

# Numerical Methods for Differential Equations

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# ORDINARY DIFFERENTIAL EQUATIONS

# 1 Ordinary Differential Equations. Basic concepts

## 1.1 Introduction and some notation

Given  $y' = f(x, y)$ , where  $\begin{cases} y(x) \in \mathbb{R}^n \\ f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \end{cases}$

**Definition.** We denote by  $y(x)$  the exact solution of the ODE system above.

**Definition.**  $y_k$  is the approximation of  $y(x_k)$  (after  $k$  steps).

**Objective.** We want to approximate  $y(x)$  within a given interval  $[x_0, x_n]$ .

We know  $\begin{cases} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{cases}$       We'd like to know  $\begin{cases} y(x_0) \\ y(x_1) \\ y(x_2) \\ \vdots \\ y(x_n) \end{cases}$       We find  $\begin{cases} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{cases}$  (given by a method)

**Definition.**  $\|y(x_n) - y_n\|$  is the **global error**.

**Definition.** We define the **local truncation error** as the error caused by one iteration, i.e.

$$LTE = \|y(x_k) - y_k\| \quad (\text{assuming the } \textit{localizing assumption}: y_{k-1} = y(x_{k-1}))$$

## 1.2 Euler's method

### 1.3 Enhanced Euler's method

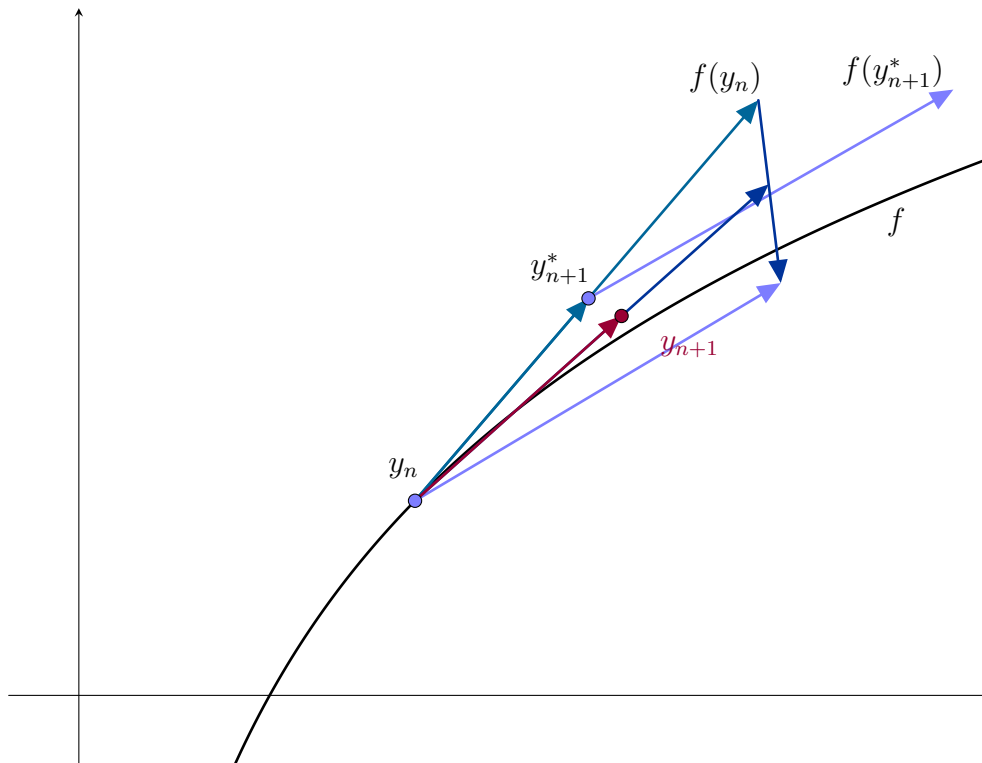


Figure 1.1: One step of the enhanced Euler's method

Given  $y'(x) = f(y(x))$ ,  $y : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we'll go through the steps to deduce the enhanced Euler's method with the help of the scheme in Figure 1.1.

The auxiliary point  $y_{n+1}^*$  can be found doing one step of the standard Euler's method, so

$$y_{n+1}^* = y_n + h \cdot f(y_n)$$

To get the point  $y_{n+1}$  we compute the average vector of  $f(y_{n+1}^*)$  and  $f(y_n)$ , and with this new vector, we can apply again a step of Euler's method, ending up with our method

$$y_{n+1} = y_n + \frac{h}{2} \cdot (f(y_{n+1}^*) + f(y_n))$$

Let's find the LTE:

The Local Truncation Error is given by:

$$LTE = \|method - exact\ solution\|$$

Then

$$LTE = \|y_{n+1} - y(x_{n+1})\| = \left\| y_n + \frac{h}{2}f(y_n) + \frac{h}{2}f(y_{n+1}^*) - \underbrace{y(x_{n+1})}_{y(x_n+h)} \right\| = (*)$$

Applying Taylor on  $f(y_{n+1}^*) = f(y_n + hf(y_n))$ , we have

$$f(y_n + hf(y_n)) = f(y_n) + hf(y_n)f'(y_n) + \mathcal{O}(h^2)$$

and on  $y(x_n + h)$  we have

$$y(x_n + h) = y(x_n) + h \cdot y'(x_n) + \frac{h^2}{2} \cdot y''(x_n) + \mathcal{O}(h^3)$$

(In this case, we expand to second order for later simplifications)

$$(*) = \left\| y_n + \frac{h}{2}f(y_n) + \frac{h}{2}(f(y_n) + hf(y_n)f'(y_n) + \mathcal{O}(h^2)) - \left( y(x_n) + h \cdot y'(x_n) + \frac{h^2}{2} \cdot y''(x_n) + \mathcal{O}(h^3) \right) \right\| \quad (1)$$

Now, given  $y'(x) = f(y(x))$ , we have

$$\begin{aligned} y''(x) &= f'(y(x)) \cdot y'(x) \\ &= f'(y(x)) \cdot f(y(x)) \end{aligned}$$

and we can rewrite the following expression as:

$$\frac{h^2}{2}f(y(x))f'(y(x)) = \frac{h^2}{2}y''(x)$$

With that and the localising assumption ( $y_n = y(x_n)$ ), we can simplify most of the terms in (1) and we end up with

$$\left\| \cancel{y_n} + \cancel{\frac{h}{2}f(y_n)} + \cancel{\frac{h}{2}f(y_n)} + \cancel{\frac{h^2}{2}f(y_n)f'(y_n)} + \mathcal{O}(h^3) - \left( \cancel{y(x_n)} + \cancel{h \cdot y'(x_n)} + \cancel{\frac{h^2}{2} \cdot y''(x_n)} + \mathcal{O}(h^3) \right) \right\| = \mathcal{O}(h^3)$$

So  $LTE = \mathcal{O}(h^3)$

**Remark.** Of course, this method also works for  $y : \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

## 1.4 Final remarks

- There's an improved Euler's method of order 2 similar to the previous one:

$$y_{n+1} = y_n + hf\left(\frac{y_{n+1}^* + y_n}{2}\right)$$

- If we have a method of order  $\geq 2$ , and we want the value of  $y(x^*)$ , with  $x^*$  off the mesh ( $x^* \neq k \cdot h$ ), we have some options:

-Step back and take a step with the right  $h$ .

-Interpolate with the right order.

-Use a continuous Runge-Kutta method.

## 2 Runge-Kutta and Linear Multistep Methods

### 2.1 General Runge-Kutta methods

#### 2.1.1 Embedded R-K

## 2.2 Linear multistep methods

### 2.2.1 Generalities

### 2.2.2 Predictor-Corrector method

### 2.2.3 Richardson's extrapolation

### 2.2.4 Convergence of a linear multistep method

Let's see an example of divergence using a linear multistep method:

**Example**

*Given the method*

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = h(b_1 f_{n+1} + b_0 f_n)$$

1) Find  $a_0, a_1, b_0, b_1$  so that the method above has the highest possible order.

2) Try it on

$$\begin{cases} y' = -y \\ y(0) = 1 \end{cases} \quad (y_0 = 1, y_1 = e^{-h})$$

*and prove the method diverges.*

1) We want  $y(x_n + 2h) - y_{n+2}$

We assume  $y_{n+1} = y(x_n + h), y_n = y(x_n)$  (localizing assumption).

$$\begin{aligned} y(x_n + 2h) - y_{n+2} &= y(x_n + 2h) - \left[ -a_1 y_{n+1} - a_0 y_n + h(b_1 f(y_{n+1}) + b_0 f(y_n)) \right] \underset{\text{loc.as.}}{=} \\ &= y(x_n + 2h) - \left[ -a_1 y(x_n + h) - a_0 y(x_n) + h b_1 \underbrace{f(y(x_n + h))}_{y'(x_n+h)} + h b_0 \underbrace{f(y(x_n))}_{y'(x_n)} \right] \end{aligned}$$

As usual, we expand in powers of  $h$ . We'll expand to order 3

$$\begin{aligned} &y(x_n) + 2h y'(x_n) + \frac{4h^2}{2} y''(x_n) + \frac{8h^3}{6} y'''(x_n) + o(h^4) - \\ &- \left[ -a_1 \left( y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + o(h^4) \right) - \right. \\ &\quad - a_0 y(x_n) \\ &\quad + h b_1 \left( y'(x_n) + h y''(x_n) + \frac{h^2}{2} y'''(x_n) + o(h^3) \right) + \\ &\quad \left. + h b_0 y'(x_n) \right] \end{aligned}$$



Let's group by powers of  $h$  and assume the right conditions to obtain the highest possible order:

$$h^0 \longrightarrow y(x_n) + a_1 y(x_n) + a_0 y(x_n) = 0$$

$$h^1 \longrightarrow 2hy'(x_n) + a_1 hy'(x_n) - hb_1 y'(x_n) - hb_0 y'(x_n) = 0$$

$$h^2 \longrightarrow 2h^2 y''(x_n) + a_1 \frac{1}{2} h^2 y''(x_n) - b_1 h^2 y''(x_n) = 0$$

$$h^3 \longrightarrow \frac{8h^3}{6} y'''(x_n) + a_1 \frac{h^3}{6} y'''(x_n) - b_1 h \left( \frac{h^2}{2} y'''(x_n) \right) = 0$$

With that, we get the system of equations

$$\begin{cases} 1 + a_1 + a_0 = 0 \\ 2 + a_1 - b_1 - b_0 = 0 \\ 2 + \frac{a_1}{2} - b_1 = 0 \\ \frac{8}{6} + \frac{a_1}{6} - \frac{b_1}{2} = 0 \end{cases}$$

And we end up with

$$a_0 = -5, \quad a_1 = 4, \quad b_0 = 2, \quad b_1 = 4$$

2) Our method is

$$y_{n+2} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n)$$

and with

$$\begin{cases} y' = -y \\ y(0) = 1, y(h) = e^{-h} \end{cases} \quad (y(x) = e^{-x})$$

we have

$$y_{n+2} + 4y_{n+1} - 5y_n = h(-4y_{n+1} - 2y_n)$$

We'll find a solution of the form

$$y_n = c_1 ( )^n + c_2 ( )^n$$

and we'll see that it diverges.

$$\begin{aligned} \lambda^2 + 4\lambda - 5 + 4h\lambda + 2h &= 0 \\ \lambda^2 + (4(1+h))\lambda + (2h-5) &= 0 \end{aligned}$$

$$\lambda = \frac{-4(1+h) \pm \sqrt{4^2(1+h)^2 - 4(2h-5)}}{2}$$

Let's expand the discriminant

$$\begin{aligned} \sqrt{4^2(1+2h+h^2) - 8h + 20} &= \sqrt{36 + 24h + 16h^2} = 6\sqrt{1 + \frac{4}{6}h + \frac{4^2}{6^2}h^2} \stackrel{\text{Taylor}}{=} \\ &= 6 \left( 1 + \frac{1}{2} \left( \frac{4}{6}h + \frac{4^2}{6^2}h^2 \right) + o(h^2) \right) = \\ &= 6 \left( 1 + \frac{1}{3}h + o(h^2) \right) \end{aligned}$$

So

$$\lambda = \frac{-4 - 4h \pm (6 + 2h + o(h^2))}{2} = \begin{cases} 1 - h + o(h^2) \\ -5 - 3h + o(h^2) \end{cases}$$

$$\implies y_n = c_1(1 - h + o(h^2))^n + c_2(-5 - 3h + o(h^2))^n$$

Let's find  $c_1$  and  $c_2$  imposing the initial conditions

$$\begin{aligned} &\begin{cases} 1 = c_1 + c_2 \implies c_1 = 1 - c_2 \\ e^{-h} = c_1(1 - h + o(h^2)) + c_2(-5 - 3h + o(h^2)) \end{cases} \\ \implies e^{-h} &= (1 - h + o(h^2)) + c_2 \underbrace{(-5 - 3h + o(h^2) - 1 + h + o(h^2))}_{-6 - 2h + o(h^2)} \\ \implies c_2 &= \frac{e^{-h} - 1 + h + o(h^2)}{-6 - 2h + o(h^2)} \stackrel{\text{Taylor } e^{-h}}{=} \frac{1 - h + o(h^2) - 1 + h + o(h^2)}{-6 - 2h + o(h^2)} \simeq \frac{-1}{6 + 2h} \\ \implies c_1 &\simeq 1 + \frac{1}{6 + 2h} = \frac{7 + 2h}{6 + 2h} \end{aligned}$$

So

$$y_n = \frac{7 + 2h}{6 + 2h}(1 - h + o(h^2))^n + \frac{-1}{6 + 2h}(-5 - 3h + o(h^2))^n$$

and the term  $(-5)^n$  will cause the solution to diverge.

### 3 Stiff Problems

# PARTIAL DIFFERENTIAL EQUATIONS

## 4 Partial Differential Equations. Generalities on their solution

## 5 Numerical Solution of PDEs with the Finite Difference Method

## 6 Introduction to Boundary Value Problems

## 7 Quality Control of Solutions