9.1 : 1, 4, 6, 8, 10, 11, 12, 19, 23, 30, 36
9.2 : 2, 5, 7, 8, 15, 26-30
9.3 10, 15, 24

9.1 1) $\left\{\frac{2n^2}{n^2+1}\right\}$ $\lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2$ (d)

the sequence converges, so it is bounded (a)

the sequence is positive (b)

$\frac{2n^2}{n^2+1} = 2(1-\underbrace{\frac{1}{n^2+1}}_{\text{decreasing to } 0})$, this sequence is increasing (c)

4) $\{\sin(\frac{1}{n})\}$ $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$ (d)

the sequence converges, so it is bounded (a)

the sequence is positive (b)

the sequence decreases (c). $\left(\frac{d}{dx}\sin(\frac{1}{x}) = -\frac{1}{x^2}\cos(\frac{1}{x}) < 0\right.$ as $\left. x\to+\infty\right)$

6) $\left\{\frac{e^n}{n}\right\}$ $\lim_{n\to\infty} \frac{e^n}{n} = +\infty$ (divergence to ∞) (d)

the sequence is positive (b), so bounded below (a)

the sequence is increasing. $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2}$
$= \frac{e^x}{x}(1-\frac{1}{x}) > 0$
if $x > 1$.

8) $\left\{\frac{(-1)^n n}{e^n}\right\}$ $\lim_{n\to\infty} \frac{n}{e^n} = 0$, so $\lim_{n\to\infty} \frac{(-1)^n n}{e^n} = 0$

$\lim_{n\to\infty} \frac{-n}{e^n} = 0$ (squeeze theorem)

the sequence alternates.

the sequence is bounded, since it converges.

10) $\left\{\frac{(n!)^2}{(2n)!}\right\}$ $\quad \frac{(n!)^2}{(2n)!} = \frac{n!\,n!}{(2n)!} = \frac{1\cdot 2 \cdots n}{(n+1)(n+2)\cdots(2n)} = \frac{1}{n+1}\cdot\underbrace{\frac{2}{n+2}}_{\leq 1}\cdots\cdots\underbrace{\frac{n}{2n}}_{\leq 1} \leq \frac{1}{n+1}$

since $0 < \frac{(n!)^2}{(2n)!} < \frac{1}{n+1}$, $\frac{(n!)^2}{(2n)!} \to 0$

the sequence is bounded (since it converges) and positive.

11) $\left\{n\cos\left(\frac{n\pi}{2}\right)\right\}$ $= 0, -2, 0, +4, 0, -6$

diverges, not increasing or decreasing, not bounded
not strictly alternating

12) $\left\{\frac{\sin(n)}{n}\right\}$ $\quad \underset{\downarrow \atop 0}{\frac{-1}{n}} \leq \underset{\downarrow \atop 0}{\frac{\sin(n)}{n}} \leq \underset{\downarrow \atop 0}{\frac{1}{n}}$ converges, bounded, not increasing, decreasing alternating

19) $\lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = 1$

23) $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(\cancel{n}+1) - \cancel{n}}{\sqrt{n+1}+\sqrt{n}} = 0$

30) $a_1 = 1$, $a_2 = \sqrt{1+2a_n}$

1) a_n is increasing

Proof by induction: • Base case $a_1 = 1$, $a_2 = \sqrt{1+2} = \sqrt{3} > 1$ ✓
$a_1 < a_2$

• Inductive step assume $a_{n-1} < a_n$
then $1+2a_{n-1} < 1+2a_n$
then $\sqrt{1+2a_{n-1}} < \sqrt{1+2a_n}$
so $a_n < a_{n+1}$ ✓

2) the sequence is bounded.
Induction • Base case : $a_1 < 3$
• Assume $a_n < 3$. Then $a_{n+1} = \sqrt{1+2a_n} < \sqrt{1+2\cdot 3} = \sqrt{7} < 3$

From 1) & 2), we find that the sequence converges.

for the limit, $a = \sqrt{1+2a} \Rightarrow a^2 = 1 + 2a \Rightarrow a^2 - 2a - 1 = 0$

$a = 1 + \sqrt{2}$
(we need the positive root)

36) (a) true (b) false (c) true
take $a_n = n$ and $b_n = -n^2$

(d) false
take $a_n = 0\ 1\ 0\ 1\ 0\ 1 \ldots$ then $a_n b_n = 0\ 0\ 0\ 0$
$b_n = 1\ 0\ 1\ 0\ 1\ 0$

(e) false
take $1\ -1\ 1\ -1\ 1\ -1$

9.2 2. $\sum_{n=1}^{\infty} 3 \cdot \left(-\frac{1}{4}\right)^{n-1} = 3 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{3}{1+\frac{1}{4}} = \frac{3 \cdot 4}{5} = \frac{12}{5}$

5. $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}} = \sum_{n=0}^{\infty} \left(\frac{-5}{64}\right)^n - \underset{n=1}{\frac{-5}{64}} - \underset{n=0}{1}$

$= \frac{1}{1+\frac{5}{64}} + \frac{5}{64} - 1 = \frac{64}{69} + \frac{5}{64} - 1 = 0{,}00566 \quad \left(\frac{25}{4416}\right)$

7. $\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}} = (2e)^3 \sum_{k=0}^{\infty} \left(\frac{2}{e}\right)^k = 8e^3 \frac{1}{1-\frac{2}{e}} = \frac{8e^4}{e-2}$

8. $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi) = \sum_{j=1}^{\infty} (\sqrt{\pi})^j (-1)^j$ diverges, since $r = -\sqrt{\pi}$
$|r| > 1$

15. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges, since $\int_1^{\infty} \frac{dx}{2x-1} = \lim_{a \to \infty} \int_1^a \frac{dx}{2x-1} = \lim_{a \to \infty} \frac{1}{2} \ln(2a+1) = +\infty$

26. false, $\sum \frac{1}{n}$ diverges.
27. true if $\sum a_n$ converges, then $a_n \to 0$, so $\frac{1}{a_n} \to \pm\infty$, so $\sum \frac{1}{a_n}$ diverges.

28. false $\sum \frac{1}{n}$ and $\sum \frac{-1}{n}$ diverge, but $\sum (a_n + b_n) = 0$

29. true $\Sigma a_n \geqslant \sum_{n=1}^{\infty} C \to \infty$

30. false. $\Sigma \frac{1}{n}$ diverges, $\frac{1}{n^3}$ is bounded, $\Sigma \frac{1}{n} \cdot \frac{1}{n^3} = \Sigma \frac{1}{n^4}$ converges.