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Collocation method with convergence for generalized fractional integro-differential equations

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HIGHLIGHTS

- Collocation method for a linear and nonlinear Generalized Fractional Integro-differential equations (GFIDEs) defined in terms of B-operators is developed.
- The convergence analysis for both forms of GFIDEs is established.
- Test examples from the literature are considered to perform the numerical simulations, and the obtained results are presented through figures and tables.

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ABSTRACT

In this paper, we study a numerical approach for some class of generalized fractional integro-differential equations (GFIDEs) defined in terms of the B-operators presented recently. We develop collocation method for linear and nonlinear forms of GFIDEs. The numerical approach uses the idea of collocation methods for solving integral equations. Legendre polynomials are used to approximate the solution in finite dimensional space with convergence analysis. The obtained approximate solution recovers the solution of the fractional integro-differential equation (FIDE) defined using Caputo derivatives in a special case. FIDEs containing convolution type kernels appear in diverse area of science and engineering applications; therefore, some test examples varying the kernel in the B-operator are considered to perform the numerical investigations. The numerical results validate the presented scheme and provide good accuracy using few Legendre basis functions.

1. Introduction

Fractional calculus describes many significant phenomena in science and engineering. The appearance of the fractional derivative in numerous applications can be found in the literature such as viscous-elasticity [1,2], bioengineering [3], electrochemistry [4], electromagnetism [5], and more could be found in [6,7]. Most commonly used fractional derivatives in developing the theories and studying various applications are centred to the Riemann–Liouville and Caputo fractional derivatives. Some other derivatives of importance are also introduced and are known as Riesz–Riemann–Liouville, Riesz–Caputo fractional, and Grünwald—Letnikov derivatives. Authors are referred to see Refs. [6–8] for a concrete comprehension of fractional derivatives. However, in a recent study [9], some new operators are presented. These operators that allow the kernel to be chosen arbitrarily are known as K-, A- and B-operators. For specific choice of the kernel (power kernel),

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B-operator reduces to the Caputo derivative. Several studies using these operators are presented in recent years such as variational problems [10,11] and more on this could be found in book [12].

Many real world problems are being modelled using fractional derivative and integral terms, and such equations are known as the fractional integro-differential equations (FIDEs). FIDEs arise in the areas of acoustic waves [8], mechanics [13], and electromagnetics in dielectric media [14]. There has been much developments on the analytical and numerical methods for solving the FIDEs in recent years. Some of them are described as follows: in [15] Saadatmandi and Dehghan applied the Legendre collocation method for numerical solution of the FIDEs. Rawashdeh [16] presented the spline collocation scheme for solving FIDEs. In [17], the application of the collocation method has been extended for solving nonlinear FIDEs. Galerkin method, and wavelet Galerkin method based numerical methods are studied by the authors, respectively, in [18] and [19] for FIDEs. In [20,21], authors presented the second kind Chebyshev wavelet based approximation and cosine and sine wavelet method, respectively, for the FIDEs. Some other methods such as least squares method [22], Tau approximation method [23], Chebyshev pseudo spectral method [24], and hybrid collocation method [25,26] are discussed by the authors in past few years. More recently in 2015, Tohidi and co-authors [27] presented the Euler function based operational matrix approach for such problems.

Here, in this paper we consider the FIDEs (Eq. (5)) defined in terms of a class of generalized derivative (B-operators) and thus name it as generalized fractional integro-differential equations (in short GFIDEs) and present the collocation approach to solve it. The Legendre polynomials are considered to approximate the unknown function in the GFIDEs. The convergence properties for GFIDEs of the presented approach are also proved. Some illustrative examples from the literature varying the kernel in B-operators are considered to perform the numerical investigations. As B-operators reduce to Riemann Liouville fractional derivative, Caputo derivative, Riesz Riemann–Liouville derivative, Riesz–Caputo fractional derivative and many other fractional derivatives, the method presented here could be easily applicable to FIDEs defined in terms of these derivatives.

2. Generalized fractional integro-differential equations

Here, first we state the definition of the K and A/B-operators and then define the FIDEs in terms of B operator. The operators namely K and A/B operators as presented recently in [9] are defined on integrable functions $h(\xi)$ as follows:

$$\left(K_P^\alpha h\right)(\xi) = r\int_a^\xi \omega_\alpha(\xi,\eta)\,h(\eta)\,d\eta + s\int_\xi^b \omega_\alpha(\eta,\xi)\,h(\eta)\,d\eta, \quad \alpha>0, \tag{1}$$

where $\xi\in\mathfrak{J}=[a,b]$, $P=\langle a,\xi,b,r,s\rangle$ is a set of parameters, and $\omega_\alpha(\xi,\eta)$ be defined on $\mathfrak{J}\times\mathfrak{J}$. We assume $h(\xi)$ and $\omega_\alpha(\xi,\eta)$ are the square integrable functions such that right side of Eq. (1) exists. It follows the linearity property, i.e. for any two integrable functions $h_1(\xi)$ and $h_2(\xi)$,

$$\left(K_P^\alpha(h_1+h_2)\right)(\xi) = \left(K_P^\alpha h_1\right)(\xi) + \left(K_P^\alpha h_2\right)(\xi). \tag{2}$$

Now we consider A and B-operators [9],

$$\left(A_P^\alpha h\right)(\xi) = \mathfrak{D}^m\left(K_P^{m-\alpha}h\right)(\xi), \tag{3}$$

$$\left(B_P^\alpha h\right)(\xi) = \left(K_P^{m-\alpha}\mathfrak{D}^m h\right)(\xi), \tag{4}$$

or, Eq. (4) can be rewritten using Eq. (1) as

$$\left(B_P^\alpha h\right)(\xi) = r\int_a^\xi \omega_{m-\alpha}(\xi,\eta)\,\mathfrak{D}^m h(\eta)\,d\eta + s\int_\xi^b \omega_{m-\alpha}(\eta,\xi)\,\mathfrak{D}^m h(\eta)\,d\eta,\ \alpha>0, \tag{4a}$$

where $m-1<\alpha<m$, m is an integer and $P=\langle a,\xi,b,r,s\rangle$ and $\mathfrak{D}$ denote the differential operator. In definition of B-operator, we assume that $\mathfrak{D}^m h(\xi)$ is once integrable on the domain. More details on these operators can be found in [9].

We now define a GFIDEs in terms of B-operator as follows:

$$\left(B_P^\alpha \gamma\right)(\xi) = (H\gamma)(\xi), \tag{5}$$

$$\gamma(0) = \gamma_0. \tag{6}$$

Here the right side of Eq. (5) is considered as

$$(H\gamma)(\xi) = \varphi(\xi) + q(\xi)\,\gamma(\xi) + \int_0^\xi \rho(\xi,\eta)\,\mathcal{G}(\gamma(\eta))d\eta, \tag{7}$$

and

$$\left(B_P^\alpha \gamma\right)(\xi) = r\int_0^\xi \omega_{m-\alpha}(\xi,\eta)\,\mathfrak{D}^m\gamma(\eta)\,d\eta + s\int_\xi^1 \omega_{m-\alpha}(\eta,\xi)\,\mathfrak{D}^m\gamma(\eta)\,d\eta,\ \alpha>0. \tag{8}$$

In Eq. (5), we consider $\mathfrak{I} = [0, 1]$, and $m = 1$ in the B-operator. Here, functions $\varphi(\xi)$ and $q(\xi)$ are known functions that belong to $L^2(\mathfrak{I})$. We assume $q(\xi) \neq 0$ for $\xi \in [0, 1]$, and $\gamma(\xi)$ is unknown function. The kernel $\rho(\xi, \eta)$ in Eq. (7) is either smooth or weakly singular of the form

$$\rho(\xi, \eta) = (\xi - \eta)^{-\mu}, \quad 0 < \mu < 1. \tag{9}$$

We study Eq. (5) under the assumption that kernel $\omega_\alpha(\xi, \eta) \in L^2(\mathfrak{I} \times \mathfrak{I})$ and $\mathcal{G}$ is some linear or nonlinear operator.

We assume that Eq. (5) with Eq. (6) possesses a unique solution for all real values of r and s. This equation is solvable for any real number r and $s = 0$ or for any real number s and $r = 0$. The particular cases of this problem given by Eq. (5) have been discussed and solved in Refs. [14–18] with $r = 1, s = 0$. The aim of this paper is to develop a numerical technique to solve Eq. (5) with initial condition given by Eq. (6) for $0 < \alpha < 1$ or $m = 1$.

3. Preliminaries: Legendre polynomials and function approximations

3.1. Definition and some properties of shifted Legendre polynomials

The shifted Legendre polynomials are well known Legendre polynomials which are shifted from $[-1, 1]$ to $[0, 1]$ by variable transformation $\xi \to 2\xi - 1$. We define shifted Legendre polynomials as follows:

$$\theta_i(\xi) = l_i(2\xi - 1), \tag{10}$$

where l_i denotes the Legendre polynomials of degree m and satisfies

$$l_0(\xi) = 1,\ l_1(\xi) = \xi \text{ and,}$$

$$l_{i+1}(\xi) = \frac{2i+1}{i+1}\xi l_i(\xi) - \frac{i}{i+1} l_{i-1}(\xi),\ i = 1, 2, 3, \ldots, \tag{11}$$

and satisfies orthogonality with respect to the weight function 1, *i.e.*,

$$\int_0^1 \theta_i(\xi)\, \theta_j(\xi)\, d\xi = \frac{1}{2i+1}\delta_{ij}, \tag{12}$$

where δ_{ij} is Kronecker Delta function. Suppose $X = L^2(\mathfrak{I})$ be the inner product space, and inner product in this space is defined by

$$\langle h_1 | h_2 \rangle = \int_0^1 h_1(\xi)\, h_2(\xi)\, d\xi, \tag{13}$$

and the corresponding norm is as follows:

$$\|h\|_2 = \left(\int_0^1 |h(\xi)|^2 d\xi\right)^{1/2}. \tag{14}$$

3.2. Function approximation

Any function $h(\xi)$ in $L^2(\mathfrak{I})$ can be approximated as

$$h(\xi) \approx \sum_{i=0}^{R} c_i \theta_i(\xi), \tag{15}$$

where C and $\theta(\xi)$ are vectors given by

$$C = [c_0, c_1, \ldots, c_R], \tag{16}$$

$$\theta(\xi) = [\theta_0(\xi), \theta_1(\xi), \ldots, \theta_R(\xi)]. \tag{17}$$

Here, C denotes a vector of some suitable coefficients.

Theorem 1 ([28]). *Let $h(\xi)$ be a real sufficiently smooth function in $L^2(\mathfrak{I})$ and $h(\xi) \approx \sum_{i=0}^{R} c_i \theta_i(\xi)$ denotes the shifted Legendre expansion of $h(\xi)$, where*

$$C = [c_0, c_1, \ldots, c_R] \text{ and,}$$

$$c_i = (2i+1)\int_0^1 h(\xi)\theta_i(\xi)\, d\xi, \tag{18}$$

then there exists a real constant α *satisfying,*

$$\|h(\xi) - h_R(\xi)\|_2 \leq \frac{\alpha}{(R+1)!2^{2R+1}}, \tag{19}$$

where $\alpha = \max\{|h^{R+1}(\xi)| \xi \in (0,1)\}$.

4. Collocation method for GFIDEs

In this section, we solve a new GFIDE given by Eq. (5) with boundary condition mentioned by Eq. (6) using collocation approach. Collocation method is based on projection method where we choose a finite dimensional family of functions to approximate exact solution and then by applying collocation method we obtain an algebraic system of linear equations and further such systems can be solved using any standard method.

We now approximate function $\gamma(\xi)$ by Eq. (15),

$$\gamma_R(\xi) \approx \sum_{i=0}^{R} c_i \theta_i(\xi). \tag{20}$$

Substituting the value of $\gamma_R(\xi)$ in (5), we get

$$\left(B_P^{\alpha}\gamma_R\right)(\xi) = (H\gamma_R)(\xi), \quad 0 < \alpha < 1, \tag{21}$$

$$\text{and } \gamma_R(0) = \gamma_0. \tag{22}$$

$$\text{or, } \left(B_P^{\alpha}(\sum_{r=0}^{R} c_i\theta_i)\right)(\xi) = \left(H\sum_{i=0}^{R} c_i\theta_i\right)(\xi), \tag{23}$$

$$\sum_{i=0}^{R} c_i\theta_i(\xi) = \gamma_0. \tag{24}$$

We pick distinct node points $\xi_k \in [0, 1]$, such that

$$\left(B_P^{\alpha}(\sum_{i=0}^{R} c_i\theta_i)\right)(\xi_k) = \left(H\sum_{i=0}^{R} c_i\theta_i\right)(\xi_k), \quad k = 0, 1, \ldots, R-1. \tag{25}$$

$$\text{And, } \sum_{i=0}^{R} c_i\theta_i(\xi_k) = \gamma_0. \tag{26}$$

Now, Eq. (5) is converted into Eqs. (25)–(26) which is a system of equations in terms of unknowns c_i and it by solving using any standard method the approximate solution is obtained. For solving system of equations numerically, we use the Mathematica software.

4.1. Convergence analysis

Lemma 1 ([29]). *Let* $X = L^2(\mathfrak{I})$*, i.e., denote the vector space of square-summable functions defined on* $\mathfrak{I} = [0, 1]$ *and* K *be a Volterra integral operator on* X *defined by*

$$K(h(\xi)) = \int_0^{\xi} \kappa(\xi, \eta) h(\eta)\, d\eta \quad \forall h \in X, \tag{27}$$

with kernel $\kappa(\xi, \eta)$ *satisfying* $\int_0^1 \int_0^1 |\kappa(\xi, \eta)| d\xi d\eta = \Omega^2$ *or* $\sup_{\xi,\eta} \kappa(\xi, \eta) = \Omega$*, and* Ω *is a constant.*

Then K *is bounded in* $L^2(\mathfrak{I})$*. That is,*

$$\|K(h(\xi))\|_2 \leq \Omega\|h\|_2. \tag{28}$$

Lemma 2. *Let* $\gamma(\xi)$ *be sufficiently smooth function in* $L^2(\mathfrak{I})$ *and* $\left(\frac{d\gamma_R}{d\xi}\right)$ *be the approximation of* $\frac{d\gamma}{d\xi}$*. Assume that* $\frac{d\gamma}{d\xi}$ *is bounded by a constant C, i.e.* $\left|\frac{d\gamma}{d\xi}\right| \leq C$*, then we have*

$$\|\frac{d\gamma}{d\xi} - \left(\frac{d\gamma_R}{d\xi}\right)\|_2^2 \leq \left(\frac{C}{\pi}\right)^2 \frac{1}{R^2}.$$

Proof. Let,

$$\frac{d\gamma}{d\xi} = \sum_{i=0}^{\infty} c_i \theta_i (\xi) \,. \tag{29}$$

Truncating it up to $R-1$ level, we get

$$\left(\frac{d\gamma_R}{d\xi}\right) = \sum_{i=0}^{R-1} c_i \theta_i (\xi) \,. \tag{30}$$

From Eqs. (29) and (30), we have

$$\frac{d\gamma}{d\xi} - \left(\frac{d\gamma_R}{d\xi}\right) = \sum_{i=R}^{\infty} c_i \theta_i (\xi) \,,$$

$$\| \frac{d\gamma}{d\xi} - \left(\frac{d\gamma_R}{d\xi}\right) \|_2^2 = \int_0^1 \left(\frac{d\gamma}{d\xi} - \left(\frac{d\gamma_R}{d\xi}\right)\right)^2 dx = \int_0^1 \left(\sum_{i=R}^{\infty} c_i \theta_i (\xi)\right)^2 d\xi \,,$$

or,

$$\| \frac{d\gamma}{d\xi} - \left(\frac{d\gamma_R}{d\xi}\right) \|_2^2 = \sum_{i=R}^{\infty} \frac{c_i^2}{2i+1}. \tag{31}$$

$$c_i = (2i+1) \int_0^1 \frac{d\gamma}{dx} \theta_i (\xi) \, d\xi \,,$$

$$c_i \leq (2i+1)\, \mathrm{C} \int_0^1 \theta_i (\xi) \, d\xi \leq (2i+1)\, \mathrm{C} \frac{\sin(i\pi)}{(i+i^2)\pi} \leq \frac{(2i+1)\, \mathrm{C}}{(i+i^2)\pi}.$$

$$|c_i|^2 \leq \left(\frac{(2i+1)\, \mathrm{C}}{(i+i^2)\, \pi}\right)^2.$$

Thus,

$$\sum_{i=R}^{\infty} \frac{c_i^2}{2i+1} \leq \sum_{i=R}^{\infty} \left(\frac{\mathrm{C}}{\pi}\right)^2 \frac{2i+1}{\left((i+i^2)\right)^2} = \left(\frac{\mathrm{C}}{\pi}\right)^2 \frac{1}{R^2}. \tag{32}$$

4.2. Error analysis

Case 1: When $\mathcal{G}$ is linear.

Let $E_R(\xi) = \gamma(\xi) - \gamma_R(\xi)$ denote the error function of the approximate solution $\gamma_R(\xi)$ to the exact solution $\gamma(\xi)$ of Eq. (5).

Substituting the approximate solution in Eq. (5), we get

$$\left(B_P^\alpha \gamma_R\right)(\xi) = (H\gamma_R)(\xi) = \varphi(\xi) + q(\xi)\gamma_R(\xi) + \int_0^\xi \rho(\xi,\eta)\, \mathcal{G}(\gamma_R(\eta)) d\eta.$$

Subtracting the above equation from Eq. (5) and rearranging the terms, we get

$$q(\xi)(E_R(\xi)) = \int_0^\xi \rho(\xi,\eta)\left(\mathcal{G}(\gamma(\xi)) - \mathcal{G}(\gamma_R(\xi))\right) d\eta - \left(B_P^\alpha(\gamma - \gamma_R)\right)(\xi) \,. \tag{33}$$

Since $\mathcal{G}$ is linear, we have

$$|q(\xi)\, E_R(\xi)| \leq \left| \int_0^\xi \rho(\xi,\eta)\, \mathcal{G}(E_R(\eta)) d\eta \right| + |\left(B_P^\alpha(\gamma - \gamma_R)\right)(\xi)|. \tag{34}$$

$$|q(\xi)\, E_R(\xi)| \leq \mathrm{Q} \left| \int_0^\xi E_R(\eta)\, d\eta \right| + |\left(B_P^\alpha(\gamma - \gamma_R)\right)(\xi)|.$$

where $\mathrm{Q} = \max \rho(\xi,\eta)$.

Now, by Gronwall's inequality,

$$\|q(\xi)\, E_R(\xi)\|_2 \leq \| \left(B_P^\alpha(\gamma - \gamma_R)\right)(\xi) \|_2. \tag{35}$$

$$\text{Now, } \| \left(B_P^{\alpha}(\gamma-\gamma_R)\right)(\xi)\|_2 \le \|T_1\|_2 + \|T_2\|_2. \tag{36}$$

where $T_1 = r\int_0^{\xi}\omega_{1-\alpha}(\xi,\eta)\,\mathfrak{D}\left(\gamma(\eta)-\gamma_R(\eta)\right)d\eta$ and $T_2 = s\int_{\xi}^{1}\omega_{1-\alpha}(\eta,\xi)\,\mathfrak{D}\left(\gamma(\xi)-\gamma_R(\xi)\right)d\xi$.

Since $\omega_{1-\alpha}(\xi,\eta)\in L^2$, then by Lemma 1 there exist constants Λ_1, Λ_2 such that

$$\|T_1\|_2 \le \Lambda_1\|\mathfrak{D}\left(\gamma(\xi)-\gamma_R(\xi)\right)\|_2,$$

and

$$\|T_2\|_2 \le \Lambda_2\|\mathfrak{D}\left(\gamma(\xi)-\gamma_R(\xi)\right)\|_2.$$

Thus, $\| \left(B_P^{\alpha}(\gamma-\gamma_R)\right)(\xi)\|_2 \le \Lambda\|\mathfrak{D}\left(\gamma(\xi)-\gamma_R(\xi)\right)\|_2$, $\Lambda = \Lambda_1+\Lambda_2$.

Using Lemma 2,

$$\| \left(B_P^{\alpha}(\gamma-\gamma_R)\right)(\xi)\|_2 \le \Lambda\left(\frac{C}{\pi}\right)^2\frac{1}{R^2}. \tag{37}$$

From Eqs. (35) and (37), we get

$$\|q(\xi)E_R(\xi)\|_2 \le \Lambda\left(\frac{C}{\pi}\right)^2\frac{1}{R^2}. \tag{38}$$

Since $q(\xi)\neq 0$, therefore $E_R(\xi)\to 0$ or $\gamma(\xi)\to\gamma_R(\xi)$ as $R\to\infty$.

Case 2: When $\mathcal{G}$ is nonlinear.

Let $\mathcal{G}$ satisfy the Lipschitz condition defined by

$$|\mathcal{G}(\gamma_1(\eta))-\mathcal{G}(\gamma_2(\eta))| \le \mathcal{L}|\gamma_1(\eta)-\gamma_2(\eta)|, \tag{39}$$

where $\mathcal{L}$ is constant independent of γ.

So by Eq. (33), we obtain

$$|q(\xi)E_R(\xi)| \le \int_0^{\xi}\mathcal{L}|\rho(\xi,\eta)|\,|E_R(\eta)|d\eta + |\left(B_P^{\alpha}(\gamma-\gamma_R)\right)(\xi)|,$$

or,

$$|q(\xi)E_R(\xi)| \le \mathcal{L}Q\left|\int_0^{\xi}|E_R(\eta)|d\eta\right| + |\left(B_P^{\alpha}(\gamma-\gamma_R)\right)(\xi)|. \tag{40}$$

Now proceeding from Eq. (35), we obtain the result given by Eq. (38).

4.3. Error estimate

Let $E_R(\xi) = \gamma(\xi)-\gamma_R(\xi)$ denotes the error function of approximate solution $\gamma_R(\xi)$ to the exact solution $\gamma(\xi)$. By substitution of $\gamma_R(\xi)$ in Eq. (5), we have

$$\left(B_P^{\alpha}\gamma_R\right)(\xi) + Y_R(\xi) = \varphi(\xi) + q(\xi)\gamma_R(\xi) + \int_0^{\xi}\rho(\xi,\eta)\,\mathcal{G}(\gamma_R(\eta))d\eta, \tag{41}$$

with $\gamma_R(0) = (\gamma_0)_R$,

where $(\gamma_0)_R$ is the approximated value of approximate solution $\gamma_R(\xi)$ at $\xi = 0$, which depends on R and $(\gamma_0)_R\to\gamma_0$ as R increases.

From Eq. (39), the perturbation function $Y_R(\xi)$ can be calculated as

$$Y_R(\xi) = \varphi(\xi) + q(\xi)\gamma_R(\xi) + \int_0^{\xi}\rho(\xi,\eta)\,\mathcal{G}(\gamma_R(\eta))d\eta - \left(B_P^{\alpha}\gamma_R\right)(\xi).$$

Subtracting Eq. (39) from Eq. (5), we get

$$\left(B_P^{\alpha}E_R\right)(\xi) + Y_R(\xi) = \varphi(\xi) + q(\xi)E_R(\xi) + \int_0^{\xi}\rho(\xi,\eta)\,\mathcal{G}(E_R(\eta))d\eta,$$

or

$$\left(B_P^{\alpha}E_R\right)(\xi) = \varphi(\xi) + q(\xi)E_R(\xi) + \int_0^{\xi}\rho(\xi,\eta)\,\mathcal{G}(E_R(\eta))d\eta - Y_R(\xi), \tag{42}$$

with the initial condition $E_R(0) = (E_0)_R$.

Now error $E_R(\xi)$ can be approximated by applying the proposed method described in Section 4.

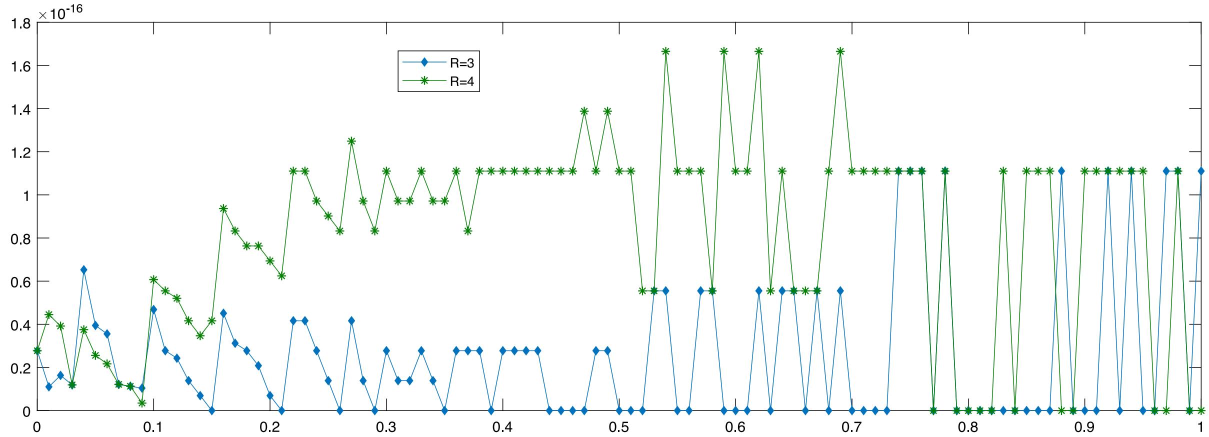

Fig. 1. Plot of Maximum absolute errors for Test example 1 for $R = 3, 4$.

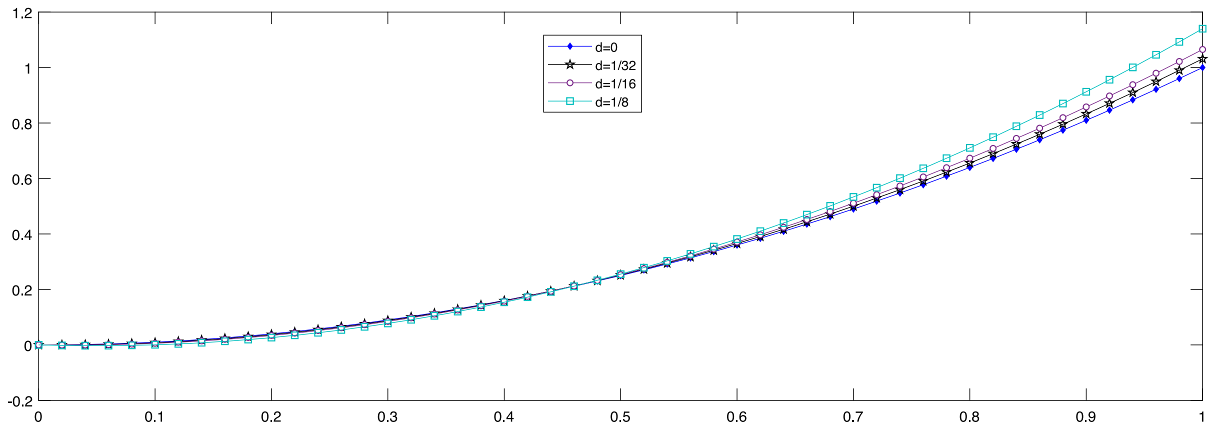

Fig. 2. Plot of approximate solutions for different values of d for Test example 1.

5. Numerical results

To perform the numerical investigations, five test examples varying the convolution type kernels in B-operator are chosen. We also present the case where the GFIDEs takes the form of FIDEs discussed in the literature. The numerical simulations are performed on the mathematical software Mathematica 10.

Test example 1. Consider Eq. (5) with $r = s = 1$ in B-operator, $\alpha = 2/3$, $\rho(\xi, \eta) = (\xi - \eta)^{-1/2}$, kernel [11] $\omega_{1-\alpha}(\xi, \eta) = \frac{\{d+(1-d)(\xi-\eta)\}^{-\alpha}}{\Gamma(\alpha)}$, $q(t) = 1$, and

$\varphi(\xi) = -\xi^2 - \frac{16\xi^{\frac{5}{2}}}{15} + \frac{9\xi^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \frac{3(1-\xi)^{1/3}(1+3\xi)}{2\Gamma\left(\frac{1}{3}\right)}$, with boundary condition $\gamma(0) = 0$.

This has exact solution ξ^2 for $d = 0$. This problem is solved using the proposed approach and numerical results with higher accuracy are obtained. Since the exact solution in this case is a second degree polynomial, therefore choice of the basis functions up to $R = 2$ would be sufficient to approximate the exact solution. The respective maximum absolute errors are also calculated and are shown in Table 1. The maximum absolute errors varying the number of polynomials ($R = 2$ and $R = 3$) are shown in Fig. 1. The numerical solutions are obtained varying the values of the parameter $d = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and are displayed through Fig. 2. We observe that as d decreases the approximate solution approaches to the exact solution for $d = 0$. Fig. 2shows the approximate solution varying the value of the parameter d.

Test example 2. Here, we choose Eq. (5) with $r = s = 1$ in the B-operator, $\alpha = 1/4$, $\rho(\xi, \eta) = (\xi - \eta)^{-1/3}$, kernel $\omega_{1-\alpha}(\xi, \eta) = (1 - \alpha)(\xi - \eta)$, $q(\xi) = 1$, and

$\varphi(\xi) = \frac{17}{16} - \frac{3x}{2} - x^2 - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{27}{440}x^{8/3}(11 + 9x)$, with boundary condition $\gamma(0) = 0$.

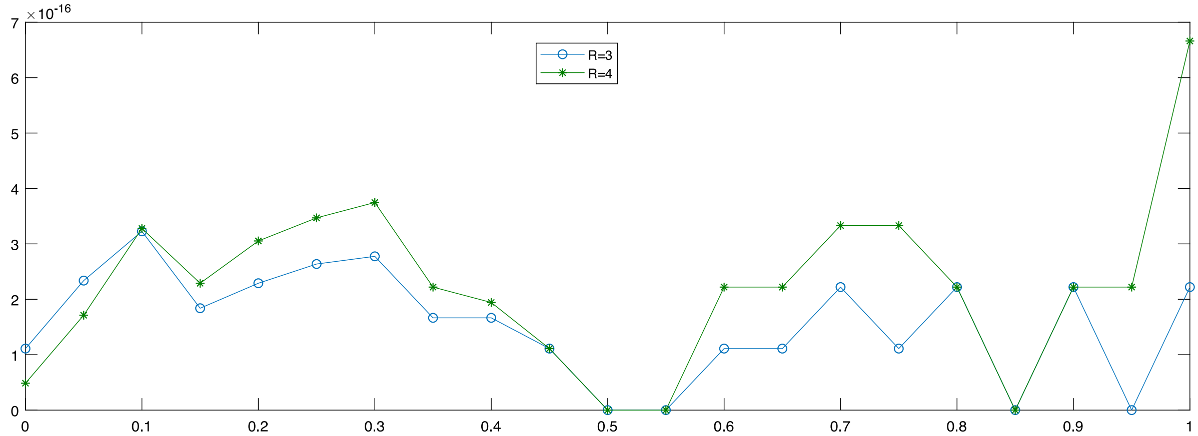

Fig. 3. Plot of maximum absolute errors for Test example 2 for $R = 3, 4$.

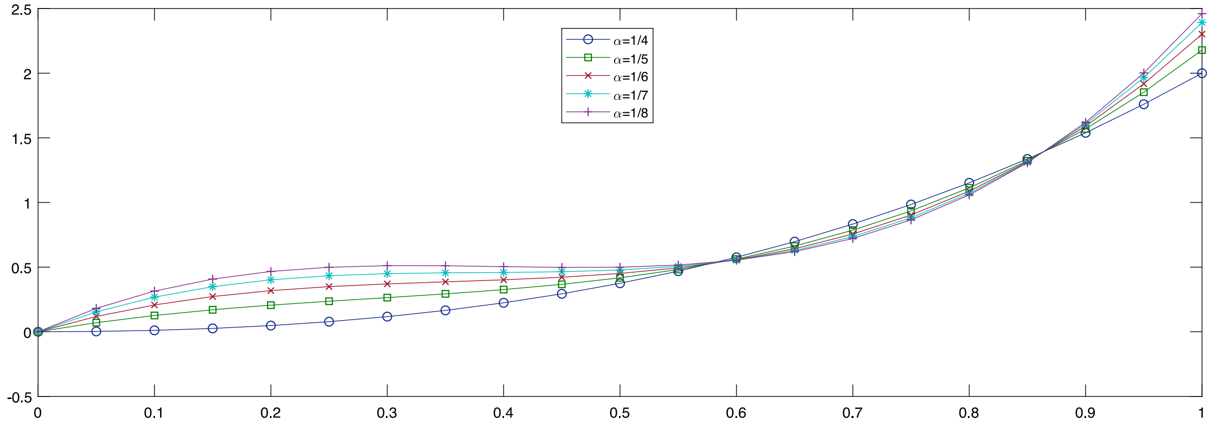

Fig. 4. Plot of approximate solutions for different values of α for Test example 2.

The exact solution in this case becomes $\xi^2 + \xi^3$. We solve this problem for $R = 2, 3, 4$ and obtained good approximate solution for $R = 3$. Maximum absolute errors for this case are listed in Table 1 for $R = 2, 3$ and are shown through Fig. 3 for $R = 3, 4$.

Also we obtain and plot the approximate solution varying the value of $\alpha = \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$. Fig. 4 ensures that the numerical solution approach to the exact solution as $\xi^2 + \xi^3$ as α increases.

Test example 3. We take Eq. (5) with $r = s = 1$ in the B-operator, $\alpha = 3/4$, $\rho(\xi, \eta) = (\xi - \eta)^{-1/2}$ kernel $\omega_{1-\alpha}(\xi, \eta) = \frac{(\xi-\eta)^{-\alpha}}{\Gamma(\alpha)}$, $q(t) = 1$, and

$$\varphi(\xi) = -e^{\xi} - e^{\xi}\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\xi}\right) + \frac{e^{\xi}(1-\xi)^{\frac{1}{4}}\left\{\Gamma\left(\frac{1}{4}\right) - \Gamma\left(\frac{1}{4}, \xi - 1\right)\right\}}{(\xi-1)^{\frac{1}{4}}\Gamma\left(\frac{1}{4}\right)} + e^{\xi}\left\{1 - \frac{\Gamma\left(\frac{1}{4}, \xi\right)}{\Gamma\left(\frac{1}{4}\right)}\right\},$$

with boundary condition $\gamma(0) = 1$.

This test example has exact solution e^{ξ}. This equation is solved for $R = 2, 3, 4, 5$ and numerical solutions are obtained. The approximated solutions appear very close to the exact solution and as the number of Legendre basis functions are increased the error decreases. The maximum absolute errors for this example are presented through Fig. 5 and also shown in Table 2. Comparison of exact and numerical solution is shown in Fig. 6 with $R = 5$.

Test example 4. Consider Eq. (5) with $r = 1, s = 1$ in the B-operator, $\alpha = 1/2$, $\rho(\xi, \eta) = (\xi - \eta)^{-1/2}$, and kernel $\omega_{1-\alpha}(\xi, \eta) = e^{-\alpha(\xi-\eta)}$, with

$$q(\xi) = 1,$$

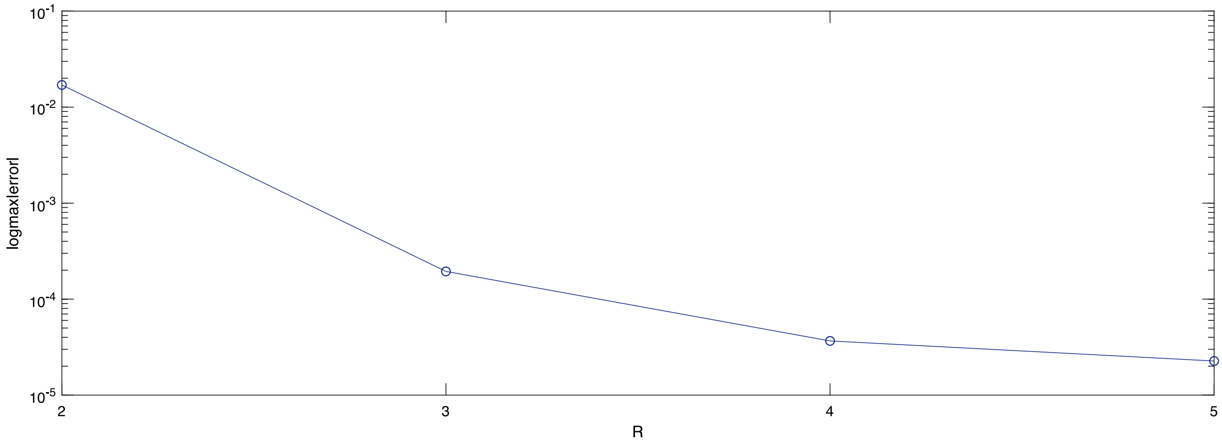

Fig. 5. Error plot for Test example 3.

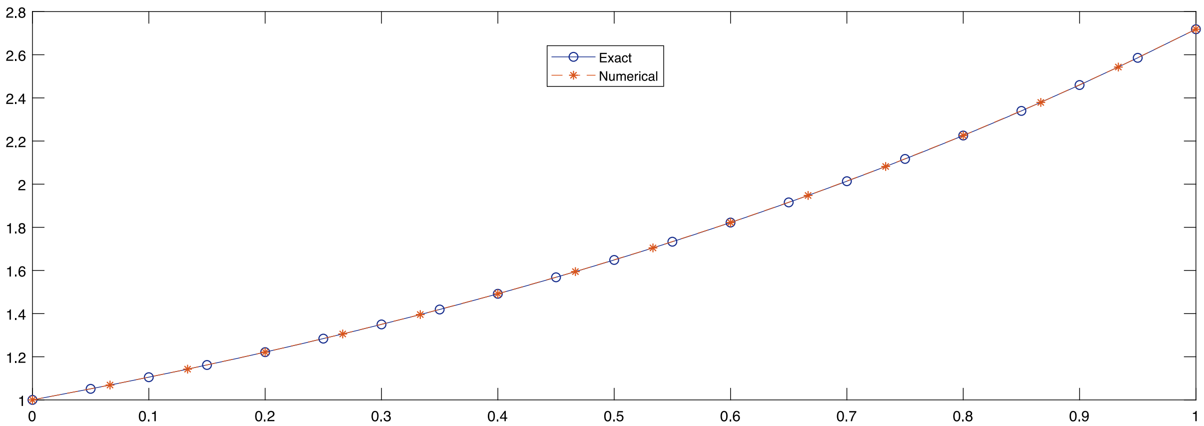

Fig. 6. Numerical versus exact solution for Test example 3 with $R = 5$.

$$\text{and, } \varphi(\xi) = -e^{\xi} + \frac{2e^{-\xi/2}(-1+e^{3\xi/2})}{3\sqrt{\pi}} + \frac{-2e^{\xi} + 2e^{\frac{1+\xi}{2}}}{\sqrt{\pi}} - e^{\xi}\sqrt{\pi}\,\mathrm{Erf}[\sqrt{\xi}],$$

with boundary condition $\gamma(0) = 1$.

In this case, the exact solution takes the form e^{ξ}. The presented approach is applied to this case and approximate numerical results are obtained with good accuracy. The calculated maximum absolute errors are shown in Table 2 and variations of maximum absolute errors are displayed through Fig. 7. Plot for different values of α is shown in Fig. 8.

Test example 5 ([30]). Consider the nonlinear case of Eq. (5) with $r = 1$, $s = 0$ in the B-operator, $\alpha = 1/2$, $\rho(\xi, \eta) = (\xi)^{1/2}$, and kernel $\omega_{1-\alpha}(\xi, \eta) = \frac{(\xi-\eta)^{-\alpha}}{\Gamma(\alpha)}$, with

$$q(\xi) = 2\sqrt{\xi} + 2\xi^{3/2} - (\sqrt{\xi} + \xi^{3/2})\ln[1+\xi],$$

$$\text{and, } \varphi(\xi) = -2\xi^{3/2} + \frac{2\mathrm{ArcSinh}[\sqrt{\xi}]}{\sqrt{\pi}\sqrt{1+\xi}},$$

with boundary condition $\gamma(0) = 0$.

The exact solution of this problem is $ln(1 + t)$. The methods discussed in Section 4 have been studied on this problem and numerical results are obtained with good accuracy. The obtained numerical errors are illustrated in Fig. 9and Table 2. Fig. 10 represents the plot of numerical and exact solution for $R = 5$. It is clear from Table 2 that numerical computational error converges to zero as the value of R increases.

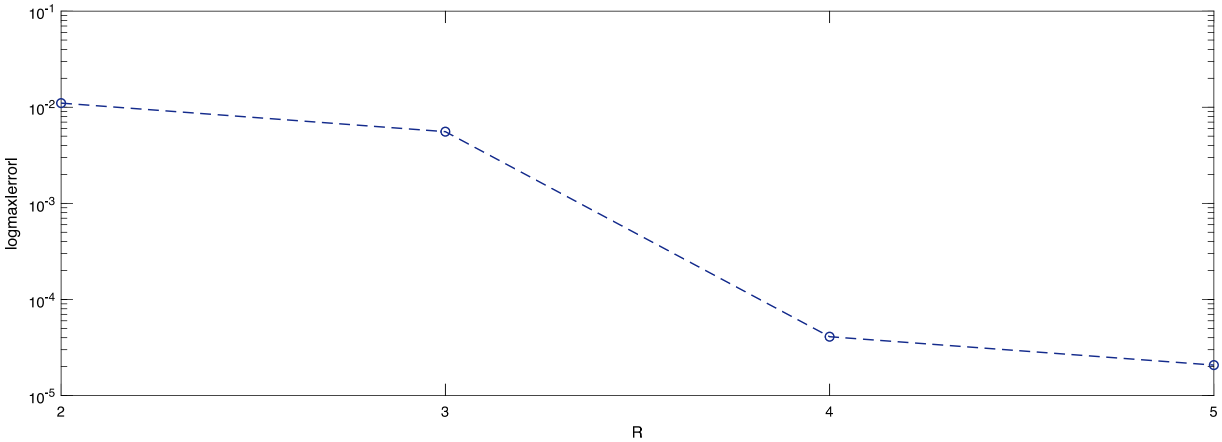

Fig. 7. Error plot for Test example 4.

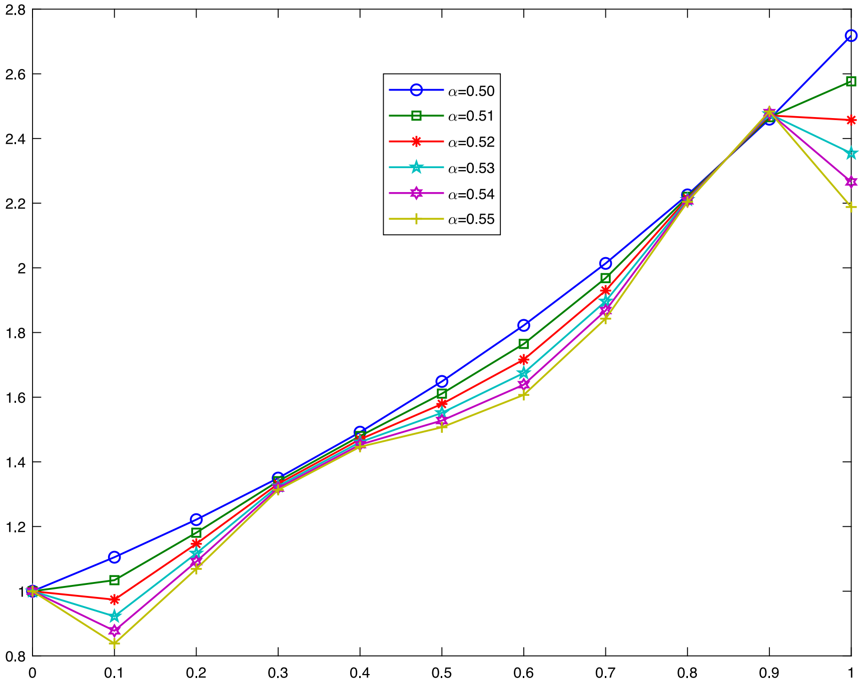

Fig. 8. Plot of approximate solutions for different values of α for Test example 4.

Table 1
Maximum absolute errors for Test examples 1 and 2.

R	Test example 1	Test example 2
2	5.2041E−17	3.8524E−2
3	6.8082E−17	2.2898E−16

6. Conclusions

A simple collocation approach is developed for the linear as well as nonlinear GFIDEs defined in terms of the B-operators. The convergence analysis of the collocation approach is also discussed and error bound of the approximation is obtained. Some numerical tests varying the kernel in the B-operator are considered, and the obtained simulations results are presented. Numerical results indicate that the presented approach for GFIDEs works well and obtains accurate results. It is observed from Test example 1 that the proposed solution recovers the solutions of the FDIEs in special case and thus it could be

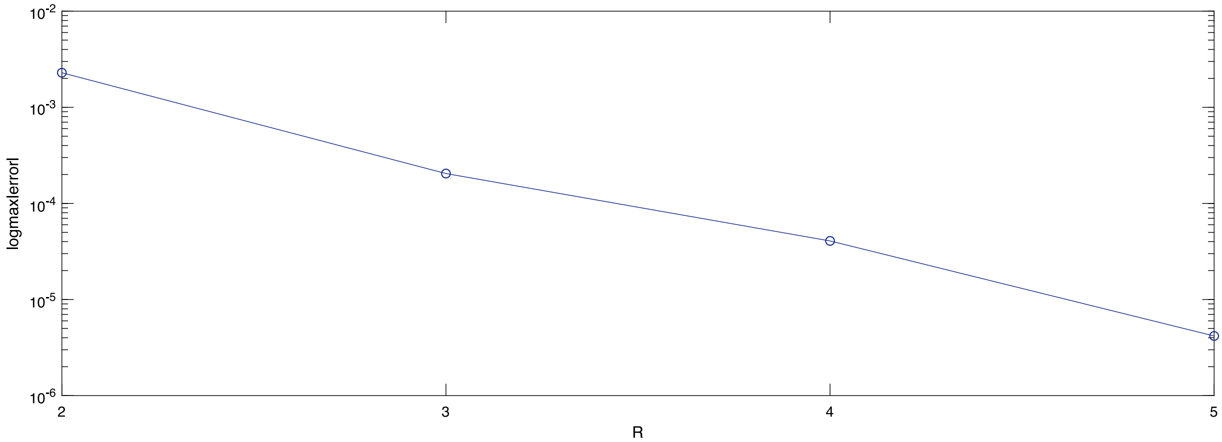

Fig. 9. Error plot for Test example 5 varying the number of polynomials R.

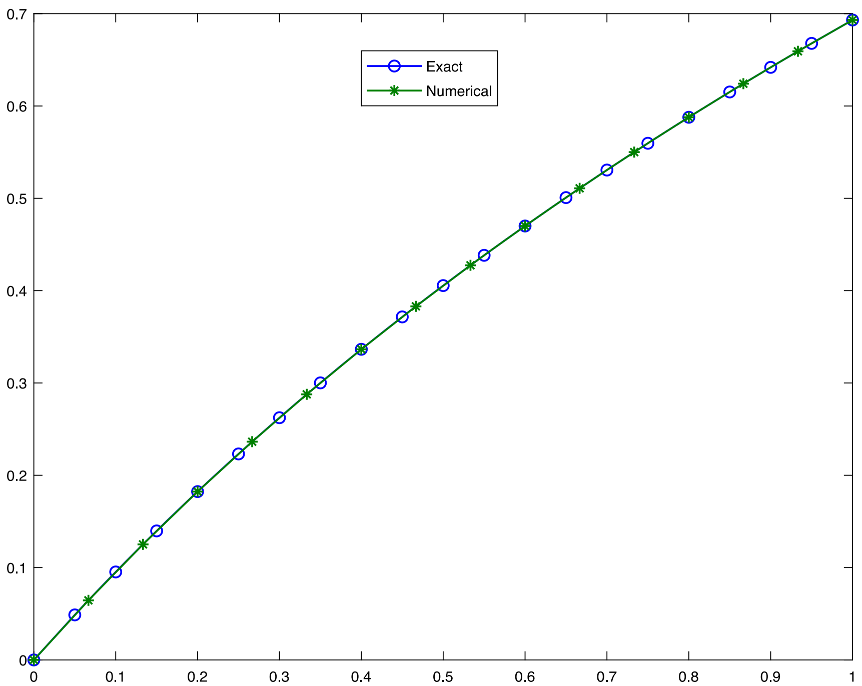

Fig. 10. Numerical versus exact solution for Test example 5 with $R = 5$.

Table 2
Maximum absolute errors for Test examples 3–5.

R	Test example 3	Test example 4	Test example 5
2	1.7019E−2	1.1104E−2	2.2891E−3
3	1.9441E−4	5.5713E−3	2.0455E−4
4	3.6687E−5	4.0970E−5	4.0699E−5
5	2.2674E−5	2.0755E−5	4.1765E−6

considered as a general approach for solving FIDEs. We observe that good approximation of exact solution can be achieved with a less number of basis polynomials. The approximate method presented here could also be applied to similar problems defined in terms of the other fractional derivatives. This could be due to fact that the B-operators reduce to Riemann–Liouville fractional derivative and Caputo fractional derivatives and many other fractional derivatives defined in the literature in special case.

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