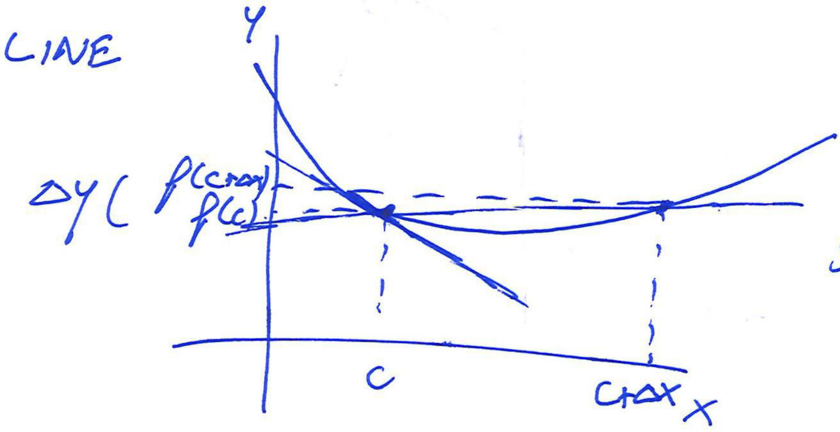
LECTURE 3: DIFFERENTIATION

TANGENT LINE



CHORD

SECANT LINE. $ax+b$.

$$a = \frac{f(c+\Delta x)-f(c)}{(c+\Delta x)-c} = \frac{f(c+\Delta x)-f(c)}{\Delta x}$$

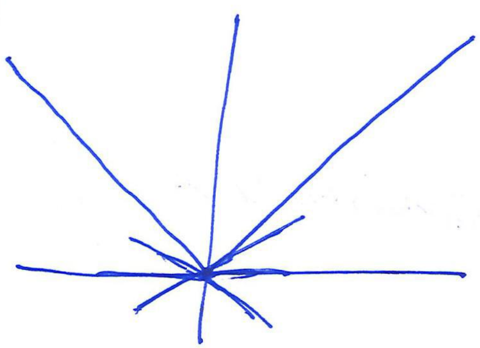
AT C

"TOUCHES" GRAPH OF f AT $x=c$.

SLOPE: $a = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \rightsquigarrow \frac{0}{0}$

WE NEED: $\lim_{\Delta x \to 0} f(c+\Delta x) = f(c)$ **f MUST BE CONTINUOUS**

EXAMPLE: NO TANGENT LINE: $f(x) = |x|$ IN $x=0$.

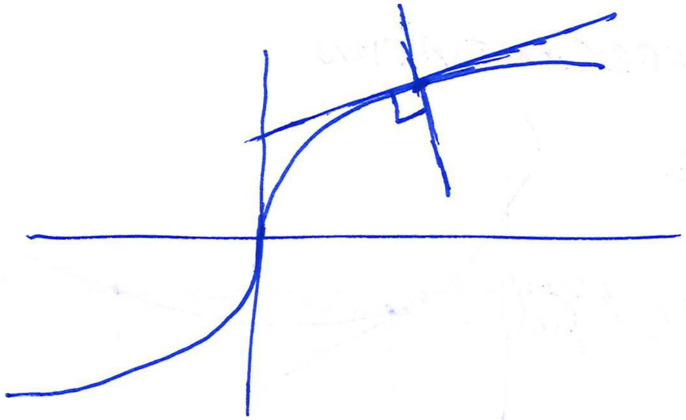


$$\lim_{\Delta x \to 0^-} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$\lim_{\Delta x \to 0^+} \frac{|0+\Delta x| - |0|}{\Delta x} = 1$$

f MUST BE "SMOOTH" IN C.

$f(x) = \sqrt[3]{x}$ AT $x = 0$.



$$\lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \quad \text{FOR } x=0$$

$$= \infty.$$

FIND TANG. LINE OF $f(x) = \sqrt{x}$ AT $x = 1$. $\quad f(1) = 1$.

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h\to 0} \frac{\cancel{1}+h-\cancel{1}}{\cancel{h}(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}.$$

$y = \frac{1}{2}x + b$.

$$\boxed{y = 1 + \tfrac{1}{2}(x-1)}$$

$= \frac{1}{2}x + \frac{1}{2}$

TANG. LINE UNIQUE $y = ax+b$ SUCH THAT 1. $y(c) = f(c)$

2. SLOPE AT C EQUAL FOR y AND f.

<u>NORMAL</u>: $y(c) = f(c)$

SLOPE: PERPENDICULAR: $y = ax + b$, $a \leftarrow -\frac{1}{a}$, ...

DERIVATIVE: AT EACH POINT c, THE SLOPE OF f HAS A VALUE.
THAT VALUE IS THE DERIVATIVE OF f IN c.

NOTATION: $$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

DO THIS FOR EVERY $c \in D$ FOR WHICH THIS LIMIT EXIST

EX: $f(x)=\sqrt{x}$ $\quad D=[0,\infty)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \dots = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}, \ x>0} \qquad D(f') \subseteq D(f)$$

TANGENT LINE: $y = f(c) + f'(c)\cdot(x-c)$.

- $f'(c)$ EXIST : f IS DIFFERENTIABLE AT c
- $f'(c)$ DOES NOT EXIST FOR SOME $c \in D$: c IS A SINGULAR POINT OF f.
- $f_-'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ (LEFT DERIVATIVE)
- $f_+'(c)$ RIGHT DERIVATIVE.

$D = [a, b]$.

$$\begin{cases} f \text{ IS DIFFERENTIABLE IN } a \text{ IF } f'_+(a) \text{ EXISTS} \\ \quad \text{"} \qquad\qquad \text{"} \qquad\quad b \qquad f'_-(b) \text{ EXISTS} \end{cases}$$

$$f'(x) \,;\; y' \,;\; \frac{d}{dx} f(x) \,;\; \frac{dy}{dx} \,;\; D_x(f)$$

$$f'(c) = \frac{d}{dx} f(x) \Big|_{x=c}$$

EXAMPLES:

- $f(x) = ax + b \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h) + b - (ax + b)}{h}$

$$= \lim_{h \to 0} \frac{a\cancel{h}}{\cancel{h}} = a.$$

- $f(x) = c \Rightarrow f'(x) = \lim_{h \to 0} \frac{c - c}{h} = 0.$

- $f(x) = |x| \Rightarrow f'(x) = \begin{cases} 1 & \text{IF } x > 0 \\ -1 & \text{IF } x < 0 \\ \text{D.N.E FOR } x = 0 \end{cases}$

- 'GENERAL POWER RULE' : $\frac{d}{dx} x^n = n \cdot x^{n-1} \quad \forall n \in \mathbb{R}$. (3)

PROOF FOR $n \in \mathbb{N}$: USE : $a^n - b^n = (a-b)(a^{n-1} + a^{n-2} \cdot b + a^{n-3} \cdot b^2 + \dots + a b^{n-2} + b^{n-1})$

$$\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\cancel{h} \cdot ((x+h)^{n-1} + (x+h)^{n-2} \cdot x + \dots + (x+h) \cdot x^{n-2} + x^{n-1})}{\cancel{h}}$$

$$= n \cdot x^{n-1}$$

- $(f+g)'(x) = f'(x) + g'(x)$
- $(k \cdot f)'(x) = k \cdot f'(x)$

EX: $f = x^5 + 5x^3 - 3\sqrt{x}$

$f'(x) = 5x^4 + 5 \cdot 3x^2 - 3 \cdot \frac{1}{2\sqrt{x}}$

PRODUCT RULE : f, g DIFFERENTIABLE $\Rightarrow (f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

PROOF: $(f \cdot g)'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x+h) \cdot g(x)}{h} + \frac{f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \cdot \boxed{\frac{g(x+h) - g(x)}{h}} - g(x) \cdot \frac{f(x+h) - f(x)}{h}$$

3a

RECIPROCAL RULE:

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}$$

PROOF:

$$\left(\frac{1}{f}\right)'(x) = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h\to 0} \frac{f(x) - f(x+h)}{h\cdot f(x+h)\cdot f(x)}$$

$$= \lim_{h\to 0} \frac{1}{f(x)\cdot f(x+h)} \cdot \frac{f(x)-f(x+h)}{h} = \frac{-f'(x)}{(f(x))^2}$$

QUOTIENT RULE:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx} f(x)\cdot\left(\frac{1}{g}\right)(x)$$

$$= f'(x)\cdot\left(\frac{1}{g}\right)(x) + f(x)\cdot\left(\frac{1}{g}\right)'(x)$$

$$= \ldots = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.$$

CHAIN RULE $f(x) = \sqrt{x^2+5} \Rightarrow f'(x) = ?$

IF $f(x) = g(h(x))$ THEN $\boxed{f'(x) = g'(h(x)) \cdot h'(x)}$

$g(x) = \sqrt{x}$; $h(x) = x^2+5$.

$g'(x) = \frac{1}{2\sqrt{x}} \Rightarrow g'(h(x)) = \frac{1}{2\sqrt{x^2+5}}$

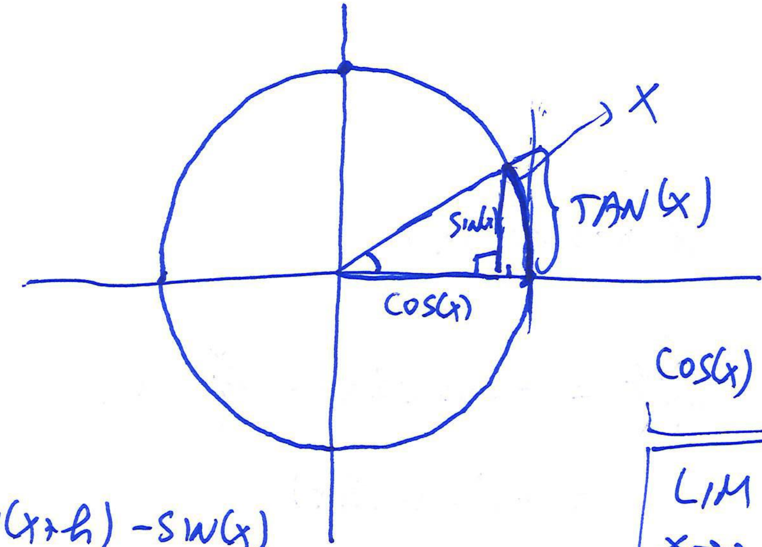
$h'(x) = 2x$

SO $f'(x) = \frac{x}{\sqrt{x^2+5}}$

OTHER NOTATION: $u = h(x) = x^2+5$

$$\frac{d}{dx} f(x) = \frac{d}{dx} g(u(x)) = \left.\frac{dg}{du}\right|_{u=h(x)} \cdot \frac{du}{dx} = \left.\frac{1}{2\sqrt{u}}\right|_{u=x^2+5} \cdot 2x = \frac{x}{\sqrt{x^2+5}}$$

$$\boxed{\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0}$$



$0 < x < \frac{1}{2}\pi$

$$\frac{\sin(x)}{x} < \frac{x}{x} < \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cos(x) < \frac{\sin(x)}{x} < 1$$

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

$$\sin(a+b) = \sin(a) \cdot \cos(b) + \sin(b) \cdot \cos(a)$$

$$\frac{d}{dx} \sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x) \cdot \cos(h) + \sin(h) \cdot \cos(x) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \cos(x) \cdot \underbrace{\frac{\sin(h)}{h}}_{1} + \sin(x) \cdot \underbrace{\frac{\cos(h) - 1}{h}}_{0} = \underline{\cos(x)}.$$

$$\frac{d}{dx} \cos(x) = \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) = \underline{-\sin(x)}$$

$$\frac{d}{dx} \tan(x) = \frac{d}{dx} \frac{\sin(x)}{\cos(x)} = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot -\sin(x)}{\cos^2(x)} = \underline{\frac{1}{\cos^2(x)}}$$

~~TRIGONOMETRIC~~:

$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

e IS "CHOSEN" SUCH THAT $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$. PROOF: LET $y = \ln(x)$.

THEN $e^y = e^{\ln(x)} = x$

$$\frac{d}{dx} e^y = \frac{d}{dx} x = 1$$

$$\| $$

$$e^y \cdot \frac{dy}{dx} = x \cdot \frac{dy}{dx} \quad \text{SO} \quad \boxed{\frac{dy}{dx} = \frac{1}{x}}$$

HIGHER ORDER DERIVATIVES:

$$f''(x) = y'' = \frac{d^2 f}{dx^2} = D_x^2 f$$

EXAMPLE:

$$\frac{d}{dx}\underbrace{e^{-\sin(3x)}}_{f}$$

$u = -\sin(3x)$

$$\frac{df}{du} = e^{u}$$

$$\frac{du}{dx} = -\cos(3x) \cdot \frac{d}{dx} 3x$$

$$= -3\cos(3x)$$

$$\frac{d}{dx} e^{-\sin(3x)} = e^{u}\Big|_{u=-\sin(3x)} \cdot -3\cos(3x) = e^{-\sin(3x)} \cdot -3\cos(3x)$$