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Numerical solution of fractional integro-differential equations by collocation method

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Abstract

This paper deals with the numerical solution of fractional integro-differential equations of the type

$$D^q y(t) = p(t)y(t) + f(t) + \int_0^t K(t,s)y(s)\,ds, \quad t \in I = [0,1]$$

by polynomial spline functions. We derive a system of equations that characterizing the numerical solution. Some numerical examples are also provided to illustrated our results.

Keywords: Fractional integro-differential equations; Spline space; Collocation method

1. Introduction

In this paper, we study the numerical solution of an integro-differential equation with fractional derivative of the type

$$D^q y(t) = p(t)y(t) + f(t) + \int_0^t K(t,s)y(s)\,ds, \quad t \in I = [0,1] \tag{1}$$

with the initial condition

$$y(0) = \alpha \tag{2}$$

by polynomial spline functions. Here, the given functions f, $p : I \to \mathbb{R}$ and $K : S \to \mathbb{R}$ (with $S = \{(t,s): 0 \leqslant s \leqslant t \leqslant 1\}$) are supposed to be sufficiently smooth, with $0 < q \leqslant 1$. Such kind of equations arises in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems (see for example [1,3,7,8]).

In recent years, the analytic results on existence and uniqueness of solutions to fractional differential equations ($K(t,s) \equiv 0$) have been investigated by many authors, (see for example [4,10,12]). Momani [6] has

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obtained local and global existence and uniqueness solution of the integro-differential equation given by (1) with the initial condition given by (2).

In Eq. (1), D^q denotes the fractional differential operator of order $q \notin \mathbb{N}$ in the sense of Riemann–Liouville, and is given by

$$D^q y(t) = D^k \frac{1}{\Gamma(k-q)} \int_0^t (t-s)^{k-q-1} y(s)\, \mathrm{d}s,$$

where $k \in \mathbb{N}$ and satisfies the relation $k-1 < q < k$.

Properties of the operators D^q can be found in [5], we mention the following:

$$D^q t^v = \frac{\Gamma(v+1)}{\Gamma(v+1-q)} t^{v-q}, \qquad q \geqslant 0, \quad t > 0, \quad v > -1.$$

In the last two decades, the numerical methods for approximating fractional derivatives have been extensively studied by many authors. A survey of some numerical methods is given by Podlubny [9]. Blank [2] proposed the collocation spline method. Rawashdeh [11] used the collocation spline method to approximate the solution of semidifferential equations.

The structure of this paper is as follows. In Section 2, we present the polynomial spline space. Section 3 contains the derivation of the collocation spline method. Some numerical examples are provided in Section 4.

2. Polynomial spline space

For $N \in \mathbb{N}$, let

$$Z_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \cdots < t_N = 1\} \tag{3}$$

be a partition of the interval [0, 1], given by the grid points

$$t_n = nh, \quad \text{with } h = \frac{1}{N} \quad \text{and} \quad n = 0, 1, \ldots, N. \tag{4}$$

Let

$$\sigma_n = [t_n, t_{n+1}], \quad n = 0, \ldots, N-1.$$

For given integers $m \geqslant 1$ and $d \geqslant 0$, let $S_{m+d}^{(d)}(Z_N)$ be the spline space of piecewise polynomial functions on the grid (3) and (4)

$$S_{m+d}^{(d)}(Z_N) = \{u(t) : u(t)|_{t\in\sigma_n} = u_n(t) \in \pi_{m+d} \text{ on } \sigma_n (n = 0, \ldots, N-1)\}$$

with

$$u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \quad \text{for } j = 0, 1, \ldots, d; \quad \text{and} \quad t_n \in Z_N - \{0, 1\},$$

where π_{m+d} denotes the set of all real polynomials of degree not exceeding $m+d$. The dimension of $S_{m+d}^{(d)}(Z_N)$ is given by

$$\dim S_{m+d}^{(d)}(Z_N) = mN + d + 1.$$

3. Derivation of the collocation method

In every subinterval $\sigma_n = [t_n, t_{n+1}]$, $(n = 0, \ldots, N-1)$, we introduce m interpolation points (called collocation points) $t_{n,1} < \cdots < t_{n,m}$, with

$$t_{n,i} = t_n + c_i h, \qquad i = 1, \ldots, m; \quad n = 0, \ldots, N-1,$$

where $c_1, \ldots, c_m$ do not depend on n and N and satisfy

$$0 < c_1 < \cdots < c_m \leqslant 1.$$

Let

$$X(N) = \bigcup_{n=0}^{N-1} X_n \quad \text{with } X_n = \{t_{n,i} = t_n + c_i h : i = 1, \ldots, m\} \subset \sigma_n.$$

The exact solution y of (1) and (2) will be approximated on I by an element $u \in S_{m+d}^{(d)}(Z_N)$ (called the collocation solution) satisfying on the set $X(N)$

$$D^q u(t) = p(t)u(t) + f(t) + \int_0^t k(t,s)u(s)\,\mathrm{d}s, \quad t \in X(N), \tag{5}$$

$$u(0) = \alpha.$$

On each subinterval σ_n the spline u can be described by a polynomial of the form

$$u(t) = u_n(t_n + \tau h) = \sum_{r=0}^{d} a_r^{(n)} \tau^r + \sum_{r=1}^{m} b_r^{(n)} \tau^{d+r}, \tag{6}$$

where $t = t_n + \tau h \in \sigma_n$ and $\tau \in [0, 1]$. Observe that the first $d + 1$ coefficients of the numerical solution u are given either by the initial conditions or the conditions of smoothness and the last m coefficients are determined by the collocation conditions given by Eq. (5). Thus, it is convenient to introduce the vectors

$$a^{(n)} = \begin{pmatrix} a_0^{(n)} \\ \vdots \\ a_d^{(n)} \end{pmatrix} \quad \text{and} \quad b^{(n)} = \begin{pmatrix} b_1^{(n)} \\ \vdots \\ b_m^{(n)} \end{pmatrix}.$$

In order to set up an equation, that can used to find the weights $a^{(n)}$ and $b^{(n)}$ in Eq. (6), differentiate relation (6) directly on the subinterval $[t_n, t_{n+1}]$ for $n \geqslant 1$, for the smooth conditions of the approximation $u \in S_{m+d}^{(d)}(Z_N)$, we get a relation between the vector $a^{(n+1)}$ and vectors $a^{(n)}$ and $b^{(n)}$

$$a^{(n+1)} = M_1 a^{(n)} + M_2 b^{(n)}, \tag{7}$$

where M_1 is the $(d + 1) \times (d + 1)$ upper triangular matrix and M_2 is the $(d + 1) \times m$ matrix, whose elements are

$$(M_1)_{j,r} = \begin{cases} 0 & \text{if } r < j \\ \binom{r}{j} & \text{if } r \geqslant j \end{cases} \quad \text{with } j = 0, 1, \ldots, d \quad \text{and} \quad r = 0, 1, \ldots, d$$

and

$$(M_2)_{j,r} = \binom{d+r}{j} \quad \text{with } j = 0, 1, \ldots, d \quad \text{and} \quad r = 1, 2, \ldots, m.$$

However, on the first subinterval we have a relation between the vector $a^{(0)}$ and the initial conditions

$$a^{(0)} = \begin{pmatrix} \ddots & & 0 \\ & \frac{h^s}{s!} & \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ y^{(s)}(0) \\ \vdots \end{pmatrix}_{s=0,\ldots,d}. \tag{8}$$

Remark 3.1. The collocation Eqs. (5) and (7) represent recursive systems for each $n = 0, 1, \ldots, N-1$, which yield the coefficients $\{b_k^{(n)}\}_{k=1,\ldots,m}$ and $\{a_k^{(n)}\}_{k=0,\ldots,d}$. Once the coefficients are known, the value of u is determined on σ_n. On each of the N subintervals of I we have to solve an $m \times m$ system of linear equations given by (5). On the first subinterval σ_0, $d + 1$ additional equations are furnished by the $d + 1$-initial conditions (8).

Now, introduce the following notations, for $i = 1,\ldots,m$

$$w_{i,0}^{(0)} = c_i^{-q},$$

$$w_{i,r}^{(0)} = c_i^{(-q+r)} \prod_{p=1}^{r} \frac{p}{p-q}, \quad r \geqslant 1$$

and for $j \geqslant 1$

$$w_{i,0}^{(j)} = (j+c_i)^{-q} - (j+c_i-1)^{-q},$$

$$w_{i,r}^{(j)} = (j+c_i)^{(-q+r)} \prod_{p=1}^{r} \frac{p}{p-q} - \sum_{v=0}^{r} (j+c_i-1)^{(-q+v)} \left\{ \prod_{p=1}^{v} \frac{r-v+p}{p-q} \right\}, \quad r \geqslant 1$$

and define the matrix

$$W^{(j)} = (w_{i,r}^{(j)}) \quad \text{with } i = 1,\ldots,m \quad \text{and} \quad r = 0,\ldots,m+d.$$

In [2], Blank proved the following result:

Theorem 3.2

$$D^q(u(t_n + c_i h)) = \frac{h^{-q}}{\Gamma(1-q)} \left[\sum_{j=0}^{n} \sum_{r=0}^{d} w_{i,r}^{(n-j)} a_r^{(j)} + \sum_{j=0}^{n} \sum_{r=1}^{m} w_{i,r+d}^{(n-j)} b_r^{(j)} \right].$$

Now, define

$$(\Psi_{n,j}^{(i)})[u_i] = \begin{cases} \sum\limits_{r=0}^{d} a_r^{(n)} \int_0^{c_i} K(t_{n,i}, t_n + vh) v^r \, \mathrm{d}v \\ \quad + \sum\limits_{r=1}^{m} b_r^{(n)} \int_0^{c_i} K(t_{n,i}, t_n + vh) v^{d+r} \, \mathrm{d}v, & \text{if } j = n, \\ \sum\limits_{r=0}^{d} a_r^{(j)} \int_0^{1} K(t_{n,i}, t_j + vh) v^r \, \mathrm{d}v \\ \quad + \sum\limits_{r=1}^{m} b_r^{(j)} \int_0^{1} K(t_{n,i}, t_j + vh) v^{d+r} \, \mathrm{d}v, & \text{if } j = 0, 1, \ldots n-1. \end{cases}$$

Then, we can get the following theorem:

Theorem 3.3

$$\int_0^{t_{n,i}} K(t_{n,i}, s) u(s) \, \mathrm{d}s = h \sum_{j=0}^{n-1} (\Psi_{n,j}^{(i)})[u_j] + h(\Psi_{n,n}^{(i)})[u_n].$$

Proof. Let $s = t_j + vh$, then

$$\begin{aligned} \int_0^{t_{n,i}} K(t_{n,i}, s) u(s) \, \mathrm{d}s &= \sum\nolimits_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} K(t_{n,i}, s) u_j(s) \, \mathrm{d}s + \int_{t_n}^{t_{n,i}} K(t_{n,i}, s) u_n(s) \, \mathrm{d}s \\ &= h \sum_{j=0}^{n-1} \int_0^1 K(t_{n,i}, t_j + vh) u_j(t_j + vh) \, \mathrm{d}v + h \int_0^{c_i} K(t_{n,i}, t_n + vh) u_n(t_n + vh) \, \mathrm{d}v \\ &= h \sum_{j=0}^{n-1} \sum_{r=0}^{d} a_r^{(j)} \int_0^1 K(t_{n,i}, t_j + vh) v^r \, \mathrm{d}v + h \sum_{j=0}^{n-1} \sum_{r=1}^{m} b_r^{(j)} \int_0^1 K(t_{n,i}, t_j + vh) v^{d+r} \, \mathrm{d}v, \\ &\quad + h \sum_{r=0}^{d} a_r^{(n)} \int_0^{c_i} K(t_{n,i}, t_n + vh) v^r \, \mathrm{d}v + h \sum_{r=1}^{m} b_r^{(n)} \int_0^{c_i} K(t_{n,i}, t_n + vh) vd + r \, \mathrm{d}v \\ &= h \sum_{j=0}^{n-1} (\Psi_{n,j}^{(i)})[u_j] + h(\Psi_{n,n}^{(i)})[u_n]. \end{aligned}$$

The proof is complete. □

From Theorems 3.2 and 3.3 Eq. (5) becomes

$$\frac{h^{-q}}{\Gamma(1-q)}\left[\sum_{j=0}^{n}\sum_{r=0}^{d} w_{i,r}^{(n-j)} a_r^{(j)} + \sum_{j=0}^{n}\sum_{r=1}^{m} w_{i,r+d}^{(n-j)} b_r^{(j)}\right]$$
$$= f(t_{n,i}) + p(t_{n,i})\sum_{r=0}^{d} a_r^{(n)} c_i^r + p(t_{n,i})\sum_{r=1}^{m} b_r^{(n)} c_i^{d+r} + h\sum_{j=0}^{n-1}(\Psi_{n,j}^{(i)})[u_j] + h(\Psi_{n,n}^{(i)})[u_n].$$

Now multiply the above equation by $h^q\Gamma(1-q)$ to get the following collocation equation:

$$Vb^{(n)} = Ua^{(n)} + F,$$

where V is the $m\times m$ matrix, U is the $m\times(d+1)$ matrix, and F is the $m\times 1$ vector, whose elements are

$$(V)_{i,r} = w_{i,r+d}^{(0)} - h^q\Gamma(1-q)c_i^{d+r}p(t_{n,i}) - h^{1+q}\Gamma(1-q)\int_0^{c_i} K(t_{n,i}, t_n + vh)v^{r+d}\,\mathrm{d}v,$$
$$(U)_{i,r} = -w_{i,r}^{(0)} + h^q\Gamma(1-q)c_i^r p(t_{n,i}) + h^{1+q}\Gamma(1-q)\int_0^{c_i} K(t_{n,i}, t_n + vh)v^{r}\,\mathrm{d}v,$$

and

$$(F)_{i,1} = \begin{cases} h^q\Gamma(1-q)f(t_{n,i}) & \text{if } n=0,\\ h^q\Gamma(1-q)f(t_{n,i}) - \sum_{j=0}^{n-1}\sum_{r=0}^{d} w_{i,r}^{(n-j)}a_r^{(j)} & \\ \quad -\sum_{j=0}^{n-1}\sum_{r=1}^{m} w_{i,r+d}^{(n-j)}b_r^{(j)} & \\ \quad +h^{1+q}\Gamma(1-q)\sum_{j=0}^{n-1}\sum_{r=0}^{d} a_r^{(j)}\int_0^1 K(t_{n,i}, t_i+vh)v^r\,\mathrm{d}v & \\ \quad +h^{1+q}\Gamma(1-q)\sum_{j=0}^{n-1}\sum_{r=1}^{m} b_r^{(j)}\int_0^1 K(t_{n,i}, t_i+vh)v^{r+d}\,\mathrm{d}v & \text{if } n>0. \end{cases}$$

Remark 3.4. For h small enough, the matrix V is invertible since for $h=0$, the determinant of V is a vandermonde type determinant.

Theorem 3.5. *The weights $a^{(n)}$ and $b^{(n)}$ in the spline function $u\in S_{m+d}^{(d)}(Z_N)$ that approximate the exact solution of (1) and (2) derived above, are determined by the following systems: on $[0, t_1)$, $a^{(0)}$ is given by (8) and $b^{(0)} = V^{-1}(Ua^{(0)} + F)$, while on $[t_n, t_{n+1})$ for $n \geqslant 1$, $a^{(n)}$ is given by (7) and $b^{(n)} = V^{-1}(Ua^{(n)} + F)$.*

4. Numerical illustration

Let $u\in S_3^{(0)}(Z_N)$ $(d=0,\ m=3)$ with collocation parameters $c_1=\frac{1}{4}$, $c_2=\frac{2}{4}$, and $c_3=\frac{3}{4}$ in the interval [0, 1]. We computed the absolute error at $t=0.1, 0.5$, and $t=1$. The following notations will be used in the presentation:

$$e_1 = |y(0.1)-u(0.1)|,\quad e_{N/2} = |y(0.5)-u(0.5)|,\quad e_N = |y(1)-u(1)|.$$

(1) Consider the following fractional integro-differential equation:

$$y^{(0.75)}(t) = \left(\frac{-t^2e^t}{5}\right)y(t) + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t s y(s)\,\mathrm{d}s \tag{9}$$

with the initial condition

$$y(0)=0 \tag{10}$$

and the exact solution of (9) and (10) is $y(t)=t^3$.

Table 1
Numerical solution of $y(t)$ in Example 1

	e_1	$e_{N/2}$	e_N
$h = 0.1$	0.1688×10^{-7}	0.21102×10^{-5}	0.168819×10^{-4}
$h = 0.05$	0.1688×10^{-7}	0.211×10^{-5}	0.168816×10^{-4}

Table 2
Numerical solution of $y(t)$ in Example 2

	e_1	$e_{N/2}$	e_N
$h = 0.1$	0.3×10^{-9}	0.1×10^{-9}	0.5×10^{-8}
$h = 0.05$	0.29×10^{-9}	0.1×10^{-9}	0.3×10^{-8}

(2) Consider the following fractional integro-differential equation:

$$y^{(0.5)}(t) = (\cos t - \sin t)y(t) + f(t) + \int_0^t t \sin s y(s)\, \mathrm{d}s, \tag{11}$$

with the initial condition

$$y(0) = 0 \tag{12}$$

and choose $f(t)$ so that the exact solution of (11) and (12) is $y(t) = t^2 + t$.

See Table 1 for Example 1 and Table 2 for Example 2.

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