Analytical physics to Automatic control : a seamless treatise

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Contents

Foreword **9**

Chap1 Preparation **12**
1.1 Single variable differential equation . . . 12
1.1.1 Energy storage devices . . . 12
1.1.2 First order differential equation . . . 13
1.1.3 Second order linear differential equation . . . 14
1.1.4 Solution of second order differential equation . . . 15
1.1.5 nth order linear differential equation . . . 15
1.1.6 First order representation of linear differential equation . . . 16
1.2 Multivariable differential equation . . . 17
1.2.1 Partial differential equation; definition . . . 17
1.2.2 Partial differential equation; solution . . . 17
1.2.3 Partial differential equation; examples . . . 18
1.3 Treatise on differential equation . . . 20
1.4 Nonlinear differential equation . . . 20
1.5 Perspective of differential equations . . . 20
1.6 Variational integral . . . 21
1.7 Lagrange multipliers . . . 24
1.7.1 Simultaneous equation . . . 24
1.7.2 Min-max problem . . . 25
1.7.3 Gradient . . . 25
1.7.4 Lagrange multipliers . . . 26
1.7.5 Lagrange multipliers by multiple constraints . . . 27
1.7.6 Lagrange multipliers; a Lagrangian . . . 28
1.7.7 Lagrange's equation: first kind . . . 28
1.8 Legendre transform . . . 29

PartI Analytical physics **31**

Chap2 Exodus from coordinates **33**
2.1 Newton's law of motion . . . 33
2.2 Newton's law of universal gravitation . . . 33
2.3 Field theory . . . 35
2.4 Coordinates . . . 36
2.4.1 Multiple coordinates . . . 36

2.4.2 Polar coordinates . . . 36
2.4.3 Problems caused by different point of view . . . 39
2.5 Generalized coordinates . . . 39
2.5.1 Extention of coordinates . . . 39
2.5.2 Overview of progress . . . 40
2.6 Energy . . . 41
2.6.1 Kinetic and potential energy . . . 41
2.6.2 Kinetic or potential ? . . . 41
2.6.3 kinetic energy . . . 42
2.6.4 potential energy . . . 42
2.6.5 gravitational potential of two objects . . . 43
2.6.6 gravitational potential by earth . . . 44
2.6.7 Complementary energy . . . 44

Chap3 Lagrangian **45**
3.1 Optimized trajectory . . . 45
3.2 Principle of least action . . . 46
3.3 Lagrangian . . . 46
3.3.1 Formulation of problem . . . 46
3.3.2 Calculation of $\mathcal{T} - \mathcal{V}$ . . . 47
3.4 Euler-Lagrange's equation . . . 48
3.5 Why Euler-Lagrange's equation? . . . 49
3.6 Generalized coordinates . . . 50
3.6.1 Euler-Lagrange's equation in two coordinates . . . 51
3.6.2 Express q by Q . . . 52
3.6.3 Express $\dot{q}$ by Q . . . 52
3.6.4 Slope of $\dot{q} - Q$ . . . 53
3.6.5 Time progress of $\partial q_i / \partial Q_k$ . . . 53
3.6.6 Proof of coordinate independence of Euler-Lagrange's equation . . . 54
3.6.7 Another proof . . . 55
3.7 What is Lagrangian? . . . 56
3.7.1 Lagrangian variables q and $\dot{q}$ . . . 56
3.7.2 Lagrangian as an intermediate state to analytical physics . . . 57
3.8 Summary . . . 58

Chap4 Hamiltonian **59**
4.1 "Definition" of Hamiltonian . . . 59
4.2 Lagrangian to Hamiltonian . . . 60
4.2.1 Introduction of new variable . . . 60
4.2.2 Applying Legendre transform . . . 60
4.2.3 Lagrangian of p_i, q_i . . . 62
4.2.4 Variational integral of $\mathcal{L}$ . . . 62
4.3 What Hamilton's equation tells . . . 64
4.4 Independence of p and q . . . 65

4.5 Reduction of order of differential equation . . . 66
4.6 Examples . . . 66
4.6.1 Newton's law expressed by Hamiltonian . . . 66
4.6.2 Newton's law in polar coordinates . . . 67
4.6.3 Free fall . . . 68
4.7 When Hamiltonian changes . . . 68
4.8 Canonical transformation . . . 69
4.8.1 Transform the "speed" ? . . . 69
4.8.2 Canonical transformation - superficial definition . . . 70
4.8.3 Transformation; Hamiltonian is retained . . . 70
4.8.4 Canonical transformation by Legendre transform . . . 71
4.8.5 Transformation; Hamiltonian is not retained . . . 71
4.9 Conserved value by canonical transformation . . . 73
4.10 Poisson bracket . . . 74
4.11 Why Hamilton deserves Hamilton's equation . . . 76

Chap5 Extending the Hamiltonian mechanics 78
5.1 Limits of Hamilton's equation . . . 78
5.2 Freed from fixed endpoint . . . 78
5.2.1 Equation of motion under time varying force . . . 78
5.2.2 Extending the Hamilton's equation : overview . . . 79
5.2.3 Extending the Hamilton's equation . . . 80
5.2.4 Value of S . . . 83
5.3 Extending the Hamiltonian mechanics . . . 83
5.4 Variables of Hamilton-Jacobi's equation . . . 84

Chap6 Hamilton-Jacobi's equation 85
6.1 Brief history . . . 85
6.2 How to change Hamiltonian . . . 85
6.2.1 Differential of W() . . . 86
6.2.2 Zero Hamiltonian . . . 86
6.2.3 Time t in Hamiltonian . . . 87
6.2.4 Canonical transformation by Hamilton-Jacobi's equation . . . 87
6.3 Generating function . . . 88
6.3.1 Variables in generating function . . . 88
6.3.2 Variables selection in generating function . . . 89
6.3.3 How generating function works . . . 90
6.4 Hamilton-Jacobi's theory . . . 90
6.4.1 Energy conserving system . . . 91
6.4.2 Process to solve Hamilton's equation . . . 91
6.4.3 Non energy conserving system . . . 93
6.5 Example of Hamilton-Jacobi's equation . . . 94
6.6 World of Hamilton-Jacobian . . . 97

PartII What is control **98**

Chap7 System expression **100**
7.1 Introduction . . . 100
7.2 System structure expression . . . 100
7.2.1 Linear or nonlinear . . . 101
7.2.2 Single variable or multivariable . . . 102
7.2.3 Known or unknown system . . . 103
7.3 Approach to nonlinear systems . . . 103
7.4 Feedback control . . . 104
7.5 Transfer function . . . 104
7.5.1 History of transfer function . . . 104
7.5.2 Definition of transfer function . . . 105
7.5.3 Transfer function of feedback loop . . . 106
7.5.4 Amplifier with feedback . . . 107
7.5.5 Conversion between parallel and series modeling . . . 109
7.6 Multi variable control vs classic control . . . 110
7.6.1 Classic control ; method and scope . . . 110
7.6.2 Multivariable control ; method and scope . . . 111
7.6.3 Potential differences in object setting . . . 111
7.6.4 Differences in signal processing system . . . 112
7.7 Potential oscillation . . . 113
7.7.1 Prevention . . . 113
7.7.2 Oscillators . . . 114

Chap8 Stability visualization and safety **117**
8.1 Bode diagram . . . 117
8.2 Bode diagram examples . . . 117
8.2.1 First order transfer function . . . 117
8.2.2 Second order transfer function -1 . . . 119
8.2.3 Second order transfer function -2 . . . 120
8.2.4 Transfer function of higher order . . . 121
8.3 Stability in practice . . . 122
8.3.1 Stability in classic control . . . 122
8.3.2 Stability in multivariable control . . . 122
8.4 Trouble shooting of loop . . . 123
8.5 Transient and malfunction . . . 123

Chap9 System identification **125**
9.1 Introduction . . . 125
9.2 Data collection and analysis . . . 126
9.2.1 Data collection and analysis for single I/O system . . . 126
9.2.2 System modeling . . . 127
9.3 Mathematical system modeling . . . 127

9.3.1 Single variable system modeling . . . 127
9.3.2 Multivariable system modeling . . . 129
9.4 System modeling method . . . 129
9.4.1 ARX model . . . 130
9.4.2 Box-Jenkins model . . . 130
9.4.3 State space model . . . 130
9.5 Model validation . . . 131

PartIII Multivariable control **132**

Chap10Overview of multivariable control **133**
10.1 First order representation . . . 134
10.2 Euler-Lagrange's equation of first order . . . 134
10.3 Implanting interface to the system . . . 135
10.4 Optimization of open loop system . . . 135
10.5 Optimization of closed loop system . . . 136
10.6 Comparison of Optimization method . . . 136
10.7 Hamilton-Jacobi's equation in control . . . 137
10.8 Ricatti equation . . . 137
10.9 Stability and system response . . . 137
10.10Design parameter . . . 138
10.11Multivariable control - the best choice ? . . . 138

Chap11Lyapunov theory of stability **139**
11.1 Introduction . . . 139
11.2 Stability of linear system . . . 140
11.3 Lyapunov theory . . . 141
11.3.1 Stability of general differential equation . . . 141
11.3.2 Lyapunov's idea . . . 141
11.3.3 Lyapunov's stability theorem . . . 142
11.4 Lyapunov theory applied to linear systems . . . 143
11.4.1 Quadratic generalized energy . . . 143
11.4.2 Value of quadratic form . . . 144
11.4.3 Example . . . 144
11.4.4 Stability of linear system . . . 145
11.5 Generalized energy . . . 146
11.6 Differential equation and physics . . . 146
11.7 Summary . . . 146

Chap12Multivariable system expression **147**
12.1 System modeling . . . 147
12.2 State variable model . . . 147
12.3 Transfer function model of multivariable system . . . 148
12.3.1 Transfer function . . . 148

12.3.2 Extention of transfer function to multivariable system 149
12.3.3 Transfer function of multivariable system 149
12.3.4 Transfer function in matrix form 150
12.3.5 Canonical form 151
12.4 Observable, controllable 152
12.5 Observability, controllability check 152
12.6 Linear transformation of system 153
12.7 Position of transfer function in multivariable system 153

Chap13Response of state variables 155
13.1 State transition matrix 155
13.2 Resolvent matrix 156
13.3 Response of the forced multivariable system 157
13.4 Summary 157

Chap14Optimal motion of a system with input 159
14.1 System assumption 159
14.2 Optimized system energy 160
14.2.1 Optimization index 160
14.2.2 System variable selection 160
14.2.3 Reduction of order; theory 160
14.3 Lagrange's undetermined multiplier 161
14.3.1 Incorporation of input 162
14.3.2 Calculation of optimal system 163
14.3.3 Lagrange's multiplier method - continued 164
14.4 Pontryagin's H function 165
14.5 Pontryagin's H function; Hamiltonian ? 166
14.6 Calculation of optimal system motion 167
14.7 Examples of optimal system with input 167
14.7.1 Example 1 167
14.7.2 Example 2 169
14.8 Summary 170

Chap15Synthesis of multivariable system by Hamilton-Jacobi's equation 171
15.1 Introduction 171
15.2 Introduction of a new performance function 171
15.3 System equation based on a new performance function 172
15.4 What became of Lagrange's multiplier? 173
15.5 Hamilton-Jacobi's equation 174
15.5.1 System expression by Hamilton-Jacobi's equation 174
15.5.2 General procedure to solve Hamilton-Jacobi's equation 175
15.6 Comparison of H function with Hamilton-Jacobi's equation 175
15.7 Design process of closed system 176
15.8 Summary of closed system 176

Chap16Matrix Ricatti equation **178**
16.1 Performance index . . . 178
16.2 Hamilton-Jacobi's equation for linear system . . . 178
16.3 Optimized Hamilton-Jacobi's equation for linear system . . . 179
16.3.1 Optimization of input . . . 179
16.3.2 Hamilton-Jacobi's equation of state variables . . . 180
16.3.3 Emerging of Ricatti equation . . . 181
16.4 Simplified process to Ricatti equation . . . 181
16.5 Linear quadratic Regulator . . . 182
16.6 Summary . . . 184

Chap17Calculation of Riccati equation **185**
17.1 Methods of calculation . . . 185
17.2 Character of solution . . . 185
17.3 Selection of $\boldsymbol{Q}, \boldsymbol{R}$ . . . 186
17.4 Non-iterative method . . . 186
17.5 Iterative method . . . 188
17.6 Summery . . . 188

Chap18Inverted pendulum **189**
18.1 Physics of inverted pendulum . . . 189
18.2 Mechanical structure . . . 190
18.3 Analysis of the system dynamics . . . 190
18.3.1 Equation of motion of inverted pendulum . . . 191
18.3.2 System equation of inverted pendulum . . . 192
18.4 Linear quadratic regulator applied to inverted pendulum . . . 193
18.5 Summary . . . 193
18.6 Appendix . . . 194

Chap19Robust and nonlinear control **195**
19.1 Robust control . . . 195
19.1.1 Norms . . . 195
19.1.2 H_∞ norm . . . 195
19.1.3 Summary . . . 196
19.2 Nonlinear control . . . 197

Chap20Lagrangian of electric circuit **198**
20.1 Coordinates in electricity . . . 198
20.2 Maxwell's equation . . . 199
20.3 Choice of coordinates . . . 200
20.3.1 Brief review of Lagrangian . . . 200
20.3.2 Electric field based choice . . . 200
20.3.3 Magnetic field based choice . . . 201
20.3.4 Summary of variables . . . 201
20.4 Lagrangian as circuit analysis . . . 201

20.5 Examples of Lagrangian in electric circuits . . . 202
20.5.1 Example 1: LC resonator . . . 202
20.5.2 Example 2: Circuit with power source . . . 203
20.5.3 Example 3: Complicated circuit . . . 206
20.6 Summary . . . 206

PartIV Classic control **208**

Chap21Modeling by transfer function **210**
21.1 Rotating disc . . . 210
21.2 Electronic circuit 1 . . . 211
21.3 Electronic circuit 2 . . . 212
21.4 Sample and hold . . . 212
21.5 Power train . . . 213
21.6 Small motors . . . 214

Chap22Classic control **216**
22.1 Rotation phase control system . . . 216
22.1.1 Cylinder of video tape recorder . . . 216
22.1.2 Speed servo control of a motor . . . 217
22.1.3 Phase servo control of a motor . . . 218
22.1.4 Ambivalence . . . 220
22.2 Phase locked loop . . . 220

Chap23DC motor control example **222**
23.1 Plan . . . 222
23.2 Speed servo . . . 222
23.3 Phase servo . . . 223
23.4 Adjustment and operation . . . 225
23.5 Design overview . . . 227
23.6 Evaluation and measurement . . . 227
23.6.1 Simple evaluation . . . 228
23.6.2 Characteristics measurement . . . 228
23.7 Improvement of characteristics . . . 229

Chap24Appendices **231**
24.1 Fourier/Laplace transform to find the solution . . . 231
24.2 Matrix calculation . . . 232
24.2.1 Basic properties . . . 232
24.2.2 Multiplication of matrices . . . 234
24.2.3 Multiplication of vector and martix . . . 235
24.2.4 Quadratic form . . . 235
24.2.5 Differential of quadratic form . . . 236
24.3 Lagrange Dirichlet stability . . . 237

References **239**

Foreword

The objective of this document is to guide readers to the entrance of robust control theory.

History of control theory can be divided into three era - classic, multivariable (or modern) and robust control. The advent of classic treatise is said to be the days of James Watt when he encountered an oscillatory behavior of steam engine as the steam technology advanced. J.C.Maxwell published a paper "On governors" in 1868 and this is the first mathematical treatment of feedback.

The application of Laplace transform to control theory is achieved by Hall in 1940 in study of anti aircraft heavy guns. The classic control theory which deals with single input/output system is established around this era. It comprises heuristic design, but easy to apply for many applications and still widely used.

After the WW 2, the focus of control shifted from classic single variable systems to multivariable systems particularly in the need of application for airborne vehicles. The multivariable control theory is based on analytical physics which is an intrinsic multivariable theory of physics and provided rigorous methodology combined with optimization. Thus, multivariable control theory progressed until some people began to realize that it depends too much on the correctness of the model of the target to be controlled.

The turn of multivariable control theory came around the middle of 1970s, when it became understood that the rigor of multivariable design could fail when applied to less rigorous reality. This is the advent of robust control, and now research and application belong to this era.

For students of automatic control, it is not an easy task to reach the entrance of this current control theory because of the underlying physics - analytical physics - is said to be not easy.

Here, the author tried very orthodox approach[1]; that is, first discuss the analytical physics so that reader can understand what it is almost fully before getting into control. In the author's opinion, there is no "elementary" analytical physics. People understand it or not. However, once understood, it is not philosophically difficult although the discussion may be cumbersome.

In the discussion of analytical physics alone, there are many amazing ideas that stimulate students - not limited to those who are interested in control. And the author tried to describe these wonders along with the logical process of analytical physics.

The author also tried to pay the same effort into the discussion of automatic control. The main objective is of course the multivariable control, but the common sense of classic control is also included, although not very comprehensively. For instance, the ease of a small DC motor control may open a door to people who may never get into control field otherwise.

Finally, author used analytical "mechanics" to those belong to pre-electric days, whereas analytical "physics" is used to comprise wider physics, although not very rigorously.

June 2015 S. Tajima

[1] Too orthodox, and thus became unique.

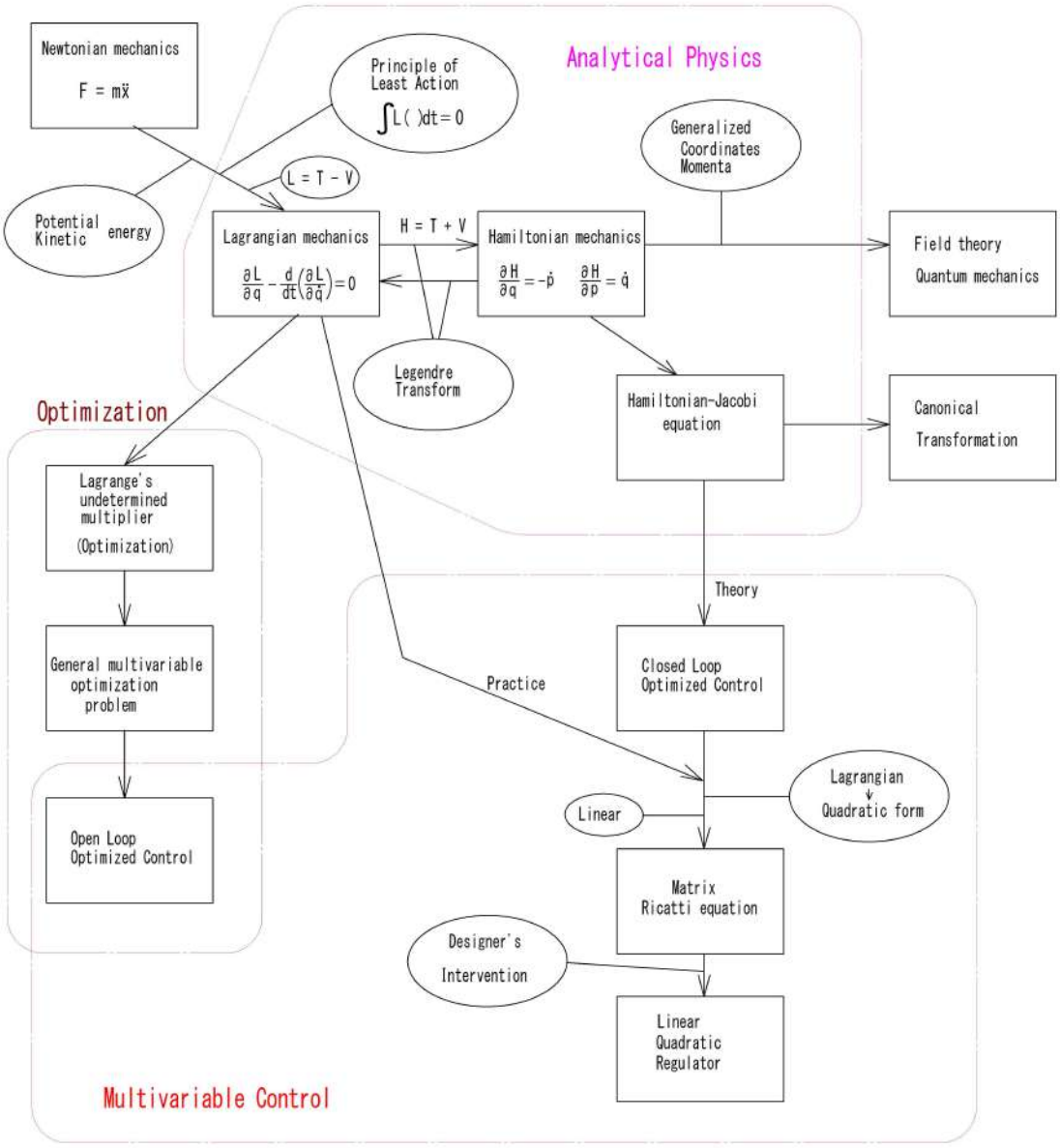

Figure 1: Relation of topics in physics and control theory

Chap1 Preparation

1.1 Single variable differential equation

In this section, we show that a dynamical system is expressed by a differential equation in two ways. One is the expression by a single nth order differential equation and the other is a set of n first order differential equations. They are convertible with each other, but the latter inherently expresses multi equations and used for discussion in multivariable control.

The differential equation discussed in this section is a **single variable differential equation**, which is generally called **ordinary differential equation**.

1.1.1 Energy storage devices

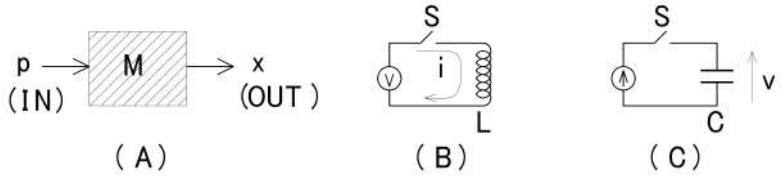

Figure 1.1: Energy storage component

Fig 1.1 shows energy storage devices in different physics. These devices have different shapes but we can generalize the characters by integral or differential operators. In case of mass, applied momentum p is integrated by time and produce the displacement of it's position.

In case of electric circuit, we have two complementary energy storage devices ; capacitors and inductors. (B) shows an inductor driven by voltage source. When the switch is turned on, voltage is integrated by the inductance L and current is observed through L. (C) shows a capacitor for which current is applied and the C integrates current to produce voltage.

They are expressed by the following equations.

$$x = \frac{1}{M}\int pdt, \quad i = \frac{1}{L}\int vdt, \quad v = \frac{1}{C}\int idt.$$

Or if we look from the other way,

$$p = M\frac{dx}{dt}, \quad v = L\frac{di}{dt}, \quad i = C\frac{dv}{dt}.$$

1.1.2 First order differential equation

The physical systems with dynamic properties are expressed by differential equations. For instance, Fig1.2 shows a simplest low pass filter used in electronics.

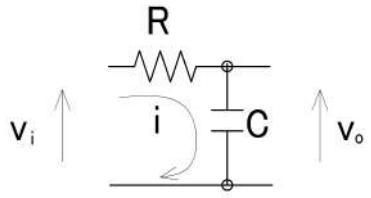

Figure 1.2: Low pass filter

The energy storage element in this circuit is the capacitor C. When voltage v_i is applied and output v_o is observed, the circuit will be written by,

$$v_i = Ri + \frac{1}{C}\int idt,$$
$$v_o = \frac{1}{C}\int idt.$$

Or if differentiate,

$$\frac{dv_i}{dt} = R\frac{di}{dt} + \frac{1}{C}i,$$
$$\frac{dv_o}{dt} = \frac{1}{C}i,$$

yielding first order differential equations.
Then,

$$\frac{dv_i}{dt} - \frac{dv_o}{dt} = \frac{d}{dt}(v_i - v_o) = R\frac{di}{dt},$$
$$v_i - v_o = Ri.$$

Therefore,

$$\dot{v}_o = \frac{dv_o}{dt} = \frac{1}{C}i = \frac{v_i - v_o}{CR}.$$
$$\dot{v}_o = -\frac{1}{CR}v_o + \frac{1}{CR}v_i.$$

In electronic circuit analysis, the above manipulation is rarely found, but the circuit of this example is expressed by the form,

$$\dot{x} = Ax + Bu.$$

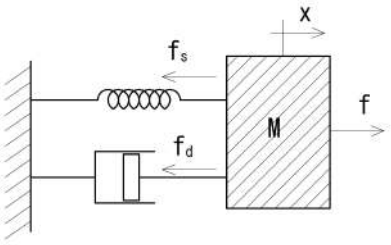

Figure 1.3: Mass with spring and damper

1.1.3 Second order linear differential equation

When a physical system comprises multiple energy storage elements - mass, springs, capacitors, inductors - then the order of the system differential equation reflects that number[1] when expressed by a single differential equation.

In Fig1.3, a mass is suspended by a spring and a damper. In this system, the energy storage elements are spring and mass, whereas the damper works as a break against the velocity of mass. This model is very commonly found in practical applications - ie. the suspension of automobiles.

The differential equation for this system is

$$M\frac{d^2x(t)}{dt^2} = f(t) - f_s(t) - f_d(t)$$

$$= f(t) - Kx(t) - D\frac{d(x)}{dt},$$

or

$$M\frac{d^2x(t)}{dt^2} + D\frac{d(x)}{dt} + Kx(t) = f(t).$$

where M is mass, D is the coefficient of damper and K is the coefficient of spring. As shown, the system is written by a second order (linear) differential equation.

This second order differential equation can be written by a set of first order equation as follows. Since we are going to have two equations, rewrite x by x_1 and using dot notation,

$$M\ddot{x}_1 + D\dot{x}_1 + Kx_1 = f.$$

Introducing new variable $x_2 = \dot{x}_1$, we obtain

$$M\dot{x}_2 + Dx_2 + Kx_1 = f.$$

Thus we have,

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\frac{K}{M}x_1 - \frac{D}{M}x_2 + \frac{1}{M}f.$$

Or,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{M} \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix},$$

which conforms to the form, $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$.

[1] Unless some components are made into one, like parallel connection of capacitors.

1.1.4 Solution of second order differential equation

In case that there is no external force, the second order differential equation is written by a homogeneous form[2].

$$\frac{d^2x(t)}{dt^2} + p\frac{d(x)}{dt} + qx(t) = 0.$$

General solution of a homogeneous second order differential equations is calculated in the following way. Assuming $x(t) = e^{\lambda x}$, we obtain

$$(\lambda^2 + p\lambda + q)e^{\lambda x(t)} = 0.$$

$$\lambda^2 + p\lambda + q = 0.$$

Thus, the solution of the differential equation becomes,

$$x_1(t) = e^{\lambda_1 x(t)}, \quad x_2(t) = e^{\lambda_2 x(t)},$$

where,

$$\lambda_1 = \frac{1}{2}(-p + \sqrt{p^2 - 4q}), \quad \lambda_1 = \frac{1}{2}(-p - \sqrt{p^2 - 4q}).$$

The general solution is

$$x_g(t) = ae^{\lambda_1 x(t)} + be^{\lambda_2 x(t)} : \quad \lambda_1 \neq \lambda_2.$$

$$x_g(t) = \big(a + bx(t)\big)e^{\lambda x(t)} : \quad \lambda_1 = \lambda_2 = \lambda.$$

The above method replaces the solution of differential equation by algebraic equation. Since more than fifth order algebraic solution is not algebraically solvable, this method has practical limitation.

1.1.5 nth order linear differential equation

By the discussions so far, we may formulate **a linear system** that can be expressed by a single nth order differential equation,

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + ... + a_2 x^{(2)} + a_1 x^{(1)} + a_0 x^{(0)} = 0.$$

Although some of the differential may well be omitted[3]. Also, in the above equation, there is no external force, showing self-sufficient autonomous system.

If we think about the process of solution, we integrate the equation once reducing the order by one providing one initial condition. In case of nth order equation, we repeat the process n times providing n initial conditions eventually yielding,

$$f(x) = 0.$$

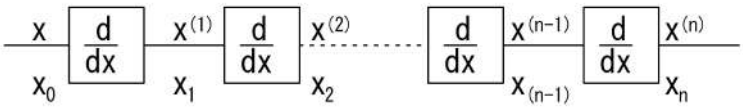

Figure 1.4: nth order differential equation

1.1.6 First order representation of linear differential equation

Referring to Fig 1.4, if we select the variables $x_0, x_1, ...x_{n-1}, x_n$ as shown, an nth order differential equation is written by a set of n simultaneous first order differential equations.

$$\begin{pmatrix} \dot{x}_1 = a_{11}x_1 \\ \dot{x}_2 = a_{22}x_2 \\ ... \\ \dot{x}_n = a_{nn}x_n \end{pmatrix}.$$

Note that

$$\dot{x}_n = x_{n-1}.$$

Or,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ ... \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & ... & 0 & 0 \\ 0 & a_{22} & ... & 0 & 0 \\ & & ... & & \\ 0 & 0 & ... & 0 & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}.$$

In the above equation, $\dot{x}_n$ is a function of only x_n, then we may generalize that $\dot{x}_n$ is a function of all other variables.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ ... \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{(1)(n-1)} & a_{1n} \\ a_{21} & a_{22} & ... & a_{(2)(n-1)} & a_{2n} \\ & & ... & & \\ a_{n1} & a_{n2} & ... & a_{(n)(n-1)} & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}.$$

Therefore, using a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{x}$, it is written by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

The beauty and advantage of this expression is that the solution $e^{\boldsymbol{A}t}$ is expressed by the real (σ) and imaginary ($\jmath\omega$) part by the following way.

- magnitude of solution attenuates or increases as times go by (σ part),
- frequency of oscillation ($\jmath\omega$ part).

The character of the system is completely determined by $\boldsymbol{A}$ and this situation cannot be observed in nth order differential formulation.

[2]When a differential equation is expressed by $\frac{dy}{dx} = f(\frac{y}{x})$, it is called homogeneous form. Thus, $\dot{x} = Ax$ is a linear homogeneous differential equation.

[3]The newton's law of motion lacks the first derivative.

1.2 Multivariable differential equation

Study of partial differential equations started in the days of Lagrange and as the name implies, this deals with intrinsic multivariable functions. **Although this type of differential equation is generally called "partial differential equation", it should better be called "multivariable differential equation"**, because the treatise of the Newtonian physics and the Lagrangian physics and later clearly corresponds to single variable and multivariable differential equations respectively. However, we will use the term partial differential equation unless we want to emphasize the multivariableness.

Since partial differential equations encountered in engineering and science are mostly first and second order, we discuss only notable points as compared with ordinary differential equations.

A common way of solving a partial differential equation is the separation of variables which eventually transforms into an ordinary differential equation.

1.2.1 Partial differential equation; definition

Partial differential equation is a function of independent variables, unknown function of those variable and partial derivatives of the unknown function. Generally, it may be expressed by,

$$f\Big(q_1, q_2, ..., u(q_1, q_2, ...), \frac{\partial u}{\partial q_1}, \frac{\partial u}{\partial q_2}, ..., \frac{\partial^2 u}{\partial q_1 \partial q_1}, \frac{\partial^2 u}{\partial q_1 \partial q_2}, ...\Big)$$

$$= f\Big(q_i, u(q_i), \frac{\partial u}{\partial q_i}, ... \frac{\partial^2 u}{\partial q_i \partial q_j}, ...\Big) = 0.$$

The equation $f()$ is defined in an n dimensional real number space where $u()$ is differentiable to required order.

Order of the partial differential equation means the highest ordered partial derivative and the solution, if it exists, is the function $u(q_i)$; that is, up to deriving the function $u(q_i)$ belongs to the territory of partial differential equation, and deriving q_i is another matter. In other word, **the solution of partial differential equation may not resolve the values of** q_i and we will need another conditions to calculate these values.

1.2.2 Partial differential equation; solution

In case of ordinary differential equations, the solution contains arbitrary constants. For the partial differential equations, the general solution contains arbitrary functions. For example, consider a partial differential equation of $u(x, y)$,

$$\frac{\partial u(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y} = 0.$$

If we assume

$$\xi = x + y, \;\; \eta = x - y,$$

then, the original equation is expressed by η [4]

$$\frac{\partial u(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y} = \frac{\partial u(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} - \frac{\partial u(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} = 2\frac{\partial u(\xi,\eta)}{\partial \eta} = 0.$$

Therefore, u can be any function that does not contain η.

$$u(\xi,\eta) = f(\xi),$$

or,

$$u(x,y) = f(x+y).$$

Thus, general solution will be an arbitrary function of $(x+y)$ which can include infinite number of functions such as $(x+y)^n, \sin n(x+y), ...$[5].

As shown above, **the general solution of partial differential equation can contain infinite number of arbitrary functions**. This freedom makes it difficult to find the solutions and it requires initial and boundary conditions. Also, as shown in this example, calculating x, y still remains - that is, we only found the solution as a function of $x+y$ and we will require further conditions to resolve this problem.

Substitution of x, y by ξ, η is one of a common techniques to convert the partial differential equation into a differential equation with one variable - an ordinary differential equation.

1.2.3 Partial differential equation; examples

(1) Maxwell's equation

In the Maxwell's equation, we have

$$rot\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

This is a three dimensional vector and written by the three components as below.

$$(rot\boldsymbol{E})_x = \nabla_y B_z - \nabla_z B_y = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\partial B_x}{\partial t},$$

$$(rot\boldsymbol{E})_y = \nabla_z B_x - \nabla_x B_z = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \frac{\partial B_y}{\partial t},$$

$$(rot\boldsymbol{E})_z = \nabla_x B_y - \nabla_y B_x = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{\partial B_z}{\partial t}.$$

In this example $\boldsymbol{E}$ (or $\boldsymbol{B}$) is a function of $x(t), y(t), x(t)$, but t does not appear in an explicit form which makes the partial differential easier to understand.

For the right hand side, $\partial \boldsymbol{B}/\partial t$ is a time varying vector.

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\frac{\partial B_x}{\partial t}, \frac{\partial B_y}{\partial t}, \frac{\partial B_z}{\partial t}).$$

[4] We also have $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} - \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} = 0$ which we can't use.

[5] This idea led to Fourier series and Fourier transform.

(2) Euler-Lagrange's equation

The Euler-Lagrange's equation for energy conserving system is written down by,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}(q_i, \dot{q}_i)}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}(q_i, \dot{q}_i)}{\partial q_i} = 0. \quad i = 0, ...n$$

This is a function of independent variables q_i and their derivatives $\dot{q}_i$. If we regard q_i and $\dot{q}_i$ collectively, then this is n variable second order differential equation of q.

The mixture of ordinary and partial differential looks strange at first, but this shall be discussed in Sec 1.6.

(3) Hamilton's equation

The Hamilton's equation is

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} = \dot{q}_i. \quad i = 0, ...n$$

This is a set of multivariable first order differential equation with $2n$ variables and is a central topic in analytical mechanics.

(4) Second order partial differential equation

We encounter more or less limited type of second order partial differential equations. They are,

- wave equation ;

$$\frac{\partial^2 u}{\partial t^2} - c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big) = 0,$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

- potential equation (Laplace equation) ;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

or

$$\nabla^2 u = 0.$$

- heat equation ; describes heat propagation by time under given initial condition.

$$\frac{\partial u}{\partial t} - \alpha\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big) = 0,$$

or

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u,$$

where $u = u(x, y, z, t)$.

1.3 Treatise on differential equation

Mechanics started to describe a motion of objects and a mathematical tool required was the differential equation. Differential equation flourished in the calculus, but in the days of Lagrange, the existence of solutions seems to have been assumed without questions. This can be understood that if a solution of a differential equation describes the path of a moving object, there must be a solution because any object moves along some trajectory.

When the calculus departed from a tool in mechanics and became a theory in mathematics, people started to rigorously discuss the region of the solution, the existence of the solution and so on. By these discussions, people realized that a differential equation **may have a solution** under some specific region, as well as there are very limited number of differential equations that can be solved analytically.

1.4 Nonlinear differential equation

Another advantage of the single order expression is that it can be extended to nonlinear system easily[6].

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

And as we will see, analytical mechanics produces nonlinear differential equation, we cannot stay within the discussion of linear differential equation only.

Moreover, a simple non linear equation like (E.N.Lorenz, 1963),

$$\frac{dx}{dt} = a(y - x),$$

$$\frac{dy}{dt} = bx - y - xz,$$

$$\frac{dz}{dt} = xy - cz,$$

produces a chaotic solution in which very small difference of initial value may cause big difference[7].

Minding about this singularity, and the tolerances and errors in real products and design may introduce nonlinearity, system synthesis (design) based on nonlinear differential equations should be avoided as much as possible.

Therefore, our stance of discussion will be,

- Discuss both linear and nonlinear differential equation (by generic expression).
- Apply linear approximation for actual system synthesis (for practicality).

1.5 Perspective of differential equations

In general, differential equations are classified into single variable differential equations (ordinary differential equations) and multivariable differential equations (partial differential equations). In field theory of physics such as electromagnetic theory, the physics is

[6] Only as far as the notation is concerned.

[7] Butterfly effect is one in which disturbance caused by a butterfly's wing results a hurricane.

expressed by vector differential equations. For instance, the Maxwell equation is expressed by two set of differential equations as shown below.

$$div\boldsymbol{E} + rot\boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{\partial \boldsymbol{B}}{\partial t},$$

$$div\boldsymbol{B} + rot\boldsymbol{B} = 0 + \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}.$$

In vector calculus, the differential "rot(or $\boldsymbol{\nabla}\times$)" is an operator applicable to virtually three dimensional vector only[8], because rot operates just like vector products.

On the other hand, "div(or $\boldsymbol{\nabla}\cdot$)" is an operator that works just like scaler products, and there is no limitation of dimension in this case. The definition of partial differential

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \frac{\partial y}{\partial x_3}dx_3 + ...,$$

clearly shows that this is also the definition of "div" of n dimension.

1.6 Variational integral

We might say that the variational integral is the main tool used in the construction of the analytical mechanics. In variational integral, we need to find a similar function to the target system, but similarity is only expressed by zero and first order derivatives[9].

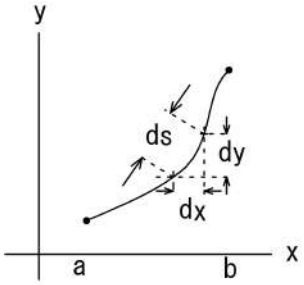

Figure 1.5: Length of curve

Suppose a curve $y = f(x)$ in two dimensional space. The length L of the curve from $x = a$ to b will be calculated by the integration of line element

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\ dx,$$

yielding

$$L = \int_a^b \sqrt{1 + y\prime^2}\, dx.$$

This can be interpreted that a function $f(x)$ specifies a corresponding value $L()$. Since $L(f(x))$ is a function of function and can be extended from length to general value, this extended function is called a "functional".

[8]The exterior algebra tells us that the exterior product for vector can be only defined for one, three and seven dimensional vector spaces.

[9]As compared with Taylor Polynomial approximation

If we try to find a function f_m that minimizes L - not necessarily length - between (a, b), we select functions, and move them around to find one that gives almost the same value; that is, the f_m will give a stationary value of L.

To find the function f_m, we assume functions

$$y = y_m + \delta y.$$

By this condition, y will give small difference δy from y_m at each x, but this condition includes functions like

$$y = y_m + \sin nx,$$

which will not give a minimal value of L.

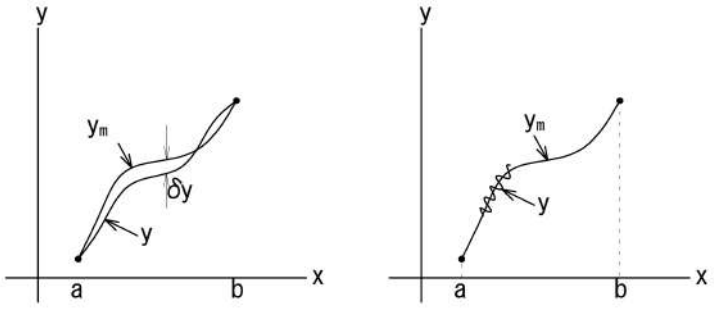

Figure 1.6: Functions that have similar curves

Therefore, we **need to incorporate some resemblance parameter** into our consideration. If shape of y resembles to y_m, the tangent of both functions should be similar value. So, we define a functional $I()$,

$$I(f(x)) = \int_a^b F\Big(x, y, \frac{dy}{dx}\Big)dx$$

$$= \int_a^b F(x, y, y\prime)dx$$

We express the small change of first derivative by $\delta y\prime$,

$$y\prime = y\prime_m + \delta y\prime.$$

($\delta y\prime$ will not be independent from δy which we will use later.)

The value of the functional by $f(x)$ becomes

$$I(f(x)) = \int_a^b F(x, y + \delta y, y\prime + \delta y\prime)dx.$$

Then, the **variation** δI of functional I is

$$\delta I = \int_a^b \Big(F(x,\ y + \delta y,\ y\prime + \delta y\prime) - F(x, y, y\prime)\Big).$$

If we obtain a function that yields a stationary value of $I()$, that shall be the answer. This is the idea of variational integral. Interesting point of this idea is that **we do not specify the function $f()$, but still we can get some further informations about $f()$.** This is the result that we incorporated first derivative $y\prime$.

For a function $F(x, y, y\prime)$, assuming that δy and $\delta y\prime$ are small,

$$F(x,\ y+\delta y,\ y\prime+\delta y\prime) = F(x,y,y\prime) + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y\prime}\delta y\prime.$$

Or, the variational integral is only applicable to the functions that satisfy the above condition - smooth and differentiable to required order.

Then, the variation of I will be,

$$\delta I = \int_a^b \Big(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y\prime}\delta y\prime\Big)dx.$$

δy is arbitrary, but $\delta y\prime$ is restricted by δy like

$$\begin{aligned}\delta y\prime &= y\prime - {y_m}\prime \\ &= \frac{d}{dx}(y - y_m) \\ &= \frac{d}{dx}\delta y.\end{aligned}$$

Therefore,

$$\delta I = \int_a^b \Big(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y\prime}\frac{d}{dx}\delta y\Big)dx.$$

$$= \int_a^b \Big(\ \frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\big(\frac{\partial F}{\partial y\prime}\big)\delta y\Big)dx + \Big[\frac{\partial F}{\partial y}\delta y\Big]_a^b$$

Here we used the integration by parts[10],

$$\int_a^b \big(\frac{\partial F}{\partial y\prime}\frac{d}{dx}\delta y\big)dx = \Big[\frac{\partial F}{\partial y}\delta y\Big]_a^b - \int_a^b \Big(\frac{d}{dx}\big(\frac{\partial F}{\partial y\prime}\big)\delta y\Big)dx.$$

Since f and f_m coincide at points a, b (or we set f, f_m meet at a, b so that the following equation holds),

$$\Big[\frac{\partial F}{\partial y}\delta y\Big]_a^b = 0.$$

$\delta I = 0$ for arbitrary value δy,

$$0 = \delta I = \int_a^b \Big(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial y\prime}\frac{d}{dx}\Big)\delta y dx.$$

Therefore,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\big(\frac{\partial F}{\partial y\prime}\big) = 0.$$

Now, we get a condition that the function $y = f(x)$ holds so that the $I()$ takes stationary value. Note that, the actual value of $I()$ doesn't matter, nor the function $f(x)$ is specified. Only the condition manifested.

This is called the **Euler's equation** and is identical to Euler-Lagrange's equation when applied to Newtonian mechanics.

[10] Method used here is frequently applied to analytical mechanics as well.

1.7 Lagrange multipliers

Lagrange has devised an ingenious way to solve optimization problem (Why Lagrange had this idea is difficult to fathom when we look at this method from optimization point of view. We discuss on this topic in Sec1.7.7.). The process itself can be executed mechanically - and because of this mechanical process, it is not easy to see what is happening in it. Here, we explore the meaning of this method by noticing that it is composed of two parts. One is how multiple constraint functions limit the area of variables, and the other is that the target function and constraint function has the normal vector with same (or inverse) direction at the min-max point.

1.7.1 Simultaneous equation

We start with the simplest linear simultaneous equation, and assume this equation is solvable. When

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1,$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2,$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3,$$

is given, we can resolve $x_1 = d_1, x_2 = d_3, x_3 = d_3$. However, if only

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1,$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2,$$

is given, x_1, x_2, x_3 is not resolved to a single set of values, but becomes "indeterminate" instead.

Therefore if we have only

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1,$$

the variables enjoy the biggest freedom.

Now we consider a multivariable function

$$f(x_1, x_2, ...x_n),$$

in which the variables have no constraints. Then, the value of the function will not be constrained as well. If we introduce an equation

$$g(x_1, x_2, ...x_n) = 0,$$

for the variables, one variable has lost freedom and there arises a situation that the area that $f(x)$ can assume is constrained. If $f(x)$ is smooth and continuous, there can be a discussion of min,max of $f(x)$.

Number of constraint function $g_k(x)$ increases, the area that $f()$ can assume is constrained more. And it will be clear that the maximum number of constraint function $g(x)$ must be

$$k < n.$$[11]

[11] If k = n, $g()$ resolves the variable to one point, thus determines the value of $f(x)$ to a single value.

Here, it should be noted that $g_k(x) = 0$ constraints the area of variables $x_i, (i = 1, n)$ and the constrained variables constrain the value of $f(x_i)$[12].

Thus, the discussion continues to min-max problem of a multivariable function under a set of constraint equations.

1.7.2 Min-max problem

Suppose a multivariable function $f(x_i)$ is given and trying to find minimum or maximum value of the function. We differentiate the function with respect to x_i and make equal to zero.

$$f\prime(x_i) = 0.$$

Then, the above equation yields the value of x_i where the function $f(x_i)$ becomes stationary. Generally it is not difficult to decide that this stationary point is local min-max, global min-max or saddle point.

This idea is further extended to optimization problems of multivariable functions under some constraints. That is, we discuss a problem to find stationary points for a function

$$f(x_1, x_2, ...x_n),$$

under constraints given by

$$g_k(x_1, x_2, ...x_n) = 0. \quad k < n.$$

If explicitly written, the constraints are,

$$g_1(x_1, x_2, ...x_n) = 0,$$

$$g_2(x_1, x_2, ...x_n) = 0,$$

$$...$$

$$g_k(x_1, x_2, ...x_n) = 0.$$

Note that the number of variables of the constraint functions is basically equal to n (if lesser than n, constraint gets weaker).

1.7.3 Gradient

Fig 1.7 shows a convex function $f(x_1, x_2)$ and it's level curve (contour) where $f = h_1$. The gradient is a vector that is directed to the steepest direction of $f(x)$ and is defined by,

$$\boldsymbol{\nabla} f(x_1, x_2) = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\Big).$$

The direction is perpendicular to contour $f = h_1$.

Gradient is most commonly used to express force $\boldsymbol{F}$ produced by potential $f(x)$.

$$\boldsymbol{F} = -grad\ f(x) = -\boldsymbol{\nabla} f(x).$$

[12]This implies that a function like $1/x_1$ shall be excluded around $x_1 = 0$.

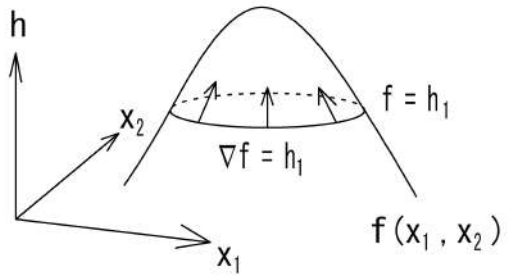

Figure 1.7: Gradient of a function

In the above example, it is 3 dimensional, but we can define a gradient of n dimensional space exactly the same way by (we write by column vector),

$$grad\ f(x) = \boldsymbol{\nabla} f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

1.7.4 Lagrange multipliers

We discuss a case under single constraint $g(x) = 0$ first.

Fig 1.8 shows the constraints by $g(x_1, x_2)$ for the path that variables can take on the surface of $f(x_1, x_2)$ - ie., the path on $f(x)$ is a projection of $g(x)$ and becomes a three dimensional line in this example. At the stationary (local min, max) points, $g(x)$ touches the contour of $f(x)$; tangents coincide at these points. This means that the normal vector of the tangent (gradient vector) of $f(x)$ becomes parallel with that of $g(x)$. Therefore, expressing the proportional coefficient by λ,

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \lambda \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \cdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}.$$

Or,

$$\boldsymbol{\nabla} f(x_i) - \lambda \boldsymbol{\nabla} g(x_i) = 0.$$

And the constraint is

$$g(x_1, x_2, ..., x_n) = 0.$$

The above two conditions are treated by a single function as follows (for the reasoning, we discuss in Sec 1.7.6).

If we define a function

$$\Lambda(x_1, x_2, ...x_n, \lambda) = f(x_1, x_2, ...x_n) - \lambda g(x_1, x_2, ...x_n),$$

the above requirements is interpreted to find

$$\nabla\Lambda(x_i, \lambda) = 0$$

with respect to x_i and λ.

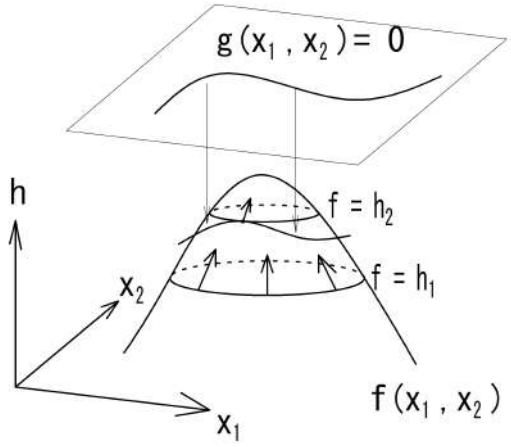

Figure 1.8: Constraint on $f()$

The process is called Lagrange undetermined multipliers method. This method can optimize - find candidates of the min, max points of - the target function $f(x_1, x_2, ...x_n) = 0$ under a constraints given by, $g(x_1, x_2, ...x_n) = 0$.

Partially differentiate Λ with respect to x_i and λ, and equaling to zero,

$$\begin{aligned}
\frac{\partial\Lambda}{\partial x_1} &= \frac{\partial}{\partial x_1}\Big\{f(x) - \lambda g(x)\Big\} = 0,\\
\frac{\partial\Lambda}{\partial x_2} &= \frac{\partial}{\partial x_2}\Big\{f(x) - \lambda g(x)\Big\} = 0,\\
&\cdots\\
\frac{\partial\Lambda}{\partial x_n} &= \frac{\partial}{\partial x_n}\Big\{f(x) - \lambda g(x)\Big\} = 0,\\
\frac{\partial\Lambda}{\partial \lambda} &= \frac{\partial}{\partial \lambda}\Big\{f(x) - \lambda g(x)\Big\} = 0.
\end{aligned}$$

The solution of this equation gives the stationary points of $f()$. Interesting point here is that the "multiplier" is treated as a variable.

1.7.5 Lagrange multipliers by multiple constraints

In case there are multiple constraints,

$$g_k(x_1, ...x_n) = 0.$$

We provide multiple variables λ_i

$$\nabla f(x) - \sum_i \lambda_i \nabla g_i(x) = 0,$$

where i is from 1 to k.

1.7.6 Lagrange multipliers; a Lagrangian

Material treated in this section is much better understood after having looked into what Lagrangian is (see Chap 3).

The Lagrange multiplier,

$$\Lambda(x_1, x_2, ...x_n, \lambda) = f(x_1, x_2, ...x_n) - \lambda g(x_1, x_2, ...x_n).$$

defines a scaler Λ by a multivariable function $f()$, under a constraint of $g()$.

The definition of the Lagrangian in analytical mechanics is given by,

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q),$$

which yields the Newton's law of motion by minimization. This can be interpreted that the Lagrangian is providing a seed which eventually yields the optimal path of kinetic energy of the target system under particular potential energy.

Thus, the Lagrangian is a special case of Lagrange multiplier where

- $\mathcal{T}$ is positive definite ($= 0$ only when $q = 0$),
- $\mathcal{T}$ is quadratic, and has a single minimum,
- $\mathcal{V}$ is positive semi-definite ($\mathcal{V}$ can be zero at $q \neq 0$).

Because of this[13], Λ is sometimes called a "Lagrangian".

1.7.7 Lagrange's equation: first kind

When solving a problem by Lagrangian, we start by writing down the kinetic and potential energy with respect to coordinates. But the selection of coordinates is a secondary matter. For instance, we might write down three equations using four variables for three dimensional problem.

$$\dot{x}_1 = f_1(x_1, x_2, x_3, x_4),$$

$$\dot{x}_2 = f_2(x_1, x_2, x_3, x_4),$$

$$\dot{x}_3 = f_3(x_1, x_2, x_3, x_4).$$

In this case, there must be the relation,

$$g(x_1, x_2, x_3, x_4) = 0.$$

That is, one variable is expressed by the other three variables and eventually eliminated. Or, one variable is constrained by the other variables and this is called **holonomic**[14] **constraint**.

For an energy conserving system, the equation of motion is given by the following Euler-Lagrange's equation.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\Big) - \frac{\partial \mathcal{L}}{\partial x_i} = 0.$$

[13] Perhaps.

[14] Condition of constraint is either coordinate or time.

In this equation coordinates are judiciously chosen so that number of variables is just necessary and sufficient. Or, holonomic constraint between variables x_i is already incorporated. This is basically the approach on Newtonian mechanics, and - as Lagrange suggested - if we try not to think about the coordinates (at least very carefully), there can be modified approach to form Euler-Lagrange's equation.

This approach is called Euler-Lagrange equation of the first kind. Suppose we have m holonomic constraints,

$$g_1(x_i) = 0, \;\; g_2(x_i) = 0, \ldots \;\; g_m(x_i) = 0.$$

the Euler -Lagrange's equation is given by,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\Big) - \frac{\partial \mathcal{L}}{\partial x_i} = \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_i}.$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\Big) - \Big\{\frac{\partial \mathcal{L}}{\partial x_i} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_i}\Big\} = 0.$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\Big) - \Big\{\boldsymbol{\nabla}\mathcal{L} - \sum_{i=1}^{m} \lambda_i \boldsymbol{\nabla} g_i\Big\} = 0.$$

In this approach, the holonomic constraint is written explicitly out of the Lagrangian. If we observe the function of x_i, we can see the Lagrange's undetermined method.

1.8 Legendre transform

Legendre[15] invented somewhat peculiar looking transform that is applicable to smooth convex (or concave) functions. Here we briefly explain how it is defined and what it means geometrically.

First derivative of convex functions increases monotonically, or second derivative is always positive. When applied to these functions, the function $F(x)$ is expressed by it's first derivative $F\prime(x)$ and intercept of the tangent.

In Fig 1.9, function $F(x)$ is a smooth convex function and the first derivative increases monotonically as shown in the lower figure. Since the value of $F(x)$ is determined by giving x, and each point of $F(x)$ has a tangent, $F(x)$ is also determined by giving **the tangent at x and it's intercept G**. The tangent $s(x)$ is defined by,

$$s(x) \equiv \frac{dF(x)}{dx} = F\prime(x).$$

Looking at the Fig 1.9,

$$s(x) \cdot x + G(s) = F(x)$$

$$G(s) = s(x) \cdot x - F(x)$$

This is called Legendre transform. When we want to determine a function $F(x)$ when $F\prime(x)$ is easily available than x, this method works well. Or, if we wish to replace a derivative

[15] 1752-1833, About fifty years senior to Hamilton.

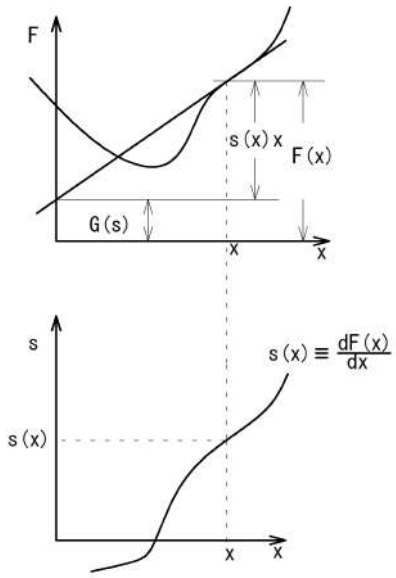

Figure 1.9: Legendre transform

with non-derivative variable, we can use this transform just as Hamilton applied to the Lagrangian.

One of the interesting characters of Legendre transform is that it is it's own inverse. If we perform Legendre transform twice, then we get the original function. This symmetric character is expressed by,

$$G(s) + F(x) = sx$$

Legendre transform is somewhat limited in application, it is not spot lighted in these days. However, this transform plays a key role to seamlessly connect Hamiltonian from Lagrangian (See Sec 4.2), and intuitively much easier than Fourier or Laplace transform, it would not be a waste of time to look into it even once in a lifetime[16].

Similarity of Legendre transform and multivariable system expression by $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ will be discussed in Sec 14.2.

[16]Some prefer to call Legendre transformation. The author detects that people feel it is light weighted than Laplace transform etc.

PartI

Analytical physics

In this part, we discuss analytical mechanics starting by the Newton's law which is a single variable mechanics. Philosophical progress of analytical mechanics is a development from single variable mechanics to multivariable mechanics which is roughly summarized in the following way.

- Description of mechanics by multivariable (or partial) differential equation; Euler-Lagrange's equation.
- Generalization of coordinates; description of mechanics by energy based variables.
- Description of mechanics by a pair of first order differential equation; Hamilton's equation.
- Clarification (or discovery) of structure of mechanics; conjugate pair of coordinate and momentum; development into generalized physics.
- Description of time varying mechanics; Hamilton-Jacobi's equation.

In view of a preparation to discuss multivariable control, following topics may not be necessary.

- Poisson bracket.
- Canonical transformation.
- Discussion on zero Hamiltonian system.
- Solution of Hamilton-Jacobi's equation.

However, there is more than enough value to understand the analytical mechanics, the author included above topics without compromise. Thus the essential part of analytical mechanics is covered to advance (hopefully) to any fields.

Chap2 Exodus from coordinates

2.1 Newton's law of motion

Newton's law of motion states,

(1) Any object stays stationary or moves by constant velocity in an inertial frame of reference.

(2) Force applied to an object with mass "m" is proportional to acceleration "a".

(3) When a body exerts force to a second body, the second body exerts force to the first body by identical magnitude with opposite direction.

The origin of the first law was in the idea by Descartes who thought that the planets moves in circular orbits by vortex of ether, and objects on the earth moves by constant velocity.

Newton only agreed with Descartes by the movement of object with constant velocity, and discarded vortex of ether completely. Because, in Newton's mind, there is no difference between celestial and terrestrial bodies. Also, Newton believed that there is only one motion which is straight, thus reasoning that a circular motion is the result of centrifugal force applied to the body.

Galileo calculated that the distance traveled by a falling object is proportional to the square of time,

$$d = \frac{1}{2}at^2.$$

The above equation suggests that a force (gravity) is constantly applied to the object. Combining the first law, Newton had an idea of the second law.

$$F = ma.$$

The third law is the result of an observation of everyday phenomena as stated by Newton " If you press a stone with your finger, the finger is also pressed by the stone".

2.2 Newton's law of universal gravitation

Newton's law of universal gravitation is derived by the result of Kepler's law which states

(1) Planetary orbit is an ellipse; the sun is located on one of the foci,

(2) Area swept in a unit time by planetary orbit is constant,

(3) The square of orbital period of the planet is proportional to the cube of the distance between the planet and the sun.

Newton demonstrates that the force between the planet and the sun is inversely proportional to the square of distance by using (2) and (3) of the Kepler's law .

$$F \propto \frac{K}{r^2}, \qquad K = constant.$$

Then he thinks that the force between the planet and the sun is equal to the force between a satellite (of Jupiter) and the Jupiter.

It was known that the orbit of the satellite of a Jupiter is a circle,

$$F = \frac{mv^2}{R}. \quad R;\ radius \quad m;\ mass \quad v;\ velocity$$

Then the period T of the revolution is

$$T = \frac{2\pi R}{v}.$$

The force is

$$F = \frac{4\pi^2 mR}{T^2}.$$

Kepler's third law states,

$$T^2 = kR^3.$$

Thus, the force exerted by Jupiter to the satellite becomes,

$$F = \frac{4\pi^2 mR}{kR^3} = \frac{4\pi^2 m}{k}\frac{1}{R^2}.$$

The satellite also exerts force to Jupiter with the same magnitude,

$$Mm \propto \frac{4\pi^2 m}{k}.$$

Therefore, by defining a constant G,

$$F = G\frac{Mm}{R^2}.$$

The genius of Newton manifests in the observation that **the law of gravitational force is universally applicable to the celestial motion as well as motion found on the earth**. For Issac Newton, there is no discrimination between divine and humane phenomena.

Thus the law of the universal gravitation is stated by replacing the celestial distance R with universal distance r,

$$F = G\frac{Mm}{r^2}.$$

One of the historical successes of the Newtonian gravity was the prediction and discovery of planet Neptune by the observed orbit of Uranus.

2.3 Field theory

Suppose a motion of a pair of objects. Newton's universal gravitation law does not restrict the size of objects for which the force is calculated, let us now make one object bigger and bigger both in size and mass, but keeping the distance small. When one object is made reasonably[1] bigger than the other, the surface of the bigger object is considered to be flat from the point of view of smaller object. And the force between them is mostly originated from the bigger object that makes natural to rename the force "gravity". It is perpendicular to the surface of the bigger object and uniform at least around the territory of the smaller object.

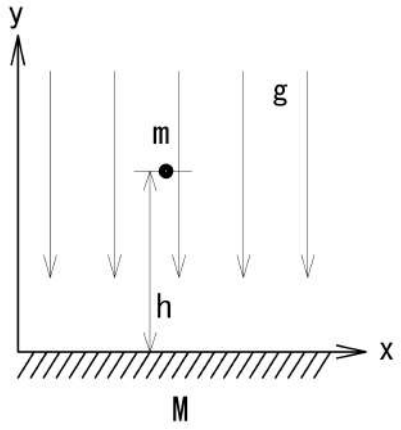

Figure 2.1: field of gravity

Now, the motion of the object can be stated by using particular coordinates that is suitable to situations - in this example, Cartesian form as shown in Fig 2.1.

Significance of the idea shown in Fig 2.1 is that the force applied to the moving object m is not considered to be originated from the other object M, but the space has "potential" mgh which eventually produces force. This is called "field theory" where the force to the object m is produced by the field nearby.

The essence of the field theory is that the force propagates through the media (in our case, air and vacuum) by the speed of light. In classical mechanics, the force is said to be present between objects without propagation, but it just exists. This is called by "force at a distance" and adopted by Newton without any further discussion. In other word, Newton stopped thinking about the cause of the force which sustained severe criticism in his days by people like Leibniz and others.

Since the field theory is adopted by electromagnetic theory and not assumed until the theory of general relativity, **the classical mechanics is not a field theory**.

However, it is convenient to utilize the idea of field theory as discussed above, we use the idea of field theory without explicit declaration.

[1] What is the reasonable value is vague, but say more than 10^4 seems to be big enough. The diameter of the earth is $1.27\times10^7 m$. Also, shape is implicitly assumed to be spherical.

2.4 Coordinates

2.4.1 Multiple coordinates

By field theory, the problem of the moving objects became practical to solve. For instance, mechanics of a thrown odd shaped object is written down by a set of two equations - one describes the movement of the center of gravity of the object and the other to describe the movement (rotation) of it's center of gravity. This presents six degrees of freedom which makes the problem difficult compared with the trajectory of a ball, but still the problem is precisely stated. Once the problem is stated, it may be solved analytically or, approximated solution shall be calculated by numerical analysis.

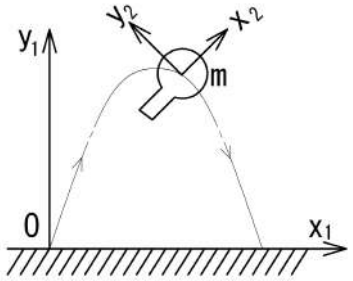

Figure 2.2: Motion and coordinates of a thrown object

As shown in Fig 2.2, selection of coordinate is crucial to solve the problem, and the mechanics has inevitably gained an element of puzzles; what is the best coordinate setting and how can we select them ?

Even if we can select good coordinates, this scheme obviously presents next problem; how can we convert one coordinate to another? Most likely transformation is from Cartesian to polar coordinate which is precisely calculated, but takes a very tedious manipulation and reveals yet another problem.

2.4.2 Polar coordinates

Here we show how the Newton's equation of motion is transformed from Cartesian to polar coordinates. The following manipulation is cited from Takahashi's book pp 16-18 of [11]. He declares that the calculation is quite uncalled for and his humors[2] are also cited here as well.

(Start citation. ; English translation by the author,)

When force F is given by the potential $\mathcal{V}(\boldsymbol{x})$, Newton's equation of motion is written by

$$m\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\mathcal{V}(\boldsymbol{x}).$$

We try to express this equation by polar coordinates.

[2]Not explicit, but still the author feels this.

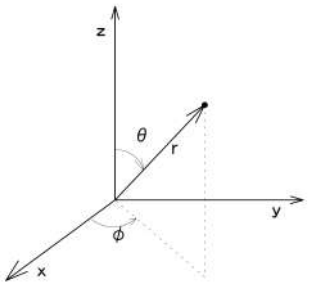

Figure 2.3: Polar coordinate

Cartesian coordinates are expressed by polar coordinates by the following equations.

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta.$$

First derivatives are,

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi,$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\theta,$$

$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta.$$

Second derivatives are,

$$\ddot{x} = \ddot{r}\sin\theta\cos\phi + r\ddot{\theta}\cos\theta\cos\phi - r\ddot{\phi}\sin\theta\sin\phi$$

$$-r\dot{\theta}^2\sin\theta\cos\phi - r\dot{\phi}^2\sin\theta\cos\phi$$

$$+2\dot{r}\dot{\theta}\cos\theta\cos\phi - 2\dot{r}\dot{\phi}\sin\theta\sin\phi - 2r\dot{\theta}\dot{\phi}\cos\theta\sin\phi,$$

$$\ddot{y} = \ddot{r}\sin\theta\sin\phi + r\ddot{\theta}\cos\theta\sin\phi - r\ddot{\phi}\sin\theta\cos\phi$$

$$-r\dot{\theta}^2\sin\theta\sin\phi - r\dot{\phi}^2\sin\theta\sin\phi$$

$$+2\dot{r}\dot{\theta}\cos\theta\sin\phi + 2\dot{r}\dot{\phi}\sin\theta\cos\phi + 2r\dot{\theta}\dot{\phi}\cos\theta\cos\phi,$$

$$\ddot{z} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta.$$

Writing the Cartesian gradients by polar coordinates,

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}.$$

Calculation of the coefficients, we use

$$r^2 = x^2 + y^2 + z^2,$$
$$\tan\phi = \frac{y}{x},$$
$$\tan^2\theta = \frac{x^2+y^2}{z^2}.$$

Then,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi,$$
$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi,$$
$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta.$$

Likewise,

$$\frac{\partial\theta}{\partial x} = \frac{1}{\tan\theta\sec^2\theta}\frac{x}{z^2} = \frac{1}{r}\cos\theta\cos\phi,$$
$$\frac{\partial\theta}{\partial y} = \frac{1}{\tan\theta\sec^2\theta}\frac{y}{z^2} = \frac{1}{r}\cos\theta\sin\phi,$$
$$\frac{\partial\theta}{\partial z} = -\frac{1}{\tan\theta\sec^2\theta}\frac{x^2+y^2}{z^2} = -\frac{1}{r}\sin\theta,$$
$$\frac{\partial\phi}{\partial x} = -\frac{1}{\sec^2\phi}\frac{y}{x^2} = -\frac{1}{r}\frac{\sin\phi}{\sin\theta},$$
$$\frac{\partial\phi}{\partial y} = \frac{1}{\sec^2\phi}\frac{1}{x} = \frac{1}{r}\frac{\cos\phi}{\sin\theta},$$
$$\frac{\partial\phi}{\partial z} = 0.$$

Then we obtain,

$$m\ddot{x} = -\sin\theta\cos\phi\frac{\partial\mathcal{V}}{\partial r} - \frac{1}{r}\cos\theta\cos\phi\frac{\partial\mathcal{V}}{\partial\theta} + \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial\mathcal{V}}{\partial\phi},$$
$$m\ddot{y} = -\sin\theta\sin\phi\frac{\partial\mathcal{V}}{\partial r} - \frac{1}{r}\cos\theta\sin\phi\frac{\partial\mathcal{V}}{\partial\theta} - \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial\mathcal{V}}{\partial\phi},$$
$$m\ddot{z} = -\cos\theta\frac{\partial V}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial V}{\partial\theta}.$$

By the above three equations, we obtain

$$m(\ddot{x}\sin\theta\cos\phi + \ddot{y}\sin\theta\sin\phi + \ddot{z}\cos\theta) = -\frac{\partial\mathcal{V}}{\partial r}.$$

(End citation.)

After above calculations, the Newton's equation of motion is written down in polar coordinates by

$$m\ddot{r} - mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) = -\frac{\partial V(r,\theta,\phi)}{\partial r}.$$

We found that the calculation is tedious enough. But also, we notice another problem that the coordinates transformation introduces mixed up expression for physical variables, for example, velocity is no longer a function of $\dot{x}$ but x as well (see Sec 2.4.3).

2.4.3 Problems caused by different point of view

In this section, we discuss what kind of inherent difficulties are presented in conjunction with coordinate transformation.

As shown in Fig 2.3, the x axis of Cartesian coordinates is expressed by polar coordinates r, θ and ϕ as follows.

$$x = r \sin\theta \cos\phi.$$

And the time derivative of the first equation becomes,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dr}{dt}\sin\theta\cos\phi + r\frac{d\theta}{dt}\cos\theta\cos\phi + r\sin\theta\frac{d\phi}{dt}(-\sin\phi) \\ &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\ sin\phi,\end{aligned}$$

Or,

$$\frac{dx}{dt} = \dot{x} = f(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}).$$

That is, $\dot{x}$ **is a function of not only first derivative of three coordinates** $(\dot{r}, \dot{\theta}, \dot{\phi}\ )$ **but also three coordinates** (r, θ, ϕ) as well. Moreover, $\dot{x}$ became a non linear function of $(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$.

If we interpret by simple words, this mixed up situation is telling us that "velocity $\dot{x}$" of an object is affected by the location (r, θ, ϕ) as well and interfered by another components$(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}\ )$.

This simple calculation clearly shows unwelcome situations in coordinate transformation; and this situation cannot be cleared as long as we are dealing with the mechanics using coordinates because these phenomena are **caused by the chain rule of differential operation**.

This mathematical mechanism tells us that the mixing up and non linearity is a "normal" situation in mechanics, and independent and linear expression is a lucky situation. The latter happened because we intentionally (with intuition) selected the coordinates that way so that the mechanics is most easily understood. Recognizing this point will help to form the philosophy of analytical mechanics.

2.5 Generalized coordinates

2.5.1 Extention of coordinates

In analytical mechanics (or some people call "Generalized mechanics"), Newtonian coordinates are extended to generalized coordinates. Parameters related to some coordinates are the result of projection of some system variables. That is, a system has a set of unique parameters which manifests in different ways once viewed from particular coordinate.

The aim of the extension of the Newtonian coordinates is to free mechanics from geometrical bind (Lagrangian mechanics) and eventually shows that the fundamental entities to describe mechanics is "coordinates[3], and momenta" (Hamiltonian mechanics).

[3]We will show that "coordinate" here means a type of variable derived from coordinate. It is a pity that physicists did not invent some appropriate term.

Generalized coordinate is rather difficult to understand, particularly when people are influenced by Newtonian mechanics - and this is generally the starting point of physics study. Even the meaning may only become clear after understanding the analytical mechanics; here we stress that the image should not be restricted by the term "coordinate". Ideas also have inertia and the Newtonian mechanics do have big inertia.

2.5.2 Overview of progress

Newtonian mechanics is extended to generalized mechanics and eventually cover different area (quantum mechanics, electric/electronic engineering, automatic control, economics, etc.) justifying the term analytical "physics". During this process, the analytical physics has been brushed up mathematically as well making it's philosophy clearer but more abstracted.

We shall discuss this abstraction and meanings later, but the essential progress of analytical mechanics may be summed up as follows.

- **Starting point = Newtonian mechanics.**

- **Description of the physical system by the kinetic and potential energy.**
  Definition of Lagrangian.
  Euler-Lagrange equation.
  Mathematically expressed by multivariable differential equation with independent and dependent variables.

- **Lagrangian → Hamiltonian.**
  Order reduction of Euler-Lagrange equation.
  (= Introduction of momentum p.)
  dependent variables $(\dot{q}, q)$ → independent variables (q, p).

- **Hamilton's canonical equation.**
  Define mechanics by two physical variables.
  (Generalized coordinates and Generalized momenta.)
  Generalization of coordinates transformation.
  Single order multivariable differential equation with independent variables.

- **Hamilton-Jacobi's equation.**
  Equation of motion under time varying force.
  Clarification of mathematical meaning of canonical transformation.
  Given Hamiltonian → transformation formula.
  Explicit inclusion of time t in the Hamiltonian mechanics.
  (→ automatic control.)
  (→ canonical transformation.)
  Another way to solve Hamilton's equation.

2.6 Energy

2.6.1 Kinetic and potential energy

In any kind of physical systems, energy is present and realizing that the characteristics of the system is written by energy, the scope of applicable world expands considerably. The expansion is not only limited by a particular field, but the combination of different fields - mechanics, fluid dynamics, electric system and electronic system and so on.

In case of mechanics, we know that there are kinetic energy and potential energy. In case of electric system, we also find the equivalents to these kinetic and potential energies.

Now the question we wish to discuss here is stated as follows.

"Is there any other kind of energy ?"

If no, this question assures us to think about only the kinetic and potential energies of a given system. Here we will not give rigorous proof[4], but instead, we will show what kind of trouble we encounter if there is the third type of energy.

Remember the oscillation of a simple pendulum. A mass is suspended by a wire, and if some force is applied to the mass, it produces an oscillation. The mechanism of oscillation is explained in energy's point of view as the interchange of kinetic and potential energy. If there is no dissipation, the interchange continues with fixed frequency for ever.

Now let us assume there is third type of energy $\mathcal{U}$. Then the energy interchange at the time of oscillation can be

$$\mathcal{T} \to \mathcal{V} \to \mathcal{U} \to ...$$

or,

$$\mathcal{T} \to \mathcal{U} \to \mathcal{V} \to ...$$

That is, there can be different order of energy exchange. This means that the nature has two intrinsically different descriptions.

Basically, we believe that the nature should present - if at all available - only a single physical representation. And the existence of the third type energy violates this philosophy.

2.6.2 Kinetic or potential ?

When we start thinking about energies in mechanics, kinetic and potential energies are introduced with somewhat vague definition.

Here we discuss a mechanics in which an object with mass exists. In Cartesian coordinates, the kinetic and potential energies are,

$$\mathcal{T} = \frac{1}{2} m \dot{\boldsymbol{x}}^2,$$

$$\mathcal{V} = V(\boldsymbol{x}),$$

whereas the $\boldsymbol{x}$ is the position of the object.

[4] Can it be proved?

The kinetic energy is related by a motion of an object with mass m and velocity[5] $\dot{\boldsymbol{x}}$. The potential energy is something related with position of the object like the hight in gravitational field. Observing these expressions, we have a sort of understanding that kinetic energy is the one in $\dot{\boldsymbol{x}}$ or $\dot{q}$ space, whereas the potential belong to $\boldsymbol{x}$ or q space. But these primitive notion is easily destroyed when we encounter that the kinetic energy in general is a function of $\dot{q}$ and q[6].

$$\mathcal{T} = \mathcal{T}(\dot{q}, q).$$

Thus, we need to discuss the kinetic and potential energy further for clarification.

2.6.3 kinetic energy

Kinetic energy is an energy that is defined only by the object's velocity. Thus, it is independent from location and requires no source of force such as another mass or electromagnetic field. In the discussion of mechanics, the kinetic energy is generally thought in conjunction with potential energy, in which the coordinate is defined.

The kinetic energy is derived,

$$\boldsymbol{F} \cdot d\boldsymbol{x} = \boldsymbol{F} \cdot dt = \frac{d\boldsymbol{p}}{dt} \cdot \boldsymbol{v}dt = \boldsymbol{v} \cdot d\boldsymbol{p} = \boldsymbol{v} \cdot d(m\boldsymbol{v})$$

$$d(\boldsymbol{v} \cdot \boldsymbol{v}) = d\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot (d\boldsymbol{v}) = 2(\boldsymbol{v} \cdot d\boldsymbol{v})$$

$$\boldsymbol{v} \cdot d(m\boldsymbol{v}) = \frac{m}{2} d(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{m}{2} dv^2 = d\Big(\frac{mv^2}{2}\Big)$$

$$E_k = \int_0^t \boldsymbol{F} \cdot d\boldsymbol{x} = \int_0^t \boldsymbol{v} \cdot d(m\boldsymbol{v}) = \int_0^v d\Big(\frac{mv^2}{2}\Big) = \frac{1}{2}mv^2$$

$\boldsymbol{F} \cdot d\boldsymbol{x}$ is the work done by small displacement x and the kinetic energy is defined by the integration of this scaler product.

Momentum is the multiplication of vector$\boldsymbol{v}$ and scaler m, and it is a vector. On the other hand, $\boldsymbol{F} \cdot d\boldsymbol{x}$ is a scaler product. Thus the energy is a scaler which can be compared with greater variety of other energies derived from other physics such as electromagnetic theory.

2.6.4 potential energy

Potential energy is defined to be an energy that is determined by the relative distance from another source of force such as mass, electromagnetic power source.

Examples of potential energies are,

* gravitational potential; mgh
* potential by spring; $\frac{1}{2}kx^2$
* potential by two masses; $-\frac{GMm}{r}$
* potential by point charge; $\frac{1}{4\pi\mu_0}\frac{q_1 q_2}{r}$

[5] If we notice that velocity $\dot{\boldsymbol{x}}$ is a vector, we have to clarify what $\dot{\boldsymbol{x}}^2$ means as well.

[6] Potential energy is a function of only q.

* potential by uniform electric field; $-qE_x$

* magnetic potential; $-\mu B_x$

Potential by spring may be defined proportional to the compression/expansion from neutral length of the spring[7].

Since potential energy depends only on the distance of two objects, there is a freedom to set the reference position. In case of two objects, the reference is selected to infinity so that the potential energy becomes zero for infinitely apart objects. On the other hand, the reference position is set to the surface of the earth in case of gravitational potential.

2.6.5 gravitational potential of two objects

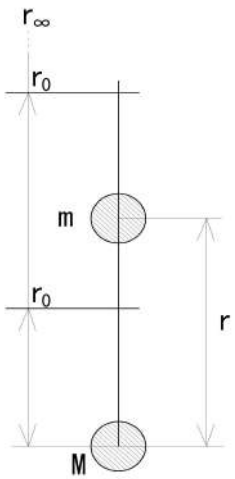

Figure 2.4: potential energy of m and M

The force between objects m and M with distance r is,

$$F = -G\frac{mM}{r^2}.$$

The negative sign means that the force is attractive.

Potential energy is only related with the relative postion, the energy required to move from r_0 to r is,

$$E_p(r-r_0) = -\int_{r_0}^{r} F\cdot dr = -\int_{r_0}^{r}\Big(-G\frac{mM}{r^2}\Big)dr = -\Big[\frac{GMm}{r}\Big]_{r_0}^{r} = GMm\Big(\frac{1}{r_0}-\frac{1}{r}\Big) = GMm\Big(\frac{r-r_0}{r_0 r}\Big).$$

When the distance is infinite, there is no force and the potential energy becomes zero. Setting the reference $r_0 \to \infty$, we obtain,

$$E_p(r) = -\frac{GMm}{r}.$$

[7]There is potential proportional to the angle in case of the mainspring.

2.6.6 gravitational potential by earth

Gravitational potential by the earth is calculated as follows. Moving the object m by the distance h from the surface of the earth whose radius is r_0,

$$E_p(h) = -\int_{r_0}^{r_0+h} \Big(-G\frac{mM}{r^2}\Big)dr = -GmM\Big[\frac{1}{r}\Big]_{r_0}^{r_0+h} = GmM\Big(-\frac{1}{r_0+h} + \frac{1}{r_0}\Big)$$

$$= GmM\frac{h}{r_0(r_0+h)} \simeq GmM\frac{h}{{r_0}^2} = \frac{GM}{{r_0}^2}mh = mgh, \quad \because g = \frac{GM}{{r_0}^2}.$$

Since lifting up from the surface of the earth, $h > 0$ always holds, and approximation by $r_0 >> h$, the potential is just proportional to the hight of the object.

2.6.7 Complementary energy

The following remarks can be only meaningful after having understood the Hamilton's theory.

In Hamiltonian mechanics, the fundamental physical variables are p and q which are independent with each other. In other words, the world is made by two "complementary[8]" variables p and q. If we ever define energy of this world, it should be related with p and q. Thus we have two different kind of (complementary) energies.

As we know that there are no fundamental difference between p and q, we may no longer call them "coordinate" nor "momentum". Then the energies related with them, what we can say is that there are two different kind of energies - there is no relevance to call them "kinetic" nor "potential" energies.

This is the ultimate state of energy classification, but it would be much more convenient and comfortable to stick to good old classification - **kinetic and potential energy using already established terminology**, remembering the abstraction.

[8]Generally called conjugate coordinates.

Chap3 Lagrangian

Analytical mechanics is established hundred years after Newton's law by Euler, Lagrange, Hamilton, Jacobi and other scholars. The power of analytical mechanics was really appreciated at the advent of quantum mechanics around 1920s and later by multivariable control theory in engineering around 1950s.

In this chapter, we discuss the advent of analytical mechanics focusing on the following points.

- Why Lagrangian was introduced?
- Physical meaning of Euler-Lagrange's equation, particularly what is different from Newton's law.
- Why Euler-Lagrange's equation is free from coordinates?
- Meaning of generalization of coordinates.

One important aspect of analytical mechanics is that the **multivariable differential equation (partial differential equation)** is used as opposed to the single variable differential equation (ordinary differential equation) in Newtonian physics.

Euler-Lagrange's equation will be further developed to Hamilton's equation which shall be treated in Chapter 4.

3.1 Optimized trajectory

From the Newton's days, philosophers often thought about the minimum-maximum problems. One of the most famous problem is the Brachistochrone curve problem which is stated as follows.

"Find the shape of the curve by which an object slides down from one point to another by minimum time. The speed of the object is zero at start and the force is given by the gravitational acceleration only."

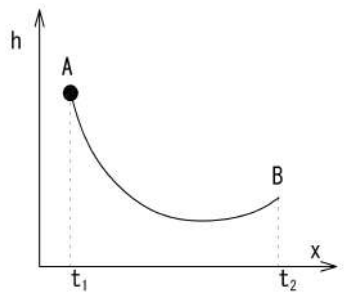

Figure 3.1: Brachistochrone curve

Solving the problem is not the purpose here[1], and the answer is cycloid by the way, but the trials to find some particular curve was one of the interesting topics in the advent of calculus.

It sounds natural that from the discussions above, people may think up new problem like this.

"Suppose a rigid body moving in a gravitational field. Find a path by which total energy is minimized."

When an object is moving from one point to the other in gravitational field, there are infinite possible trajectories. And the above problem is asking ;

"What is the most energy economic trajectory ?"

This is also a standard problem in Newtonian mechanics which we will discuss in Sec 3.3, but the point is the methodology to treat this old problem.

3.2 Principle of least action

When we imagine a moving object, infinite number of paths is possible from point A to point B. But in real environment, the actual path such as that of thrown ball becomes predictable which means that the nature always selects some particular path; by the experience of this, we throw things without knowing mechanics. The Newtonian mechanics described this path using mathematics - geometry - with no ambiguity. The basis of this formulation are the mass, force and acceleration. We also know that energies are necessary to move objects and eventually found two complementary energies; kinetic and potential energies.

Then, it may be natural to think that the nature is working under the least energy consumption - otherwise, the path should be arbitrary, or if nature selects the maximum energy consumption, the thrown object will never reach it's destination; the origin of this idea is philosophical, rather than scientific but provides a good point to start.

This must have been a motive for Euler, Lagrange and other people who studied "variational integral" and it's "minimum points". We know well by elementary calculus that when a given function is differentiated and equaled to zero, the answer gives points which are either local minimum or maximum. The calculation of variational integral seeking to the min-max function is considered to be an extension of local min-max problem in elementary calculus - **the solution being a function**, but not particular values.

3.3 Lagrangian

3.3.1 Formulation of problem

The problem stated in Sec 3.1 shall be interpreted to find a curve that minimizes the following integral

[1] See ref [12] for instance.

$$S = \int_{t_0}^{t_1} \Big\{ \frac{1}{2} m \big(\frac{dx}{dt}\big)^2 - mgh \Big\} dt.$$

If we express kinetic energy by $\mathcal{T}$

$$\mathcal{T} = \frac{1}{2} m \big(\frac{dx}{dt}\big)^2 = \frac{1}{2} m \dot{x}^2,$$

and potential energy by $\mathcal{V}$, the above integral becomes

$$S = \int_{t_0}^{t_1} (\mathcal{T} - \mathcal{V}) dt.$$

Then, minimizing this integral was what Lagrange thought when he started to discuss this problem.

3.3.2 Calculation of $\mathcal{T} - \mathcal{V}$

Looking into the process of the calculation of variational integral will provide a good example how minimizing problem is treated ([13]).

When we want to find a minimum (or maximum) value of a function $f(x)$, we would utilize a character at the minimum point - the value of function $f(x+h)$ will not change by changing x by a small value h.

This should happen for functions as well, so we substitute x in the integral by $x(t)+\eta(t)$, that is, a variable is substituted by a function, yielding

$$S = \int_{t_0}^{t_1} \Big\{ \frac{1}{2} m \big(\frac{dx}{dt} + \frac{d\eta}{dt}\big)^2 - \mathcal{V}(x+\eta) \Big\} dt.$$

And we use,

$$\begin{aligned} \big(\frac{dx}{dt} + \frac{d\eta}{dt}\big)^2 &= \big(\frac{dx}{dt}\big)^2 + 2\frac{dx}{dt}\frac{d\eta}{dt} + \big(\frac{d\eta}{dt}\big)^2 \\ &\simeq \big(\frac{dx}{dt}\big)^2 + 2\frac{dx}{dt}\frac{d\eta}{dt}. \end{aligned}$$

Because η is a small value, we ignore the square of the η.

For the potential energy $\mathcal{V}$, we expand by Tailor series yielding,

$$\mathcal{V}(x+\eta) = \mathcal{V}(x) + \eta \mathcal{V}\prime(x) + \frac{\eta^2}{2} \mathcal{V}\prime\prime(x) + \dots .$$

Again, ignoring the terms of η higher than second power, we get,

$$S = \int_{t_0}^{t_1} \Big\{ \frac{1}{2} m \big(\frac{dx}{dt}\big)^2 - \mathcal{V}(x) + m\frac{dx}{dt}\frac{d\eta}{dt} - \eta \mathcal{V}\prime(x) \Big\} dt.$$

Here, the first two terms

$$\frac{1}{2} m \big(\frac{dx}{dt}\big)^2 - \mathcal{V}(x)$$

are the value for the original trajectory, and what we need is the difference from the original. So we totally omit these terms getting,

$$\delta S = \int_{t_0}^{t_1} \Big\{ m\frac{dx}{dt}\frac{d\eta}{dt} - \eta \mathcal{V}\prime(x) \Big\} dt.$$

We know that $\delta S = 0$ for any η. However, $\eta(t)$ will be a concrete path and $d\eta(t)/dt$ becomes dependent to η. Therefore, we cannot conclude dx/dt and $\mathcal{V}\prime(x)$ are both zero simultaneously[2]. Remembering the rule of integration by parts[3] for $fd\eta/dt$,

$$\int f\frac{d\eta}{dt}dt = f\eta - \int \frac{df}{dt}\eta dt.$$

Then,

$$\begin{aligned}\delta S &= m\left[\frac{dx}{dt}\eta(t)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\Big(m\frac{dx}{dt}\Big)\eta(t)dt - \int_{t_0}^{t_1}\mathcal{V}\prime(x)\eta(t)dt \\ &= -\int_{t_0}^{t_1}\Big\{m\frac{d^2x}{dt^2} + \mathcal{V}\prime(x)\Big\}\eta(t)dt.\end{aligned}$$

Because $\eta(t_0) = \eta(t_1) = 0$. And since $\eta(t) \not\equiv 0$,

$$m\big(\frac{dx}{dt}\big)^2 - \mathcal{V}\prime(x) = 0.$$

This is the Newton's equation of motion. Note that $\mathcal{V}\prime(x)$ is a derivative by variable x, that is, a gradient of potential $\mathcal{V}$.

It should be clear now that $\mathcal{T} - \mathcal{V} = 0$ provides the most economical path from t_0 to t_1, and this lead to define his famous Lagrangian by,

$$\mathcal{L} = \mathcal{T} - \mathcal{V}.$$

3.4 Euler-Lagrange's equation

Having understood the background of Lagrangian, we can write down Newton's equation of motion by using Lagrangian. In doing this, we apply the Euler's equation described in Sec 1.6. As we know that $\mathcal{T}, \mathcal{V}$ express kinetic and potential energies respectively, they are,

$$\mathcal{T} = \frac{1}{2}m\big(\frac{dx_i}{dt}\big)^2 = \frac{1}{2}m\big(\dot{x}_i\big)^2,$$

$$\mathcal{V} = \mathcal{V}(x_i).$$

Then, the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}m\big(\dot{x}_i\big)^2 - \mathcal{V}(x_i).$$

Here, $i = 1$ to 3 expressing three dimensional world, but this shall be extended to arbitrary dimensions.

Partially differentiate the Lagrangian by x_i and $\dot{x}_i$, we get

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\nabla\mathcal{V}(x_i),$$

[2]Given $ax + by = 0$, we can only conclude $a = b = 0$ when x and y are linearly independent with each other.

[3]This situation is often encountered in analytical mechanics.

$$\frac{\partial \mathcal{L}}{\partial \dot{x_i}} = m\dot{x_i}.$$

Note that $\nabla \mathcal{V}(x_i)$ is a function of variables x_i which belong to the same phase space q only[4], and was introduced by Lagrange. Then

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\Big) - \frac{\partial \mathcal{L}}{\partial x_i} = m\ddot{x_i} + \nabla \mathcal{V}(x_i).$$

Newton's law is written by

$$m\ddot{x_i} = -\nabla \mathcal{V}(x_i).$$

Therefore,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\Big) - \frac{\partial \mathcal{L}}{\partial x_i} = 0.$$

The above equation is called Euler-Lagrange's equation and is identical to the Newton's law of motion.

3.5 Why Euler-Lagrange's equation?

If we look at the Euler-Lagrange's equation, it looks much more abstract because mass and acceleration disappeared (at least from the surface of the equation) and partial differential appeared.

Common explanation of merits using Euler-Lagrange's equation is the **freedom from particular coordinates**. Because the Euler-Lagrange's equation describes the relationship of energy related to the target objects. And these energies are kinetic and potential energy; meaning that the mechanics should be written by field theory.

Writing down mechanics by field theory worked to prepare potential extension of the scope of the original Newtonian mechanics[5], for instance, the electromagnetic theory is also treated by the same idea.

This should be noted as a drastic step to get out of not only particular coordinate, but from geometrical coordinates. The electron flow in electronic circuits cannot be expressed by coordinates, but the behavior of this circuit is treated by Euler-Lagrange's equation as well.

In Newtonian mechanics, the equation of motion becomes specific to the coordinates. That is, for a thrown object,

$$m\frac{d^2x}{dt^2} = 0, \quad m\frac{d^2y}{dt^2} = 0, \quad m\frac{d^2z}{dt^2} = -mgh.$$

If we have to think about the rotation around the center of gravity, the situation becomes much more complicated. However, in Lagrangian mechanics, the equation of motion is written by

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q_i}}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

[4]Not belonging to q and $\dot{q}$ space.

[5]Real extension of mechanics to wider physics is only recognized by the Hamilton's equation. This point shall be discussed in detail in Chapter 4.

where,

$$\mathcal{L} = \mathcal{L}(q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...).$$

Freeing from spatial coordinates is accompanied with the number of dimensions as well - seamlessly extended to multi dimensional motion.

By this philosophy, the **coordinate dependent equation of motion is freed from particular coordinate** extending the scope of application, with the cost of abstraction and difficulties of approaches and solutions (partial differential equations are more difficult than ordinary differential equations).

The abstraction of the Lagrangian is most noticeably found in the expression itself - it only says that the Lagrangian is a function of q_i and $\dot{q}_i$; neither specific coordinate nor particular relation in that coordinate is specified. We only have to supply the systems' energy in solving particular problems. **No "Law" is given** but instead, **a frame work is provided**.

3.6 Generalized coordinates

Discussions in Sec 3.3 and Sec 3.4 does not show why Euler-Lagrange's equation can treat any coordinates.

Questions and explanations given in Sec 3.5 in mind, let us start the discussion of generalized coordinates. The discussion is cumbersome, but careful tracing will show how the coordinates are generalized.

Let us start by writing the Euler-Lagrange's equation by manifestly showing the multiple number of coordinates. Remember that the Lagrangian $\mathcal{L}$ is a function of $q, \dot{q}$ pairs. That is, a function of $2n$ variables in n dimensions (there is dependence between q_i and $\dot{q}_i$).

$$\mathcal{L} = \mathcal{L}(q_1, q_2, ...q_n, \dot{q}_1, \dot{q}_2, ...\dot{q}_n).$$

The following equations hold for a system that conserves energy.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\Big) - \frac{\partial \mathcal{L}}{\partial q_1} = 0,$$
$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\Big) - \frac{\partial \mathcal{L}}{\partial q_2} = 0,$$
$$.....,$$
$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q_n}}\Big) - \frac{\partial \mathcal{L}}{\partial q_n} = 0.$$

The point we have to prove is;

"q_n can be any variable and not restricted to certain coordinates." which is rather amazing because the discussion so far followed the logic stream below - although superficial.

To write down a mechanics equation, use kinetic energy $(1/2)m\dot{\boldsymbol{x}}^2$, and potential energy $\mathcal{V}(\boldsymbol{x})$ instead of position $(\boldsymbol{x})$, velocity $(\dot{\boldsymbol{x}})$ or acceleration$(\ddot{\boldsymbol{x}})$

(Select energy related coordinates.)

↓

Energy expression (= field theory approach) enables to include wider physics such as electric system and can use different coordinates.

(Coordinates are expanded to n.)

↓

A set of Euler-Lagrange's equation express the target physical system precisely by a set of variables q_n.

(We are still bound by original coordinates.)

↓

Variables q_n can be any variable in a system !!!

(Selection of coordinates doesn't matter !!!)

Now the last step looks like a jump which we did not expect nor easily believe. But, deeper discussion would reveal that when we selected "energy", we were freed from coordinates. Because energies other than gravitational potential have quadratic form. And the gravitational potential should be thought in conjunction with the distance rather than particular coordinate.

However, we discuss the logical processes to prove this amazing jump in the next sections.

3.6.1 Euler-Lagrange's equation in two coordinates

We start by writing down the Euler-Lagrange's equation which is formed under a certain coordinate (coordinate q) with $2n$ variables (relation of q and $\dot{q}$ is discussed in Sec 3.7.1).

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q_n}}\Big) - \frac{\partial \mathcal{L}}{\partial q_n} = 0.$$

The other Lagrangian viewed from different coordinate (coordinate Q) will be,

$$\mathcal{L} = \mathcal{L}(Q_1, Q_2, ...Q_n, \dot{Q}_1, \dot{Q}_2, ...\dot{Q}_n).$$

If we can prove the following equation,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q_n}}\Big) - \frac{\partial \mathcal{L}}{\partial Q_n} = 0,$$

then **mechanics expressed by different coordinates satisfies the Euler-Lagrange's equation as well - that is, the mechanics is free from coordinates**.

Referring to Fig 3.2, followings are what we will show.

(1) Calculate spatial slope of $\dot{q}, Q$ which is a function of time.

$$\frac{\partial \dot{q}_i}{\partial Q_k}$$

(2) Calculate time progress of $\partial q_i / \partial Q_k$

$$\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k}\Big)$$

(3) We will find that (1) and (2) becomes identical.
(4) As a conclusion,
Once structure is given and coordinates are selected to precisely describe the system, the different coordinates produces the same Euler-Lagrange's equation.

The process of proof is to express Euler-Lagrange's equation in Q coordinate by q coordinate. This principally means that the **"ratio" of the two physical systems are formed and Euler-Lagrange's equation is derived.** This can be comparable that a Newtonian mechanics is expressed in differential form. Instead of describing the physics in solved form by coordinates, the physics is written by the ratio by time (differential form by time).

During the process, we use the relations between $q-Q$ and $\dot{q}-Q$. ($\dot{q}-\dot{Q}$ is unnecessary because this is identical to $q-Q$.)

3.6.2 Express q by Q

We write down the relation between q and Q in most generalized way as,

$$q_1 = q_1(Q_1, Q_2, ...Q_n),$$

$$q_2 = q_2(Q_1, Q_2, ...Q_n),$$

$$.....,$$

$$q_n = q_n(Q_1, Q_2, ...Q_n).$$

Note that in this expression, the relationship is quite general - it may be linear or nonlinear - between q and Q.

3.6.3 Express $\dot{q}$ by Q

Time differential of variable q_i will be

$$\dot{q}_i = \frac{\partial q_i}{\partial Q_1}\dot{Q}_1 + \frac{\partial q_i}{\partial Q_2}\dot{Q}_2 + ... + \frac{\partial q_i}{\partial Q_n}\dot{Q}_n.$$

By using $\sum$,

$$\dot{q}_i = \sum_{k=1}^{n} \frac{\partial q_i}{\partial Q_k}\dot{Q}_k.$$

Since $\dot{q}_i$ is expressed by $Q_k, \dot{Q}_k$, the Lagrangian $\mathcal{L}(q_1, ...q_n, \dot{q}_1, ...\dot{q}_n)$ is a function of $Q1, ..., Qn$, $\dot{Q}_1, ..., \dot{Q}_n$. The variable q is defined to be a function of Q only, but as we have derived here, $\dot{q}$ is a function of $Q, \dot{Q}$ which is shown in Fig 3.2.

****

If we take partial differential of the following equation along with the axis $\dot{Q}_k$.

$$\dot{q}_i = \frac{\partial q_i}{\partial Q_1}\dot{Q}_1 + \frac{\partial q_i}{\partial Q_2}\dot{Q}_2 + ... + \frac{\partial q_i}{\partial Q_n}\dot{Q}_n,$$

Then,

$$\frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \frac{\partial q_i}{\partial Q_k}.$$

It says slope of $\dot{q}_i$ by $\dot{Q}_i$ is equal to that of q_i by Q_i. This is interesting, but not necessary for the discussions in this section.

However, philosophically, this may give the proof of the freedom of coordinates without much ado, because the Euler-Lagrange's equation is the function of $q, \dot{q}$ and $Q, \dot{Q}$ and if the ratio is equal as above, the only difference in expressions q and Q should be a scaler. The difference of the scaler wouldn't matter because the right hand side of Euler-Lagrange's equation = 0.

Rigorous proof is given in Section3.6.7.

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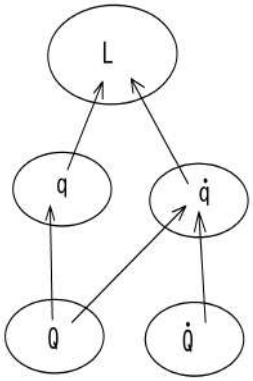

Figure 3.2: Relation of variables in Lagrangian

3.6.4 Slope of $\dot{q} - Q$

Referring to Fig 3.2, we calculate the slope between $\dot{q}_i - Q_k$ which is,

$$\frac{\partial \dot{q}_i}{\partial Q_k} = \frac{\partial^2 q_i}{\partial Q_k \partial Q_1}\dot{Q}_1 + \frac{\partial^2 q_i}{\partial Q_k \partial Q_2}\dot{Q}_2 + ... + \frac{\partial^2 q_n}{\partial Q_k \partial Q_n}\dot{Q}_n$$

$$= \sum_{j=1}^{n} \frac{\partial^2 q_i}{\partial Q_k \partial Q_j}\dot{Q}_j.$$

Note that this is a function of time.

3.6.5 Time progress of $\partial q_i / \partial Q_k$

Noting that $\partial q_i / \partial Q_k$ is a function of Q only,

$$\frac{\partial q_1}{\partial Q_k} = f_1(Q_1, Q_2, ...Q_n),$$

$$\frac{\partial q_2}{\partial Q_k} = f_2(Q_1, Q_2, ...Q_n),$$
$$.....,$$
$$\frac{\partial q_n}{\partial Q_k} = f_n(Q_1, Q_2, ...Q_n).$$

Taking the time differential of $\partial q_i/\partial Q_k$,

$$\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k}\Big)$$
$$= \frac{d}{dt}\Big(f_i(Q_1, Q_2, ..., Q_n)\Big)$$
$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial Q_j}\dot{Q}_j$$
$$= \sum_{j=1}^{n} \frac{\partial}{\partial Q_j}\Big(\frac{\partial q_i}{\partial Q_k}\Big)\dot{Q}_j$$
$$= \sum_{j=1}^{n} \frac{\partial^2 q_i}{\partial Q_j \partial Q_k}\dot{Q}_j.$$

Comparing with the result of Sec 3.6.4,

$$\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k}\Big) = \frac{\partial \dot{q}_i}{\partial Q_k}.$$

****

By the formalism of differential calculation,

$$\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k}\Big) = \frac{d}{dt}\Big(\frac{\partial \dot{q}_i}{\partial \dot{Q}_k}\Big) = \frac{d}{dt}\Big(\frac{\partial \dot{q}_i}{\frac{dQ_k}{dt}}\Big)$$
$$= \frac{d}{dt}\Big(\frac{\partial \dot{q}_i}{\partial Q_k}dt\Big) = \frac{\partial \dot{q}_i}{\partial Q_k}.$$

****

The final proof of coordinate independence is given by using above result in the next section.

3.6.6 Proof of coordinate independence of Euler-Lagrange's equation

Now we are going to show that if the following equation

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\Big) - \frac{\partial \mathcal{L}}{\partial q_n} = 0,$$

holds, then the following equation also holds.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}_n}\Big) - \frac{\partial \mathcal{L}}{\partial Q_n} = 0.$$

In general,

$$\mathcal{L} = \mathcal{L}(Q_1, Q_2, ...Q_n, \dot{Q}_1, \dot{Q}_2, ...\dot{Q}_n).$$

Remembering that Q is a function of q and $\dot{q}$ (See Fig 3.2),

$$\frac{\partial \mathcal{L}}{\partial Q_k} = \sum_i \Big(\frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial Q_k} \Big).$$

$\dot{Q}_k$ is only dependent to $\dot{q}$,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{Q}_k} &= \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{Q}_k} \\ &= \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial q_i}{\partial Q_k}. \end{aligned}$$

We take time differential of $\partial \mathcal{L} / \partial \dot{Q}_k$,

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}_k} \Big) &= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Big) \frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k} \Big) \Big\} \\ &= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Big) \frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial Q_k} \Big\}. \end{aligned}$$

Here we used the result of Sec 3.6.5,

$$\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k} \Big) = \frac{\partial \dot{q}_i}{\partial Q_k}.$$

Then

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}_k} \Big) - \frac{\partial \mathcal{L}}{\partial Q_k} \\ &= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Big) - \frac{\partial \mathcal{L}}{\partial q_i} \Big\} \frac{\partial q_i}{\partial Q_k} \\ &= 0. \end{aligned}$$

Now we know that the **Euler-Lagrange's equation in coordinate Q is identical with that of coordinate q** which means that the coordinates really don't matter.

3.6.7 Another proof

Here is another slightly different proof using the relationship,

$$\frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \frac{\partial q_i}{\partial Q_k}.$$

Since $\partial \dot{q}_i / \partial \dot{Q}_k, \partial q_i / \partial Q_k$ are functions of t,

$$\frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \frac{\partial q_i}{\partial Q_k} = f_i(t).$$

And,

$$\frac{\partial \dot{q}_i}{\partial Q_k} = \frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k} \Big).$$

Remembering that Q is a function of q and $\dot{q}$ (See Fig 3.2),

$$\frac{\partial \mathcal{L}}{\partial Q_k} = \sum_i \Big\{ \frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_k} \Big\}.$$

$$= \sum_i \Big\{ \frac{\partial \mathcal{L}}{\partial q_i} f_i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big(\frac{d}{dt}\frac{\partial q_i}{\partial Q_k}\big) \Big\}$$

$$= \sum_i \Big\{ \frac{\partial \mathcal{L}}{\partial q_i} f_i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big(\frac{d}{dt} f_i(t)\big) \Big\}$$

$\dot{Q}_k$ is only dependent to $\dot{q}$,

$$\frac{\partial \mathcal{L}}{\partial \dot{Q}_k} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial q_i}{\partial Q_k}.$$

Time differential of $\partial \mathcal{L}/\partial \dot{Q}_k$ becomes,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}_k}\Big) = \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big)\frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\Big(\frac{\partial q_i}{\partial Q_k}\Big) \Big\}$$

$$= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big)\frac{\partial q_i}{\partial Q_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_k} \Big\}.$$

$$= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) f_i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big(\frac{d}{dt} f_i(t)\big) \Big\}$$

Then Euler -Lagrange's equation in Q yields,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}_k}\Big) - \frac{\partial \mathcal{L}}{\partial Q_k}$$

$$= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) f_i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big(\frac{d}{dt} f_i(t)\big) \Big\} - \sum_i \Big\{ \frac{\partial \mathcal{L}}{\partial q_i} f_i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big(\frac{d}{dt} f_i(t)\big) \Big\}$$

$$= \sum_i \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) f_i(t) - \frac{\partial \mathcal{L}}{\partial q_i} f_i(t) \Big\}$$

$$= \sum_i f_i(t) \Big\{ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} \Big\}$$

$$= 0$$

3.7 What is Lagrangian?

3.7.1 Lagrangian variables q and $\dot{q}$

In Sec 3.6.1, we wrote down the Euler-Lagrange's equation under n coordinates[6] with $2n$ variables (q and $\dot{q}$). Although dependence/independence between q and $\dot{q}$ is not required during the process to prove coordinate independence of Lagrangian, it was not discussed explicitly how many dimension the Lagrangian has.

[6] Answer is actually given here, but we wish to clear the reasoning.

In Sec 3.6.3, we obtained

$$\dot{q}_i = \frac{\partial q_i}{\partial Q_1}\dot{Q}_1 + \frac{\partial q_i}{\partial Q_2}\dot{Q}_2 + ... + \frac{\partial q_i}{\partial Q_n}\dot{Q}_n.$$

Then,

$$\dot{q}_i = \dot{Q}_1\frac{\partial q_i}{\partial Q_1} + \dot{Q}_2\frac{\partial q_i}{\partial Q_2} + ... + \dot{Q}_n\frac{\partial q_i}{\partial Q_n},$$

which clearly shows that $\dot{q}_i$ is dependent to q_i via Q. Since Q can be any (valid) coordinate, this dependence is permanent. Thus, the **Lagrangian is an n dimensional vector**.

Therefore, when we solve the Euler-Lagrange's equation, we have to reduce the number of variables from $2n$ to n. This is implicitly done when forming the Lagrangian by giving momenta $m\dot{q}$, because the scope of Lagrangian (at this moment) is still limited within Newtonian mechanics, and there is no need to think another equivalent momenta. In case of Hamiltonian, this process becomes explicit as shall be discussed in Sec 4.4.

3.7.2 Lagrangian as an intermediate state to analytical physics

Euler-Lagrange's equation and Lagrangian are a great step from Newtonian mechanics, but still looks like an intermediate phase to the analytical mechanics or Hamiltonian mechanics.

Lagrangian $\mathcal{L}$ is a function of variables q_i and $\dot{q}_i$. However, transformation of variables are restricted between q_i and Q_i. Because if we suppose,

$$q = q(Q, \dot{Q}),$$

then

$$\dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial \dot{Q}}\ddot{Q}.$$

The new Lagrangian shall include $\ddot{Q}$ and cannot be expressed by,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\Big) - \frac{\partial \mathcal{L}}{\partial Q} = 0.$$

That is, the transformation between different phase variables (q and $\dot{Q}$ for instance) inherently yields higher order of differential and it destroys the Newtonian mechanics.

This is the limitation of the Euler-Lagrange's equation. It can treat different coordinates - Cartesian and polar coordinates for instance - in a uniform way, but **coordinates transformation is limited within the q phase variables**[7].

In Lagrangian mechanics, coordinates selection became easier, but the transformation of coordinates is limited between q and Q. **The merit of the Lagrangian should be found in coordinate free construction of equation, but not in the transformation of coordinates**.

The variables in the Lagrangian reflect real world physics directly - location, speed, voltage, current and so on - but further coordinates transformation is not the objective in the Lagrangian mechanics. In other words, the Lagrangian mechanics treat each problems

[7] Q and $\dot{Q}$ work asymmetrically. Refer to Fig 3.2.

as they are, but does not explore any further[8]. Only the importance of the "energy" is noticed.

The Hamiltonian mechanics will make clear what the crucial physical variables are and will formally define the generalized coordinates and generalized momenta. These variables can be further transformed to open up a new world. That is, the real philosophical progress will be found in the Hamiltonian mechanics; the structure of mechanics will become the target of discussions. **A question "What is mechanics?" will be posed in the Hamiltonian mechanics**, and the Lagrangian mechanics is the starting point.

"You ain't seen nothing yet."

3.8 Summary

What Newton has done in establishing the Newtonian mechanics was to write down the physical law independent from absolute time. The resultant equation is a **single variable 2nd order differential equation** with variable x and parameter t. The integration by time t yields a function $x(t)$ which is the trajectory of the object in space.

Lagrange expressed the Newton's equation of motion form the perspective of energy in the target system. Since there are kinetic and potential energies, the resultant differential equation must comprise at least two different variables. This effectively means that **the differential equation adopted by Lagrange is "multi variable" differential equation** - though generally called partial differential equation.

However, the Euler-Lagrange's equation is a set of second order equations of dependent variables $\dot{q}$ and q. The number of differential equation is n, and the number of $\dot{q}$ and q are both n. This obviously presents food for thought.

[8]This is very practical and useful point when we discuss the automatic control where coordinates transformation certainly makes things difficult.

Chap4 Hamiltonian

William Rowen Hamilton was said to have been deeply impressed by the treatise of Newtonian mechanics by Euler and Lagrange. Recognizing the energy as an interface between different physical field, and coordinate free treatise are indeed great factors that contributed to theoretical physics as well as engineering.

In some textbooks and literatures, the discussion of Hamiltonian mechanics starts by "defining" the Hamiltonian. The amazingly simple result of the Hamilton's equation may have effects to let people forget to explore how it came up. But a discussion without a thought on how Hamiltonian flourished from Lagrangian will leave logical discontinuity that makes Hamiltonian mechanics difficult to understand.

4.1 "Definition" of Hamiltonian

In many literatures, Hamiltonian $\mathcal{H}$ is "defined" by the following equation,

$$\mathcal{H}(q_i, p_i) = \sum p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i),$$

or, is the starting point of further discussion.

Since

$$\sum p_i \dot{q}_i = \sum m \dot{q}_i \dot{q}_i = \sum \frac{1}{2} m \dot{q}_i^2,$$

and this is equal to $2\mathcal{T}$ of the system, the above equation eventually yields

$$\begin{aligned} \mathcal{H} &= 2\mathcal{T} - (\mathcal{T} - \mathcal{V}) \\ &= \mathcal{T} + \mathcal{V}. \end{aligned}$$

If we call the $\mathcal{H}$ as the "definition" of Hamiltonian without discussing the logical processes, one might ask questions;

"Why did Hamilton get the idea to subtract Lagrangian from $\sum p_i \dot{q}_i$?

"What is new and significant[1] about $\mathcal{H}$ which just changed the sign of $\mathcal{V}$ in the Lagrangian ?"

"Why does it deserve the name Hamiltonian ? "

or

"Why $p_i \dot{q}_i$ and not $(1/m)p_i^2$ nor $m\dot{q}_i^2$?[2] "

We intend to answer all the questions above and make the process to Hamiltonian continuous from Lagrangian (and eventually, show that the Lagrangian is same with the Hamiltonian in Sec 4.8.4).

[1] One is even tempted to call Hamiltonian a plagiarism.

[2] This question is caused by having prior knowledge that $p = m\dot{q}$.

4.2 Lagrangian to Hamiltonian

4.2.1 Introduction of new variable

We have shown why Lagrangian is written down by,

$$\mathcal{L} = \mathcal{T}(q_i, \dot{q}_i) - \mathcal{V}(q_i).$$

Then the Euler-Lagrange's equation is

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} = 0,$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{T}(q_i, \dot{q}_i)}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{T}(q_i, \dot{q}_i)}{\partial q_i} + \frac{\partial \mathcal{V}(q_i)}{\partial q_i} = 0.$$

Observing the above equation, we have phase variables q_i and $\dot{q}_i$. And we have made clear that the Euler-Lagrange's equation is defined in an n dimensional space with $2n$ variables.

That is, the number of equations to express mechanics is n, but we have $2n$ variables which means something is still unsatisfactory. And once the Euler-Lagrange's equation is made into a concrete form[3] by supplying energy $\mathcal{T}, \mathcal{V}$, variable $\ddot{q}$ manifests.

These uneasiness seems to suggest hints what Hamilton saw in the Euler-Lagrange's equation and start thinking about further clarification of the Lagrangian mechanics, because these points is fundamental to Euler-Lagrange's equation.

Then, it sounds a good idea to reduce the order of Euler-Lagrange's equation. To this end, one may observe $\ddot{q}_i$ and seek a way to replace with some other variable. Because $\ddot{q}_i$ is brought from $\dot{q}_i$ and we know $m\dot{q}_i$ is a momentum from Newtonian mechanics, it is natural to think that the replacement is based on the equation $p_i = m\dot{q}_i$.

Although, what Hamilton really did was different from the way as described in this chapter, the derivation of Hamilton's equation is much logically understood by the following explanation. (Refer to the Hamilton's original paper. [18], [19])

4.2.2 Applying Legendre transform

Euler-Lagrange's equation is a second order differential equation of q_i. Techniques to reduce the order by increasing the number of variable are already known in the days of Poisson, and Lagrange recognized the importance ([20],[21]).

Although why Hamilton advanced to this direction is unknown, he applied Legendre transform which can express the target function by it's tangent and intercept thus replacing a derivative (See 1.8).

However, we know that the Legendre transform can be only applied to concave (or convex) functions and if we apply to the Lagrangian, we have to think about this first.

Lagrangian is a function of kinetic energy $\mathcal{T}$ (convex function) and potential energy $\mathcal{V}$ but the latter is generally not a convex function. Therefore,

$$\mathcal{L}(q_i, \dot{q}_i) = \mathcal{T}(q_i, \dot{q}_i) - \mathcal{V}(q_i)$$

[3] In an object oriented programming language, an "instance" is generated.

may not be a convex function overall, but if we look at how Legendre transform takes effect when applied to $\mathcal{L}$, we can see that the **Legendre transform affects** $\mathcal{T}(q_i, \dot{q}_i)$ **only**. Because $\mathcal{V}(q_i)$ does not contain variable $\dot{q}_i$.

So, we apply Legendre transform without fear, but still there are some implications.

We will apply the Legendre transform to the Lagrangian, but real target (of which we wish to reduce the order of differential) is the Euler-Lagrange's equation.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

The convex function is $\mathcal{T}(q, \dot{q})$, hence the primary variable in applying the Legendre transform is $\dot{q}$. And the derivative (tangent) which we are going to replace by new variable p is $\partial \mathcal{T}/\partial \dot{q}$.

Since,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\Big(\frac{1}{2}m\dot{q}^2\Big) = m\dot{q},$$

the replacement of $\partial \mathcal{T}/\partial \dot{q}$ by p is equal to replacement of $m\dot{q}$ by p.

By the Legendre transform, a function $F(x)$ is expressed by the tangent $s(x) = F\prime(x)$ and it's intercept $G(s)$ by,

$$F(x) = s(x) \cdot x - G(s).$$

Then, after the transformation we get

$$G(s) = s(x) \cdot x - F(x).$$

Therefore, variable correspondence is,

$$x_i \to \dot{q}_i,$$

$$F(x_i) \to \mathcal{L}(q_i, \dot{q}_i) = \mathcal{T}(q_i, \dot{q}_i) - \mathcal{V}(q_i),$$
$$s_i = F\prime(x_i) \to p_i = \frac{\partial \mathcal{T}(q_i, \dot{q}_i)}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}(q_i, \dot{q}_i)}{\partial \dot{q}_i}.$$

Note that partial differential is used because $x \to F\prime(x)$ corresponds to $\dot{q} \to \partial \mathcal{T}/\partial \dot{q}$. Assigning $\mathcal{H}(p_i, q_i)$ to the function resulted by the Legendre transform, we obtain

$$\mathcal{H}(p_i, q_i) = \sum p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i).$$

Here, $\mathcal{L}$ is still a function of q_i and $\dot{q}_i$ whereas $\mathcal{H}$ is defined to be a function of p_i, q_i. $\sum p_i \dot{q}_i$ is a function of p_i, q_i. To complete the replacement we write,

$$\mathcal{L}(q_i, \dot{q}_i) = \sum p_i \dot{q}_i - \mathcal{H}(p_i, q_i).$$

What we have done so far are,

(1) $\partial \mathcal{L}/\partial \dot{q}_i$ is selected as new parameter p_i.

(2) By the Euler-Lagrange's equation

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} = 0,$$

second derivative $\ddot{q}_i$ is produced by

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) \to \ddot{q}_i$$

but by the replacement of $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, only the first derivative $\dot{p}_i$ is produced, thus reducing the order.

(3) Euler-Lagrange's equation whose variables are $q_i, \dot{q}_i, p_i, \dot{p}_i$ is obtained.
(but still, necessary and sufficient number of variables are unclear.)

4.2.3 Lagrangian of p_i, q_i

In Sec 4.2.2, we have applied Legendre transform and yielded the following equation,

$$\mathcal{L}(q_i, \dot{q}_i) = \mathcal{L}(p_i, q_i) = \sum p_i\dot{q}_i - \mathcal{H}(p_i, q_i).$$

Left hand side can be a **Lagrangian expressed by variables $p_i, \dot{p}_i$ and $q_i, \dot{q}_i$.**

We performed a variational integral to Lagrangian in Sec 3.3 and obtained Newton's law of motion. Likewise, if we perform a variational integral for this Lagrangian, **we shall get equivalent of Newton's law of motion with two variables q_i and p_i.** Although, we will yet to know if a single equation yields two equations[4].

4.2.4 Variational integral of $\mathcal{L}$

We calculate the variational integral seeking for the stationary (local minimum) value of $\mathcal{L}$. As $\mathcal{V}(q_i)$ works just a bias to $\mathcal{T}(q_i, \dot{q}_i)$, we can omit $\mathcal{V}(q_i)$ from the calculation of local minimum.

Our objective here is the calculation of the variational integral of $\mathcal{L}$ which is equaled to zero.

$$0 = \delta\int_{t_0}^{t_1} \mathcal{L}dt = \int_{t_0}^{t_1} \delta\mathcal{L}dt$$

$$= \int_{t_0}^{t_1} \Big(\delta\sum(p_i\dot{q}_i) - \delta\mathcal{H}(p_i, q_i)\Big)dt$$

$$= \int_{t_0}^{t_1} \sum \delta(p_i\dot{q}_i)dt - \int_{t_0}^{t_1} \delta\mathcal{H}(p_i, q_i)dt.$$

For further calculation we prepare,

$$\delta(p_i\dot{q}_i) = (p_i + \delta_p p_i)(\dot{q}_i + \delta_q\dot{q}_i) - p_iq_i = \delta_p p_i\dot{q}_i + p_i\delta_q\dot{q}_i + \delta_p p_i\delta_q\dot{q}_i$$

$$\doteq \delta_p p_i\dot{q}_i + p_i\delta_q\dot{q}_i.$$

Note that $\delta(\xi)$ means a small value of (ξ) and δ can assume independent amount when applied to multiple variables. To emphasize this point, we use δ_p, δ_q. **This independence of value plays a crucial role in the following discussion.**

[4]Hamilton's equation is formed by two equations

Then,

$$\int_{t_0}^{t_1} \sum \delta(p_i \dot{q}_i) dt = \int_{t_0}^{t_1} \sum (\delta_p p_i \dot{q}_i + p_i \delta_q \dot{q}_i) dt$$

$$= \int_{t_0}^{t_1} \sum \delta_p p_i \dot{q}_i dt + \int_{t_0}^{t_1} \sum p_i \delta_q \dot{q}_i dt.$$

Performing the integral by parts for $p_i \delta_q \dot{q}_i$,

$$= \int_{t_0}^{t_1} \sum \delta_p p_i \dot{q}_i dt + \sum \Big([p_i \delta_q q_i]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}_i \delta_q q_i dt \Big)$$

$$= \int_{t_0}^{t_1} \sum \delta_p p_i \dot{q}_i dt - \int_{t_0}^{t_1} \sum \dot{p}_i \delta_q q_i dt,$$

because $\delta_q q_i = 0$ at t_0 and t_1. And,

$$\delta \mathcal{H}(p_i, q_i) = \frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} \delta_p p_i + \frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} \delta_q q_i.$$

****

The above equation is not self evident. What it means is that when the function $F(x_1, x_2, ..., x_n)$ holds the following equation,

$$\delta F(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \delta x_i.$$

Then, for this set of functions we can eventually derive the Hamilton's equation.
****

Now we resume the original calculation,

$$0 = \delta_p \int_{t_0}^{t_1} \mathcal{L} dt$$

$$= \int_{t_0}^{t_1} \Big(\sum (\delta_p p_i \dot{q}_i) - \delta \mathcal{H}(p_i, q_i) \Big) dt$$

$$= \int_{t_0}^{t_1} \Big(\sum (p_i \delta_q \dot{q}_i + \delta_p p_i \dot{q}_i) - \delta \mathcal{H}(p_i, q_i) \Big) dt$$

$$= \int_{t_0}^{t_1} \Big(\sum (\delta_p p_i \dot{q}_i - \dot{p}_i \delta_q q_i) - \delta \mathcal{H}(p_i, q_i) \Big) dt$$

$$= \int_{t_0}^{t_1} \sum \Big(\delta_p p_i \dot{q}_i - \dot{p}_i \delta_q q_i - \frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} \delta_p p_i - \frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} \delta_q q_i \Big) dt$$

$$= \int_{t_0}^{t_1} \sum \Big(\big(- \frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} - \dot{p}_i \big) \delta_q q_i + \big(- \frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} + \dot{q}_i \big) \delta_p p_i \Big) dt.$$

Here, $\delta_q q_i$ **and** $\delta_p p_i$ **are arbitrarily chosen and hence independent values**.

Then, in order that the above integral always becomes zero, the following set of first order differential equations must hold.

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} = -\dot{p}_i,$$

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} = \dot{q}_i.$$

This is amazing. Single equation yielded two equations with simplest and symmetrical form. These are called **Hamilton's canonical equations**[5].

Note that the key requirement in the above discussion is the independence of small values of $\delta p, \delta q$ - relation between p and q does not affect to the discussion (however, we will see that p is perfectly independent from q in Sec 4.4).

4.3 What Hamilton's equation tells

The Hamilton's equation is derived by

- Introducing new variables p_i,
- Assuming the principle of least action,

and it revealed the relation of energy-momenta and energy-coordinates.

As an example, free motion of an object with mass m in potential $\mathcal{V}$ is expressed in the following way. Using the Newtonian momenta $p_i = m\dot{q}_i$, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_z^2 + \mathcal{V}(q_x, q_y, q_z).$$

Then, one part of the Hamilton's equation yields the Newton's law,

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \quad \rightarrow \quad grad\, \mathcal{V}(q_x, q_y, q_z) = -f.$$

And the the other part yields

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \quad \rightarrow \quad \frac{1}{m}p_x + \frac{1}{m}p_y + \frac{1}{m}p_z = m\dot{q}_i$$

which is the relation we assumed to write down the Hamiltonian. **This is a tautology or, the second Hamilton's equation looks like to be redundant (!?)**.

If we look at the structure of the Hamilton's equation (Fig 4.1), the equation may be considered as a filter to Hamiltonian, and the images filtered out are $\dot{q}_i$ and $\dot{p}_i$. The filter or the Hamilton's equation reflects the other counterpart's elements thus making it a complementary equation[6]. That is, the Hamilton's equation refers to each other. To calculate the Hamiltonian mechanics, **we need to cut this cross reference.**

Therefore, in an application to actual mechanics, we have to specify the relation between p_i and q_i. In case of Newtonian mechanics and if we use the Cartesian coordinates,

$$p_i = m\dot{q}_i.$$

[5]The name canonical was by Jacobi in 1837.

[6]If the filter shows it's own image, there can be no connection between momenta and coordinates - no physics will exist.

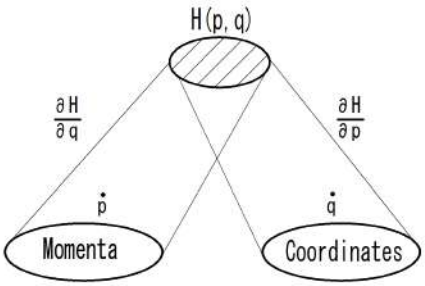

Figure 4.1: Relation in the Hamilton's equation

In polar coordinates

$$p_r = m\dot{q}_r, \quad p_\phi = mq_r\dot{q}_\phi.$$

From other point of view, the equation

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i$$

is asking us,

- **determine what physics we are going to discuss**,
- **give an appropriate relation between coordinates and momenta**.

Without recognizing the above mentioned fundamental mechanism, we can be easily caught into a trap of tautology.

The discussion above shows that the Lagrangian mechanics is generalized during the derivation of the Hamilton's equation. This enabled wider selection of momenta and coordinates which extended the mechanics to "generalized physics".

In the days of Hamilton, there was no electromagnetic theory nor quantum mechanics. Therefore, it is hard to imagine that Hamilton himself was aware that he had "generalized" the physics, or, had revealed the structure of the mechanics which is hidden in the Lagrangian (=Newtonian) mechanics.

Hamilton's equation comprises much more than it's simple form tells us.

4.4 Independence of p and q

The reason that two fundamental variables (coordinates and momenta) manifested goes back to the introduction of

$$p_i = m\dot{q}_i.$$

Since mass is independent from position, p_i and q_i becomes independent regardless of the dependence/independence between q_i and $\dot{q}_i$ (although we discussed in Sec 3.7.1 that q is dependent to $\dot{q}$).

Hence, the Hamilton's equation becomes an equation of multi independent variables p_i and q_i. The dimension of Hamilton's equation is $2n$ and we have $2n$ equations and uneasiness of Euler -Lagrange's equation is cleared.

This independence is directly reflected by the Hamilton's equation, and this is the reason that we have to supply the relation of coordinate and momenta in actual problems of physics. Or, **independence between p_i and q_i works to distinguish coordinate and momenta as a key philosophy to generalize the Newtonian mechanics to "Generalized Physics"**.

4.5 Reduction of order of differential equation

In Sec 1.1.6, we have discussed that an nth order differential equation is expressed by a set of first order differential equations.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

In view of order reduction of derivatives, this method is much more general and systematic compared with the Legendre transform. Although we have discussed only linear differential equations, it is not difficult to extend the idea to non-linear differential equations at least philosophically.

In multivariable control theory, relation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is used to **reduce the order of Lagrangian yielding Hamilton's equation** (see Chap 14). Thanks to the coefficient $\boldsymbol{A}$, variable $\dot{\boldsymbol{x}}$ is independent from $\boldsymbol{x}$ and Hamilton's equation manifests.

4.6 Examples

4.6.1 Newton's law expressed by Hamiltonian

The following equation

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} = \dot{q}_i$$

shows the relation between energy and "generalized" momenta. First thing we have to do is the selection of physics - the Newtonian mechanics in this case.

And if we use the Cartesian coordinates (and this process means to remove the term "generalized"),

$$p_i = mv_i = m\dot{q}_i \quad (i = x, y, z).$$

That is, we specify the definition of momenta by,

$$\dot{q}_x = \frac{1}{m}p_x, \;\; \dot{q}_y = \frac{1}{m}p_y, \;\; \dot{q}_z = \frac{1}{m}p_z.$$

Now we can write the Hamiltonian,

$$\mathcal{H}(p_i, q_i) = \mathcal{T}(p_i, q_i) + \mathcal{V}(q_i) = \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_z^2 + \mathcal{V}(q_x, q_y, q_z).$$

Where $\mathcal{V}(q_x, q_y, q_z)$ is the potential energy.
Then,

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} = -\dot{p}_i = \frac{\partial \mathcal{V}(q_x, q_y, q_z)}{\partial q_x} + \frac{\partial \mathcal{V}(q_x, q_y, q_z)}{\partial q_y} + \frac{\partial \mathcal{V}(q_x, q_y, q_z)}{\partial q_z}$$

$$= grad\, \mathcal{V}(q_x, q_y, q_z)$$
$$= -f.$$

The above equation means

$$-\dot{p}_i = -m\ddot{q}_i = grad\, \mathcal{V}(q_x, q_y, q_z) = -f,$$

or

$$f = m\ddot{q}_i,$$

which is the Newton's law.

By the above example, the **potential energy $\mathcal{V}$ in the Hamiltonian turns into $grad\, \mathcal{V}$ in the Hamilton's equation**.

Since $f = -grad\, \mathcal{V}$, the difference of sign in

$$\mathcal{L} = \mathcal{T} - \mathcal{V},$$

$$\mathcal{H} = \mathcal{T} + \mathcal{V},$$

is adjusted by the "sign of gradient" yielding the identical Newton's law.

4.6.2 Newton's law in polar coordinates

The discussion is in the Newtonian mechanics under polar coordinates. The Newton's law is expressed by the polar coordinates (q_r, q_ϕ) (for simplicity's sake, we use two dimensional space),

$$q_r = \frac{dq_r}{dt}, \quad q_\phi = q_r \frac{dq_\phi}{dt}.$$

Then the momenta are expressed by

$$p_r = m\dot{q}_r, \quad p_\phi = mq_r\dot{q}_\phi.$$

If we write the potential energy by $\mathcal{V}(q_r, q_\phi)$, the Hamiltonian will be

$$\mathcal{H} = \frac{1}{2}m(\dot{q}_r^2 + {q_r}^2\dot{{q_\phi}}^2) + \mathcal{V}(q_r, q_\phi).$$

The Hamilton's equation is,

$$\frac{\partial \mathcal{H}}{\partial q_r} = -\dot{p}_r = mq_r\dot{{q_\phi}}^2 + \frac{\partial \mathcal{V}(q_r, q_\phi)}{\partial q_r},$$

$$\frac{\partial \mathcal{H}}{\partial q_\phi} = -\dot{p}_\phi = \frac{\partial \mathcal{V}(q_r, q_\phi)}{\partial q_\phi}.$$

Now the Hamilton's equation has a concrete form and we can solve it.

4.6.3 Free fall

We discuss the Newtonian mechanics under Cartesian coordinates.

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q} = \frac{p}{m}.$$

Then we can discuss further Newtonian motion.

Suppose an object is falling down with gravitation g only. Then kinetic and potential energies are,

$$\mathcal{T} = \frac{1}{2}m\dot{z}^2, \;\; \mathcal{V} = -mgz.$$

Since we have specified $p = m\dot{z}$,

$$\mathcal{H} = \frac{1}{2m}p^2 - mgz.$$

Hamilton's equation is

$$\frac{\partial \mathcal{H}}{\partial z} = -\dot{p} = -mg.$$

Then, assuming p_0 as the initial value,

$$p = mgt + p_0.$$

Since $\dot{z} = p/m$ and assuming z_0 as the initial value,

$$\begin{aligned} z = \frac{1}{m}\int p dt &= \frac{1}{m}\int (mgt + p_0)dt \\ &= \frac{1}{m}(mg\frac{1}{2}t^2 + p_o t + z_0) \\ &= \frac{1}{2}gt^2 + c_1 t + c_2, \end{aligned}$$

where $c_1 = p_0/m, c_2 = z_o/m$.

4.7 When Hamiltonian changes

When Hamiltonian does not contain parameter t explicitly,

$$\begin{aligned} \frac{\partial \mathcal{H}(p,q)}{\partial t} &= \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial t} \\ &= \frac{\partial \mathcal{H}}{\partial p}\dot{p} + \frac{\partial \mathcal{H}}{\partial q}\dot{q} \\ &= \dot{q}\dot{p} - \dot{p}\dot{q} = 0. \end{aligned}$$

In this case the Hamiltonian cannot change and the total energy is always conserved.

However, if the Hamiltonian includes variable t explicitly,

$$\mathcal{H} = \mathcal{H}(p, q, t),$$

the Hamiltonian can be changed in two ways.

One is found in automatic control in which system energy is lost/supplied by external intervention[7] so that the system moves as we wish. In this case, the change of total energy is reflected via p or q. Also, the coordinate transformation will not be performed, and hence the Hamiltonian will be equal to the total energy of the system.

Second example is found in the canonical transformation of the coordinates described in Sec 4.8.5. In this case, even a system with no interaction with outer world - a system that conserves total energy - the Hamiltonian can be changed. The cause of the change is the change of view point (view from different time flow) to the target system and the total energy of the system also changes; that is, **the Hamiltonian is always equal to the total energy**.

This discussion goes deeper into physics, and the reasoning that the Hamiltonian is equal to total energy is explained by Noether's theorem. Noether's theorem states that "every differentiable symmetry of a physical system has a corresponding conservation law" and it can be shown that the Hamiltonian is such a conservation value.

Further discussion will be found in Sec 5.3 and Sec 5.4, because we are not enough equipped yet.

4.8 Canonical transformation

4.8.1 Transform the "speed" ?

In the Newtonian mechanics, variables are tightly bound to our daily life; mass, speed, acceleration etc.. It would have been difficult to imagine transformation of variables. Also, Newton's description was based on geometry and selection of coordinates can be cumbersome - this is said to be one of the motives that Lagrange explored analytical description. Lagrange succeeded to "transform" the Newtonian mechanics into mathematics which eventually yielded Hamilton's equation.

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q}, \qquad \frac{\partial \mathcal{H}}{\partial q} = -\dot{p}.$$

Although who actually thought further transformation of the variables was not clear, but Poisson disclosed his Poisson bracket in the early days of Lagrange's equation, transformation might have been a common daily tool for those scholars. However, they were well aware that the original topic was based on nature, and sought a transformation that retains the original physics. This is the fundamental character of the "canonical transformation".

The existence of the canonical transformation shows that the Newtonian or Lagrangian coordinates are just a temporary view point of the target system and intrinsic physical property is the **conjugate pair of coordinates and momenta**. The meaning of "the conjugate pair" becomes clear in Sec 5.2.3.

[7] This energy may be equipped in the system by fuel or batteries.

4.8.2 Canonical transformation - superficial definition

The Hamilton's - canonical - equation is

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} = -\dot{p}_i,$$

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} = \dot{q}_i.$$

If we think about a replacement of variables,

$$q \to Q, \;\; \dot{q} \to \dot{Q},$$

and when new Hamilton's equation has the form,

$$\frac{\partial \mathcal{H}(P_i, Q_i)}{\partial Q_i} = -\dot{P}_i,$$

$$\frac{\partial \mathcal{H}(P_i, Q_i)}{\partial P_i} = \dot{Q}_i,$$

then we call this "canonical transformation". Superficially, a canonical transformation is a particular coordinate transformation that retains the "style" of the canonical equation. But of course, this is far from convincing explanation and shall be discussed in Sec 4.10 in detail.

We will see the canonical transformation examples before getting into fundamental discussions.

4.8.3 Transformation; Hamiltonian is retained

Suppose new coordinates Q, P is expressed by

$$Q = p, \;\; P = -q.$$

That is, in new variables Q, P, coordinates and momenta are exchanged. The Hamiltonian becomes,

$$\mathcal{H}(p, q) \to \mathcal{K}(Q, -P).$$

Then

$$\frac{\partial \mathcal{H}(p, q)}{\partial q} = -p \to -\frac{\partial \mathcal{K}}{\partial P} = -Q$$

$$\frac{\partial \mathcal{H}(p, q)}{\partial p} = q \to \frac{\partial \mathcal{K}}{\partial Q} = -P.$$

This is a canonical transformation. As shown here , **in Hamiltonian mechanics, coordinates and momenta lose their original meanings**.

4.8.4 Canonical transformation by Legendre transform

We have shown that the Hamiltonian is converted from Lagrangian by applying Legendre transform.

$$\mathcal{H}(q_i, p_i) = \sum p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i).$$

Now, we discuss a replacement from q_i to Q_i by an **arbitrary monotonic function** $f()$,

$$Q_i = f(q_i).$$

Then, time derivative of Q_i will be

$$\dot{Q}_i = \frac{\partial f(q_i)}{\partial q_i}\frac{\partial q_i}{\partial t} = \frac{\partial f(q_i)}{\partial q_i}\dot{q}.$$

The (generalized) momenta corresponding Q_i cannot be given a priori because we are discussing under new variables (coordinates) Q_i. So we define the momenta P_i as,

$$\begin{aligned} P_i &= \frac{\partial \mathcal{L}(q_i, \dot{q}_i)}{\partial \dot{Q}_i} \\ &= \frac{\partial q_i}{\partial f(q_i)}\frac{\partial \mathcal{L}(q_i, \dot{q}_i)}{\partial \dot{q}_i} \\ &= \frac{\partial q_i}{\partial f(q_i)} p_i. \end{aligned}$$

because $\partial f(q_i)/\partial q_i$ is a constant in regard with $\dot{Q}_i$ and $\dot{q}_i$.

Then, the new Hamiltonian $\mathcal{H}'$ will be,

$$\mathcal{H}'(P_i, Q_i) = P_i \dot{Q}_i - \mathcal{L}(q_i, \dot{q}_i) = p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i).$$

That is, the new Hamiltonian $\mathcal{H}'$ is philosophically identical with original $\mathcal{H}$ meaning that the above transformation is a canonical transformation. Important point is that this transformation **will not mix up coordinate and momenta** which is the character of applying Legendre transform.

Hence, **the Lagrangian is philosophically identical to the Hamiltonian**, just showing different aspects of the target physics.

(Once we know that the Legendre transform is a canonical transform, we see that the Lagrangian and the Hamiltonian is the same one because the inverse of the Legendre transform is the Legendre transform.)

4.8.5 Transformation; Hamiltonian is not retained

For simplicity's sake, we discuss a single variable sinusoidal oscillation given by,

$$q = A\sin(\omega t + \phi).$$

Then variable p will be[8],

$$p = m\dot{q} = m\omega A\cos(\omega t + \phi).$$

[8]Here, we declared that we are going to discuss the Newtonian mechanics. See Sec 4.3.

Potential of the simple oscillation is

$$V(q) = \frac{1}{2}m\omega^2 q^2.$$

Then the Hamiltonian will be

$$\mathcal{H} = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

$$= \frac{m\omega^2}{2}\Big(q^2 + (\frac{1}{m\omega})^2 p^2\Big).$$

Now we will try a transformation by the variables Q, P with the following way (don't ask why at this moment),

$$Q = q^2 + (\frac{1}{m\omega})^2 p^2,$$

$$P = tan^{-1}(m\omega\frac{q}{p}) - \omega t.$$

Then new Hamiltonian $\mathcal{H}\prime$ is,

$$\mathcal{H}\prime = \frac{m\omega^2}{2}Q.$$

Therefore,

$$\dot{Q} = \frac{\partial \mathcal{H}\prime}{\partial P} = 0,$$

$$-\dot{P} = \frac{\partial \mathcal{H}\prime}{\partial Q} = \frac{m\omega^2}{2}.$$

Since $\dot{Q} = 0$, Q becomes constant, and by $P = (\omega^2/2)t$, P has constant speed; that is, the sytem is moving in the phase space at certain point by constant speed (Fig 4.2).

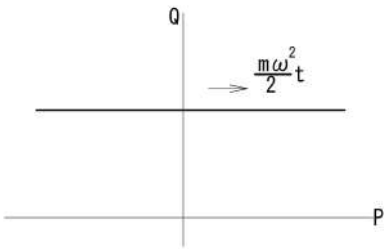

Figure 4.2: Reduced to constant speed motion

If we calculate $\{P, Q\}$ (see Sec 4.10), substituting

$$m\omega q p^{-1} = \eta,$$

we obtain

$$\frac{\partial \eta}{\partial q} = m\omega p^{-1}, \quad \frac{\partial \eta}{\partial p} = -m\omega q p^{-2}.$$

Then,

$$\frac{\partial P}{\partial \eta} = \frac{1}{1+\eta^2} = \frac{p^2}{m^2\omega^2 q^2 + p^2},$$

$$\frac{\partial P}{\partial p} = \frac{\partial P}{\partial \eta}\frac{\partial \eta}{\partial p} = -\frac{m\omega q}{m^2\omega^2 q^2 + p^2}.$$

Likewise,

$$\frac{\partial P}{\partial q} = \frac{\partial P}{\partial \eta}\frac{\partial \eta}{\partial q} = \frac{m\omega p}{m^2\omega^2 q^2 + p^2}.$$

Therefore,

$$\{P, Q\} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$

$$= -\frac{m\omega q}{m^2\omega^2 q^2 + p^2}\ 2q - \frac{m\omega p}{m^2\omega^2 q^2 + p^2}\ \frac{2p}{m^2\omega^2}$$

$$= -\frac{2}{m\omega}.$$

As the result shows, the transformation is not canonical, but this can be readily modified to canonical transformation by

$$P = -\frac{m\omega}{2} tan^{-1}(m\omega\frac{q}{p}) + \frac{m\omega^2}{2}t,$$

and

$$Q = q^2 + (\frac{1}{m\omega})^2 p^2.$$

Now,

$$-\dot{P} = \frac{\partial \mathcal{H}\prime}{\partial Q} = 0.$$

This is interpreted that in new coordinate, the Hamiltonian becomes zero and there is no dynamics, just stay still. So are the variables Q, P. Nothing moves in this coordinate.

Why this happened is that the changed new coordinate is explicitly time dependent and coordinate itself is moving. If we think we will be discussing in the new coordinates hereafter, this is a bit too much - particularly to the applications of automatic control in which total energy[9] looks like to be zero, our reference position is moving and the variables are neither measurable nor understandable any longer in our common engineering sense.

The result of coordinates transformation is interesting, but we should clearly recognize the motive of this transformation (this shall be discussed in Chapter 6), and should not be mislead that we will be living in that strange world for good.

4.9 Conserved value by canonical transformation

We have shown some examples for which canonical transformation are applied. Since it is a bit confusing which values are conserved by canonical transformation and which are not, we summarize the position of the canonical transformation in analytical mechanics.

Analytical mechanics progressed from "Euler-Lagrange's equation" → "Hamilton's equation" → "Hamilton-Jacobi's equation".

Applying the Legendre transform to the Euler-Lagrange's equation yields Hamilton's equation. Legendre transform is a canonical transformation that retains th equation of

[9] Of course, what is zero is the Hamiltonian, total energy will not be zero.

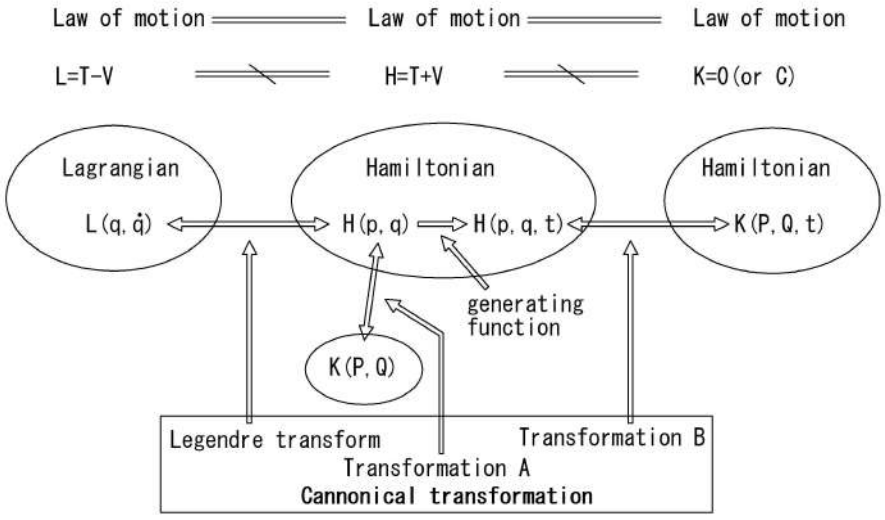

Figure 4.3: Conserved value by canonical transformation

motion in the original system. Although the Hamiltonian becomes different from the Lagrangian, total energy is conserved, and no mixing up of $q, \dot{q}$ variables.

Applying canonical transformation to Hamiltonian, the total energy is conserved as long as the Hamiltonian does not contain t explicitly.

Adding the generating function which comprises t explicitly to Hamilton's equation yields Hamilton-Jacobi's equation. **Application of canonical transformation to the Hamilton-Jacobi's equation changes Hamiltonian**, although the equation of motion is retained.

The change of Hamiltonian is used effectively in two ways.

(1) To solve Hamilton's equation. Application in mathematics.
Hamiltonian is made zero intentionally.

(2) Application for multivariable control theory.
System Hamiltonian changes by itself because of energy application/loss.

4.10 Poisson bracket

(We omit suffix i.)

Suppose we perform a transformation of variables from p, q to P, Q, that is,

$$\mathcal{H}_1(p, q) \to \mathcal{H}_2(P, Q).$$

Since functions $\mathcal{H}_1, \mathcal{H}_2$ are continuous and smooth, small change in $\mathcal{H}_1$ corresponds to another small change in $\mathcal{H}_2$. Therefore,

$$\delta P = \frac{\partial P}{\partial p}\delta p + \frac{\partial P}{\partial q}\delta q,$$

$$\delta Q = \frac{\partial Q}{\partial p}\delta p + \frac{\partial Q}{\partial q}\delta q.$$

Or,

$$\begin{pmatrix} \delta P \\ \delta Q \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial p}, \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p}, \frac{\partial Q}{\partial q} \end{pmatrix} \times \begin{pmatrix} \delta p \\ \delta q \end{pmatrix}.$$

The same idea is applicable to a replacement from P, Q to p, q.

$$\begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial P}, \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P}, \frac{\partial q}{\partial Q} \end{pmatrix} \times \begin{pmatrix} \delta P \\ \delta Q \end{pmatrix}.$$

The above relation holds for arbitrary small changes,

$$\begin{pmatrix} \frac{\partial p}{\partial P}, \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P}, \frac{\partial q}{\partial Q} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial p}, \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p}, \frac{\partial Q}{\partial q} \end{pmatrix}^{-1}.$$

If we calculate inverse matrix,

$$\begin{pmatrix} \frac{\partial P}{\partial p}, \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p}, \frac{\partial Q}{\partial q} \end{pmatrix}^{-1} = \Big(\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\Big)^{-1} \times \begin{pmatrix} \frac{\partial Q}{\partial q}, -\frac{\partial P}{\partial q} \\ -\frac{\partial Q}{\partial p}, \frac{\partial P}{\partial p} \end{pmatrix}.$$

If we write

$$\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \{P, Q\},$$

then,

$$\begin{pmatrix} \frac{\partial p}{\partial P}, \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P}, \frac{\partial q}{\partial Q} \end{pmatrix} = \{P, Q\}^{-1} \times \begin{pmatrix} \frac{\partial Q}{\partial q}, -\frac{\partial P}{\partial q} \\ -\frac{\partial Q}{\partial p}, \frac{\partial P}{\partial p} \end{pmatrix}.$$

From the above equation, we get,

$$\frac{\partial P}{\partial q} = -\{P, Q\}\frac{\partial p}{\partial Q},$$

$$\frac{\partial P}{\partial p} = \{P, Q\}\frac{\partial q}{\partial Q},$$

$$\frac{\partial Q}{\partial p} = -\{P, Q\}\frac{\partial q}{\partial P},$$

$$\frac{\partial Q}{\partial q} = \{P, Q\}\frac{\partial p}{\partial P}.$$

Time derivative of $P(p, q)$ is,

$$\begin{aligned} \dot{P} &= \dot{p}\frac{\partial P}{\partial p} + \dot{q}\frac{\partial P}{\partial q} \\ &= -\frac{\partial \mathcal{H}}{\partial q}\frac{\partial P}{\partial p} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial P}{\partial q} \\ &= -\{P, Q\}\Big(\frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q}\Big) \\ &= -\{P, Q\}\Big(\frac{\partial \mathcal{H}}{\partial Q}\Big). \end{aligned}$$

Time derivative of $Q(p, q)$ is,

$$\dot{Q} = \dot{q}\frac{\partial Q}{\partial q} + \dot{p}\frac{\partial Q}{\partial p}$$

$$= \frac{\partial \mathcal{H}}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial \mathcal{H}}{\partial q}\frac{\partial Q}{\partial p}$$
$$= \{P, Q\}\Big(\frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P}\Big)$$
$$= \{P, Q\}\Big(\frac{\partial \mathcal{H}}{\partial P}\Big).$$

Now it is clear that the relation $\{P, Q\}$ is a coefficient of transformation between p, q and P, Q. If this value is not unity, then the transformation may amplify/attenuate or distort between two coordinates. This $\{P, Q\}$ is called **Poisson bracket**.

Also, the better understanding of canonical transformation is reached here; a **transformation that retains the physics**.

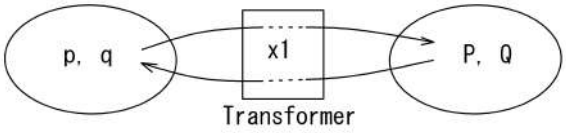

Figure 4.4: Bi-directional transformer with unity coefficient

Then, **non-canonical transformation** will have a form,

$$\frac{\partial \mathcal{H}(P, Q)}{\partial Q} = f(p, q)\dot{P},$$
$$\frac{\partial \mathcal{H}(P, Q)}{\partial P} = -g(p, q)\dot{Q}.$$

If $f(p, q)$ or $g(p, q)$ becomes a constant other than unity, the transformation is still not canonical, but should be easily scaled into canonical transformation (see Sec 4.8.5). In other cases, the original physics is destroyed because the **Hamilton's canonical form declares the coordinates and momenta in the target system** as discussed in Sec 4.3. The $f(p, q) \neq constant$ or $g(p, q) \neq constant$ irretrievably destroys original coordinates or momenta.

4.11 Why Hamilton deserves Hamilton's equation

The Lagrangian and Euler-Lagrange's equation were born from the Newtonian mechanics. Euler-Lagrange's equation is a second order differential equation and writing down into a pair of first order differential equations was done by Poisson of which Lagrange recognized the importance.

Then, the reduction of the order of the Euler-Lagrange's equation into a pair of first order differential equations - which eventually yielded Hamilton's equation - cannot be Hamilton's original.

However, the pair of first order differential equations abstracted the mechanics by

- relation between energy and momenta,

- relation between energy and coordinates.

By this abstraction, the Hamiltonian mechanics is extended so that wider physics can be described by the same philosophy.

Therefore, to actual application of the Hamiltonian mechanics, we are requested to choose "physics" and supply appropriate momenta - in case of mechanics, we supply (if Cartesian coordinates are used) $p = m\dot{q}$, and in case of electromagnetism, we would supply $L\dot{q}_e$, $C\dot{\phi}$ and so on.

The origin of the long discussion goes back to the Newton's law, and it is surprising that a descriptive form of the force and material yielded a philosophy how a physics should be constructed. That is, an answer to the question "What is physics?" was given by Hamilton.

Euler, Lagrange generalized coordinates, Hamilton generalized physics.

Chap5 Extending the Hamiltonian mechanics

5.1 Limits of Hamilton's equation

We have worked out the Hamilton's canonical equation for energy conserving systems. Although the canonical equation

$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial q_i} = -\dot{p}_i,$$
$$\frac{\partial \mathcal{H}(p_i, q_i)}{\partial p_i} = \dot{q}_i,$$

looks like an **almost ultimate form of the mechanics**, no scope is extended from the Newtonian mechanics[1].

If we review the process how Hamilton's equation is derived (see Sec 4.2.4), we notice that the main part is the calculation of variational integral

$$\delta \int_{t_0}^{t_1} \mathcal{L} dt = 0,$$

by assuming the Newton's law. Also, the integral

$$\int_{t_0}^{t_1} p_i \delta \dot{q}_i dt = [p_i \delta q_i]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}_i \delta q_i dt$$
$$= -\int_{t_0}^{t_1} \dot{p}_i \delta q_i dt,$$

is carried out by assuming $\delta q_i = 0$ at t_0 and t_1. That is, **the start and end points are fixed**.

These assumptions clearly shows that the Hamilton's equation - although the expression is very much different from that of Newton's law - is another form of Newton's law.

If we think about the motion of real world, it will be a clear limitation that the end point is fixed, even if we compromise the fixed start point. And it should have been quite natural that scholars tried to remove this constraint.

5.2 Freed from fixed endpoint

5.2.1 Equation of motion under time varying force

If we look into the Newtonian mechanics, the primary application is for energy conserving systems under constant force. For instance, the free fall in gravitational field is

[1] It is already extended into generalized physics, but we think of only mechanics here.

expressed by,

$$m\ddot{z} = -mgz.$$

This yields,

$$z = \frac{1}{2}gt^2 + c_1 t + c_2.$$

In real applications, objects can move under time varying force - either in time varying field and/or by active power sources like engines. In either case, the total energy of the system changes. This obviously changes the Hamiltonian. Then, how can we formulate a mechanics that can deal with the changing Hamiltonian ?

5.2.2 Extending the Hamilton's equation : overview

(We omit suffix i.)

Now we wish to **extend the Hamilton's equation to time varying systems by removing the constraint of the endpoint in both time and space.**

The process is somewhat cumbersome, especially traced by the manipulation of equations[2].

Hence we state the strategy first.

In Sec 4.2.4, we calculated the stationary value of Lagrangian by the following integral.

$$\delta \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}) dt.$$

This is the variational integral for energy conserving system from t_0 to t_1. To free the endpoint t_1, we introduce t explicitly into the Lagrangian which means that we are supplying/consuming energy and move the end point. A function $S(q, \dot{q}, t)$ is defined for a Lagrangian with t,

$$S(q, \dot{q}, t) = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t) dt.$$

Then we think about the following integral.

$$\delta S(q, \dot{q}, t) = \delta \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t) dt.$$

This is a variational integral in which the "variation" is sought into variable q only. Because variation of q affects the value $\dot{q}$ (dependent to q) and we have no control for time.

By this approach, δS will not be zero, and function $S(t)$ is expected to manifest.

We assume the start point t_0 fixed, and path of the motion q_1 moved to q_2 in time δt_1. We try to calculate the variational integral of path q_2 (area A+B, Fig5.1) which has changed from path q_1 (area C, Fig5.1). The difference between those paths is $\delta t_1, \delta q_1$ which we eventually **normalize by the value at** t_1.

Referring to Fig 5.1, the total variation δS which is the result of $q_1 \to q_2$ is

$$\delta S = \int (A + B) - \int C = \int B + \int (A - C).$$

[2]Details described in Sec 5.2.3 are mostly the manipulation, hence if the logical process of the birth of Hamilton-Jacobi's equation is roughly understood, that will be sufficient.

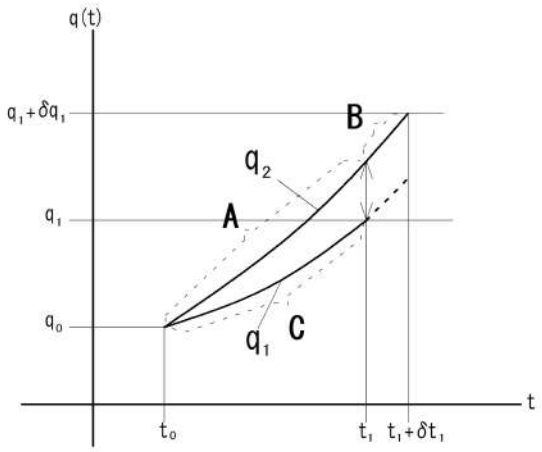

Figure 5.1: Process of calculation

By this process, calculation of area B is separated from the calculation up to t_1 which is calculated without difficulties. Then, the integral of q_2 from t_1 to $t_1 + \delta t_1$ (area B) is approximated by using the values of Lagrangian at t_1 and small time δt_1 as,

$$\int B \simeq \mathcal{L}(q, \dot{q}, t)\big|_{t_1} \delta t_1.$$

Finally, δt_1 is made smaller and t_1 is regarded as general time t.

Since the path q_2 is generally different from the path q_1 which satisfies Newton's law, the δS is generally not zero. (If it happens to be zero, the new path is identical to that calculated by the Newton's law for energy conserving system.) And hence, we obtain an equation of motion under dynamic force.

5.2.3 Extending the Hamilton's equation

The variation δS of $q + h$ from q is,

$$\delta S = \int (A + B) - \int C = \int_{t_0}^{t_1+\delta t_1} \mathcal{L}(q + h, \dot{q} + \dot{h}, t)dt - \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t)dt$$

$$= \int B + \int (A - C)$$

$$= \int_{t_1}^{t_1+\delta t_1} \mathcal{L}(q + h, \dot{q} + \dot{h}, t)dt + \int_{t_0}^{t_1} \Big(\mathcal{L}(q + h, \dot{q} + \dot{h}, t) - \mathcal{L}(q, \dot{q}, t)\Big)dt.$$

(1) From t_1 to $t_1 + \delta t_1$:
The expected increase of q_2 from q_1 after δt_1 is extrapolated by using the tangent of q at t_1.

$$\delta q_1 = \dot{q}(t_1)\delta t_1 + h(t_1),$$

where $h(t_1)$ is the difference of q_2 from q_1 at time t_1 and works as a bias. On the other hand, mean value theorem of integration states,

$$\int_a^b f(x)dx = (b - a)f(\xi), \qquad a < \xi < b.$$

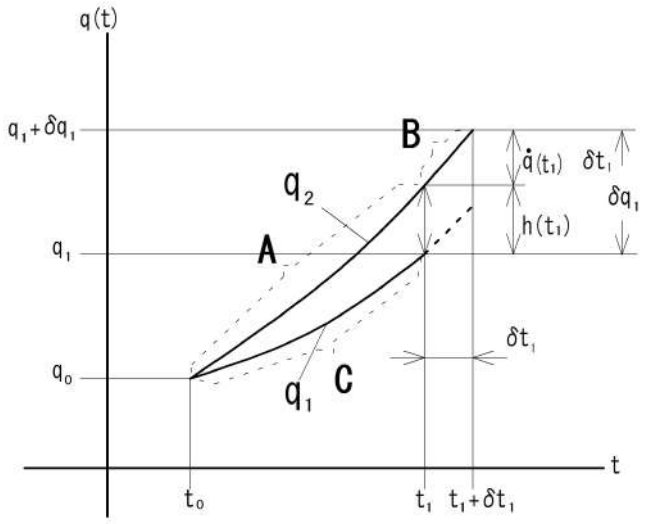

Figure 5.2: Moving of the endpoint

Then, using the value of Lagrangian at t_1

$$\int_{t_1}^{t_1+\delta t_1} \mathcal{L}(q+h, \dot{q}+\dot{h}, t)dt \simeq \mathcal{L}(q, \dot{q}, t)\big|_{t_1} \delta t_1.$$

Thus we obtain,

$$\begin{aligned} \int B &= \int_{t_1}^{t_1+\delta t_1} \mathcal{L}(q+h, \dot{q}+\dot{h}, t)dt \\ &\simeq \mathcal{L}(q, \dot{q}, t)\big|_{t_1} \delta t_1 \\ &= \dot{q}(t_1)\delta t_1 + h(t_1). \end{aligned}$$

(2) From t_0 to t_1:
The path A has changed from path C and we have to adjust the difference. The small change of variables $\delta q, \delta \dot{q}$ affects the Lagrangian,

$$\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} = \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d}{dt}\delta q.$$

Then the variation between t_0 and t_1 is,

$$\int (A-C) = \int_{t_0}^{t_1} \Big(\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q} h + \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} \dot{h} \Big) dt.$$

Applying the integral by parts to the second term,

$$= \int_{t_0}^{t_1} \Big(\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} \Big) h dt + \Big[\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} h \Big]_{t_0}^{t_1}.$$

Since the Newton's law holds, only the following part remains.

$$\int (A-C) = \Big[\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} h \Big]_{t_0}^{t_1},$$

$$= \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} h(t_1). \qquad \because h(t_0) = 0$$

(3) From t_0 to $t_1 + \delta t_1$:

Total δS is,

$$\delta S = \int B + \int (A - C) = \mathcal{L}(q,\dot{q},t)\big|_{t_1} \delta t_1 + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} h(t_1)$$

$$= \mathcal{L}(q,\dot{q},t)\big|_{t_1} \delta t_1 + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \big(\delta q_1 - \dot{q}(t_1)\delta t_1\big)$$

$$= \Big(\mathcal{L}(q,\dot{q},t)\big|_{t_1} - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \dot{q}(t_1)\Big)\delta t_1 + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \delta q_1.$$

Remembering that δt and δq is independent, we fix δq and make δt_1 infinitesimally small. Then, for the results of variational integral δS,

$$\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \delta q_1 = 0,$$

and $\delta t_1 \to \partial t$, $\delta S \to \partial S$.

$$\partial S = \Big(\mathcal{L}(q,\dot{q},t) - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \dot{q}\Big)\partial t,$$

$$\frac{\partial S}{\partial t} = \mathcal{L}(q,\dot{q},t) - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} \dot{q}.$$

If we remember,

$$\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}} = \frac{\partial \mathcal{T}(q,\dot{q},t)}{\partial \dot{q}} = p(t).$$

Then,

$$\frac{\partial S}{\partial t} = \mathcal{L}(q,\dot{q},t) - p(t)\dot{q} = -\mathcal{H}(p,q,t).$$

Or,

$$\mathcal{H}(p,q,t) + \frac{\partial S(p,q,t)}{\partial t} = 0.$$

Note that the S is a function of p, q, t now.

The replacement of

$$\mathcal{L}(q,\dot{q},t) - p(t)\dot{q} = -\mathcal{H}(p,q,t),$$

means more than just a replacement of Lagrangian by Hamiltonian. Because, this replacement means **dependent variables q, $\dot{q}$ are now replaced by independent variables q and p.** Discussion in Sec 5.3 will be only possible when we are dealing with Hamiltonian.

Starting from the Lagrangian with explicit t yielded Hamiltonian (with explicit t) which drastically clarifies the relation between $\partial S/\partial t$ and $q(t), p(t)$ in the Hamiltonian.

In Sec 4.8.1 , we called p, q a conjugate pair of variables and now the meaning has become clear. They are the energy source/sink pairs corresponding to kinetic and potential energies.

5.2.4 Value of S

As stated, we expect $\delta S \neq 0$, we explore what it amounts to. Using the following relation again.

$$\delta S = \Big(\mathcal{L}(q,\dot{q},t)\big|_{t_1} - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}}\dot{q}(t_1)\Big)\delta t_1 + \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}}\delta q_1$$

Substituting by $p(t)$,

$$= \Big(\mathcal{L}(q,\dot{q},t)\big|_{t_1} - p(t)\dot{q}(t_1)\Big)\delta t_1 + p(t)\delta q_1.$$

This time, we make δq_1 infinitesimally small while fixing time. That is, $\delta t_1 \to 0$, $\delta q_1 \to \partial q$, $\delta S \to \partial S$. We obtain,

$$\frac{\partial S}{\partial q} = p(t).$$

5.3 Extending the Hamiltonian mechanics

The equation that holds when the endpoint changes by time

$$\mathcal{H}(p,q,t) + \frac{\partial S(p,q,t)}{\partial t} = 0,$$

is a clear extension from Hamiltonian (= Newtonian) mechanics that **expresses a time varying physical system**. For final touch, $\partial S/\partial t = p(t)$ is a small change of energy applied to the target system and the target Hamiltonian has to respond accordingly. Hence we replace variable p in the Hamiltonian by $\partial S/\partial q$.

$$\mathcal{H}(\frac{\partial S}{\partial q}, q, t) + \frac{\partial S(p,q,t)}{\partial t} = 0.$$

This is called a Hamilton-Jacobi's equation.

In the discussion so far, we implicitly assumed q, p as coordinate (position) and momentum, but from the discussion in Sec 4.8.3, q and p can be interchanged without affecting the Hamilton's equation. Therefore, the variable that receives energy from $\partial S/\partial t$ can be q. In this case,

$$\mathcal{H}(\frac{\partial S}{\partial p}, p, t) + \frac{\partial S(p,q,t)}{\partial t} = 0.$$

And,

$$\frac{\partial S}{\partial p} = q(t).$$

Furthermore, because of the independence of p and q,

$$\mathcal{H}(\frac{\partial S}{\partial q}, \frac{\partial S}{\partial p}, t) + \frac{\partial S(p,q,t)}{\partial t} = 0.$$

So far, we made clear that the Hamilton-Jacobi's equation can express time varying system in which parameter t is explicitly comprised. The original usage of this inclusion must have been for the description of energy non-conserving systems. However, **this explicit inclusion of parameter t opened a new paradigm to change t - change the flow of time** to solve the Hamilton's equation.

5.4 Variables of Hamilton-Jacobi's equation

We have shown that the Hamilton-Jacobi's equation is expressed in general,

$$\mathcal{H}(\frac{\partial S}{\partial q}, \frac{\partial S}{\partial p}, t) + \frac{\partial S(p,q,t)}{\partial t} = 0.$$

That is, the variables in the Hamiltonian are,

$$\frac{\partial S}{\partial q} = p(t), \qquad \frac{\partial S}{\partial p} = q(t), \qquad t.$$

The energy applied to the system affects the generalized coordinates and momenta of the Hamiltonian $p(t), q(t)$ and thus the Hamiltonian changes. In case of energy, Hamiltonian increases and in case of brake, it decreases. This is quite consistent with the behavior of the physical system.

Now, if we look at the variable "t", it has been working just a parameter and in the actual energy/brake discussion, it is left untouched. The "t" in the Hamiltonian is so far a symbolic existence to show that the Hamiltonian is no longer constant.

This is where **an unusual idea was born**[3].
Suppose if we replace "t" with ωt what happens?
(1) $p(t), q(t)$ becomes $p(\omega t), q(\omega t)$, and what could it mean ?
(2) Value of Hamiltonian changes because, for an arbitrary function $f(t)$, the parameter changes from t to ωt.

On this point, we discuss in Chap 6.

[3]Or, can be a result of canonical transformation of $\mathcal{H} \to \mathcal{K}$ in which t is transformed, but anyhow, who had this idea ?

Chap6 Hamilton-Jacobi's equation

6.1 Brief history

Lagrangian mechanics was established in around 1788, Poisson disclosed his Poisson bracket in 1809 as well as the first order expression of the Lagrange's equation. Poisson didn't pay much attention to that equation, but Lagrange noticed the importance. These works fruited into Hamiltonian mechanics which was established around 1834.

Jacobi seems to have been the first physicist who encountered a physical system in which Hamiltonian includes t explicitly. He calculated a trajectory of a comet under the influence of the force from the Sun and a planet. That is, the comet is moving in a gravitational potential of the Sun with an attractive force from the planet which is revolving around the Sun. This was in 1836. And eventually, the Hamilton-Jacobi's equation was formed.

However, the various meanings of the Hamilton-Jacobi's equation were researched by many scholars later - Poincaré (1854-1912), Hilbert(1862-1943), Carathéodory(1873-1950) - and logical progress is difficult to trace by now. Clear appreciation of the Hamilton-Jacobi's theory had to wait for the advent of quantum mechanics followed by multivariable automatic control and economics in 20th century.

6.2 How to change Hamiltonian

To change Hamiltonian, there are two ways. One obvious way is to supply/deprive energy to the system - $p(t), q(t)$ changes. Another way is the one applied to energy conserving system as shown in Sec 4.8.5, where Hamiltonian was changed, but total energy kept intact. Here, we are now equipped to discuss "how ?" by exploiting the Hamilton-Jacobi's equation.

We assume the Hamilton-Jacobi's equation in the following form. (We omit suffix i unless necessary.)

$$\mathcal{H}\Big(\frac{\partial W}{\partial q}, \frac{\partial W}{\partial p}, t\Big) + \frac{\partial W(p,q,t)}{\partial t} = 0.$$

$$\frac{\partial W}{\partial q} = p(t), \qquad \frac{\partial W}{\partial p} = q(t).$$

(We use W instead of S because the same function is going to be interpreted from "variational integral" to "generating function".)

As we shall see, the change of Hamiltonian is caused by the real change of energy and/or modification of parameter t. But here, we only discuss the change of Hamiltonian caused by t.

6.2.1 Differential of W()

In sec 6.2, we expressed the generating function by partial differential form by $\partial W(p,q,t)/\partial t$ rather than $dW(p,q,t)/dt$.

Since q and p are independent, if we express the variable by vector $\boldsymbol{x}$, and applying the chain rule of differential, the generating function $dW(\boldsymbol{x},t)/dt$ becomes,

$$\frac{dW(\boldsymbol{x},t)}{dt} = \sum_{i=1}^{n} \frac{\partial W(\boldsymbol{x},t)}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial W(\boldsymbol{x},t)}{\partial t}$$

$$= \sum_{i=1}^{n} \frac{\partial W(\boldsymbol{x},t)}{\partial x_i}\dot{x}_i + \frac{\partial W(\boldsymbol{x},t)}{\partial t}.$$

In case of analytical mechanics, the target system is time invariant, that is, $\dot{\boldsymbol{x}} = 0$. Thus ,

$$\frac{\partial W(\boldsymbol{x},t)}{\partial x_i}\dot{x}_i = 0.$$

Therefore, the **generating function becomes $\partial W(p,q,t)/\partial t$ for analytical mechanics.**

However, in case of multivariable control theory, we modify (at least the part of) the system variable $\boldsymbol{x}$ in which case $\dot{\boldsymbol{x}} \neq 0$. Therefore, **the generating function in multivariable control theory must be expressed by $dW(p,q,t)/dt$.**

6.2.2 Zero Hamiltonian

In Sec 4.7 and 4.8, we have introduced the canonical transformation of the Hamilton's equation. In this section, we further investigate the effect by clearly separating "Hamiltonian" from "total energy".

If we think about a physical system in real world, zero total energy system is not interesting at all - there is no physics there; the Hamiltonian of a zero energy system is also zero.

However, the **Hamiltonian of an energy conserving system which has non zero total energy can be made zero**. Let $\mathcal{H}(p,q)$ be the original Hamiltonian, we can make a new Hamiltonian $\mathcal{K}(P,Q,t)$ zero by Hamilton-Jacobi's equation.

$$\mathcal{K}(P,Q,t) = \mathcal{H}(p,q) + \frac{\partial W(p,q,t)}{\partial t} = 0.$$

The explicit parameter t in the new Hamiltonian $\mathcal{K}$ is supplied by the term $\partial W(p,q,t)/\partial t$. Then, it results,

$$\frac{\partial \mathcal{K}}{\partial P} = \dot{Q} = 0,$$
$$\frac{\partial \mathcal{K}}{\partial Q} = -\dot{P} = 0.$$

Thus, solutions are obtained readily,

$$P = constant, \quad Q = constant.$$

The system stays still even if the system has non zero total energy - that is, we didn't apply brake and the total energy is not made zero.

The interpretation of this paradoxical situation is that the observation reference is synchronously moving with the target physics. Because **changing the Hamiltonian for energy conserving system means changing the observation flow of time** - uniform but multiplied, or sinusoidal like the case discussed in Sec 4.8.5. However, this different flow of time happens only in $\mathcal{K}$ system, and when the Hamilton's equation (for $\mathcal{H}$) is solved, time returns to our normal time.

The original reason of explicit inclusion t seemed to be a necessity to deal with time varying system (by Jacobi), but people realized (later by somebody) that zero Hamiltonian transformation will yield the solution of the Hamilton's equation, and canonical transformation became one of the main topics in the Hamiltonian mechanics.

6.2.3 Time t in Hamiltonian

The Hamiltonian $\mathcal{H}$ of an energy conserving system can be changed by Hamilton-Jacobi's equation to $\mathcal{K}$ which contains parameter t explicitly. Therefore, when we talk about the change of Hamiltonian (canonical transformation), we assume that the target Hamiltonian explicitly contains t all the time. That is,

$$\mathcal{H} = \mathcal{H}(p, q, t).$$

Since the purpose of the transformation is to make the solution of $\mathcal{K}$ trivial under the variables P, Q and parameter t, we are going to seek transformation formula of parameter t so that the equation below holds.

$$\mathcal{K}(P, Q, t) = \mathcal{H}(p, q, t) + \frac{\partial W(p, q, t)}{\partial t} = 0.$$

However $\mathcal{K}$ also includes $P(t), Q(t)$ which are functions of t, what happens when t is modified? Very conveniently,

$$\begin{aligned}\frac{\partial \mathcal{K}(P, Q, t)}{\partial t} &= \frac{\partial \mathcal{K}}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial \mathcal{K}}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial \mathcal{K}}{\partial t}\frac{\partial t}{\partial t} \\ &= \frac{\partial \mathcal{K}}{\partial P}\dot{P} + \frac{\partial \mathcal{K}}{\partial Q}\dot{Q} + \frac{\partial \mathcal{K}}{\partial t} \\ &= \dot{Q}\dot{P} - \dot{P}\dot{Q} + \frac{\partial \mathcal{K}}{\partial t} = \frac{\partial \mathcal{K}}{\partial t}.\end{aligned}$$

That is, change of $P(t)$ by the change of t is canceled out by that of $Q(t)$ and change of total Hamiltonian of $\mathcal{K}$ is only caused by parameter t - we can forget about $P(t), Q(t)$.

6.2.4 Canonical transformation by Hamilton-Jacobi's equation

Introduction of $\mathcal{K}$,

$$\mathcal{K}(P, Q, t) = \mathcal{H}(\frac{\partial W}{\partial q}, \frac{\partial W}{\partial p}, t) + \frac{\partial W(p, q, t)}{\partial t}$$

means a definition of transformation from one energy conserving system $\mathcal{H}$ to another energy conserving system $\mathcal{K}$. **The Hamiltonian $\mathcal{H}$ is an energy conserving system, and W is the energy source/sink to $\mathcal{K}$ which again conserves energy**. This sounds quite paradoxical and the key to this idea is that the energy from $\partial W/\partial t$ is pseudo-energy.

That is, we will treat $\partial W/\partial t$ as a function that will modify the parameter t in $\mathcal{K}^1$ so that the observation time flow is changed (Hamiltonian is changed), but total energy stays as before.

However, the transformation must not change the physics. That is, for $\mathcal{K}$,

$$\frac{\partial \mathcal{K}}{\partial Q} = -\dot{P}, \quad \frac{\partial \mathcal{K}}{\partial P} = \dot{Q},$$

and of course, we assume,

$$\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}, \quad \frac{\partial \mathcal{H}}{\partial p} = \dot{q}.$$

6.3 Generating function

6.3.1 Variables in generating function

The Hamiltonian $\mathcal{H}$ is defined as a function of 2n variables,

$$\mathcal{H} = \mathcal{H}(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n).$$

The Hamilton-Jacobi's equation yields $\mathcal{K}$,

$$\mathcal{K} = \mathcal{K}(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n).$$

Then, observing the Hamilton-Jacobi's equation from generating function,

$$\frac{\partial W()}{\partial t} = \mathcal{K}(P, Q, t) - \mathcal{H}(p, q),$$

we can readily see the generating function has $4n$ variables p, q, P, Q. However, the variables will be expressed

$$Q_1 = Q_1(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$$

$$...$$

$$Q_n = Q_n(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$$

$$P_1 = P_1(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$$

$$...$$

$$P_n = P_n(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$$

That is, the number of variables of generating function can be (or should be) reduced to $2n$ by using above dependencies.

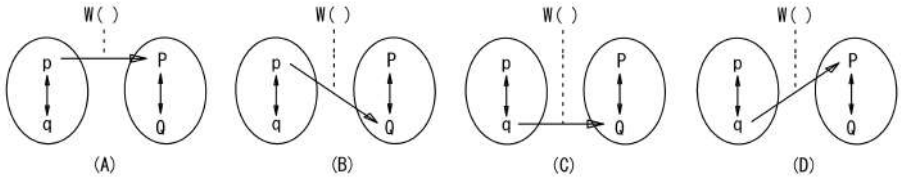

Figure 6.1: Selection of parameters

6.3.2 Variables selection in generating function

There are four choices of variables as depicted in Fig 6.1.

$$W = W(p, P, t), \;\; W(p, Q, t), \;\; W(q, Q, t), \;\; W(q, P, t),$$

Once the variables in generating function are selected, there are relations between the generating function and non selected variables. These relations are shown as follows.

Generating function W is

$$W = \int \Big(\mathcal{K}(P, Q, t) - \mathcal{H}(p, q, t) \Big) dt$$

Then,

$$\frac{\partial W}{\partial q} = \int \Big(\frac{\partial \mathcal{K}}{\partial q} - \frac{\partial \mathcal{H}}{\partial q} \Big) dt = p.$$

Likewise,

$$\frac{\partial W}{\partial Q} = \int \Big(\frac{\partial \mathcal{K}}{\partial Q} - \frac{\partial \mathcal{H}}{\partial Q} \Big) dt = -P,$$

and so on.

These relations play a crucial role in solving the Hamilton-Jacobi's equation . (see Sec 5.3 for discussion.)

In Hamiltonian physics, there is no difference between coordinates and momenta, but here - for convenience's sake - we call p, P momenta, and q, Q coordinates. Then the above four cases are interpreted,

(A) $W = W(p, P, t)$: momenta p in $\mathcal{H} \rightarrow$ momenta P in $\mathcal{K}$,

$$q = -\frac{\partial W}{\partial p}, \quad Q = \frac{\partial W}{\partial P}.$$

(B) $W = W(p, Q, t)$: momenta p in $\mathcal{H} \rightarrow$ coordinates Q in $\mathcal{K}$,

$$q = -\frac{\partial W}{\partial p}, \quad P = -\frac{\partial W}{\partial Q}.$$

(C) $W = W(q, Q, t)$: coordinates q in $\mathcal{H} \rightarrow$ coordinates Q in $\mathcal{K}$,

$$p = \frac{\partial W}{\partial q}, \quad P = -\frac{\partial W}{\partial Q}.$$

(D)$W = W(q, P, t)$: coordinates q in $\mathcal{H} \rightarrow$ momenta P in $\mathcal{K}$.

$$p = \frac{\partial W}{\partial q}, \quad Q = \frac{\partial W}{\partial P}.$$

[1] $P(t), Q(t)$ are also affected, but they cancel each other.

6.3.3 How generating function works

Suppose a Hamiltonian $\mathcal{H}$ is transformed into $\mathcal{K}$ by a generating function W.

$$\mathcal{K}(P,Q,t) = \mathcal{H}(p,q) + \frac{\partial W()}{\partial t} = 0.$$

From the discussion in Sec 5.4 we know that,

$$\frac{\partial W}{\partial q} = p(t), \quad \frac{\partial W}{\partial p} = q(t),$$
$$\frac{\partial W}{\partial Q} = P(t), \quad \frac{\partial W}{\partial P} = Q(t).$$

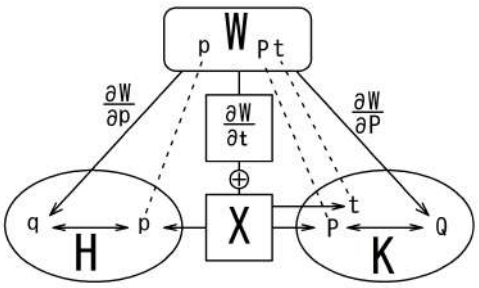

Figure 6.2: How generating function works

If we choose p in $\mathcal{H}$ and P in $\mathcal{K}$, the X performs the transformation of variables (see Fig 6.2). Whereas the relation p, q and P, Q are defined within system $\mathcal{H}, \mathcal{K}$ respectively.

The Hamilton-Jacobi's equation is solved by using the knowledge collected so far which are,

(1) W is a function of p, P, t (we chose variables this way).
(2) We don't know W (this is what we are going to obtain).
(3) We don't know X, but we know it will be the solution of W.
(4) We know $\partial W/\partial p = q(t)$ and $\partial W/\partial P = Q(t)$.
(5) $\partial W/\partial t$ yields (pseudo) energy thus producing parameter t in $\mathcal{K}$.

That is, the generating function W is solved using the following known relation

$$\frac{\partial W}{\partial p} = q(t), \quad \frac{\partial W}{\partial P} = Q(t),$$

and transformation to zero Hamiltonian is defined to be,

$$\frac{\partial \mathcal{K}}{\partial P} = constant, \quad \frac{\partial \mathcal{K}}{\partial Q} = constant.$$

6.4 Hamilton-Jacobi's theory

There are two main fields that the Hamilton-Jacobi's equation is applied. One is the description of time varying physics, or energy non conserving systems. Good example is found in multivariable control. The other is energy conserving systems where the purpose is to solve the original Hamilton's equation.

6.4.1 Energy conserving system

Suppose a canonical transformation produces $\mathcal{K}$ from $\mathcal{H}$ so that $\mathcal{K} = 0$, solving the Hamilton's equation,

$$\frac{\partial \mathcal{K}}{\partial P} = \dot{Q} = 0,$$
$$\frac{\partial \mathcal{K}}{\partial Q} = -\dot{P} = 0,$$

becomes trivial. Since canonical transformation retains the original physics, solution of $\mathcal{K}$ satisfies the original system. That is, **solving the Hamilton's equation is identical to solve Hamilton-Jacobi's equation by setting new Hamiltonian zero**.

In the process of Hamilton-Jacobi's equation, time t is transformed automatically and unless we worry about the physics in $\mathcal{K}$ intentionally, we obtain the solution with consistent t.

The reason to explore this method is in the philosophy that a **system Hamiltonian can be made zero, but still retaining it's physics**, providing a possibility to solve the equation of motion.

6.4.2 Process to solve Hamilton's equation

We try to find a generating function that transforms the current system to a zero Hamiltonian system. In an energy conserving system, the Hamiltonian does not contain t explicitly, and the Hamilton-Jacobi's equation can be **solved by separation of variables**.

Here is a standard process.

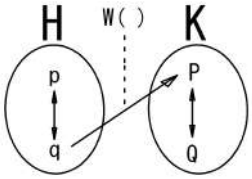

Figure 6.3: Parameter mapping

(1) Parameter mapping

In the Hamilton-Jacobi's equation

$$\mathcal{K}(P, Q) = \mathcal{H}(p, q) + \frac{\partial W()}{\partial t} = 0,$$

we select a single pair of variables for W that maps $\mathcal{H} \to \mathcal{K}$. For the new Hamiltonian $\mathcal{K}$, if we choose,

$$W = W(q, P, t).$$

Then,

$$p = \frac{\partial W}{\partial q}, \quad Q = \frac{\partial W}{\partial P}.$$

(2) Setting a new Hamiltonian

We make the new Hamiltonian $\mathcal{K}$ so that,

$$\frac{\partial K}{\partial Q} = -\dot{P} = 0, \;\; \frac{\partial K}{\partial P} = \dot{Q} = 0.$$

Then, both momenta P and coordinates Q eventually become constants, but during the calculation process, treat as variables.

(3) Forming a Hamilton-Jacobi's equation

The Hamilton-Jacobi's equation is,

$$\mathcal{H}\Big(\frac{\partial W(q,P,t)}{\partial q}, q\Big) + \frac{\partial W(q,P,t)}{\partial t} = 0.$$

For the energy conserving system, the $\mathcal{H}$ is constant, and the $\partial W/\partial t$ becomes constant as well which we assign by Q.

$$\mathcal{H}\Big(\frac{\partial W(q,P,t)}{\partial q}, q\Big) = -\frac{\partial W(q,P,t)}{\partial t} = -Q.$$

(4) Separation of variables in W

We only focus on the solution of W.

Observing

$$\frac{\partial W(q,P,t)}{\partial t} = Q,$$

we can see that in W, the coordinates are separated from time. Because,

$$\frac{\partial F(x,t)}{\partial t} = C \rightarrow F(x,t) = F(x) + Ct.$$

If we express time independent part by $W(q,P)$, the W is,

$$W(q,P,t) = W(q,P) - Qt.$$

Then,

$$\frac{\partial W(q,P,t)}{\partial q} = \frac{\partial W(q,P)}{\partial q}.$$

Thus separation of variables is complete.

****

Substituting W without time "t" yields,

$$\mathcal{H}\Big(\frac{\partial W(q,P)}{\partial q}, q\Big) = Q.$$

This is also called a Hamilton-Jacobi's equation, but we are not interested in this one.

****

(5) Calculation of p and q

Since,

$$Q = \frac{\partial W(q,P,t)}{\partial P} = \frac{\partial W(q,P)}{\partial P} - \frac{\partial Q}{\partial P}t,$$

Then,

$$\frac{\partial W(q,P)}{\partial P} = \frac{\partial Q}{\partial P}t + Q.$$

which will be integrated to yield W. Thus, q is calculated by using constants P, Q,

$$q = q(P, Q, t).$$

And p is,

$$p = \frac{\partial W(q,P,t)}{\partial q} = \frac{\partial W(q,P)}{\partial q}.$$

Substituting q with the above results,

$$p = p(P, Q, t).$$

The important points are the separation of variables and usage of P, Q which are treated as variables and finally regard them as constants.

6.4.3 Non energy conserving system

As in the case of real physical system, we think about a system whose total energy varies by time. The total energy is changed by $p(t), q(t)$ in the original Hamiltonian $\mathcal{H}(p, q)$, even without explicit t. (However, without t, we have no ways to observe the change of Hamiltonian.)

Here, we discuss the case when the Hamiltonian explicitly includes t, and total energy is changed by the energy source (or sink). If we define a new Hamiltonian by $\mathcal{K}$,

$$\mathcal{K}(P, Q, t) = \mathcal{H}(p, q, t) + \frac{\partial W(p,q,t)}{\partial t} = 0,$$

then, solving the generating function W by p, q yields the solution for $\mathcal{H}$.
The Hamilton's equation for the $\mathcal{K}$ is,

$$\frac{\partial K}{\partial Q} = -\dot{P} = 0, \quad \frac{\partial K}{\partial P} = \dot{Q} = 0.$$

Then, P, Q become constants, but that shall be done at the end of process.
If we map $q \to P$ (see Fig 6.3)

$$W = W(q, P, t).$$

Then,

$$p = \frac{\partial W(q,P,t)}{\partial q}, \quad Q = \frac{\partial W(q,P,t)}{\partial P}.$$

The Hamilton-Jacobi's equation is,

$$\mathcal{K}(P, Q, t) = \mathcal{H}(p, q, t) + \frac{\partial W(q,P,t)}{\partial t}$$

$$= \mathcal{H}\Big(\frac{\partial W(q,P,t)}{\partial q}, q, t\Big) + \frac{\partial W(q,P,t)}{\partial t} = 0.$$

The above equation is a partial differential equation of W with variables q, P and if solved, it yields.

$$W = W(q, P, t) + C.$$

The C is a constant spawned by the integration of $\partial W/\partial t$ and not important. The $W(q, P, t)$ is called Hamilton's principal function because the solution of this function is what we want.

In case of energy conserving system, the variables of W is separated by q and t (P is eventually a constant, we don't care), but that may not be expected.
Since

$$Q = \frac{\partial W(q, Q, t)}{\partial P},$$

is a relation that W satisfies, the solution will be yielded if the following equation is solved.

$$Q = \frac{\partial}{\partial P}\big(W(q, P, t) + C\big).$$

When it is solved, q shall be expressed by a function of constants P, Q and parameter t,

$$q = q(P, Q, t).$$

And putting the above q to the following equation,

$$p = \frac{\partial W(q, P, t)}{\partial q},$$

p shall be solved as,

$$p = p(P, Q, t).$$

Theoretically, solution of the time varying system shall be derived as in the case of energy conserving system. But in practice, the Hamilton-Jacobi's equation is mostly unsolvable.

6.5 Example of Hamilton-Jacobi's equation

We discuss one dimensional harmonic oscillation.
(1) Hamiltonian.
The Newton's equation is,

$$\frac{1}{2}m\dot{q}^2 = kq$$

where m is the mass, and k is force proportional from the origin. The Hamiltonian is,

$$\mathcal{H}(p, q) = \frac{1}{2m}p^2 + \frac{k}{2}q^2.$$

Using angular frequency ω (this simplifies the calculation; we know this happens by different method.)

$$\omega = \sqrt{\frac{k}{m}},$$

the Hamiltonian is,

$$\mathcal{H}(p, q) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2.$$

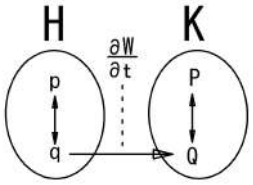

Figure 6.4: Parameter mapping

(2) Hamilton-Jacobi's equation.
If we choose $q \to Q$, the Hamilton-Jacobi's equations becomes,

$$\mathcal{K}(P,Q,t) = \mathcal{H}(p,q) + \frac{\partial W(q,Q,t)}{\partial t} = 0.$$

In the above equation, P, Q are constants, but we use this at the last moment. Replacing with concrete Hamiltonian,

$$\mathcal{K}(P,Q,t) = \frac{1}{2m}\big(\frac{\partial W(q,Q,t)}{\partial q}\big)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial W(q,Q,t)}{\partial t} = 0.$$

(3) Relation of generating function W and variables.
Because we mapped $q \to Q$,

$$\frac{\partial W}{\partial q} = p, \quad \frac{\partial W}{\partial Q} = -P.$$

We will use this relation to calculate p and q from W.
(4) Solution of W.
Now, we observe the Hamilton-Jacobi's equation carefully and use a frequently used technique. That is,

$$\frac{1}{2m}\big(\frac{\partial W(q,Q,t)}{\partial q}\big)^2 + \frac{1}{2}m\omega^2 q^2$$

is a function of q only, and

$$\frac{\partial W(q,Q,t)}{\partial t}$$

is a function of t only. Conveniently, the variables q, t are separated in the equation, we assume the solution W has a following structure.

$$W(q,Q,t) = W(q,Q) - f(Q)t = W(q,Q) - Qt.$$

Then,

$$\frac{\partial W}{\partial Q} = \frac{\partial W(q,Q)}{\partial Q} - t = -P - t, \quad \frac{\partial W}{\partial t} = -Q.$$

Therefore, our target equation becomes,

$$\frac{1}{2m}\big(\frac{\partial W}{\partial q}\big)^2 + \frac{1}{2}m\omega^2 q^2 - Q = 0,$$

$$\Big(\frac{\partial W}{\partial q}\Big)^2 = 2mQ - m^2\omega^2 q^2$$

$$W = \pm \int \left(2mQ - m^2\omega^2 q^2\right)^{\frac{1}{2}} dq = \pm m\omega \int \left(\frac{2Q}{m\omega^2} - q^2\right)^{\frac{1}{2}} dq,$$

To solve this equation for q, we use $\partial W/\partial Q = -P - t$,

$$\frac{\partial W}{\partial Q} = \pm m\omega \frac{\partial}{\partial Q} \int \left(\frac{2Q}{m\omega^2} - q^2\right)^{\frac{1}{2}} dq$$

$$= \pm m\omega \int \frac{\partial}{\partial Q} \left(\frac{2Q}{m\omega^2} - q^2\right)^{\frac{1}{2}} dq$$

$$= \pm \frac{1}{\omega} \int \left(\frac{2Q}{m\omega^2} - q^2\right)^{-\frac{1}{2}} dq$$

****

Here we need calculation of integral,

$$y = \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

If we put

$$x = a\cos z \quad (a > 0), \qquad z = \cos^{-1}\frac{x}{a}.$$

$$y = \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{-a\sin z}{\sqrt{a^2(1 - \cos^2 z)}} dz = -z = -\cos^{-1}\frac{x}{a}.$$

Note that even for a simple case like this example, we have to cope with complicated integral. Thus, the application of Hamilton-Jacobi's equation to simple mechanics does not make sense.

****

$$= \mp \frac{1}{\omega} \cos^{-1}\left(\sqrt{\frac{m\omega^2}{2Q}} q\right) = -P - t.$$

Therefore,

$$q = \sqrt{\frac{2Q}{m\omega^2}} \cos\omega(t + P) = \sqrt{\frac{2Q}{k}} \cos(\omega t + \phi), \quad \phi = \omega P,$$

where

$$\mp \cos^{-1} x = -y \to x = \cos y.$$

and,

$$p = \frac{\partial W}{\partial q} = \pm\sqrt{2mQ - m^2\omega^2 q^2}$$

$$= \pm\sqrt{2mQ - m^2\omega^2\left(\frac{2Q}{m\omega^2}\right)\cos^2(\omega t + \phi)}$$

$$= \pm\sqrt{2mQ}\sin(\omega t + \phi).$$

P and Q are constants and the actual value is calculated by initial conditions.

6.6 World of Hamilton-Jacobian

From Hamiltonian to Hamilton-Jacobian mechanics, there is a big jump; **treatise of time varying systems and energy conserving system in a unified way** which is the direct result of Hamilton-Jacobi's equation.

From a philosophical aspect, once the Hamiltonian is expressed by,

$$\mathcal{H} = \mathcal{H}\Big(p(t), q(t), t\Big),$$

then the description of the target physics becomes complete, in the sense it is now equipped full physical variables and a parameter. This naturally defines

$$\frac{\partial \mathcal{H}}{\partial p}, \quad \frac{\partial \mathcal{H}}{\partial q}, \quad \frac{\partial \mathcal{H}}{\partial t},$$

which are independent with each other and we observe the system $\mathcal{H}$ by these values. Also, when we think real physical systems, $\partial\mathcal{H}/\partial t$ is non zero for most cases. But we have no ways to treat until $\partial W/\partial t$ manifested.

In more practical aspect, the Hamilton-Jacobi's equation started to include t by a necessity to describe time varying physics, then it spawned two main streams in the application of analytical physics.

(1) Description of time varying physical system. This is directly connected to multivariable automatic control theory, because in this theory, description is inherently multivariable and the control inputs are generated from the outputs which are time varying.

(2) Solution of Hamilton's equation (= canonical transformation of energy conserving system).

However, it is well known that the Hamilton-Jacobi's equation is mostly unsolvable. This is caused by the structure of the generating function. When the variables are separated in the generating function, the integral shall be calculated analytically, but if this is not the case, the integral is generally not obtained by analytical method.

Then numerical approximation by computer shall be the only way for solution. But even in this case, the generating function can be of any order with any dimension - multiple dimensional matrix calculation is expected.

Thus, in the application of multivariable control, trial to solve Hamilton-Jacobi's equation is very cleverly avoided - replaced by simpler, but still laborious Ricatti equation. Ricatti equation is a "second order" matrix equation, this is about the level we can handle at this moment.

However, throughout the discussion from Newton to Jacobi, we finally reached to an equation that can describe much wider physics, and we know how to describe problems.

PartII

What is control

When we say control, we may mean controlling the climate, inflation, rocket, robot and so on. In any case, we hope to move the target system to the direction we wish by setting a value. We generally agree that the climate is beyond our control and are skeptical if inflation can ever be controlled. On the other hand, rockets and robots are designed to be controlled. This difference that we can control or not is originated how precisely the target system is expressed - physically and mathematically, system variables are measured (or observable) and if some means to affect the target system are equipped or not (or controllable).

Progress of control theory may be divided into three eras.

Classic control era ; Started by the Watt's speed regulation of steam motor to the end of 1940s. In this era, studies are implementation (products) driven and almost heuristic. Main interests were stability researches based on Hurwitz (1859-1919), Routh (1831-1907) and Nyquist (1889-1976). Formulation of transfer function basically completed the theory which enabled the unification of electronic and mechanical system together.

Modern (multivariable) control era; After the WW2 when control and tracking of airborne and space vehicles became national (U.S.A and Soviet union) interests as well as the advent of digital computers. Until around 1980s.

Post modern control era; Multivariable control presented rigorous analysis and design. But the rigorous design is depending on the correctness of the model which may be difficult to construct in reality. For example, in the simulation of controller, Trident submarine responded unexpectedly under moderate sea condition or disappointing results of F8 aircraft are experienced [37] which clearly showed limitation of multivariable control approach. These led to the notion of "robust" control and progressed to H_∞ control, for instance. As of 2015, the robust control is the main topic of research and implementation (particularly, to mission critical controls).

We only treat classic and multivariable control because without understanding these ideas, there can be no meanings for further discussion.

In this part we discuss common factors/ideas for both classic and multivariable control stressing on the people's experience; hence not systematically, but try to give continuous understanding from daily works. Also, we will discuss how common ideas/objectives are treated by different ways in classic and multivariable control.

Chap7 System expression

7.1 Introduction

In implementation of controls, there can be roughly two different ways. One is manual control and the other is automatic control. Manual means "by hands of the people", and always requires the intervention of human existence. But the effect or result of manual control does not contradict the ideas discussed in the automatic control - in case of driving a car (see Sec 7.4), for instance. Thus the fundamental properties of control exists in time domain, where human being inherently cannot fulfill - the control requires too fast response, too long duration of intervention or too many to take care of. These factors correspond to the limitation of human beings' physical ability and the effort to extend this ability is the origin of automatic control. On the other hand, human beings are very good at subtle, context aware controls and in this area - often called "art" - the automatic control will not be able to take over.

Since the origin of automatic control is based on human beings' laziness, we generally wish to make our effort and expenditures minimal. We only give the system our intention or the objective value and how it should be achieved is done by the system itself. This is the origin of "automatic" control.

An autonomous system is a self complete energy conserving system; mathematically, a system that is expressed by a homogeneous differential equation. Here we extend this definition to a system with control input, which is generated from the output - thus the basis of autonomous system is feedback control. Also, we wish the system consumes minimal energy to achieve the goal and this is the basis of optimal control.

Therefore, an autonomous system is equipped with an information database[1] how it should operate to realize self completeness. Automatic control system is an autonomous system with interface to it's database. That is, it can communicate with external world for it's operation when necessary.

7.2 System structure expression

We start discussion of control by the simplest cases. Thus, we compare linear and nonlinear systems first.

[1] Simplest database is the setting of a potentiometer for a voltage regulator.

7.2.1 Linear or nonlinear

Suppose a system whose output y is a function of input x.

$$y = f(x).$$

The system is defined to be linear when,

$$y_1 = f(x_1), \quad y_2 = f(x_2),$$

and if,

$$y_1 + y_2 = f(x_1 + x_2)$$

holds. This is generally written by,

$$y = Ax,$$

where A is a constant.

In the linear systems, principle of superposition holds and linear differential equations have well defined solutions. These provide fundamental basis to express the target system mathematically; hence analysis and design get practicable. For the response of linear system, see Sec.11.2.

However, most of the existing physical systems are actually nonlinear and the approximation by linearity is the first thing we have to worry about.

Some of many examples of non linear systems are

$$y = e^x, \quad y = \sin x.$$

Some of the characteristics of the nonlinear systems can be,

(1) For a small change of input, the output changes more than expected - magnitude and direction.

(2) For the same input, there are multiple different outputs.

(3) For different inputs, there is one output.

These results alone are enough to make nonlinear systems difficult to control.

Since we deal with a system that can be expressed mathematically, the linear or nonlinear system is written down by linear or nonlinear differential equations. However, we cannot know the general behavior of the solution of nonlinear differential equations [9] except some special cases.

Even when we are dealing with linear systems, the resultant differential equation which we have to solve may become nonlinear. In the case of optimal control of multivariable system, we obtain Ricatti equation which is nonlinear. Although this nonlinearity is caused by the optimization of system's total energy, but not by the system's nonlinearity. We will discuss further in Sec 16.4 on this point.

In case of non-linear systems, the characteristics are against our long cultivated common sense and we feel difficult to control. Also, non linear response can have any style and if we deal with a non linear system, that necessarily becomes system specific.

Thus, the first thing we have to do for nonlinear system is linear approximation. This is the first step for practical control system design[2].

[2]In academia, there seems to be a tendency to research on nonlinear systems, because they can provide endless amount of topics.

7.2.2 Single variable or multivariable

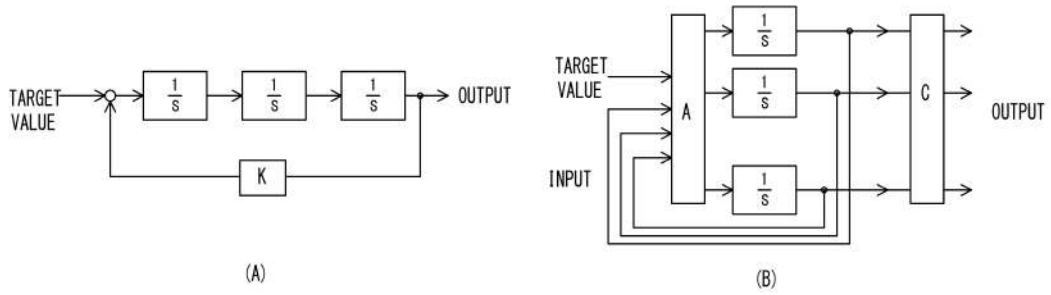

Figure 7.1: Single input-output (A) and multiple inputs-outputs (B)

Next item of simplification is to minimize the number of variables of the target system - express physical property with minimal variables. When we can express the system by single input and output as shown in Fig 7.1 (A), this is generally treated by classic control. By the result of simplified system description, we may still have multiple observation and stimulating points as shown in Fig 7.1 (B), then multivariable control should be applied.

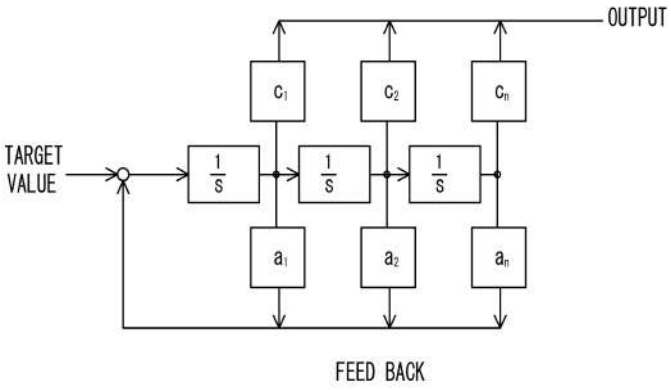

Figure 7.2: Autonomous system and target value

As shown in Fig 7.2, a multivariable system may be expressed by a single nth order differential equation with multiple input and output. This expression would reflect the practical hardware more realistically, but it can be converted to the block diagram shown in Fig 7.1 (B). One may find the reason in that a set of single order differential equations have been researched much better than a single nth order differential equation.

Stimulation points are generally called input which shall be generated by the system output. Target value is a reference for the system so that the system works as we wish. Observation points work as the source of control signals as well as the systems' objectives - when some combination of the observation points trace our wish, the system is under control.

Historically, the control started by a single input/output in late 18th century and progressed to multivariables in the middle of 20th century. Mathematical representation precisely reflects this number of variables we deal with - single nth order differential equation

(classic control) or a set of n first order differential equations (multivariable control).

Our scope of discussion only includes cases when the target systems are described mathematically and control interfaces are equipped - observable and controllable. In particular, the electro-mechanical systems are the first targets, but not limited by physical systems.

7.2.3 Known or unknown system

From another viewpoint, there can be roughly two situations; one is that the system is already existing with some measurable points and controls, and the other is that we have to (or can) design from the scratch under some specifications. In former case, finding the mathematical description which is called "system identification" is the first thing we have to do. In latter case, we will be able to draw a system block diagram based on the characteristics of the components making the mathematical expression available from the start.

We generally focus on the latter cases, but we will discuss system identification briefly in Chap 9.

7.3 Approach to nonlinear systems

When a system is working around one operating point, it is generally considered to be linear - this is the essence of linear approximation for nonlinear system. However, if we have to deal with wide range of operation, many system start showing nonlinearity. For instance, output of a simple amplifier can work linearly within it's power supply range[3] but it will saturates at the power supply voltage. The ultimatum of this example is the switching circuit in which the gain of the amplifier is set huge so that the output always saturates either bottom level (generally zero volt, or "0") or high level (or "1").

Therefore, we have to admit that most of the existing system has nonlinear aspects even if they are designed to work linearly, and also, there are many inherently nonlinear systems.

To control a nonlinear system, there is basically only one way in which target nonlinear system is expressed somehow by linear models. The linear approximation at the operating point relies on a time independent linear model for smaller operating area. This is the most commonly used and the simplest way. In order to control wide range operation, the range may be approximated by piece wise linearity. If the system state parameters change as time goes by (nonlinearity to time direction, or time variant), this may be coped with by changing the feedback by solving the "State Dependent Ricatti Equation"; that is, time wise linear approximation for each sampling (See sec 19.2). The adaptive control tries to change the control law according to the varying system parameters or uncertainty, which may incorporate nonlinear control when applicable, but systematic design is difficult.

[3]To be precise, slightly less than power supply range.

7.4 Feedback control

Since we are interested in control of a system, observing the system behavior and applying external signal is the key operation. Generally, the system behavior is picked up at the output of the system and the control is added to the input. And this should be done autonomously; the input is generated by the output so as to correct the output. If output becomes bigger than the desired value, reduce the input - this is the principle of the negative feed back.

Hence, fundamental part of the automatic control system is the negative feedback which actually is not so exotic from our daily life. Suppose you are driving a car, you steer the wheel by observing the road to adjust the direction of the car. If you feel the car is off the course you readjust by turning the steering wheel so as to trace the right track. You and the car is forming an automatic control system. Interestingly, a novice driver might overturn the steering wheel causing zig-zag moving. This is the "hunting " of the automatic control in which you are one of the malfunctioning components - either your are responding too late or too much.

As we have discussed, a dynamic system is expressed by a differential equation and **in classic control theory, single nth order differential equation is used**. This is the biggest difference from **multivariable control theory in which the target system is expressed by a set of n simultaneous first order differential equations**. This mathematical approach signifies the difference of the "classic" and "multivariable (modern)" control, and should not think that the "modern" is inherently better than "classic". In fact, both yield the same design in case of optimal regulators[4].

Because of this approach, the classic control treats a single input/output system as well as a linear system in which the feedback signal is picked up at the output and fed to input with some processing (scaling, integration and differentiation etc.).

7.5 Transfer function

7.5.1 History of transfer function

System analysis based on transfer function was started by A.C.Hall for the control of anti aircraft heavy guns in 1940s. He suggested to use transfer function $KG(j\omega)$ to describe the output/input relationship of a gun control.

$$\frac{\theta_o}{\theta_i} = \frac{KG(j\omega)}{1 + KG(j\omega)}.$$

Where θ_i and θ_o is the input and output angle respectively. Although, the above equation is a function of $j\omega$ which shows a frequency response, Hall used Laplace transform for the analysis.

Transfer function is a core of the classic control theory which was established at around the end of the WW2. Control theory progress in post war era is said to be driven by two factors, the government will - control and tracking of airborne and space vehicles - and

[4]There are cases that a system can be controlled only by multivariable control.

the advent of the digital computer (A. J. G. MacFarlane; 1979). This led to the eventual establishment of modern control theory in which a set of first order differential equations and time domain approach play main role.

However, in 1970s, people realized that there are many difficult cases, particularly in industrial problems, to apply the state space model of the modern control theory. Because already existing industrial plants are too complex to describe by state variables, but the frequency response is easily obtained. This lead to use transfer function (frequency response method) in multivariable control as well (H. Rosenbrock; 1966).

7.5.2 Definition of transfer function

When Laplace transform is applied to a linear differential equation, the time derivative dx/dt, integral $\int dt$ is replaced by the complex variable s and $1/s$ respectively. That is, differential equation is converted into arithmetic equation of s.

Transfer function $G(s)$ is defined by the ratio of output and input in s domain as,

$$G(s) = \frac{O(s)}{I(s)}.$$

For example, suspension of automobiles shown in Fig 7.3 is expressed by the following differential equation.

$$M\frac{d^2x_o}{dt^2} = K_s(x_i - x_o) + D\frac{d}{dt}(x_i - x_o)$$

Applying the Laplace transform, we get

$$Ms^2X_o = K_s(X_i - X_o) + Ds(X_i - X_o),$$

where X is a function of s corresponds to x which is a function of t. The transfer function is,

$$G(s) = \frac{K_s + Ds}{Ms^2 + Ds + K_s}.$$

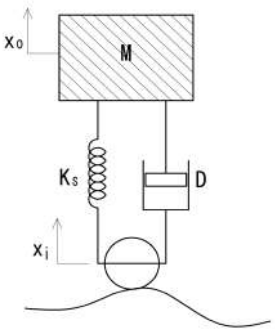

Figure 7.3: Block diagram of a suspension

Why and how a transfer function describes a system response is not obvious when encountered for the first time - it took many scholars' and engineers' effort to reach this conclusion. But, for a system,

$$G(s) = \frac{K_s + Ds}{Ms^2 + Ds + K_s},$$

if

$$Ms^2 + Ds + K_s \to 0,$$

the output goes to infinity (called poles) and if

$$K_s + Ds \to 0,$$

the output becomes zero (called zeros). That is, on those points, the system response becomes extreme - either infinite or zero.

Even if we do not know the precise response, the knowledge of extreme response are very much helpful particularly for engineers. They can think that the proper system operation should be somewhere in between.

This primitive notion is extended to a complex variable $s = \sigma + j\omega$ by which system's extreme response is mapped into complex plane. Thus the system response analysis is interpreted to find the location of zeros and poles in s plane and the synthesis is how we can locate zero and poles appropriately on s plane[5].

7.5.3 Transfer function of feedback loop

As shown in Fig 7.4, if the open loop transfer function of the system is $G(s)$ and the feedback loop is $H(s)$,

$$\big\{I(s) - H(s)O(s)\big\}G(s) = O(s),$$

then the closed loop transfer function becomes

$$F(s) = \frac{O(s)}{I(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

The above calculation is straight forward and quite mechanical. But if we recall that the original system is expressed by a time differential equation (comprises energy storage elements like mass, capacitor and inductor), feedback the output to input involves time delay in a recursive way presenting a chicken - egg situation. This recursive aspect in time is cut by using a function of s; applying the Laplace transform may be interpreted to collect all information from $-\infty$ to $+\infty$[6], and assumption of the zero initial condition cuts the recursive situation.

For a single variable application, feedback is simple not only in analysis but in design as well. However, in case of multivariable system with n state variables, all n variables must be used. These variables shall be further processed to generate control signals making the situation quite complicated. This complexity is resolved by using a digital computer. Since **multivariable control requires a digital computer**, not only the multivariable processing is taken care of, but multiplication is easily utilized. In classic control, implementing a system without processor is possible (and has been done this way), but in this case, multiplication is only done by amplifiers or attenuators in analog way, with some frequency characteristics. This is another big difference in practical design.

[5]Historically, electronic circuit response has long been analyzed by frequency response and mechanical servo system has been treated in time response. By the introduction of transfer function, they gained a common language which enabled to treat electro mechanical system uniformly.

[6]Engineers should be equipped to interpret this nonsense from "the time you feel like it" to "the time you fed up".

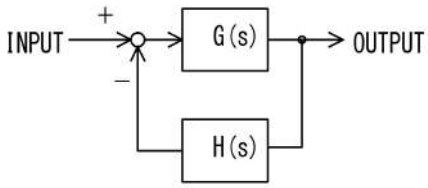

Figure 7.4: Transfer function with feed back

7.5.4 Amplifier with feedback

The effect of feedback can be better depicted by the classic approach because in case of multivariable controls, the feedback signal is also multiple and it is difficult to show the overall effects of the feedback. Also, a simpler element will tell the effects of feedback more intuitively.

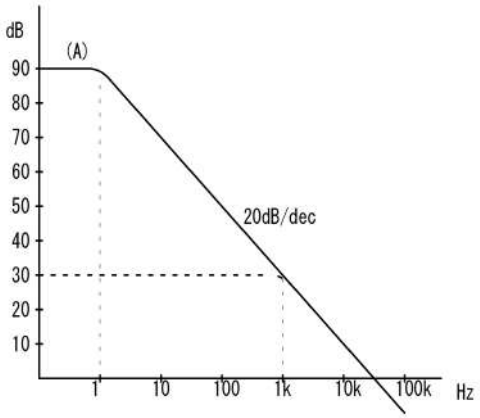

Figure 7.5: Frequency characteristics of an amplifier without feedback

Operational amplifiers have a single order low pass filter (LPF) characteristic. In case of open loop, the turnover frequency is very low (less than 1Hz) and the gain decreases by 6dB/oct (20dB/dec) as frequency increases (Fig 7.5). That is, the operational amplifiers are not designed to be used at open loop for (linear) amplifications.

If we apply 60dB negative feedback (total gain 30dB), the overall frequency response becomes (B) in Fig 7.6 - constant gain range is extended to $\simeq 1kHz$. If we apply full feedback[7], the total gain becomes 0dB and constant frequency range is extended to $\simeq 30kHz$[8].

The area shown by hatched line is used to attenuate the external disturbances and noises. That is, bigger feedback level (smaller overall gain) will give more robust system against disturbances and noise as well as wider frequency bandwidth for constant gain.

The transfer function $G(s)$ of open loop is,

$$G(s) = \frac{K_o}{1+Ts}.$$

[7]Often called voltage follower.

[8]One may ask why use amplifier for no gain ? The input impedance becomes almost ∞ with very small output impedance; current gain exists.

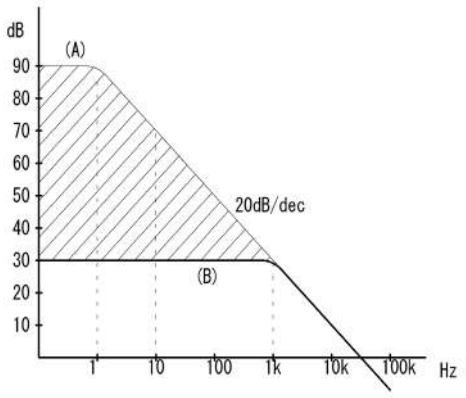

Figure 7.6: Frequency characteristics of an amplifier with feedback

When negative feedback K_n is applied,

$$G_c(s) = \frac{G(s)}{1 + K_n G(s)} = \frac{K_o}{1 + Ts + K_n K_o}$$

Since $K_n K_o >> 1$,

$$= \frac{K_o}{K_n K_o + Ts}$$

$$= \frac{\frac{1}{K_n}}{1 + \frac{1}{K_n K_o} Ts}.$$

That is, the form of transfer function does not change (still first order) and the system is always stable. The gain is decreased by $1/K_n$, time constant T is also decreased by $1/(K_n K_o)$.

If we have a low pass filter $H(s)$ in the feed back loop,

$$H(s) = \frac{K_n}{1 + T_2 s}$$

the overall transfer function becomes.

$$G_c(s) = \frac{G(s)}{1 + H(s)G(s)} = \frac{\frac{K_o}{1+T_1 s}}{1 + \frac{K_n}{1+T_2 s}\frac{K_o}{1+T_1 s}}$$

$$= \frac{\frac{1}{K_n}(1 + T_2 s)}{\frac{T_1 T_2}{K_n K_o} s^2 + \frac{T_1 + T_2}{K_n K_o} + 1}$$

$$= \frac{K\prime}{T\prime_1 s^2 + T\prime_2 s + 1}.$$

The closed loop transfer function becomes 2nd order and can be unstable depending of the value of the coefficients.

7.5.5 Conversion between parallel and series modeling

In a system design, it looks strange that a serially connected blocks can be expressed by "parallel" blocks and vice versa. The question is common in both classic and multivariable control and may worth paying attention to. In Fig 7.7 (A), a system is constructed by a series connection of two transfer functions. And the overall transfer function is,

$$F(s) = \frac{2}{s+1} \cdot \frac{6}{s+3} = \frac{8s+12}{s^2+4s+3}.$$

Parallel system expression is obtained by the following way. If we observe the process of partial fraction decomposition,

$$\begin{aligned}\frac{8s+12}{s^2+4s+3} &= \frac{a}{s+1} + \frac{b}{s+3} \\ &= \frac{a(s+3)+b(s+1)}{(s+1)(s+3)} \\ &= \frac{6(s+3)}{(s+1)(s+3)} + \frac{-6(s+1)}{(s+1)(s+3)} \\ &= \frac{6}{s+1} + \frac{-6}{s+3},\end{aligned}$$

we have introduced common terms to both denominator and numerator in the process of conversion from products (series) to addition (parallel). That is, the common terms in the parallel expression cancels out.

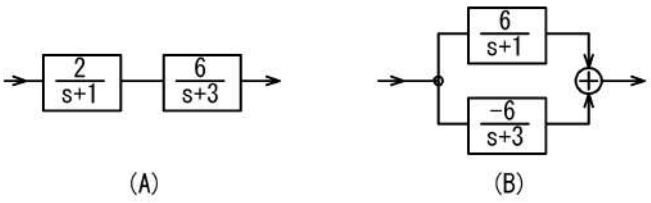

Figure 7.7: Series and parallel expression of transfer function

Conversion from series connection to parallel connection raises yet another question in case of practical applications. In Fig 7.8, (A) shows a cascaded simple low pass filter and the overall transfer function becomes second order. (B) shows a parallel of two low pass filters without thinking anything, but recognizing the value of resistance and capacitance should be changed. Obviously, parallel of $R1\prime$ and $R2\prime$, $C1\prime$ and $C2\prime$ yield just another resistance and capacitance respectively. And the overall transfer function is first order which is different from (A).

If we really wish to construct by parallel form, we shall need buffers so that the two single order RC filters can work without disturbing with each other as shown in (C).

As shown in the example above, the mathematical calculation assumes implicitly that there is no interferences between formulas both in case of series or parallel connection. This point should better be remembered in practical design.

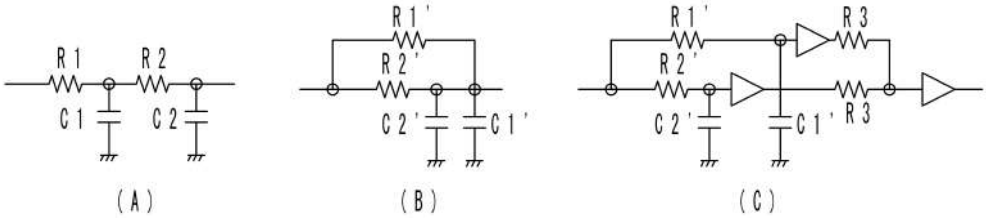

Figure 7.8: Series and parallel expression of low pass filters

7.6 Multi variable control vs classic control

There are distinct differences between multivariable control theory from classic (single variable) control theory. The first and clear difference is the methodology.

7.6.1 Classic control ; method and scope

In case of classic control theory, the system is expressed by a single variable nth order differential equation.

$$a_n \frac{d^n f(t)}{dt^n} + a_{n-1} \frac{d^{n-1} f(t)}{dt^{n-1}} + ...a_1 \frac{df(t)}{dt} + a_0 = i(t)$$

where $i(t)$ denotes input signal.
Then, by applying Laplace transform we get an arithmetic equation of nth order.

$$a_n s^n F(s) + a_{n-1} s^{n-1} F(s) + ...a_1 F(s) + a_0 = I(s)$$

$$(a_n s^n + a_{n-1} s^{n-1} + ...a_1) F(s) + a_0 = I(s)$$

$$F(s) = \frac{I(s) - a_0}{a_n s^n + a_{n-1} s^{n-1} + ...a_1}$$

The transfer function G(s) becomes

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + ...b_1}{a_n s^n + a_{n-1} s^{n-1} + ...a_1}$$

where $n > m$.

If we look how this transfer function is composed,

- **denominator is originated from open loop transfer function**,
- **numerator is originated from the feed back components**; particularly, variable s came from energy storage components in the feedback loop.

That is, if we have a system with first order open loop, and feedback loop that has energy storage components (in case of electric circuit, capacitance and inductance), the closed loop transfer function becomes at least 2nd order. The conditions stated here apply almost all cases. Hence almost all feedback system can cause hunting.

The nth order arithmetic equation is decomposed into partial fraction which eventually will be the product of first order function $1/(s + a_1)$ and second order function $1/(s^2 + a_{21}s + a_{22})$.

That is,

$$G(s) = \frac{O(s)}{I(s)} = \frac{1}{s + a_1} \frac{1}{s^2 + a_{21}s + a_{22}} ... \frac{1}{s^2 + a_{m1}s + a_{m2}}$$

When the system transfer function becomes second order, it responds as we expect intuitively, as below.

- stable but slow response,
- slight overshoot but quick response,
- too much overshoot and eventual oscillation (hunting),

Therefore, if the system has a second order transfer function, the system can satisfy almost all what we want. This is one of the reasons that the second order system has been well analyzed in classic control. And also, this is about the scope of the classic automatic control theory along with the application of the Laplace transform limits the system being linear or linearly approximated.

Because of these accumulation of techniques and simplicities of the target system - second order, single variable - the classic control is relatively[9] easy to implement and still widely used.

7.6.2 Multivariable control ; method and scope

Implementation of multivariable control, rigorous mathematics and theory is necessary and theory itself is not only difficult, but takes a long way to the final destination. But if we have to cope with multi variables, or non linear systems, the multivariable theory provides no-nonsense methodology.

Physical system is expressed by a single nth order differential equation in the Newtonian mechanics, there is an alternative approach - a set of n first order differential equations. This is the way in which multivariable (or modern) control is treated. At the first encounter with multivariable control, particularly after having learned a bit about classic control, the system expression looks like quite baffling. This is caused by the difference of treatise on differential equations.

In the middle of 20th century, differential equations are very comprehensively researched by the then Soviet union mathematicians; Pontryagin, Lyapunov, Cetaev to name but a few. Their contributions are forming a foundation of modern control theory and it should be strongly recommended to look into their books when start learning modern control - particularly when one is stuck with the equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.

7.6.3 Potential differences in object setting

Because of formulation of classic control theory, the classic automatic control system basically tries to minimize the error between current output and objective value. For example, if we wish to control the rotation phase of a motor, we provide a reference signal and the phase difference (error) is minimized under changing load, external disturbances and so on.

[9] Relatively simple, but automatic control involves feedback, and trouble shooting is very cumbersome compared with straight forward signal processing.

If the reference signal has a constant value, and the system is controlled so that output traces the reference (try to keep output constant), then the system is called a "regulator".

DC regulated power supply is a good example by which voltage (or current) is kept constant under changing load.

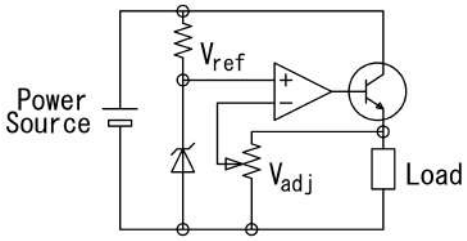

Figure 7.9: DC constant voltage regulator

In these examples, the objective value is given by us. Therefore, if a rocket is controlled by the classic theory, we have to provide the information until it's destination (Fig 7.10). Or, the path information is required. The rocket is always comparing it's current position and if the trajectory is disturbed by weather etc., the feedback loop minimizes this error.

In contrast to the classic theory, the multivariable control which is based on analytical physics has another potential at least in theory. That is, the multivariable control may enable the system to calculate the optimized path from the start to the destination - we only provide the location. Because the rocket calculates the optimized path from here to the destination by solving the Hamilton-Jacobi's equation. However, most of the Hamilton-Jacobi's equation are unsolvable, and we have to convert this problem into a solvable problem by introducing optimization of quadratic function. That is where Ricatti equation shows up and the controller becomes **Linear Quadratic Regulator**.

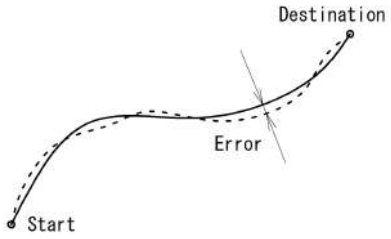

Figure 7.10: Path and error

7.6.4 Differences in signal processing system

In classic control system, the electronic parts can be constructed by analog circuits and basically a digital computer is not necessary. The main (electronic) components are sensors, amplifiers and actuators. No memories are required in regulator application (potentiometer to set the operating point, or output voltage adjust potentiometer is the only memory). Even for rockets, the memory of path can be simple.

On the other hand, in multivariable control, digital computers are necessary in many aspects. To solve multivariable differential equation numerically, to control actuators synchronously, to manage information equipments and so on. These computers are equipped in the target systems particularly autonomously moving systems. Also, computers for off line calculation, simulation and design are necessary. Hence, for the progress of multivariable control, digital computers have been playing a crucial role.

7.7 Potential oscillation

7.7.1 Prevention

Figure 7.11: Crane in operation ; from Hirano crane industry home page.
http://www.hiranogr.jp/body0402.html

The target system shall be constructed with mechanical, electrical, electronic components and programs - hardwares and softwares - to realize some functionalities. And the result shall be observed by some means so that we decide that the system is working properly or not. That is, right/wrong behavior of the target system entirely depends on our intention; the target system behaves only according to the physics law.

For example, when a crane is lifting a material without unwanted oscillation - even under wind - we think that the crane is working properly. But if we observe the material (mass) and the lifting wire, the system is obviously a pendulum and may well start oscillation. If the crane or crane operator started oscillation[10] , the phenomenon is called "hunting" which certainly is not wanted. But for a pendulum, the oscillation is what the pendulum is designed for[11].

In electronic circuit, the physical mass does not exist in circuit operation since what matters are only electrons[12]. But we have equivalent energy storage mechanisms. The flow of electrons is a current and the stock of electrons is a voltage. And both are stored

[10]People on the ground generally support the object to direct and suppress oscillation.
[11]Of course, there are exceptions. See Poe's "Pit and Pendulum".
[12]We do not think about the mass of the electron here.

as energy in inductors and capacitors respectively. In real circuitry, inductance and capacitance (and resistance in most cases) are unavoidable - meaning that some oscillation is always possible.

So, we are now aware that any system can cause oscillation. Good control enables the system operate as we want without causing this oscillation.

7.7.2 Oscillators

In order to prevent oscillation, it will be helpful to know the converse - how to cause continuous stable oscillation.
Electronic systems quite often require oscillators - either nonlinear or linear.
Wien bridge is an example of linear oscillator and the oscillation is analyzed by the point of view of automatic control. Since the output is sinusoid, the differential operator is substituted by $j\omega$.

Frequency is determined by the CR connected in the positive feedback loop of an amplifier. This type of oscillator generates reasonably good sinusoid and used often in the old days[13]. The continuous oscillation is expressed by the solution of differential equation that makes (mathematically) stable at the critical point. Since the solution is critical, it is not easy to keep the constant oscillation. And automatic gain control is generally equipped. The AGC is constructed by using an FET whose resistance is modified by the output level, or using a thermistor[14]. The thermistor is a variable resistance device whose resistance increases as the current through the thermistor produces heat thus forming a negative feedback loop.

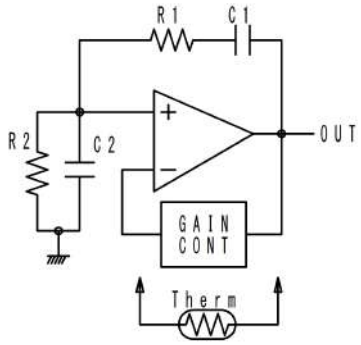

Figure 7.12: Wien bridge

Therefore, the solution of the system differential equation is set to a critical point to generate continuous sinusoid by the feedback control (AGC). By referring to Fig 7.12, the frequency of oscillation is calculated as follows. Assuming z_1 the series impedance by R_1, C_1, and z_2 the parallel impedance by R_2, C_2

$$z_1 = R_1 + \frac{1}{j\omega C_1}$$

$$z_2 = \frac{R_2}{1 + j\omega C_2 R_2}$$

[13] As of 2014, sinusoid is easily synthesized by a microprocessor
[14] Different device from ceramic thermistors for sensing temperature.

Then the signal ratio at the + input of OP amp is,

$$\frac{z_1}{z_1+z_2}=\frac{R_2\Big(R_1+R_2(1+\frac{C_2}{C_1})\Big)-jR_2\Big(\omega R_1R_2C_2-\frac{1}{\omega C_1}\Big)}{\Big(R_1+R_2(1+\frac{C_2}{C_1})\Big)^2+\Big(\omega R_1R_2C_2-\frac{1}{\omega C_1}\Big)^2}$$

Oscillation starts when the phase of + input is equal to the output of the OP amp; that is, the imaginary part = 0.

$$R_2\Big(\omega R_1R_2C_2-\frac{1}{\omega C_1}\Big)=0$$

$$\omega=\frac{1}{\sqrt{C_1C_2R_1R_2}}$$

We choose $C_1=C_2, R_1=R_2$ in practice, then

$$f=\frac{1}{2\pi CR}$$

The attenuation by z_1, z_2 at the + input is,

$$\frac{z_1}{z_1+z_2}=\frac{R_2\Big(R_1+R_2(1+\frac{C_2}{C_1})\Big)}{\Big(R_1+R_2(1+\frac{C_2}{C_1})\Big)^2}=\frac{1}{3}$$

Therefore, the AGC should control the overall gain to $\times 3$. This mathematical result shows that the Wien bridge is operating indeed at a critical point and stabilizing the output (or system state) is achieved by the feedback control.

Another good example of linear oscillator is LC resonant oscillator which includes most common X'tal oscillators. Fig7.13 shows a circuit diagram of an LC oscillator constructed by a CMOS gate IC. The kernel of oscillation is generated by LC resonant circuit and the CMOS gate is supplying power so that the circuit keeps oscillation.

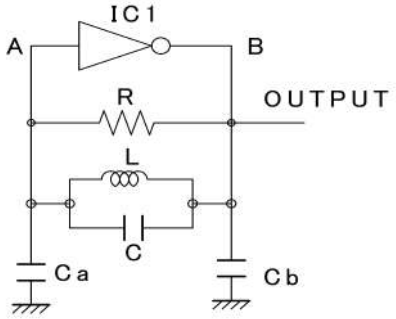

Figure 7.13: LC tuning oscillator

The frequency is given by

$$f_0=\frac{1}{2\pi\sqrt{LC}}.$$

When the LC resonant circuit is replaced by a quarts oscillating device, it forms a X'tal oscillator. The principle of oscillation is the resonance of L and C which exchange energy at frequency f_o, that is, the oscillation is the character of the resonant circuit. Thus,

to disable oscillation, either reduce the gain of the amplifier or damp the resonance or both. However, it is not possible to remove the characteristics of resonance once this is incorporated within a loop of automatic control system, avoiding resonance is most crucial in control.

For example, if a load is connected to a shaft of motor, and the shaft is thin compared with the rotor of the motor and the inertia of load, it may present a nice oscillating mechanism. The trouble shall be avoided with stronger shaft because it will shift the resonant frequency higher over the control frequency range.

A good example of nonlinear oscillator is an astable multivibrator(Fig7.14). In this case, frequency of oscillation is determined by the CR pairs (R3,C1 and R4,C2) and the mechanism of oscillation is the alternate switching of the transistors S1 and S2, and not analyzed by linear circuit analysis. That is, for a component of automatic control, this kind of circuit (or mechanism) should be avoided to make the system linear.

Since this type of oscillation is very much intentional by the designer of the system, it may be easily avoided in automatic control system design. But unintentional nonlinearity similar to this oscillator such as backlash of gear wheels or play between levers may exist in the system. If these nonlinearity is significant, remove those nonlinearity first.

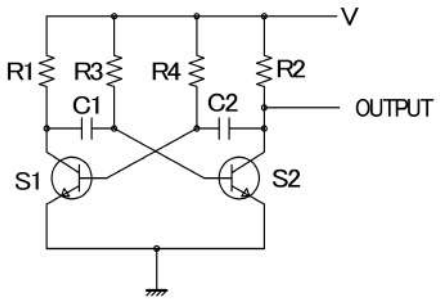

Figure 7.14: Astable multivibrator

It is interesting to observe that to keep the oscillation going and to avoid oscillation, we need careful design and/or control. Understanding the mechanism of oscillation will help avoiding and suppressing unwanted oscillation.

Chap8 Stability visualization and safety

In discussion of automatic control, stability has priority to everything else. Stability is a character of automatic control system to trace control without tendency of oscillatory movements. This character is visualized by a graph which enables to discuss stability numerically.

Since closed loop feedback system is the basic assumption, the input is generated from the output. The input signal flows through a system which has a frequency/phase characteristics, the ratio of input vs. output in gain and phase expresses the system response visually. One of these examples is the Bode diagram. Here we discuss Bode diagram only because it is simple to understand and practical to measure.

There is an idea of gain in matrices used in multivariable control, but here classic control is in mind.

8.1 Bode diagram

In linear systems, the principle of superposition holds. Thus for any input/output signals of a system, we can decompose the signal into sum of sinusoids. For a single frequency sinusoidal input, the output is always a sinusoid with the same frequency because the system is linear. However, the amplitude and phase of the output change as compared to the input. Observing the behavior of the output/input in regard with gain and phase will provide a very good information of the system characteristic. The graph which shows the gain/phase behavior of the system is called Bode diagram.

The Bode diagrams can be drawn by observing the system transfer functions or measured by the existing systems. We start discussion by drawing the Bode diagrams for several transfer functions and discuss about the measurement in Sec 23.6 .

8.2 Bode diagram examples

8.2.1 First order transfer function

First order transfer function has the form,

$$G(s) = \frac{K}{1+Ts},$$

where K is DC gain and T is time constant. The first order system is the basis of all the transfer functions, because all physically possible structure has "inertia" (mass in mechanism, capacitance/inductance in electric circuit) which works as an integration

element. The term "s" in the denominator is supplied by this integration element. Fig 8.1 shows two of typical examples.

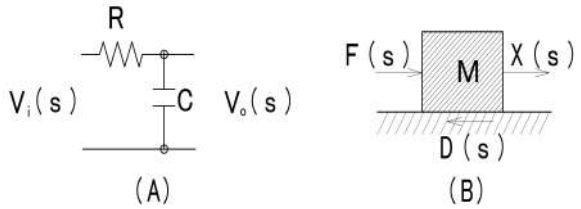

Figure 8.1: 1st order transfer function

For a low pass filter shown in Fig 8.1 (A), the transfer function is,

$$G(s) = \frac{K}{1+Ts}, \quad K = 1, \quad T = CR.$$

(B) shows a system with mass M and friction D to which force f is applied. This is a model to slide a mass put on a floor. If the floor is slippery, D is small.

$$M\frac{d^2x}{dt^2} = f - D\frac{dx}{dt}$$

If we observe the output by velocity $v = dx/dt$,

$$MsV(s) = F(s) - DV(s)$$

$$G(s) = \frac{V(s)}{F(s)} = \frac{1}{D+Ms} = \frac{\frac{1}{D}}{1+\frac{M}{D}s}$$

$$K = 1/D, \quad T = M/D.$$

Thus, a first order transfer function is obtained in both cases.

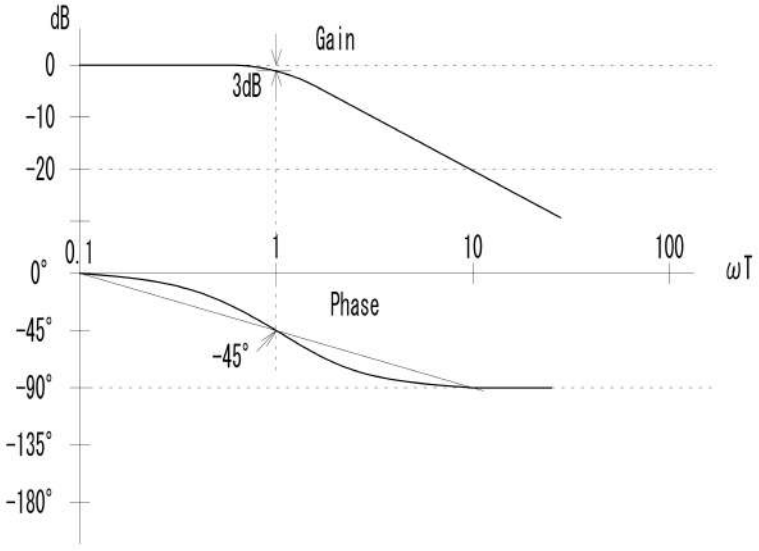

Figure 8.2: Bode diagram of 1st order transfer function

Bode diagram shows both magnitude and phase response to the input frequency. As shown in Fig8.2, the magnitude response of the first order system is flat until normalized frequency $\omega T = 1$ and then decreases by $20dB/$decade. The gain attenuation at $\omega T = 1$ is $-3dB$. Phase lag becomes 45 degree at $\omega T = 1$ and approaches 90 degree as ωT becomes bigger.

System with the first order transfer function is always stable since the phase lag never exceeds 90 degree.

8.2.2 Second order transfer function -1

The second order transfer function is given by

$$G(s) = \frac{{\omega_n}^2}{s^2 + \zeta\omega_n s + {\omega_n}^2},$$

where ω_n and ζ are resonant (angular) frequency and damping factor respectively. Fig 8.3 shows the normalized frequency response of the second order transfer function when $\zeta \simeq 0.3$. As shown, there is a possibility of oscillatory response at $\omega/\omega_n = 1$ where the phase lag becomes 90 degree. When $\zeta \simeq 0.7$, the response is said be optimal with no overshoot, whereas $\zeta \simeq 0.3$, the response becomes oscillatory at frequency ω_n and if ζ gets smaller it eventually cause hunting[1].

Good example of a second order control is found in the application of motor rotation phase control which will be discussed in Chap 22.

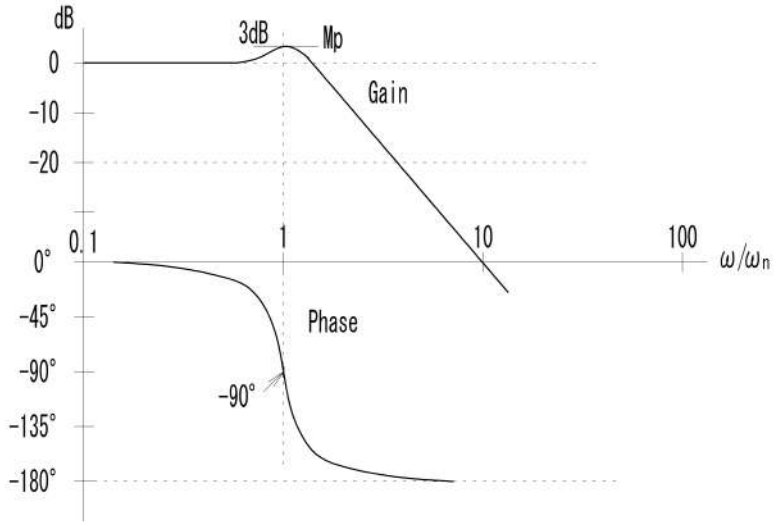

Figure 8.3: Bode diagram of 2nd order transfer function

System transfer function could be 3rd order or more, but the overall transfer function is expressed by cascading

$$1st\ order + 2nd\ order,$$

[1]By Merriam-Webster, hunting is defined by "a continuous attempt by an automatically controlled system to find a desired equilibrium condition".

the resonant response of the 2nd order comes up first in the system behavior, and if the response of the 2nd order is not stabilized, there will be no sense to worry about higher order part. This is one of the reasons that many of the (classic) system is expressed by a second order transfer function - or, the major area that the classic control can deal with is the second order system.

In other word, a higher order components are treated as a final "seasoning"[2] to the major control system which is at most second order.

8.2.3 Second order transfer function -2

We discuss a second order transfer function which has totally different characteristics from the one discussed in Sec 8.2.2. Suppose an LC parallel circuit shown in Fig 8.4. This is a most common LC resonant circuit which can be found anywhere.

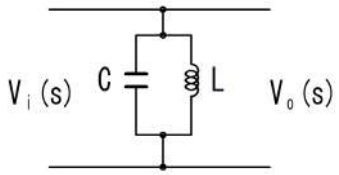

Figure 8.4: Transfer function of resonant circuit

Observing the LC resonant circuit, inductance does not matter for higher frequency beyond some point. On the other hand, the capacitance wouldn't affect for lower frequency. Thus, the resonant circuit looks like inductive to some point and then capacitive beyond ; and at the "certain point"(resonant frequency) it looks just resistive. This is why the phase changes 180 degree at the resonant frequency.

If the resonant circuit is incorporated within an automatic control system, phase of the control signal at both side of the resonant frequency is completely inverted and wouldn't make the system stable. We wouldn't know which phase the control signal should be around this point.

The impedance of this parallel circuit is (consider it in "s" plane from the start)

$$Z(s) = \frac{sL}{s^2LC + 1}.$$

Since

$$V_o(s) = Z(s)V_i(s),$$

The transfer function is

$$G(s) = \frac{sL}{s^2LC + 1}.$$

The Bode diagram of this transfer function is shown in Fig 8.5. The gain at the normalized frequency becomes infinity (in practice, because of resistance components, it is limited) where phase changes from +90 to -90. The transfer function has zeros at 0(DC) and ∞

[2]For instance, phase error of a rotation can be compensated by adding integration of phase error loop which makes the system a third order.

frequencies - in other word, the L and C produces a short circuit for 0 and ∞ frequency respectively.

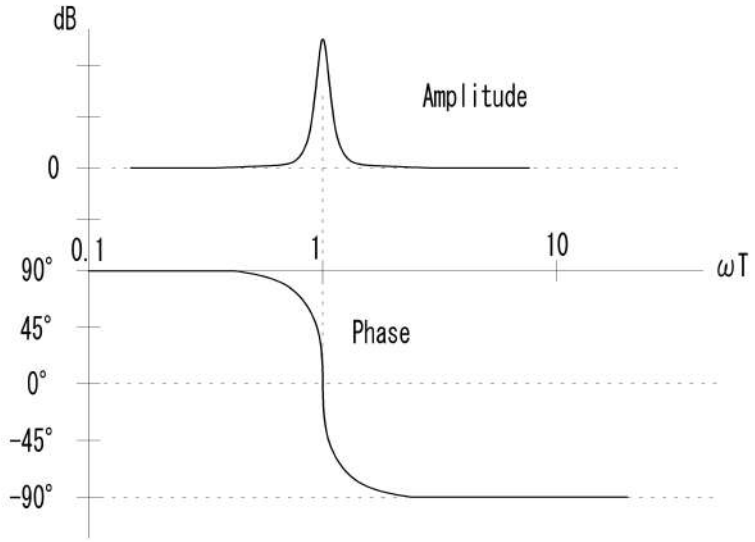

Figure 8.5: Frequency/phase response of resonant circuit

Although this characteristics cannot be used for control systems, but essential for oscillating circuits, particularly resonant oscillators. A good example is a pendulum. A pendulum oscillates at the resonant frequency which is determined by the mass of weight and the length of string. The applied force is caused by gravity whose direction (phase) changes when the weight goes through lowest point. The pendulum forever hunts for the point of equilibrium.

The resonant frequency is calculated by (replacing s by $j\omega$),

$$\omega^2 LC = 1$$

$$\omega = \frac{1}{\sqrt{LC}}.$$

Or,

$$f = \frac{1}{2\pi\sqrt{LC}}.$$

In a sense of differential equation or resonant oscillators, sinusoidal oscillation is considered to be "stable", whereas in automatic control, this is hunting and "unstable".

8.2.4 Transfer function of higher order

In case of classic control, analysis and design of higher order system gets difficult. However, transfer function has wide applicability - one of which is a filter design in electronic circuit field. In this field, particularly in digital filter design, higher order of transfer functions are used quite often.

In filter design, the order is determined by the required frequency characteristics (sharpness of cutoff, flatness of pass band and phase characteristics etc,) and the pole-zero

locations are calculated rigorously. In case of analog filters, these locations are converted to the value of inductance and capacitance for final construction - but higher the order, tolerances of these components gets severer and difficult to build. In case of digital filters, the implementation will not be difficult particularly the limitation of data length[3] presents no problems as of 2015.

8.3 Stability in practice

We have stated system response by using Bode diagrams. However, there are some strange aspects in stability discussions in control theories.

8.3.1 Stability in classic control

In actual control systems, it is generally sufficient if the system's output position (or phase for rotating system) is controlled. Then, the system can be expressed by a second order transfer function by which position (or coordinate q) is controlled. If we control the position error, the third order system may be used, but this is basically a small extension of second order system.

For the second order system, the optimal response is available at $\zeta = 0.7$, most of the systems are designed to meet this value. However, there is no rigorous methodology during the design process to set ζ to prescribed value and this is one of the reasons that the approach of classic control becomes heuristic.

Since automatic control is almost always accompanied by real working hardware, it is generally easier to adjust by observing the system response[4]. By this adjustment process, the optimal value will be found which makes the system automatically stable - that is, the optimized system cannot be oscillatory.

By heuristic approach, even there are many variables that are not precisely resolved, the result becomes more or less satisfactory. The reason that even this kind of approach is effective is based on the big open loop gain and effect of feedback. Feedback is not only effective to noise and disturbances but to rough design[5] as well.

8.3.2 Stability in multivariable control

In case of multivariable control, the stability issue is discussed in conjunction with the system differential equation. Stability of solution of differential equation was very comprehensively analyzed by Lyapunov, and condition has been well known regardless that the differential equation is solvable or not.

However, in multivariable control design, the target system equation is written by a Hamilton-Jacobi's equation which is not solvable. To meet this end, optimization is incorporated and the stability issue is hidden in this optimization. That is, **the optimum**

[3] unsigned char, int, long, etc.

[4] The author has no experience of big plant control which require precise operation from the start.

[5] Author is not recommending rough design. Starting with rough design so that system works without hunting, and then seek optimal control will be a general process.

system will not waste energy, thus it will not be oscillatory. We discuss Liapunove theory in Chap 11, but the result is not directly referred at the time of design.

8.4 Trouble shooting of loop

Since in a closed loop system, the control signal is circulating and basically identical, it is difficult to shoot troubles when the system response is still unsatisfactory or smaller malfunction is encountered - that is, the system seems to be working more or less, but desired character is not available.

One of the ways to shooting troubles, we may cut the closed loop to find the broken or unsatisfactory points. But generally, the overall gain of the open system is huge, and even a small test signal would saturate the output. Therefore, we have to check the system in the closed loop, but tracing the signal through the components returns to the starting point - this is the closed loop.

In coping with this situation, FFT analysis of error signal (control signal) is one of the ways. Suppose a motor speed control system is now working as far as the servo control is concerned, but still has some unwanted characteristics (wow, flutter, jitter etc.), observing the error signal and transform into frequency domain by FFT analyzer will provide many informations.

For instance, the motor's cogging frequency, ball bearing's number of balls[6], vibration of chassis of the motor etc. can be observed as frequency components of the FFT output.

These items are out of control theory, but the engineer in charge has to do something for it. And the origin of these troubles can only be identified by the engineer in charge of automatic control because of the knowledge of control and the expected system response.

8.5 Transient and malfunction

There can be two different kind of automatically controlled system - automatic tracking system and regulators. The automatic tracking system can be found in radar antennas or human face tracking in video camera applications. Constant voltage DC power supply is a most fundamental application of regulators. In either case, the overshoot will not be favorable. In DC power supplies, if DC voltage overshoots at power on, then the load may be damaged and this can never be accepted. However, some overshoot may be acceptable to increase the tracking ability in some applications. These characters are basically included in design specifications and not the topic of theoretical discussions.

In case of servo control of robotics arm, the arm may settle to the initial position with full power/speed at power on unless some precaution is incorporated. Also, this dangerous movement is expected if the feedback loop is cut off. In practical wirings, sensor cables are thinner than power cables but the effect will be much worse when accidentally cut off. This can be relaxed by providing bypass circuit as shown in Fig 8.6. The RV is attached to the actuator whose signal is connected by a signal wire. The resistances $R1, R2$ give

[6] If smooth rotation is required, the number of balls shouldn't be observable within error signal. Replace the ball bearing.

some level to the operational amplifier in a controller even when the signal wire is cut off. Without this kind of precaution, the actuator will move to the extreme position with the maximum power. There can be many ideas for safety, and these should not be forgotten even if the main purpose is the servo mechanism design.

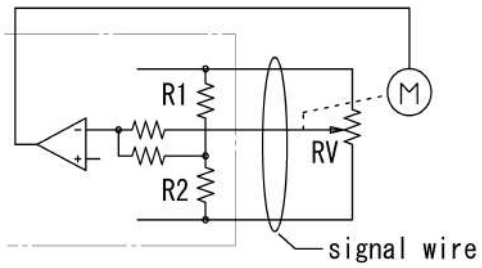

Figure 8.6: Saftey precaution

These dangers[7] are inherent with automatic control or servo systems and cannot be covered by theories.

[7]It claimed people's lives from time to time.

Chap9 System identification

Scope of system identification treated here is restricted within automatic control. That is, we not only investigate the system characteristic, but we have intention to control.

There are basically two approaches for system identification - one is parametric and the other is nonparametric. Former is assuming the system model from the start by transfer function or state variable and the latter does not assume model a priori, but in practice, both shall be used; first using nonparametric approach and having obtained rough idea, then try to express the system in detail, for instance. Therefore, we shall not pay attention to the difference in the following discussions.

9.1 Introduction

Imagine that a system is already existing, but the performance is not quite as expected and improvement of controller is necessary, or - for a long working system - performance improvement is planned, and original design documents are lost.

In these cases, system characteristics should be first obtained to model the system by transfer function (frequency domain) or state variables (time domain). For simplicity's sake, signal input and output are assumed to be observable. Also, the system is assumed to be linear.

In practice, the physical system structure is reviewed with available design plans so that block diagram can be drawn. The system transfer function or state equation are assumed based on the real system and data are analyzed to determine coefficients - trying to make the system a white box.

Once the model is chosen, we determine the coefficient of the model and check if the model fits to the real working system. If the result is unsatisfactory, we will choose another model and repeat the process.

Thus the identification process is roughly divided into the following steps.

(1) System data collection and structure analysis (data analysis).

(2) System model selection.

(3) Choice of model fit criteria.

(4) Parameter estimation.

(5) Validation.

Research of system identification tends to focus mathematical data analysis and the target is abstracted as a mathematical model. However, in practice, system identification require wide knowledge and experience, and talent required for an engineer is quite similar to that of a hacker. He shall collect as much information available about the target, apply stimuli and observe the response, with assumed knowledge about the target and obtain a

way to control the target, the only difference may be that he shall be responsible or not to what he does.

9.2 Data collection and analysis

We start a discussion by single input/output system first.

9.2.1 Data collection and analysis for single I/O system

The most straight forward test signals to apply to the working system are either sinusoid or random signal (pseudo-random binary sequence is commonly used). By observing the output, system transfer function is measured (See 23.6). If the working system permits, higher level of test signal will provide better S/N ratio, but be careful not to saturate the target.

The output data may have DC component; that is, data has some DC bias. Then, DC component may better be eliminated first for further process, because DC component may show just the operating point in which case elimination will improve S/N ratio.

An alternative to apply continuous signal, an impulse may be applied. Since the Laplace transform of the impulse is unity, the measured output is the transfer function. However, the target system is generally not designed for impulse - imagine to hit a flying helicopter - application of this method is limited.

As yet another alternative, applying square wave may sometimes work well. For instance, if we wish to know the response of a DCDC converter, practical and easy method is to switch load between R_{low} and R_{high} and observe the voltage response at the load. When we choose R_{low} and R_{high} within DCDC converter's specifications (as recommended), this will clearly show the frequency characteristic of the DCDC converter.

If we work with a system with few prior information, it is important to know the rough system response first such as rough frequency bandwidth, time constant, static gain, resonant frequency etc. Since knowing the control bandwidth helps later analysis very much even roughly (in this sense, observing the system's scale by the naked eyes is very much preferable; system observation via telecommunication media should be avoided as much as possible).

Series data analysis can provide some of the information by just observing the data - that is, without stimuli.

For data collection, the sampling period becomes another issue to worry about. Simplest method is to use high sampling rate first and estimate the critical frequency (cut off frequency) area as stated above, and use approximately 10 times higher sampling rate.

By the data collected as above, Bode diagram can be drawn relatively easily at least as a start of identification process. Based on that Bode diagram, order of a transfer function can be assumed.

9.2.2 System modeling

In case of control, the overall frequency response shall be that of the low pass filter, and knowing the cutoff frequency is important. Because, for the very low frequency the target system is expected to be under control - by big DC gain, small phase lag, and for higher frequency, control is no longer effective and system response is just decreasing by the systems characteristics. In other word, frequency range that the control becomes most critical will be around the cut off frequency area.

Once hypothetical transfer function is made, it should be compared to the system by fathoming the original system designer's intention.

For instance, if the system is a regulator of position (or phase of rotation), the transfer function becomes at least second order. And if the position control error is important, there can be position error compensator making the system third order. However, major control should be the position regulation and compensator part shall be an additional control - because position compensation makes sense after position regulation is realized.

If the target is a tracking system, the speed and accuracy shall be the most important parameters. Speed is mostly governed by the torque of actuator (motor) and (if equipped) speed reduction gears. Accuracy is determined by the actuator structure and sensor granularity (sensor frequency response would be enough). If very fine accuracy is required, mechanical gears are not favorable because of back lash and wear. On the other hand, speed or system mass may be compromised. That is, high speed, high accuracy heavy tracking system is hard to construct - not only in engineering sense, but by expenditure as well.

These observation from original designer's point of view will help to construct a model of the target system.

9.3 Mathematical system modeling

Here we will only show a modeling that introduces stochastic process to system description. Because in any system, we have noise which often cannot be ignored and difficult to treat by deterministic ways. Even a semi periodic noise is difficult to compensate because of it's unstable period, magnitude, spectrum and phase. Thus stochastic treatise becomes necessary. For detailed discussion refer to Ljung [32].

9.3.1 Single variable system modeling

The observed signal (input, output) is always accompanied by noise, we take account of this into the modeling. The noise can be caused from anywhere, we may assume that the noise is added to the output of the system as shown in Fig 9.1. (Noise at the input is regarded as a part of input and is observed as lumped output (see Chap 2 of Ljung [32])).

For a discrete system (sampled system), the output $y(t)$ is expressed by,

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) + v(t).$$

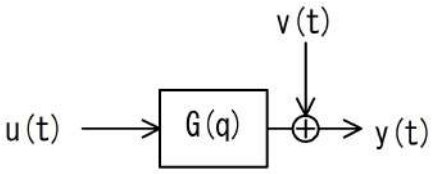

Figure 9.1: System with noise

The noise can be of any form, but if we assume $v(t)$ a sequence of independent random variable with a certain probability density function, it is written by

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k).$$

The choice of probability density function can tailor the noise from almost deterministic (impulse etc.,) to random form.

The mean of the noise is (we express mean of x by Ex and covariance of x, y by Exy)

$$Ev(t) = \sum_{k=0}^{\infty} h(k)Ee(t-k) = 0,$$

that is, the noise does not produce DC bias to the system. Or, does not produce pushing (or pulling) force to the system in average.

The covariance (relation of output and noise) is

$$\begin{aligned} Ev(t)v(t-\tau) &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} h(k)h(s)Ee(t-k)e(t-\tau-s) \\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} h(k)h(s)\delta(k-\tau-s)\lambda \\ &= \lambda\sum_{k=0}^{\infty} h(k)h(k-\tau) \end{aligned}$$

where $h(r) = 0$ if $r < 0$.

Note that both mean and covariance is independent of time, the system is called stationary. Also, note that the $G(q)$ shows the system's deterministic part, where as $H(q)$ shows the stochastic part.

If a forward shift operator q[1] is defined by,

$$qu(t) = u(t+1).$$

The backward shift operator is

$$q^{-1} = u(t-1).$$

The transfer function of a linear discrete system is defined by[2].

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}.$$

[1]This q has nothing to do with generalized coordinates.

[2]Refer to z transform of time series $g(k); k = 1\ to\ \infty$

Similarly,

$$H(q) = \sum_{k=0}^{\infty} h(k) q^{-k}.$$

Then,

$$v(t) = H(q)e(t).$$

Eventually the system output with noise is expressed by,

$$y(t) = G(q)u(t) + H(q)e(t).$$

9.3.2 Multivariable system modeling

In case of multivariable system,

$$\boldsymbol{y}(t) = \boldsymbol{G}(q)\boldsymbol{u}(t) + \boldsymbol{H}(q)\boldsymbol{e}(t).$$

where $\boldsymbol{y}(t)$ is an n dimensional output vector and $\boldsymbol{u}(t)$ is an m dimensional input vector. Noise $\boldsymbol{e}(t)$ is also an n dimensional vector. $\boldsymbol{G}(q), \boldsymbol{H}(q)$ are $n \times m$ and $n \times n$ matrices respectively.

In an explicit notation,

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \cdots & G_{1m} \\ G_{21} & G_{22} \cdots & G_{2m} \\ & \cdots & \\ G_{n1} & G_{n2} \cdots & G_{nm} \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{pmatrix}$$

$$+ \begin{pmatrix} H_{11} & H_{12} \cdots & H_{1n} \\ H_{21} & H_{22} \cdots & H_{2n} \\ & \cdots & \\ H_{n1} & H_{n2} \cdots & H_{nn} \end{pmatrix} \times \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_n \end{pmatrix}$$

The noise $\boldsymbol{e}(t)$ has zero mean values and covariance matrix $\boldsymbol{\Lambda}$ is expressed by,

$$\boldsymbol{E}\boldsymbol{e}(t)\boldsymbol{e}^T(t) = \boldsymbol{\Lambda}.$$

9.4 System modeling method

In the following sections, we show two of classic models based on the block diagram in Fig 9.1, and state space model.

In either case, determining the order and delay requires trial-error process. Some of the hints are shown below.

- For order estimation, observe the number of peaks in Bode diagram. As shown in Sec 8.2.3, one peak is expressed by two energy storage components.
  Thus, the order is double of the number of peaks.
- Obtain time delay by using correlation analysis. Combined with information obtained by another means (inspection by eye, etc), set this to reasonable value.

- By plotting the zeros and poles, they may be canceled to reduce the order. The zero and pole are generally not too near in real controlled system.[3]

9.4.1 ARX model

By this model, system transfer function is expressed by a polynomial $B(q)/A(q)$. This is the most efficient of the polynomial estimation methods, but the deterministic part ($G()$) and stochastic part ($H()$) share the common pole which is generally not correct. However, when the SN ratio is high, this disadvantage is reduced.

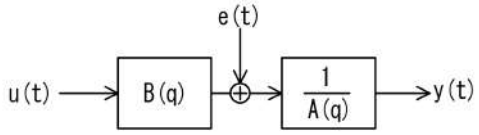

Figure 9.2: ARX(autoregressive with exogenous input) model

$$y(t) = \frac{B(q)}{A(q)}u(k) + \frac{1}{A(q)}e(k)$$

9.4.2 Box-Jenkins model

By this model, the system (deterministic) and noise (stochastic) transfer functions are separated.

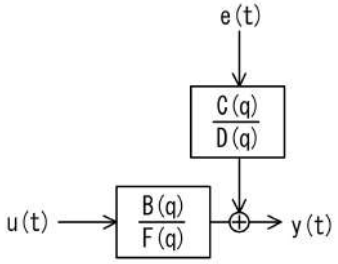

Figure 9.3: Box-Jenkins model

$$y(t) = \frac{B(q)}{F(q)}u(k) + \frac{C(q)}{D(q)}e(t)$$

9.4.3 State space model

By the state space model, the system is expressed by,

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{K}\boldsymbol{e}(t),$$

$$\boldsymbol{Y}(t) = \boldsymbol{C}(t) + \boldsymbol{D}\boldsymbol{u}(t) + \boldsymbol{e}(t).$$

[3]This situation happens for X'tal oscillators which aims at stable oscillation.

As shown, the differential equation in continuous time is simply substituted by a difference equation. This equation describes a multivariable system by linear equation, the estimation converges to a unique solution regardless of the initial parameter guess. The order or number of state variables must be chosen, with the prior knowledge and/or probes tried as shown in Sec 9.2.

The state space model approach for a large number of state variables requires high computational cost, thus the sampling number is selected to reasonable value.

9.5 Model validation

Since we have made a model, and data from real system at hand, input these data to the model and compare the output with the real output.

PartIII

Multivariable control

Chap10 Overview of multivariable control

Here we mostly focus on **Linear Quadratic Regulator** design, because this is the best utilized multivariable control technology to actual systems.

One of the difficulties in learning multivariable control theory[1] is that analysis and synthesis (design) follows different way.

As depicted in Fig 10.1, a target system is analyzed, of structure, stability and time response. Since the target is multivariable system, stability analysis is much more complicated than classic control and one common way is to use Lyapunov's method. By this method, we can tell if the system is stable or not, but how stable is another matter.

To check the detailed stability, the system response can be analyzed using state transition matrix by which we are able to know the behavior of each state variables.

System synthesis does not trace the inverse of the system analysis in multivariable control theory. System is modeled by state variables using block diagrams etc., which is expressed by Euler-Lagrange's equation. Then, by implanting control interface to the target system and optimize the whole variables. Optimization of the design process is not a bonus, but a necessary procedure to obtain a solution for LQR design.

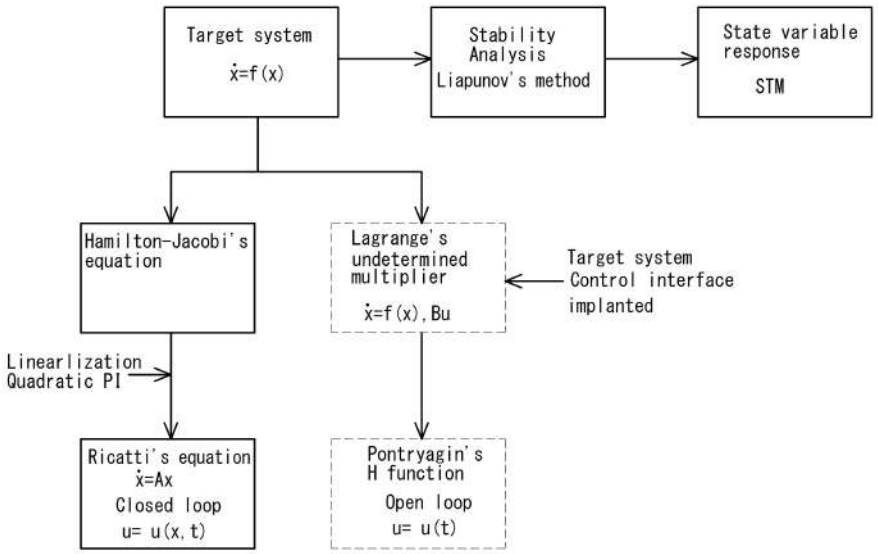

Figure 10.1: Process overview

In this chapter we will show the overall process of the discussion to provide bird's eye

[1]This character is more or less common with classic control theory

view.

10.1 First order representation

We know that a linear system can be expressed by a single nth order linear differential equation

$$a_0 + a_1 x + ... + a_n x^{(n)} = 0$$

or a set of n first order differential equations.

$$\begin{pmatrix} \dot{x}_1 = a_1 x_1 \\ ... \\ \dot{x}_n = a_n x_n \end{pmatrix}$$

If we use vector notation, the equation shall be,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

If the system linearity is relaxed, we have an equation,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

One of the superior points of multivariable theory to classic control theory is the ability that the former can be applied to non linear systems[2], we begin with this most generic equation.

10.2 Euler-Lagrange's equation of first order

Euler-Lagrange's equation is the start of the physical representation in multivariable control.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}(x_i, \dot{x}_i)}{\partial \dot{x}_i}\Big) - \frac{\partial \mathcal{L}(x_i, \dot{x}_i)}{\partial x_i} = 0$$

where

$$\mathcal{L}(x_i, \dot{x}_i) = \mathcal{T}(x_i, \dot{x}_i) - \mathcal{V}(x_i)$$

If we recall the process of the treatise of the Euler-Lagrange's equation,
(1) Form a Lagrangian $\mathcal{L}(x, \dot{x}) = \mathcal{T}(x, \dot{x}) - \mathcal{V}(x)$,
(2) Apply variational integral to $\mathcal{L}$ and make equal to zero,
(3) Newton's law $f = m\ddot{x}$ is revealed.

The variational integral applied to Euler-Lagrange's equation produces a second order differential equation which is inconvenient because we wish to express the system by a set of first order differential equations. In the orthodox process in analytical mechanics, applying the Legendre transform reduced the order and produced first order Hamilton's equation.

[2]Although, we will not go into non linear cases.

We see the first clever point here. Since we are dealing with a system that is expressed by,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}),$$

or

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x},$$

we can replace $\dot{x}$ by x in the Euler-Lagrange's equation without using any transform. Also, **the coordinates or physical variables of the target system are kept intact** which is another big point because we still can deal with our conventional variables such as velocity, voltage and current.

****

In Euler-Lagrange's equation, we have dependent $2n$ variables q and $\dot{q}$ and n equations. In Hamilton's equation, the variables become independent ($2n$), and number of equations are doubled (n to $2n$).

By applying the relation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, Euler-Lagrange's equation will have only n equations with n variables. This effectively transforms Lagrangian physics ($2nd$ order differential equation) to Hamiltonian physics.

In other words, **the multivariable control theory progressed by Hamiltonian physics in which only first order differential equations can fit. And reduction of order in Lagrangian is achieved by the system equation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.**

****

10.3 Implanting interface to the system

Even if we can write down the system physics by a set of n first order differential equations, there is no ways to influence the system motion since there is no interface yet. Therefore, **we need to implant an interface to the system and this is done by providing new variables - say u_i - to system Lagrangian.**

This means that the target system comprises additional input u_i which is assumed to take any value. The **maximum number of input shall be** n and thus we have an Euler-Lagrange's equation with $2n$ (maximum) variables with n equations. Therefore, if the Euler-Lagrange's equation is to be solved, we need some condition to reduce the number of variables to n, or increase the number of equation. Both of the following method provide neat treatise.

(1) For open loop system, use Lagrange's undetermined multiplier method.

(2) For closed loop, use Hamilton-Jacobi's equation.

10.4 Optimization of open loop system

The system Lagrangian is expressed as a function of $\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}, t$ in which $\boldsymbol{u}$ is the input. $\dot{\boldsymbol{x}}$ is replaced by $f(\boldsymbol{x})$ effectively eliminating the first derivative in the Lagrangian. The

inclusion of input increased the number of variable. The situation is resolved by applying Lagrange's undetermined multiplier method which optimizes system Lagrangian. For the optimization, **system equation** $\dot{x}_i = f(x_i)$ **provides constraints**. We might think that our control constrains the system movements, but it it the other way around.

The Lagrange's undetermined multiplier method eventually yields a function

$$H() = \boldsymbol{\lambda} \boldsymbol{f} + \mathcal{L}.$$

This is called Pontryagin's state function and is a Lagrangian which will yield first order differential equation when optimized.

10.5 Optimization of closed loop system

The optimization process by Lagrange's multiplier simply calculate optimized value and will not bind the relation between $\boldsymbol{u}$ and $\boldsymbol{x}$. That is, Lagrange's multiplier cannot produce solution for closed loop system.

Therefore, we need to choose another process. To this end, we assume that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and apply optimization so that $\boldsymbol{u}$ becomes a function of $\boldsymbol{x}$ and t. This process yields a function $J()$ using gradient $\boldsymbol{\nabla} W$ of state variables $\boldsymbol{x}$,

$$J() = \boldsymbol{\nabla} W \boldsymbol{f}() + \mathcal{L} + \frac{\partial W}{\partial t}$$

In this way, we reach **Hamilton-Jacobi's equation to describe a closed loop control system**.

10.6 Comparison of Optimization method

Closed loop optimization is derived under the assumption that the control input is expressed by state variables.

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t).$$

There seems to be no logical uncertainty during this process, but comparison with the case of open loop system may provide deeper understanding of what we have done and justify the above assumption.

In the optimization process of open loop, we derived a function $H()$

$$H() = \boldsymbol{\lambda} \boldsymbol{f}() + \mathcal{L}.$$

When we compare $H()$ and $J()$, we will see that time progress of $\boldsymbol{\lambda}$ is equal to time progress of

$$\boldsymbol{\nabla} W \boldsymbol{f}() + \frac{\partial W}{\partial t}.$$

That is, the time progress of $\boldsymbol{\lambda}$ is split into spacial part $\boldsymbol{\nabla} W$ and pure time part $\partial W / \partial t$. Since gradient is a function of all x_i, calculation of $W()$ means a calculation of control that is generated by state variables x_i.

This part is just for the understanding of theory, and quite unnecessary for practical design procedure.

10.7 Hamilton-Jacobi's equation in control

In analytical mechanics, one of the most frequent use of the Hamilton-Jacobi's equation is to calculate conversion formula specifying the value of new Hamiltonian (generally 0) to solve the original Hamilton's equation. Then, it is accompanied by coordinates transformation by which the original meaning of coordinates (and momenta) are generally lost.

In case of automatic control, original physical meaning of variables must be kept, otherwise we will encounter practical troubles. This is assured in control by the fact, that **we do not apply any transformation of the coordinates** because system expression $\dot{\boldsymbol{x}} = \boldsymbol{f}()$ plays the trick.

Having reduced the order of differential equations, we encounter new problem which does not exist in analytical mechanics - how can we generate control inputs $\boldsymbol{u}$ from $\boldsymbol{x}$? This is solved by introducing a Hamilton-Jacobi's equation.

The Hamilton-Jacobi's equation is generally analytically unsolvable and said to be difficult to solve even using a computer. However, if we assume the system being linear and choose the total energy as performance index (minimize the total energy for optimization), the equation results a matrix Ricatti equation which is solved by a computer.

10.8 Ricatti equation

Since Ricatti equation is a nonlinear differential equation, investigating it's characters etc., does not make much sense, because nonlinear differential equations are generally treated case by case. We simply focus on a program to solve matrix Ricatti equation which has been already worked out. However, we should remember that the Ricatti equation is quadratic, the number of solution is multiple.

10.9 Stability and system response

By Fig 10.1, we stated the different ways that analysis takes from synthesis. The discussion of system stability goes back to the behavior of solution of differential equations which was comprehensively researched by Lyapunov. Particularly, Lyapunov's second method by which we will be able to know the stability without solving the target equation is our choice.

The essence of Lyapunov's second method is described in Chap 11, here we simply state that the stability issue is incorporated into the choice of performance index and optimization.

That is, the system stability becomes hidden under optimization and not explicitly calculated - only the calculation of optimization is focused.

If the system is optimized, it cannot be oscillatory - hunting is the ultimatum of unstable system - because the energy consumption cannot be minimal when unnecessary movement is accompanied.

10.10 Design parameter

We have stressed that the multivariable theory (Linear Quadratic Regulator, in particular) provides no-nonsense rigorous method. However, we are dealing with multivariable optimization problem, question **"what is optimal ?" cannot be answered automatically**, because there is no mathematical rationale to judge weighting for those variables.

Therefore, at the last point, there are parameters that the designer has to provide. There is a guideline on how to choose those variables, but here we have to admit that **human choice becomes explicit**.

These are the coefficient of quadratic performance indices $\boldsymbol{Q}$ and $\boldsymbol{R}$ of which we discuss later.

10.11 Multivariable control - the best choice ?

Rigorous theory and no nonsense methodology is one of the features of multivariable control. But because of this, it is not a simple task to find eventual solution. This is proved by the fact that classic control is still widely used.

Then, multivariable control shall be most suitably applied where "multivariable" is the most requiring condition - control of humanoid robot is one. Also mission critical applications where budget and complexity have secondary importance. Thirdly, non linear system which classic control theory cannot handle.

Classic control theory was completed at around 1950s, multivariable control in 1980s and many computer simulation programs seems to have made control theory a desktop work - at least in simulation, choice is not in "classic or multivariable" but control system design by understanding physics well or not. This point will give rationale that we discussed "analytical physics" first.

Chap11 Lyapunov theory of stability

The most important point in automatic control is the stability. When we travel in an aeroplane, if the flight is unstable we would be scared. If an automobile moves in a zig zag way, we inevitably feel something is wrong. That is, we have a common feeling of "stable" and "stability".

However, in discussion of automatic control, the "stability" must be formally defined, and under that definition, the stability can be further discussed.

11.1 Introduction

Since every physical system is described by a differential equation, the stability is also discussed mathematically. The stability of the solution of a differential equation is discussed by Hurwitz, Routh, Lyapunov and other scholars[1].

Hurwitz and Routh discussed a single nth order differential equation (expressed by Laplace transformed polynomial of s),

$$a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0.$$

First we form the following determinant[2],

$$\Delta = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & ... & 0 \\ a_0 & a_2 & a_4 & a_6 & ... & 0 \\ 0 & a_1 & a_3 & a_5 & ... & 0 \\ 0 & a_0 & a_2 & a_4 & ... & 0 \\ & & .... & & & \\ 0 & 0 & 0 & 0 & ... & a_n \end{vmatrix}.$$

If the sign of the following determinants are all positive, then the system is stable.

$$\Delta_1 = a_1 > 0, \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, ...$$

This methodology clearly shows that it is only applicable to linear systems, or systems that can be expanded to polynomials.

In contrast to Routh-Hurwitz approach, Lyapunov published a paper in 1892 (see [27]) in which he analyzed the stability of solution of a set of single order differential equations.

[1] Hurwitz:1859-1919, Routh:1831-1907, Lyapunov:1857-1918

[2] Looks like quite artificially constructed for stability discussion.

Because of the difference of approach, **Lyapunov's method can cover nonlinear systems as well**.

In this chapter, we start with the stability of linear differential equation by very intuitive manner, and describe the condition for linear multivariable systems without proof. Then we discuss Lyapunov's theory without mathematical rigor, but trying to keep logical process in mind. Although Lyapunov's theory can cover nonlinear system, we apply to linear system only.

As discussed in Chap 10, analysis and synthesis takes different path in a study of automatic control. And stability issue is hidden within the process of optimization; that is, in design of an automatic control system, the contents of this chapter will not be explicitly utilized.

11.2 Stability of linear system

Single variable linear system is expressed by the following differential equation

$$\dot{x} = Ax,$$

which has a solution

$$x = e^{At}.$$

Generally, A becomes a complex value.

$$A = \sigma + j\omega.$$

We know well that the the solution becomes one of the followings.

(1) $\sigma > 0$; increases to infinity

(2) $\sigma < 0$; decreases to zero

(3) $\sigma = 0$; sinusoidally oscillates

The states are generally called, "unstable", "asymptotically stable" and "stable[3]" respectively.

In case of multivariable linear system, it is expressed by n simultaneous differential equations,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$

Since we are dealing with multivariable system, the coefficient $\boldsymbol{A}$ becomes a matrix and we cannot discuss the "value" of $\boldsymbol{A}$ any longer, but we strongly feel that the stability must be related with this matrix.

Here we only state that stability condition of linear multivariable system is that all the component of the matrix must have negative real part.

[3]From the control point of view, this corresponds to hunting and is not regarded "stable", but here we view it mathematically.

11.3 Lyapunov theory

11.3.1 Stability of general differential equation

Lyapunov discussed the stability of solution for general differential equation

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

Since $f(\boldsymbol{x})$ can be of any form, we may not be able to solve the equation first of all. Lyapunov started to treat the stability problem without solving the differential equation (called Lyapunov's second method). The idea is that the total energy of the system decreases as time goes by, the system eventually gets still and we can call this system (asymptotically) stable. Lyapunov's idea is the generalization of this intuitive idea with mathematical rigor. But because of mathematical rigor, literatures on Lyapunov theory is difficult to trace in a sense that treatise is too divergent from physics or engineering sense.

We try to overview the Lyapunov theory within our common sense so as to show that the materials are not so exotic.

11.3.2 Lyapunov's idea

The system we discuss is expressed,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

Since the system can be linear or non-linear, the expression is very general. The real system generally operates at some operating point, but this operating point can be shifted in space, or we can choose the operating point as the origin. Thus we discuss the stability problem at the origin hereafter.

One way to recognize the system stability is the behavior of the system's total energy in time t. There can be only three ways which are,

- the energy increases,
- the energy is kept constant,
- the energy decreases.

Therefore, if we think about an energy function V with system variables x_i - system's generalized coordinates and momenta - the time differential of V becomes

$$\begin{aligned}\dot{V}(x_i) &= \frac{\partial V(x_1)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V(x_2)}{\partial x_2}\frac{dx_2}{dt} + ... \\ &= gradV(\boldsymbol{x})\dot{\boldsymbol{x}} \\ &= gradV(\boldsymbol{x})f(\boldsymbol{x}).\end{aligned}$$

Because ,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

Now, if $\dot{V}(x)$ is negative, the total energy of the system decreases and the system becomes asymptotically stable.

The unusual point of Lyapunov's idea is that the energy function V is not necessarily limited within kinetic or potential energy known in mechanics. It is sufficient that the function $V(x_i)$ is positive definite which is defined by,

(1) $V(x_i) > 0$ where $x_i \neq 0$.

(2) $V(x_i)$ is differentiable to space and time.

(3) $V(x_i) = 0$ when $x_i = 0$.

In the discussion of Lyapunov theory, the positive semi-definite function is also treated, but here - for simplicity's sake - we only treat positive definite function. Then the positive definite function becomes an extension of energy function.

This extension of energy function may be explained as follows. Referring to Fig 11.1, the condition (1) means that the energy should always be positive, (2) assures that we can discuss $\dot{V}(x_i)$ and (3) means that for all zero variables, the energy is zero. If there were energy for all zero variables (B), where did it come from ? Also, if there are zero area as shown in (C), for certain variables $x_i \neq 0$, this is inconsistent particularly when x_i happen to be momenta. In case of (D), it is not differentiable at the origin[4].

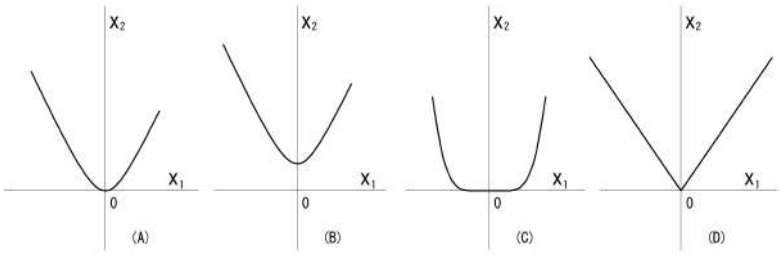

Figure 11.1: Lyapunov function(A) and non-Lyapunov functions(B,C,D)

11.3.3 Lyapunov's stability theorem

By the discussion of the positive definite function, Lyapunov generalized the idea of energy and stated his theorem as

"If a positive definite function $V(x_i)$ is found, and $\dot{V}(x_i)$ is negative or zero, then the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is stable."

Note that "stable" in this case include continuous oscillation, just excluding the divergence to infinity.

By his theorem, only the existence of a function $V(x_i)$ is required. The function may not have the quadratic form like kinetic energy. This is the reason that $V(x_i)$ is called **generalized energy**.

However, in regard of automatic control, we restrict Lyapunov's V function by a quadratic form, but even this restriction require further discussion (see Sec 11.5).

One of the difficulties in understanding Lyapunov's theory is the relation between $V(x)$

[4](D) is linear energy. See related topics in Sec 24.3.

which is called Lyapunov function and the system function $f(x)$ is not explicitly stated. Since $f(x)$ provides no conditions to $V(x)$ and $V(x)$ can be chosen as you like, how we should understand $V(x)$ may cause confusion.

This point may be clarified if we notice that Lyapunov stability is stated by the framework of $\epsilon-\delta$ definition of limit which is most commonly found in the discussion of calculus.

11.4 Lyapunov theory applied to linear systems

When Lyapunov's idea is applied to linear systems, we obtain the same results as described in Sec 11.2.

11.4.1 Quadratic generalized energy

We assume $V(x)$ to be quadratic and try not to discuss the "generalized" aspects of the energy any further.
Quadratic form of multivariable system is defined as follows.
For n dimensional system, the variable is given by a vector,

$$\boldsymbol{x}^T = (x_1 \quad x_2 \quad ... \quad x_n)$$

and the quadratic form is written by

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x},$$

in which $\boldsymbol{R}$ is $n \times n$ matrix.

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & ... & r_{1n} \\ r_{21} & r_{22} & ... & r_{2n} \\ & & ... & \\ r_{n1} & r_{n2} & ... & r_{nn} \end{pmatrix}$$

Although we cannot define positive/negative value of matrix $\boldsymbol{R}$, we can discuss the value for quadratic form, and more importantly, the positive/negative of matrix quadratic form is not at all obvious by inspection. Since we need to know the positive/negative of quadratic form eventually, we investigate the character of matrix quadratic form.

The matrix $\boldsymbol{R}$ is expressed by symmetrical and skew symmetrical matrix.

$$\boldsymbol{R} = \boldsymbol{R}_s + \boldsymbol{R}_{sk},$$

where

$$\boldsymbol{R}_s = \frac{\boldsymbol{R} + \boldsymbol{R}^T}{2}, \quad \boldsymbol{R}_{sk} = \frac{\boldsymbol{R} - \boldsymbol{R}^T}{2}.$$

Then,

$$\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{R}_s \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{R}_{sk} \boldsymbol{x}.$$

Since the quadratic form has the following character,

$$\boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{R}^T \boldsymbol{x}.$$

Then

$$\boldsymbol{x}^T\boldsymbol{R}_{sk}\boldsymbol{x} = \frac{\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{R}^T\boldsymbol{x}}{2} = \frac{\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}}{2} = 0.$$

Therefore,

$$\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{R}_s\boldsymbol{x}.$$

That is, in case of quadratic form, only symmetric part of the matrix $\boldsymbol{R}$ is effective. Thus we can assume that **the $\boldsymbol{R}$ is symmetric in the discussion of quadratic form**.

11.4.2 Value of quadratic form

The value that a quadratic form can take is determined by the coefficient matrix $\boldsymbol{R}$. To calculate, we use the "Sylvester's theorem". Calculating the following determinants of $\boldsymbol{R}$,

$$|r_{11}|, \begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix}, \ldots \begin{vmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ & & \ldots & \\ r_{n1} & r_{n2} & \ldots & r_{nn} \end{vmatrix}.$$

If all the value of determinants are positive, the quadratic form is positive definite. If any one of the determinants is zero, the quadratic form is semi-positive definite. For negative definite, negative semi-definite, we check $-\boldsymbol{R}$.

11.4.3 Example

Suppose $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \begin{pmatrix} 15 & 2 & -4 \\ 6 & 1 & 0 \\ 3 & -5 & 7 \end{pmatrix}$$

Then

$$\boldsymbol{R}_s = \begin{pmatrix} 15 & 4 & -0.5 \\ 4 & 1 & -2.5 \\ -0.5 & -2.5 & 7 \end{pmatrix} \quad \boldsymbol{R}_{sk} = \begin{pmatrix} 0 & -2 & -3.5 \\ 2 & 0 & 2.5 \\ 3.5 & -2.5 & 0 \end{pmatrix}$$

Quadratic form by $\boldsymbol{R}$ is,

$$\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} = (x_1 \;\; x_2 \;\; x_3) \begin{pmatrix} 15 & 2 & -4 \\ 6 & 1 & 0 \\ 3 & -5 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 15{x_1}^2 + {x_2}^2 + 7{x_3}^2 + 8x_1x_2 - x_1x_3 - 5x_2x_3.$$

Quadratic form by $\boldsymbol{R}_s$ is,

$$\boldsymbol{x}^T\boldsymbol{R}_s\boldsymbol{x} = (x_1 \;\; x_2 \;\; x_3) \begin{pmatrix} 15 & 4 & -0.5 \\ 4 & 1 & -2.5 \\ -0.5 & -2.5 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 15{x_1}^2 + {x_2}^2 + 7{x_3}^2 + 8x_1x_2 - x_1x_3 - 5x_2x_3.$$

The value of determinants are,

$$|15| = 15, \quad \begin{vmatrix} 15 & 4 \\ 4 & 1 \end{vmatrix} = -1, \quad \begin{vmatrix} 15 & 4 & -0.5 \\ 4 & 1 & -2.5 \\ -0.5 & -2.5 & 7 \end{vmatrix} = -91.$$

Thus the quadratic form is neither positive nor negative definite. This corresponds to the simplest case shown in Fig 11.2.

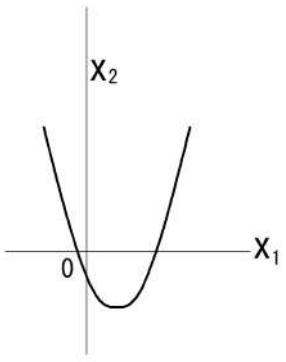

Figure 11.2: Quadratic function that takes positive and negative value

11.4.4 Stability of linear system

When Lyapunov's V function is given by

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x},$$

the time differential becomes,

$$\dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T \boldsymbol{R} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{R} \dot{\boldsymbol{x}}.$$

Since $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$,

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}^T (\boldsymbol{A}^T \boldsymbol{R} + \boldsymbol{R}\boldsymbol{A}) \boldsymbol{x} = -\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$$

If $\boldsymbol{Q}$ is positive definite, $\dot{V}(\boldsymbol{x})$ becomes negative definite and the system becomes stable. That is, if we choose arbitrary positive definite matrix $\boldsymbol{Q}$ so that,

$$\boldsymbol{Q} = -(\boldsymbol{A}^T \boldsymbol{R} + \boldsymbol{R}\boldsymbol{A}) > 0.$$

Then the condition that the solution $\boldsymbol{R}$ becomes positive definite yields the (necessary and sufficient) condition that the target system is stable. This condition is obtained by applying the Sylvester's theorem to $\boldsymbol{R}$.

The arbitrariness of $\boldsymbol{Q}$ will be exploited further by optimization.

11.5 Generalized energy

When we discuss "energy", it can be viewed by two different aspects. The first observation is focused on the variable that construct energy function. By the Lagrangian mechanics, we know that kinetic energy is a function of $\dot{q}$ and q, whereas the potential energy is a function of q (see Sec 2.6.2, 20.3). However, if we assume $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, energy definition by q or $\dot{q}$ is not fundamental any longer.

From different aspect, the mechanical energy is expressed by a quadratic form like $\frac{1}{2}m\dot{x}^2$ or linear form mgh as found in gravitational potential. In both cases, they satisfy the positive definiteness of Lyapunov's V function.

If we think about the simplest form of energy functions in engineering, variables being q or $\dot{q}$, that shall be quadratic form. Particularly, **when the quadratic form is used for minimization**, the value itself does not matter - the value that minimizes the performance function is what we want. Thus, the value of linear energy (gravitational potential h) is replaced by h^2 without problem - minimum of h^2 is equal to the minimum of $h, (h \geq 0)$.

Therefore, quadratic function of variable $\boldsymbol{x}$

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x},$$

becomes a performance function for multivariable control.

11.6 Differential equation and physics

Either single variable or multivariable, the physics is generally written down by differential equations. Thus we are tempted to think that every differential equation - solvable or unsolvable - should have corresponding physical counter parts. But this is not the case, or at least by our current technology, there can be movements that is beyond our imagination which is a solution of a differential equation. UFO's movements are said to defy our physics, but still these can be a solution of a differential equation.

And Lyapunov researched those differential equations in general.

11.7 Summary

Stability of a system which is expressed by $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is discussed by Lyapunov's idea. In the process of discussion, Lyapunov showed that the quadratic form of energy such as $\frac{1}{2}mv^2$ can be further extended to generalized energy by defining positive definite energy function.

As a method to analyze stability, the Routh-Hurwitz method checks the stability for the given equation, whereas the Lyapunov method opens a way to synthesis because of the freedom of the $\boldsymbol{Q}$ which is introduced during the stability discussion.

Chap12 Multivariable system expression

We discuss state variable model and transfer function model taking advantage of well established merits of classic control. Thus, in this chapter, we discuss only linear systems.

12.1 System modeling

The nth order system can be written by a set of n first order differential equations. In multivariable control, the target system is not necessarily be linear, but can be any form. However, in practical control, the best starting point of discussion is linear systems. Because,

- linear systems are easy for analysis and synthesis (principle of superposition holds),
- many real systems can be made linear or approximated by linear systems,
- can be discussed in general (discussion of a non linear system is system specific),
- non linear systems are mostly unsolvable (see Chap 16),

and so on.

For linear systems, Laplace transform is applicable and we exploit the transfer function to multivariable systems to investigate the system structure and response.

12.2 State variable model

Multivariable state variable model is a straight forward expression of the system and it describes by a set of first order ordinary differential equations. Thus, it becomes an expression in time.

The nth order linear system is written by,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t).$$

The above equation shows a system that has no interaction to outer world (an autonomous system). When we deal with a system in conjunction with control, we must apply input to the the system. Then the system shall be expressed by,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

where

$\boldsymbol{x}$: n dimensional state vector,
$\boldsymbol{u}$: r dimensional input vector,
$\boldsymbol{A}$: $n \times n$ state matrix,
$\boldsymbol{B}$: $n \times r$ control matrix.

Also, we need to observe the output of the system. Since each variables x_i are independent with each other, it would be natural to form the output of the system by the linear combination of x_i as,

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t).$$

where

$\boldsymbol{y}$: m dimensional output vector,

$\boldsymbol{C}$: $m \times n$ output matrix.

Or writing down each elements,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2n} \\ & \cdots & \\ a_{n1} & a_{n2} & \ldots a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \ldots b_{1r} \\ b_{21} & b_{22} & \ldots b_{2r} \\ & \cdots & \\ b_{n1} & b_{n2} & \ldots b_{nr} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_r \end{pmatrix}.$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \ldots c_{1n} \\ c_{21} & c_{22} & \ldots c_{2n} \\ & \cdots & \\ c_{m1} & c_{m2} & \ldots c_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}.$$

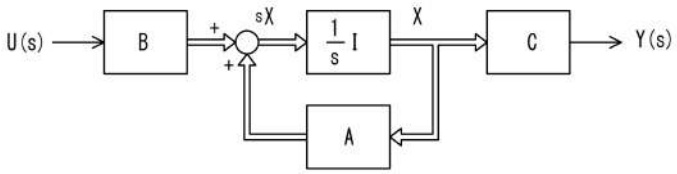

Figure 12.1: State variable expression

12.3 Transfer function model of multivariable system

12.3.1 Transfer function

Transfer function is an established tool to analyze systems in classic control theory. In classic control theory, the system is expressed by a single nth order differential equation which is converted to nth order arithmetic function of s by Laplace transform.

We try to extend this transfer function to multivariable system via discussion of an example.

Suppose a system that is expressed by the following differential equation.

$$\frac{d^3y}{dt^3} + c\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = u.$$

Applying the Laplace transform, we obtain

$$(s^3 + cs^2 + bs + a)Y = U,$$

which yields a transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + cs^2 + bs + a}.$$

This idea is applied in the next section.

12.3.2 Extention of transfer function to multivariable system

Multivarible control systems are expressed by state variables. Suppose a system is given by,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Or,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -ax_1 - bx_2 - cx_3 + u_3.$$

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3$$

We apply Laplace transform to state variable expression, obtaining

$$sX_1 = X_2, \quad sX_2 = X_3, \quad sX_3 = -aX_1 - bX_2 - cX_3 + U_3.$$

In this example, the input is only a scaler, the transfer function for each output y_1, y_2, y_3 becomes,

$$G_1(s) = \frac{Y_1(s)}{U_3(s)} = \frac{1}{s^3 + cs^2 + bs + a},$$

$$G_2(s) = \frac{Y_2(s)}{U_3(s)} = \frac{s}{s^3 + cs^2 + bs + a},$$

$$G_3(s) = \frac{Y_3(s)}{U_3(s)} = \frac{s^2}{s^3 + cs^2 + bs + a}.$$

We see that the transfer function of input u_3 to output y_1 becomes identical to that obtained in Sec 12.3.1.

That is, the transfer function for multivariable system can be defined between a pair of certain input and output.

12.3.3 Transfer function of multivariable system

By generalizing the discussion above, we define,

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \\ ... \\ Y_m(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) & ... & G_{1r}(s) \\ G_{21}(s) & G_{22}(s) & ... & G_{2r}(s) \\ & ... & & \\ G_{m1}(s) & G_{m3}(s) & ... & G_{mr}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \\ ... \\ U_r(s) \end{pmatrix}.$$

Or,

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s),$$

where $\boldsymbol{G}(s)$ is called a transfer function matrix. Note that a transfer function - one element of the transfer function matrix - is defined by one input/output pair. That is,

$$G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}.$$

In other word, to obtain G_{22}, we measure output y_2 applying u_2 only.

$$\begin{pmatrix} - \\ Y_2(s) \\ ... \\ - \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) & ... & G_{1r}(s) \\ G_{21}(s) & G_{22}(s) & ... & G_{2r}(s) \\ & & ... & \\ G_{m1}(s) & G_{m3}(s) & ... & G_{mr}(s) \end{pmatrix} \begin{pmatrix} 0 \\ U_2(s) \\ ... \\ 0 \end{pmatrix}.$$

12.3.4 Transfer function in matrix form

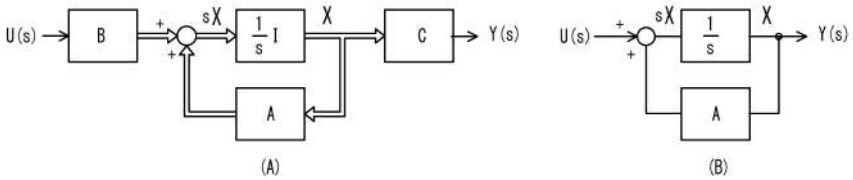

Figure 12.2: Comparison of multivariable and single variable system

Transfer function is calculated by the system equation of matrix form.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t).$$

In s plane,

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{u}(s).$$

$$s\boldsymbol{X}(s) - \boldsymbol{A}\boldsymbol{X}(s) = s\boldsymbol{I}\boldsymbol{X}(s) - \boldsymbol{A}\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{B}\boldsymbol{U}(s).$$

Therefore,

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s).$$

Since

$$\boldsymbol{Y}(s) = \boldsymbol{C}(s)\boldsymbol{X}(s) = \boldsymbol{C}(s)(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s),$$

and the input/output relation is,

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s).$$

The transfer function becomes,

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}.$$

Matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ is called a resolvent matrix.

Transfer function of a single variable becomes (Fig 12.2 (B)),

$$G_c(s) = \frac{G(s)}{1 - G(s)H(s)} = (s - A)^{-1}.$$

Then the significance of multivariable case becomes clear. In case of multivariable, $\boldsymbol{B}$ and $\boldsymbol{C}$ become necessary because these coefficient determines the weight of inputs and outputs respectively. Also, the resolvent matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ is the kernel of transfer function and deserves specific term.

12.3.5 Canonical form

Matrix notation is a compact form of a set of n first order differential equations. When the coefficient matrix is expressed by a diagonal form, the n differential equations become independent with each other. This is called "canonical[1]" form and the variables are called canonical variables.

We try to write down a system in matrix notation when a transfer function is given decomposed by each poles. The transfer function is written by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B_1}{s - \lambda_1} + \frac{B_2}{s - \lambda_2} + ... + \frac{B_n}{s - \lambda_n}.$$

That is,

$$\begin{aligned} Y_i(s) &= \frac{B_i}{s - \lambda_i} U_i(s) \\ &= B_i \frac{U_i(s)}{s - \lambda_i}. \end{aligned}$$

Now, we wish to express the transfer function in the following state variable form

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t).$$

The output equation in time domain,

$$y_i(t) = C_i x_i(t),$$

is expressed in s plane,

$$Y(s) = C_i X_i(s).$$

Therefore,

$$C_i X_i(s) = B_i \frac{U_i(s)}{s - \lambda_i}.$$

B_i is recognized as a corresponding value to C_i,

$$X_i(s) = \frac{U_i(s)}{s - \lambda_i} = U_i(s)(s - \lambda_i)^{-1},$$

$$sX_i(s) - \lambda_i X_i(s) = U_i(s).$$

Or,

$$s\boldsymbol{X}(s) - \boldsymbol{\Lambda X}(s) = (s\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{X}(s) = \boldsymbol{U}(s).$$

[1]No direct connection to canonical equation in Hamiltonian mechanics.

In time domain,

$$\dot{x}_i(t) = \lambda_i x_i(t) + u_i(t).$$

Or,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ ... \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} + \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix}.$$

For the output

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + ... + C_n x_n(t).$$

The matrix $(s\boldsymbol{I} - \boldsymbol{\Lambda})$ is,

$$(s\boldsymbol{I} - \boldsymbol{\Lambda}) = \begin{pmatrix} s-\lambda_1 & 0 & ... & 0 \\ 0 & s-\lambda_2 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & s-\lambda_n \end{pmatrix}.$$

Determinant of this matrix is the denominator of the transfer function and thus, equal to the poles of the system.

12.4 Observable, controllable

In contrast to classic control, state variable approach tries to make the system a white box. State variable expression by first order differential equation assumes that the system is expressed by a white box which of course is not generally true.

Thus, in reality, we have to admit that there are state variables that we cannot detect but the system is somehow disturbed by that invisible entity. Also, there is a dual situation; we can detect the output of the entity, but we have no ways to do something to it - noise may be a good example.

These situation are expressed by an idea of "observable", "controllable" which was first introduced by Kalman [2](1960). Here we do not go into rigorous definition of these words, but roughly

Observable : a system variable which can be measured from outside.

Controllable: a system variable to which we can stimulate.

This situation is best understood by the Fig 12.3(cited from [26]) for which no explanation shall be necessary.

12.5 Observability, controllability check

Observability and controllability are checked by the following way (we only show the result without discussion).

For a system written by,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

[2]Rudolf Emil Kalman: 1930-2016

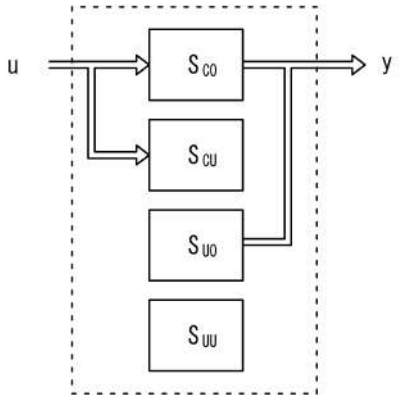

Figure 12.3: Relation of system variables to outer world

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t).$$

(1) Observable

Only if the following $n \times nm$ matrix has rank n.

$$\Big[\boldsymbol{C}^T \mid \boldsymbol{A}^T\boldsymbol{C}^T \mid (\boldsymbol{A}^T)^2\boldsymbol{C}^T \mid .... \mid (\boldsymbol{A}^T)^{(n-1)}\boldsymbol{C}^T\Big].$$

(2) Controllable

Only if the following $n \times nr$ matrix has rank n.

$$\Big[\boldsymbol{B} \mid \boldsymbol{A}\boldsymbol{B} \mid \boldsymbol{A}^2\boldsymbol{B} \mid .... \mid \boldsymbol{A}^{(n-1)}\boldsymbol{B}\Big].$$

12.6 Linear transformation of system

Linear system expressed by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t).$$

can be transformed by nonsingular matrix $\boldsymbol{P}$ to new variables. This is a standard topic of linear transformation and we only note the following two characters.

(1) By linear transformation, the transfer function does not change.

(2) Eigenvalue of transfer function does not change.

These mathematical results are used to transform non canonical form expression to canonical form; that is, mutually dependent set of system equations are converted to mutually independent set of equations. (This corresponds to the transformation of coordinates discussed in Chap 2 in analytical mechanics.)

12.7 Position of transfer function in multivariable system

Transfer function describes the output/input relation between single variable in s plane. Also, Laplace transform is only applicable to linear systems. Thus, transfer function

approach is established as an extension of linear single variable systems; the classic control systems.

However, once it's meaning and visualization such as Bode, Nyquist diagram are established, simplicity of philosophy, operation - does not necessarily require a computer - and visual capability were too valuable to discard in the application of multivariable control.

Taking advantage of this character, there were methods that treated a multivariable system by decomposed set of transfer functions. That is, the system is considered as a multiloop feedback system and the design was carried out by one loop at a time basis. However, simultaneous gain variations was not taken into account which may yield unsatisfactory results. This is one of the reasons that the robust control took over eventually, but that does not mean that the transfer functions are useless.

For a multivariable system designed by state variable approach, the internal behavior can be expressed very effectively by the visual of transfer function. Particularly, control that was not possible in classic way maybe realized by multivariable control. In this case, we should not be satisfied by the overall outcome only, because some variables may assume critical values making the whole system potentially dangerous. Of course, this point is common with classic theory, but mind that in classic control, the number of variables is one and if we take care not to saturate the system, then this point is relatively easily managed.

We shall see later in Sec 17.3, the range of state variables can be suppressed by giving weight to coefficient matrices, thus we not only have the visualization method, but design method - although not very precise - as well.

Chap13 Response of state variables

In this chapter, we assume linear systems.

13.1 State transition matrix

A linear multivariable system is written by,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t).$$

We begin with a single variable system,

$$\dot{x}(t) = Ax(t) + Bu(t).$$

The equation above is a system equation with input, but time response is characterized with no input system.

$$\dot{x}(t) = Ax(t).$$

The solution of above equation is

$$x(t) = e^{At}x(0),$$

and the pattern of response itself is quite simple and they are the combination of

- increase, decrease or stay as it is,
- oscillation with some frequency.

Then, it would be easy to imagine that the solution of

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

becomes

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0).$$

However, we have to clarify what a matrix power of exponential is. The scaler function e^{At} is defined by a Tailor series expansion,

$$e^{At} = 1 + At + \frac{A^2}{2!}t^2 + ...+ = \sum_{i=0}^{\infty} \frac{A^i}{i!}t^i.$$

It is proved that a similar way is applicable for exponential $e^{\boldsymbol{A}t}$,

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2}{2!}t^2 + ...+ = \sum_{i=0}^{\infty} \frac{\boldsymbol{A}^i}{i!}t^i.$$

Noting that,

$$\boldsymbol{A}^2 = \begin{pmatrix} a_{11} & a_{12}... & a_{1n} \\ a_{21} & a_{22}... & a_{2n} \\ & ... & \\ a_{n1} & a_{n2}... & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}... & a_{1n} \\ a_{21} & a_{22}... & a_{2n} \\ & ... & \\ a_{n1} & a_{n2}... & a_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12}... & c_{1n} \\ c_{21} & c_{22}... & c_{2n} \\ & ... & \\ c_{n1} & c_{n2}... & c_{nn} \end{pmatrix}$$

where,

$$c_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} + ... + a_{in}a_{nj},$$

and the resultant matrix is still $n \times n$, thus $\boldsymbol{A}^i$ becomes an $n \times n$ matrix. That is, the exponential is defined by addition of $n \times n$ matrices.

The matrix $e^{\boldsymbol{A}t}$ is called state transition matrix and if we denote by $\boldsymbol{\Phi}(t)$,

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0).$$

Now, we will show how the state of the system moves. The system state at time t_0 is

$$\boldsymbol{x}(t_0) = \boldsymbol{\Phi}(t_0)\boldsymbol{x}(0).$$

Multiplying $\boldsymbol{\Phi}(t_0)^{-1}$ from left,

$$\boldsymbol{x}(0) = \boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{x}(t_0)$$

$$= \boldsymbol{\Phi}(-t_0)\boldsymbol{x}(t_0).$$

Because, $\boldsymbol{\Phi}(t)^{-1} = \boldsymbol{\Phi}(-t)$. Therefore,

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(-t_0)\boldsymbol{x}(t_0)$$

$$= \boldsymbol{\Phi}(t - t_0)\boldsymbol{x}(t_0).$$

That is, the system state is determined by the state transition matrix under given initial value $x(t_0)$. This is the reason why $\boldsymbol{\Phi}(t)$ is called "state transition matrix".

13.2 Resolvent matrix

Since $\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$ is expressed by addition of terms, applying the Laplace transform presents no problems. Therefore, applying Laplace transform to

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x},$$

we obtain,

$$s\boldsymbol{x}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{x}(s).$$

Noting that $s\boldsymbol{x}(s) = s\boldsymbol{I}\boldsymbol{x}(s)$,

$$s\boldsymbol{I}\boldsymbol{x}(s) - \boldsymbol{A}\boldsymbol{x}(s) = \boldsymbol{x}(0)$$

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}(s) = \boldsymbol{x}(0),$$

$$\boldsymbol{x}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0),$$
$$= \boldsymbol{\Phi}(s)\boldsymbol{x}(0).$$

Thus we know $\boldsymbol{\Phi}(s)$ becomes the resolvent matrix (see Sec12.3.4).

The resolvent matrix is the Laplace transform of state transition matrix, and the state transition matrix is the inverse Laplace transform of the resolvent matrix. That is,

$$\boldsymbol{\Phi}(s) = \mathcal{L}\big(\boldsymbol{\Phi}(t)\big), \qquad \boldsymbol{\Phi}(t) = \mathcal{L}^{-1}\big(\boldsymbol{\Phi}(s)\big).$$

13.3 Response of the forced multivariable system

Meanings of state transition matrix and resolvent matrix are clarified, we discuss the response of a system that is driven by external force,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t).$$

As imagined with the case of classic control, discussion of the response of the above system only makes sense for some specific inputs. Since we are thinking about time progress, the inputs are basically step, ramp and impulse[1].

The solution of the system differential equation with input $\boldsymbol{u}(t)$ is given by (without discussing the process to this solution),

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau.$$

$$= \boldsymbol{\Phi}(t)\boldsymbol{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)d\tau.$$

That is, the response is determined by the initial value $\boldsymbol{x}(0)$ and the convolution integral of input $\boldsymbol{u}(t)$ with input matrix $\boldsymbol{B}$.

$\boldsymbol{\Phi}$ is an $n \times n$ matrix, $\boldsymbol{B}$ is an $n \times r$ matrix and $\boldsymbol{u}$ is an $n \times 1$ vector, the resultant integrand yields an $n \times 1$ vector. Thus, one convolution integral must be calculated for each state variable.

13.4 Summary

We have derived that the system response of the multivariable system is calculated by using state transition matrix. By this calculation we know the time progress of each state variable. Although dealing with multivariable system, this idea and basic procedure is quite similar to that of classic control. For actual calculation, computers shall be used under some specific input. And observing the response of one particular variable, it is quite similar to the case of classic control - rise time, overshoot, oscillatory response etc.

However, hunting of even a single variable make the system unstable - that is, focusing on a single variable and design the target multivariable system step by step is not a practical

[1]Since we are dealing with linear system, sinusoidal input always yield sinusoid with same frequency.

way[2]. And the process discussed in this chapter shall better be applied to visualize the internal behavior of the state variables.

[2]In the history of control, this was the early method to design multivariable system influenced by the methodology of classic control.

Chap14 Optimal motion of a system with input

Mechanics started to investigate the motion of object with no external controls. In automatic control, the object is described based on physics, but always has control, or external intervention. The purpose of "automatic control" is to produce this control from the object's output. However, as an intermediate step, we discuss optimal motion of a system with the input which is not produced by the output - that is, optimized motion of an open loop system.

Method discussed in this chapter may be recognized as one of the **standard approaches of multivariable optimization problem**, but not that of multivariable automatic control system.

14.1 System assumption

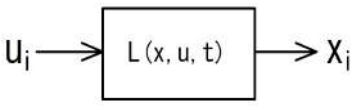

Figure 14.1: Open loop system with input

Generalized n dimensional autonomous system is expressed by,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}).$$

We prepare additional system variable $\boldsymbol{u}$ which is an r dimensional input vector to the system, and we assume that this input is independent from system variable $\boldsymbol{x}$; this is the definition of an open system. Since we have intention to apply inputs, the system is no longer time invariant (no longer energy conserving system), it explicitly includes parameter t,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t).$$

Also, we assume the input is not limited by magnitude nor frequency bandwidth. This assumption enables us to apply variational integral[1].

Another point to note is the number of input. We assume the maximum number of input is equal to the number of state variable. For the open loop system, this is not necessary - because we will not control anyway - but continuity to closed system, we assume this. Therefore, the dimension of $f(\boldsymbol{x}, \boldsymbol{u}, t)$ is $n+r+1$ and $r \leq n$. However, time is a parameter

[1]Variational integral is defined assuming that the variables of the function can take any value.

on which we have no control, we ignore the dimension for time hereafter. And we regard the dimension of $f(\boldsymbol{x}, \boldsymbol{u}, t)$ just $n + r$ unless dimension for time is explicitly necessary.

14.2 Optimized system energy

14.2.1 Optimization index

We discuss optimal motion of a system. Thus, we have to define performance index to show what is optimal. There can be many ideas, but if we remember the process to obtain Euler-Lagrange's equation, **minimization of Lagrangian is how nature works and will be an agreeable one.** Thus, we use a performance index defined by

$$PI = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt,$$

where $\mathcal{L}$ is a Lagrangian, but it is assumed to contain explicit t. This is identical to mimic natural law - Newton's law, electromagnetic theory - and we can exploit the results of analytical physics.

Since $\boldsymbol{u}$ is assumed to be independent from $\boldsymbol{x}$, it can be just included into $\boldsymbol{x}$, but if we identify the input,

$$PI = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}, t)dt.$$

Here we do not include $\dot{\boldsymbol{u}}$ because $\boldsymbol{u}$ is input which is under our control, and there is no reason to provide differential term[2].

14.2.2 System variable selection

We know that energy is a function of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$, the Lagrangian is a function of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$.

We also know that the target system is expressed by $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Then we have an option to express the Lagrangian by $\boldsymbol{x}$ (or, $\dot{\boldsymbol{x}}$) only. We use this option for theoretical study, whereas in practice, we use both $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ as found in Sec 18.3.2.

If we recall how Euler-Lagrange's equation is calculated by the Lagrangian (Sec 1.6), explicit usage of both $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ is fundamental - not only for the trajectory but for tangent of trajectory is minimized.

Then, we might wonder that if we use single type of variable (either $\boldsymbol{x}$ or $\dot{\boldsymbol{x}}$) only, how does the optimization work? This is assured that the system has a relation $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$, and all the variables are within scope of optimization.

14.2.3 Reduction of order; theory

Since, multivariable control theory is treated by first order differential equations, we get rid of $\dot{\boldsymbol{x}}$ from the Lagrangian.

[2]In practice, differential of signal should be avoided because it becomes noise sensitive.

This is simply achieved by applying the equation $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$, effectively reducing the order[3]. Therefore, the performance index is expressed by,

$$PI = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)dt.$$

This is a very significant thing because the **Lagrangian we are going to use will only yield the first order differential equations after applying variational integral which in fact is a Hamiltonian**, as noted below.

****

When deriving the Hamiltonian, we applied Legendre transform to the Lagrangian (see Sec 4.2.2). The fundamental property of Legendre transform is to express function $F(x)$ by tangent $F\prime(x)$ and it's intercept u.

Then, application of Legendre transform to a function $f(x, y) = 0$ means a transformation into

$$f(x, \dot{x}) = 0.$$

Note that x in the above equation corresponds to the intercept. Solving by the derivative,

$$\dot{x} = g(x).$$

This is identical with the system function we are discussing above.

In case of Legendre transform, the function $F(x)$ is restricted as a convex (concave) function to ensure one to one correspondence of $F\prime(x)$ and intercept x. However, in case of system function, it can be shown as in (C) of Fig 14.2 which obviously has lost one to one correspondence. When we are interested in the path of an object, it really doesn't matter if one by one correspondence is lost. It is a daily matter that we are at the same location at different times (but not vice versa).

In this context, application of system expression (first order differential equation) works as an extension of Legendre transform.

When we think the system relation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, it is not very clear, but if we assume linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, it is clear enough that **the variable $\dot{\boldsymbol{x}}$ is independent from $\boldsymbol{x}$.** Thus application of system equation not only reduces the order of Lagrangian, but actually changes the system equation into Hamilton's equation.

****

14.3 Lagrange's undetermined multiplier

Lagrange's undetermined multiplier method optimizes a multivariable function by the following two steps.

(1) Increase the number of variables (λ_i).

(2) Increase the number of equation (by partial differential), then optimize (partial differential = 0).

[3]Analytical physics is based on first order differential equations, and it is a necessary process to reduce higher order differential equations into first order.

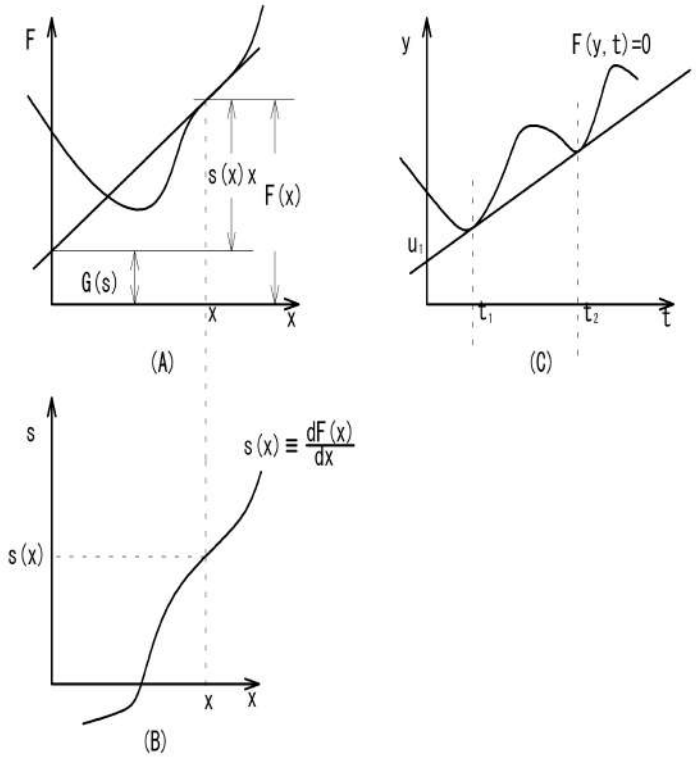

Figure 14.2: Legendre transform and system expression

14.3.1 Incorporation of input

Once we have incorporated $\boldsymbol{u}$ satisfying,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t),$$

system variables $\boldsymbol{x}$ and $\boldsymbol{u}$ are constrained by the system mechanics. Our purpose is to minimize the total Lagrangian under this system constraint. Mind that the number of Euler-Lagrange's equation is n whereas the number of variable is $n + r$; the number of equation is not sufficient for solution. We have to think something either decrease the number of variable or increase the number of equation. That is, **even if we have formed a Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)$ we need to adjust the number of variable and number of equation before applying optimization**.

We adopt a very clever way called Lagrange's undetermined multiplier method devised by Lagrange (for details, see Sec 1.7); **increase the number of equation so that they can be solved**.

Core of the Lagrange multiplier method states for minimizing[4] a function $f(x_n)$ under k constraints of $g_k(x_n) = 0$, form a function $\Lambda(x_n, \lambda_k)$ in the following way.

$$\Lambda(x_1, ...x_n, \lambda_1, ..., \lambda_k) = f(x_1, ..., x_n) - \lambda_1 g_1(x_1, ..., x_n) - \lambda_2 g_2(x_1, ..., x_n), ...,$$

$$-\lambda_k g_k(x_1, ..., x_n) = f(x_1, ..., x_n) - \sum_{j=1}^{k} \lambda_j g_j(x_1, ..., x_n).$$

[4]To be precise, to find a stationary point, but here we state this way.

Note that $k \leq n$.

Then, partially differentiate Λ with respect to x_i and λ_j (this increases the number of equation to $n+k$) which are equaled to 0.

$$\frac{\partial \Lambda}{\partial x_1} = 0, ..., \frac{\partial \Lambda}{\partial x_n} = 0,$$

$$\frac{\partial \Lambda}{\partial \lambda_1} = 0, ..., \frac{\partial \Lambda}{\partial \lambda_k} = 0.$$

The solution will yield the required x_i. (We will have solution of λ_j, but we don't need.)

Applying the Lagrange multiplier above, we replace $f(x)$ by $\mathcal{L}()$ and $g(x)$ by $f(x)$ (target system equation). Thus we increase the number of equation.

Next point we have to note is that we are not seeking for a value, but a function instead. That is, simply differentiate and equal to zero will not provide solution, but **we have to apply variational integral** to the target equation. In this way, we obtain $\boldsymbol{x}$ and $\boldsymbol{\lambda}$. In other words, the optimization is not an option, but a necessary process to get final solution $\boldsymbol{x}$.

The system function,

$$\dot{x}_j = f_j(\boldsymbol{x}, \boldsymbol{u}, t), \quad j = 1, n$$

$$f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j = 0,$$

works as the constraint for x_j and u_j. Note that suffix j refers to constraints caused by the system, whereas i refers to the variables in the Lagrangian. The target function to which we apply variational integral becomes,

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = \sum_{j=1}^{n} \lambda_j \{f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\} + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t).$$

(The sign of λ is inverted from the original multiplier method, but this is not fundamental; the original meaning of the sign is to show that two vectors f and g are in parallel. The change of sign means that two vectors are also parallel, but inverse direction. see Sec1.7.)

14.3.2 Calculation of optimal system

Now we have done the first part of Lagrange's multiplier method. The current status is summed up by,

(1) Target Lagrangian H is formed by variables $\boldsymbol{x}$ and $\boldsymbol{u}$.

(2) The equation has $n + r$ variables.

(3) Ready to apply optimization.

(Increasing the number of equation and optimization at one time.)

Here is one note. In original Lagrange's multiplier, λ_j is a number. However, in the function H, the multiplier becomes $\lambda_j(t)$ because we are dealing with $\boldsymbol{x}(t), \boldsymbol{u}(t)$ (Elsgolc, 1962). We will use this point without proof.

14.3.3 Lagrange's multiplier method - continued

We follow the Lagrange's multiplier method. The target function we optimize is expressed by,

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = \sum_{j=1}^{n} \lambda_j \{f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\} + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t).$$

where $\boldsymbol{x}, \boldsymbol{u}$ are the system and input variables respectively. Then we apply variational integral to H with respect to x_i and u_i.

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\Big(\frac{\partial H}{\partial \dot{x}_i}\Big) = 0, \quad i = 1, ...n,$$

$$\frac{\partial H}{\partial u_i} - \frac{d}{dt}\Big(\frac{\partial H}{\partial \dot{u}_i}\Big) = 0, \quad i = 1, ...r.$$

On the other hand, λ is only a function of t (not a state variable), and thus simple differential by λ is sufficient,

$$\frac{\partial H}{\partial \lambda_i} = 0, \quad i = 1, ...n.$$

These are the equations to solve this optimization problem.

(1) Calculation of $\partial H/\partial \boldsymbol{x}$: yields n equations.
For the variables $x_i (i = 1 \text{ to } n)$,

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\Big(\frac{\partial H}{\partial \dot{x}_i}\Big)$$

$$= \frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j \big[f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\big]\Big\} = 0.$$

Noticing neither $\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)$ nor $\lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t)$ are function of $\dot{x}_i$,

$$\frac{\partial H}{\partial x_i} = -\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\sum_{j=1}^{n} \lambda_j \dot{x}_j\Big\}.$$

$\partial/\partial \dot{x}_i$ operates only for $i = j$,

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\sum_{j=1}^{n} \lambda_j \dot{x}_j\Big\}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\lambda_1 \dot{x}_1 + ... + \lambda_i \dot{x}_i + ... + \lambda_n \dot{x}_n\Big\}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\lambda_i \dot{x}_i\Big\}$$

$$= \frac{d}{dt}\lambda_i = \dot{\lambda}_i.$$

Therefore,

$$\frac{\partial H}{\partial \boldsymbol{x}} = -\sum_{i=1}^{n} \dot{\lambda}_i = -\dot{\boldsymbol{\lambda}}.$$

(2) Calculation of $\partial H/\partial \boldsymbol{u}$: yields r equations.

For the variables $u_i (i = 1 \text{ to } r)$,

$$\frac{\partial H}{\partial u_i} - \frac{d}{dt}\Big(\frac{\partial H}{\partial \dot{u}_i}\Big)$$

$$= \frac{\partial H}{\partial u_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{u}_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{r} \lambda_j \big[f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\big]\Big\} = 0.$$

$\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)$ and $\big[\lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\big]$ are not functions of $\dot{u}_i$,

$$\frac{\partial H}{\partial u_i} = 0.$$

Or,

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0.$$

(3) Calculation of $\partial H/\partial \boldsymbol{\lambda}$: yields n equations.
For the variables $\lambda_i (i = 1 \text{ to } n)$,

$$\frac{\partial H}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j \big[f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\big]\Big\}$$

$$= \frac{\partial}{\partial \lambda_i}\lambda_i \big[f_i(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_i\big] = 0.$$

$$f_i(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_i = 0.$$

This is the system equation and calculation of $\partial H/\partial \boldsymbol{\lambda}$ adds no information. (Optimization of λ yielded the target system equation.)

14.4 Pontryagin's H function

During the process of calculation of the following equation,

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j \big[f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\big]\Big\} = 0,$$

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t)\Big\} = -\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(\lambda_i \dot{x}_i).$$

H was originally defined by,

$$H = \sum_{j=1}^{n} \lambda_j \{f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\} + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t),$$

but the term $\dot{x}_j$ does not affect for $\partial H/\partial x_i$, then we can write

$$\frac{\partial}{\partial x_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t)\Big\} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\Big\{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) + \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t)\Big\}$$

$$= -\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(\lambda_i \dot{x}_i)$$

$$= -\sum_{i=1}^{n} \dot{\lambda}_i = -\dot{\boldsymbol{\lambda}}.$$

The same idea is applicable to variable u_i yielding,

$$\frac{\partial}{\partial u_i}\Big\{\mathcal{L}(\boldsymbol{x},\boldsymbol{u},t) + \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x},\boldsymbol{u},t)\Big\} - \frac{d}{dt}\frac{\partial}{\partial \dot{u}_i}\Big\{\mathcal{L}(\boldsymbol{x},\boldsymbol{u},t) + \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x},\boldsymbol{u},t)\Big\}$$

$$= 0.$$

Then we can define a Lagrangian (also write by H)

$$H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\lambda},t) = \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x},\boldsymbol{u},t) + \mathcal{L}(\boldsymbol{x},\boldsymbol{u},t).$$

This is called **Pontryagin's H function**. The significance of this form is that the $H()$ does not contain $\dot{x}_i$ any longer, and we can use this function as a target of optimization.

14.5 Pontryagin's H function; Hamiltonian ?

By the result of Sec 14.3.3 and 14.4 we have,

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0 \quad or \quad \frac{\partial H}{\partial u_i} = 0, \quad (i = 1...r)$$

$$\frac{\partial H}{\partial \boldsymbol{x}} = -\dot{\boldsymbol{\lambda}} \quad or \quad \frac{\partial H}{\partial x_i} = -\dot{\lambda}_i, \quad (i = 1...n)$$

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\boldsymbol{x}} \quad or \quad \frac{\partial H}{\partial \lambda_i} = \dot{x}_i. \quad (i = 1...n)$$

If we look at the following equations,

$$\frac{\partial H}{\partial \boldsymbol{x}} = -\dot{\boldsymbol{\lambda}}, \qquad \frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\boldsymbol{x}},$$

we notice that they are quite similar to the Hamilton's equation. **Pontryagin's H function is a Lagrangian whose order of differential is reduced by the application of system function $\dot{\boldsymbol{x}} = f(\boldsymbol{x},\boldsymbol{u},t)$** and **Lagrange's multiplier method produced the solvable number of differential equations**.

Whereas in analytical mechanics, Hamiltonian is derived from Lagrangian by applying the Legendre transform. The Legendre transform works to reduce the order of differential equation, increasing the variables in the resultant Hamiltonian.

However, $\boldsymbol{\lambda}$ is a coefficient and has no physical meaning. Therefore, Pontryagin's H function is not a Hamiltonian in physical sense. But still, because of this similarity, some people call this H function as control Hamiltonian. (The sign of λ was intentionally inversed in Sec 14.3.1 to yield this similarity.)

14.6 Calculation of optimal system motion

What the equation

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0,$$

is telling us is that we have to minimize the system Lagrangian with respect to $\boldsymbol{u}$. And the equation is not a differential equation but rather just an arithmetic equation[5], we can simply use this equation to calculate $\boldsymbol{u}$ which we expect, at most,

$$\boldsymbol{u} = g(\boldsymbol{x}, \boldsymbol{\lambda}, t).$$

However, we assumed $\boldsymbol{u}$ to be completely independent from $\boldsymbol{x}$, we expect $\boldsymbol{u}$ is not a function of $\boldsymbol{x}$.

$$\boldsymbol{u} = \boldsymbol{u}_o = g(\boldsymbol{\lambda}, t).$$

Then, the optimized input $\boldsymbol{u}_o$ is used to calculate $\boldsymbol{x}$ and $\boldsymbol{\lambda}$.

During the process of calculation, the number of variables are increased to $2n + r$ when H is formed. Then the optimization of the $\boldsymbol{u}$,

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0,$$

reduced the number of variables to $2n$. And we have $2n$ equations with $2n$ variables, x_i and λ_i.

$$\frac{\partial H}{\partial \boldsymbol{x}} = -\dot{\boldsymbol{\lambda}}, \quad \frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\boldsymbol{x}}.$$

14.7 Examples of optimal system with input

14.7.1 Example 1

Suppose a system given by the following linear differential equation.

$$\dot{x} = -x + u.$$

This may be interpreted by an electronic circuit of an integrator with full feedback.

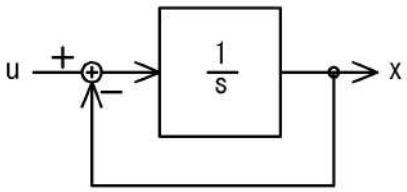

Figure 14.3: Integrator with full feedback

We use performance index,

$$PI = \int_0^1 (x^2 + u^2) dt.$$

[5]Remember that the purpose of canonical transformation is to produce $\partial \mathcal{H}/\partial x = 0$.

Then H function is,

$$H(x, u, \lambda, t) = \lambda(-x + u) + x^2 + u^2.$$

For the variable u,

$$\frac{\partial H}{\partial u} = \lambda u + 2u = 0.$$

The optimal input u_o is

$$u_o = -\frac{\lambda}{2}.$$

Note that the input u is not a function of x. The optimal H function becomes,

$$H_o(x, \lambda, t) = -x\lambda - \frac{\lambda^2}{2} + x^2 + \frac{\lambda^2}{4} = -x\lambda - \frac{\lambda^2}{4} + x^2.$$

Then,

$$\frac{\partial H_o}{\partial x} = -\dot{\lambda} = -\lambda + 2x.$$

$$\frac{\partial H_o}{\partial \lambda} = \dot{x} = -x - \frac{\lambda}{2}.$$

From the latter equation,

$$\ddot{x} = -\dot{x} - \frac{\dot{\lambda}}{2} = -\dot{x} - \frac{1}{2}\lambda + x.$$

Since

$$-\frac{1}{2}\lambda = \dot{x} + x,$$

we obtain

$$\ddot{x} - 2x = 0.$$

The solution is,

$$x = K_1 e^{-\sqrt{2}t} + K_2 e^{\sqrt{2}t},$$

$$\lambda = -2K_1(1 - \sqrt{2})e^{-\sqrt{2}t} - 2K_2(1 + \sqrt{2})e^{\sqrt{2}t}.$$

K_1, K_2 are determined by the given boundary condition. If we assume the target moves from $x(0) = 1$ to $x(1) = 0$, we obtain

$$K_1 = \frac{1}{1 - e^{-2\sqrt{2}}}, \qquad K_2 = \frac{1}{1 - e^{2\sqrt{2}}}.$$

In this case, the trajectories for $x(t)$ and $u(t)$ are shown in Fig 14.4.

****

In the above example, we used $\partial H/\partial \lambda = \dot{x}$, but we know this equation yields system equation in the discussion of Sec 14.3.3. Then,

$$\dot{x} = -x + u = -x + \frac{\lambda}{2},$$

is readily obtained.

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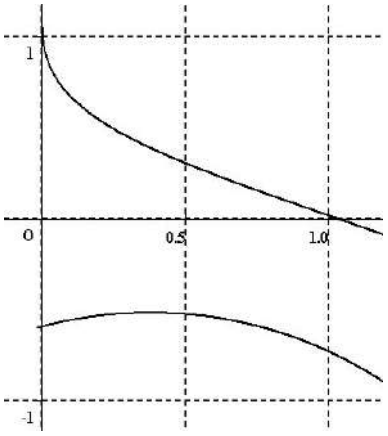

Figure 14.4: Responce of x(upper) and u(lower)

14.7.2 Example 2

For the same target as in Sec 14.7.2,

$$\dot{x} = -x + u.$$

the weight of the performance index is chosen,

$$PI = \int_0^1 \Big((10x)^2 + u^2\Big)dt.$$

We obtain the same optimal $u_o = \lambda/2$, Then,

$$\ddot{x} - 101x = 0,$$

which we approximate by,

$$\ddot{x} - 100x = 0.$$

The solution is,

$$x = K_1 e^{-10t} + K_2 e^{10t},$$

$$\lambda = -2K_1(-9)e^{-10t} - 2K_2(11)e^{10t}$$

If we assume the target moves from $x(0) = 1$ to $x(1) = 0$, we obtain

$$K_1 = \frac{1}{1 - e^{-20}}, \qquad K_2 = \frac{1}{1 - e^{20}}.$$

In this case, the trajectories for $x(t)$ and $u(t)$ are shown in Fig 14.5.

As the figure shows, the wight to the x is equivalent that we need less input power which happens quite often. For instance, if we think about CMOS inverter, the input impedance is very large (almost ∞) where output is inverted by the input. This tendency is shown by Fig 14.5.

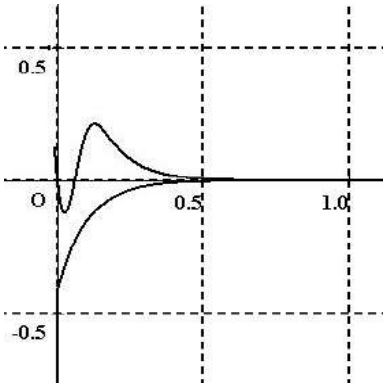

Figure 14.5: Responce of x(upper) and u(lower)

14.8 Summary

Optimization of an open loop system is identical to minimization of total system Lagrangian. Incorporation of input $\boldsymbol{u}$ to the target system increased the number of variables. Thus, the total Lagrangian of the system has excess number of variables than the number of equations, and makes equations unresolvable.

The application of Lagrange's undetermined variable method increased the number of equation so that the system become solvable again, yielding,

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = \sum_{j=1}^{n} \lambda_j \{f_j(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_j\} + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t).$$

The solution we are looking for is obtained by optimization - applying variational integral and simple differential to this H function. Eventually, we found out that $f_i(\boldsymbol{x}, \boldsymbol{u}, t) - \dot{x}_i$ can be replaced by $f_i(\boldsymbol{x}, \boldsymbol{u}, t)$, we defined the Pontryagin's H function

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = \sum_{j=1}^{n} \lambda_j f_j(\boldsymbol{x}, \boldsymbol{u}, t) + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t).$$

Although Pontryagin's H function behaves just like a Hamiltonian, this is a Lagrangian with zero order state variables only. Thus applying optimization to H function will yield the optimal motion of the system just as a Lagrangian yielded Newton's law in mechanics.

Optimization can be done in two ways. One is use variables $\boldsymbol{x}, \dot{\boldsymbol{x}}$ as found in actual design, or use $\boldsymbol{x}$ only by applying $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$ as found in theoretical discussion.

Chap15 Synthesis of multivariable system by Hamilton-Jacobi's equation

In Chap 14, we have discussed optimization of a system with input by Lagrange's undetermined multiplier method. In this chapter, we discuss optimized closed loop control systems by defining a performance function $W(\boldsymbol{x})$[1] which leads to Hamilton-Jacobi's equation. We will also show that **gradient of $W(\boldsymbol{x})$ will replace Lagrange's multiplier $\boldsymbol{\lambda}$**, the fundamental identicalness of Pontryagin's H function with Hamilton-Jacobi's equation, and the difference of $W(\boldsymbol{x})$ from $\boldsymbol{\lambda}$ yields a closed system solution.

We focus on linear systems eventually, but for generality's sake, we keep that the systems is expressed by $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ in this chapter.

Discussion in this chapter is based on Hamilton-Jacobi's equation, but starts discussing a Lagrangian based on system differential equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$. As stated in Section 14.2.3, the **Lagrangian is already converted into Hamiltonian.** This is the missing link between Lagrangian applied to a Hamilton-Jacobi's equation which should be based on Hamiltonian. We stick to the term Lagrangian hereafter.

15.1 Introduction

In Chap 14 we minimized the open loop system's Lagrangian. However, the input is not produced by the output and cannot make a closed loop control system.

The reason that we can only obtain open loop solution is in it's process. By the Lagrange's multiplier, implanting input and optimization is performed simultaneously. However, **generation of the input from the output is another matter which Lagrange's multiplier cannot provide** - since we didn't ask, we were not given. We need to ask by some contrivance.

15.2 Introduction of a new performance function

In Sec 14.2, we used a performance index

$$PI = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)dt.$$

This is interpreted to be consumed energy from time t_i to time t_f. Since we are dealing with real time control, initial and final time must not be fixed. Either initial or final time

[1] We will see that $\partial W/\partial t$ works as a generating function.

of integration must be replaced by variable t. Here, we change initial time from t_i to t and integrate to the final time t_f - the reason will be discussed below. Also we assume that we can **obtain control as a function of output**. Thus, we define a new performance function $W(\boldsymbol{x}, t)$ by,

$$W(\boldsymbol{x}, t) = \int_t^{t_f} \mathcal{L}\big\{\boldsymbol{x}(\tau), u(\boldsymbol{x}, \tau), \tau\big\} d\tau$$

$$= \mathcal{L}\big\{\boldsymbol{x}(\tau), u(\boldsymbol{x}, \tau), \tau\big\}\Big|_{\tau=t_f} - \mathcal{L}\big\{\boldsymbol{x}(\tau), u(\boldsymbol{x}, \tau), \tau\big\}\Big|_{\tau=t}.$$

Then, time differential of $W(\boldsymbol{x}, t)$ is

$$\dot{W}(\boldsymbol{x}, t) = -\mathcal{L}\big\{\boldsymbol{x}(t), u(\boldsymbol{x}, t), t\big\}.$$

Since the Lagrangian is a positive definite function, $\dot{W}(\boldsymbol{x}, t)$ becomes negative. This means that the magnitude of $W(\boldsymbol{x}, t)$ decreases as time goes by. If the time derivative of the performance index is positive, there cannot be optimal W. This is the reason that we set the start of integration by variable t.

What we had done here is,

"defined a new performance function W so that it contains variable x and the magnitude decreases as time goes by."

Since the performance function is defined based on a Lagrangian, we will find a new system equation by which we can eventually solve the closed loop optimal control.

15.3 System equation based on a new performance function

We defined a performance function W and calculated $\dot{W}$ in Sec15.2.

If we apply chain rule of differential to $W(\boldsymbol{x}, t)$,

$$\dot{W}(\boldsymbol{x}, t) = \frac{dW(\boldsymbol{x}, t)}{dt} = \sum_{i=1}^{n} \frac{\partial W(\boldsymbol{x}, t)}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t}$$

$$= \sum_{i=1}^{n} \frac{\partial W(\boldsymbol{x}, t)}{\partial x_i} \dot{x}_i + \frac{\partial W(\boldsymbol{x}, t)}{\partial t}.$$

The reason why we calculate time derivative of performance function $W(\boldsymbol{x}, t)$ instead of just using $\partial W(\boldsymbol{x}, t)/\partial t$, refer to Section6.2.1.

If we observe,

$$\sum_{i=1}^{n} \frac{\partial W(\boldsymbol{x}, t)}{\partial x_i},$$

we see that it is a gradient of function $W(\boldsymbol{x}, t)$. Also, substituting $\dot{x}_i = f_i()$, we obtain,

$$\dot{W}(\boldsymbol{x}, t) = \boldsymbol{\nabla} W(\boldsymbol{x}, t) \cdot \boldsymbol{f}\big\{\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t}.$$

Equating the result of Sec15.2 and that of chain rule,

$$\boldsymbol{\nabla} W(\boldsymbol{x}, t) \cdot \boldsymbol{f}\big\{\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t} = -\mathcal{L}\big\{\boldsymbol{x}(t), u(\boldsymbol{x}, t), t\big\}.$$

Or,

$$-\boldsymbol{\nabla} W(\boldsymbol{x},t)\cdot \boldsymbol{f}\big\{\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x},t),t\big\} = \mathcal{L}\big\{\boldsymbol{x}(t),u(\boldsymbol{x},t),t\big\} + \frac{\partial W(\boldsymbol{x},t)}{\partial t}.$$

If we observe

$$\mathcal{L}\big\{\boldsymbol{x}(t),u(\boldsymbol{x},t),t\big\} + \frac{\partial W(\boldsymbol{x},t)}{\partial t},$$

and recognize the similarity with

$$\mathcal{H}\big\{\boldsymbol{x}(t),\boldsymbol{u}(t),t\big\} + \frac{\partial \mathcal{W}(\boldsymbol{x},t)}{\partial t},$$

we can see that the equation we derived is a Lagrange's equation with external power. Further, both Lagrangian and Hamiltonian express energy of the system - fundamental meaning is identical - we can say that what we have obtained is a Hamilton-Jacobi's equation based on Lagrangian[2].

15.4 What became of Lagrange's multiplier?

In this section, we show that new performance function $W()$ works just like $\boldsymbol{\lambda}$ in Pontryagin's H function. In essence, Hamilton-Jacobi's equation is equal to Pntryagin's H function - **the former explicitly expresses spacial and time information separately whereas the latter mix them up into $\boldsymbol{\lambda}$**. If this is accepted, the following cumbersome discussion may be skipped.

Expressing the Hamilton-Jacobi's equation by J, we get

$$J(\boldsymbol{x},\boldsymbol{u},W,t) = \boldsymbol{\nabla} W(\boldsymbol{x},t)\cdot \boldsymbol{f}\big\{\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x},t),t\big\} + \mathcal{L}\big\{\boldsymbol{x}(t),u(\boldsymbol{x},t),t\big\} + \frac{\partial W(\boldsymbol{x},t)}{\partial t} = 0.$$

We first analyze spacial property of $J()$.
Thus, partially differentiate $J()$by x_i,

$$\begin{aligned}\frac{\partial J}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(\boldsymbol{\nabla} W(\boldsymbol{x},t)\cdot \boldsymbol{f}\big\{\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x},t),t\big\}\Big) + \frac{\partial^2 W}{\partial x_i \partial t} + \frac{\partial \mathcal{L}}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}\Big(\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}f_i\Big) + \frac{\partial^2 W}{\partial x_j \partial t} + \frac{\partial \mathcal{L}}{\partial x_i}\\ &= \sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_j \partial x_i}f_i + \sum_{i=1}^{n}\frac{\partial W}{\partial x_i}\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 W}{\partial x_j \partial t} + \frac{\partial \mathcal{L}}{\partial x_j} = 0.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_j \partial x_i}f_i + \frac{\partial^2 W}{\partial x_j \partial t} = -\Big(\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}\frac{\partial f_i}{\partial x_i} + \frac{\partial \mathcal{L}}{\partial x_j}\Big).$$

On the other hand, if we explore the behavior of $\nabla W_j = \partial W/\partial x_j$ into time direction,

$$\frac{d}{dt}\frac{\partial W}{\partial x_j} = \frac{d\boldsymbol{\nabla} W}{dt} = \sum_{i=1}^{n}\frac{\partial^2 W}{\partial x_i \partial x_j}f_i + \frac{\partial^2 W}{\partial t \partial x_j}$$

[2]But the variables in the Lagrangian is no longer $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$, but variables $\boldsymbol{x}$ and $\boldsymbol{u}(\boldsymbol{x},t)$.

$$= \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_j \partial x_i} f_i + \frac{\partial^2 W}{\partial x_j \partial t}$$

$$= -\Big(\sum_{i=1}^{n} \frac{\partial W}{\partial x_i} \frac{\partial f_i}{\partial x_i} + \frac{\partial \mathcal{L}}{\partial x_j} \Big).$$

Pontryagin's H function is defined by using Lagrange's multiplier $\boldsymbol{\lambda}$,

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = \sum_{j=1}^{n} \lambda_i f_i(\boldsymbol{x}, \boldsymbol{u}, t) + \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t).$$

The time progress of $\boldsymbol{\lambda}$ is expressed (see Sec 14.5),

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}}.$$

$$\dot{\lambda}_j = -\Big(\sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j} + \frac{\partial \mathcal{L}}{\partial x_j} \Big).$$

Thus, the gradient of $W()$ satisfies the differential equation of $\boldsymbol{\lambda}$, that is, time progress of $\boldsymbol{\lambda}$ is identical to that of gradient W. Or $\boldsymbol{\lambda}$ implicitly contains time progress of variables $\boldsymbol{x}$, but it can be only made explicit by $\boldsymbol{\nabla} W$. Therefore, we can replace $\boldsymbol{\lambda}$ by $\boldsymbol{\nabla} W$. The remarkable difference is that the $\boldsymbol{\nabla} W$ contains $\boldsymbol{x}$ explicitly.

15.5 Hamilton-Jacobi's equation

15.5.1 System expression by Hamilton-Jacobi's equation

Target function of which we minimize is written by,

$$H = \dot{W}(\boldsymbol{x}, t) = \boldsymbol{\nabla} W(\boldsymbol{x}, t) \cdot \boldsymbol{f}\big\{\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t}$$

$$= H\big\{\boldsymbol{x}(t), \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t} = 0.$$

The above function is a Hamilton-Jacobi's equation, but this is still a Lagrangian[3], and if we minimize - solve $\dot{W}(\boldsymbol{x}, t) = 0$, or solve this Hamilton-Jacobi's equation - the solution is what nature would do; move most energy economic way.

If we recall why Hamilton-Jacobi's equation was derived and what it means (see Sec 6.4), it is almost obvious that a control system is described by this way. Control systems are inherently time open, that is, we cannot specify the time of finish. The Hamilton's equation can only describe energy conserving systems **in which time t is an implicit parameter**. Hamilton-Jacobi's equation can describe non-energy conserving system **in which time t becomes an explicit parameter**.

The real control system must have the following characters.

(1) control signal is a function of time t and no final point is assumed.

[3]In the discussion of analytical mechanics, we reached to a conclusion that a Lagrangian and corresponding Hamiltonian are same physical entity, but just looked from different angle. In this sense, argument weather if it is Lagrangian or Hamiltonian is not important.

(2) control signal is generated by the system's output.

These situations are only coped with the Hamilton-Jacobi's equation. Both of the points are related with the generating function $W()$ - the introduction of the generating function is the key point in Hamilton-Jacobi's equation - and generating function does not affect the result of Hamilton's equation

$$\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}, \quad \frac{\partial \mathcal{H}}{\partial p} = \dot{q}.$$

This may be interpreted that the generating function is the energy source (or sink) to the system. **Since energy source is independent from the system, system equation should have the free choice of generating function.** The Hamilton-Jacobi's equation clearly states this relation; sum of system physics + energy source.

15.5.2 General procedure to solve Hamilton-Jacobi's equation

Since it is not practical to solve Hamilton-Jacobi's equation for general control system, here we only show the process of solution. The process is comprised by two steps. The Hamilton-Jacobi's equation

$$H\big\{\boldsymbol{x}(t), \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t} = 0,$$

yields optimal values for both $\boldsymbol{u}$ and $\boldsymbol{x}$ when it is solved.

In case of Lagrange's multiplier method, we obtained,

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0, \quad \frac{\partial H}{\partial \boldsymbol{x}} = \dot{\lambda}, \quad \frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\boldsymbol{x}}.$$

Comparing these equations, $\boldsymbol{\nabla} W$ replaces $\boldsymbol{\lambda}$, but $\partial H / \partial \boldsymbol{u} = 0$ still holds. Therefore, we first calculate optimal value $\boldsymbol{u}_o$ by the same procedure. Then replacing $\boldsymbol{u}$ by $\boldsymbol{u}_o$ in the Hamilton-Jacobi's equation yielding,

$$H\big\{\boldsymbol{x}(t), \boldsymbol{u}_o(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t} = 0.$$

If this equation is solved, we get $W(\boldsymbol{x}, t)$ and eventually,

$$\boldsymbol{u}(\boldsymbol{x}, t) = u_o(\boldsymbol{x}, \boldsymbol{\nabla} W(\boldsymbol{x}, t)).$$

15.6 Comparison of H function with Hamilton-Jacobi's equation

In case of open loop system, we used Pontryagin's H function which is derived by the optimization scheme by the Lagrange's undetermined multiplier. On the other hand, the Hamilton-Jacobi's equation is derived for closed loop control by defining a performance function based on Lagrangian.

In either case, optimization by the same condition ($\boldsymbol{x}$ and $\boldsymbol{u}$) is applied to the same target system. Thus it is natural that we obtained identical equation (Pontryagin's H function and Hamilton-Jacobi's equation). The only difference is that the relation of variable $\boldsymbol{u}$ and $\boldsymbol{x}$ is explicitly manifested or not - although this makes all the difference for control.

.

15.7 Design process of closed system

So far, we have shown that closed loop control system is analyzed by Hamilton-Jacobi's equation. Since Hamilton-Jacobi's equation is written by first order differential equation, the order of the variables in the original Lagrangian is reduced by the system expression $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Optimal control is realized by optimizing the Lagrangian, but we will show in Chap 16 that the Lagrangian is replaced by a quadratic performance function (index) which eventually leads to find a solution of Ricatti equation.

It may be said that long theoretical discussion to Hamilton-Jacobi's equation was intended to show an exit via Ricatti equation.

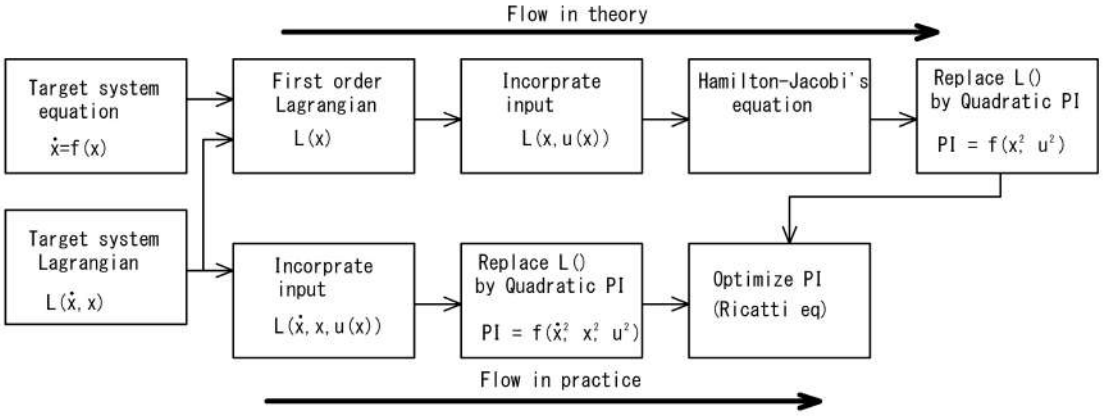

Figure 15.1: Designe process of closed system

Threfore, in actual design process, we can reach to the final Ricatti equation without taking trouble to worry about Hamilton-Jacobi's equation. As shown in Fig 15.1, we just form system Lagrangian of variables $\dot{\boldsymbol{x}}, \boldsymbol{x}$ and $\boldsymbol{u}$. Then replace the Lagrangian by quadratic performance function[4] and optimize. The target of optimization becomes a multivariable quadratic function, thus the work becomes finding a solution of Ricatti equation.

15.8 Summary of closed system

We have seen that a closed loop system is expressed by,

$$J(\boldsymbol{x}, \boldsymbol{u}, W, t) = \boldsymbol{\nabla} W(\boldsymbol{x}, t) \cdot \boldsymbol{f}\big\{\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}, t), t\big\} + \mathcal{L}\big\{\boldsymbol{x}(t), u(\boldsymbol{x}, t), t\big\} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t} = 0.$$

If we interpret $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, W, t)$ expresses current status, and Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)$ the system asset, what Hamilton-Jacobi's equation is telling us are,

(1) Function $W(\boldsymbol{x}, t)$ is working as a basis to determine next movement.
$\boldsymbol{\nabla} W(\boldsymbol{x}, t)$ determines spacial movement based on current location $\boldsymbol{f}$.
$\partial W(\boldsymbol{x}, t)/\partial t$ determines time movement based on universal time.
(time is shift invariant ; there is no base time.)

(2) Initial spacial point is provided by $\boldsymbol{f}$.

[4]Mathematically, it may not be identical, but for optimization it works.

(3) When the system is non linear, the non-linearity manifests by spacial value calculation ($\boldsymbol{\nabla} W \boldsymbol{f}$).

We assumed $\boldsymbol{u}$ is generated by $\boldsymbol{x}$ - that is, calculate $\boldsymbol{u}$ requires $\boldsymbol{x}$ and vice versa. This chain is cut by first calculating optimal $\boldsymbol{u}$ and then calculate $\boldsymbol{x}$ - that is, we take advantage that the system function

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0, \quad \frac{\partial H}{\partial \boldsymbol{x}} = \dot{\lambda}, \quad \frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\boldsymbol{x}},$$

has three equations. One excessive equation which does not express physics of the system

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0,$$

is used to calculate optimized input.

Chap16 Matrix Ricatti equation

In this chapter, we discuss linear, time invariant and controllable system in which performance index is quadratic. As the result shows, a linear system control problem yields a nonlinear equation ; Ricatti equation. We will show where this happened in the following section as well.

The following discussion includes matrix calculation exploiting matrix characteristics. The manipulation process is to ensure the logical continuity, and hence the process of deriving Ricatti equation is understood, the details can be forgotten.

16.1 Performance index

In Sec 11.5, we discussed that a quadratic function

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x},$$

can express generalized energy of a system with variables $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}$ where $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.

Since we have input $\boldsymbol{u}$, we use

$$PI = \int (\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})dt,$$

as a performance function of a generalized system.

Therefore, if PI is minimized, we call the control optimal.

Now, we are going to replace the Lagrangian in Hamilton-Jacobi's equation by a quadratic form of $\boldsymbol{x}$ and $\boldsymbol{u}$,

$$\mathcal{L} = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}.$$

Note that $\boldsymbol{Q}$ and $\boldsymbol{R}$ are square matrices by $n \times n$ and $m \times m$ respectively.

16.2 Hamilton-Jacobi's equation for linear system

In Chap 15, we derived the following Hamilton-Jacobi's equation.

$$J\big\{\boldsymbol{x}(t), \boldsymbol{u}(t), W(x,t), t\big\} = \boldsymbol{\nabla} W(\boldsymbol{x},t) \cdot \boldsymbol{f}\big\{\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x},t), t\big\} + \mathcal{L}\big\{\boldsymbol{x}(t), \boldsymbol{u}(\boldsymbol{x},t), t\big\}$$

$$+\frac{\partial W(\boldsymbol{x},t)}{\partial t}.$$

Since we assume linear systems,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$$

where $\boldsymbol{x}$, $\boldsymbol{u}$ are n, m ($n \geq m$) dimensional vectors respectively; $\boldsymbol{A}, \boldsymbol{B}$ are $n \times n, n \times m$ matrices respectively. By an explicit notation,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ ... \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11}, a_{12}, ..., a_{1n} \\ a_{21}, a_{22}, ..., a_{2n} \\ ... \\ a_{n1}, a_{n2}, ..., a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11}, ..., b_{1m} \\ b_{21}, ..., b_{2m} \\ ... \\ b_{n1}, ..., b_{nm} \end{pmatrix} \begin{pmatrix} u_1 \\ ... \\ u_m \end{pmatrix}.$$

Replacing both the system equation $\boldsymbol{f}\{\}$ and Lagrangian $\mathcal{L}\{\}$, the Hamilton-Jacobi's equation becomes,

$$J() = \boldsymbol{\nabla} W^T(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}) + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + \frac{\partial W(\boldsymbol{x}, t)}{\partial t}.$$

Note that $\boldsymbol{\nabla} W^T$ is an $n \times 1$ vector[1], $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ is a $1 \times n$ vector, then $\boldsymbol{\nabla} W^T(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u})$ is a scaler function of $\boldsymbol{x}, \boldsymbol{u}$. The quadratic form is a scaler function of $\boldsymbol{x}, \boldsymbol{u}$ and $\partial W(\boldsymbol{x}, t)/\partial t$ is a scaler function of t, thus $J()$ is a scaler function.

16.3 Optimized Hamilton-Jacobi's equation for linear system

Optimized solution is calculated by the following way.
(1) Optimize for input $\boldsymbol{u}$ (elimination of $\boldsymbol{u}$).
(2) Form Hamilton-Jacobi's equation of state variable $\boldsymbol{x}$.
(3) Apply quadratic performance index to Hamilton-Jacobi's equation.
(4) Solve Ricatti equation.

16.3.1 Optimization of input

As we have seen in Chap 15, Hamilton-Jacobi's equation is a function of $\boldsymbol{x}, \boldsymbol{u}$ and $\boldsymbol{\nabla} W$. And we need to reduce the number of variables for final solution. First variable we reduce is $\boldsymbol{u}$ by optimization.

$$\frac{\partial J}{\partial \boldsymbol{u}} = \frac{\partial(\boldsymbol{\nabla} W^T\boldsymbol{B}\boldsymbol{u})}{\partial \boldsymbol{u}} + \frac{\partial(\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})}{\partial \boldsymbol{u}} = 0.$$
$$\boldsymbol{B}^T\boldsymbol{\nabla} W + 2\boldsymbol{R}\boldsymbol{u} = 0.$$

Then, the optimal input $\boldsymbol{u}_o$ is,

$$\boldsymbol{u}_o = -\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\nabla} W.$$

Since $\boldsymbol{R}$ is a symmetric matrix, the transpose of $\boldsymbol{u}_o$ is,

$$(\boldsymbol{u}_o)^T = -\frac{1}{2}\boldsymbol{\nabla} W^T(\boldsymbol{R}^{-1}\boldsymbol{B}^T)^T = -\frac{1}{2}\boldsymbol{\nabla} W^T\boldsymbol{B}(\boldsymbol{R}^{-1})^T = -\frac{1}{2}\boldsymbol{\nabla} W^T\boldsymbol{B}\boldsymbol{R}^{-1}.$$

Therefore,

$$\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} = -\frac{1}{2}\boldsymbol{\nabla} W^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{R}\big(-\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\nabla} W\big)$$

[1]Row vector. To distinguish row and column vectors, we use "T".

$$= \frac{1}{4}\nabla W^T \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{B}^T\nabla W$$

$$= \frac{1}{4}\nabla W^T \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\nabla W.$$

Thus we have eliminated $\boldsymbol{u}$ and derived a function of $\boldsymbol{x}$ for optimized input $\boldsymbol{u}_o$.

16.3.2 Hamilton-Jacobi's equation of state variables

The optimal Hamilton-Jacobi's equation is expressed by

$$J_o() = \nabla W^T(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}) + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + \frac{\partial W(\boldsymbol{x},t)}{\partial t}$$

$$= \nabla W^T\boldsymbol{A}\boldsymbol{x} - \frac{1}{2}\nabla W^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\nabla W + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$$

$$+\frac{1}{4}\nabla W^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\nabla W + \frac{\partial W(\boldsymbol{x},t)}{\partial t}$$

$$= \nabla W^T\boldsymbol{A}\boldsymbol{x} - \frac{1}{4}\nabla W^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\nabla W + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{\partial W(\boldsymbol{x},t)}{\partial t} = 0.$$

Since we know we can use quadratic performance index (Sec 16.1), we assume $W()$ is quadratic form of $\boldsymbol{x}$.

$$W(t) = \boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{x}.$$

Where, $\boldsymbol{P}(t)$ is a coefficient matrix. Then,

$$\nabla W = \frac{\partial W}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{x}^T}{\partial \boldsymbol{x}}\boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{x}^T\frac{\partial}{\partial \boldsymbol{x}}\big(\boldsymbol{P}(t)\boldsymbol{x}\big)$$

$$= \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{P}(t)^T\boldsymbol{x} = 2\boldsymbol{P}(t)\boldsymbol{x}.$$

And $\partial W()/\partial t$ becomes also quadratic (see Sec 24.2.5).

$$\frac{\partial W}{\partial t} = \boldsymbol{x}^T\dot{\boldsymbol{P}}(t)\boldsymbol{x}.$$

Then, the Hamilton-Jacobi's equation becomes,

$$2\boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^T\dot{\boldsymbol{P}}(t)\boldsymbol{x} = 0,$$

$$\boldsymbol{x}^T\{2\boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} + \dot{\boldsymbol{P}}(t)\}\boldsymbol{x} = 0.$$

Since this equation holds for all $\boldsymbol{x}$,

$$2\boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} + \dot{\boldsymbol{P}}(t) = 0.$$

16.3.3 Emerging of Ricatti equation

We exploit characters of quadratic matrix here. In quadratic form, only symmetrical part matters. Observing the following equation

$$2\boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} + \dot{\boldsymbol{P}}(t) = 0,$$

the first term is not symmetric. Therefore, if we manipulate the first term,

$$sym\{2\boldsymbol{P}(t)\boldsymbol{A}\} = 2\frac{\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t)}{2} = \boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t).$$

Then, we have

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} + \dot{\boldsymbol{P}}(t) = 0.$$

This equation is a first order differential equation of $\boldsymbol{P}(t)$ in which quadratic form of $\boldsymbol{P}(t)$ is comprised. This is called a **Ricatti equation**.

In practical control design, the performance index has a character

$$\lim_{t\to\infty} \dot{\boldsymbol{P}}(t) = 0,$$

because we design the system to be stable; if the performance index still fluctuates after some time, it means the system fluctuates as well. Therefore, the differential Ricatti equation becomes algebraic Ricatti equation by equaling $\dot{\boldsymbol{P}}(t) = 0$.

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} = 0.$$

Once the solution of the algebraic Ricatti equation is obtained, the optimal input is generated by $\boldsymbol{x}$.

$$\boldsymbol{u}_0(\boldsymbol{x}, t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x}.$$

For the closed loop system, the coefficient matrix $\boldsymbol{K}(t)$ of the feedback component is,

$$\boldsymbol{u}_0(\boldsymbol{x}, t) = -\boldsymbol{K}^T(t)\boldsymbol{x},$$

$$\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}.$$

16.4 Simplified process to Ricatti equation

For most of the people, matrix manipulation is not a daily routine, and manipulation above may have dazzling effect. We have stated that a linear system produced a nonlinear equation. In order to clarify where and why this happened, we show a simplified (single variable, no vector, no matrix) version of the process here, but mind that this simplification is to provide bird's eye view only.

The system Hamilton-Jacobi's equation is,

$$J() = \frac{\partial W}{\partial x}(Ax + Bu) + Qx^2 + Ru^2 + \frac{\partial W}{\partial t}.$$

We first calculate the optimized input u_o.

$$\frac{\partial J}{\partial u} = B\frac{\partial W}{\partial x} + 2Ru = 0,$$

$$u_o = -\frac{1}{2R}B\frac{\partial W}{\partial x}.$$

Then the optimized Hamilton-Jacobi's equation is,

$$J_o() = \frac{\partial W}{\partial x}Ax - \frac{1}{4R}\Big(B\frac{\partial W}{\partial x}\Big)^2 + Qx^2 + \frac{\partial W}{\partial t} = 0.$$

Assuming that W is a quadratic function, we write,

$$W = P(t)x^2.$$

Then,

$$\frac{\partial W}{\partial x} = 2P(t)x,$$
$$\frac{\partial W}{\partial t} = \dot{P}(t)x^2.$$

Replacing $\partial W/\partial x, \partial W/\partial t$ in the Hamilton-Jacobi's equation, we obtain,

$$J_o() = 2AP(t)x^2 - \frac{1}{4R}\Big(B2P(t)x\Big)^2 + Qx^2 + \dot{P}(t)x^2 = 0,$$

$$2AP(t) - \frac{1}{R}B^2P(t)^2 + Q + \dot{P}(t) = 0.$$

Now, by this simplified process we can see what happened. During the process to arrive at[2] Ricatti equation, system variable u and input x disappears completely[3]. And **the purpose of the resultant non linear differential equation is to solve coefficient $P(t)$ of quadratic performance index.**

We were not solving the system differential equation !

16.5 Linear quadratic Regulator

We started from a general expression of a system by $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, avoiding to solve Hamilton-Jacobi's equation by constraining the system to be linear $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, and finally arrived at an algebraic Ricatti equation.

These results can be very succinctly written down as a linear quadratic regulator problem. A linear quadratic regulator is a linear multivariable feed back system using a quadratic cost function. As shown in Fig 16.1, this is a multivariable version of simple feedback control. The difference is that the output is multivariable, and all the outputs are fed back in which determining the coefficient of each variable is the problem.

In this discussion of linear quadratic regulators, we assume that all the state variables are observable, and all the inputs can affect the system (controllable).

The problem is stated as follows.

[2]Can one foresee that we eventually get Ricatti equation from Hamilton-Jacobi's equation ?
[3]u disappeared by optimization, but why we get x^2 in common ?

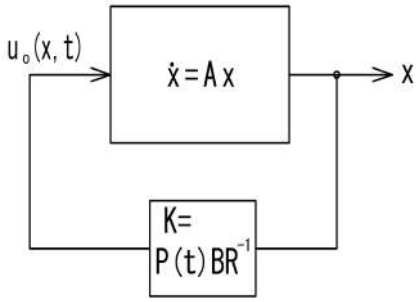

Figure 16.1: Linear quadratic regulator

Given a linear system,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0.$$

find a feed back vector $\boldsymbol{K}$

$$\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1},$$

which minimizes the quadratic cost function[4]

$$\boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{x} = \int_0^\infty (\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})dt.$$

Since $\boldsymbol{B}, \boldsymbol{R}$ are design parameters, the problem is to find $\boldsymbol{P}(t)$, where $\boldsymbol{P}(t)$ is a unique solution of,

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} = 0.$$

The optimized input u_o is,

$$u_0 = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x}(t) = -\boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1} = -\boldsymbol{K}_o\boldsymbol{x}(t)$$

Then,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{x}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o)\boldsymbol{x}(t).$$

which has the general solution,

$$\boldsymbol{x}(t) = e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}_o)t}\boldsymbol{x}_0.$$

We want to make the system stable, state variables $\boldsymbol{x}$ must stay within some limits; not in the sense of mathematical stable, state variables must decay even if oscillatory. For this requirement the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o$ must be negative, that is, all the eigenvalues must have negative real parts.

As a consequence,

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_o,$$

$$\lim_{t\to\infty} e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}_o)} = 0.$$

[4]To be precise, functional.

16.6 Summary

What have been done in the tedious calculations are,

- Assume the target system to be linear,
- Performance index is generally a concave function, and we assume it to be quadratic,
- Focus on the energy consumption of system and assume input variables quadratic,
- Form Hamilton-Jacobi's equation for the total energy consumption,
- Eliminate input variables by optimization (normalize into $W(\boldsymbol{x}, t)$),
- Obtain Hamilton-Jacobi's equation with system variable $\boldsymbol{x}$ only,
- Exploit matrix characters (symmetry),
- Ricatti equation (quadratic differential equation) is yielded.
- By the requirement that the system is stable, differential Ricatti equation turns into algebraic Ricatti equation.

Since $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ tells nothing concrete, linear system assumption has a great effect to move the discussion forward by which we obtained a solvable Ricatti equation.

Chap17 Calculation of Riccati equation

In this chapter, we focus on a computer calculation of algebraic matrix Ricatti equation. By computer calculation, continuous Ricatti equation is converted to discrete Ricatti equation by sampling. Here, we do not get into the discussion how sampling period is determined, nor the algorithm itself. For practical calculation, MATLAB[1] or SLICOT(Subroutine Library In COntrol Theory) [35] are generally used.

17.1 Methods of calculation

Algorithms to compute algebraic Ricatti equation belong to two groups. One is numerically robust algorithms that uses Hamiltonian matrix. By this algorithm, it is assured that we can get the solution if it exists, although computationally expensive - takes time. It may not be suitable for the real time applications, but it is valuable to provide a reference to iterative method. Schur method belongs to this type.

The other type is to compute the solution by iteration. In this type, we have to give a guess of the solution, and the algorithm calculates iteratively starting by the guess. The computational cost is much smaller and can be applied to real time control. However, there is a possibility that the algorithm does not converge to the solution. Newton's method, Newton's method with exact line search and Kleinman algorithm belong to this type.

For time invariant multivariable control (when Ricatti equation is time invariant), calculation of Ricatti equation can be done off line and speed will present no problem. However, for a (nonlinear) system that has time varying Ricatti equation, real time solution becomes necessary. For those applications, both methods may be used because of the assured solution availability and computational speed. That is, initial value is computed by non-iterative method and iterative algorithm provides solution for changed state variables.

As of computational speed, it is reported [41] that the Kleinman algorithm performs best.

There is another research that by applying Saidi's successive approximation, linear quadratic optimal control problem is solved for any initial control law. In this method, a Lyapunov equation is successively solved instead of Ricatti equation [38].

Here we only focus basic aspects for solution of the Ricatti equation.

17.2 Character of solution

There are several points that should be understood before the discussion of solutions.

[1] A registered trademark of The MathWorks, Inc.

Ricatti equation has multiple solutions. Since we use the solution to control, the solution must make the target system stable - that is, a symmetric positive definite solution[2] of the Ricatti equation is required. And it is known that there is one unique solution that gives this condition if the solution exists [29]. This solution makes the eigenvalue of the closed loop system state matrix in the left s-plane.

Therefore, the algorithm to solve Ricatti equation is accompanied with functions to choose the right answer. Also, in case of iterative method, functions to detect non-convergence and iteration cycle limiter (this can improve the efficiency of the algorithm) as well.

In iterative method, we have to provide a guess as the starting value. But the iteration may fail in two ways. One is the non-convergence and the other is convergence to local minimum (convergence to wrong value). Non-convergence failure is easy to detect and cope with. Wrong value is detected by comparing the right value which is calculated by non-iterative method.

Computational speed is proportional to the number of state variable n by Kn^3 where K is a coefficient. By the Schur method, K is about 75, whereas by the Kleinman's method, K is 6 [41].

17.3 Selection of $\boldsymbol{Q}, \boldsymbol{R}$

The coefficient matrices $\boldsymbol{Q}, \boldsymbol{R}$ of assumed performance index

$$PI = \int_0^\infty (\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}) dt,$$

are design parameters and left designers to select. Since $\boldsymbol{Q}$, $\boldsymbol{R}$ are weighting coefficients for state variables and inputs respectively, they shall reflect the designer's intention[3]. However, there is a guideline [41],

(1)Select $\boldsymbol{R}$ diagonal simplifies calculation,
(weighting to each variable becomes independent with each other.)

(2) $\boldsymbol{Q}, \boldsymbol{R}$ to be diagonal in which,
(2-1) i_{th} diagonal entry of $\boldsymbol{Q}^{-1}$ as n times the maximum acceptable value of $x_i^2(t)$,
(2-2) i_{th} diagonal entry of $\boldsymbol{R}^{-1}$ as m times the maximum acceptable value of $u_i^2(t)$,
where, n and m are the number of states and inputs respectively.

17.4 Non-iterative method

Detailed description on how to solve Ricatti equation is beyond our scope. Here we only describe rough process to show what kind of matrix calculation is necessary.

Continuous time Ricatti differential equation is written by,

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T \boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P}(t) + \boldsymbol{Q} = -\dot{\boldsymbol{P}}(t)$$

[2]Since we are seeking for the minimum of performance index, this is obvious.
[3]For instance, imagine gain assignment for a multistage amplifier, either open loop or feedback types.

where $\boldsymbol{A}; (n \times n)$, $\boldsymbol{B}; (n \times m)$, $\boldsymbol{Q}; (n \times n)$ and $\boldsymbol{R}; (m \times m)$ are constant matrix.

We define a $2n \times 2n$ matrix M

$$M = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{pmatrix},$$

which is called a Hamiltonian matrix.

We define two $n \times n$ square matrices $\boldsymbol{F}, \boldsymbol{G}$ by

$$\begin{pmatrix} \dot{\boldsymbol{F}}(t) \\ \dot{\boldsymbol{G}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{F}(t) \\ \boldsymbol{G}(t) \end{pmatrix}.$$

Or,

$$\dot{\boldsymbol{F}}(t) = \boldsymbol{A}\boldsymbol{F}(t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{G}(t),$$
$$\dot{\boldsymbol{G}}(t) = -\boldsymbol{Q}\boldsymbol{F}(t) - \boldsymbol{A}^T\boldsymbol{G}(t).$$

If

$$\boldsymbol{G}(t) = \boldsymbol{P}(t)\boldsymbol{F}(t),$$

then, by chain rule of differential,

$$\dot{\boldsymbol{G}}(t) = \dot{\boldsymbol{P}}(t)\boldsymbol{F}(t) + \boldsymbol{P}(t)\dot{\boldsymbol{F}}(t)$$
$$= \dot{\boldsymbol{P}}(t)\boldsymbol{F}(t) + \boldsymbol{P}(t)\Big(\boldsymbol{A}\boldsymbol{F}(t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{G}(t)\Big)$$
$$= \dot{\boldsymbol{P}}(t)\boldsymbol{F}(t) + \boldsymbol{P}(t)\boldsymbol{A}\boldsymbol{F}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{G}(t)$$
$$= \dot{\boldsymbol{P}}(t)\boldsymbol{F}(t) + \boldsymbol{P}(t)\boldsymbol{A}\boldsymbol{F}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{F}(t).$$

We obtain,

$$-\boldsymbol{Q}\boldsymbol{F}(t) - \boldsymbol{A}^T\boldsymbol{G}(t) = -\boldsymbol{Q}\boldsymbol{F}(t) - \boldsymbol{A}^T\boldsymbol{P}(t)\boldsymbol{F}(t)$$
$$= \dot{\boldsymbol{P}}(t)\boldsymbol{F}(t) + \boldsymbol{P}(t)\boldsymbol{A}\boldsymbol{F}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{F}(t).$$

which is the Ricatti equation,

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} = -\dot{\boldsymbol{P}}(t).$$

Thus, the solution will be

$$\boldsymbol{P}(t) = \boldsymbol{G}(t)\boldsymbol{F}(t)^{-1}.$$

Schur decomposition method is most commonly used to solve Ricatti equation in which the Hamiltonian matrix is used.

17.5 Iterative method

Here we describe the Kleinman's method only. The Kleinman's iterative method is shown in his theorem[39] which is an improvement of Newton's method.

[Kleinman 's Theorem]

Let $\boldsymbol{V}_k,\ k = 0, 1, 2, ...$ be the $n \times n$ (unique) positive definite matrix solution of the linear algebraic matrix equation,

$$0 = \boldsymbol{A}_k^T \boldsymbol{V}_k + \boldsymbol{V}_k \boldsymbol{A}_k + \boldsymbol{Q} + \boldsymbol{L}_k^T \boldsymbol{R} \boldsymbol{L}_k,$$

where, recursively,

$$\boldsymbol{L}_k = \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{V}_{k-1}. \qquad k = 1, 2, 3, ...$$

$$\boldsymbol{A}_k = \boldsymbol{A} - \boldsymbol{B} \boldsymbol{L}_k. \qquad k = 0, 1, 2, ...$$

and where L_o is chosen such that the matrix $\boldsymbol{A}_o = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{L}_o$ has eigenvalues with negative real parts. Then,

$$(1) \boldsymbol{P} \leq \boldsymbol{V}_{k+1} \leq \boldsymbol{V}_k \leq ... \qquad k = 0, 1, 2, ...$$

$$(2) \lim_{k \to \infty} \boldsymbol{V}_k = \boldsymbol{P}$$

For an actual calculation of solution, several items are checked [40].

(1) The system controllability : check $\boldsymbol{A}, \boldsymbol{B}$ are controllable.

(2) Initial guess stability check : check eigenvalues of the closed loop system control matrix.

(3) Positive definite solution check : the solution should be positive definite.

17.6 Summery

Solution of Ricatti equation is easily calculated by using MATLAB or SLICOT for simulation or off line in the present days. Also, for relatively small dimensional systems (say, up to 10), the Ricatti equation can be solved real time by microprocessors. This means that applications to autonomous nonlinear multivariable systems - like autonomous air born vehicles [43] - are getting feasible by using State Dependent Ricatti Equation.

However, requirements of control never get smaller and the number of variables to be controlled is increasing. And small scale air born systems like "drones" are expanding which means the research of algorithm to calculate Ricatti equation will be continued.

Chap18 Inverted pendulum

Inverted pendulum presents a standard problem in automatic control theory either in classic or multivariable control. Segway[1] is one of the good examples of an application of inverted pendulum, and humanoid robot is another.

The control design of an inverted pendulum can be treated by classic control, but here we apply the multivariable control theory. Also, the problem can be treated by root locus problem of the transfer function or LQR (linear quadratic regulator) problem. We discuss the problem by the LQR method focusing on the process and philosophy rather than actual implementation.

As discussed in Sec 15.7, we express the system Lagrangian by variables $\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{u}$, replace the Lagrangian by quadratic form and solve the problem by optimization ([46],[47] etc.).

18.1 Physics of inverted pendulum

Tops can spin by either upright position or inverted position. They have two dynamically stable position either of which is attained only while spinning.

Figure 18.1: Upright spin

Figure 18.2: Inverted spin

This is the same with pendulum with rigid rod. They have two stable points; one is where the mass is lowermost point and the other is uppermost point. The latter cannot be stable without control but we know that by supporting with our hand, it can be kept on standing - although maybe not for a long time.

A system that realizes this control automatically is called an inverted pendulum.

Interesting difference of normal pendulum and inverted pendulum is that the former basically realizes an inherently oscillatory system - a system with continuous energy exchange between potential and kinetic energy and thus, it would be impossible to move fulcrum without causing oscillation if there is no friction.

[1] Trademark of Segway, inc.

For the inverted pendulum, the potential energy may be transfered to kinetic energy but not the other way round without control. However, once control makes upright position stable, energy exchange between potential energy and kinetic energy becomes bi-directional. Or, the purpose of the control is to produce normal pendulum environment - bi-directional energy exchange between kinetic and potential energy - in an inverted environment.

18.2 Mechanical structure

Typical inverted pendulum is constructed with a rod which is attached to a cart as shown in Fig18.3. M, m denote the mass of cart and rod respectively. The gravitational force works at center of gravity G of the rod to where the length is L. Force f is applied to the cart so that the rod stays still at the upright position. Important assumption is that the rod is rigid and will not cause resonance within control frequency range. The cart is driven by a motor, and we assume $f = K_m u$ where K_m is a coefficient of motor and u is input voltage[2].

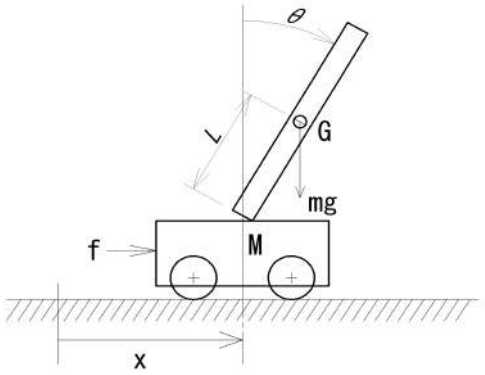

Figure 18.3: Structure of inverted pendulum

18.3 Analysis of the system dynamics

The objective is to control x by measuring the angle θ of the rod so that $\theta = 0$ is maintained.

The process of analysis is

- Assume that the system is lossless and apply Euler-Lagrange's equation to derive equation of motion in regard with x and θ,
- Linearize the system,
- Introduce friction factor to the cart (linear motion) and rod (angular motion).

[2]We assume common DC motor which has this simple characteristic.

18.3.1 Equation of motion of inverted pendulum

We use both Cartesian and polar coordinates for the inverted pendulum; x, y and r, θ. The cart has x component only, and the rod has y and θ component, however none has component r.

The Lagrangian of the inverted pendulum is

$$\mathcal{L} = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\Big(\frac{d}{dt}(x + L\sin\theta)\Big)^2 + \frac{1}{2}m\Big(\frac{d}{dt}(L\cos\theta)\Big)^2 + \frac{1}{2}M\dot{x}^2 - mgL\cos\theta$$

$$= \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}L\cos\theta\cdot\dot{\theta} + L^2\dot{\theta}^2) - mgL\cos\theta,$$

where J is moment of inertia at the center of gravity G. Euler-Lagrange's equation becomes,

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\Big) - \frac{\partial\mathcal{L}}{\partial\theta} = 0,$$

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big) - \frac{\partial\mathcal{L}}{\partial x} = f.$$

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\Big) = \frac{d}{dt}\Big(J\dot{\theta} + m\dot{x}L\cos\theta + mL^2\dot{\theta}\Big)$$

$$= J\ddot{\theta} + mL^2\ddot{\theta} + m\ddot{x}L\cos\theta - m\dot{x}\sin\theta\cdot\dot{\theta}.$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -m\dot{x}\sin\theta\cdot\dot{\theta} + mgL\sin\theta,$$

Thus, we obtain,

$$(J + mL^2)\ddot{\theta} + m\ddot{x}L\cos\theta - mgL\sin\theta = 0.$$

Similarly,

$$(M + m)\ddot{x} + mL\cos\theta\cdot\ddot{\theta} - mL\sin\theta\cdot\dot{\theta}^2 = K_m u.$$

Assuming that $\theta << 1$ ($\cos\theta = 1, \sin\theta = \theta$, $\dot{\theta}^2 = 0$) and introducing friction term $K_r\dot{\theta}$ and $K_c\dot{x}$,

$$mL\ddot{x} + (J + mL^2)\ddot{\theta} + K_r\dot{\theta} - mgL\theta = 0,$$

$$(M + m)\ddot{x} + mL\ddot{\theta} + K_c\dot{x} = K_m u.$$

From the second equation, $\ddot{\theta}$ is

$$\ddot{\theta} = \frac{1}{mL}\Big(K_m u - K_c\dot{x} - (M + m)\ddot{x}\Big).$$

Substituting $\ddot{\theta}$ in the first equation, we obtain

$$\ddot{x} = \frac{1}{W}\Big((J + mL^2)K_m u - (J + mL^2)K_c\dot{x} + mLK_r\dot{\theta} - m^2L^2g\theta\Big),$$

where $W = (M + m)J + mML^2$.

Likewise,

$$\ddot{\theta} = \frac{1}{W}\Big(-mLK_m u + mLK_c\dot{x} - (M + m)K_r\dot{\theta} + (M + m)mLg\theta\Big).$$

In matrix form,

$$\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \frac{1}{W}\left\{ \begin{pmatrix} -m^2L^2g & -(J+mL^2)K_c & mLK_r \\ (M+m)mgL & mLK_c & -(M+m)K_r \end{pmatrix} \begin{pmatrix} \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} (J+mL^2)K_m \\ -mLK_m \end{pmatrix} u \right\}.$$

18.3.2 System equation of inverted pendulum

In Sec 18.3.1, a second order differential equation is derived. But we need to express by a set of first order differential equation so that the system is framed in

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}.$$

To that end, we increase the number of state variables from x, θ to $x, \dot{x}, \theta, \dot{\theta}$ resulting,

$$\dot{\boldsymbol{x}}(\dot{x}, \dot{\theta}, \ddot{x}, \ddot{\theta}) = \boldsymbol{A}\boldsymbol{x}(x, \theta, \dot{x}, \dot{\theta}) + \boldsymbol{B}\boldsymbol{u}.$$

In our concrete case,

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \end{pmatrix} \begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{pmatrix} u,$$

where,

$$a_1 = -\frac{m^2L^2g}{W}, \quad a_2 = -\frac{(J+mL^2)K_c}{W}, \quad a_3 = \frac{mLK_r}{W},$$
$$a_4 = \frac{(M+m)mgL}{W}, \quad a_5 = \frac{mLK_c}{W}, \quad a_6 = -\frac{(M+m)K_r}{W},$$
$$b_3 = \frac{(J+mL^2)K_m}{W}, \quad b_4 = -\frac{mLK_m}{W}.$$

In the above first order[3] differential equation, trivial equations,

$$\dot{x} = \dot{x}, \;\; \dot{\theta} = \dot{\theta},$$

are incorporated.

What is done so far is to express the second order differential equation by the framework of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$, because we need the matrix $\boldsymbol{A}$ to calculate optimal control. Note that no order reduction of differential equation is applied.

[3]Regarding $\dot{x}$ and $\dot{\theta}$ as independent variables, it fits in to first order differential equation.

18.4 Linear quadratic regulator applied to inverted pendulum

When the optimization (variational integral) is applied to Lagrangian, we obtained Euler-Lagrange's equation which shows what nature does - most energy economic movement. Thus, we expect an optimized motion by applying variational integral to the Lagrangian of inverted pendulum, but when the Lagrangian is replaced by quadratic performance index, we know the result is the same (Sec 11.5).

Then, using the performance index

$$PI = \int_0^\infty (\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}) dt,$$

the objective of control is to find optimal $\boldsymbol{u}_o$ as a function of $\boldsymbol{x}(t)$.

$$\boldsymbol{u}_o = -\boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P}(t) \boldsymbol{x}(t).$$

We also know, the solution $\boldsymbol{P}(t)$ is the only one positive definite solution of the following Ricatti equation,

$$\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T \boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P}(t) + \boldsymbol{Q} = 0.$$

$$\boldsymbol{P} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_1 & a_4 \\ 1 & 0 & a_2 & a_5 \\ 0 & 1 & a_3 & a_6 \end{pmatrix} \boldsymbol{P} +$$

$$-\boldsymbol{P} \begin{pmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{pmatrix} \boldsymbol{R}^{-1} \begin{pmatrix} 0 & 0 & b_3 & b_4 \end{pmatrix} + \boldsymbol{Q} = 0.$$

The value $\boldsymbol{R}$ and $\boldsymbol{Q}$ are given by the designer, and the actual calculation shall be done by a computer.

18.5 Summary

Major part of the control design is forming of the Lagrangian, system expression by $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ and calculation of Ricatti equation by computer. That is, the solution of control maybe obtained without knowing the background of the optimized control, why Ricatti equation is solved and so on.

However, what is made clear by the discussion of an inverted pendulum is the difference of theoretical analysis and practice as discussed in Sec 15.7 which provides much deeper understanding of both analytical physics and multivariable control. That is, for theoretical treatise we need to go as far as Hamilton-Jacobi's equation, but in practical design, we use only Lagrangian.

18.6 Appendix

It is known that a double inverted pendulum is also controllable to keep upright position. Because the direction of control to keep the arm A upright coincides with that of the arm B as shown in Fig18.4. On the other hand, triple inverted pendulum cannot be controlled to keep upright position. The reason is simple since there is a situation that controlling the arms A and B upright contradicts to keep the arm C upright as shown in Fig 18.5.

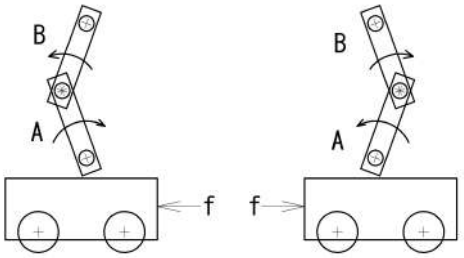

Figure 18.4: Double inverted pendulum

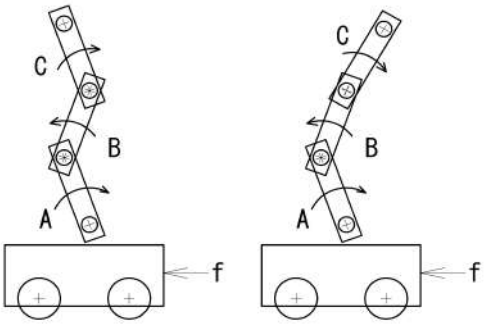

Figure 18.5: Triple inverted pendulum

Chap19 Robust and nonlinear control

In this chapter we very briefly describe which direction the control theory is heading from the multivariable control theory.

19.1 Robust control

Starting from analytical mechanics to the solution of Ricatti equation, we have shown the logical process of multivariable control. System modeling is based on analytical mechanics and once coordinate is properly chosen, there is no ambiguity to derive the equation of motion for the system.

19.1.1 Norms

Progress of multivariable control is most remarkably shown by the Linear Quadratic Regulator synthesis. In this treatise, the optimization is given by a quadratic performance index,

$$PI = \int_0^\infty (\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}) dt,$$

in which the value is given by the root mean square

$$\| \boldsymbol{x} \| = \sqrt{{x_1}^2 + {x_2}^2 +, ..., {x_n}^2}.$$

The $\| \boldsymbol{x} \|$ gives a length of the vector and called Euclidean norm or L_2 norm.

There are different definition of norm (length) and a familiar one is,

$$\| \boldsymbol{x} \| = |x_1| + |x_2| +, ..., +|x_n|,$$

which is an absolute value or L_1 norm.
Thus we can define L_∞ norm which is roughly the maximum of the elements of vector $\boldsymbol{x}$.

19.1.2 H_∞ norm

Fig 19.2 shows a response of a closed loop of certain multivariable system, and suppose curve B is observed overall. However B is an average of curves A,C,D,E,... and as a result, B is yielded. That is, even there is a peak at A, this is mostly disappeared by the average. And even if peak A causes trouble to this particular system, this is not counted because we used average - or Euclidean norm - to estimate the performance.

The fundamental reason that we cannot count peak A is the usage of average, and it is natural to use maximum value of the response incorporated into the performance index, for better control.

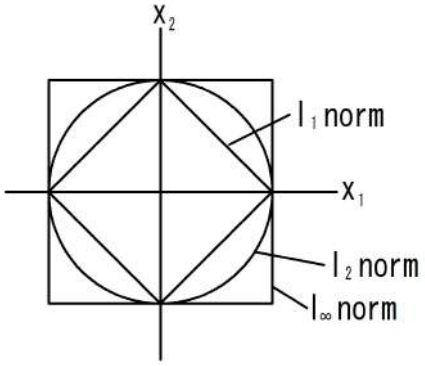

Figure 19.1: l_1, l_2, l_∞ norm

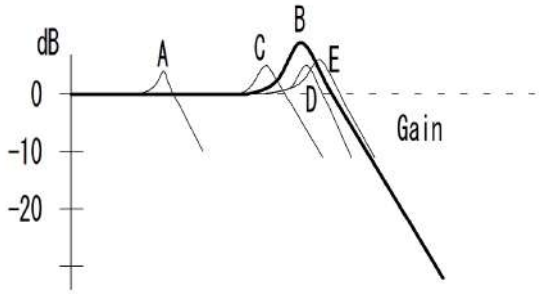

Figure 19.2: Closed loop response

Now the (mathematical) problem is that this "maximum norm" cannot be used in multi (or infinite) dimensional Euclidean space which is also called Hilbert space, and we have to seek for appropriate space. And the mathematicians have already thought about this space and it is called "Hardy space". Thus if we choose L_∞ norm to estimate performance index, we can keep the discussion going. This norm is called H_∞ norm and the essence of H_∞ control is the use of this norm for optimization. Then the yielded system obviously has better margin than the system derived by Euclidean norm.

This is called H_∞ control.

19.1.3 Summary

We have given a very rough sketch of H_∞ control which sounds baffling to many engineers. One of the reason is the term "Hardy space" which certainly is unfamiliar to most of the engineers[1], but if we look into the treatise and recognize it as an operation corresponding to "peak hold" as compared with "average" of a spectrum analyzer, then the meaning becomes much simpler.

Also note, that the Fig 19.2 is drawn by L_2 norm and cannot express mathematical rigor, but that can be (hopefully) permitted.

[1]Even farther away from Hilbert or Banach space.

19.2 Nonlinear control

The best and most common strategy to tackle nonlinear system is to express the target by linear systems. **Never get far away from linear** seems to be the motto.

Here we show one of those approaches by state dependent Ricatti equation([43][44]).

A nonlinear system can be represented by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(x)\boldsymbol{x} + \boldsymbol{B}(x)\boldsymbol{u}.$$

That is, the coefficient $\boldsymbol{A}$ of linear system is replaced by $\boldsymbol{A}(x)$ which is a coefficient dependent of variable x. Then the above equation is optimized under the performance index,

$$PI = \int_0^\infty (\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})dt,$$

yielding state dependent algebraic Ricatti equation,

$$\boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}(t)^T\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} = 0.$$

As shown above, this nonlinear approach is a direct extension of linear quadratic regulator, but further discussion is out of scope of this document.

Chap20 Lagrangian of electric circuit

In the days of Lagrange, electric circuit[1] was far beyond and how electric components would be expressed by using generalized coordinates was completely out of scope. It was as late as 1950s that people started to express electric circuits by using state variables and apply analytical physics to multivariable automatic control.

Then, obvious question is how can we interpret electric variables in analytical physics' sense. That is, people may have following perplexing questions.

- In analytical physics, variable q and $\dot{q}$ corresponds position and velocity. In electronics, which variable corresponds to q or $\dot{q}$ and why?
- One of the correspondence in electronic circuit is

$$q \rightarrow electric\ \ charge \quad q_e \ ^2$$

$$\dot{q} \rightarrow current \quad i$$

but there are another correspondence

$$q \rightarrow magnetic\ \ flux \quad \phi$$

$$\dot{q} \rightarrow voltage \quad v$$

Why?

- In electronic circuits, voltage and current are generally used. How can we form Lagrangian form these circuits ?

20.1 Coordinates in electricity

The advent of electricity may go far earlier than the Lagrange's days, but electricity became applicable to our life in the days of Edison and Tesla. From those days until today, electric circuits are generally analyzed by using voltage, current and impedance. This is a very straight forward way because the values are easily measured and the behavior of the circuit is observed easily for analysis and design.

However, the approach and purpose is different for automatic control from circuit design. In latter case, the objective is generally the calculation of voltage and current related with components' impedance in steady state. Also, the system is assumed to be linear. For this reason, the driving power is considered to be a sinusoid[3] with some frequency.

[1] Here, the author do not distinguish electric circuits from electronic circuits.

[2] Unfortunately, electric charge is often expressed by q. To prevent ambiguity, we use q_e for electric charge.

[3] There is a DC analysis, but here we ignore.

And differential or integration properties are replaced by the phase shift of sinusoid (or multiplication of $j\omega$; "$j^2 = -1$").

For the automatic control, the system physics is in interest. Thus, more fundamental differential equation approach is necessary by which system components' dynamics are described by using energy. In this approach, components' topology is of secondary interest. Topology becomes a way to clarify the relations of system constraints.

The energy approach enabled to discuss different physical systems in a uniform way. When the electric circuit is incorporated into mechanical system - electric motor driven humanoids for instance - the interface between electricity and mechanism is the energy. Electricity (voltage and current) is supplied to a motor which is converted to mechanical torque and eventually moves the target object which has mass.

In electric systems, there are variables that correspond to coordinates, velocity and so on in Newtonian mechanics. Retaining the term "coordinates" is rather misleading for modern applications in engineering, but there seems to be no good alternative and hence, we will use it as necessary.

20.2 Maxwell's equation

We will see in Sec 20.3 that there are two ways to define coordinates in electricity. The origin of coordinates is in the Maxwell's equation which is,

$$
\begin{aligned}
div\boldsymbol{E} &= \frac{\rho}{\epsilon_0}, \\
rot\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\
div\boldsymbol{B} &= 0, \\
rot\boldsymbol{B} &= \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}.
\end{aligned}
$$

Or more fundamentally,

$$
\begin{aligned}
Dif(\boldsymbol{E}) &= div\boldsymbol{E} + rot\boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{\partial \boldsymbol{B}}{\partial t}, \\
Dif(\boldsymbol{B}) &= div\boldsymbol{B} + rot\boldsymbol{B} = 0 + \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}.
\end{aligned}
$$

where $Dif()$ means differential of a three dimensional vector. In other words, **the electromagnetic field is expressed by a set of two differential equations of three dimensional vectors**[4] $\boldsymbol{E}$ **and** $\boldsymbol{B}$. That is, in electromagnetic theory, what we have - can observe - are the electric field and magnetic field. Both fields are orthogonal with each other, where physical parameters like voltage, current, magnetic flux etc., are defined and luckily, measurable.

Two ways of coordinates choice precisely reflect this situation - one starts at the electric field, the other from magnetic field.

[4]If we differentiate three dimensional vector, we get a scaler and a three dimensional vector. In most cases, the Maxwell's equation is written by four equations, but fundamental physical meaning is better understood by a set of two differential equations.

20.3 Choice of coordinates

20.3.1 Brief review of Lagrangian

The Euler-Lagrange's equation

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = 0,$$

is an equation of variable q and it's derivatives. Therefore, expressing a physical system by Euler-Lagrange's equation, we have to choose one "coordinate" and it's derivatives. In case of Newtonian mechanics, we have only one coordinate (q) and question posed here didn't arise. However, in electric system, we have to choose either electric charge based expression or magnetic flux based expression. Because the electro-magnetic system is expressed by two orthogonal coordinates.

This situation is the fundamental origin of the problem discussed in this chapter.

20.3.2 Electric field based choice

In this choice, we start by the electric field $\boldsymbol{E}$. Since the static electric field is originated by the electric charges q_e, this is chosen as a variable in phase space q. We have only two complimentary energy storage components[5] which are inductance and capacitance.

When electric charge q_e changes by time, it produces current i,

$$i = \frac{dq_e}{dt} = \dot{q}_e.$$

For the electric field based approach, the current is the parameter in $\dot{q}$ space (equivalent to velocity).

Then by inductance L;

- kinetic energy $\mathcal{T}_L$ stored in the inductor is,

$$\mathcal{T}_L = \frac{1}{2}Li^2$$

$$= \frac{1}{2}L\dot{q}_e^2.$$

- momentum p is,

$$p = L\dot{q}_e.$$

For the capacitance C, electric charge q_e produces voltage across the capacitor;

$$v = \frac{q_e}{C}.$$

- potential energy $\mathcal{V}_C$ in the capacitor is,

$$\mathcal{V}_C = \frac{1}{2}Cv^2 = \frac{1}{2C}q_e^2.$$

- force f (voltage) is,

$$f = \frac{1}{C}q_e.$$

[5]Batteries may store energy, but it is stored chemically.

20.3.3 Magnetic field based choice

For the magnetic field based approach, the magnetic flux is the parameter in q space and voltage v belongs to $\dot{q}$ space.

$$v = \frac{d\phi}{dt} = \dot{\phi}.$$

Since magnetic flux is a duel of electric charge, symmetric equations hold.

For the capacitance C;

- kinetic energy $\mathcal{T}_C$ is,

$$\begin{aligned}\mathcal{T}_C &= \frac{1}{2}Cv^2 \\ &= \frac{1}{2}C\dot{\phi}^2.\end{aligned}$$

- momentum p is,

$$p = C\dot{\phi}.$$

For the inductance L, flux ϕ produces current i,

$$i = \frac{1}{L}\phi.$$

- potential energy $\mathcal{V}_L$ in the inductor is,

$$\mathcal{V}_L = \frac{1}{2L}\phi^2.$$

- force f (current) is,

$$f = \frac{1}{L}\phi.$$

20.3.4 Summary of variables

We summarize the generalized coordinates and corresponding variables in table 20.1. Note that the kinetic energy is a function of only $\dot{q}$ - no q is included as in the case of polar coordinates expression of Newtonian motion. This is a lucky situation and we regard the **variable which belongs to phase space q is considered to be potential energy, and variable belongs to $\dot{q}$ space is considered to be kinetic energy**.

Also note, that we have no linear energy like the one caused by gravitational potential - all energy has quadratic form.

20.4 Lagrangian as circuit analysis

Electric circuit analysis can be done by a couple of well known methods - use Kirchhoff' law, principle of superposition and so on. For AC response analysis, driving signal is assumed to be sinusoid, and the energy storage components are expressed by the reactance - $j\omega L, 1/j\omega C$. By this way, we do not have to describe the circuit dynamics by a differential equation, but by an arithmetic equation.

system physics	generalized coordinate	generalized velocity	kinetic energy	potential energy	generalized momentum	generalized force
	q	$\dot{q}$	$\mathcal{T}$	$\mathcal{V}$	p	$\mathcal{F}$
mechanical translation	position	velocity	$\frac{1}{2}M\dot{x}^2$	$\frac{1}{2}Kx^2$	momentum	force
	x	$\dot{x}=v$			$M\dot{x}$	Kx
mechanical rotation	angle	angular velocity	$\frac{1}{2}J\dot{\theta}^2$	$\frac{1}{2}K\theta^2$	angular momentum	torque
	θ	$\dot{\theta}=\omega$			$J\dot{\theta}$	$K\theta$
electric field base $\boldsymbol{E}$	charge	current	$\frac{1}{2}L\dot{q}_e^2$	$\frac{1}{2C}q_e^2$	flux	voltage
	q_e	$\dot{q}_e=i$			$L\dot{q}_e$	$\frac{1}{C}q_e$
magnetic field base $\boldsymbol{B}$	flux	voltage	$\frac{1}{2}C\dot{\phi}^2$	$\frac{1}{2L}\phi^2$	charge	current
	ϕ	$\dot{\phi}=v$			$C\dot{\phi}$	$\frac{1}{L}\phi$

Table 20.1: Generalized coordinates and variables

From the view point of Lagrangian, what we need is the energy term and they are self evident by observing the circuit diagram; that is, writing down the kinetic and potential energy is straight forward. The circuit topology works as holonomic constraints by which we reduce the number of variables that are independent with each other.

20.5 Examples of Lagrangian in electric circuits

Euler-Lagrange's equation or Lagrangian must be expressed either by electric charge or magnetic flux base. Here we will discuss some examples of electric circuits. Since the energy storage components prevail the behavior of the system, we focus only on these elements, but note that circuits with L, C only do not exist in real world.

20.5.1 Example 1: LC resonator

LC resonator shown in Fig 20.1 is the simplest circuit with energy storage components. In real world, there is no circuit without resistance and this may be understood as the extreme case when resistance made smaller[6].

If there happen to be current in this circuit, we know that current oscillates by a single frequency sinusoid and keeps on going for ever.

By the electric field approach, the correspondence of the coordinates and variables becomes,

$q \to q_e$ (electric charge of the capacitor)

$\dot{q} \to \dot{q}_e$ (current through the inductor and the capacitor)

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}L\dot{q}_e^2 - \frac{1}{2C}q_e^2.$$

[6]Note that we cannot call this structure neither series nor parallel circuit. If we observe by current basis, the circuit becomes a series circuit, and by voltage base, it becomes a parallel circuit. Series or parallel is defined by the ways of interaction with outside world - driving or drawing power to/from the circuit.

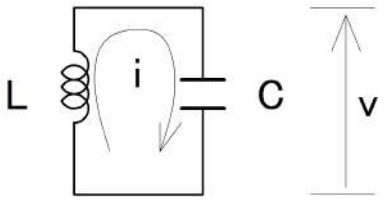

Figure 20.1: Simplest circuit by L and C

Then we have the following Euler-Lagrange equation.

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_e}\Big) - \frac{\partial \mathcal{L}}{\partial q_e} = 0.$$

If we solve this differential equation we get,

$$L\ddot{q}_e + \frac{q_e}{C} = 0.$$

This can be written by,

$$\ddot{q}_e + \omega^2 q_e = 0.$$

where, $\omega^2 = 1/(LC)$. We know the solution of this equation is

$$q_e(t) = q_{e0}\cos\omega t,$$

$$\dot{q}_e(t) = -\omega q_{e0}\sin\omega t.$$

By the magnetic field approach, we choose the variables as,
$q \to \phi$ (magnetic flux of the inductor)
$\dot{q} \to \dot{\phi}$ (voltage between the inductor or the capacitor)
The Lagrangian is,

$$\mathcal{L} = \frac{1}{2}C\dot{\phi}^2 - \frac{1}{2L}\phi^2.$$

Then we have a second order equation of variable ϕ and will yields the same solution.

20.5.2 Example 2: Circuit with power source

In circuits with series and parallel components, there can be different choice of current flow allocation at the time of analysis. This causes different Euler-Lagrange's equation, but they are identical. We clarify this point by the following discussion.

(1) Current allocation 1

Fig 20.2 shows an electric circuit with (sinusoidal) power source. The Lagrangian is defined by excluding this power source - $E(t)$ is shorted to make a closed circuit.

If we select electric charge as q, the energy of each components is,

$$\frac{1}{2}L_1\dot{q}_{e1}^2,\ \ \frac{1}{2}L_2\dot{q}_{e2}^2,\ \ \frac{1}{2C_2}(q_{e2}-q_{e1})^2.$$

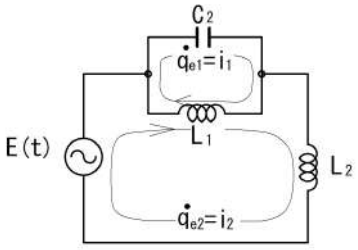

Figure 20.2: Electric circuit example 2-1

Then the Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}L_1\dot{q}_{e1}^2 + \frac{1}{2}L_2\dot{q}_{e2}^2 - \frac{1}{2C_2}(q_{e2} - q_{e1})^2.$$

The Euler-Lagrange's equation of energy conserving system is

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_e}\Big) - \frac{\partial \mathcal{L}}{\partial q_e} = 0.$$

Therefore, the equation of motion becomes,

$$L_1\ddot{q}_{e1} + \frac{1}{C_2}(q_{e1} - q_{e2}) = 0,$$

$$L_2\ddot{q}_{e2} + \frac{1}{C_2}(q_{e2} - q_{e1}) = 0,$$

$$\begin{pmatrix} L_1, & 0 \\ 0, & L_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_{e1} \\ \ddot{q}_{e2} \end{pmatrix} + \begin{pmatrix} \frac{1}{C_2}, & -\frac{1}{C_2} \\ -\frac{1}{C_2}, & \frac{1}{C_2} \end{pmatrix}\begin{pmatrix} q_{e1} \\ q_{e2} \end{pmatrix} = 0.$$

We have ignored the power source $E(t)$, the circuit becomes equivalent to (A) in Fig 20.3. (A) is easily converted to (B) which is identical to Fig 20.1.

By the equations,

$$L_1\ddot{q}_{e1} + \frac{1}{C_2}(q_{e1} - q_{e2}) = 0, \quad L_2\ddot{q}_{e2} + \frac{1}{C_2}(q_{e2} - q_{e1}) = 0,$$

we obtain,

$$L_1L_2\ddot{q}_{e1} + \frac{L_2}{C_2}(q_{e1} - q_{e2}) = 0,$$

$$L_1L_2\ddot{q}_{e2} + \frac{L_1}{C_2}(q_{e2} - q_{e1}) = 0.$$

Then,

$$L_1L_2\ddot{q}_{e1} - L_1L_2\ddot{q}_{e2} + \frac{L_2}{C_2}(q_{e1} - q_{e2}) - \frac{L_1}{C_2}(q_{e2} - q_{e1}) = 0.$$

By a bit of manipulation,

$$\frac{L_1L_2}{L_1 + L_2}(\ddot{q}_{e1} - \ddot{q}e2) + \frac{1}{C_2}(q_{e1} - q_{e2}) = 0.$$

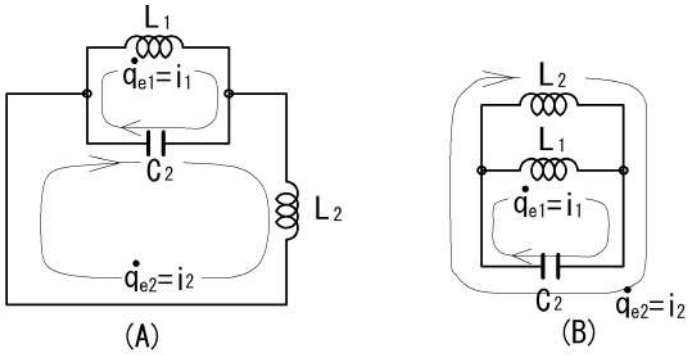

Figure 20.3: Equivalent circuit of example 2

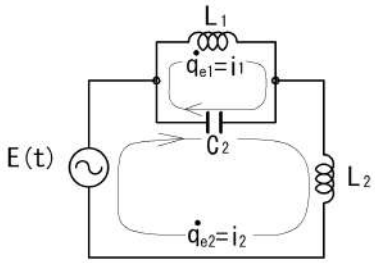

Figure 20.4: Electric circuit example 2-2

Since the parallel inductance of L_1 and L_2 is equal to $L_1L_2/(L_1+L_2)$, we merge them into L.

$$L(\ddot{q}_{e1}-\ddot{q}_{e2})+\frac{1}{C_2}(q_{e1}-q_{e2})=0.$$

This is a resonant circuit of L and C_2 with current $q_{e1}-q_{e2}$ which flows into C_2.

(2) Current allocation 2

In this example, we changed C_2 and L_1 (current flow) from the above example.

Then the Lagrangian becomes,

$$\mathcal{L}=\frac{1}{2}L_1(\dot{q}_{e2}-\dot{q}_{e1})^2+\frac{1}{2}L_2\dot{q}_{e2}^2-\frac{1}{2C_2}q_{e1}^2.$$

The equation of motion is,

$$L_1\ddot{q}_{e1}-L_1\ddot{q}_{e2}+\frac{1}{C_2}q_{e1}=0,$$

$$-L_1\ddot{q}_{e1}+(L_1+L_2)\ddot{q}_{e2}=0,$$

$$\begin{pmatrix} L_1, & -L_1 \\ -L_1, & L_1+L_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_{e1} \\ \ddot{q}_{e2} \end{pmatrix}+\begin{pmatrix} \frac{1}{C_2}, & 0 \\ 0, & 0 \end{pmatrix}\begin{pmatrix} q_{e1} \\ q_{e2} \end{pmatrix}=0.$$

We have obtained a different equation of motion by allocating different current path. However, by the following equation,

$$-L_1\ddot{q}_{e1}+(L_1+L_2)\ddot{q}_{e2}=0,$$

we get

$$\ddot{q}_{e2}=\frac{L_1}{L_1+L_2}\ddot{q}_{e2}.$$

Then by substituting q_{e2}, we obtain

$$\frac{L_1 L_2}{L_1 + L_2}\ddot{q}_{e1} + \frac{1}{C_2}q_{e1} = 0,$$

which shows the same conclusion.

20.5.3 Example 3: Complicated circuit

In Sec 20.5.2, we have seen that the allocation of current path may produce different equation of motion for a same circuit, or allocation of current path does not matter as long as we observe the Kirchhoff's law.

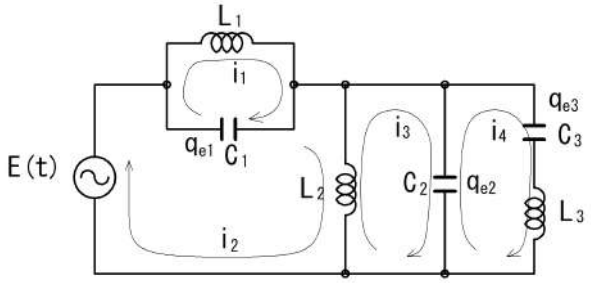

Figure 20.5: Electric circuit example 3

If we select current as shown in Fig 20.5, That is, the charge of three capacitors $C1, C2, C3$ becomes the "coordinates". Then,

$$\dot{q}_{e1} = i_1, \quad \dot{q}_{e2} = i_3 - i_4, \quad \dot{q}_{e3} = i_4.$$

The kinetic energies stored in L_1, L_2, L_3 become

$$\frac{1}{2}L_1\dot{q}_{e1}^2, \quad \frac{1}{2}L_2(\dot{q}_{e2} - \dot{q}_3 e)^2, \quad \frac{1}{2}L_3\dot{q}_{e4}^2.$$

The potential energies stored in C_1, C_2, C_3 become

$$\frac{1}{2C_1}(q_{e1} - q_{e2})^2, \quad \frac{1}{2C_2}(q_{e3} - q_{e4})^2, \quad \frac{1}{2C_3}q_{e4}^2$$

Then, the Lagrangian is,

$$\mathcal{L} = \frac{1}{2}L_1\dot{q}_{e1}^2 + \frac{1}{2}L_2(\dot{q}_{e2} - \dot{q}_3 e)^2 + \frac{1}{2}L_3\dot{q}_{e4}^2.$$

$$-\big(\frac{1}{2C_1}(q_{e1} - q_{e2})^2 + \frac{1}{2C_2}(q_{e3} - q_{e4})^2 + \frac{1}{2C_3}q_{e4}^2\big).$$

20.6 Summary

In electric circuit theory, circuit analysis is carried out by using voltage, current and impedance. The main objective becomes the calculation of those variables, because the

waveform of voltage or current is assumed to be sinusoid (of commercial frequency) and the system is linear.

For linear electronic circuits, the target of interest becomes the wave form of voltage or current, because - generally speaking - they process superposed sinusoid to required output signal.

Lagrangian approach lies lower than the electric/electronic analysis which derives equation of motion of the electric/electronic system. Thus, objective becomes the treatment of energy storage components and have to clarify kinetic/potential energy. This is directly reflected by the selection of "coordinates" - either magnetic flux or electric charge based. In either case, the multiple expression is possible.

The Lagrangian is based on kinetic and potential energy, the former only related with $\dot{q}$ and the latter with q, the analysis of physical system is generally easier than electric circuit analysis.

PartIV

Classic control

In this part, no comprehensive discussions are given concerning automatic control theory, but instead we discuss an example of motor control system to show how easily people can start designing automatic control system.

The simplicity is based on the fact that many of servo control systems are described by second order transfer function which we are very much accustomed to. Of course, for precise control, it will require works on many technical issues - may have to start designing the motor and sensors, but rough control systems are implemented much easier than people think.

Particularly in Chap 23, we will show that a small motor can be easily controlled to construct a phase locked system with minimal components. From this example, it is not difficult to design a remote controlled tracking system such as used in cameras in TV conference system or servo controllers for toy planes.

Although simple, the discussion presented in this part was the base of video tape recorder once dominated the entertainment at home in 1980s to early 21st century.

Chap21 Modeling by transfer function

In this chapter we will discuss how basic components are expressed by transfer function. The topic in this chapter is focused on rotating mechanism.

21.1 Rotating disc

One of the most common physical components we encounter in daily life are rotating discs - wheels of vehicles, DVD players and so on. Discs are made by physical materials and have mass. To rotate the disc, we apply force - which is generally called torque. And the rotating object with mass once started rotation, it keeps rotation. This is caused by rotational inertia, or just inertia. Fig 21.1 shows a physical model of disc rotation. Applied torque is represented by τ, inertia by J, rotational angle by Θ and friction by D.

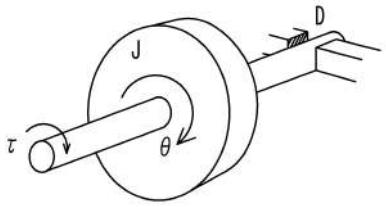

Figure 21.1: Inertia with friction

The equation of motion is

$$J\frac{d^2\Theta}{dt^2} = \tau - D\frac{d\Theta}{dt},$$

whereas the angular velocity ω is

$$\omega = \frac{d\Theta}{dt}.$$

Therefore

$$J\frac{d\omega}{dt} + D\omega = \tau.$$

The differential equation above completely determines the rotating angle Θ or angular velocity ω at the time t under the known torque τ and friction D. That is, if we wish to control Θ or angular velocity ω, we could modify τ and D by sensing[1] Θ and ω. For the control example, see Chap 23.

Note that in Fig 21.1, the driving shaft has also mass and hence inertia, but this part is merged into inertia J. This is a common practice in analyzing a system, but only possible when the shaft is rigid and connected to the disc without elasticity. If elasticity exists in

[1]To control ω, sensing ω is sufficient, but controlling Θ, sensing both ω and Θ is necessary.

the coupling, it introduces another oscillatory element into the system and it should be get rid of first[2].

21.2 Electronic circuit 1

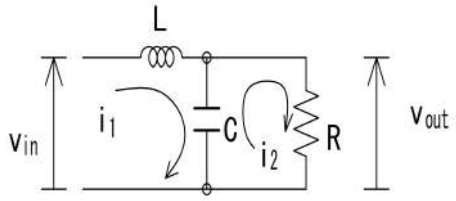

Figure 21.2: Electric circuit with I/O

Fig 21.2 shows a simple 2nd order low pass filter with input and output. Inductance L and capacitance C are the energy storage elements that determine the characteristics of the LPF. In practice, resistance R is generally large so that the LPF characteristics is not greatly affected. Here, we start by describing the circuit by a differential equation. Assign current i_1 and i_2 as depicted.

$$v_{in} = L\frac{di_1}{dt} + \frac{1}{C}\int (i_1 - i_2)dt,$$

$$\frac{1}{C}\int (i_1 - i_2)dt + Ri_2 = 0.$$

$$v_{out} = -Ri_2.$$

The concrete solution is only available when we specify the inputs. If the input is sinusoid, the output is sinusoid with different amplitude and phase. If input is a step voltage, then some peak or oscillation is expected at the rising/falling edge.

On the other hand, Laplace transform converts differential equation into arithmetic function of s, and makes the analysis independent from some specific inputs. In this case, the above equations yields,

$$V_{IN} = sLI_1 + \frac{1}{sC}(I_1 - I_2),$$

$$\frac{1}{sC}(I_1 - I_2) + RI_2 = 0.$$

$$V_{OUT} = -RI_2.$$

Calculating the input/output response, we calculate V_{OUT}/V_{IN} as a function of complex variable s.
From the first equation,

$$I_1 = \frac{V_{IN}Cs + I_2}{LCs^2 + 1}.$$

[2]Increase rigidity and shift the resonant frequency high where feedback loop gain is small enough.

From the second equation,

$$I_1 = (1 - CRs)I_2.$$

Substituting I_1 will yield

$$I_2 = \frac{V_{IN}}{-LCRs^2 + LS - R}.$$

Substituting by $I_2 = -V_{OUT}/R$,

$$G(s) = \frac{V_{OUT}}{V_{IN}} = \frac{R}{LCRs^2 - Ls + R}.$$

The $G(s)$ is a transfer function which shows that the input output representation of Fig 21.2 becomes a second order equation of complex variable s. If we restrict the moving path of s on the imaginary axis, the result gives the frequency response of the system[3].

The frequency plot will visually show the characteristics of the circuit.

We will see that the phase controlled motor system has a second order transfer function and thus we can discuss the response in the same way as discussed above.

21.3 Electronic circuit 2

Active electronic components such as transistors, diodes are generally nonlinear devices. However as a circuit, it is generally made into a linear circuit. Operational amplifier is a good example. It can be used as a linear amplifier, integrator, differentiator etc., which are all linear components. And the frequency characteristics shall be wide enough comparing with the expected frequency range of the target control system. Therefore, it is expressed by simple transfer function (constant, first order low pass filter etc.,) and shall not bring troubles into the system.

21.4 Sample and hold

Rotation speed of a motor may be observed continuously, but in many cases, it is sampled. The sampled value is held until the next sample keeping the same value. This is called zero order hold.

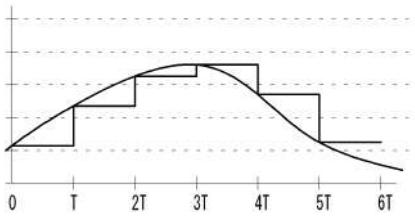

Figure 21.3: Zero order sample/hold

For sampling, delta function $\delta(t)$ is defined by

$$\delta(t) = 0, \quad t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

[3]Put σ to zero in $s = \sigma + j\omega$.

Laplace transform of delta function is

$$\mathcal{L}(\delta(t)) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1.$$

Suppose a continuous signal $v(t)$ is sampled by delta function $\delta(t)$

$$v^* = \sum_{n=0}^{\infty} v(nT)\delta(t - nT).$$

Applying Laplace transform,

$$V^*(s) = \sum_{n=0}^{\infty} v(nT) e^{-nsT}.$$

The Laplace transform of the sample/hold output is,

$$U^*(s) = \int_0^T v(0)e^{-st}dt + \int_T^{2T} v(T)e^{-st}dt + \int_{2T}^{3T} v(2T)e^{-st}dt + ...$$

$$= \frac{v(0) - v(0)e^{-sT}}{s} + \frac{v(T)e^{-sT} - v(T)e^{-2sT}}{s} + \frac{v(2T)e^{-2sT} - v(2T)e^{-3sT}}{s} + ...$$

$$= \frac{1 - e^{-sT}}{s}\Big(v(0) + v(T)e^{-sT} + v(2T)e^{-2sT} + ...\Big)$$

$$= \frac{1 - e^{-sT}}{s} \sum_{n=0}^{\infty} v(nT)e^{-nsT} = \frac{1 - e^{-sT}}{s} V^*(s).$$

Thus the transfer function $H(s)$ is,

$$H(s) = \frac{1 - e^{-sT}}{s}.$$

The frequency response is,

$$H(j\omega) = \frac{1}{j\omega}(1 - e^{-j\omega T}) = \frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}} T e^{\frac{-j\omega T}{2}}.$$

The gain curve is shown in Fig 21.4. Phase at sampling period T is $-180°$, the sampling frequency should be preferably ten times of the frequency bandwidth. Then the transfer function is approximated by a constant[4].

21.5 Power train

In many cases, the speed of the actuator driven by a motor is decreased by using gears or pully/belt. When using gears, most concerned character is play and backlash because these character inherently presents nonlinearity.

Another speed reduction by using a set of pullies and elastic belt, the transfer function of the power train becomes second order[5]. Because the elasticity of the belt and the inertia

[4]Figure 21.4 doesn't look like constant, but we only use area near the origin.

[5]Note that the whole system will have higher order of transfer function.

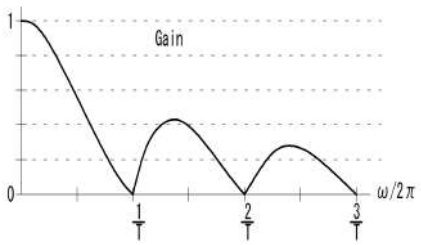

Figure 21.4: Gain of sample/hold

of motor (rotor), pully on the rotor shaft, and pully attached to driven components make a resonant circuit just like simple pendulum. The friction of pully and belt is the only damping factor and if the resonant frequency is within the range of servo signal, hunting shall be expected.

This should most certainly be avoided, because this is well anticipated at the time of structure design.

For precision rotation control circuit, if DC motor is selected, it should be brush less, low cogging and has reasonably big inertia like that found in hard disc drives[6] (in case of Hard disc drive, the inertia of the disc is included).

If the above conditions are satisfied, the transfer function of power train becomes just a constant K.

21.6 Small motors

Most suitable motor for servo system has long been the DC motors. Speed/voltage and speed/load characteristics are linear as shown in Fig 21.5. Reflecting these characteristics, the transfer function becomes identical with that of rotating disc,

$$G_m(s) = \frac{K_m}{1 + T_m s}.$$

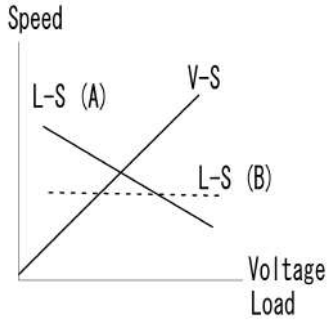

Figure 21.5: Speed v.s. voltage, load characteristics of DC motor

In Figure21.5, L-S (A) shows the speed-load characteristic of the motor. As the load increases, the speed decreases. L-S (B) is the same characteristic with speed servo control.

[6]In Winchester drive in 1970s, it was driven by an AC motor with pully and belt.

The speed servo tries to make this line horizontal; speed is kept constant under changing load.

The demerits of conventional DC motor is brush and commutator which are the cause of almost all troubles with this type of motor - noise and life. Therefore, for precision application, brushless motors are used. There are variety of designs, but the one used frequently has a rotor with permanent magnet and a stator with three set of windings. The position of rotor is sensed by a sensor (Hall device) which are located on the stator side. The stator voltage is switched by semiconductor switch according to the rotor position thus eliminating mechanical switch (brush - commutator)[7].

However, AC servo motor that utilizes vector control are now widely used in industrial applications.

[7]This brush less DC motor is actually a 3 phase synchronous motor, and it is generally incorporated with a driver so that the designer can just consider is as a simple DC motor.

Chap22 Classic control

22.1 Rotation phase control system

Now we discuss electro mechanical system by which not only the rotation speed, but also the phase of the disc is synchronized to an external reference signal.

People who have played with a toy motor[1], this sounds a very sophisticated control but it was once very commonly used in home equipments - video tape recorders. Until around early 21st century, television is made using a CRT and no picture memories are incorporated. Therefore, showing a visual data, electron beams scanned the CRT and this beam is completely in sync with broadcasting station or video tape recorder in case of prerecorded tapes.

For the video tape recorder, picture information written on the tape is read out by magnetic heads mounted on a cylinder which is driven by a motor. Since the field[2] frequency is specified ($60Hz$ for NTSC, $50Hz$ for PAL), the motor must very stably lock to this reference frequency; otherwise the reproduced picture is distorted or jittery. In engineering this is called phase lock technique, which is one of the common applications of the automatic control.

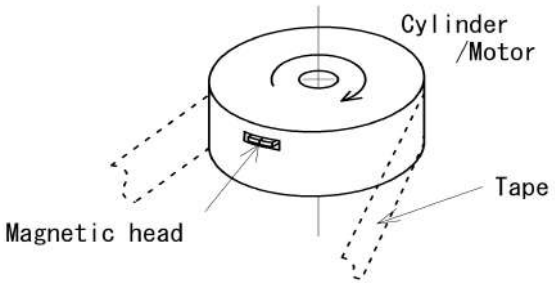

Figure 22.1: Cylinder of video tape recoder

22.1.1 Cylinder of video tape recorder

The cylinder of a video tape recorder is driven by a motor, and here we assume a DC motor[3]. Speed of DC motor changes by the applied voltage, and load. In this case the load is the contact with video tape and friction of bearings, which are mostly negligible.

[1]By experience, the speed changes easily by load and voltage and thus controlling the speed alone looks difficult.

[2]One frame is made of two interleaving fields.

[3]In the early days, AC induction motors are also used. In this case speed servo control is omitted because motor speed is determined by the frequency of the power source; either 50 or 60Hz.

The speed-voltage characteristics of DC motor is basically linear but it changes by voltage, load, ambient temperature, external noise and vibration, stiffness of the tape etc., the speed fluctuates under a constant voltage. Also, the speed characteristics is inversely proportional to load, speed servo control is required before phase locked loop is achieved.

22.1.2 Speed servo control of a motor

Because of the reason given above, we start discussing the speed servo control. For the speed servo control, we have to know the physical character of the cylinder with a motor as shown in Fig 22.2.

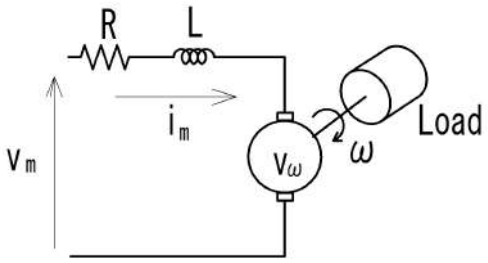

Figure 22.2: motor drive diagram

Assuming that the resistance of motor winding R, inductance L, the applied voltage v_m, motor current i_m and angular velocity ω, the system is represented by

$$L\frac{di_m}{dt} + Ri_m + v_\omega = v_m.$$

Where v_ω is the counter electromotive voltage of the motor caused by rotation. Assuming that the magnetic field is produced by permanent magnet and hence constant, the angular velocity ω is directly proportional to the v_ω.

$$v_\omega = K_\omega \cdot \omega.$$

Torque of the motor is proportional to the current i_m,

$$\tau = K_t \cdot i_m.$$

Applying the Laplace transform, we get

$$sLI_m(s) + RI_m(s) + V_\omega(s) = V_m(s),$$
$$V_\omega(s) = K_\omega \cdot \Omega(s),$$
$$T(s) = K_t \cdot I_m(s),$$
$$sJ\Omega(s) + D\Omega(s) = T(s).$$

Then,

$$sJ\Omega(s) + D\Omega(s) = K_t I_m(s).$$

Therefore,

$$I_m(s) = \frac{1}{K_t}\Omega(s)(sJ + D).$$

By substitution,

$$sL\Big(\frac{1}{K_t}\Omega(s)(sJ + D)\Big) + R\Big(\frac{1}{K_t}\Omega(s)(sJ + D)\Big) + K_\omega\Omega(s) = V_m(s),$$

$$\Omega(s)\Big(\frac{sL}{K_t}(sJ + D) + \frac{R}{K_t}(sJ + D) + K_\omega\Big) = V_m(s).$$

The transfer function is

$$\begin{aligned} G(s) &= \frac{\Omega(s)}{V_m(s)} \\ &= \frac{1}{(sJ + D)\Big(\frac{sL}{K_t} + \frac{R}{K_t}\Big) + K_\omega} \\ &= \frac{K_t}{(sJ + D)(sL + R) + K_\omega K_t}. \end{aligned}$$

i_m is DC current and fluctuating component is small, L can be ignored. Then the transfer function is simplified as,

$$\begin{aligned} G(s) &= \frac{K_t}{(sJ + D)R + K_\omega K_t} \\ &= \frac{K_t}{sRJ + DR + K_\omega K_t} \\ &= \frac{\frac{K_t}{DR+K_\omega K_t}}{\frac{RJ}{DR+K_\omega K_t}s + 1} \\ &= \frac{K_m}{T_m s + 1}. \end{aligned}$$

Where,

$$T_m = \frac{RJ}{DR + K_\omega K_t}, \qquad [rad \cdot sec]$$

$$K_m = \frac{K_t}{DR + K_\omega K_t}.$$

The form of the resulted transfer function is identical with the one obtained in Sec 21.1 and the system still has first order transfer function; always stable and no hunting is expected.

22.1.3 Phase servo control of a motor

After the motor speed is controlled with constant angular velocity, the phase of the motor can be synchronized to external reference signal. The reference frequency and the motor phase pulse may be the multiple of the one rotational frequency[4], but generally it is equal to one rotational frequency. That is, the sampling period of phase information is much longer than speed control, thus the frequency range becomes much lower.

[4]Phase comparison is done by higher frequency, thus it requires some mechanism to normalize to one rotation.

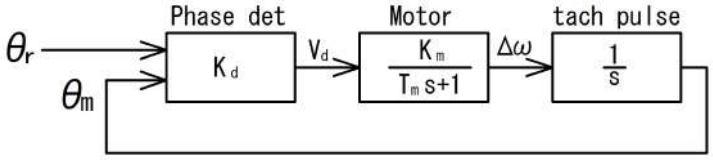

Figure 22.3: Block diagram of phase servo control

The error voltage of the reference signal and motor tach pulse is given by,

$$V_d = K_d(\Theta_r - \Theta_m),$$

where Θ_r and Θ_m are the phase of reference and motor tach pulse respectively. K_d is the coefficient and has the dimension $[V/rad]$. The phase detector produces voltage proportional to the phase difference as shown in Fig 22.4[5].

The motor is already speed controlled (workings as a Voltage Controlled Oscillator) whose transfer function is $K_m/(T_m s+1)$. The VCO is controlled by the output of the phase detector and the deviation of angular frequency from the center value $\Delta\omega$ is integrated by tach pulse yielding,

$$\Theta_m(s) = \frac{1}{s}\frac{K_m}{T_m s+1}V_d = \frac{K_m}{s(T_m s+1)}K_d[\Theta_r(s) - \Theta_m(s)].$$

The transfer function is,

$$\begin{aligned}\frac{\Theta_m(s)}{\Theta_r(s)} &= \frac{\frac{K_m K_d}{T_m}}{s^2 + \frac{s}{T_m} + \frac{K_m K_d}{T_m}} \\ &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.\end{aligned}$$

where

$$\omega_n = \sqrt{\frac{K_m K_d}{T_m}}, \quad \zeta = \frac{1}{2}\sqrt{\frac{1}{T_m K_m K_d}}.$$

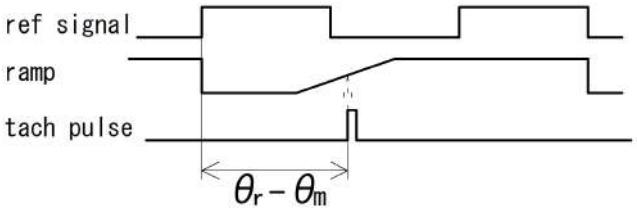

Figure 22.4: Time chart of phase servo

Now, we have the second order system in which ω_n is angular frequency of the role off point (could yield peak at this point) and the ζ is the damping factor. It is generally understood that $\zeta = 0.7$ gives best performance.

[5]To be precise, not proportional for all the rotational period. But around the operating range.

22.1.4 Ambivalence

In case of position control, the speed control comes first. And the speed is corrected by the position information. From the point of view of speed control, this correction is just an external disturbance. That is, if the position correction control is big, the system becomes oscillatory - hunting starts. In this sense, position error is small compared with speed error and the frequency components of position error is much lower; since position error is the integration (average) of speed, this is clear. That is, bigger the phase gain, system tends to oscillate but with small phase gain, it won't lock in.

22.2 Phase locked loop

Phase locked loop or PLL is one of the most commonly used techniques in electronic and electro-mechanical systems. The essence of this technology is to lock the generator's phase to reference signal. If applied to rotating system, the phase or position of rotating object is locked to the external reference signal. If the reference signal is very stable (like the output from quarts oscillator), the rotation of the mechanical object is stabilized to almost utmost extent[6].

Fig 22.5 shows a block diagram of a PLL system. The PLL system is one of the best examples of the classic automatic control theory, because the transfer function becomes second order and the second order system is best worked out for analysis and design[7].

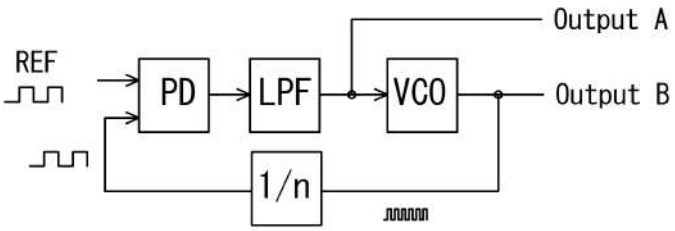

Figure 22.5: Block diagram of PLL

In Fig 22.5, "Output B" is produced by the voltage controlled oscillator and can be n times higher frequency than "REF" signal because we have $1/n$ counter in the loop. "Output B" is the averaged phase difference of the reference and frequency divided VCO. If reference signal is a frequency modulated signal, then the Output A becomes the demodulated signal.

Since PLL has a second order transfer function, the behavior of the system covers all possibilities like,

- dull response system (over damped)
- quickest response system with slight overshoot (critical damping)

[6]Fluctuation of cylinder of video tape recorder was around several microseconds as compared with one period; 16 mSec

[7]Or to be precise, classic control seems to be comfortably applicable up to second order only.

- oscillates (hunting)

Even for higher order system, the response will not be far different and a study of second order system is very much informative.

Chap23 DC motor control example

In this chapter, we describe a DC motor speed and phase control circuit for better understanding of the automatic control. Here, we design a system using classic control with completely analog technologies which may be more intuitive.

Unlike other chapters, there is no calculation appears. This is because,

(1) For speed servo control of a DC motor, the transfer function becomes first order. Therefore, the gain does not affect the stability of the system - open loop gain is not critical.

(2) When phase control is incorporated, the system becomes second order and the required response (over damped - critically damped - hunting) is easily achieved by adjusting the ratio of phase error to speed error (in Fig 23.7, the ratio of R3 and R4).

We intend to show that the **basic design of DC motor control is much easier than many people expect**.

23.1 Plan

Using a very common DC motor, we design a rotation phase control system. Rotation speed depends on the motor we use, but around 1200 rpm will be easily achievable. The power supply voltage we used is 6V, and the rating of 2 - 6V motor will work well. (That is, motor should reach 1200 rpm between, say, 2 - 5 V.)

In order to construct phase controlled system, speed of the motor must be controlled first, because the speed of a DC motor is easily disturbed by the fluctuation of load. For precision speed control, speed sensor is critical, but here we use simplest method, but still works.

Although there is no practical load attached to the system discussed here, it can be scaled up to usable system.

For the overall circuit diagram, see Fig 23.3.

23.2 Speed servo

For the speed servo control, we need a speed detector. There can be variety of methods, but the sensor's accuracy is very important, because the accuracy of speed control greatly depends on the quality of the sensor. One of the sensors can be a disc with radially placed uniform holes with a photo coupler (see Fig 23.1), by which pulses with inversely proportional period to speed are generated. Another better example is a frequency gen-

erator with a pair of winding and magnet that produces AC signal by integrating whole circumference - sensor errors are averaged.

Figure 23.1: Photo coupler

However, those kind of sensors are generally not available, and tachometers will be much more expensive than the DC motor, we use so called "electronic governor" in this example. Governor is first used by Watt to control the speed of steam engine and mostly the same idea is used in DC motors as well. A mechanical governor applied to DC motor is very bold in design. A switch in series to the rotor winding is equipped on the rotor which turns OFF when the motor exceeds some speed. That is, weight is applied to the switch and it works with centrifugal force (Fig 23.2).

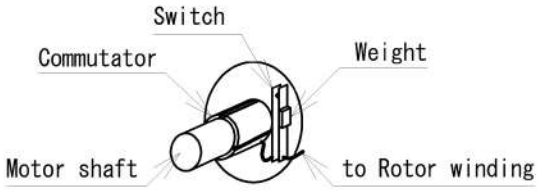

Figure 23.2: Mechanical governor motor

Electronic governor is an improved version of the mechanical governor, and it senses the counter electromotive voltage across the commutator. This sounds somewhat difficult method if we think the motor and supply voltage in ideal condition, because we might think that the voltage across the commutator is equal to supply voltage. But in reality, we can sense the voltage that is (locally[1]) proportional to the speed with some ripple caused by the rotor. Therefore, if we filter this voltage, we can use as the speed information of the motor.

In Fig 23.3, C5 works as a spark killer and L1 is an impedance looking into the power source Q4, thus the counter electromotive voltage is divided from driving voltage of the motor.

23.3 Phase servo

For the rotation phase sensor, we use a disc with a single hole on it. This is easy to make because we don't have to care about the absolute accuracy of the hole. Motor and phase sensor used are shown in Fig 23.4.

[1]Only proportional at around the operating point

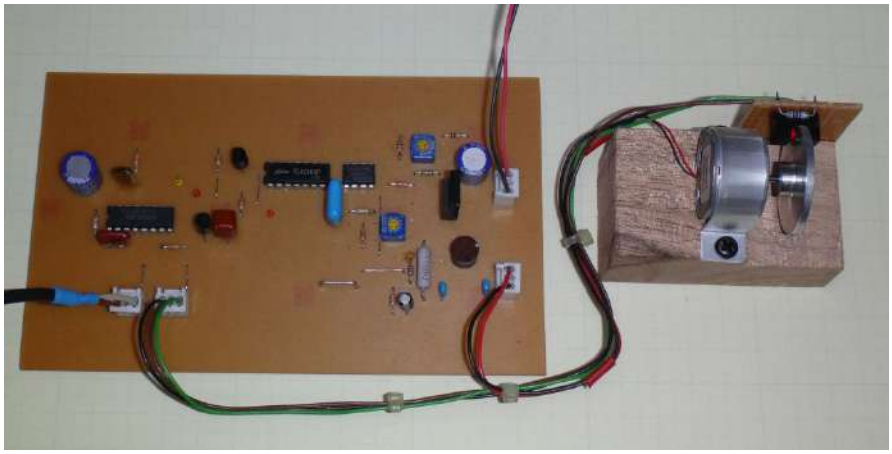

Figure 23.3: Motor speed/phase servo system

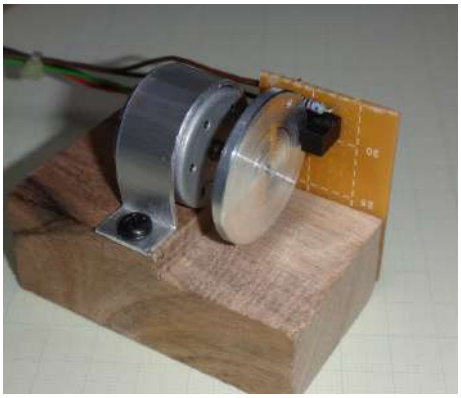

Figure 23.4: Motor with rotation phase sensor

By referring to the time chart, system operation is expected as follows. (A) shows the reference signal to which the motor rotation phase shall be locked. By the trailing edge of (A), we generate pulse (B) which resets the charge in capacitor C1. When (B) becomes "0", the ramp wave (C) starts. If we use a constant current source instead of R1, the inclination of ramp becomes linear (Fig 23.5 shows liniar ramp), but simplified circuits like this example works well[2]. The ramp is sampled by the motor's tach pulse (D) and the level of the t_s is held by the capacitor C2. This sampled level is buffered by IC3-1/2 to become phase error signal. This phase error signal is mixed with speed error signal by R4 and R5.

Fig 23.6 shows the ramp signal and sampling pulse in actual operation. The ramp signal is not linear because just a CR charging circuit is used.

Note that the phase error signal becomes a disturbance looking from speed servo signal. Because of this, the mix ratio of phase signal is much smaller than speed error.

[2]Since system is a closed loop and the loop gain is big, the small non linearity is negligible.

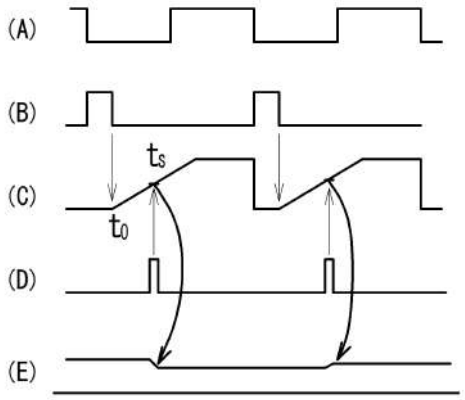

Figure 23.5: Time chart

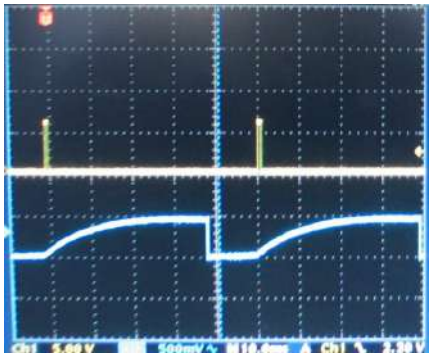

Figure 23.6: Ramp and sampling pulse

23.4 Adjustment and operation

Basis of phase control is the speed control. Therefore, by cutting off the phase loop, we adjust the speed by RV2. The best way to do this is monitoring the reference signal and motor phase signal by an oscilloscope, and adjust the speed so that the both frequency becomes the same - even with the same frequency, two signals will not be locked.

Then turn ON the phase loop. First, set the RV1 maximum (biggest resistance) and see if the phase locks with each other. By changing the level of phase error component, it will be easily observed the lock-in operation. And as the magnitude of phase error increases, the system must become oscillatory ; hunting will start.

By the reasonable setting (that is, there is a lock range) of the RV1, the phase locked operation will be observable, particularly, when the rotation is disturbed by touching with a finger[3].

Now, this is a fine example of 2nd order automatic control and will give a very good idea of what servo control and regulator means.

In this example, motor driver basically works unidirectional; that is, acceleration works fine, but breaking is only effective by the resistance R5. For improvement see Sec 23.7.

[3]Warning, we used a small toy motor.

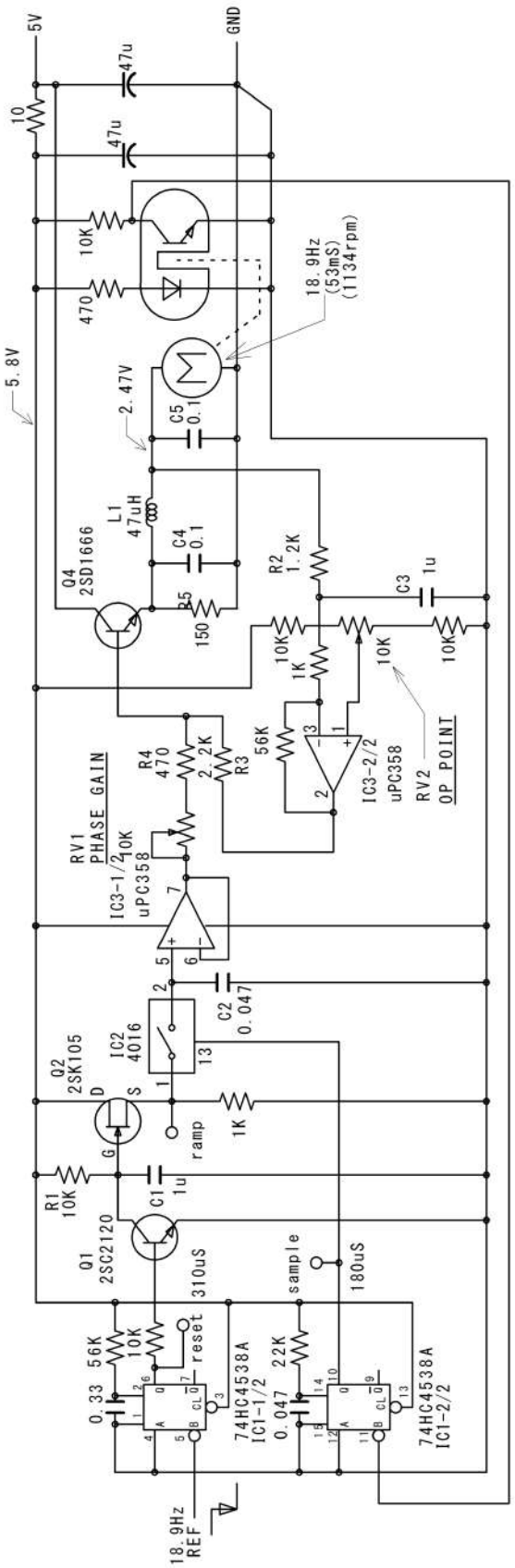

Figure 23.7: Circuit diagram

23.5 Design overview

In the above design procedure, we didn't care about the stability issue much. And a good response will be found by adjusting the phase error level. If the phase lock never settles, we might use so called phase compensation. If speed servo is poor (this is found that the robustness against load fluctuation is weak), we might increase the gain of the operational amplifier IC3-2/2. The maximum gain is available when the local feedback resistance is removed (and this is often done; that is, maximum feedback is applied via motor speed sensor like a voltage follower of OP amp application).

The overall best operating point is rather easily found heuristically. This is one of the reasons that the classic control is still widely used in the field - or if the control is satisfied by this kind of simple way, there isn't much sense to design the system by modern control.

23.6 Evaluation and measurement

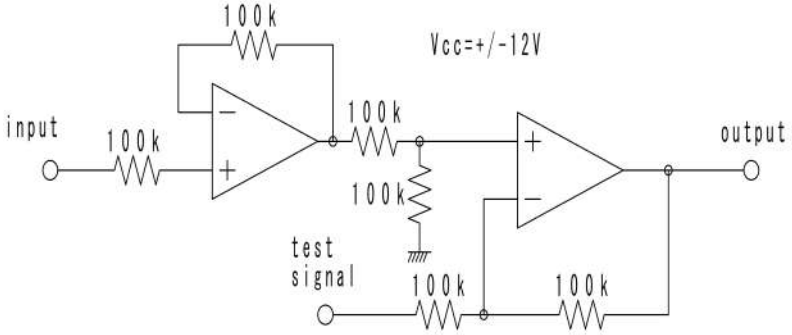

Figure 23.8: Test signal mixer

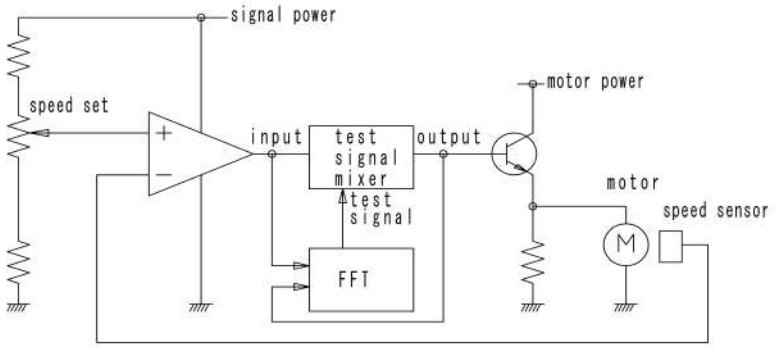

Figure 23.9: Measurement of transfer function by FFT

23.6.1 Simple evaluation

Widely used and simplest method to evaluate a servo control system is to observe a step response. By this method, low frequency square wave is applied to an appropriate point of the loop. Application shall be experimented roughly by the following criteria.

(1) Choose application point where control signal is not too small, but does not require power - that is, avoid the sensor input, or power output stage.

(2) Disturbance should be small so that the overall linear operation is maintained.

By observing the output - or intermediate signal - we can see rough but very important system responses. Theoretical basis is that we assume - and generally is - that the system is second order whose system response is well known. If we increase the phase error and intentionally cause hunting, we will also know the system resonance frequency. Although, we can only see the lowest resonance frequency, we will not be able to achieve higher frequency response over the resonance point in any case.

23.6.2 Characteristics measurement

Another more systematic way to evaluate the target will be the measurement of the system transfer function by using an FFT or specific equipments. There are two different transfer functions to be measured - open loop and closed loop - but the open loop transfer function may be more difficult because of the high gain of the system. Suppose the system has a 60dB DC gain, the 1mV input produces 1V output which may well saturate the system, or even if it does not, obtaining good S/N ratio is difficult.

The system response and potential resonance shall be known by observing the closed loop transfer function. And the merit of the measurement of the open loop transfer function may be that the low frequency gain (which shows the robustness to the disturbance of the system) is obtained better. In either measurement, the result will not be as clear like the measurement of an audio amplifier frequency characteristics.

Transfer function is measured basically in two ways. One is applying sinusoidal test signal sweeping from low frequency to appropriate frequency[4]. Another way is to use an FFT analyzer in which case the applied signal is random noise generated by the analyzer.

In either case, test signal is applied to the target system without changing the overall target transfer function. Fig 23.8 shows a circuit to apply test signal. Since the frequency range of the target system is low, almost any standard operational amplifies can be used. Fig 23.9 shows a practice to measure closed loop transfer function of a motor speed control system. As shown, the test signal mixer is inserted in the loop where target signal level is reasonably large to obtain better S/N ratio, and power is not flowing.

[4]For most of the automatic control system, the highest frequency would be $1kHz - 10kHz$, that is, relatively low. On the other hand low frequency could be 0.01Hz or even lower.

23.7 Improvement of characteristics

Phase servo system shown in Fig 23.3 is a simple and rough example and if we wish to improve the characteristics (particularly accuracy) or have to design more mechanically complicated system, at least the following points shall be considered.

(1) Use accurate speed sensor with higher sampling - 6 times sampling per rotation or more.

(2) Choose appropriate DC motor (brush less for longevity, core less for lower torque ripple, bigger in size for higher torque, sleeve bearing for less noise).

(3) Brushing up of overall mechanism is required; not only the motor, but driving mechanism including bearings (if ball bearing is used, the system noise will surely have components caused by the balls), play of driving mechanism (conventional gears can be harmful because of backlash), stiffness of belts (resonance caused stiffness and mass will introduce higher order transfer function to the system), elasticity of mechanism (cause resonance to the system) and so on.

(4) For better temperature stability and uniform production quality, use micro processor and employ digital servo. Because one of the weakness of analog servo system is drift under environmental condition, particularly most of the servo systems should require DC operation.

(5) If speed (and phase) tracking is necessary, simple emitter follower (Q4) used in Fig 23.7 does not work well because the drive is basically unidirectional (acceleration only,

break depends on the mechanical friction and R5). The same situation is applicable to the application of position control by a motor. It will require bi-directional driver. See Fig 23.10.

(6) For less power consumption, D class motor driver shall be used. (This is quite similar to D class amplifier in audio application, but the frequency range should cover DC.)

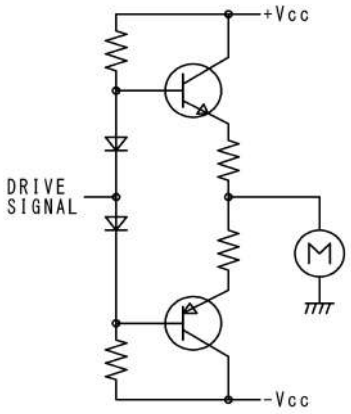

Figure 23.10: Bi-directional motor driver

Since the characteristics of the target system varies for each design, the servo control system design also becomes target specific. Automatic control is inherently a part of a system and if we wish to construct a good system, people are required to have wider

knowledge - not only electricity, programming, signal processing but mechanical design and good sense.

Chap24 Appendices

24.1 Fourier/Laplace transform to find the solution

Solution of linear differential equation can be found by applying the Laplace transform.

$$F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt.$$

That is, the target differential equation is first Laplace transformed into a function of s. Then, the function $F(s)$ will be expressed by $G(s)$ which is a polynomial of output/input showing the system characteristics. Since the solution of the original differential equation becomes the pole of $G(s)$, finding the points of peak response of $G(s)$ means to find the solution. Time response will be found by applying inverse Laplace transform; replacing by time function referring to a table[1] in practice.

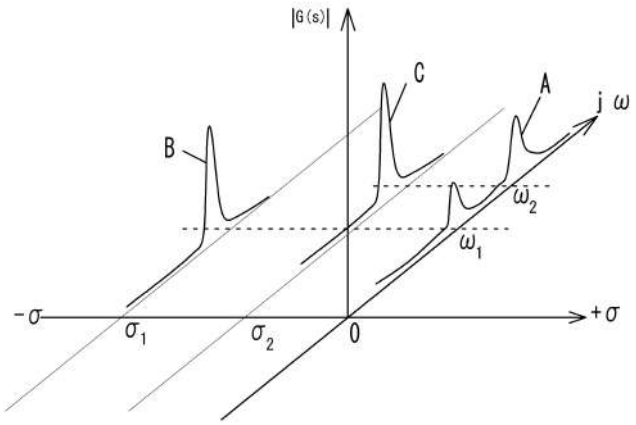

Figure 24.1: Fourier and Laplace transform

By the above process, differential equation is solved mechanically, but what is happening here may be explained as follows.

Laplace transform is an extension of Fourier transform[2] since the variable $s = \sigma + j\omega$ is the extension of $j\omega$. When Fourier transform is applied to the target system (frequency response is measured), we obtain peaks at some frequencies ω_1, ω_2.... (Fig 24.1, A) These peaks on $j\omega$ axis reflects the points where denominator of $G(s)$ becomes zero(Fig 24.1, B, C) (poles of transfer function, see Sec 7.6.1), but the peaks may not be significant by $j\omega$ axis depending on how far σs are located from $j\omega$ axis. If these poles happens to be on

[1]Laplace transform table which is a list of corresponding functions F(s) and f(t).
[2]Historically not, but mathematically yes.

$j\omega$ axis[3], the peak goes to infinity at those points.

Now, we are on the s plane, if the pole is far away from $j\omega$ axis, the Fourier transform (frequency response method) will not yield clear peak at corresponding frequency. Then, what we will do is shift the sweep by σ and apply the Fourier transform along with that line from 0[4] to $+\infty$ seeking sharp peaks. The range of value of σ is from $-\infty$ to 0[5].

This operation is not possible in engineering, (if it were, we would have FLT by now) but mathematically this is how the Laplace transform is defined.

The Laplace transform is the conventional tool in (classic) control, and it searches the solutions of the differential equation by sweeping all over the s plane.

Comparing with the method shown in Sec 1.1.4, the order of differential equation is not limited, but it is assumed that the differential equation is decomposed by eigenvalues.

24.2 Matrix calculation

In the final part of the design process of multivariable control, we need to solve matrix Ricatti equation. Not only for tracing the logical progress by matrix treatise, but for practical computer calculation as well, some basic rules of matrix operation is necessary.

24.2.1 Basic properties

(1) Rank

Matrix $\boldsymbol{A}$ is a coefficient of multiple, simultaneous linear equation $y_i = \sum_{j=1}^{m} a_{ij} x_j$.

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{pmatrix}$$

Notation of matrix form can include linear equations that are not independent with each other - or can include redundant equations, we need to worry about the maximum independent number of equations. This is the rank [6] of a matrix, and can be defined,

- Maximum number of independent vectors in column vectors of matrix $\boldsymbol{A}$,

or,

- Maximum number of independent vectors in row vectors of matrix $\boldsymbol{A}$.

Rank is calculated mechanically either by hand or computer by established procedures.

(2) Transpose

[3] This is a case of oscillator.

[4] Integration from -∞ to 0 becomes symmetrical results.

[5] For $\sigma > 0$, the differential equation diverges, and no solution available.

[6] Mathematically, rank is not ambiguous, but when we assume approximation and computer calculation, rank may be determined based on the numerical precision and may become uncertain.

Given an $n \times m$ matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ & ... & & \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix}$$

Transpose of $\boldsymbol{A}$ is defined by,

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & ... & a_{n1} \\ a_{12} & a_{22} & ... & a_{n2} \\ & ... & & \\ a_{1m} & a_{2m} & ... & a_{nm} \end{pmatrix}$$

If $\boldsymbol{A}$ is a vector,

$$\boldsymbol{A} = \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix}$$

we define a transpose of vector by

$$\boldsymbol{A}^T = (a_1 \ \ a_2 \ \ ... \ \ a_n).$$

(3) Square matrix

A matrix with equal number of row and column $(n \times n)$, it is called square matrix. If $a_{ij} = a_{ji}$ it is called symmetric matrix.

$$\begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & 7 \\ 5 & 7 & 4 \end{pmatrix}$$

For symmetry matrix,

$$\boldsymbol{A}_s = \boldsymbol{A}_s^T.$$

If $a_{ij} = -a_{ji}$ it is called skew symmetric matrix.

$$\begin{pmatrix} 1 & 3 & 5 \\ -3 & 2 & -7 \\ -5 & 7 & 4 \end{pmatrix}$$

For skew symmetric matrix,

$$\boldsymbol{A}_{sk}^T = -\boldsymbol{A}_{sk}.$$

Arbitrary square matrix is expressed by an addition of symmetric and skew-symmetric matrix.

$$\boldsymbol{A} = \boldsymbol{A}_s + \boldsymbol{A}_{sk}$$

where,

$$\boldsymbol{A}_s = \frac{\boldsymbol{A} + \boldsymbol{A}^T}{2}, \quad \boldsymbol{A}_{sk} = \frac{\boldsymbol{A} - \boldsymbol{A}^T}{2}$$

(4) Identity matrix

If $a_{ij} = 0$, and $a_{ii} = 1$, it is called an identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

24.2.2 Multiplication of matrices

If $\boldsymbol{A}$ is an $n \times m$ matrix, and $\boldsymbol{B}$ is $m \times r$ matrix, multiplication $\boldsymbol{C}$ $(n \times r)$ is defined.

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ & ... & & \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & ... & b_{1r} \\ b_{21} & b_{22} & ... & b_{2r} \\ & ... & & \\ b_{m1} & b_{m2} & ... & b_{mr} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & ... & c_{1r} \\ c_{21} & c_{22} & ... & c_{2r} \\ & ... & & \\ c_{n1} & c_{n2} & ... & c_{nr} \end{pmatrix}$$

Definition is better explained by an example, like

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & -3 & 6 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ 2 & 3 \\ -5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (1 \times 3) + (3 \times 2) + (5 \times (-5)) & (1 \times (-2)) + (3 \times 3) + (5 \times 7) \\ (2 \times 3) + (-3 \times 2) + (6 \times (-5)) & (2 \times (-2)) + (-3 \times 3) + (6 \times 7) \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 42 \\ -30 & 29 \end{pmatrix}.$$

Significant characters in matrix calculation are,

$$\boldsymbol{A} \times \boldsymbol{B} \neq \boldsymbol{B} \times \boldsymbol{A}.$$

$$\boldsymbol{AB} = 0 \quad \textit{does not imply} \quad \boldsymbol{A} = 0 \ \textit{or} \ \boldsymbol{B} = 0$$

$$\boldsymbol{AB} = \boldsymbol{AC} \quad \textit{does not imply} \quad \boldsymbol{B} = \boldsymbol{C}$$

Devision is not defined in matrix, but for square matrices,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I},$$

provided $\det \boldsymbol{A} \neq 0$.

For square, non-singular matrices,

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

$$(\boldsymbol{AB})^{T} = \boldsymbol{B}^{T}\boldsymbol{A}^{T}$$

24.2.3 Multiplication of vector and martix

Given $\boldsymbol{A}$ an n dimensional vector,

$$\boldsymbol{A} = \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix}$$

and $\boldsymbol{B}$ an $n \times n$ matrix,

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ & & ... & \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix}$$

$$\boldsymbol{A}^T\boldsymbol{B} = \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ & & ... & \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix}$$

$$= (a_1b_{11} + a_2b_{21} + ... + a_nb_{n1} \quad a_1b_{12} + a_2b_{22} + ... + a_nb_{n2} \quad ...$$

$$a_1b_{1n} + a_2b_{2n} + ... + a_nb_{nn})$$

$$\boldsymbol{B}^T\boldsymbol{A} = \begin{pmatrix} b_{11} & b_{21} & ... & b_{n1} \\ b_{12} & b_{22} & ... & b_{n2} \\ & & ... & \\ b_{1n} & b_{2n} & ... & b_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_{11} + a_2b_{21} + ... + a_nb_{n1} \\ a_1b_{12} + a_2b_{22} + ... + a_nb_{n2} \\ ... \\ a_1b_{1n} + a_2b_{2n} + ... + a_nb_{nn} \end{pmatrix}$$

Therefore,

$$\boldsymbol{A}^T\boldsymbol{B} = (\boldsymbol{B}^T\boldsymbol{A})^T$$

24.2.4 Quadratic form

Quadratic form of multivariable function is expressed by

row vector $(1 \times n)\times$ matrix $(n \times n)\times$ column vector $(n \times 1)$

such as,

$$f(x_1, x_2, x_3) = {x_1}^2 + 2{x_2}^2 + 4{x_3}^2 + 3x_1x_2 + 4x_1x_3 + 8x_2x_3$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 7 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In general, n dimensional vector $\boldsymbol{x}$ with coefficient Q is written by

$$Q = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}.$$

where,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

$$\boldsymbol{x}^T = (x_1 \quad x_2 \quad ... \quad x_n)$$

and $n \times n$ matrix $\boldsymbol{Q}$,

$$\boldsymbol{Q} = \begin{pmatrix} Q_{11} & Q_{12} & ... & Q_{1n} \\ Q_{21} & Q_{22} & ... & Q_{2n} \\ & & ... & \\ Q_{n1} & Q_{n2} & ... & Q_{nn} \end{pmatrix}$$

In case when $n = 3$, $\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$ is,

$$\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= Q_{11}{x_1}^2 + Q_{12}x_1x_2 + Q_{13}x_1x_3$$

$$+Q_{21}x_1x_2 + Q_{22}{x_2}^2 + Q_{23}x_2x_3$$

$$+Q_{31}x_1x_3 + Q_{32}x_2x_3 + Q_{33}{x_3}^2.$$

It is known that square matrix is transformed into diagonal matrix, then

$$\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= R_{11}{x_1}^2 + R_{22}{x_2}^2 + R_{33}{x_3}^2.$$

This is the form we are accustomed to, deleting the coordinates interference such as x_1x_2.

24.2.5 Differential of quadratic form

Suppose $f(x)$ is a quadratic function in diagonal form,

$$f(\boldsymbol{x}) = f(x_1, ..., x_n) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}.$$

Then,

$$f(\boldsymbol{x} + \Delta\boldsymbol{x}) = (\boldsymbol{x} + \Delta\boldsymbol{x})^T \boldsymbol{Q} (\boldsymbol{x} + \Delta\boldsymbol{x}).$$

$$= \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \Delta\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{Q} \Delta\boldsymbol{x} + \Delta\boldsymbol{x}^T \boldsymbol{Q} \Delta\boldsymbol{x}$$

$\Delta \boldsymbol{x}^T \boldsymbol{Q} \Delta \boldsymbol{x}$ is a scaler, and $\boldsymbol{Q} = \boldsymbol{Q}^T$.

$$\Delta \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} = (\Delta \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x})^T = \boldsymbol{x}^T \boldsymbol{Q}^T (\Delta \boldsymbol{x}^T)^T = \boldsymbol{x}^T \boldsymbol{Q} \Delta \boldsymbol{x}$$

Therefore,

$$f(\boldsymbol{x} + \Delta \boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + 2\boldsymbol{x}^T \boldsymbol{Q} \Delta \boldsymbol{x} + \Delta \boldsymbol{x}^T \boldsymbol{Q} \Delta \boldsymbol{x}.$$

Thus,

$$f(\boldsymbol{x} + \Delta \boldsymbol{x}) - f(\boldsymbol{x}) = (2\boldsymbol{x}^T \boldsymbol{Q} + \Delta \boldsymbol{x}^T \boldsymbol{Q}) \Delta \boldsymbol{x},$$

$$\lim_{\Delta \boldsymbol{x} \to 0} \frac{f(\boldsymbol{x} + \Delta \boldsymbol{x}) - f(\boldsymbol{x})}{\Delta \boldsymbol{x}} = \lim_{\Delta \boldsymbol{x} \to 0} (2\boldsymbol{x}^T \boldsymbol{Q} + \Delta \boldsymbol{x}^T \boldsymbol{Q}).$$

$$f\prime(\boldsymbol{x}) = 2\boldsymbol{x}^T \boldsymbol{Q}.$$

24.3 Lagrange Dirichlet stability

In discussion of Sec 11.5, it is clear how the linear energy is incorporated into quadratic performance function. However, linear energy like the gravitational potential energy presents some uneasiness - can become negative without defining the area, how zero point should be defined and so on. This uneasiness must have been shared by Lagrange because we have a theorem called "Lagrange Dirichlet stability".

Gravitational potential energy $\mathcal{T}(h)$ of an object with mass m is,

$$\mathcal{T}(h) = mgh.$$

If we define $\mathcal{T}(h) = 0$ at h (height h) is 0, the energy function looks like as shown in Fig 24.2. Then, the obvious question is that "How can we interpret the gravitational potential under generalized energy?", particularly in comparison with (D) in Fig 11.1.

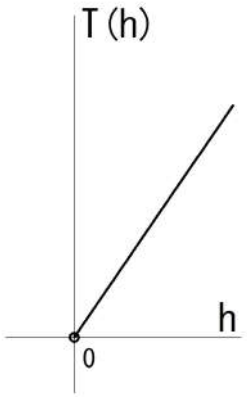

Figure 24.2: Gravitational potential energy

The fundamental difference of Fig 24.2 from (D) of Fig 11.1 is it's area of definition. The area is $h \geq 0$ in which $T(h)$ is differentiable and positive definite. If we think about free fall of an object, the total energy is

$$H(h) = \frac{1}{2} m \dot{h}^2 - mgh$$

and if we assume the energy is conserved,

$$\frac{d}{dt}H(h) = 0.$$

Thus, the system is stable (in Lyapunov's sense) at $h = 0$ which is quite consistent with our notion. If we extend the same discussion to simple pendulum, it is stable at the lowest point of weight[7].

This is posed by Lagrange as "an energy conservative system with zero kinetic energy and minimum potential energy is stable" and proved by Dirichlet. This is referred to "Lagrange Dirichlet stability".

[7]Further extend to inverted pendulum, we can also see that the upper most point of weight is also stable, provided, it is controlled so that energy exchange between kinetic and potential is kept on going.

References

[1] Tyn Myint-U, Lokenath Debnath
Linear Partial Differential Equations for Scientists and Engineers Fourth Edition
Birkhäuser 2007
ISBN-10:0-8176-4393-1, e-ISBN-10: 0-8176-4560-8
ISBN-13: 978-0-8176-4393-5, e-ISBN-13: 978-0-8176-4560-1

[2] Erich Miersemann
Partial Differential Equations Lecture Notes
Department of Mathematics, Leipzig University
Version October, 2012

[3] 吉川 敦
解析学C講義ノート, 偏微分方程式入門
九州大学, 大学院数理学研究院
平成 14 年 2 月 9 日

[4] Dan Klein
Lagrange Multipliers without Permanent Scarring
http://www.cs.berkeley.edu/ klein/papers/lagrange-multipliers.pdf

[5] Jorge Nocedal, Stephen J. Wright
Numerical Optimization
Springer 2006

[6] R. K. P. Zia, Edward F. Redish, and Susan R. McKay
Making Sense of the Legendre Transform
January 30, 2009
http://arxiv.org/pdf/0806.1147.pdf

[7] ポントリヤーギン
常微分方程式
千葉　克裕　訳
共立出版　1987

[8] 矢嶋　信男
常微分方程式
岩波　書店　1989

[9] 武部　尚志
ソリトンの数学
講座：　数学の発見　2008

[10] 高橋　康
量子力学を学ぶための解析力学入門
増補第 2 版
講談社　2000

[11] 小出　昭一郎
解析力学
岩波書店　1989

[12] Yutaka Nishiyama
The Brachistochrone Curve: The Problem of Quickest Descent
Osaka Keidai Ronshu, Vol.61 No6 March 2011 pp.309-316

[13] Richard P. Feynman The Feynman Lectures On Physics Vol.2 Addison-Wesley, 1965

[14] 十河　清
解析力学　講義ノート
北里大学・理学部・物理学科
http://www.kitasato-u.ac.jp/sci/resea/buturi/hisenkei/sogo/physics.pdf

[15] 十河　清
解析力学と交換子
数理科学　NO. 576, JUNE 2011

[16] 井上　順一郎
解析力学講義
http://ocw.nagoya-u.jp/files/39/Mechanics4.pdf

[17] 津江　保彦
高知大学理学部理学科物理科学
解析力学・量子力学
http://www.cc.kochi-u.ac.jp/ tsue/text.pdf

[18] William Rowan Hamilton
Editied by David R. Wilkins, 2000
ON A GENERAL METHOD IN DYNAMICS
Philosophical Transactions of the Royal Society ,part II for 1834, pp. 247308.

[19] William Rowan Hamilton
Editied by David R. Wilkins, 2000
SECOND ESSAY ON A GENERAL METHOD IN DYNAMICS
Philosophical Transactions of the Royal Society, part I for 1835, pp. 95144.)

[20] 中根　美知代
物理学から数学へ：Hamilton-Jacobi 理論の誕生
数理解析研究所講究録 1130 巻 2000 年 pp. 58-71

[21] Michiyo Nakane, Craig G. Fraser
The Early History of Hamilton-Jacobi Dynamics 1834-1837
Centaurus 2002: Vol.44; pp. 161-227

[22] S. Bennet
A history of control engineering 1930-1955
IEE Control engineering series 47
Peter Peregrinus Ltd, 1993.

[23] S. Bennet
A brief history of Automatic Control
IEEE Control Systems, 1966, pp. 17- 25

[24] Joseph La Salle, Solomon Lefschetz
Stability by Liapunov's direct method
Academic Press, 1961

[25] J. ラ　サール、　S. レフシェッツ
リヤプノフの方法による安定性理論
産業図書、1975

[26] Donald G. Schultz, James L. Melsa
State Functions and Linear Control Systems
McGraw-Hill Book Company, 1967

[27] The General Problem of the Stability of Motion
A. M. Lyapunov
Taylor and Francis, 1992

[28] シュルツ、メルサ. 久村 富持 訳
状態関数と線形制御系
学献社、 1970

[29] Graham C. Goodwin, Stefan F. Graebe, and Mario E. Salgado.
Control System Design. Prentice Hall, 2001.

[30] 水上 憲夫
自動制御
朝倉電気工学講座 11、 朝倉書店、 1968

[31] 福島 弘毅
自動制御理論
東北大学基礎電子工学入門講座

[32] Lennart Ljung
System Identification: Theory for the User
Prentice Hall 1987

[33] Bruce A. Chubb
Modern Analytical Design of Instrument Servomechanisms
Addison-Wesley

[34] http://octave.sourceforge.net/control/index.html

[35] http://slicot.org/

[36] What is System identification
https://indico.cern.ch/event/11749/contribution/17/material/slides/0.pdf

[37] Michael G. Safnov
The Origins of Robust Control in the 1970's
Special Issues on Multivariable Stability Magazine
Intl. J. Robust and Nonlinear Control, 7(2): 97-103 1997

[38] Randal Beard, George Saridis, John Wen
An Iterative Solution to the Finite-Time Linear Quadratic Optimal Feedback Control Problem
Rensselaer Polytechnic Institute

[39] D.L. Kleinman,
On an Iterative Technique for Ricatti Equation Computations
IEEE Transactions on Automatic Control, Vol. 13, p 114-115, 1968

[40] Donald Edward Heath

A Fortran Program to Solve the Steady-state Matrix Ricatti Equation of Optimal Control

Naval Postgraduate School, Master thesis, 1972

[41] Sergey Katsev

Streamlining of the state-dependent Riccati equation controller algorithm for an embedded implementation

Thesis. Rochester Institute of Technology, 2006

[42] Peter Benner, Vasile Sima

Solving Algebraic Riccati Equations with SLICOT

[43] Alexander Bogdanov, et. al

State-Dependent Ricatti Equation Control of a Small Unmanned Helicopter

American institute of Aeronautics and Astronautics

[44] Bradley A. Steinfeldt, Panagiotis Tsiotras

A State-Dependent Riccati Equation Approach to Atmospheric Entry Guidance

Georgia Institute of Technology

[45] Adnan Klempic, Andreas Jansson

One axle light weight truck

Chalmers University of Technology

Gothenburg, Sweden, 2012

[46] Andrew K. Stimac

Stand up and Stabilization of the Inverted Pendulum

Massachusetts Institute of Technology, 1999.

[47] 宮部　道夫、阿知波　重春

倒立振子の研究-システム制御教育実験として

中部自動車短期大学論叢　1996