Trigonometric Interpolation

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Overview

DEFINITIONS

MOTIVATION

FUNCTION APPROXIMATION (EVENLY SPACED)

GENERAL APPROXIMATION (LAGRANGE)

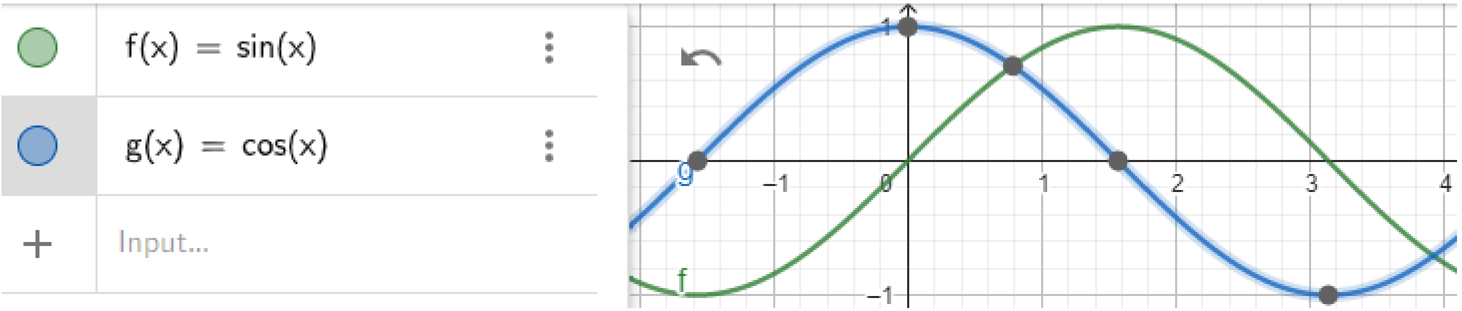
CODE DEMO

Trigonometric Functions

"…a set of functions which relate angles to the sides of a right triangle"

-> Elementary: sin(x), cos(x)

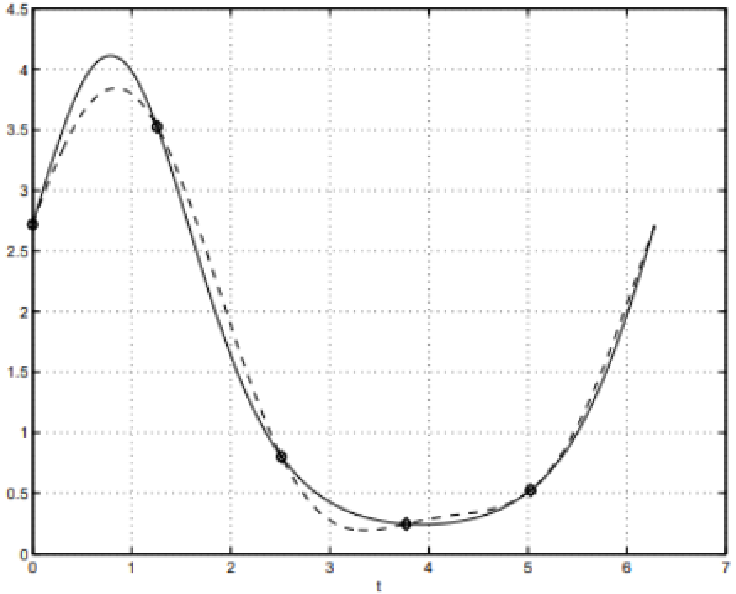
-> Representable as complex numbers (later)



Interpretations of „Interpolation“

„Interpolation“: Construct/Find new points based on known ones

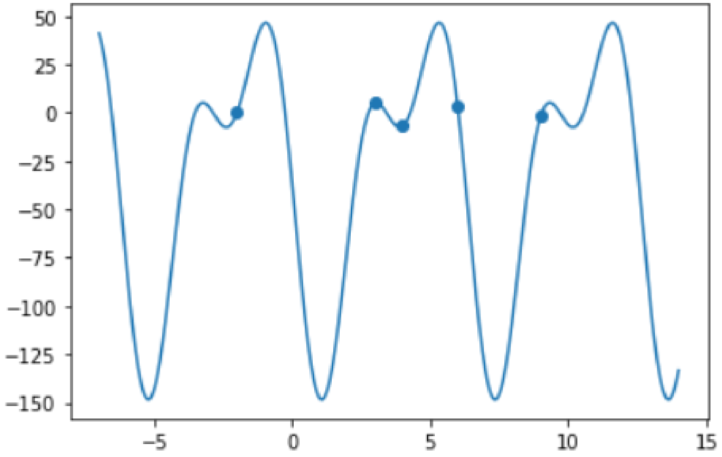
Evenly-Spaced approximation of a different function:



Only based on a set of points:

```
y_vals = trig_interpolate(points = {(-2,0),(3,5),(4,-7),(6,3), (9, -2)}, plot= True)
```

✓ 0.1s



Motivation

Engineering design and analysis:

- Modeling periodic signals in electrical, mechanical and acoustical systems
- Analysis of vibrations, resonances and stability in structures and machines

Image and Signal Processing:

- Compression of digital audio and video signals
- Removing noise from images

Geographical Information Systems:

- Interpolation of elevation data for digital terrain models
- Modelling ocean tides and wave patterns

Medical imaging and analysis:

- Reconstruction of 2D and 3D images from tomographic scans
- Analysis of ECG and EEG signals

Evenly-Spaced Function Approx.

- Every set of compositions of *sin* and cos (Trigonometric Polynomial) can be described as follows:

$$p_n(t) = a_0 + \sum_{\ell=1}^{n} \left[a_\ell \cos(\ell t) + b_\ell \sin(\ell t)\right]$$

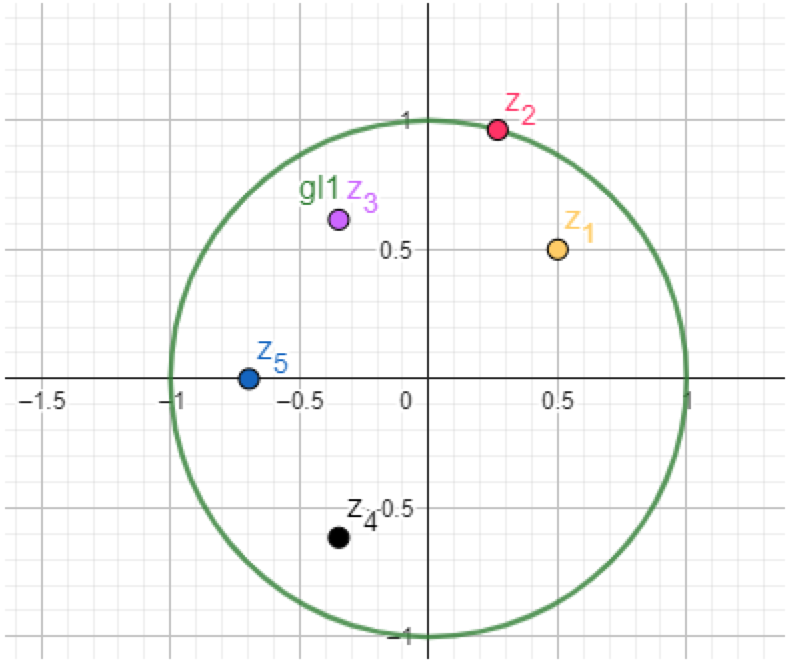
$n \ldots\ Degree\ (2n + 1\ points\ can\ be\ modeled)$

$a_i, b_i \ldots coefficient \in \mathbb{R},\ 1 \leq \mathrm{i} \leq n$

Modeling Trig. Functions in $\mathbb{C}$

GeoGebra Demo-Time:

https://www.geogebra.org/calculator/c4hgvp5f



Interpolation Task

- Simplify Formula with complex numbers:

$$p_n(t) = a_0 + \sum_{\ell=1}^{n} [a_\ell \cos(\ell t) + b_\ell \sin(\ell t)]$$

$$p_n(t) = \sum_{\ell=-n}^{n} c_\ell e^{i\ell t}$$

where $c_0 = a_0 \quad c_\ell = \frac{1}{2}(a_\ell - ib_\ell) \quad c_{-\ell} = \frac{1}{2}(a_\ell + ib_\ell)$, for $1 \leq \ell \leq n$

Interpolation Task

- Simplify Formula by hiding Euler's Number:

$$p_n(t) = \sum_{\ell=-n}^{n} c_\ell e^{i\ell t}$$

- Parameter is now a point on the **unit circle**:

$$z = e^{it}$$

$$p_n(z) = \sum_{\ell=-n}^{n} c_\ell z^\ell$$

-> Polyinomial of Degree $\leq 2n$

Define matching points

- Non-Continuous, Evenly Spaced in $[0, 2\pi)$
- From 0 (constant) until 2n (because of complex conjugates)

$$p_n(z_\ell) = f(t_\ell) \quad \ell = 0, 1, \cdots, 2n.$$

$$t_\ell = \frac{2\pi}{2n+1}\ell$$

- GeoGebra Demo-Time

https://www.geogebra.org/calculator/yjed3dv4

Solution to the problem

- For complex coefficient k:
  - Match given function f at the supplied timings t_l

$$c_k = \frac{1}{2n+1} \sum_{\ell=0}^{2n} e^{-ikt_\ell} f(t_\ell) \qquad k = -n, \ldots, n$$

- Compute real/imaginary parts directly:

$$a_\ell = c_\ell + c_{-\ell} = \frac{1}{2n+1} \sum_{k=0}^{2n} \left(e^{-i\ell t_k} + e^{i\ell t_k}\right) f(t_k)$$

$$c_\ell = \tfrac{1}{2}(a_\ell - ib_\ell)$$

$$c_{-\ell} = \tfrac{1}{2}(a_\ell + ib_\ell)$$

$$b_\ell = i(c_{-\ell} - c_\ell) = \frac{2}{2n+1} \sum_{k=0}^{2n} f(t_k) \sin \ell t_k$$

Proof

Proof: Want $p_n\left(t_\ell\right)=f\left(t_\ell\right) \quad$ for $\ell=0,1, \ldots, 2 n$. Hence

$$\sum_{m=-n}^{n} c_m e^{i m t_\ell}=f\left(t_\ell\right) \quad \text { for } \ell=0,1, \ldots, 2 n .$$

Multiply both sides by $e^{-i k t_\ell}$ and sum over $0 \leq \ell \leq 2 n$ (as this will cancel out conveniently later),

$$\sum_{\ell=0}^{2 n} \sum_{m=-n}^{n} c_m e^{i(m-k) t_\ell}=\sum_{\ell=0}^{2 n} e^{-i k t_\ell} f\left(t_\ell\right)$$

and interchange the order of summations on the left hand side:

$$\sum_{m=-n}^{n} c_m \underbrace{\sum_{\ell=0}^{2 n} e^{i(m-k) t_\ell}}_{\text {look at this now }}=\sum_{\ell=0}^{2 n} e^{-i k t_\ell} f\left(t_\ell\right) .$$

Proof cont.

When $m = k$

$$\sum_{\ell=0}^{2n} e^{i(m-k)t_\ell} = \sum_{\ell=0}^{2n} 1 = 2n + 1.$$

When $m \neq k$, the result is 0:

First note that $i(m-k)t_\ell = \frac{i\ell(m-k)2\pi}{2n+1}$. Let $r = e^{i(m-k)2\pi/(2n+1)}$ and note $r^{2n+1} = 1$. Thus

$$\sum_{\ell=0}^{2n} e^{i(m-k)t_\ell} = \sum_{\ell=0}^{2n} r^\ell = \frac{r^{2n+1} - 1}{r - 1} = 0.$$

Thus, we conclude from the double summation and the case of m=k:

$$c_k = \frac{1}{2n+1} \sum_{\ell=0}^{2n} e^{-ikt_\ell} f\left(t_\ell\right) \quad k = -n, \cdots n$$

Example from Paper

Find the trigonometric polynomial interpolation of degree 2 to $f(t) = e^{\sin t + \cos t}$ $on\ [0,2\pi)$

$$p_2(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + b_1 \sin t + b_2 \sin 2t$$

$$t_\ell = \tfrac{2\pi}{5}\ell \text{ where } \ell = 0, 1, 2, 3, 4.$$

$$p_2\left(t_\ell\right) = e^{\sin t_\ell + \cos t_\ell}$$

Example from Paper

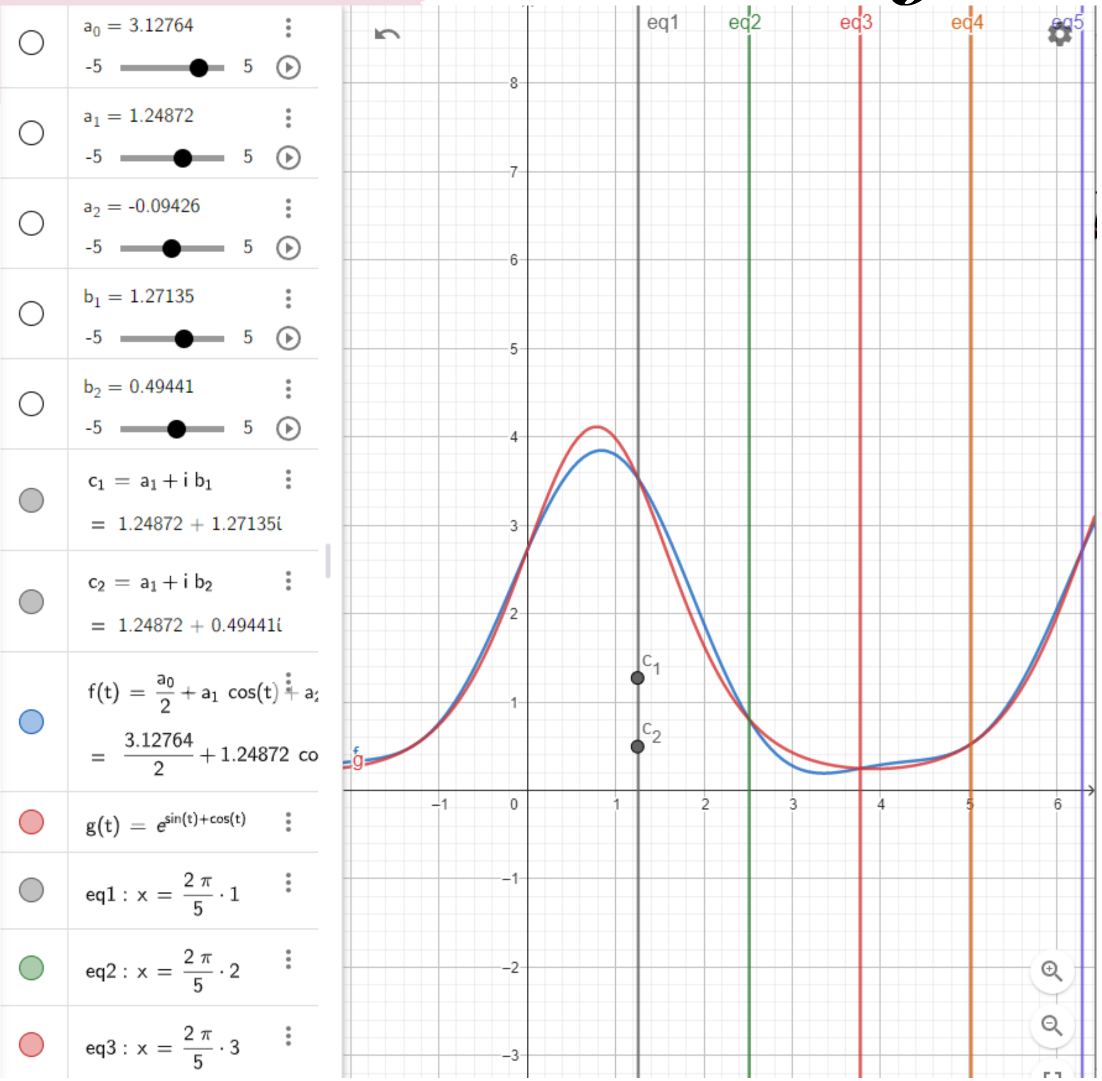
Find the trigonometric polynomial interpolation of degree 2 to $f(t) = e^{\sin t + \cos t}$ on $[0,2\pi)$

$$a_\ell = \frac{2}{2n+1} \sum_{k=0}^{2n} f(t_k) \cos \ell t_k$$

$$b_\ell = \frac{2}{2n+1} \sum_{k=0}^{2n} f(t_k) \sin \ell t_k.$$

$$\Rightarrow a_0 = 3.12764 \quad a_1 = 1.24872 \quad a_2 = -0.09426 \quad b_1 = 1.27135 \quad b_2 = 0.49441.$$

Visualization of the Example



$$f(t) = \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + b_1 \sin(t) + b_2 \sin(2t)$$

$$= \frac{3.12764}{2} + 1.24872 \cos(t) - 0.09426 \cos(2t) + 1.27135 \sin(t) + 0.49441 \sin(2t)$$

https://www.geogebra.org/calculator/c8pdmtqv

General Case – Task Description

- Given:
  - Set of points
  - $\{(x_0, y_0), (x_1, y_1), \dots\}\ where\ \ i \neq j \Rightarrow x_i \neq x_j$ (functional)

- Compute some trigonometric polynomial precisely „hitting“ every point

Lagrange Interpolation

- Polynomial formulation (Complex Plane with N points):

$$p(x) = \sum_{k=0}^{N} y_k t_k(x)$$

- Basis function (1 for point to hit, 0 for every other one):

$$t_k(x) = e^{-iN/2x + iN/2x_k} \prod_{\substack{m=0 \\ m \neq k}}^{2N} \frac{e^{ix} - e^{ix_m}}{e^{ix_k} - e^{ix_m}}$$

Simplifying the Basis function

- This ignores that N can be even (only relevant for a proof, more precise formulation linked in Notebook):

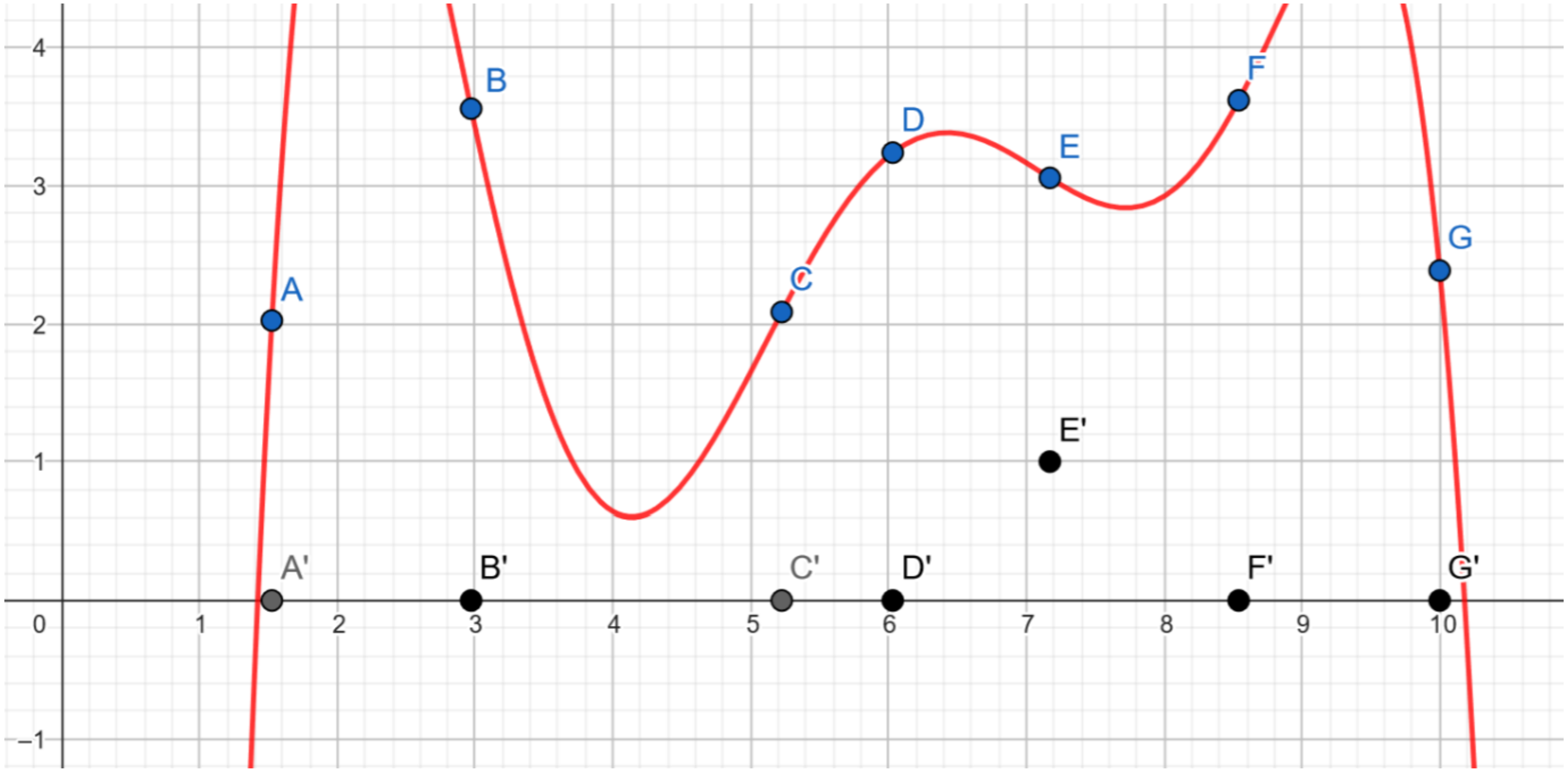
$$t_k(x) = e^{-i(N/2)x + i(N/2)x_k} \prod_{\substack{m=0 \\ m \neq k}}^{N} \frac{e^{ix} - e^{ix_m}}{e^{ix_k} - e^{ix_m}}$$

- By using this identity: $e^{iz_1} - e^{iz_2} = 2i \sin \frac{1}{2}(z_1 - z_2)\, e^{(z_1+z_2)i/2}$
- We rewrite:

$$t_k(x) = \prod_{\substack{m=0 \\ m \neq k}}^{N} \frac{\sin \frac{1}{2}(x - x_m)}{\sin \frac{1}{2}(x_k - x_m)}$$

Visualizing Non-Trigonometric Func.

GeoGebra Demo-Time:



Python Implementation

```python
import numpy as np
import matplotlib.pyplot as plt
import math
```
✓ 0.4s

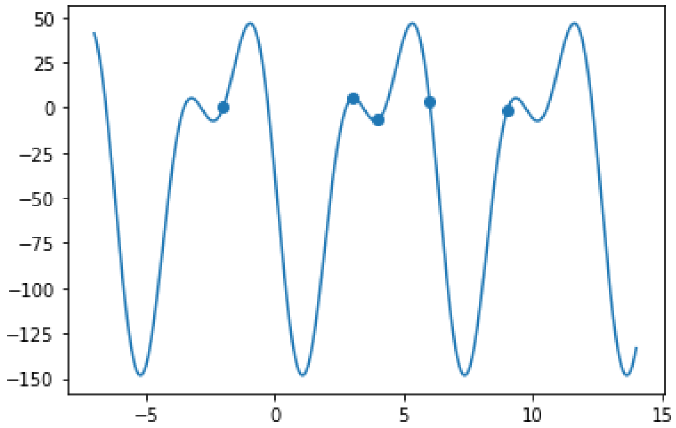
```python
> def basis_fun(k, x, xlist):···

> def lagrange_interpolation(curr_x, x, y):···

> def trig_interpolate(points, plot = False):···
```
✓ 0.3s

Interactive Demo-Time:

```python
y_vals = trig_interpolate(points = {(-2,0),(3,5),(4,-7),(6,3), (9, -2)}, plot= True)
```
✓ 0.1s



Questions?

Sources, Code and Presentation: https://github.com/S0urC10ud/Trigonometric-Interpolation/