Model Comoparison

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1 LR model

$$\sum_{g=1}^{m}\sum_{\ell=1}^{b}\sum_{i=1}^{n}\frac{1}{nb}\rho_{\tau_\ell}(Y_i^{(g)}-Z_i^{(g)})+\lambda||Z||_*$$

$$\begin{aligned}L_\delta(z,e,u)&=\sum_{g=1}^{m}\sum_{\ell=1}^{b}\sum_{i=1}^{n}\frac{1}{nb}\rho_{\tau_\ell}(e_i^{(\ell)(g)})+\sum_{g=1}^{m}\sum_{\ell=1}^{b}u^{(\ell)(g)T}(Y^{(g)}-Z^{(g)}-E^{(\ell)(g)})\\&+\frac{\delta}{2}\sum_{g=1}^{m}\sum_{\ell=1}^{b}||Y^{(g)}-Z^{(g)}-E^{(\ell)(g)}||_2^2+\lambda_1||Z||_*\end{aligned}$$

1.1 update Z

$$Z^{k+1}=\arg\min_Z\underbrace{-\sum_\ell\sum_g u^{(\ell)(g)T}Z^{(g)}+\frac{\delta}{2}\sum_\ell\sum_g||Y^{(g)}-Z^{(g)}-E^{(\ell)(g)}||_2^2}_{(*)}+\lambda||Z||_*$$

$$\begin{aligned}(*)&=-\sum_\ell\mathrm{tr}(U^{(\ell)T}Z)+\frac{\delta}{2}\sum_\ell||Y-Z-E^{(\ell)}||_F^2\\&=\sum_\ell\left\{\frac{\delta}{2}||Y-Z-E^{(\ell)}||_F^2-\mathrm{tr}(U^{(\ell)T}Z)\right\}\\&=\sum_\ell\mathrm{tr}\left(\frac{\delta}{2}(Y-Z-E^{(\ell)})^T(Y-Z-E^{(\ell)})-U^{(\ell)T}Z\right)\\&=\frac{\delta}{2}\mathrm{tr}\left(\sum_\ell(Y-Z-E^{(\ell)})^T(Y-Z-E^{(\ell)})-\frac{2}{\delta}U^{(\ell)T}Z\right)\\&\propto\frac{\delta}{2}\mathrm{tr}\left(\sum_\ell(Y-E^{(\ell)}+\frac{1}{\delta}U^{(\ell)}-Z)^T(\sum_\ell(Y-E^{(\ell)}+\frac{1}{\delta}U^{(\ell)}-Z)\right)\\&=\frac{\delta\cdot b}{2}\mathrm{tr}\left(\left[\frac{1}{b}\sum_\ell(Y-E^{(\ell)}+\frac{1}{\delta}U^{(\ell)})-Z\right]^T\left[\frac{1}{b}\sum_\ell(Y-E^{(\ell)}+\frac{1}{\delta}U^{(\ell)})-Z\right]\right)\\&=\frac{\delta\cdot b}{2}||\frac{1}{b}\sum_\ell(Y-E^{(\ell)}+\frac{1}{\delta}U^{(\ell)})-Z||_F^2\end{aligned}$$

$$\begin{aligned}\therefore Z^{k+1} &= \arg\min_Z \frac{1}{2}||\frac{1}{b}\sum_\ell (Y - E^{(\ell)} + \frac{1}{\delta}U^{(\ell)}) - Z||_F^2 + \frac{\lambda}{\delta \cdot b}||Z||_* \\ &= PD_{\lambda/(\delta\cdot b)}(\Sigma)Q^T\end{aligned}$$

where $P\Sigma Q^T$ is the SVD of $\frac{1}{b}\sum_\ell(Y - E^{(\ell)} + \frac{1}{\delta}U^{(\ell)})$
and $D_{\lambda/(\delta\cdot b)}(\Sigma) = \mathrm{diag}(\{\sigma_i - \frac{\lambda}{\delta\cdot b}\}_+)$

1.2 update e

$$e_i^{(\ell)(g)} = \arg\min \frac{1}{nb}\rho_{\tau_\ell}(e_i^{(\ell)(g)}) - u_i^{(\ell)(g)}e_i^{(\ell)(g)} + \frac{\delta}{2}(Y_i^{(g)} - Z_i^{(g)} - e_i^{(\ell)(g)})^2$$

$$\frac{\partial Q}{\partial e_i^{(\ell)(g)}} = -\delta(Y_i^{(g)} - Z_i^{(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}\frac{\partial \rho_{\tau_\ell}(e_i^{(\ell)(g)})}{\partial e_i^{(\ell)(g)}}$$

Let $\epsilon_i^{(\ell)(g)} = Y_i^{(g)} - Z_i^{(g)}$.
If $e_i^{(\ell)(g)} < 0$,

$$-\delta(\epsilon_i^{(\ell)(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}(\tau_\ell - 1) = 0$$

$$e_i^{(\ell)(g)} = \epsilon_i^{(\ell)(g)} + \frac{u_i^{(\ell)(g)}}{\delta} - \frac{\tau_\ell - 1}{nb\delta} < 0 \quad \text{by hypo}$$

If $e_i^{(\ell)(g)} > 0$,

$$-\delta(\epsilon_i^{(\ell)(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}\tau_\ell = 0$$

$$e_i^{(\ell)(g)} = \epsilon_i^{(\ell)(g)} + \frac{u_i^{(\ell)(g)}}{\delta} - \frac{\tau_\ell}{nb\delta} > 0 \quad \text{by hypo}$$

If $e_i^{(\ell)(g)} = 0$,

$$-\delta\epsilon_i^{(\ell)(g)} - u_i^{(\ell)(g)} + \frac{c}{nb} = 0$$

$$\frac{\tau_\ell - 1}{nb\delta} \le (\epsilon_i^{(\ell)(g)}) + \frac{u_i^{(\ell)(g)}}{\delta} = \frac{c}{nb\delta} \le \frac{\tau_\ell}{nb\delta}$$

$$\therefore e_i^{(\ell)(g)^{k+1}} = \begin{cases} \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta - (\tau_\ell - 1)/nb\delta & \text{if } \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta < (\tau_\ell - 1)/(nb\delta) \\ 0 & \text{if } (\tau_\ell - 1)/(nb\delta) \le \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta \le \tau_\ell/(nb\delta) \\ \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta - \tau_\ell/nb\delta & \text{if } \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta > \tau_\ell/(nb\delta) \end{cases}$$

1.3 update u

$$u^{(\ell)(g)^{k+1}} = u^{(\ell)(g)^k} + \delta(Y^{(g)} - Z^{(g)} - e^{(\ell)(g)})$$

2 SP model

$$\sum_g \sum_\ell \sum_i \frac{1}{nb} \rho_{\tau_\ell}(Y_i^{(g)} - V^{(\ell)}\Theta) + \lambda \sum_g \sum_j ||\theta_j^{(g)}||_2$$

$$\begin{aligned} L_\delta(\theta, \eta, e, u, w) = &\sum_g \sum_\ell \sum_i \frac{1}{nb} \rho_{\tau_\ell}(e_i^{(\ell)(g)}) + \sum_g \sum_\ell u^{(\ell)(g)T}(Y^{(g)} - V^{(\ell)}\eta^{(g)} - e^{(\ell)(g)}) \\ &+ \frac{\delta}{2} \sum_g \sum_\ell ||Y^{(g)} - V^{(\ell)}\eta^{(g)} - e^{(\ell)(g)}||_F^2 + \sum_g w^{(g)T}(\theta^{(g)} - \eta^{(g)}) \\ &+ \frac{\delta}{2} \sum_g ||\theta^{(g)} - \eta^{(g)}||_2^2 + \lambda \sum_g \sum_j ||\theta_j^{(g)}||_2 \end{aligned}$$

2.1 update η

$$\begin{aligned} \eta^{(g)^{k+1}} = \arg\min_{\eta^{(g)}} &- \sum_g \sum_\ell u^{(\ell)(g)T} V^{(\ell)} \eta^{(g)} + \frac{\delta}{2} \sum_g \sum_\ell ||Y^{(g)} - V^{(\ell)}\eta^{(g)} - e^{(\ell)(g)}||_F^2 \\ &- \sum_g w^{(g)T} \eta^{(g)} + \frac{\delta}{2} \sum_g ||\eta^{(g)} - \theta^{(g)}||_2^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial Q}{\partial H} &= -\sum_\ell V^{(\ell)T} U^{(\ell)} - \delta \sum_\ell V^{(\ell)T}(Y - V^{(\ell)}H - E^{(\ell)}) - W + \delta(H - \Theta) \\ &= 0 \end{aligned}$$

$$\therefore H^{k+1} = \frac{1}{\delta}(\sum_\ell V^{(\ell)T} V^{(\ell)} + I)^{-1}(W + \delta\Theta + \sum_\ell V^{(\ell)T} U^{(\ell)} + \delta \sum_\ell V^{(\ell)T}(Y - E^{(\ell)})$$

2.2 update θ

$$\theta_j^{(g)^{k+1}} = \arg\min_{\theta_j^{(g)}} w_j^{(g)T} \theta_j^{(g)} + \frac{\delta}{2} ||\theta_j^{(g)} - \eta_j^{(g)}||_2^2 + \lambda ||\theta_j^{(g)}||_2$$

$$\frac{\partial Q}{\partial \theta_j^{(g)}} = w_j^{(g)} + \delta(\theta_j^{(g)} - \eta_j^{(g)}) + \lambda \frac{\partial ||\theta_j^{(g)}||_2}{\partial \theta_j^{(g)}}$$

if $\theta_j^{(g)} \neq 0$

$$w_j^{(g)} + \delta(\theta_j^{(g)} - \eta_j^{(g)}) + \lambda \frac{\theta_j^{(g)}}{||\theta_j^{(g)}||_2} = 0$$

$$(1 + \frac{\lambda}{\delta ||\theta_j^{(g)}||_2}) \theta_j^{(g)} = \underbrace{\eta_j^{(g)} - \frac{w_j^{(g)}}{\delta}}_{r_j^{(g)}}$$

$$||r_j^{(g)}||_2 = (1 + \frac{\lambda}{\delta||\theta_j^{(g)}||_2})||\theta_j^{(g)}||_2$$
$$\to ||\theta_j^{(g)}||_2 = ||r_j^{(g)}||_2 - \lambda/\delta \geq 0$$

$$\therefore \theta_j^{(g)} = r_j^{(g)}(1 + \frac{\lambda}{\delta||\theta_j^{(g)}||_2})^{-1}$$
$$= (1 - \frac{\lambda}{\delta||r_j^{(g)}||_2})r_j^{(g)}$$

if $\theta_j^{(g)} = 0$

$$w_j^{(g)} - \delta\eta_j^{(g)} + \lambda \cdot u = 0$$
$$\frac{\lambda}{\delta}u = \eta_j^{(g)} - \frac{w_j^{(g)}}{\delta} = r_j^{(g)}$$
$$\frac{\lambda}{\delta}||u||_2 = ||r_j^{(g)}||_2 \leq \frac{\lambda}{\delta}$$

$$\therefore \hat{\theta}_j^{(g)} = \begin{cases} (1 - \frac{\lambda}{\delta||r_j^{(g)}||_2})r_j^{(g)} & \text{if } ||r_j^{(g)}||_2 \geq \lambda/\delta, \\ 0 & \text{if } ||r_j^{(g)}||_2 < \lambda/\delta \end{cases}$$

2.3 update e

$$e_i^{(\ell)(g)} = \text{arg min } \frac{1}{nb}\rho_{\tau_\ell}(e_i^{(\ell)(g)}) - u_i^{(\ell)(g)}e_i^{(\ell)(g)} + \frac{\delta}{2}(Y_i^{(g)} - V_i^{(\ell)}\eta^{(g)} - e_i^{(\ell)(g)})^2$$

$$\frac{\partial Q}{\partial e_i^{(\ell)(g)}} = -\delta(Y_i^{(g)} - V_i^{(\ell)}\eta^{(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}\frac{\partial\rho_{\tau_\ell}(e_i^{(\ell)(g)})}{\partial e_i^{(\ell)(g)}}$$

Let $\epsilon_i^{(\ell)(g)} = Y_i^{(g)} - V_i^{(\ell)}\eta^{(g)}$.
If $e_i^{(\ell)(g)} < 0$,

$$-\delta(\epsilon_i^{(\ell)(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}(\tau_\ell - 1) = 0$$
$$e_i^{(\ell)(g)} = \epsilon_i^{(\ell)(g)} + \frac{u_i^{(\ell)(g)}}{\delta} - \frac{\tau_\ell - 1}{nb\delta} < 0 \quad \text{by hypo}$$

If $e_i^{(\ell)(g)} > 0$,

$$-\delta(\epsilon_i^{(\ell)(g)} - e_i^{(\ell)(g)}) - u_i^{(\ell)(g)} + \frac{1}{nb}\tau_\ell = 0$$
$$e_i^{(\ell)(g)} = \epsilon_i^{(\ell)(g)} + \frac{u_i^{(\ell)(g)}}{\delta} - \frac{\tau_\ell}{nb\delta} > 0 \quad \text{by hypo}$$

If $e_i^{(\ell)(g)} = 0$,

$$-\delta\epsilon_i^{(\ell)(g)} - u_i^{(\ell)(g)} + \frac{c}{nb} = 0$$

$$\frac{\tau_\ell - 1}{nb\delta} \leq (\epsilon_i^{(\ell)(g)}) + \frac{u_i^{(\ell)(g)}}{\delta} = \frac{c}{nb\delta} \leq \frac{\tau_\ell}{nb\delta}$$

$$\therefore e_i^{(\ell)(g)^{k+1}} = \begin{cases} \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta - (\tau_\ell - 1)/nb\delta & \text{if } \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta < (\tau_\ell - 1)/(nb\delta) \\ 0 & \text{if } (\tau_\ell - 1)/(nb\delta) \leq \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta \leq \tau_\ell/(nb\delta) \\ \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta - \tau_\ell/nb\delta & \text{if } \epsilon_i^{(\ell)(g)} + u_i^{(\ell)(g)}/\delta > \tau_\ell/(nb\delta) \end{cases}$$

2.4 update u and w

$$u^{(\ell)(g)^{k+1}} = u^{(\ell)(g)^k} + \delta(Y^{(g)} - V^{(\ell)}\eta^{(g)} - e^{(\ell)(g)})$$

$$w^{(g)^{k+1}} = w^{(g)^k} + \delta(\theta^{(g)} - \eta^{(g)})$$