Dynamic Modeling Assignment
Ball Regulator

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1. The set of generalized coordinates is (r, θ, ϕ). Give the geometric constraint(s) of the system (S). How many independent degrees of freedom does the system have? Give a brief description of them.

Answer:

Given that the shape is a rhombus, every side has the same length:

$$\overline{OA_1} = \overline{A_1C} = \overline{OA_2} = \overline{A_2C}$$

According to the cosine law, the length of spring with respect to XZ plane, r_{xz} can be written as:

$$r_{xz}^2 = a^2 + a^2 - 2a^2 \cos\theta$$

$$r_{xz} = \sqrt{2a^2(1 - \cos\theta)}$$

Since r_{xz} depends on θ, r_{xz} is the geometric constraint.

2. Now, the new set of coordinates is $q = (q_1, q_2) = (\theta, \phi)$. Give the direct geometric model that expresses the Cartesian coordinates in R_0 of the ball center B_2 as a function of the generalized coordinates.

Answer:

Given the problem, we aim to express the coordinates x, y, and z in terms of the angles θ and ϕ. θ is the angle between vector $\overline{OA_2}$ and $\overline{OC}$. ϕ is the angle of projection along the z-axis. The projection of $\overline{OB_2}$ defined as L along the z-axis is given as:

$$z = -L\cos\theta$$

The negative sign indicates the downward direction.

From the graphic, the radius r in the xy-plane is determined as:

$$r = L\sin\theta$$

From the xy-plane:

$$x = r\cos\phi = L\sin\theta\cos\phi,$$

$$y = r\sin\phi = L\sin\theta\sin\phi.$$

In the B_2 frame, the coordinates are:

$$x_{B_2} = L \sin\theta \cos\phi,$$

$$y_{B_2} = L \sin\theta \sin\phi,$$

$$z_{B_2} = -L \cos\theta.$$

3. Calculate the Jacobian matrix that gives the variations of the Cartesian coordinates of B_2 as a function of the variations of the generalized coordinates.

Answer:

To transform the coordinates (x, y, z) to the angles (θ, ϕ), the Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} L\cos\theta\cos\phi & -L\sin\theta\sin\phi \\ L\cos\theta\sin\phi & L\sin\theta\cos\phi \\ L\sin\theta & 0 \end{bmatrix}$$

The partial derivatives for the Jacobian are computed as follows:

$$\frac{\partial x}{\partial \theta} = L\cos\theta\cos\phi, \quad \frac{\partial x}{\partial \phi} = -L\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = L\cos\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = L\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial \theta} = L\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

The above equations provide the elements of the Jacobian matrix for the transformation.

4. Express the generalized forces/torques, namely $Q_\theta(\vec{W})$ and $Q_\phi(\vec{W})$ relative to the weight $\vec{W}$ of each of the spheres.

Answer:

The concept of virtual work is given by:

$$\delta W = Q_\theta \delta\theta + Q_\phi \delta\phi = \overrightarrow{w} \cdot \delta \overrightarrow{r}_{B_2}$$

where:

$$\delta \overrightarrow{r}_{B_2} = \begin{bmatrix} L\cos\theta\cos\phi\,\delta\theta + (-L\sin\theta\sin\phi)\,\delta\phi \\ L\cos\theta\sin\phi\,\delta\theta + L\sin\theta\cos\phi\,\delta\phi \\ L\sin\theta\,\delta\theta \end{bmatrix}$$

and:

$$\overrightarrow{w} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

The negative sign indicates the downward direction of the force.
Thus, the virtual work becomes:

$$\delta W = \overrightarrow{w} \cdot \delta \overrightarrow{r}_{B_2} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \cdot \begin{bmatrix} L\cos\theta\cos\phi\,\delta\theta + (-L\sin\theta\sin\phi)\,\delta\phi \\ L\cos\theta\sin\phi\,\delta\theta + L\sin\theta\cos\phi\,\delta\phi \\ L\sin\theta\,\delta\theta \end{bmatrix}$$

Simplifying this expression:

$$\delta W = (-mg \cdot L\sin\theta)\delta\theta$$

The generalized force and torque become:

$$Q_\theta = -mg \cdot L\sin\theta, \quad Q_\phi = 0$$

5. Express the generalized forces/torques, namely $Q_\theta(\vec{S})$ and $Q_\phi(\vec{S})$ relative to the spring force $\vec{S}$.

Answer:

The virtual work of the spring is given as:

$$\delta W = \vec{S} \cdot \delta r, \quad \text{where} \quad \vec{S} = -k\vec{r}_{\text{spring}}$$

Since OA_1A_2C is a rhombus, the spring displacement can be expressed as:

$$r_{\text{spring}} = 2a\cos\theta$$

Taking the derivative of r_{spring} with respect to θ:

$$\frac{\partial r}{\partial \theta} = -2a \sin\theta \qquad \Longrightarrow \qquad \delta r = -2a \sin\theta \, \delta\theta$$

Substituting δr and $\vec{S}$ into δW:

$$\delta W = -kr_{\text{spring}} \cdot (-2a \sin\theta \, \delta\theta) = 4ka^2 \sin\theta \cos\theta \, \delta\theta$$

From the virtual work equation $\delta W = Q_\theta \delta\theta + Q_\phi \delta\phi$,
we identify:

$$Q_\theta = 4ka^2 \sin\theta \cos\theta, \quad Q_\phi = 0$$

6. What are the expressions of $Q_\theta(\vec{R}_O)$ and $Q_\phi(\vec{R}_O)$, where $\vec{R}_O$ is the reaction force at O of the vertical axis on the system?

Answer:

Let R_0 represent the reaction force at point O exerted by the vertical axis. The force is influenced by the weight of two balls and the spring. At equilibrium, the reaction force is:

$$R_0 = 2mg - kr = 2mg - kL\sin\theta$$

The virtual work is given by:

$$\delta W = Q_\theta \delta\theta + Q_\phi \delta\phi$$

Since there is no displacement in the ϕ direction ($\delta\phi = 0$), we have:

$$\delta W = R_0 \cdot (-L\sin\theta\delta\theta)$$

Substituting R_0 into the equation:

$$\delta W = (2mg - kL\sin\theta)(-L\sin\theta\delta\theta)$$

From the above, the generalized forces are:

$$Q_\theta(\vec{R}_O) = -R_0 L\sin\theta = -(2mg - kL\sin\theta)L\sin\theta$$

$$Q_\phi(\vec{R}_O) = 0$$

7. What are the expressions of $Q_\theta(\vec{R}_C)$ and $Q_\phi(\vec{R}_C)$, where $\vec{R}_C$ is the reaction force at C of the vertical axis on the system?

Answer:

The reaction force $\overline{R}_C$ acts along A_1C or A_2C, where the perpendicular terms cancel out. The magnitude of $\overline{R}_C$ is given by:

$$R_C = 2mg - kr_{\text{spring}} = 2mg - k(2a\cos\theta)$$

The generalized forces are calculated as follows:

$$Q_\theta(\vec{R}_C) = -R_C L \sin\theta = -(2mg - k \cdot 2a\cos\theta)L\sin\theta$$

For the ϕ direction:

$$Q_\phi(\vec{R}_C) = 0$$

8. Identify the conservative forces exerted on the device, and give the potential energy they derive from. The total potential energy U will be set to 0 for $r = 0$ and $\theta = \frac{\pi}{2}$.

Answer:

The conservation forces include:

1. Weight of the mass due to gravity:

$$U_g = mgz_{B_2} + mgz_{B_1}$$

2. Spring force:

$$U_s = \frac{1}{2}kr^2 = \frac{1}{2}k(2a\cos\theta)^2$$

The total potential energy is:

$$U = U_g + U_s$$

Substituting the expressions:

$$U = -2mgL\cos\theta + 2ka^2\cos^2\theta$$

When applying $\theta = \frac{\pi}{2}$ and $r = 0$, the potential energy becomes:

$$U = 2mg(-L \cdot 0) + 2k \cdot 0^2$$

$$U = 0$$

9. Calculate the translational kinetic energy T_t.

Answer:

The velocity components are given as:

$$v_x = \frac{d}{dt}(L \sin\theta \cos\phi) = L(\dot{\theta} \cos\theta \cos\phi - \dot{\phi} \sin\theta \sin\phi)$$

$$v_y = \frac{d}{dt}(L \sin\theta \sin\phi) = L(\dot{\theta} \cos\theta \sin\phi + \dot{\phi} \sin\theta \cos\phi)$$

$$v_z = \frac{d}{dt}(-L \cos\theta) = L\dot{\theta} \sin\theta$$

The total translational energy is:

$$T_t = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \times 2 \quad \text{(for } B_1 \text{ and } B_2\text{)}$$

Substituting the velocity components:

$$T_t = m\left[L^2(\dot{\theta}^2 \cos^2\theta + \dot{\phi}^2 \sin^2\theta) + L^2\dot{\theta}^2 \sin^2\theta\right]$$

Simplifying:

$$T_t = mL^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

10. Calculate the rotational kinetic energy T_r (use the moment of inertia of the sphere with respect to its diameter).

Answer:

The moment of inertia of the sphere is denoted as I_s.

The total rotational kinetic energy is given by:

$$T_r = \left(\frac{1}{2}I_s\dot{\phi}^2 + \frac{1}{2}I_s\dot{\theta}^2\right) \times 2 \quad \text{(for } B_1 \text{ and } B_2\text{)}$$

Simplifying:

$$T_r = I_s\dot{\phi}^2 + I_s\dot{\theta}^2$$

11. Prove that the Lagrangian $L = (ML^2 + I_s)\dot{\theta}^2 + (ML^2\sin^2\theta + I_s)\dot{\phi}^2 + 2MgL\cos\theta - 2ka^2\cos^2\theta$.

Answer:

We need to prove that the Lagrangian is given by:

$$L = (ML^2 + I_s)\dot{\theta}^2 + (ML^2\sin^2\theta + I_s)\dot{\phi}^2 + 2MgL\cos\theta - 2ka^2\cos^2\theta$$

From questions 8–10, we have the total kinetic energy T:

$$T = T_t + T_r$$

$$T = mL^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + I_s\dot{\phi}^2 + I_s\dot{\theta}^2$$

Simplifying:

$$T = mL^2\dot{\theta}^2 + mL^2\dot{\phi}^2\sin^2\theta + I_s\dot{\phi}^2 + I_s\dot{\theta}^2$$

$$T = (mL^2 + I_s)\dot{\theta}^2 + (mL^2\sin^2\theta + I_s)\dot{\phi}^2$$

The potential energy U is:

$$U = -2mgL\cos\theta + 2ka^2\cos^2\theta$$

The Lagrangian L is defined as:

$$L = T - U$$

Substituting T and U:

$$L = (mL^2 + I_s)\dot{\theta}^2 + (mL^2\sin^2\theta + I_s)\dot{\phi}^2 + 2mgL\cos\theta - 2ka^2\cos^2\theta$$

Thus, the Lagrangian is proven as required.

12. Calculate the partial derivatives of the Lagrangian to be used for the differential equations of Lagrange.

Answer:

The partial derivatives of the Lagrangian L are calculated as follows:

$$\frac{\partial L}{\partial \theta} = 2mL^2 \sin\theta \cos\theta\, \dot{\phi}^2 - 2mgL\sin\theta + 4ka^2 \cos\theta \sin\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2(mL^2 + I_s)\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\phi}} = 2(mL^2 \sin^2\theta + I_s)\dot{\phi}$$

These partial derivatives are used to form the Lagrange equations for the system.

13. Express the differential equations of Lagrange.

Answer:

The Lagrange equations are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

For θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2(mL^2 + I_s)\ddot{\theta}$$

Substituting into the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2(mL^2 + I_s)\ddot{\theta} - \big(2mL^2 \sin\theta\cos\theta\dot{\phi}^2 - 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta\big) = 0$$

Simplifying:

$$2(mL^2 + I_s)\ddot{\theta} - 2mL^2\sin\theta\cos\theta\dot{\phi}^2 + 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta = 0$$

For ϕ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 2(mL^2\sin^2\theta + I_s)\ddot{\phi} + 4mL^2 cos\theta sin\theta\dot{\phi}\dot{\theta}$$

Substituting into the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 2(mL^2\sin^2\theta + I_s)\ddot{\phi} + 4mL^2 cos\theta sin\theta\dot{\phi}\dot{\theta} - 0 = 0$$

Simplifying:

$$2(mL^2\sin^2\theta + I_s)\ddot{\phi} + 4mL^2 cos\theta sin\theta\dot{\theta}\dot{\phi} = 0$$

14. Use the matrix format to present the equations of Lagrange.

Answer:

The equations of motion are given as follows:

For θ:

$$2(mL^2 + I_s)\ddot{\theta} - 2mL^2\sin\theta\cos\theta\dot{\phi}^2 + 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta = 0$$

For ϕ:

$$2(mL^2\sin^2\theta + I_s)\ddot{\phi} + 4mL^2 cos\theta sin\theta\dot{\theta}\dot{\phi} = 0$$

The equations can be expressed in the matrix form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = 0$$

where:

$$\mathbf{q} = \begin{bmatrix}\theta \\ \phi\end{bmatrix}, \quad \dot{\mathbf{q}} = \begin{bmatrix}\dot{\theta} \\ \dot{\phi}\end{bmatrix}, \quad \ddot{\mathbf{q}} = \begin{bmatrix}\ddot{\theta} \\ \ddot{\phi}\end{bmatrix}.$$

For mass matrix:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix}2(mL^2 + I_s) & 0 \\ 0 & 2(mL^2\sin^2\theta + I_s)\end{bmatrix}$$

For Coriolis and centrifugal matrix:

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix}\dot{\theta} & \dot{\phi}\end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2ML^2\cos\theta\sin\theta \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 4ML^2\sin\theta\cos\theta & 0 \end{bmatrix} \end{bmatrix}$$

And for generalized force:

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta \\ 0 \end{bmatrix}$$

Therefore, by substituting these matrices, the dynamic model can be expressed as:

$$\begin{bmatrix} 2(mL^2+I_s) & 0 \\ 0 & 2(mL^2\sin^2\theta + I_s) \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix}\dot{\theta} & \dot{\phi}\end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & -2ML^2\cos\theta\sin\theta \end{bmatrix} \atop \begin{bmatrix} 0 & 0 \\ 4ML^2\sin\theta\cos\theta & 0 \end{bmatrix} \right) \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$$+ \begin{bmatrix} -2mgL\sin\theta + 4ka^2\cos\theta\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

15. The system is driven in pivoting rotation with constant angular velocity Ω_0. Express $\cos\theta$ as a function of Ω_0, a, k, M, L, and g.

Answer:

The system is driven in pivoting rotation with constant angular velocity Ω_0. Under constant angular velocity, there is no angular acceleration, so $\ddot{\theta} = 0$. From the θ equation in question 13:

$$2(mL^2 + I_s)\ddot{\theta} - 2mL^2\sin\theta\cos\theta\dot{\phi}^2 + 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta = 0$$

Substitute $\ddot{\theta} = 0$ and $\dot{\phi} = \Omega_0$:

$$-2mL^2\sin\theta\cos\theta\Omega_0^2 + 2mgL\sin\theta - 4ka^2\cos\theta\sin\theta = 0$$

Factorizing $\sin\theta$:

$$\sin\theta\big(-2mL^2\cos\theta\Omega_0^2 + 2mgL - 4ka^2\cos\theta\big) = 0$$

For $\sin\theta = 0$:

$$\theta = 0 \quad \text{or} \quad \theta = \pi$$

These solutions are unstable or impossible for the system.
For $\sin\theta \neq 0$:

$$-2mL^2\cos\theta\Omega_0^2 + 2mgL - 4ka^2\cos\theta = 0$$

Rearranging:

$$\big(-2mL^2\Omega_0^2 - 4ka^2\big)\cos\theta = -2mgL$$

$$\cos\theta = \frac{2mgL}{2mL^2\Omega_0^2 + 4ka^2}$$

Thus, $\cos\theta$ is expressed as:

$$\cos\theta = \frac{mgL}{mL^2\Omega_0^2 + 2ka^2}$$

16. From the previous motion state, imagine that the pivoting velocity is suddenly decreased to an angular velocity Ω_0^- due to some kind of disturbance. How will the system evolve? Same question if the velocity is set to Ω_0^+. Explain.

Answer:
From the previous question, the equilibrium condition is:

$$\cos\theta = \frac{2mgL}{2mL^2\Omega_0^2 + 4ka^2}$$

If Ω_0 decreases:

- The denominator of $\cos\theta$ decreases, causing $\cos\theta$ to increase.
- As $\cos\theta$ increases, θ decreases (since $\cos\theta$ and θ are inversely related within $0 \leq \theta \leq \pi/2$).

Thus:

$$\Omega_0^- \implies \theta^-$$

If Ω_0 increases:

- The denominator of $\cos\theta$ increases, causing $\cos\theta$ to decrease.

- As $\cos\theta$ decreases, θ increases.

Thus:

$$\Omega_0^+ \implies \theta^+$$

The system's evolution is directly dependent on the change in angular velocity Ω_0, as it alters the equilibrium condition for θ through $\cos\theta$.

17. The system is now pivoting at constant angular speed Ω_0, which enables the device to be in equilibrium in the vertical plane. Represent the forces exerted on the right part of the system on the figure in the case of equilibrium in the vertical plane of the device, including the inertial centrifugal force due to the pivoting rotation, the weight, and the reaction forces from the vertical axis. Make sure the force polygon is closed.

Answer: