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Matlab Code for Lyapunov Exponents of Fractional-Order Systems

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In this paper, the Benettin–Wolf algorithm to determine all Lyapunov exponents for a class of fractional-order systems modeled by Caputo's derivative and the corresponding Matlab code are presented. First, it is proved that the considered class of fractional-order systems admits the necessary variational system necessary to find the Lyapunov exponents. The underlying numerical method to solve the extended system of fractional order, composed of the initial value problem and the variational system, is the predictor-corrector Adams–Bashforth–Moulton for fractional differential equations. The Matlab program prints and plots the Lyapunov exponents as function of time. Also, the programs to obtain Lyapunov exponents as function of the bifurcation parameter and as function of the fractional order are described. The Matlab program for Lyapunov exponents is developed from an existing Matlab program for Lyapunov exponents of integer order. To decrease the computing time, a fast Matlab program which implements the Adams–Bashforth–Moulton method, is utilized. Four representative examples are considered.

Keywords: Lyapunov exponents; Benettin–Wolf algorithm; fractional-order dynamical system.

1. Introduction

Despite a long history, the doubts that fractional-order (FO) derivatives have no clear geometrical interpretations (see e.g. [Podlubny, 2002]), was one of the several reasons that fractional calculus was not used in physics or engineering. However, during the last more than ten years, fractional calculus has started to attract increasing attention. There are nowadays more and more works on FO systems and their related applications in physics, engineering, mathematics, finance, chemistry, and so on. For the theory on the existence, uniqueness, continuous dependence on parameters and asymptotic stability of solutions of FDEs with general nonlinearities see, for example, [Oldham & Spanier, 1974; Caputo, 1967], and [Diethelm & Ford, 2002; Kilbas & Trujillo, 2001; Podlubny, 1999].

The Lyapunov exponents (LEs) measure the average rate of divergence or convergence of orbits starting from nearby initial points. Therefore, they can be used to analyze the stability of limits sets and to check sensitive dependence on initial conditions, that is, the presence of potential chaotic attractors. On the other hand, in [Cvitanović *et al.*, 2016] the authors do not recommend the evaluation of the LEs and recommend: "Compute stability exponents and the associated covariant vectors instead. Costs less and gets you more insight. [...] we are doubtful of their utility as means of predicting any observables of physical significance." Moreover, we additionally note here, Perron's counterexample [Leonov & Kuznetsov, 2007] which shows actually that the use of LEs, obtained via the linearization procedure, for the study of the behavior of nonlinear system requires a rigorous justification. However, determining LEs remains the subject of many works and has grown into a real software industry for modern nonlinear physics (see, e.g. [Hegger *et al.*, 1999; Barreira & Pesin, 2001; Skokos, 2010; Czornik *et al.*, 2013; Pikovsky & Politi, 2016; Vallejo & Sanjuan, 2017] and others).

Nowadays there are two widely used definitions,[1] of the LEs: via the exponential growth rates of norms of the fundamental matrix columns [Lyapunov, 1892] and via the exponential growth rates of the singular values of fundamental matrix [Oseledets, 1968]. Corresponding approaches for the LEs computations and their difference are discussed, e.g. in [Kuznetsov *et al.*, 2018, 2016].

Remark that in numerical experiments we can consider only finite time, and, thus, the numerically computed values of LEs can differ significantly from the limit values (e.g. if the considered trajectory belongs to a *transient chaotic set*), and are often referred to as *finite-time LEs*.

Applying the statistical physics approach and assuming the ergodicity (see, e.g. [Oseledets, 1968]), the LEs for a given dynamical system are often estimated by local LEs along a "typical" trajectory. However, in numerical experiments, the rigorous use of the ergodic theory is a challenging task (see, e.g. [Cvitanović *et al.*, 2016, p. 118]). If the LEs are the same for any trajectory, then Frederickson *et al.* [1983, p. 190] suggested to call them as *absolute* ones and wrote that such absolute values rarely exist. For example, substantially different values of the local LEs can be obtained along trajectories on coexisting nonsymmetric attractors in the case of multistability[2] (see, e.g. such corresponding examples for the classical Lorenz system and Henon map [Leonov *et al.*, 2016; Kuznetsov *et al.*, 2018]).

In order to study the chaoticity of an attractor in numerical experiments, one has to consider a grid of points covering the attractor and compute corresponding finite-time local LEs for a certain time. We remark that while the time series obtained from a *physical experiment* are assumed to be reliable on the whole considered time interval, the time series produced by the integration of *mathematical dynamical model* can be reliable on a limited time interval only due to computational errors.

Taking into account the above discussion further, we consider finite-time local LEs and their computation in Matlab by the analog of the Benettin–Wolf algorithm.

2. Benettin–Wolf Algorithm for LEs of FO

The determination of LEs of a system of integer or fractional order with the Benettin–Wolf algorithm, requires the numerical integration of differential equations of integer or fractional order. Because the purpose of this paper is to present a Matlab code for LEs of systems of FO, in the next subsections, only the most important steps (such as the existence of variational equations of FO) used to implement the algorithm in Matlab language are presented (theoretical details can be found in the related references).

2.1. *Numerical integration of FDEs*

The autonomous FO systems considered in this paper are modeled by the following Initial Value Problem (IVP) with Caputo's derivative

$$D_*^q x = f(x), \quad x(0) = x_0, \tag{1}$$

for $t \in [0,T]$, $q \in (0,1)$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and D_*^q, Caputo's differential operator is of order q with

[1]Relying on the Oseledets ergodic theorem [Oseledets, 1968] the above definitions often do not differ (see, e.g. [Eckmann & Ruelle, 1985, p. 620, p. 650; Wolf *et al.*, 1985, p. 286, p. 290–291; Abarbanel *et al.*, 1993, pp. 1363–1364]), however in general case, they may lead to different values [Kuznetsov *et al.*, 2018; Bylov *et al.*, 1966, p. 289; Leonov & Kuznetsov, 2007, p. 1083].

[2]While trivial attractors (stable equilibrium points) can be easily found analytically or numerically, the search for all periodic and chaotic attractors for a given system is a challenging problem. See, e.g. famous 16th Hilbert problem [Hilbert, 1901–1902] on the number of coexisting periodic attractors in two-dimensional polynomial systems, which was formulated in 1900 and is still unsolved, and its generalization for multidimensional systems with chaotic attractors [Leonov & Kuznetsov, 2015].

starting point 0[3]

$$D_*^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} x'(\tau) d\tau,$$

with Γ the known Euler function.

Properties of the Caputo's differential operator, D_*^q, are discussed in [Podlubny, 1999; Gorenflo & Mainardi, 1997].

Under Lipschitz continuity of the function f, the IVP (1) admits a unique solution [Diethelm *et al.*, 2002].

Remark 2.1

(i) In the case of an integer order dynamical system, denoting the solution of the underlying IVP as $x(t, x_0)$, one has $x_s \circ x_t = x_{t+s}$ (see e.g. [Zhou, 2016]). Due to the memory dependence of the derivatives, this does not hold in the case of systems modeled by FDEs. However, motivated by the numerical character of this paper, the definition of integer order dynamical systems states that if the underlying IVP admits unique solutions existing on infinite time interval, the problem defines a dynamical system (see [Stuart & Humphries, 1998, Definition 2.1.2]) is adopted.

(ii) Even fractional-order dynamics describes a real object more accurately than classical integer order dynamics, systems modeled by the IVP (1) cannot have any nonconstant periodic solution (see e.g. [Tavazoei & Haeri, 2009]). However, a solution may be asymptotically periodic [Danca *et al.*, 2018a]. These trajectories are called *numerically periodic*, in the sense that the trajectory, from numerical point of view, can be extremely-near periodic with respect to, e.g. Euclidean norm [Danca *et al.*, 2018a]. A numerically periodic trajectory is referred to as a closed trajectory in the phase space in the sense that the closing error is within a given bound of $1E-n$, with n being a sufficiently large positive integer.

The numerical integrations required by the LEs algorithm for FO systems are performed in this paper with the predictor-corrector Adams–Bashforth–Moulton (ABM) method for FDEs, proposed by Diethelm *et al.* [2002], which is constructed for the fully general set of equations without any special assumptions, being easy to implement in any language.

Let us next assume that we are working on a uniform grid $\{t_j = jh : n = 0, 1, \ldots, N+1\}$ with some integer N with the step-size h and the case of $q \in (0,1)$. Then, the predictor form, x^P, at the point t_{j+1}, is the fractional variant of the Adams–Bashforth method

$$x^P(t_{n+1}) = x_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(x(t_j)),$$

while the corrector formula (the fractional variant of the one-step implicit Adams–Moulton method) reads

$$x(t_{n+1}) = x_0 + \frac{h^q}{\Gamma(q+2)} f(x^P(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(x(t_j)),$$

where a and b are the corrector and predictor weights respectively given by the following formula

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q (n-j+2)^{q+1} + (n-j)^{q+1} & \text{if } j = 0, \\ -2(n-j+1)^{q+1} & \text{if } 1 \le j \le n, \\ 1 & \text{if } j = n+1 \end{cases}$$

and

$$b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q).$$

[3]Based on philosophical arguments rather than a mathematical point of view, some researchers questioned the appropriateness of using initial conditions of the classical form in the Caputo derivative [Hartley *et al.*, 2013]. However, it should be emphasized that, in practical (physical) problems, physically interpretable initial conditions are necessary and Caputo's derivative is a fully justified tool [Diethelm, 2014].

```
for $k := 1$ to $N$ do
    $a[k] := k^q - (k-1)^q$
    $b[k] := (k+1)^{q+1} - 2k^{k+1} + (k-1)^{q+1}$
end
```

The predictor-corrector ABM method has an error which is roughly proportional to h^2. Thus, to obtain an error of, e.g. $1.0E-6$, a step-size close to $h = 1.0E-3$ should be considered.

Due to the long memory processes, the utilized ABM method as described in [Diethelm *et al.*, 2002], is time consuming. Therefore, a fast optimized ABM method, `FDE12.m` [Garrappa, 2012] is used.

`FDE12.m` is called by the following command line:

```
[t,x] = FDE12(q,fcn,t0,tf,x0,h);
```

where `q` represents the commensurate fractional order, `fcn.m` is the file with the function to be integrated, `t0` and `tf` define the time span, `x0` are the initial conditions, and `h` represents the integration step-size.

For example, consider the integration of the FO Rabinovich–Fabrikant (RF) system [Danca, 2016], which has the following Matlab function, `RF.m`

```
function dx = RF(t,x)
 dx=[x(2)*(x(3)-1+x(1)*x(1))+0.1*x(1);
    x(1)*(3*x(3)+1-x(1)*x(1))+0.1*x(2);
    -2*x(3)*(p+x(1)*x(2))];
```

with the bifurcation parameter $p = 0.98$.

With the following parameters, one obtains the chaotic attractor in Fig. 1(a):

```
[t,x]=FDE12(0.999,@RF,0,1500,[0.1;0.1;0.1],...
0.01);
```

Take note that `FDE12.m` requires the entry, `x0`, as a column vector `[x10,x20,x30]'` or, similarly, `[x10;x20;x30]`. Also, the returned exit, `x`, is a column vector.

2.2. *Algorithm for LEs of the FO system (1)*

Notation 2.1. Hereafter the finite-time local LEs of FO are called LEs.

The existence of the variational equations necessary to determine LEs is ensured by the following theorem [Li *et al.*, 2010].

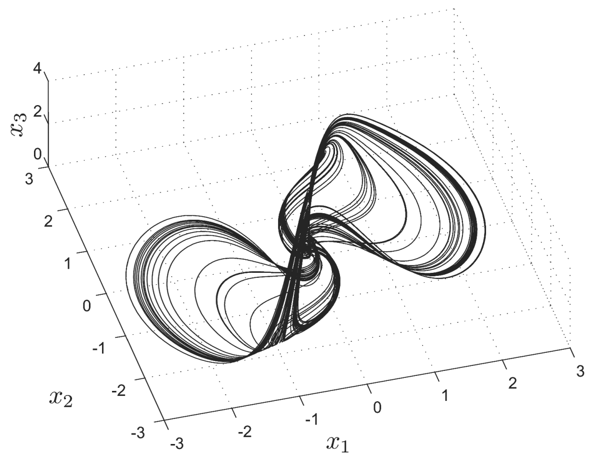
Theorem 1. *System (1) has the following variational equations which define the LEs*

$$D_*^q \Phi(t) = D_x f(x)\Phi(t), \quad \Phi(0) = I, \qquad (2)$$

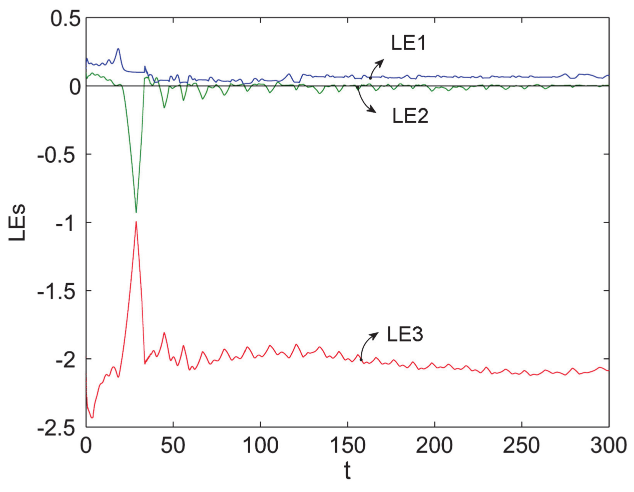
where Φ is the matrix solution of the system (1), D_x is the Jacobian of f and I is the identity matrix.

Further we assume that for the matrix $\Phi(t)$ the cocycle property takes place. Therefore, the algorithm to determine the LEs of the system (1) becomes similar to the case of integer order.

Lyapunov exponents measure the exponential growth, or decay, of infinitesimal phase space perturbations of a chaotic dynamical system.



(a)



(b)

Fig. 1. (a) A chaotic attractor of the RF system of FO, for $q = 0.999$ and (b) dynamics of the LEs.

Algorithm 1. Algorithm for LEs of FO system (1)

```
Input:
-$ne$                                          ▷ number of equations
-$x\_start$                                    ▷ $n_e$ initial conditions of (1)
-$t\_start$, $t\_end$                          ▷ time span
-$h\_norm$                                     ▷ Normalization step-size
$n\_it \leftarrow (t\_end - t\_start)/h\_norm$ ▷ iterations number
for $i \leftarrow ne+1$ to $ne(ne+1)$ do
    $x(i) = 1.0$                               ▷ initial conditions of (2)
end
$t \leftarrow t\_start$
for $i \leftarrow 1$ to $n\_it$ do
    $x \leftarrow$ integration of FO systems (1)–(2)
    $t \leftarrow t + h\_norm$
    $zn(1), \ldots, zn(ne) \leftarrow$ Gram–Schmidt procedure
    $s(1) \leftarrow 0$
    for $k \leftarrow 1$ to $ne$ do
        $s(k) \leftarrow s(k) + \log(zn(k))$   ▷ vector magnitudes
        $LE(k) \leftarrow s(k)/(t - t\_start)$ ▷ LEs
    end
end
Output: LE
```

The algorithm for numerical evaluation of LEs utilized in this paper, was proposed in the seminal works of Benettin *et al.* [1980] (see also [Shimada & Nagashima, 1979]), one of the first works to propose a Gram–Schmidt orthogonalization procedure to compute LEs for continuous systems of integer order, as described in [Eckmann & Ruelle, 1985], and by Wolf *et al.* [1985] (see also [Eckmann *et al.*, 1986]).

The algorithm to find all LEs, described as a Fortran code by Wolf *et al.* [1985], and also as a Basic code in [Baker & Gollub, 1990], solves the equations of motion under perturbations and periodic orthonormalization.

To note, the accuracy and reliability of numerically determined LEs depend on initial conditions, on the selection of the perturbations, performances of the utilized integration numerical method and also on orthonormalization step size. A long-time numerical calculation of the leading Lyapunov exponent requires rescaling the distance between nearby trajectories, in order to keep the separation within the linearized flow range. To avoid overflow, one calculates the divergence of nearby trajectories for finite timesteps and renormalizes to unity after a finite number of steps (Gram–Schmidt procedure [Eckmann & Ruelle, 1985; Christiansen & Rugh, 1997]).

Therefore, the main steps to determine numerically the LEs are: numerical integration of the FO system (1) together with the variational system (2) (i.e. the extended system), Gram–Schmidt procedure and picking up the exponents during the renormalization procedure, the LEs being determined as the average of the logarithm of the stretching factor of each perturbation, steps presented in Algorithm 1.

3. The Matlab Code for LEs

Consider the following general assumptions:

- The considered systems, modeled by the IVP (1), are autonomous;
- The system (1) is of commensurate order: $q_1 = q_2 = \cdots = q_{ne} = q$;[4]

[4]The incommensurate case can be treated similarly, the only difference being to refer to the utilized numerical method for FDEs.

- In the case of chaotic behavior, the fractional order has been chosen close to 1, such that chaos is significant.
- The right-hand side of system (1), f, is smooth enough.
- Because of the space restrictions, only the main program's code is presented, and indications on how to write the other ones.

A simple way to build in Matlab language the algorithm for FO systems, the program `FO_Lyapunov.m` (Appendix A), was to modify either some existing program, e.g. the program `lyapunov.m` [Govorukhin, 2004], which is a Matlab variant of the original LEs algorithm proposed in [Benettin *et al.*, 1980] or [Wolf *et al.*, 1985] or, similarly, to translate in Matlab the BASIC program [Baker & Gollub, 1990], a close variant of the original algorithm or, also, the Fortran code proposed by Wolf *et al.* [1985], and modify it for FDEs.

The program, called `FO_Lyapunov.m`, is launched with the following command line:

```
[t,LE]=FO_Lyapunov(ne,@ext_fcn,t_start,h_norm,...
t_end,x_start,h,q,out);
```

where `ne` represents the equations (and state variables) number, `ext_fcn.m` the function containing the extended system (1)–(2), `t_start` and `t_end` the time span, `h_norm` the normalization step, `x_start` the initial condition (as column vector), `h` the step size of FDE12.m, and `out` indicates the steps number when intermediate values of time and LEs are printed (for `out=0`, no intermediate results will be printed out).

As shown in the algorithm for LEs (Algorithm 1), it is necessary to solve the extended system (1)–(2) of FO (yellow line) which is given in the function `ext_fcn.m`. In this file, besides the right-hand side function `f` of the system (1), the Jacobi matrix `J` should be included (see Appendix B where the function for the RF system is presented).

The program plots the time evolutions of the LEs.

4. Numerical Tests

Beyond numerical artifacts that might occur when numerically integrating a system of ODEs of integer order, notions such as "shadowing time" and "maximally effective computational time" reveal that it is possible to have reliable numerical simulations only on a relatively finite-time interval (see, e.g. [Sarra & Meador, 2011; Wang *et al.*, 2012]). The case of FO systems is even more delicate. In this paper, we have considered generally $t \in [0, 300]$ and, for the Lorenz system, $t \in [0, 500]$.

(1) Let us consider the function for the RF system, `LE_RF.m`, which include the extended system (1)–(2) (Appendix B). Because `ne=3`, beside the `3` variables `x(1),x(2),x(3)` required by the numerical solution of the original system (1), the matrix solution of the system (2) requires other more `ne`×`ne=9` variables from the total of `ne(ne+1)=12` variables: `x(1:12)`, `f=zeros(size(x))=zeros(12)`, where `f` loads the first `ne=3` right-hand side expressions of system (1), and the `ne`×`ne=9` right-hand side expressions of the variational system (2).

For example, for the RF system, with the following command line:

```
[t,LE]=FO_Lyapunov(3,@LE_RF,0,0.02,300,...
[0.1;0.1;0.1],0.005,0.999,1000);
```

one obtains the intermediary results printed every `out=1000 h_norm` steps, presented in Table 1 [see also Fig. 1(b) where the dynamics of the LEs are drawn].

(2) If one considers the Lorenz system

$$\begin{aligned} D_*^q x_1 &= \sigma(x(2) - x(1)), \\ D_*^q x_2 &= -x(1)x(3) + px(1) - x(2), \qquad (3) \\ D_*^q x_3 &= x(1)x(2) - \beta x(3); \end{aligned}$$

with $q = 0.985$ and the standard parameters $\sigma = 10$, $\beta = 8/3$ and the bifurcation parameter $p = 200$, after some neglected transients, one obtains an apparently stable cycle [see Fig. 2(a) and Remark 2.1(ii)].

To obtain the LEs one writes the following command line:

```
[t,LE]=FO_Lyapunov(3,@LE_Lorenz,0,5,500,...
[0.1;0.1;0.1],0.001,0.985,10);
```

which gives `LE=(-0.0026, -0.0870, -1.6225)`. The function `LE_Lorenz.m` can be obtained similarly with `LE_RF.m`. The time evolution of the LEs is presented in Table 2 [see also Fig. 2(b)].

Table 1. For $t \in [0, 300]$, the RF system has `LE = (0.0749, 0.0018, -2.0850)` (last line, blue).

10.00	0.1611	0.0660	-2.1614
20.00	0.1923	0.0069	-2.0503
30.00	0.0984	-0.7397	-1.1817
40.00	0.0248	0.0542	-1.9761
50.00	0.0440	-0.0168	-1.9867
60.00	0.0697	-0.0006	-2.0701
70.00	0.0354	-0.0116	-2.0167
80.00	0.0331	-0.0618	-1.9360
90.00	0.0201	0.0125	-1.9754
100.00	0.0429	0.0112	-1.9794
110.00	0.0337	0.0252	-1.9698
120.00	0.0262	-0.0213	-1.9038
130.00	0.0624	-0.0043	-1.9537
140.00	0.0660	-0.0029	-1.9811
150.00	0.0645	-0.0010	-2.0008
160.00	0.0743	0.0018	-2.0304
170.00	0.0710	-0.0009	-2.0394
180.00	0.0583	0.0139	-2.0548
190.00	0.0678	0.0042	-2.0665
200.00	0.0662	-0.0217	-2.0498
210.00	0.0628	-0.0155	-2.0622
220.00	0.0602	-0.0125	-2.0715
230.00	0.0570	-0.0222	-2.0666
240.00	0.0643	-0.0222	-2.0813
250.00	0.0639	-0.0079	-2.1020
260.00	0.0623	0.0045	-2.1121
270.00	0.0630	0.0006	-2.0987
280.00	0.0551	-0.0036	-2.0771
290.00	0.0761	0.0009	-2.0937
300.00	0.0749	0.0018	-2.0850

(3) Consider next the nonsmooth four-dimensional system [Danca *et al.*, 2018a]

$$\begin{aligned} D_*^q x_1 &= -x_1 + x_2, \\ D_*^q x_2 &= -x_3 \operatorname{sgn}(x_1) + x_4, \\ D_*^q x_3 &= |x_1| - a, \\ D_*^q x_4 &= -b x_2, \end{aligned} \tag{4}$$

with $a = 1$, $b = 0.5$ and $q = 0.98$.

With the command line:

```
[t,LE]=FO_Lyapunov(4,@LE_4d,0,0.02,300,...
[0.1;0.1;0.1;0.1],0.005,0.98,1000);
```

one obtains `LE = (0.1262, 0.0846, 0.0778, -1.5244)` (Fig. 3).

For this example, `ne=4`, the number of variables is 20 and, therefore, compared with systems with `ne=3`, the function file, `LE_4d.m`, must be modified accordingly (compare the red line in `LE_RF.m`). Also, in order to obtain the printed intermediated values

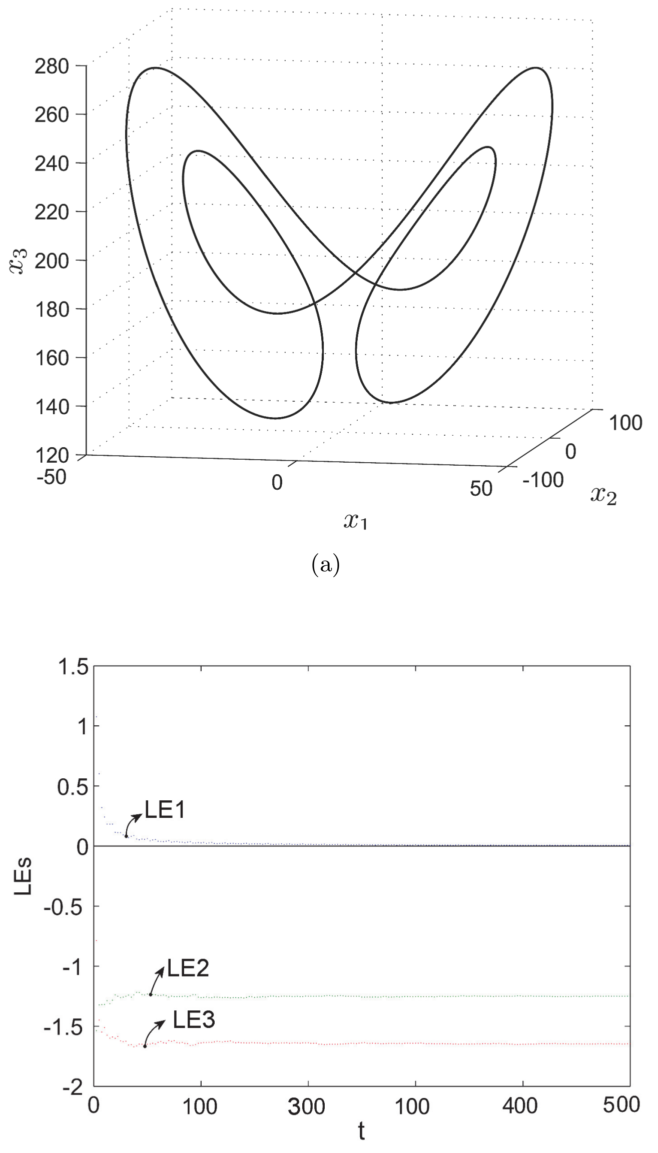


Fig. 2. (a) An apparently stable cycle of the generalized Lorenz system of FO, for $q = 0.985$ and (b) dynamics of the LEs.

Table 2. Time evolution of the LEs for the Lorenz system. `LE=(-0.0026, -0.0870, -1.6225).`

50.00	0.1759	-0.1591	-1.5683
100.00	0.0611	-0.1108	-1.6050
150.00	0.0346	-0.0927	-1.6300
200.00	0.0215	-0.0877	-1.6288
250.00	0.0135	-0.0866	-1.6269
300.00	0.0082	-0.0865	-1.6255
350.00	0.0043	-0.0866	-1.6244
400.00	0.0014	-0.0867	-1.6236
450.00	-0.0008	-0.0869	-1.6230
500.00	-0.0026	-0.0870	-1.6225

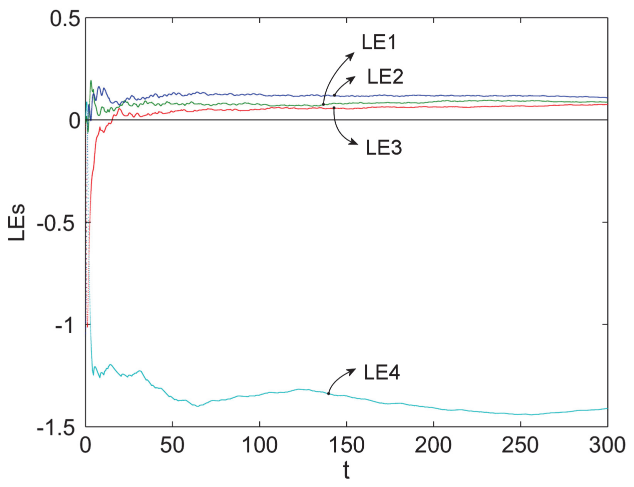


Fig. 3. Dynamics of the LEs of four-dimensional system of FO (4).

of LEs, in the `FO_Lyapunov.m`, the `fprintf` command, must be modified by adding one supplementary specifier `%10.3f`.

From the four LEs, the system admits three positive LEs, so it can be considered as hyperchaotic (see [Danca *et al.*, 2018b] for a discussion about the number of positive LEs of hyperchaotic systems).

Remark 4.1. Note that because the system is not smooth and also discontinuous, the correct numerical integration required for this system cannot be done without a previous smooth approximation [Danca *et al.*, 2018a]. Thus, like all numerical methods for FDEs which are designed for continuous dynamical systems, the integrator `FDE12` cannot be utilized in this case without the mentioned approximation. Also, without a smooth approximation, the Jacobian matrix, utilized in the Benettin–Wolf algorithm, cannot be determined.

(4) LEs can be also plotted as a function of the bifurcation parameter p for $p \in [p_{\min}, p_{\max}]$ (program `run_Lyapunov_p.m`, Appendix C). The program uses a slightly modified variant of `FO_Lyapunov.m`, which is called `FO_Lyapunov_p.m`. To obtain `FO_Lyapunov_p.m` from `FO_Lyapunov.m`, the following modifications have to be done:

(a) The header line of the code `FO_Lyapunov.m` is replaced with the following line (to note that the output `t` and the input `out` are no more necessary)

```
function LE=FO_Lyapunov_p(ne,ext_fcn_p,...
t_start,h_norm,t_end,x_start,h,q,p);
```

(b) the printing and plotting lines `(*)` (red color) are deleted;
(c) the rest of the lines remain the same.

The m-file containing the extended system, `ext_fcn_p.m`, for the RF system, is presented in Appendix D. Passing the parameter p between these codes, can be easily realized due to the facilities of the program FDE12 (see FDE12).

For example, for the RF system, for $p \in [1.1, 1.3]$, for 1000 values of p ($n = 1000$), with the following command

```
run_FO_Lyapunov_p(3,@LE_RF_p,0,0.02,200,...
[0.1;0.1;0.1],0.002,0.998,1.1,1.3,800)
```

one obtains the evolution of the LEs drawn in Fig. 4(a).

(5) LEs can be plotted also as a function of the fractional order `q` (program `run_Lyapunov_q.m` in Appendix E). The used program `FO_Lyapunov_q`, is obtained from `FO_Lyapunov` with the following modifications:

(a) The header line of the code is replaced by the following line (to note that the input `out` is no longer necessary);

```
function [t,LE]=FO_Lyapunov_q(ne,ext_fcn,...
    t_start,h_norm,t_end,x_start,h,q);
```

(b) the printing and plotting lines `(*)` (red color) are deleted;
(c) the rest of the lines remain the same.

The function containing the extended system, `ext_fcn.m`, does not require any modification.

For example, for the RF system, with the command

```
run_FO_Lyapunov_q(3,@LE_RF,0,0.05,150,...
[0.1;0.1;0.1],0.002,0.9,1,800)
```

one obtains the LEs plotted in Fig. 4(b).

Remark 4.2. The presented programs can be optimized especially in the cases of `FO_Lyapunov_p.m` and `FO_Lyapunov_q.m`. A simple improvement was to use `while` loop (instead of `for`) which for these two programs reduce substantially the computational time. However, at every parameter p (or order q) step, several constant parameters are shared between programs, which slows significantly the

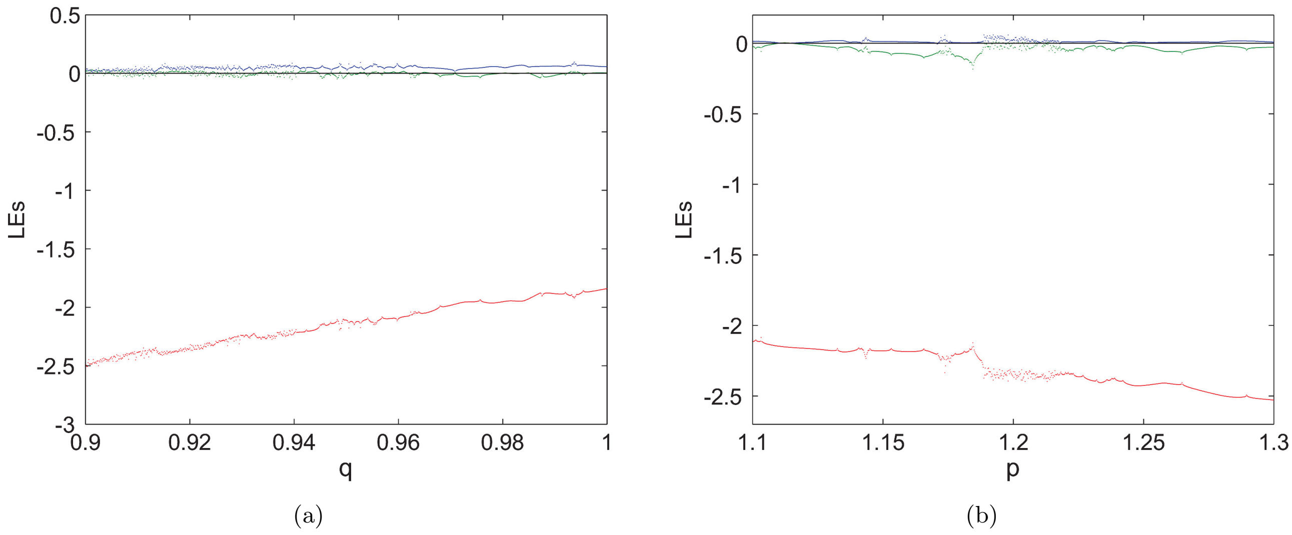


Fig. 4. LEs of RF system. (a) LEs as function of q for $q \in [0.9, 1]$ and (b) LEs as function of p for $p \in [1.1, 1.3]$.

programs. Therefore, supplementary optimization should be done.

(6) Another potential application of the proposed LEs algorithm, is to represent the LEs as function of two variables: the order q and the bifurcation parameter p.

Let the Chen system

$$D_*^q x_1 = a(x_2 - x_1),$$

$$D_*^q x_2 = (p - a)x_1 - x_1 x_3 + c x_2,$$

$$D_*^q x_3 = x_1 x_2 - b x_3,$$

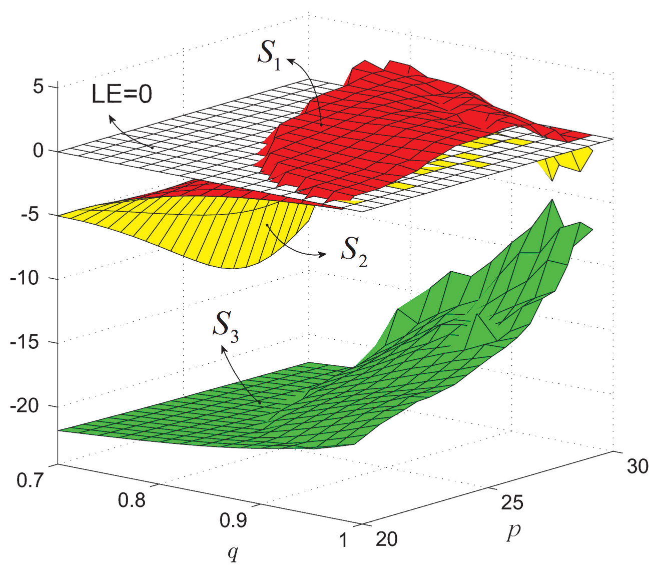


Fig. 5. LEs of Chen's system of FO represented by the function of two variables: q and parameter p. Surface S_i, for $i = 1, 2, 3$, represents $LE(q, p)$.

with parameters $a = 35$, $b = 3$ and q and p variables. By considering $S_i := LE(q, c)$, for $i = 1, 2, 3$, for $q \in [0.9, 1]$ and $p \in [20, 30]$, the obtained surfaces are plotted in Fig. 5 (see [Danca *et al.*, 2018b] where the algorithm to obtain LEs as surfaces is described). One can see that there exists a unique positive LE (red surface, S_1) for all values of the considered parameter c but only for some fractional order values q, for which S_1 is situated over the horizontal plane LE $= 0$, where LEs are zero. Also, one can see that S_2, which is almost identical to S_1 when the underlying LE1,2, are negative, becomes zero for relatively large values of q and p, when S_2 separates from S_1 and identify (with the underlying numerical error) with the plane LE $= 0$.

5. Conclusion and Discussion

Starting from an existing variant for integer order system, in this paper, based on the Benettin–Wolf algorithm, we proposed a Matlab program to determine numerically finite-time local LEs for FO systems.

Due to inherent numerical errors, the algorithm should be utilized with precaution. As known, among the numerical errors, the results depend strongly on the initial conditions, time integration interval and, especially, on the renormalization step size `h_norm`.

The relatively large numerical errors of the Benettin–Wolf algorithm, which for the considered examples were of order of $1.E-2$, can be observed in the cases when one knows that the system presents a numerically periodic cycle (see Remark 2.1(ii)), when the maximum LE should be zero. For example, the Lorenz system, which for $p = 200$ presents such apparently stable cycle [see Fig. 2(a)], has the maximum LE zero only with two precise decimals. Actually, in general, three precise decimals for zero LEs are extremely rare. Note that this zero value, appears only for `h_norm=5` and a smaller integration step size, $h = 0.001$ and only for a larger time interval `[0,500]`. Therefore, if no special improvements of numerical integration are implemented, with Benettin–Wolf algorithm, for integer but also for fractional order systems, two or three decimals are the maximal expected number of decimals.

One of the most important algorithm variable, is `h_norm`, which determines the normalization moments which influences the results. This dependence can be also deduced from the example of the RF system. With `h=0.005` and `h_norm=0.005` one obtains `LEs=(0.0631, 0.0031, -2.0774)`, while with the same step-size, `h=0.005`, but `h_norm=0.5`, `LEs=(0.0643, 0.0026, -1.8254)`.

Since to our best knowledge, there is not criterion to choose precisely `h_norm`, the recommended way would be to realize several tests to choose the value for which slight modifications do not change the results significantly.

In the case of fixed step-size integration numerical methods, like the considered `FDE12`, `h_norm` can be chosen as depending on the step-size `h`. Therefore, as Fig. 6 shows, `h_norm` could be multiples of integration step-size `h` (ε represents the perturbation between nearly two trajectories x and $\bar{x}$).

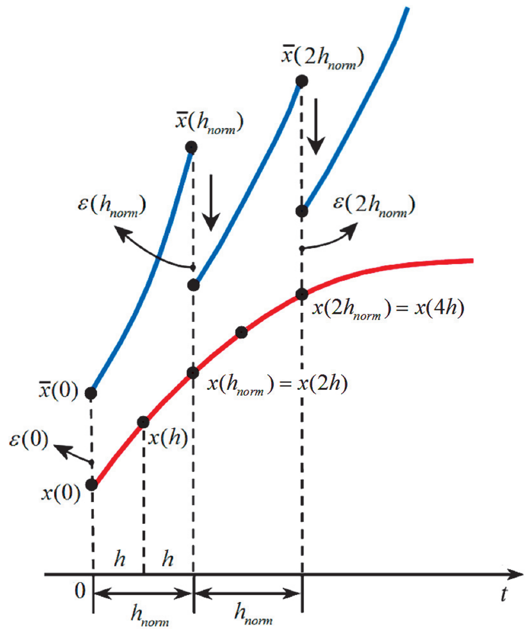


Fig. 6. Perturbation and rescaling of a nearby trajectory, after every $h_{\rm norm}$ steps, considered as multiple of h (here $h_{\rm norm} = 2h$, sketch).

Regarding the Gram–Schmidt procedure, in order to speed up the code, one can use the QR decomposition based on the Householder transformation: `[Q, R] = qr(A)`. To have matrix R with positive diagonal elements, one can additionally use `Q = Q*diag(sign(diag(R))); R = R * diag (sign(diag(R)))` (see e.g. [Ramasubramanian & Sriram, 2000]).

The integration step-size of `FDE12`, h, plays an important role. As the documentation of the program specifies, there exists the possibility to increase the performances of the numerical integration, but in time integration becomes detrimental.

Acknowledgment

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Appendices

A. Program for LEs of FO

```
function [t,LE]=FO_Lyapunov(ne,ext_fcn,t_start,
    h_norm,t_end,x_start,h,q,out);

%
%   Program to compute LEs of systems of FO.
%
%   The program uses a fast variant of the
%   predictor-corrector Adams-Bashforth-Moulton,
%   "FDE12.m" for FDEs, by Roberto Garrappa:
%
%   https://goo.gl/XScYmi
%
%   m-file required: FDE12 and
%   the function containing the extended system
%   (see e.g. LE_RF.m).
%
%   FO_Lyapunov.m was developed, by
%   modifying the program Lyapunov.m,
%   by V.N. Govorukhin:
%
%   https://goo.gl/wZVCtg
%
%   FO_Lyapunov.m, FDE12.m and LE_RF.m
%   must be in the same folder.
%
%   How to use it:
%     [t,LE]=FO_Lyapunov(ne,ext_fcn,t_start,...
%     h_norm,t_end,x_start,h,q,out);
%
%   Input:
%     ne - system dimension;
%     ext_fcn - function containing the extended
%     system;
%     t_start, t_end - time span;
%     h_norm - step for Gram-Schmidt
%     renormalization;
%     x_start - initial condition;
%     outp - priniting step of LEs;
%     ioutp==0 - no print.
%
%   Output:
%     t - time values;
%     LE Lyapunov exponents to each time value.
%
%   Example of use for the RF system:
%   [t,LE]=FO_Lyapunov(3,@LE_RF,0,0.02,300,...
%   [0.1;0;1;0.1],0.005,0.999,1000);
%
%   The program is presented in:
%
%   Marius-F. Danca and N. Kuznetsov,
%   Matlab code for Lyapunov exponents of
%   fractional order systems
%
%
hold on;

% Memory allocation
x=zeros(ne*(ne+1),1);
x0=x;
c=zeros(ne,1);
gsc=c; zn=c;
n_it = round((t_end-t_start)/h_norm);

% Initial values
   x(1:ne)=x_start;
   i=1;
   while i<=ne
      x((ne+1)*i)=1.0;
      i=i+1;
   end
   t=t_start;
% Main loop
it=1;
while it<=n_it
%       Solution of extended ODE system
        [T,Y] = FDE12(q,ext_fcn,t,t+h_norm,x,h);
        t=t+h_norm;
        Y=transpose(Y);
        x=Y(size(Y,1),:); %solution at t+h_norm
        i=1;
        while i<=ne
            j=1;
            while j<=ne;
                x0(ne*i+j)=x(ne*j+i);
                j=j+1;
            end;
            i=i+1;
        end;
%       orthonormal Gram-Schmidt basis
        zn(1)=0.0;
        j=1;
        while j<=ne
            zn(1)=zn(1)+x0(ne*j+1)*x0(ne*j+1);
            j=j+1;
        end;
        zn(1)=sqrt(zn(1));
        j=1;
        while j<=ne
            x0(ne*j+1)=x0(ne*j+1)/zn(1);
            j=j+1;
        end
        j=2;
        while j<=ne
            k=1;
            while k<=j-1
                gsc(k)=0.0;
                l=1;
                while l<=ne;
                    gsc(k)=gsc(k)+x0(ne*l+j)*x0(ne*
                        l+k);
                    l=l+1;
                end
                k=k+1;
            end
            k=1;
            while k<=ne
                l=1;
                while l<=j-1
                    x0(ne*k+j)=x0(ne*k+j)-gsc(l)*x0
                        (ne*k+l);
                    l=l+1;
                end
                k=k+1;
            end;
            zn(j)=0.0;
            k=1;
            while k<=ne
                zn(j)=zn(j)+x0(ne*k+j)*x0(ne*k+j);
                k=k+1;
            end
            zn(j)=sqrt(zn(j));
            k=1;
            while k<=ne
                x0(ne*k+j)=x0(ne*k+j)/zn(j);
                k=k+1;
            end
            j=j+1;
        end
%       update running vector magnitudes
        k=1;
        while k<=ne;
            c(k)=c(k)+log(zn(k));
            k=k+1;
        end;
```

```
%       normalize exponent
        k=1;
        while k<=ne
            LE(k)=c(k)/(t-t_start);
            k=k+1;
        end
        i=1;
        while i<=ne
            j=1;
            while j<=ne;
                x(ne*j+i)=x0(ne*i+j);
                j=j+1;
            end
            i=i+1;
        end;
        x=transpose(x);
        it=it+1;
%       print and plot the results
        if (mod(it,out)==0) %     (*)
            fprintf('%10.2f %10.4f %10.4f...
             %10.4f\n',[t,LE]); % (*)
        end; %                    (*)
        plot(t,LE) %              (*)
end
% displays the box outline around axes
xlabel('t','fontsize',16) %       (*)
ylabel('LEs','fontsize',14) %     (*)
set(gca,'fontsize',14)%           (*)
box on;%                          (*)
line([0,t],[0,0],'color','k')%    (*)
```

B. Function LE_RF.m

```
function f=LE_RF(t,x)

%Output data must be a column vector
f=zeros(size(x));

%variables allocated to the variational equations
X= [x(4), x(7), x(10);
    x(5), x(8), x(11);
    x(6), x(9), x(12)];

%RF equations
f(1)=x(2)*(x(3)-1+x(1)*x(1))+0.1*x(1);
f(2)=x(1)*(3*x(3)+1-x(1)*x(1))+0.1*x(2);
f(3)=-2*x(3)*(0.98+x(1)*x(2));

%Jacobian matrix
J=[2*x(1)*x(2)+0.1, x(1)*x(1)+x(3)-1, x(2);
     -3*x(1)*x(1)+3*x(3)+1,0.1,3*x(1);
     -2*x(2)*x(3),-2*x(1)*x(3),-2*(x(1)*x(2)+0.98)
        ];

%Righthand side of variational equations
f(4:12)=J*X; % To be modified if ne>3
```

C. Program for LEs as Function on *p*

```
function run_Lyapunov_p(ne,ext_fcn,t_start,h_norm
    ,t_end,x_start,h,q,p_min,p_max,n);
hold on;
p_step=(p_max-p_min)/n
p=p_min;
while p<=p_max
   LE=FO_Lyapunov_p(ne,ext_fcn,t_start,h_norm,
       t_end,x_start,h,q,p);
   p=p+p_step
   plot(p,LE);
end
```

D. Function LE_RF_p.m

```
function f=LE_RF_p(t,x,p)
%p is the parameter
f=zeros(size(x));
X= [x(4), x(7), x(10);
x(5), x(8), x(11);
x(6), x(9), x(12)];
%RF equations
f(1)=x(2)*(x(3)-1+x(1)*x(1))+0.1*x(1);
f(2)=x(1)*(3*x(3)+1-x(1)*x(1))+0.1*x(2);
f(3)=-2*x(3)*(p+x(1)*x(2));
%Jacobian matrix
J=[2*x(1)*x(2)+0.1, x(1)*x(1)+x(3)-1, x(2);
-3*x(1)*x(1)+3*x(3)+1,0.1,3*x(1);
-2*x(2)*x(3),-2*x(1)*x(3),-2*(x(1)*x(2)+p)];
f(4:12)=J*X; % To be modified if ne>3
```

E. Program for LEs as Function of *q*

```
function run_FO_Lyapunov_q(ne,ext_fcn,t_start,
    h_norm,t_end,x_start,h,q_min,q_max,n);
hold on;
q_step=(q_max-q_min)/n;
q=q_min;
while q<q_max
[t,LE]=FO_Lyapunov_q(ne,ext_fcn,t_start,h_norm,
    t_end,x_start,h,q);
q=q+q_step;
fprintf('q=%10.4f\n %10.4f', q);
plot(q,LE);
end
```