EP-204 OPTICS

Manipulating Light Trace and creating an Acoustic Black Hole using GRIN optics

IV Semester Project

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Contents

Abstract

Light propagation obeys Fermat's principle, and an important inference of Fermat's principle is the optical Lagrange equation, from which the light trace can be determined with a given refractive index. Here, we consider the inverse problem of how to derive the refractive index distribution of a planar geometric optical system once the trace is predetermined. Based on the optical Lagrange equation, we propose a dynamic equation model which associates the refractive index with the light trace. With the consideration of a certain trace, we illustrate the process of solving the partial differential equation of refractive index through first integral method. By setting the distribution function of a gradient-refractive-index (GRIN) medium, one can control the light traveling along a desirable curve, adjust the incoming and outgoing rays, and also use the trace to paint geometrics. This method develops the Lagrangian optics in the application of ray dynamic system design, such as lens, beam splitter, meta-surface and optical waveguide. It provides a theoretical guidance to manipulate the ray in a GRIN medium.

1. Introduction

The desire to control the propagation of light by using gradient-refractive-index (GRIN) medium has received considerable attention during recent years. Because the GRIN elements can remarkably improve the properties of optical elements while reducing the size, weight, and the number of optical components, they can greatly promote the integrated level of optical systems and are widely used in the manufacture of lens, meta-surface, biological optics, and visible systems. In addition, because of the similar principles between light and acoustic wave, its extraordinary performance is also introduced in the design of phononic crystals.

Due to its wide application, it is fundamentally important to study how to properly design a GRIN system, so that the light trace can exhibit some expected characters. Actually, this is a geometric optics problem. As known, in geometric optics, light trace can be determined by the Fermat's principle. Its counterpart in mechanics is the famous Hamilton's principle. Because of the similarity between these two principles, there is a very close analogy between analytical mechanics and geometric optics. There are two direct inferences of Fermat's principle, known as the Snell's law and the optical Lagrange (or Hamilton) equations. To anticipate the light trace, the former is applicable for the refraction in a piecewise constant refractive index, while the latter is convenient to handle the continuous cases, that is, GRIN medium. Aiming to the design of on-chip lens and bio-photonics applications, previous studies mostly focused on the study of optical system with a given refractive index function. These systems are usually designed by experience. Since the line shape of refractive index is already known, the only procedure is to modify the undetermined coefficients in a given function to match the technical requirement. But usually, what we expect is a more purposeful design for optical systems. That is to say, we expect the ray can travel as imagined (its predetermined path may be an arbitrary curve), rather than to solve the light trace with a given refractive index function. Therefore, we propose such a problem: to design a certain distribution function (it is beforehand unknown and thus cannot be obtained by fitting parameters), so that the ray can travel along a predetermined path. However, the general method for this inverse process, to the best of our knowledge, has not been paid much attention.

In this paper, we will seek for a general method on how to design the GRIN optical system when the light trace is predetermined, so that we can precisely manipulate the light trace and let it travel along a desirable geometric curve. This may provide some reference models for the design of optical systems. The situation we consider here is a pure geometric optical system, that is, we ignore the dispersion, dissipation, interference, diffraction and other factors, only need to consider the beam's refraction in a GRIN medium. At some point, this model is similar to the corresponding classical mechanics system. Based on variation principle and optical Lagrange equation, we will illustrate the general procedure of solving the refractive index function with a given trace through several examples.

2. Mathematics

We begin our discussion by reviewing the basic Fermat principle: the optical path reaches the extreme value when light propagates through a background medium, which is mathematically presented as the variation of optical path functional equals to zero, that is

$$\delta \int_{l} \mu \, ds = 0 \qquad \dots (\mathbf{1})$$

For simplicity, we consider the two-dimensional case, that is, we assume the medium is z-direction homogeneous and the ray is launched in x-y plane. Using Cartesian coordinates, **eq1** can be rewritten as

$$\delta \int_{x_1}^{x_2} \mu(x,y)\sqrt{dx^2 + dy^2} = \delta \int_{x_1}^{x_2} \mu(x,y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\delta \int_{x_1}^{x_2} \mu(x,y)\sqrt{1 + y'^2}\, dx = \delta \int_{x_1}^{x_2} \mathcal{L}(x,y,y') = 0 \;\dots (\mathbf{2})$$

where $\mathcal{L}(x,y,y') = \mu(x,y)\sqrt{1 + y'^2}$ which satisfies the Euler – Lagrange's equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \dots (\mathbf{3})$$

As shown below, the Cartesian-coordinate form is convenient for the derivation of the refractive index distribution once the light trace is predetermined

$$\mathcal{L}(x,y,y') = \mu(x,y)\sqrt{1 + {y'}^2} =: \frac{\mu}{\gamma}$$

$$where\; \gamma = \frac{dx}{ds} = \frac{1}{\sqrt{1 + {y'}^2}} \leq 1$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mu}{\partial y}\sqrt{1 + y'^2} \; and \; \frac{\partial \mathcal{L}}{\partial y'} = \frac{\mu\, y'}{\sqrt{1 + y'^2}}$$

On substituting above values in **eq3** and simplifying we get

$$y'\frac{\partial \mu}{\partial x} - \frac{\partial \mu}{\partial y} + \frac{y''}{1 + y'^2}\,\mu \; = \; 0 \quad \dots (\mathbf{4})$$

or simply we can write it compactly as

$$\frac{d}{dx}\left(\mu(x,y)\;\frac{dy}{dx}\right) = \frac{\partial \mu}{\partial y} \qquad \dots (\mathbf{4}\,*)$$

Theorem 1 : (Fermat's principle implies eikonal equation). *Stationarity of the optical path or action* $\boldsymbol{A}$, *under variations of the ray paths*

$$\delta \boldsymbol{A} = \delta \int_A^B \mu(\boldsymbol{r}(s))\sqrt{\frac{d\boldsymbol{r}}{ds}\cdot\frac{d\boldsymbol{r}}{ds}}\,ds = 0,$$

defined using arc-length parameter s, satisfying $ds^2 = d\mathbf{r}(s)\cdot d\mathbf{r}(s)$ *and* $|\dot{\mathbf{r}}| = 1$, *implies the equation for the ray path* $\boldsymbol{r} \in \mathbb{R}^{\mathbf{3}}$ *is*

$$\frac{d}{ds}\left(\mu(\boldsymbol{r})\,\frac{d\boldsymbol{r}}{ds}\right) = \frac{\partial \mu}{\partial \boldsymbol{r}}$$

In ray optics, this is called ***eikonal equation.***

This equation is the foundation of what we are going to do further in this report. As seen, if a certain distribution for GRIN medium $\mu = \mu(x,y)$ is given, **eq. 4** is a second-order ordinary differential equation (ODE) about the light trace y=(x). Principally, despite the existence of analytical solution, a certain $\mu(x,y)$ can determine a family of propagation traces (integral curves). On the contrary, a specific light trace, assumed as y=f(x), can be straightforwardly obtained by properly designing the distribution function of refractive index, $\mu = \mu(x, y)$, which is solved from **eq. 4**. In such a case, **eq. 4** is a first-order linear partial differential equation (PDE) about the refractive index. Actually, because of the mathematical difficulty that the analytical solution for nonlinear ODE does not always exist, the shape of the light trace is hard to anticipate under an arbitrary non-constant refractive index, and only a few situations are achievable. However, it is relatively simple to design the refractive index so that the light trace can paint what we imagine, and as stated in Sec. I, this work is more practical. Therefore, we are more interested in the latter case.

It should be emphasized that, for a predetermined trace y=(x), although the refractive index function $\mu = \mu(x, y)$ can be solved from **Eq. 4**, it does not ensure that the light must travel along the trace y=$f(x)$, because when we substitute $\mu = \mu(x, y)$ back into **Eq. 4**, y=$f(x)$ is not surely the unique solution. Physically, a specific $\mu(x, y)$ may support more than one propagation mode, and it depends on the position and the direction of light source (initial conditions). In addition, once we substitute solution $\mu = \mu(x, y)$ back to **Eq. 4**, the initial conditions of Cauchy problem $y(x_0) = y_0$, $y'(x_0) = y'_0$ may conflict with the predetermined trace y=$f(x)$. In such a case, the variation problem has no solution, and the predetermined light trace just means the asymptotic line. The light will travel, to some extent, close to this curve, but not coincide. Therefore, in order to obtain predetermined trace y= (x), one need not only to find a certain $\mu = \mu(x, y)$ satisfies **Eq. 4** (this may not be unique), but also to properly set the light source so that its launch condition (initial condition) satisfies y= (x).

Generally, to derive the PDE about $\mu(x, y)$, the light equation should be considered. This indicates that the solution of the PDE about $\mu(x, y)$ may be added an additional condition y= (x). In other words, $\mu(x, y)$ only needs to satisfy **Eq. 4** on the predetermined light trace y=(x), and no need to be valid on the whole plane area. This condition is physically weak, but for **Eq. 4**, it is a constraint condition and thus mathematically strong. Nevertheless, for $\mu(x, y)$ the constraint condition y= (x) is not necessary, since a given $\mu(x, y)$ which satisfies the equation in the whole area is certainly valid on a specific curve. That is to say, we can also physically strengthen the condition, so that $\mu(x, y)$ in **Eq. 4** is valid in the whole plane area, and the conditions that $\mu(x, y)$ should satisfy are mathematically weakened. Making use of the trace equation may be, sometimes, beneficial to solve the problem.

Based on the first-order linear PDE theory, we can "play with the beam" and depict some common geometrics by properly setting the refractive index. Below are some simple examples.

3. Examples

Here are some examples to solve **Eq. (4)** under a predetermined light trace, and simply introduce their applications. In order to conveniently illustrate this method, the chosen examples are all analytically solvable. For some more complicated light traces, we can also refer to numerical method to solve **Eq. (4)** and get the refractive index function.

3.1. Straight line with a certain slope

As it is familiar to us, light travels along a straight line in a homogeneous medium. This can be also seen in **Eq. (4)** as $\mu = \mu(x, y) \equiv const$. However, can we add some restrictions so that only light propagates along a certain direction can go straight? This can be answered from **Eq. (4)**. Assume the trace equation is y=mx+c and substitute it into **Eq. (4)**

$$y = mx + c$$

$$y' = m \; and \; y'' = 0$$

putting these values in **eq 4** we get

$$m\frac{\partial \mu}{\partial x} - \frac{\partial \mu}{\partial y} = 0 \quad \dots (\mathbf{5})$$

the above equation is called *linear Lagrange's differential equation* of the form $Pp + Qq = R$ where P, Q and R are the function of x, y, z or constant and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. Hence has the characteristic equation as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

But in our case, third variable is missing hence we only have P = m and Q = -1. Therefore we have

$$\frac{dx}{m} = \frac{dy}{-1}$$

Hence our solution takes the form $x + my = C$. Therefore the refractive index would be

$$\mu(x, y) = \phi(x + my) \quad \dots (\mathbf{6})$$

Where ϕ is undetermined arbitrary function that can be chosen as per convenience. In such a case, if the initial condition of y=f(x) satisfies $y'(x0)$=k, the light will travel in straight line. In other words, if the light source's inclination with respect to horizontal axis satisfies tan(α)=k, the light trace is straight. However, if the source does not satisfy this condition, the initial conditions will contradict with the light equation (whatever c is), and the variation problem has no solution, which means $y = mx + c$ is not the light trace but an asymptotic line of the light trace. That is to say, if the refractive index is a non-constant function of $x + my$, only light with slope m can travel in straight line, as the simulation shown in **Fig. 1(a)**. These rays will be approximately parallel after a certain propagation distance. This is in contrast with homogeneous case.

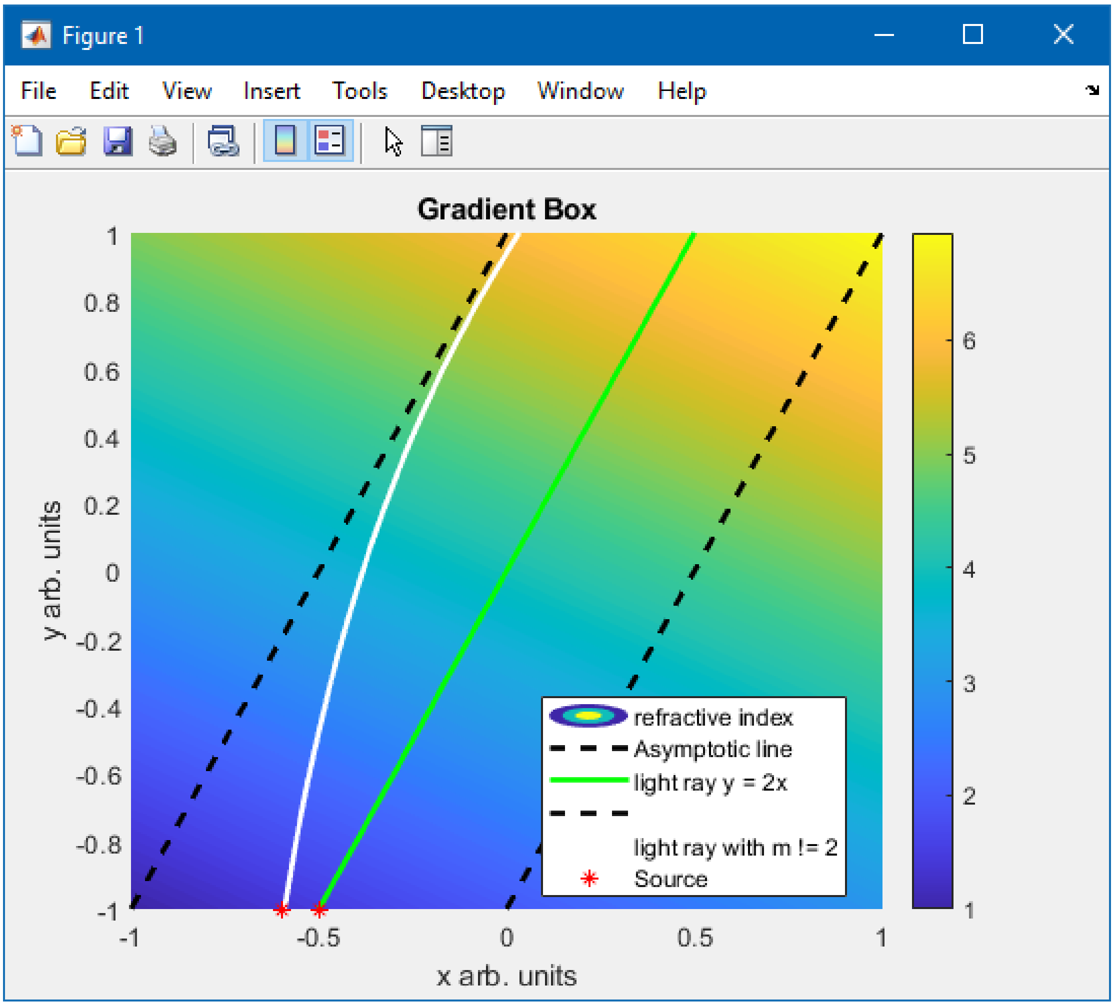


Fig. 1 Light propagating in a GRIN medium with refractive index characterized by **Eq. (6)**

$\mu(x, y) = \phi(x + my) = x + 2y + 4.$ If the launch condition satisfies $y'(x0)$=m=2, the ray will travel straightly. Otherwise, it will get close to the asymptotic line with slope m = 2.

Hence this gradient box can be used to transform the divergent light into directional light. If the GRIN is set as a function of x+my, ray with launch condition satisfies $y'(x_0) \neq$ m will travel close to an asymptotic line y = m x + c. This is also the direction of refractive index gradient. Therefore, if the GRIN medium is thick enough, the inclination of emergent rays is almost the same and thus approximately parallel.

This way is more convenient than traditional convex lens and concave mirror, since there is no need to place the source at the focus, only need that the GRIN medium is thick enough. Also, compared with the Wood lens, the design of the medium's parameter has more choices, because we only need to design the refractive index as the form of **Eq. (6)**, and specially, it can be a linear function of coordinates. Noticeably, when we take advantage of this effect, the medium should become denser along the propagation direction of light, so that it can collimate the beam. However, if the medium becomes rarer, the beam will be divergent, since the light path is invertible. This inverse process can be also used to focus the parallel beam. Compared with the traditional lens, if the thickness of GRIN medium is adjustable, it can realize a changeable focus.

Actually, this result is explainable from the perspective of refraction. From **Eq. (6)**, the direction of refractive index gradient $\nabla\mu = \frac{\partial\mu}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\mu}{\partial y}\hat{\boldsymbol{y}} = \phi'(x+my)(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})$ is in accordance with the propagation direction. This can be understood as the light always points toward the "normal incidence" direction from one medium to another. Refraction does not happen along the normal incidence direction and thus its direction does not change. Therefore, if we set the initial direction of source as **i**+m**j**, the light trace is apparently straight.

3.2. Circle

The previous example is, in a sense, to seek for the general condition under which the light can travel in straight line. From this one we begin to discuss how to control the light propagating along a certain geometric shape. Assume the light propagates as a trace of circle $x^2 + y^2 = R^2$ and therefore $y' = -\frac{y}{x}$ *and*

$y'' = -R^2/y^3$. On putting these values in **Eq. 4** the equation

$$y' \frac{\partial \mu}{\partial x} - \frac{\partial \mu}{\partial y} + \frac{y''}{1 + y^2} \mu = 0$$

takes the form

$$x \frac{\partial \mu}{\partial x} + y \frac{\partial \mu}{\partial y} + \mu = 0 \quad \dots (\mathbf{7})$$

Now using polar coordinates ,

$$\mu = \mu(r, \theta), \qquad x = rcos\, \theta, \qquad y = rsin\, \theta$$

$$\frac{\partial \mu}{\partial x} = \frac{\partial \mu}{\partial \theta} \frac{d\theta}{dx} + \frac{\partial \mu}{\partial r} \frac{dr}{dx} \quad and \quad \frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial \theta} \frac{d\theta}{dy} + \frac{\partial \mu}{\partial r} \frac{dr}{dy}$$

$$\therefore \frac{\partial \mu}{\partial x} = \cos\theta \frac{\partial \mu}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \mu}{\partial \theta} \quad and \quad \frac{\partial \mu}{\partial y} \sin\theta \frac{\partial \mu}{\partial r} - \frac{\cos\theta}{r} \frac{\partial \mu}{\partial \theta}$$

Using these values, **Eq. 7** becomes

$$r \frac{\partial \mu}{\partial r} + \mu = 0 \quad \dots (\mathbf{8})$$

So the Lagrange’s coefficients are P = r, Q = 0, R = -n. Hence the characteristic equation will be written as follows

$$\frac{dr}{r} = \frac{d\mu}{-\mu}$$

On integrating the above equation

$$\int \frac{dr}{r} = -\int \frac{d\mu}{\mu}$$

$$\ln r = -\ln \mu + c$$

$$\ln r\mu = c$$

$$r\mu = e^c$$

$$\mu = \frac{r_0}{r} \quad \ldots \mathbf{(9)}$$

$$\mu(x, y) = \frac{r_0}{\sqrt{x^2 + y^2}} \quad \ldots \mathbf{(10)}$$

Considering the fact that the refractive index is usually greater than 1 and in this case it approaches infinity, so let's modify **Eq. 10**

$$\mu(x, y) = \begin{cases} 1, \quad x^2 + y^2 > r_0^2 \\ \dfrac{r_0}{\sqrt{x^2 + y^2}}, \quad \dfrac{r_0^2}{25} \leq x^2 + y^2 \leq r_0^2 \\ 5, \quad x^2 + y^2 < \dfrac{r_0^2}{25} \end{cases} \quad \ldots \mathbf{(11)}$$

Thus, the refractive index distribution in **Eq. (11)** is valid for **Eq. (7)** in the annulus $\frac{r_0^2}{25} \leq x^2 + y^2 \leq r_0^2$. Under such a circumstance, when $\frac{r_0}{5} \leq r \leq r_0$ and some appropriate initial conditions are satisfied, the light will travel along a circle trace.

We choose the Radii of circles as R = a, 2a, 3a. To ensure the circles are located in the GRIN medium, $r_0 < a < \frac{5}{3} r_0$ is required. We show the result in **Fig. 2**. The initial conditions, according to the trace equation, should be $y'|_{x=x_0} = -\frac{x_0}{y_0}$ and $x_0^2 + y_0^2 = R^2$. This means, to obtain a circle trace, the launch direction of the source should be perpendicular to the radial direction, and the radius of circle is determined by the launch point.

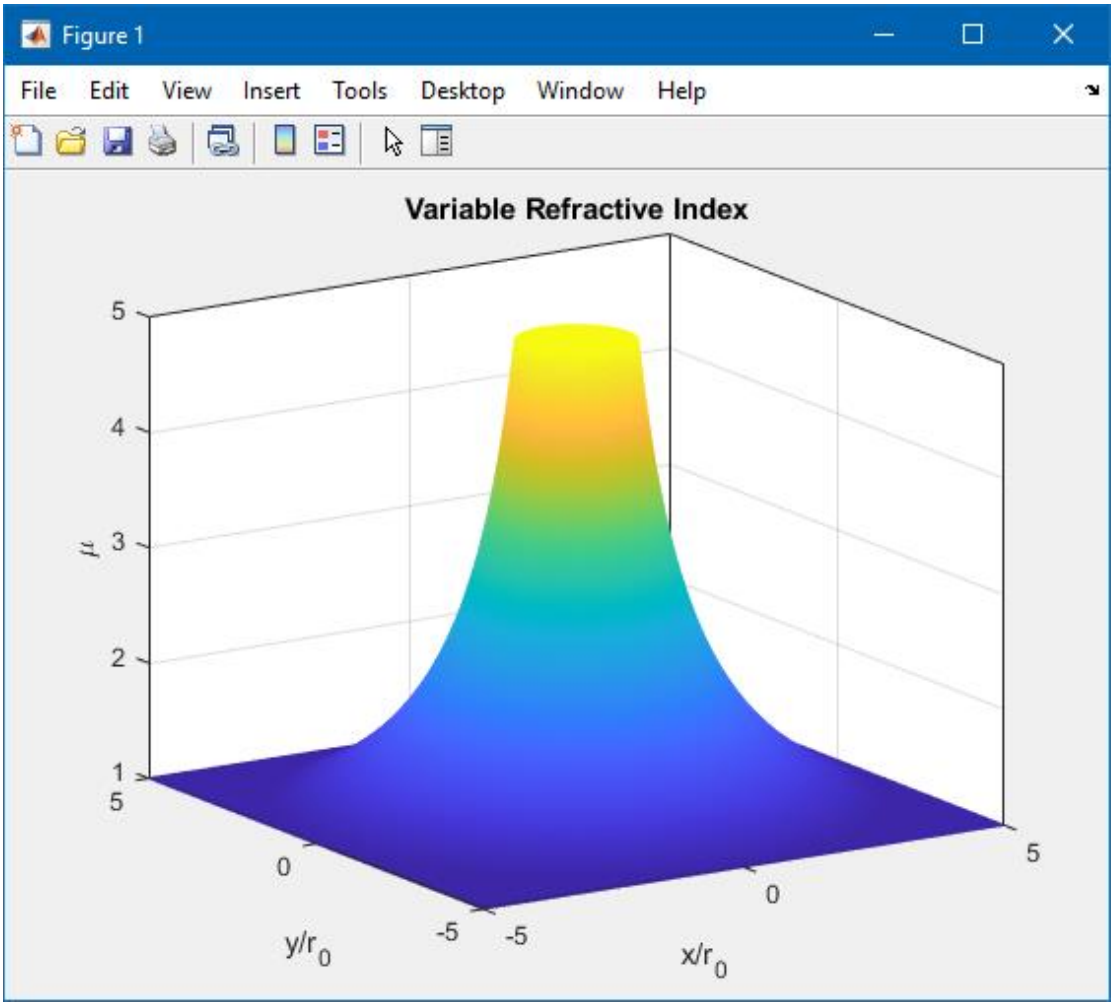


Fig 2. (a). Refractive index distribution described by **Eq. (11)**

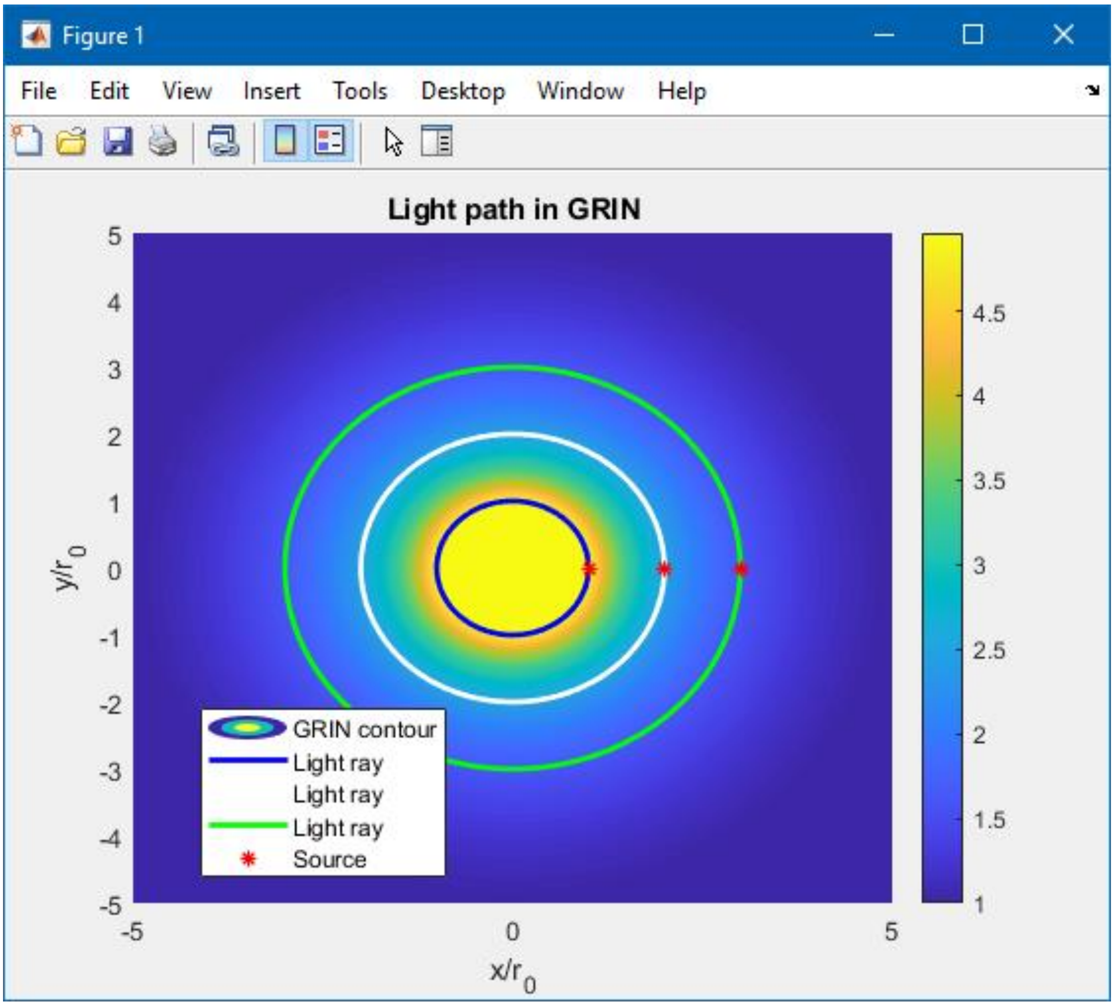


Fig 2. (b) Light propagating in a GRIN medium with refractive index characterized by **Eq. (11)**. If the ray is launched along the tangential direction, the trace is a circle.

Undoubtedly, this result can be also explained from the perspective of refraction. The gradient of refractive index $\nabla\mu = -\frac{1}{r^2}\,\hat{\boldsymbol{e}}_{\boldsymbol{r}}$ is along the radial direction, and the light propagating along the tangential direction $\mathbf{e}_\theta$ (perpendicular to the radial direction) can be understood as "grazing incidence". Because the angular distribution of refractive index is uniform, the deflection of light should be equal everywhere when rr is fixed. This means, if the light initially travels towards the tangential direction $\mathbf{e}_\theta$, it will point to this direction forever, indicating that the trace is a circle. This is similar to a particle's circular motion under a centripetal force (or a gravitational potential), since the gradient field always points to the center. As the gradient gets smaller with the increase of distance, the refraction is weakened and the trace's curvature is also getting smaller, resulting in a larger circle trace, as expected.

3.3. Equilateral Hyperbola

Assume the light travels as a trace of equilateral hyperbola $x^2 - y^2 = \pm c^2$ then we have

$$y' = \frac{y}{x} \quad and \; y'' = \mp\frac{c^2}{y^3}$$

Hence the **Eq. 4** becomes

$$x\frac{\partial y}{\partial x} - y\frac{\partial y}{\partial x} - \frac{x^2 - y^2}{x^2 + y^2}\mu = 0 \qquad \dots(\mathbf{12})$$

So the characteristic equation is written as

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{x^2 + y^2}{x^2 - y^2}\frac{d\mu}{\mu}$$

The first integral can be obtained as $xy = C_1$ and second integral as $\frac{\mu^2}{x^2+y^2} = C_2$. The implicit general solution of **Eq.12** is $F\left(xy, \frac{\mu^2}{x^2+y^2}\right) = 0$. And refractive index is solved as

$$\mu(x, y) = \sqrt{x^2 + y^2}\phi(xy) \quad \dots(\mathbf{13})$$

where $\phi(xy)$ is a positive function which can be arbitrarily determined. For simplicity, we set $\phi(xy) \equiv \frac{1}{r_0}$, and as above, in case the refractive index is abnormal, we modify **Eq. (13)** as

$$\mu(x,y) = \begin{cases} 1, & x^2+y^2<r_0^2 \\ \dfrac{\sqrt{x^2+y^2}}{r_0}, & r_0^2 \leq x^2+y^2 \leq 25r_0^2 \\ 5, & x^2+y^2>25r_0^2 \end{cases} \quad \ldots(\mathbf{14})$$

The hyperbolic trace holds in the annulus $r_0^2 \leq x^2+y^2 \leq 25r_0^2$. We depict the result in **Fig. 3**. As seen, the plane area is separated into four parts by two asymptotic lines y=±x, and the four branches of $x^2-y^2=\pm c^2$ are respectively located in the four parts. Which branch the ray goes is determined by the launch point. Once the initial condition $y'(x=x_0)=\frac{x_0}{y_0}$ is satisfied, the ray will travel along a hyperbolic trace.

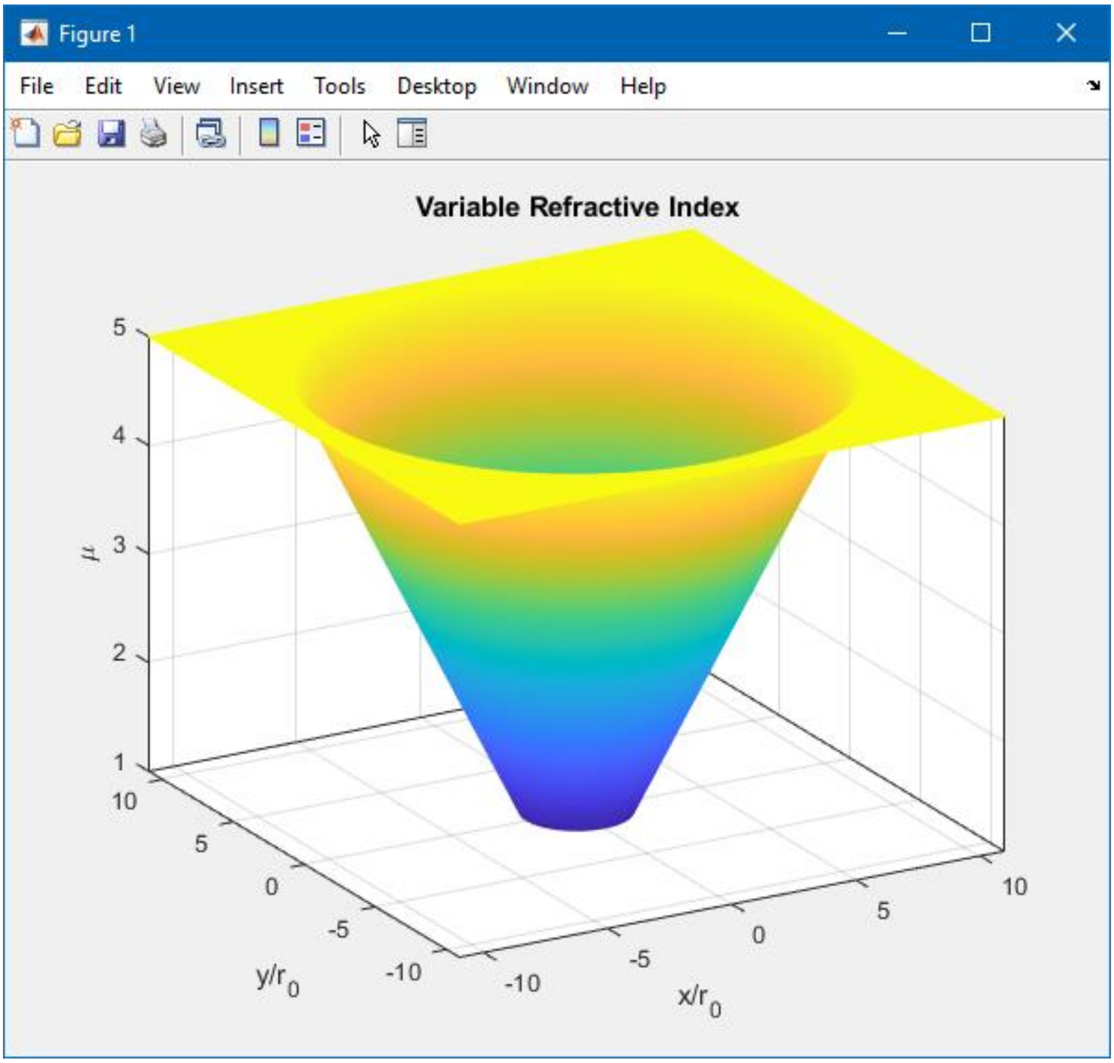


Fig. 3 (a) Refractive index distribution described by **Eq.(14)**

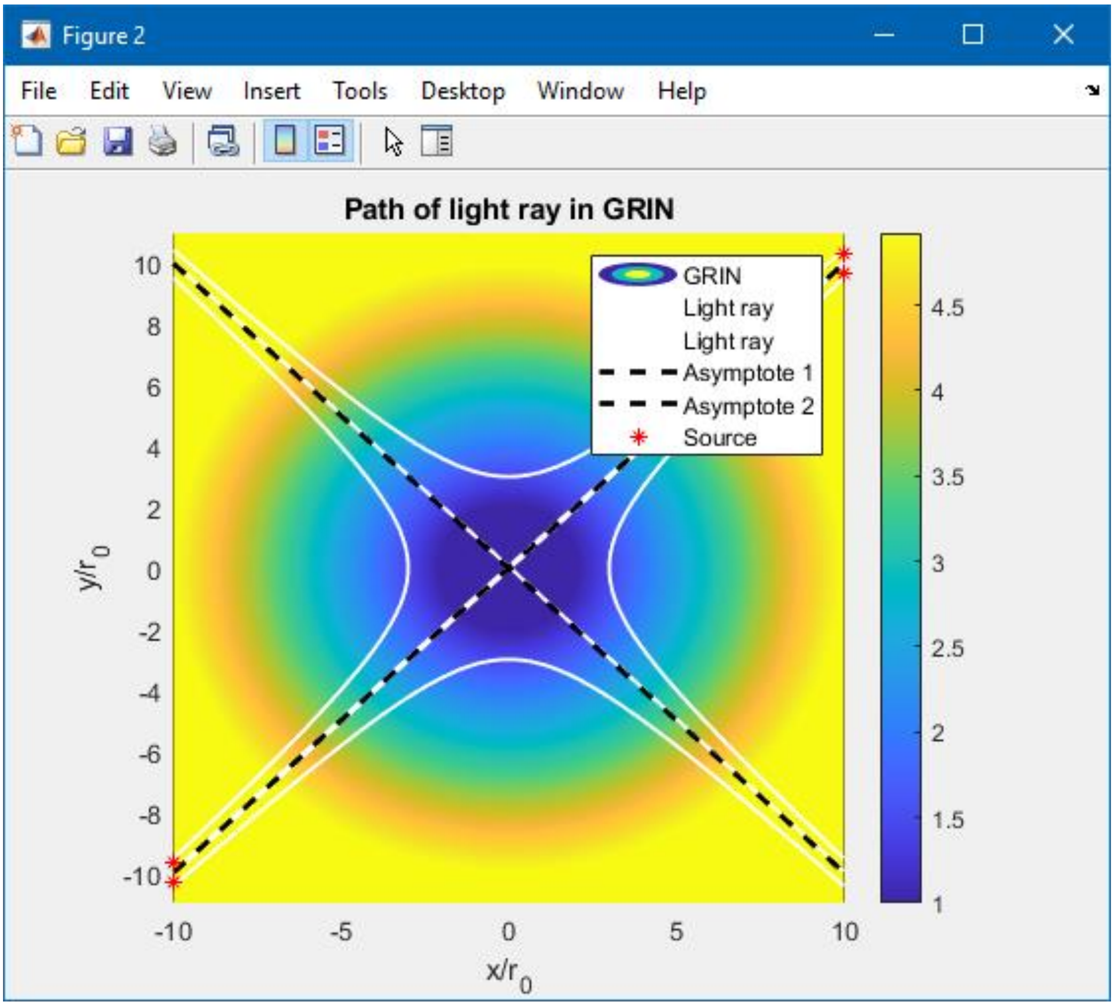


Fig 3 (b) Light propagating in a GRIN medium with refractive index characterized by **Eq. (14)**. If the launch condition satisfies $y'(x = x_0) = \frac{x_0}{y_0}$, the trace is a branch of the equilateral hyperbola.

We can simply illustrate the application of this example. Actually, this optical system can be used as a beam splitter. Imagine a wide beam is incident along one asymptotic line from infinity, and it will be split into two beams. Compared with **Fig. 3**, the optical system here is rotated by 45 degrees. If the GRIN medium is thick and wide enough, they are close to the other asymptotic line according to the above result. We can externally connect the homogenous medium to the GRIN medium, so that they will finally go to opposite directions.

3.4. Parabola

We set the predetermined light trace as a parabola $y=ax^2$ ($a>0$), and we have $y'=2ax$, $y''=2a$. **Eq. (4)** becomes

$$2ax\frac{\partial\mu}{\partial x}-\frac{\partial\mu}{\partial y}+\frac{2a}{1+4a^2x^2}\mu=0 \qquad \dots(\mathbf{15})$$

Although **Eq. (15)** can be directly solved, the expression of $\mu(x,y)$ is a little complicated. Therefore, we try to seek for a simpler distribution function. Multiplying **Eq.(15)** by x and using $y=ax^2$, we recast **Eq. (15)** as (16)

$$2y\frac{\partial\mu}{\partial x}-x\frac{\partial\mu}{\partial y}+\frac{2ax}{1+4ay}\mu=0 \qquad \dots(\mathbf{16})$$

It should be pointed out, the transformation of **Eq. (15)** is not unique, indicating that the solution of refractive index is not unique either. As mentioned above, the solution only need to satisfy **Eq. (4)** on the predetermined trace $y=ax^2$. Therefore, this procedure is only to simplify the equation so that its solution will be more concise. The characteristic equation of **Eq.(16)** is

$$\frac{dx}{2y}=-\frac{dy}{x}=-\frac{1+4ay}{2ax}\frac{d\mu}{\mu}$$

and its first integral is

$$x^2 + 2y^2 = C_1 \ , \quad \frac{\mu^2}{1 + 4ay} = C_2$$

Therefore the implicit general solution is

$$F\left(x^2 + 2y^2, \frac{\mu^2}{1 + 4ay}\right) = 0$$

which gives the general solution

$$\mu = \sqrt{1 + 4ay}\, \phi(x^2 + 2y^2) \qquad \dots (\mathbf{17})$$

As above, we set $\phi \equiv 1$ for simplicity, and we can get the distribution function

$$\mu(x, y) = \sqrt{1 + 4ay} \qquad \dots (\mathbf{18})$$

We depict the result in **Fig. 4**. With the increase of y, the refractive index increases while the gradient ∇n decreases. This indicates that, when the light direction is inclined to the + y direction, it is equivalent that the light continuously travels from the denser medium to the thinner one, and the slope of the trace will increase. Because $\nabla n \to 0$ when $y \to +\infty$, the medium is approximately homogeneous and the light approximately rises in a straight line when y is large. Definitely, if the light direction is inclined to the *-y* direction, we can derive the opposite conclusion. This is in accordance with the characters of parabolic trace.

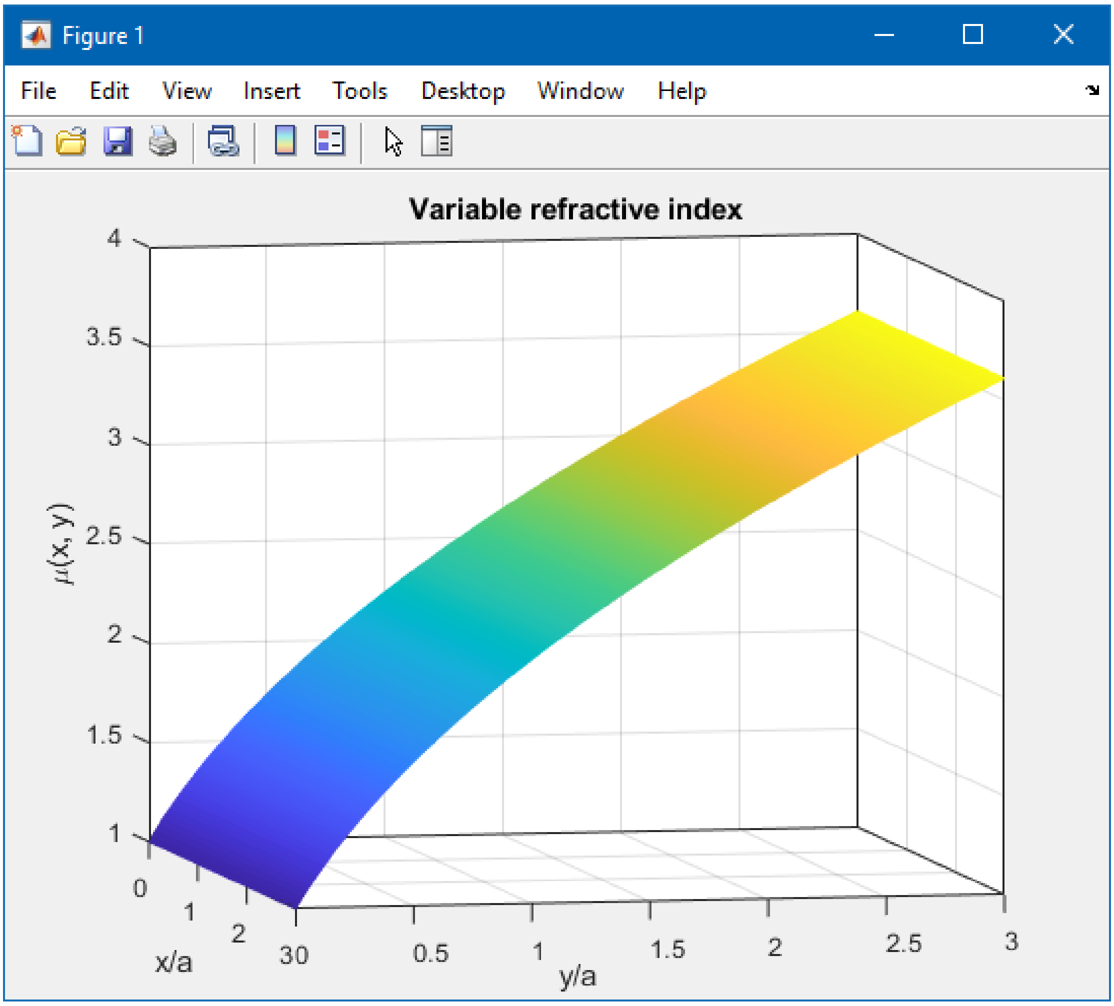


Fig. 4 (a) Refractive index distribution described by **Eq.(18)**.

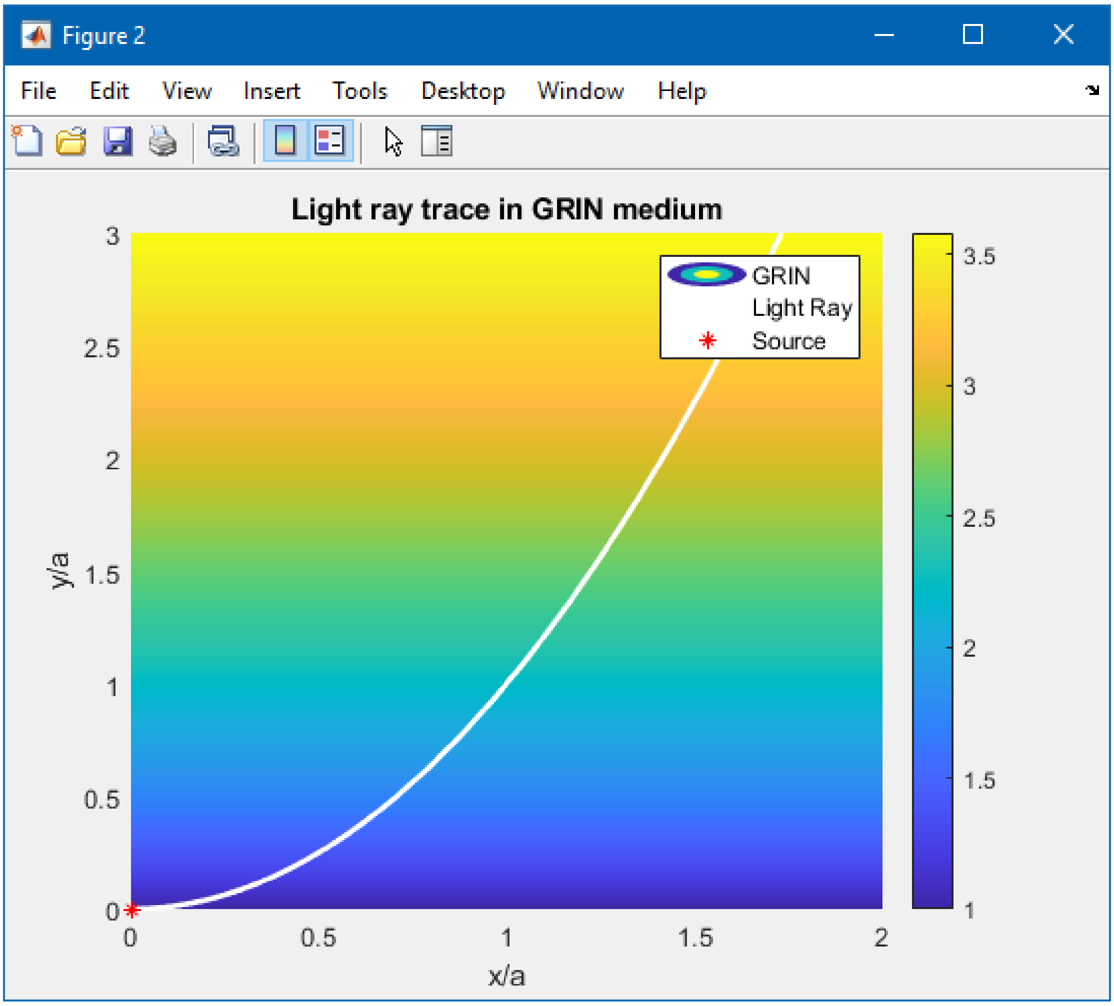


Fig 4 (b) Light propagating in a GRIN medium with refractive index characterized by **Eq. (18)**. If the launch condition satisfies $y(x = x_0) = ax_0^2$ *and* $y'(x = x_0) = 2ax_0$, the trace is a parabola.

3.5. Exponential (Logarithmic) curve

Through similar procedures, we can also manipulate the ray to travel as an exponential curve. Substituting y=y'=y''=e^x into **Eq. (4)** yields

$$y\frac{\partial\mu}{\partial x} - \frac{\partial\mu}{\partial y} + \frac{y}{1+y^2}\mu = 0 \qquad \dots(\mathbf{19})$$

The characteristic equation of the above linear partial differential equation is

$$\frac{dx}{y} = -dy = -\frac{1 + y^2}{\mu}\frac{d\mu}{\mu}$$

So the first integral is determined as

$$\frac{dx}{y} = -dy$$

$$dx + ydy = 0$$

$$\int dx + \int ydy = C$$

$$x + \frac{y^2}{2} = C$$

$$2x + y^2 = C_1 \quad and$$

$$dy = \frac{1 + y^2}{y}\frac{d\mu}{\mu}$$

$$\int \frac{y}{1 + y^2} dy = \int \frac{d\mu}{\mu}$$

$$\frac{1}{2}\ln(1 + y^2) + C = \ln \mu$$

$$\ln(1 + y^2) + C' = \ln \mu^2$$

Taking exponent of both sides

$$C_2(1 + y^2) = \mu^2$$

$$C_2 = \frac{\mu^2}{1 + y^2}$$

So the implicit general solution of the partial linear differential equation is

$$F\left(2x + y, \frac{\mu^2}{1 + y^2}\right) = 0$$

Which gives

$$\mu(x, y) = \sqrt{1 + y^2}\phi(2x + y^2) \quad \dots(\mathbf{20})$$

As above we put $\phi \equiv 1$and obtain refractive index function

$$\mu(x, y) = \sqrt{1 + y^2} \qquad \dots(\mathbf{21})$$

When the refractive index is set as **Eq. (21)**, it is easy to see $y=e^{-x}$,$=-e^{x}$ and $y=-e^{-x}$ also satisfy **Eq. (4)** because of the symmetry. Therefore, depending on the initial conditions, we can obtain four exponential curves. If we exchange *x* and *y* coordinates, the corresponding logarithmic curves can be derived. We depict the result in **Fig. 5**.

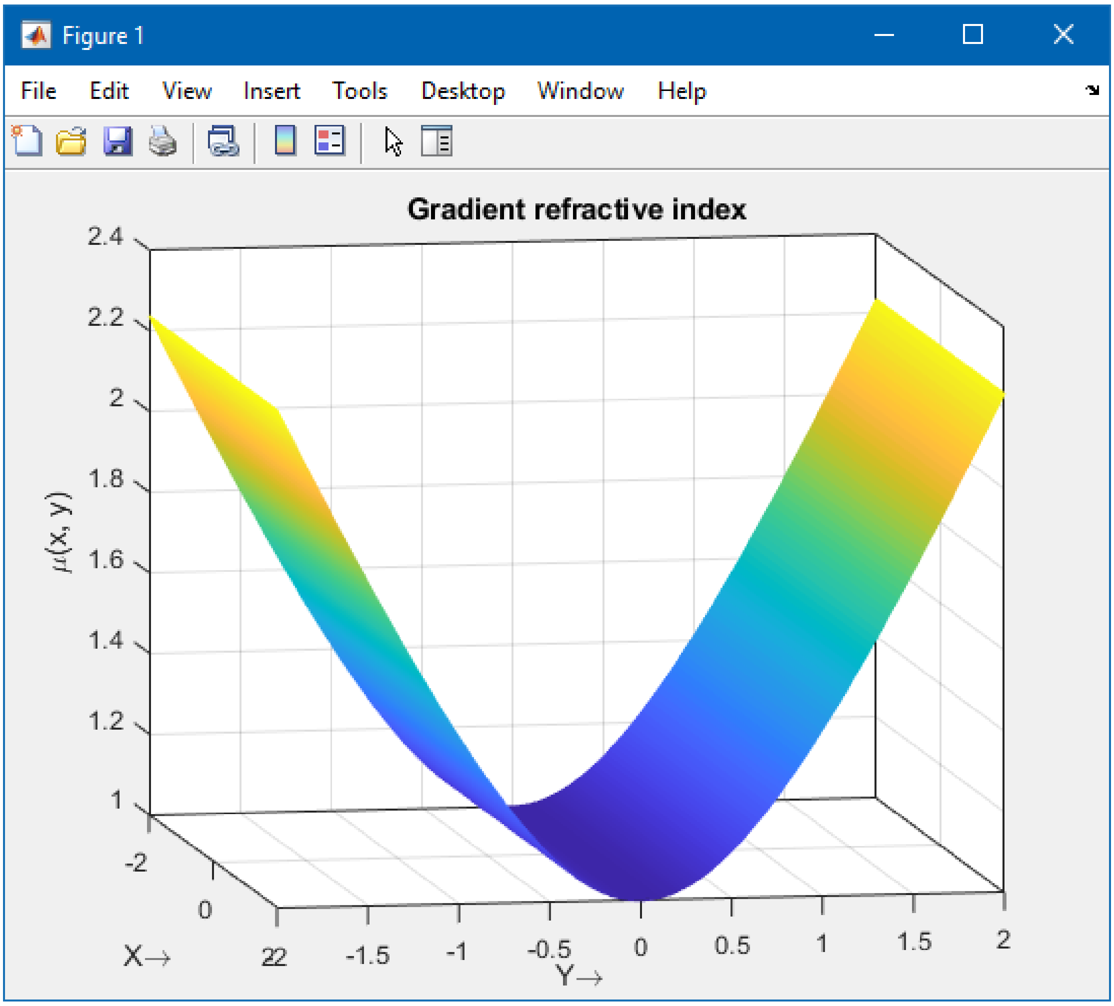


Fig. 5 (a) Refractive index distribution described by **Eq.(21)**.

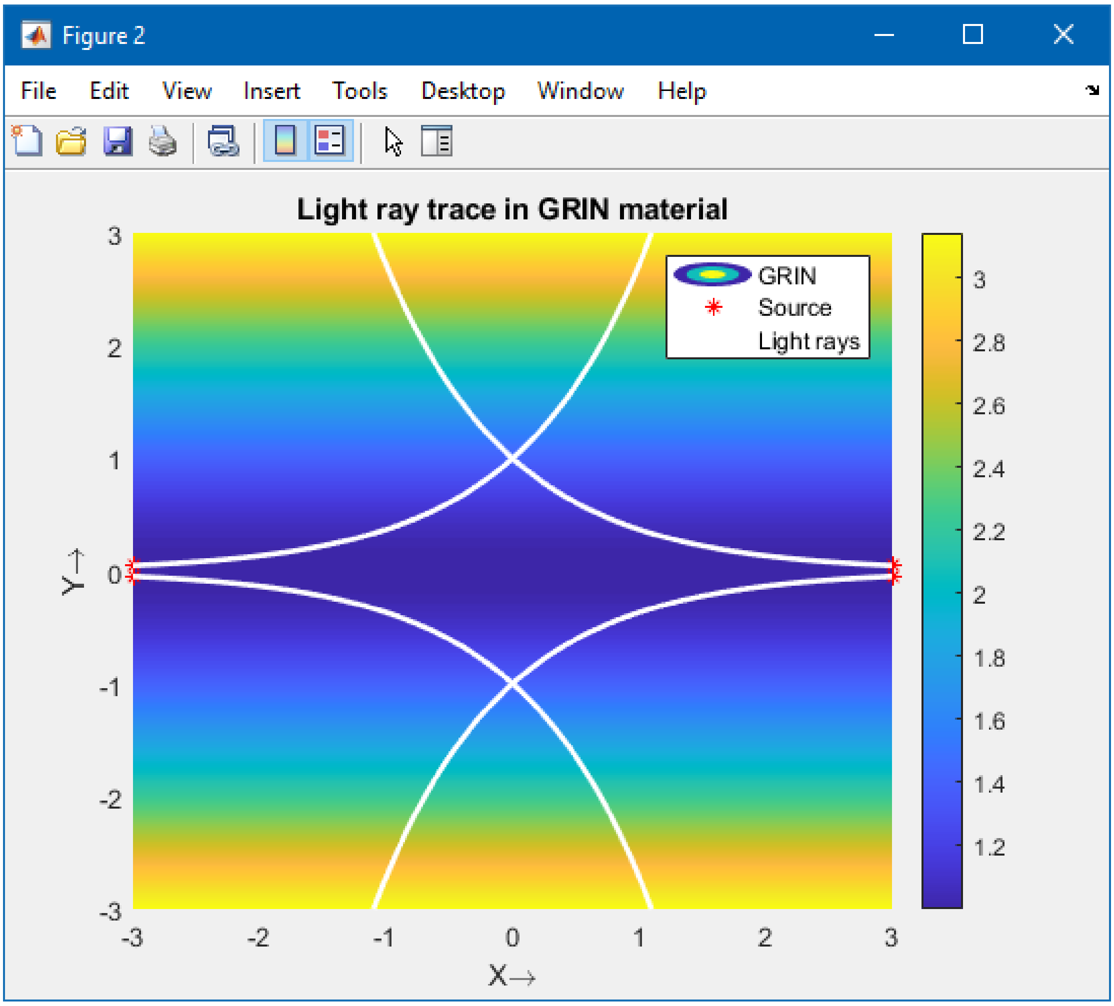


Fig 5 (b) Light propagating in a GRIN medium with refractive index characterized by **Eq. (21)**. If the launch condition satisfies $y(x = x_0) = y'(x = x_0) = e^{x_0}$, the trace is an exponential curve.

Similar to the Example 3.3, this system can be also used as a beam splitter. If the light is incident along the x axis from infinity, the original beam will split into two branches $|y| = e^{|x|}$ and $|y| = e^{-|x|}$. Compared with hyperbolic trace used in **Fig. 4**, these two branches will be separated in a shorter distance, since the exponential function varies more rapidly than hyperbola. In contrast with the traditional beam splitter, the beam-splitting angle between two branches can be adjusted by properly setting the thickness of GRIN medium. If we truncate the GRIN medium at x=x_0, we can compute that the beam-splitting angle is $\theta = \tan^{-1} e^{x_0}$. From the above examples, we see that, the gradient of refractive index is very similar to the force imposed on a moving particle. It always deflects the light trace towards its direction, as a particle's movement always bends to the direction of force. These examples indicate that the geometric optics and the classical mechanics are intrinsically related. As they can both be described by Lagrangian dynamics, they share the same essence. Therefore, we can understand the gradient of refractive index as a generalized force which can act on the light.

4. Application of GRIN optics

In response to the advances in GRIN fabrication over the last three decades, there has been increasing interest in exploring novel GRIN materials to develop compact, lightweight and robust optics. GRIN optics offer appealing form factors as well as additional degrees of freedom in controlling the propagation of light and have found application in telecommunications and compact imaging. In the former application, a refractive index profile that varies radially from the centre of an optical fibre can be chosen properly so that the rays that transit the fibre are guided by refraction, rather than by total internal reflection in step-index optical fibres, for instance. In addition, GRIN fibres can be designed so that all modes propagate with the same velocity, reducing modal dispersion and thereby increasing the bandwidth and the repeater distance of optical communication systems [Moore, 1980]. Furthermore, the cylindrical form factor of GRIN optics simplifies coupling between optical fibres and sources. The latter application of GRIN optics is primarily based on exploiting the optical power provided by GRIN materials. It follows that the optical power of a lens is not only determined by its surface geometry but also by its refractive index distribution. By combining the two effects, new approaches to chromatic as well as spherical aberration correction become possible. Furthermore, GRIN optics can be designed to redistribute irradiance in an optical beam and perform coherent mode conversion in beam shaping applications.

Mirage Effect

Air adjacent to a hot surface rises in temperature and becomes less dense (Figure 6).

Theorem 2 : (Fermat's principle for axial eikonal equation).
Stationary under variation of action A,

$$\delta \boldsymbol{A} = \delta \int_{z_A}^{z_B} \mathcal{L}\big(\boldsymbol{q}(\boldsymbol{z}), \dot{\boldsymbol{q}}(z)\big)\, dz = 0,$$

For the optical Lagrangian,

$$\mathcal{L}\big(\boldsymbol{q}(\boldsymbol{z}), \dot{\boldsymbol{q}}(z)\big) = \mu(\boldsymbol{q}, z)\sqrt{1 + |\dot{\boldsymbol{q}}|^2} =: \frac{\mu}{\gamma},$$

with,

$$\gamma := \frac{dz}{ds} = \frac{1}{\sqrt{1 + |\dot{\boldsymbol{q}}|^2}} \leq 1,$$

implies the ***axial eikonal equation***

$$\gamma \frac{d}{dz}\left(\mu(\boldsymbol{q}, z)\gamma \frac{d\boldsymbol{q}}{dz}\right) = \frac{\partial \mu}{\partial \boldsymbol{q}}, \quad with \quad \frac{d}{ds} = \gamma \frac{d}{dz}$$

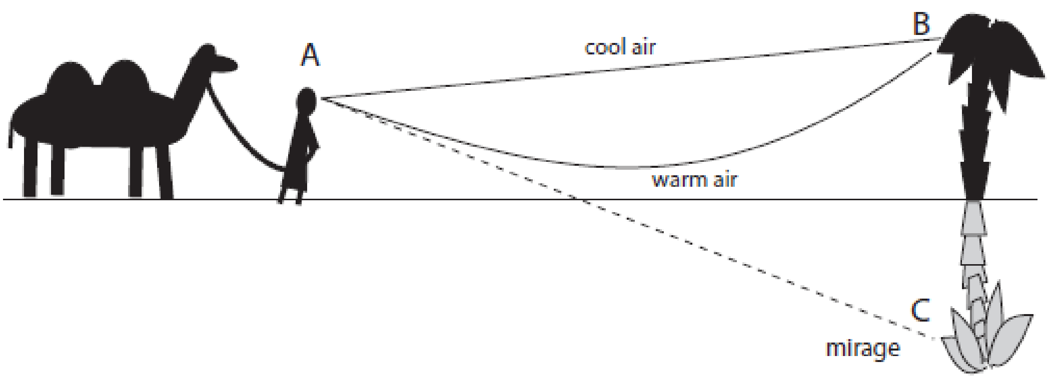


Figure 6: Fermat's principle states that the ray path from an observer at A to point B in space is a stationary path of optical length. For example, along a sun-baked road, the temperature of the air is warmest near the road and

decreases with height, so that the index of refraction, n, increases in the vertical direction. For an observer at A, the curved path has the same optical path length as the straight line. Therefore, he sees not only the direct line-of-sight image of the tree top at B, but it also appears to him that the tree top has a mirror image at C. If there is no tree, the observer sees a direct image of the sky and also its mirror image of the same optical length, thereby giving the impression, perhaps sadly, that he is looking at water, when there is none.

Thus over a at hot surface, such as a desert expanse or a sun-baked roadway, air density locally increases with height and the average refractive index is approximated by linear variation of the form

$$\mu = \mu_0(1 + \kappa y),$$

where x is the vertical height above the planar surface, μ_0 is the refractive index at ground level and κ is a positive constant. We may use the eikonal equation to find an equation for the approximate ray trajectory. This will be an equation for the ray height y as a function of ground distance x of a light ray launched from a height y_0 at an angle θ_0 with respect to the horizontal surface of the earth.

In this geometry, the eikonal equation

$$\gamma \frac{d}{dz}\left(\mu(\boldsymbol{q}, z)\gamma \frac{d\boldsymbol{q}}{dz}\right) = \frac{\partial \mu}{\partial \boldsymbol{q}}$$

Becomes

$$\frac{1}{\sqrt{1 + y'^2}} \frac{d}{dx}\left(\frac{1 + \kappa y}{\sqrt{1 + y'^2}} \cdot y'\right) = \kappa$$

For nearly horizontal rays $y'^2 \ll 1$ and if refractive index is also small i.e. $\kappa y \ll 1$. In this case the eikonal equation simplifies considerably to

$$\frac{d^2 y}{dx^2} \approx \kappa \;\; for \;\; \kappa y \ll 1 \;\; \& \;\; y'^2 \ll 1$$

Thus, the ray trajectory can be obtained by integrating both sides w.r.t x

$$\int \frac{d^2 y}{dx^2}\, dx = \kappa \int dx$$

$$y' = \kappa x + c_0$$

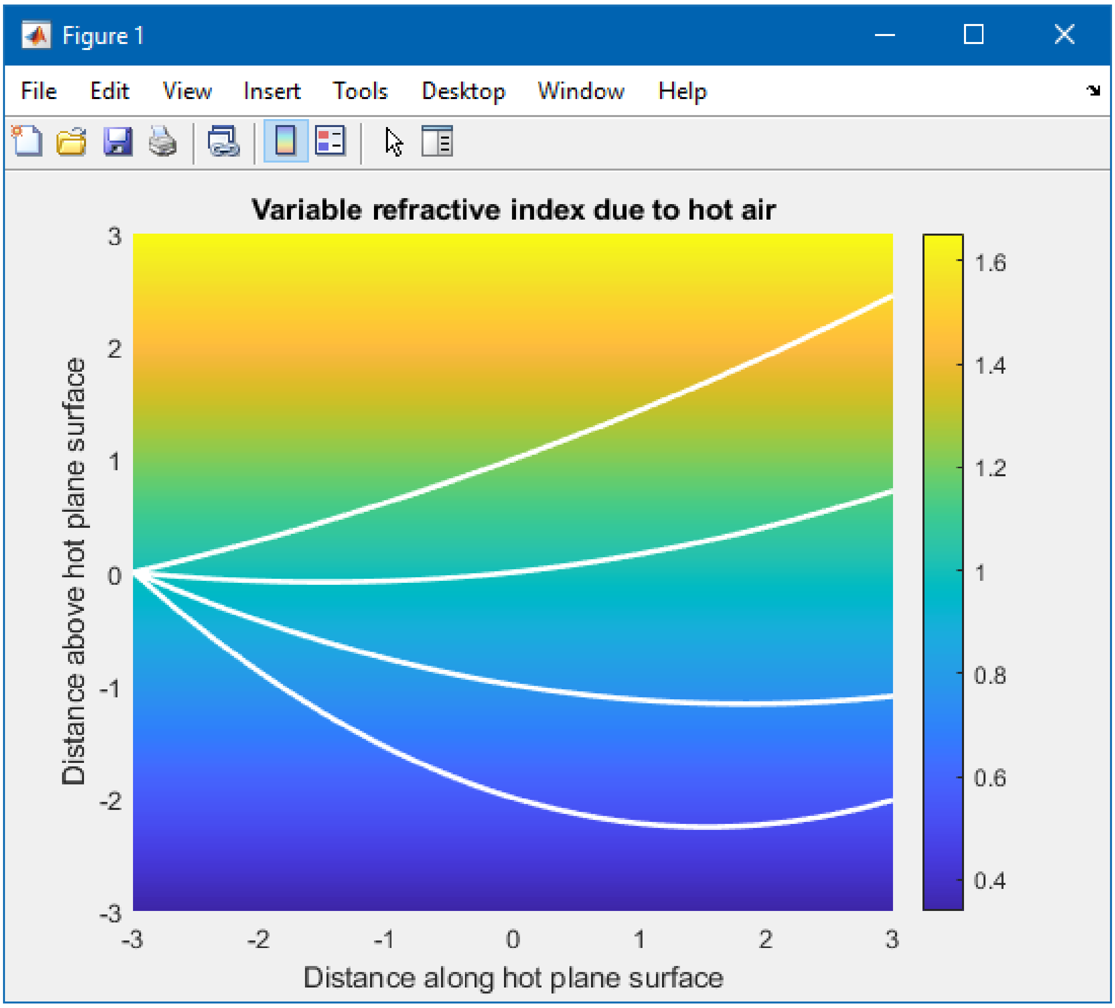
again integrating w.r.t x

$$y(x) = \frac{\kappa}{2}x^2 + c_0 x + y_0$$

On applying initial conditions we get

$$y(x) = \frac{\kappa}{2}x^2 + \tan\theta_0\, x + y_0$$

is the required equation of the path traced by the light due to hot air just above the hot surface of dessert or road.



The resulting parabolic divergence of rays above the hot surface is shown in Figure.

Acoustic Black Holes

1. The notion of curved space

To get the ball rolling, we will need to set up some mathematical machinery; we will build it up slowly starting from elementary notions. Start with Fermat's principle in optics: Take a medium with a position-dependent refractive index $\mu(\vec{x})$, and in that medium consider an arbitrary path γ described by $\vec{x}(s)$, with the parameter s being the physical distance along the path. Then as light travels through the medium it is slowed down to a speed

$$c(\vec{x}) = \frac{c_0}{\mu(\vec{x})} \tag{1.1}$$

So the total time taken for light to travel along the path is

$$T_\gamma = \int \frac{ds}{c(\vec{x}(s))} = \int \frac{ds\, \mu(\vec{x}(s))}{c_0} \tag{1.2}$$

Fermat's principle then states that the actual path light takes in crossing the medium is that path which minimizes the total transit time. Equivalently we want to minimize the so-called "optical distance"

$$L_\gamma = \int dl = \int ds\, \mu\big(\vec{x}(s)\big). \tag{1.3}$$

Once it is phrased as a minimization problem, it is clear that everything can be reformulated mathematically in terms of the geometrical notion of "geodesies", looking for the "shortest" path through a curved space. The notion of distance in this curved space is described by a 3 x 3 tensor called the metric:

$$g_{ij}(\vec{x}) = \mu^2(\vec{x})\delta_{ij}. \tag{1.4}$$

In this situation, we have two distance functions, the optical metric and the physical metric, with the optical distance and the physical distance between two nearby points related by

$$dl^2 = g_{ij}(\vec{x})dx^i dx^j = \mu^2(\vec{x})\delta_{ij}dx^i dx^i = \mu^2(\vec{x})ds^2 \tag{1.5}$$

We can now write the total optical distance as

$$L_\gamma = \int dl = \int ds \sqrt{g_{ij}(\vec{x}(s)) \frac{dx^i}{ds}\frac{dx^j}{ds}}, \qquad (1.6)$$

and apply the usual Euler-Lagrange equations to find the geodesic. That is applying brute force,

$$\frac{d}{ds}\left(\frac{g_{ij}(\vec{x})}{\sqrt{g_{kl}(\vec{x}(s)) \frac{dx^k}{ds}\frac{dx^l}{ds}}} \frac{dx^j}{ds}\right)$$
$$= \frac{\partial g_{kl}(\vec{x})}{\partial x^i}\frac{dx^k}{ds}\frac{dx^l}{ds}\left(g_{mn}(\vec{x})\frac{dx^m}{ds}\ \frac{dx^n}{ds}\right). \qquad (1.7)$$

Rearranging, this can be shown to be equivalent to the equation of geodesic motion. It is often convenient to re-parameterize the path in terms of optical length *l* instead of physical length s and write

$$\frac{d^2x^i}{dl^2} + \Gamma^i_{jk}\frac{dx^i}{dl}\frac{dx^k}{dl} = 0, \qquad (1.8)$$

where Γ^i_{jk} is called “Christoffel symbols”

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right). \qquad (1.9)$$

Here, as usual g^{ij} with both indices up denotes the inverse of the matrix g_{ij} with both indices down, and $g_{ij,k}$ denotes the partial derivative, i.e.

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$$

Despite the abundance of indices the physical ideas are rather straightforward, and the upshot is that anything that can in optics be described by using so-called "index gradient" methods can be turned into statements about three-dimensional differential geometry. Indeed by inspecting the form of the metric it is clear that "index gradient" methods are completely equivalent to considering three-dimensional conformaly flat Riemannian geometry.

In particular, the focussing of initially parallel light rays in optics is driven by gradients in the refractive index, while in Riemannian geometry focusing of nearby geodesies is governed by Jacobi's equation of geodesic separation: If two initially parallel geodesies are separated by a transverse distance Δx^i then

$$\frac{d^2}{dx^2}\left(\Delta x^i\right) = R^i_{jkl}\frac{dx^j}{dl}\Delta x^k\frac{dx^l}{dl}. \qquad (1.10)$$

Here $\frac{dx^l}{dl}$ is the tangent along either geodesic (they start out parallel to each other); Δx^k is the transverse separation between the geodesies, and R^i_{jkl} is the Riemann curvature tensor. The relevant observation regarding the Riemann tensor is that it is defined in terms of second derivatives of the metric (and so in terms of first derivatives of the Christoffel symbols)

$$R^i_{jkl} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}. \qquad (1.11)$$

So in the same way that gradients in the refractive index drive focussing in optics, gradients in the metric drive focussing when you view the situation in terms of curved space.

There are situations in which the refractive index is itself not a scalar — this occurs when the refractive index depends on direction, and is characterized by a 3 x 3 real symmetric matrix

$$[\mu^2]_{ij}. \qquad (1.12)$$

The eigenvectors of this matrix define the three principal directions, and the eigenvalues the principal values of the refractive index in three mutually orthogonal directions. If now this matrix-valued refractive index depends on position, then

$$g_{ij}(\vec{x}) = [\mu^2]_{ij}(\vec{x}) \qquad (1.13)$$

and you are dealing with a general three-dimensional Riemannian geometry, not just a conformally flat geometry.

Now the analogy is two-way — you can describe a refractive index in terms of a metric, or you can (sometimes) describe a metric in terms of a refractive index. In particular gravitational lensing in general relativity can often be characterized by an effective refractive index, where the gravitational field of a star or galaxy warps space and modifies the coordinate speed of light (not the locally measured invariant speed of light) as it passes by.

2. Adding a dimension: Curved space-time

The most important step in describing a gravitational field is to add a dimension, by adding the time coordinate and so going to space-time instead of space. In the case of the index gradient methods of geometrical optics this is accomplished by defining augmented coordinates $x^{\mu} \coloneqq (t; \vec{x}) = \left(t; x^{i}\right)$ simply setting

$$g_{\mu\nu}^{index-gradient} := \begin{bmatrix} -c_0^2 & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \mu^2(\vec{x})\delta_{ij} \end{bmatrix}. \qquad (1.14)$$

We can then extend the definitions of Christofel symbol and Riemann tensor simply by replacing Latin indices with Greek ones. In optics the r_{00} component of the metric is trivial. The paths of light-rays through space-time are characterized by the equation

$$g_{\mu\nu} \frac{dX^{\mu}}{dt} \frac{dX^{\nu}}{dt} = 0. \qquad (1.15)$$

Unwrapping this statement in terms of coordinates simply says that light moves with coordinate speed c = c_0/n. In Lorentzian geometry this is the statement that light propagates along null curves of the metric $g_{\mu\nu}$.

A particularly nice example where g_{00} is not trivial is the Schwarzschild geometry corresponding to the gravitational field of a point particle in general relativity. From standard textbooks (working in so-called isotropic coordinates so that we keep the space-space part of the metric quasi-Cartesian)

$$g_{\mu\nu}^{Schwarzchild} = \begin{bmatrix} -c_0^2\left(\dfrac{1-\dfrac{M}{2r}}{1+\dfrac{M}{2r}}\right)^2 & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \left(1+\dfrac{M}{2r}\right)^4 \delta_{ij} \end{bmatrix} \qquad (1.16)$$

The fact that 500 is not trivial affects the rate at which clocks run in a gravitational field — deep in a gravitational field clocks slow down. However, if the only thing you are interested in is the paths light rays follow, it turns out that you can multiply the four-dimensional metric by any position-dependent function without affecting those paths. For the purposes of seeing how gravity focuses light you might as well take

$$g_{\mu\nu}^{optical-Schwarzchild} = \begin{bmatrix} -c_0^2 & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \dfrac{\left(1+\dfrac{M}{2r}\right)^6}{\left(1-\dfrac{M}{2r}\right)^2}\delta_{ij} \end{bmatrix}. \qquad (1.17)$$

That is — as far as the propagation of light is concerned the Schwarzschild gravitational field is equivalent to a position-dependent refractive index

$$\mu(r) = \frac{\left[1+\dfrac{M}{2r}\right]^3}{1-\dfrac{M}{2r}} \approx 1 + \frac{2M}{r}. \qquad (1.18)$$

An example that goes in the other direction, from optics to general relativity, is the so-called "Maxwell fisheye". This is a spherically symmetric lens in which the refractive index falls off as

$$\mu(r) = \frac{\mu_0}{1 + \frac{r^2}{a^2}}. \tag{1.19}$$

In optics this is useful toy model in that it is an example of an "absolute instrument", a lens that universally produces stigmatic (sharp) images. If you interpret this as a space-time metric in Lorentzian geometry

$$g_{\mu\nu}^{fisheye} = \begin{bmatrix} -c_0^2 & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \mu_0^2 \left[1 + \frac{r^2}{a^2}\right]^{-2} \delta_{ij} \end{bmatrix}. \tag{1.20}$$

The spatial slices are simply hyper-spheres S^3, which makes it obvious that there is a hidden symmetry under hyper-rotations. Additionally we know without calculation that the (spatial) geodesics are "great circles" which wrap all the way around the universe and smoothly connect back on themselves7 — this is less than obvious if you work directly with the refractive index $\mu(r)$. Another particularly simple example is the acoustic metric. Let a point source flowing along with a moving fluid medium emit a spherical pulse of sound. Since the sound is dragged along by the fluid, a short time dt after the pulse is emitted the pulse will form a spherical shell of radius $c_{sound}dt$ surrounding the point to which the point source has been dragged by the fluid. That is, the spherical shell has been shifted a distance $\vec{v}_{fluid}dt$ and so compared to its initial emission point (taken to be $\vec{0}$) the location of the shell (call it $d\vec{x}$) is found by solving the equation

$$\left\| d\vec{x} - \vec{v}_{fluid}dt \right\| = c_{sound}dt. \tag{1.21}$$

On squaring both sides we get

$$\left(d\vec{x} - \vec{v}_{fluid}dt\right)^2 = (c_{sound}dt)^2. \tag{1.22}$$

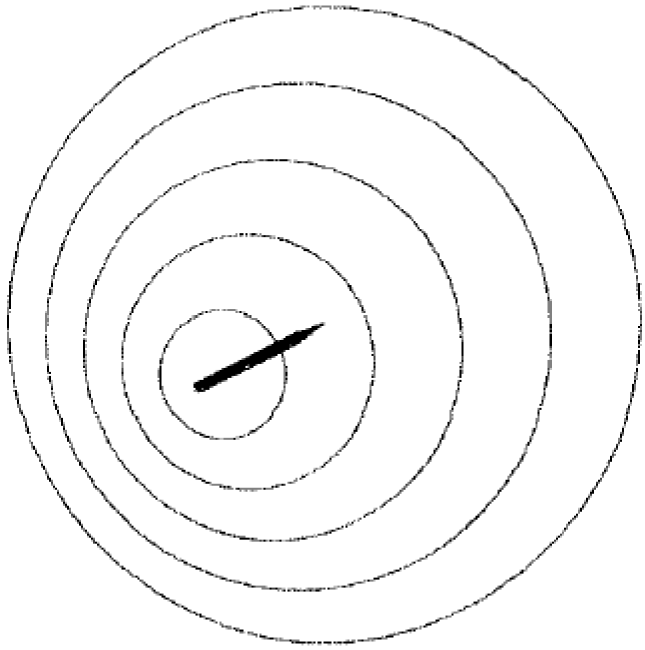


Fig 1: Spherical pulses swept along by the fluid flow

We can re-arrange this to give

$$-\left(c_{sound}^{2}-v_{fluid}^{2}\right)^{2}dt^{2}-2\,\vec{v}_{fluid}\cdot d\vec{x}dt+(d\vec{x})^{2}=0. \tag{1.23}$$

This strongly suggests we define a space-time acoustic metric

$$g_{\mu\nu}^{acoustic} \propto \begin{bmatrix} -\left(c_{sound}^{2}-v_{fluid}^{2}\right) & \vdots & -\left[v_{fluid}\right]_{j} \\ \cdots & \cdot & \cdots \\ -\left[v_{fluid}\right]_{i} & \vdots & \delta_{ij} \end{bmatrix}. \tag{1.24}$$

In terms of this metric, sound rays follow paths defined by

$$g_{\mu\nu}^{acoustic}\frac{dX^{\mu}}{dt}\frac{dX^{\nu}}{dt}=0. \tag{1.25}$$

This says that sound rays propagate along null curves of the acoustic metric. When we unwrap this statement in terms of coordinates it simply says that sound moves with coordinate speed c_{sound} with respect to the moving fluid.