LECTURE 7: SEQUENCES & SERIES

SEQUENCE: ORDERED LIST OF NUMBERS : INFINITE

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NOTATION $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$

EXAMPLE $\{n\} = 1, 2, 3, 4, 5, \ldots \Rightarrow$ DIVERGES TO ∞

$a_1 = 1$
$a_{n+1} = a_n + 1$

$\{1\} = 1, 1, 1, 1, 1, \ldots \Rightarrow$ CONVERGES TO 1

SEQUENCE AS FUNCTION $f: \mathbb{N} \to \mathbb{R}$ WITH $f(n) = a_n$

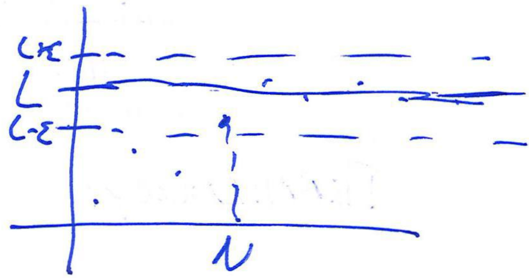
$\{\frac{1}{n}\} \xrightarrow{n\to\infty} 0$

$\{(-1)^n\} = -1, 1, -1, 1, -1, 1, \ldots \rightsquigarrow$ DIVERGES

MAIN QUESTION:
DOES IT CONVERGE?

RECURSIVELY DEFINED SEQUENCE: $a_1 = a_2 = 1$

$\boxed{a_{n+1} = a_n + a_{n-1}}$ $\forall n \geq 2$



$1, 1, 2, 3, 5, 8, 13 \ldots$ FIBONACCI-SEQUENCE

EX: $a_1 = 10$, $a_{n+1} = \sqrt{6a_n + 4}$; $10, 8, \sqrt{52}, \sqrt{6\cdot\sqrt{52} + 4}, \ldots$

$\{a_n\}$ CONVERGES TO L IF $\forall \varepsilon > 0\ \exists N > 0\ \forall n \geq N : |a_n - L| < \varepsilon$

NOTATION: $\lim_{n\to\infty} a_n = L$

<u>THM</u>: IF A FUNCTION f EXISTS SUCH THAT $a_n = f(n)$ AND $\lim_{x\to\infty} f(x) = L$,

THEN $\lim_{n\to\infty} a_n = L$.

$$\{n \cdot \sin(2\pi n)\} = 0, 0, 0, 0, 0 \to 0$$

$\lim_{x\to\infty} \sin(2\pi x)$ D.N.E.

$$\left\{\frac{3n^2-5n+2}{n+1}\right\} \xrightarrow{n\to\infty} \infty$$

$$\left\{\frac{3n^2-5n+2}{n^2+1}\right\} \xrightarrow{n\to\infty} 3$$

TERMINOLOGY: $\{a_n\}$ IS BOUNDED ABOVE IF $a_n \le M$ FOR SOME $M \in \mathbb{R}$ $\forall n$

UPPER BOUND

BELOW $a_n \ge m$

LOWER BOUND

$\{a_n\}$ IS BOUNDED IF BOTH.

$\{\frac{1}{n}\}$ $\frac{1}{n} \le 10$ $\forall n$

$\frac{1}{n} \ge -\pi$

$\{a_n\}$ IS INCREASING IF $\boxed{a_{n+1} \ge a_n}$ $\forall n$

DECREASING $\le$

MONOTONIC IF IT IS INCR. OR DECR.

$\{\frac{1}{n}\}$ IS DECREASING

IF $a_n = f(n)$ AND f IS CONTINUOUS ON $[1,\infty)$ AND DIFFERENTIABLE ON $(1,\infty)$

IF $f'(x) > 0 \ \forall x \in (1,\infty)$ THEN $\{a_n\}$ IS INCREASING

DE- "

EXAMPLE: $\left\{ \frac{3n^2-5n+2}{n^2+n+8} \right\}$ IS INCREASING.

$$f(x) = \frac{3x^2-5x+2}{x^2+\cancel{}x+8} \ ; \quad f'(x) = \frac{8x^2+44x-42}{(x^2+\cancel{}x+8)^2} > 0$$

$\{a_n\}$ IS ALTERNATING IF $a_n > 0 \Rightarrow a_{n+1} < 0$

$a_n < 0 \Rightarrow a_{n+1} > 0$

SO $a_n \cdot a_{n+1} < 0 \ \forall n.$

$\underline{\{(-1)^n\}}$

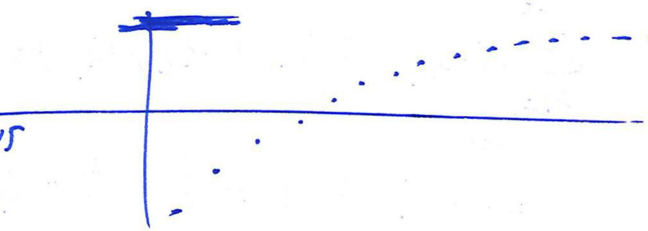
POSITIVE IF $a_n \geq 0 \ \forall n$

NEGATIVE IF $a_n \leq 0 \ \forall n$

THM $\{a_n\}$ CONVERGENT $\Rightarrow$ $\{a_n\}$ BOUNDED

$\{a_n\}$ INCREASING & BOUNDED ABOVE $\Rightarrow$ $\{a_n\}$ CONVERGENT

$\{a_n\}$ MONOTONIC & BOUNDED $\Rightarrow$ CONVERGENT



$a_1 = 10$; $a_{n+1} = \sqrt{6a_n + 4}$ DECREASING.

1 $a_2 = \sqrt{6 \cdot 10 + 4} = 8 < a_1$

ASSUME THAT $\underline{a_n < a_{n-1}}$

2 $a_{n+1} = \sqrt{6a_n + 4} < \sqrt{6a_{n-1} + 4} = a_n$

$\Rightarrow$ DECREASING.

BOUNDED BELOW (SHOW THAT $a_n > 0 \ \forall n$)

1 $a_1 > 0$

2 ASSUME $a_n > 0$. THEN $a_{n+1} = \sqrt{6a_n + 4} > \sqrt{6 \cdot 0 + 4} = 2 > 0$ $\Rightarrow$ BOUNDED BELOW

FINDING THE LIMIT L:

$$\Rightarrow \quad L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6a_n+4} = \sqrt{6 \cdot \lim_{n\to\infty} a_n + 4} = \sqrt{6L+4}$$

SOLVE FOR L. $\rightsquigarrow L = 3 \pm \sqrt{13}$, SO $L = 3+\sqrt{13}$

(INFINITE) SERIES: SUM OF TERMS IN A SEQUENCE.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots = ?$$

$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \ldots\ldots = ?$$

SEQUENCE OF <u>PARTIAL SUM</u>:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

ONLY POSSIBLE IF $\{a_n\} \to 0$.

IF $\boxed{S_n \to S}$, THE SERIES CONVERGES TO S

$$\sum_{n=1}^{\infty} \frac{n}{2n+5} = \infty .$$

$$\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2}$$

EX: $\boxed{\text{GEOMETRIC SERIES}}$

RATIO

$\uparrow$

$$\left.\begin{aligned} a_n &= C\cdot R^{n-1} \\ a_{n+1} &= C\cdot R^n \end{aligned}\right\} \Rightarrow \frac{a_{n+1}}{a_n} = R \quad \forall n.$$

$$S_n = C + \cancel{cr} + \cancel{cr^2} + \cancel{\ldots} + \cancel{cr^{n-1}}$$

$$r\cdot S_n = \quad \cancel{cr} + \cancel{cr^2} + \cancel{cr^3} + \cancel{\ldots} + cr^n$$

$$(1-r)\cdot S_n = c - cr^n \Rightarrow S_n = \frac{c\cdot(1-r^n)}{1-r}$$

$a_n \to 0$ IF $-1 < r < 1$

$S_n \to \boxed{\frac{c}{1-r}}$ IF $-1 < r < 1$

S_n DIVERGES IF $r \leq -1$ OR $r \geq 1$.

$$\text{EX: } \sum_{n=1}^{\infty} \frac{3\cdot 2^n + 5\cdot 3^n}{4^n} = \sum_{n=1}^{\infty} 3\cdot\left(\frac{1}{2}\right)^n + 5\cdot\left(\frac{3}{4}\right)^n = 3\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + 5\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$$

$$= \frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} + \frac{15}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}$$

$$= \frac{3}{2}\cdot\frac{1}{1-\frac{1}{2}} + \frac{15}{4}\cdot\frac{1}{1-\frac{3}{4}} = \underline{\underline{18}}$$

P-SERIES:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

CONV. IF $p > 1$

DIV. IF $p \leq 1$

AS LONG AS $p > 0$ $\frac{1}{n^p} \to 0$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1}_{} + \underbrace{\tfrac{1}{2}}_{} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{} + \underbrace{\tfrac{1}{9} + \tfrac{1}{10} + \dots}_{}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \infty.$$

HARMONIC SERIES ($p=1$) DIVERGES

INTEGRAL TEST: SUPPOSE THAT $a_n = f(n)$ WHERE f IS A POSITIVE, CONT. FUNCTION ON $[N, \infty)$

THEN: $\sum_{n=1}^{\infty} a_n$ AND $\int_N^{\infty} f(t)\,dt$ EITHER BOTH CONVERGE OR BOTH DIVERGE.

$\int_1^{\infty} \frac{1}{x}\,dx = \infty$, SO $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES

$$\sum_{n=1}^{\infty} \left[\frac{n-1}{10n^2+n} \approx \frac{n}{10n^2} = \frac{1}{10n} = \frac{1}{10} \cdot \frac{1}{n}\right] \text{ DIVERGES.}$$