{-1}{LM}$) shows how the system can be controlled

This linearized model is commonly used for designing linear controllers such as:

- Linear Quadratic Regulator (LQR)
- Pole placement controllers
- State feedback controllers