Available online at www.sciencedirect.com

ScienceDirect

Journal of Ocean Engineering and Science 3 (2018) 127–132

www.elsevier.com/locate/joes

Numerical studies for solving fractional integro-differential equations

A.M.S. Mahdy[a,b,*]

[a] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
[b] Department of Mathematics, Faculty of Science, Taif University, Saudi Arabia

Received 8 May 2018; accepted 14 May 2018
Available online 19 May 2018

Abstract

In this paper, we give a new numerical method for solving a linear system of fractional integro-differential equations. The fractional derivative is considered in the Caputo sense. The proposed method is least squares method aid of Hermite polynomials. The suggested method reduces this type of systems to the solution of systems of linear algebraic equations. To demonstrate the accuracy and applicability of the presented method some test examples are provided. Numerical results show that this approach is easy to implement and accurate when applied to integro-differential equations. We show that the solutions approach to classical solutions as the order of the fractional derivatives approach.

Keywords: Least squares method; Caputo fractional; Hermite polynomials; Linear fractional integro-differential equations.

1. Introduction

In this paper, least squares method with the aid of the Hermite collocation method is applied to solving system of fractional integrals-differential equations [1–6, 19–21]. Least squares the method has been studied in [4].

In this paper, we present a numerical solution of the system of integral-differential equation with fractional derivative of the type [2]:

$$D^{\alpha}u_n(y) = \phi_n(y) + \int_0^1 k_n(y,\ r)\left(\sum_{k=1}^{i}\alpha_{nk}u_k(r)\right)dr.$$
$$n = 1,\ 2,\ ...,\ i,\quad 0 \le y,\ r \le 1, \tag{1}$$

With initial conditions

$u_n^{(j)}(y_0) = u_{nj} n = 1,\ 2,\ ...,\ i,$

Where $D^{\alpha}u_n(y)$ indicates the αth Caputo fractional derivative of $u_n(y).\phi_n(y)$ and $k_n(y,r)$ are given functions, y, r are real varying in the interval $[0,1]$ and $u_n(y)$ is the unknown functions to be determined.

* Correspondence to: Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.
E-mail address: amr_mahdy85@yahoo.com.

2. Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries on the fractional calculus theory required for our subsequent development.

Definition 2.1. The Caputo fractional derivative operator D^{ν} of order ν is defined in the following form [7–9]:

$$D^{\nu}\phi(y) = \frac{1}{\Gamma(m-\nu)}\int_0^y \frac{\phi^{(m)}(r)}{(y-r)^{\nu-m+1}}dr,\quad y > 0,$$

Where, $m-1 < \nu \le m,\ m \in \mathrm{N}$.

Similar to integer-order differentiation, the Caputo fractional derivative operator is a linear operation:

$D^{\nu}(\lambda\phi(y) + \mu g(y)) = \lambda D^{\nu}\phi(y) + \mu D^{\nu}g(y),$

Where,λ and $\mu$are constants. For the Caputo's derivative we have [10,19]

$$D^{\nu}C = 0,\ C is a constant, \tag{2}$$

$$D^{\nu}y^i = \begin{cases} 0, & for\ i \in \mathrm{N}_0\ and\ i < \lceil\nu\rceil; \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\nu)}y^{i-\nu} & for\ i \in \mathrm{N}_0\ and\ i \ge \lceil\nu\rceil. \end{cases} \tag{3}$$

We use the ceiling function $\lceil\nu\rceil$ to denote the smallest integer greater than or equal to ν, and $\mathrm{N}_0 = \{0,\ 1,\ 2, ...\}$. Recall

https://doi.org/10.1016/j.joes.2018.05.004

that for $\nu \in \mathrm{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.[10,19]

3. Some properties of the Hermite polynomials

Definition 3.1. The Hermite polynomials are given by [11–18]:-

$$H_i(y) = (-1)^i e^{x^2} \frac{d^i}{dy^i} e^{-y^2}.$$

Some main properties of these polynomials are:

The Hermite polynomials evaluated at zero argument $H_i(0)$ and have called Hermite number as follows [12,13]:-

$$H_i(0) = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ (-1)^{\frac{i}{2}} 2^{\frac{i}{2}} (n-1)!, & \text{if } i \text{ is even,} \end{cases} \tag{4}$$

Where $(i-1)!$ is the factorial.

The polynomials $H_i(y)$ are orthogonal with respect to the weight function $\omega(y) = e^{-y^2}$ with the following condition [12]:-

$$\int_{-\infty}^{\infty} H_i(y)\, H_m(y)\, \omega(y)\, dy = \sqrt{\pi} 2^i i! \delta_{im}.$$

4. An approximate formula of the integral derivative

The Hermite polynomials are defined on R and can be determined with the aid of the following recurrence formula [12,13]

$$H_{i+1}(y) = 2yH_i(y) - 2iH_{i-1}(y),\ H_0(y) = 1,$$
$$H_1(y) = 2y,\ i = 1,\ 2, \ldots.$$

The analytic form of the Hermite polynomials of degree i is given by [19]

$$H_i(y) = i! \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^k 2^{i-2k}}{(k)!(i-2k)!} y^{i-2k}. \tag{5}$$

In consequence, for the pth derivatives of Hermite polynomials the following relation hold:

$$H_i^{(p)}(y) = 2^p \frac{i!}{(i-p)!} H_{i-p}(y) = v_{i,\,p} H_{i-p}(y),$$
$$v_{i,\,p} = 2^p \frac{i!}{(i-p)!}. \tag{6}$$

The function $u(y) \in L^2_{\omega(y)}(\mathrm{R})$, may be expressed in terms of Hermite polynomials as follows:

$$u(y) = \sum_{k=0}^{\infty} c_k H_k(y) \tag{7}$$

Where the coefficients c_i are given by

$$c_i = \frac{1}{\sqrt{\pi} 2^i i!} \int_{-\infty}^{\infty} u(y)\, H_i(y)\, \omega(y)\, dy,\ i = 0,\ 1, \ldots. \tag{8}$$

In practice, only the first $(m+1)-$ terms of Hermite polynomials are considered. Then we have [13]

$$u_m(y) = \sum_{i}^{m} c_i H_i(y). \tag{9}$$

Theorem 4.1. [13] *Let $u(y)$ be approximated by Hermite polynomials as (9) and also suppose $\alpha > 0$, then*

$$D^{\alpha}(u_m(y)) \cong \sum_{i=\lceil \alpha \rceil}^{m} \left[i! \sum_{k=\lceil \alpha \rceil}^{\ell} c_i b_{i,\,k}^{(\alpha)} y^{i-2k-\alpha} \right], \tag{10}$$

Where $\ell = \frac{i - \lceil \alpha \rceil}{2}$ and $b_{i,\,k}^{(\alpha)}$ is given by

$$b_{i,\,k}^{(\alpha)} = \frac{(-1)^k 2^{i-2k} \Gamma(i-2k+1)}{(k)!(i-2k)! \Gamma(i-2k-\alpha+1)}.$$

5. Solution of system of linear fractional integro-differential equation

In this section, the least squares method with the aid of Hermite polynomial is applied to study the numerical solution of these systems of fractional integro-differential (1).

The method is based on approximating the unknown functions $u_n(y)$ as

$$u_n(y) = \sum_{j=0}^{m} \alpha_j^n H_j(y), \qquad 0 \le y \le 1, \tag{11}$$

Where $H_j(y)$ is Hermite polynomial, and α_j^n, $n = 1,\ 2, \ldots,\ i$, are constants.

Substituting (11) into (1), we obtain in [4]

$$D^{\alpha} \sum_{j=0}^{m} \alpha_j^n H_j(y) = \phi_n(y)$$
$$+ \int_0^1 k_n(y,\ r) \left(\sum_{k=1}^{i} \alpha_{nk} \left[\sum_{j=0}^{m} \alpha_j^n H_j(r) \right] \right) dr. \tag{12}$$

Hence the residual equation is defined as

$$R_n\big(y,\ \alpha_0^n,\ \alpha_1^n, \ldots,\ \alpha_m^n\big) = D^{\alpha} \sum_{j=0}^{m} \alpha_j^n H_j(y) - \phi_n(y)$$
$$- \int_0^1 k_n(y,\ r) \left(\sum_{k=1}^{i} \alpha_{nk} \left[\sum_{j=0}^{m} \alpha_j^n H_j(r) \right] \right) dr. \tag{13}$$

Let

$$S_n\big(\alpha_0^n,\ \alpha_1^n, \ldots,\ \alpha_m^n\big) = \int_0^1 \big[R_n\big(y,\ \alpha_0^n,\ \alpha_1^n, \ldots,\ \alpha_m^n\big)\big]^2 w(y)\, dy, \tag{14}$$

where $w(y)$ is the positive weight function defined on the interval $[0,1]$, in this work we take, then $w(y) = 1$, then

$$S_n\left(\alpha_0^n, \alpha_1^n, \ldots, \alpha_m^n\right) = \int_0^1 \left\{ \begin{array}{c} \sum_{j=0}^{m} \alpha_j^n D^\alpha H_j(y) \\ - \int_0^1 k_n(y, r)\left(\sum_{k=1}^{i} \alpha_{nk}\left[\sum_{j=0}^{m} \alpha_j^n H_j(y)\right]\right) dr - \phi_n(y) \end{array} \right\}^2 dy. \tag{15}$$

So finding the values of α_j^n, $j = 0, 1, \ldots, m$ which minimize S_n is equivalent to finding the best approximation for the solution of the SLFIDE (1).

The minimum value is S_n is obtained by setting [4]

$$\frac{\partial S_n}{\partial \alpha_j^n} = 0, \qquad j = 0, 1, \ldots, m, \tag{16}$$

$$\int_0^1 \left\{ \sum_{j=0}^{m} \alpha_j^n D^\alpha H_j(y) - \int_0^1 k_n(y, r)\left[\sum_{k=1}^{i} \alpha_{nk} \sum_{j=0}^{m} \alpha_j^n H_j(r)\right] dr - \phi_n(y) \right\} \times \left\{ D^\alpha H_j(y) - \int_0^1 k_n(y, r)\left[\sum_{k=1}^{i} \alpha_{nk} \sum_{j=0}^{m} \alpha_n H_j(r)\right] dr \right\} dy = 0. \tag{17}$$

By evaluating the above equation for $j = 0, 1, \ldots, i$ we can obtain a system of $(i+1)$ linear equations with $(i+1)$ unknown coefficients α_j^n. This system can be formed by using matrices form as follows:

$$A = \begin{pmatrix} \int_0^1 R_n\left(y, \alpha_0^n\right) h_0^n dy & \int_0^1 R_n\left(y, \alpha_1^n\right) h_0^n dy \ldots & \int_0^1 R_i\left(y, \alpha_i^n\right) h_0^n dy \\ \int_0^1 R_n\left(y, \alpha_0^n\right) h_1^n dy & \int_0^1 R_n\left(y, \alpha_1^n\right) h_1^n dy \ldots & \int_0^1 R_n\left(x, \alpha_i^n\right) h_1^n dy \\ \vdots & \vdots & \vdots \\ \int_0^1 R_n\left(y, \alpha_0^n\right) h_i^n dy & \int_0^1 R_n\left(x, \alpha_1^n\right) h_i^n dy \ldots & \int_0^1 R_n\left(y, \alpha_i^n\right) h_i^n dy \end{pmatrix}, \tag{18}$$

$$B = \begin{pmatrix} \int_0^1 \phi_n(y) h_0^n dy \\ \int_0^1 \phi_n(y) h_1^n dy \\ \vdots \\ \int_0^1 \phi_n(y) h_i^n dy \end{pmatrix}, \tag{19}$$

Where

$$R_n\left(y, \alpha_j^n\right) = \sum_{j=0}^{m} \alpha_j^n D^\alpha H_j(y) - \int_0^1 k_n(y, r)\left[\sum_{k=1}^{i} \alpha_{nk} \sum_{j=0}^{m} \alpha_j^n H_j(r)\right] dr, \tag{20}$$

$$h_j^n = D^\alpha H_j(y) - \int_0^1 k_n(y, r)\left[\sum_{k=1}^{i} \alpha_{nk} \sum_{j=0}^{m} H_j(r)\right] dr\, j = 0, 1, \ldots, m,\ n = 1, 2, \ldots, i. \tag{21}$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solutions of (1).

6. Numerical examples

In this section, we have applied Hermite polynomials for solving system of linear fractional integro-differential equations with known exact solution. All results are obtained by using Maple 15 prgramming.

Example 6.1. Consider the following system of fractional integro-differential equations [2]

$$D^{\frac{3}{4}} u_1(y) = -\tfrac{1}{20} - \tfrac{y}{12} + \frac{4y^{\frac{1}{4}}(15-23y^2)}{15\Gamma\left(\frac{1}{4}\right)} + \int_0^1 (y+r)[u_1(r) + u_2(r)]\, dr,$$

$$D^{\frac{3}{4}} u_2(y) = \tfrac{5y^3}{6} + \frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 \sqrt{yr^2}[u_1(r) - u_2(r)]\, dr. \tag{22}$$

Subject to initial conditions $u_1(0) = 0$, $u_2(0) = 0$ with exact solution $u_1(y) = y - y^3$, $u_2(y) = y^2 - y$.

First By assuming the approximate of the solution of $u(y)$ with $m=3$ as:

$$u_1(y) = \sum_{i=0}^{3} c_i H_i(y), \quad u_1(r) = \sum_{i=0}^{3} c_i H_i(r)$$
$$u_2(y) = \sum_{i=0}^{3} a_i H_i(y), \quad u_2(r) = \sum_{i=0}^{3} a_i H_i(r) \tag{23}$$

Where $H_i(y)$ is the Hermite polynomials and a_i, c_i are constant

Second by Substituting (23) into (22) we obtain

$$D^{\frac{3}{4}} \sum_{i=0}^{3} c_i H_i(y) = -\tfrac{1}{20} - \tfrac{y}{12} + \frac{4y^{\frac{1}{4}}(15-23y^2)}{15\Gamma\left(\frac{1}{4}\right)} + \int_0^1 (y+r)\left[\sum_{i=0}^{3} c_i H_i(r) + \sum_{i=0}^{3} a_i H_i(r)\right] dr,$$

$$D^{\frac{3}{4}} \sum_{i=0}^{3} a_i H_i(y) = \tfrac{5y^3}{6} + \frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 \sqrt{yr^2}\left[\sum_{i=0}^{3} c_i H_i(r) - \sum_{i=0}^{3} a_i H_i(r)\right] dr \tag{24}$$

Hence the residual equation is defined as:

$$R(y,\ c_0,\ c_1,\cdots,\ c_i)=D^{\frac{3}{4}}\sum_{i=0}^{3}c_iH_i(y)+\tfrac{1}{20}+\tfrac{y}{12}-\frac{4y^{\frac{1}{4}}(15-23y^2)}{15\Gamma\left(\frac{1}{4}\right)}-\int_0^1(y+r)\left[\sum_{i=0}^{3}c_iH_i(r)+\sum_{i=0}^{3}a_iH_i(r)\right]dr,$$

$$R(y,\ a_0,\ a_1,\cdots,\ a_i)=D^{\frac{3}{4}}\sum_{i=0}^{3}a_iH_i(y)-\tfrac{5y^3}{6}-\frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)}-\int_0^1\sqrt{yr^2}\left[\sum_{i=0}^{3}c_iH_i(r)-\sum_{i=0}^{3}a_iH_i(r)\right]dr \quad (25)$$

By substituting $H_i(y)$, $H_i(r)$ and Eq. (3) in Eq. (25) and second let

$$S(y,\ c_0,\ c_1,\cdots,\ c_i)=\int_0^1\left[R(y,\ c_0,\ c_1,\cdots,\ c_i)\right]^2\omega(y)\,dy$$
$$S(y,\ a_0,\ a_1,\cdots,\ a_i)=\int_0^1\left[R(y,\ a_0,\ a_1,\cdots,\ a_i)\right]^2\omega(y)\,dy \quad (26)$$

where $\omega(y)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(y)=1$ for simplicity. Thus

$$S(y,\ c_0,\ c_1,\cdots,\ c_i)=\int_0^1\left\{\begin{array}{c}D^{\frac{3}{4}}\sum_{i=0}^{3}c_iH_i(y)+\frac{1}{20}+\frac{y}{12}-\frac{4y^{\frac{1}{4}}(15-23y^2)}{15\Gamma\left(\frac{1}{4}\right)}\\-\int_0^1(y+r)\left[\sum_{i=0}^{3}c_iH_i(r)+\sum_{i=0}^{3}a_iH_i(r)\right]dr\end{array}\right\}^2dy,$$

$$S(y,\ a_0,\ a_1,\cdots,\ a_i)=\int_0^1\left\{\begin{array}{c}D^{\frac{3}{4}}\sum_{i=0}^{3}a_iH_i(y)-\frac{5y^3}{6}-\frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)}\\-\int_0^1\sqrt{yr^2}\left[\sum_{i=0}^{3}c_iH_i(r)-\sum_{i=0}^{3}a_iH_i(r)\right]dr\end{array}\right\}^2dy \quad (27)$$

The minimum value of S is obtained by setting

$$\frac{\partial S}{\partial a_i}=0,\ \frac{\partial S}{\partial c_i}=0,\ i=0,\ 1,\ 2 \quad (28)$$

From the initial condition $u_1(0)=0$, $u_2(0)=0$ and from Eq. (4) we get

$$c_0-2c_2=0,\ a_0-2a_2=0 \quad (29)$$

By solving the Equations produced from (28) with (29) we get the constants $c_0,c_1,c_2,c_3,a_0,a_1,a_2,a_3$ as:

$$a_0=13.4673,\ a_1=-15.9927,\ a_2=6.7336,\ a_3=-2.0099$$
$$c_0=4.8004,\ c_1=-0.5108,\ c_2=2.4002,\ c_3=-0.7457$$

By substitute about constants in (23) we get solutions as series:

$$u_1(y)=7.9268y+9.6008y^2-5.9656y^3+\cdots$$
$$u_2(y)=-7.8666y+26.9344y^2-16.0792y^3+\cdots$$

Applying the least squares method with the aid of Hermite polynomials $H_j(y)$, $j=0,\ 1,\ ...,\ i$, at $i=3$ for a system of the linear fractional integro-differential Eq. (22). The numerical results are showing in fig. 1, we obtain a system of linear equations with unknown coefficients. The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (11). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results, we can see that these solutions are in more accuracy of those obtained in [2]

Example 6.2. Consider the following system of fractional integro-differential equations [2]

$$D^{\frac{4}{5}}u_1(y)=\frac{83y}{80}+\frac{25y^{\frac{6}{5}}(11+15y)}{33\Gamma\left(\frac{1}{5}\right)}+\int_0^1 2yr[u_1(r)+u_2(r)]\,dr,$$
$$D^{\frac{4}{5}}u_2(y)=\frac{5y^3}{6}+\frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)}+\int_0^1(y+r)[u_1(r)-u_2(r)]\,dr. \quad (30)$$

Subject to initial conditions $u_1(0)=0$, $u_2(0)=0$ with exact solution $u_1(y)=y^3-y^2$, $u_2(y)=\frac{15}{8}y^2$.

First

By assuming the approximate of the solution of $u(y)$ with m = 3 as:

$$u_1(y)=\sum_{i=0}^{3}c_iH_i(y),\quad u_1(r)=\sum_{i=0}^{3}c_iH_i(r)$$
$$u_2(y)=\sum_{i=0}^{3}a_iH_i(y),\quad u_2(r)=\sum_{i=0}^{3}c_iH_i(r) \quad (31)$$

Where $H_i(y)$ are the Hermite polynomials and a_i,c_i are constant Second by Substituting (31) into (30) we obtain

$$D^{\frac{4}{5}}\sum_{i=0}^{3}c_iH_i(y)=\frac{83y}{80}+\frac{25y^{\frac{6}{5}}(11+15y)}{33\Gamma\left(\frac{1}{5}\right)}+\int_0^1 2yr\left[\sum_{i=0}^{3}c_iH_i(r)+\sum_{i=0}^{3}c_iH_i(r)\right]dr,$$
$$D^{\frac{4}{5}}\sum_{i=0}^{3}a_iH_i(y)=\frac{5y^3}{6}+\frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)}+\int_0^1(y+r)\left[\sum_{i=0}^{3}c_iH_i(r)-\sum_{i=0}^{3}c_iH_i(r)\right]dr \quad (32)$$

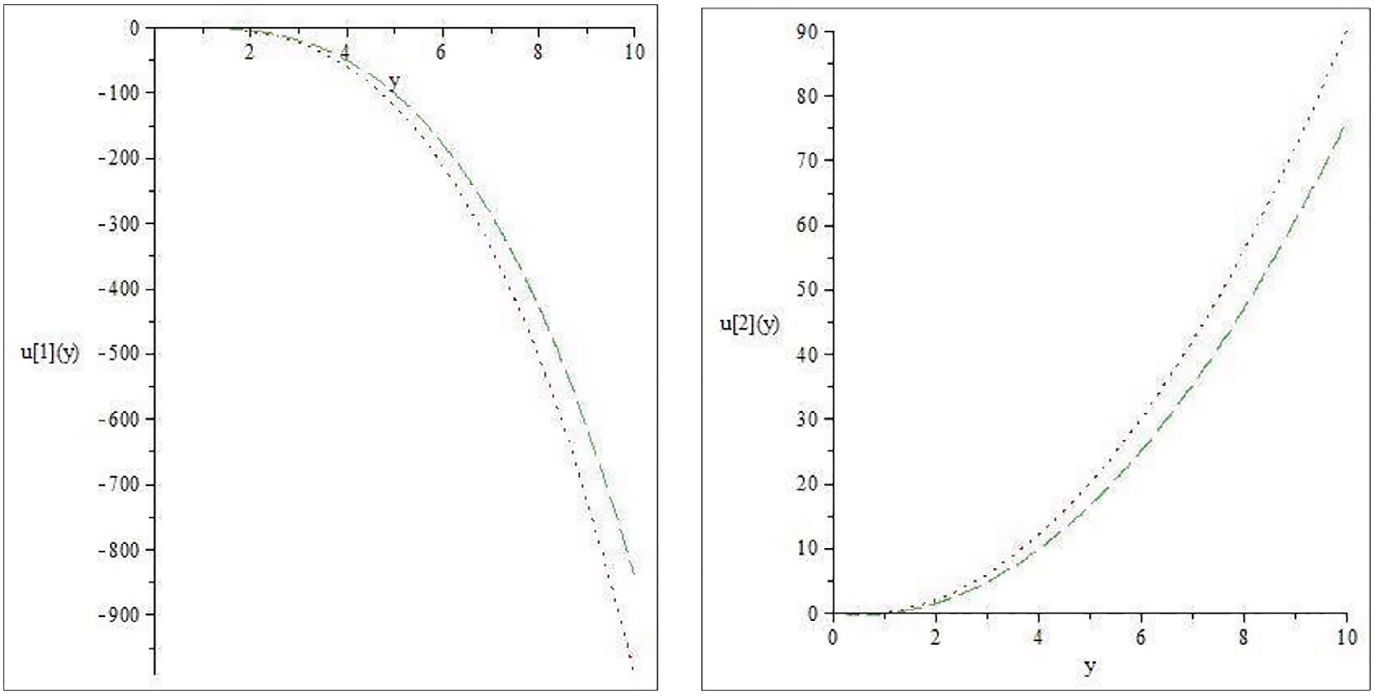

Fig. 1. Comparison between the approximate solution and the exact solution.

Hence the residual equation is defined as:

$$R(y,\ c_0,\ c_1, \cdots,\ c_i) = D^{\frac{4}{5}} \sum_{i=0}^{3} c_i H_i(y) - \frac{83y}{80} - \frac{25y^{\frac{6}{5}}(11+15y)}{33\Gamma\left(\frac{1}{5}\right)} - \int_0^1 2yr\left[\sum_{i=0}^{3} c_i H_i(r) + \sum_{i=0}^{3} c_i H_i(r)\right] dr,$$

$$R(y,\ a_0,\ a_1, \cdots,\ a_i) = D^{\frac{4}{5}} \sum_{i=0}^{3} a_i H_i(y) - \frac{5y^3}{6} - \frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} - \int_0^1 (y+r)\left[\sum_{i=0}^{3} c_i H_i(r) - \sum_{i=0}^{3} c_i H_i(r)\right] dr \tag{33}$$

By substituting $H_i(y)$, $H_i(r)$ and Eq. (3) in Eq. (33) and second let

$$S(y,\ c_0,\ c_1, \cdots,\ c_i) = \int_0^1 \left[R(y,\ c_0,\ c_1, \cdots,\ c_i)\right]^2 \ \omega(y)\, dy$$

$$S(y,\ a_0,\ a_1, \cdots,\ a_i) = \int_0^1 \left[R(y,\ a_0,\ a_1, \cdots,\ a_i)\right]^2 \ \omega(y)\, dy \tag{34}$$

Where $\omega(y)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(y) = 1$ for simplicity. Thus

$$S(y,\ c_0,\ c_1, \cdots,\ c_i) = \int_0^1 \left\{ \begin{array}{c} D^{\frac{4}{5}} \sum_{i=0}^{3} c_i H_i(y) - \frac{83y}{80} - \frac{25y^{\frac{6}{5}}(11+15y)}{33\Gamma\left(\frac{1}{5}\right)} \\ - \int_0^1 2yr\left[\sum_{i=0}^{3} c_i H_i(r) + \sum_{i=0}^{3} c_i H_i(r)\right] dr, \end{array} \right\}^2 dy,$$

$$S(y,\ a_0,\ a_1, \cdots,\ a_i) = \int_0^1 \left\{ \begin{array}{c} D^{\frac{4}{5}} \sum_{i=0}^{3} a_i H_i(y) - \frac{5y^3}{6} - \frac{9y^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} \\ - \int_0^1 (y+r)\left[\sum_{i=0}^{3} c_i H_i(r) - \sum_{i=0}^{3} c_i H_i(r)\right] dr \end{array} \right\}^2 dy \tag{35}$$

The minimum value is S is obtained by setting

$$\frac{\partial S}{\partial a_i} = 0,\ \frac{\partial S}{\partial c_i} = 0,\ i = 0,\ 1,\ 2 \tag{36}$$

From the initial condition $u_1(0) = 0$, $u_2(0) = 0$ and from Eq. (4) we get

$$c_0 - 2c_2 = 0,\ a_0 - 2a_2 = 0 \tag{37}$$

By solving the Equations produced from (36) with (37) we get the constants $c_0, c_1, c_2, c_3, a_0, a_1, a_2, a_3$ as:

$a_0 = 42.5603,\ a_1 = -55.4141,\ a_2 = 21.2801,\ a_3 = -6.7384$
$c_0 = 1.9048,\ c_1 = 0.5290,\ c_2 = 0.9524,\ c_3 = 0.0743$

By substituting about constants in (31) we get solutions as series:

$u_1(y) = 0.1664y + 3.8096y^2 + 0.5944y^3 + \cdots$
$u_2(y) = -29.9674y + 85.1204y^2 - 53.9072y^3 + \cdots$

Applying the least squares method with the aid of Hermite polynomials $H_j(y)$, $j = 0, 1, ..., i$, at $i = 3$ for a system of the linear fractional integro-differential Eq. (30). The numerical results are showing in Fig. 2, we obtain a system of linear equations with unknown coefficients The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach

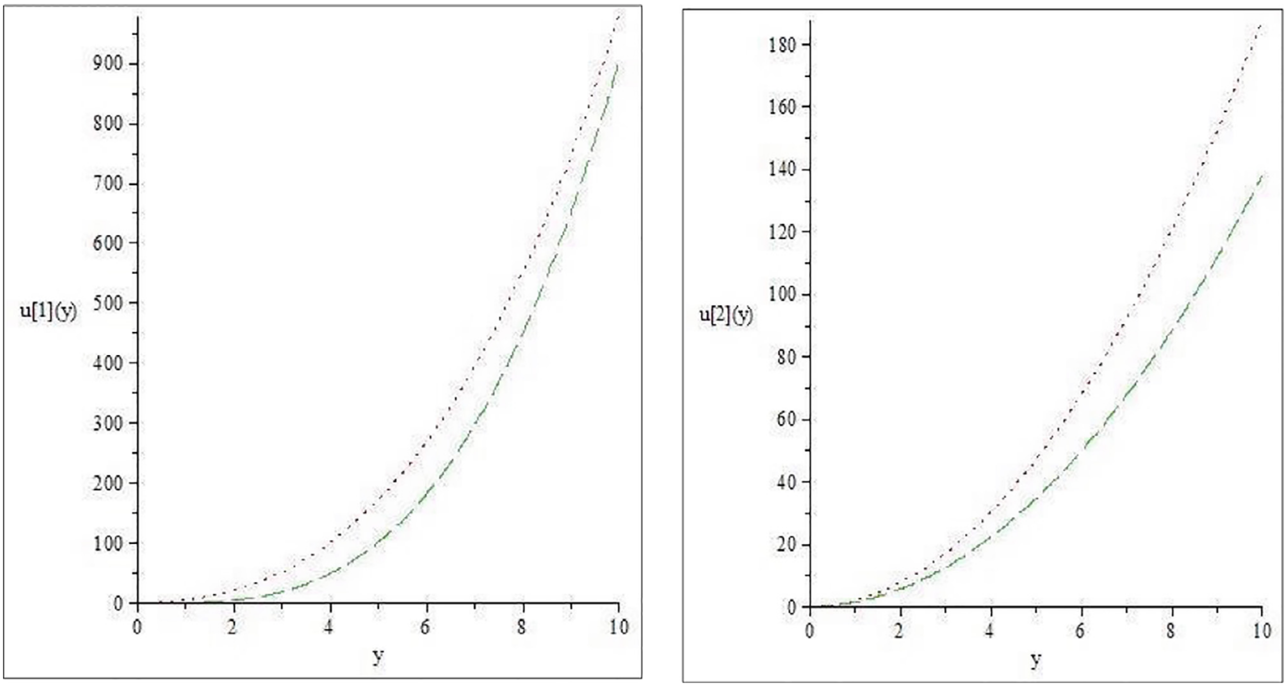

Fig. 2. Comparison between the approximate solution and the exact solution.

can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (11). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results, we can see that these solutions are in more accuracy of those obtained in [2].

7. Conclusion and remarks

In this article, we introduced an accurate numerical technique for solving a system of linear fractional integro-differential equations. We have introduced an approximate formula for the Caputo fractional derivative of the Hermite polynomials in terms of classical Hermite polynomials. The results show that the proposed algorithm converges as the number of terms is increased. Some numerical examples are presented to illustrate the theoretical results, and compared with the results obtained by other numerical methods. We have computed the numerical results using the Mathematica programming 10.

References

[1] S. Irandoust-Pakchin, S. Abdi-Mazraeh, Int. J. Adv. Math. Sci. 1 (3) (2013) 139–144.
[2] M.H. Saleh, D.S. Mohamed, M.H. Ahmed, M.K. Marjan, Int. J. Comput. Appl. 121 (24) (2015) 9–19.
[3] Y. Yang, Y. Chen, Y. Huang, J. Korean Math. Soc. 51 (1) (2014) 203–224.
[4] D.S. Mohammed, Math. Probl. Eng. 2014 (2014) 5 431965.
[5] A.M.S. Mahdy, R.T. Shwayyea, Int. J. Sci. Eng. Res. 7 (4) (2016) 1589–1596.
[6] A.M.S. Mahdy, E.M. Mohamed, J. Abstract Comput. Math. 1 (2016) 24–32.
[7] A.M.A. El-Sayed, S.M. Salman, J. Fract. Calc. Appl. 4 (2) (2013) 251–259.
[8] R.P. Agarwal, A.M. El-Sayed, S.M. Salman, Adv. Differ. Equ. 2013 (1) (2013) 320.
[9] A.A. Elsadany, A.E. Matouk, J. Appl. Math. Comput. 49 (1–2) (2015) 269–283.
[10] I. Podlubny, Fractional Differential Equations, 25, Academic Press, New York, 1999.
[11] D. Funaro, Polynomial Approximation of Differential Equations, Springer-Verlag, 1992.
[12] L.C. Andrews, Appl. Opt. 25 (1986) 3096.
[13] M.M. Khader, E.M. Solouma, J. Comput. Theor. Nanosci. 12 (11) (2015) 4579–4583.
[14] M. Bagherpoorfard, F.A. Ghassabzade, J. Appl. Math. Phys. 1 (05) (2013) 58.
[15] Z.B. Kalateh, S. Ahmadi, A. Aminataei, J. linear topol. algeb 2 (2) (2013) 91–103.
[16] S.H. Brill, Int. J. Diff. Equ. Appl 4 (2002) 141–155.
[17] B. Bialecki, SIAM J. Numer. Anal. 30 (2) (1993) 425–434.
[18] W.R. Dyksen, R.E. Lynch, Math. Comput. Simul. 54 (4–5) (2000) 359–372.
[19] Y.A. Amer, A.M.S. Mahdy, E.S.M. Youssef, CMC 54 (2) (2018) 161–180.
[20] A. Arikoglu, I. Ozkol, Chaos Solitons Fractals 40 (2) (2009) 521–529.
[21] R.K. Saeed, H.M. Sdeq, Aust. J. Basic Appl. Sci. 4 (4) (2010) 633–638.