For part (b), pick any x and let N be an integer such that $N > |x|$. If $n > N$ we have

$$\begin{aligned}\left|\frac{x^n}{n!}\right| &= \frac{|x|}{1}\frac{|x|}{2}\frac{|x|}{3}\cdots\frac{|x|}{N-1}\frac{|x|}{N}\frac{|x|}{N+1}\cdots\frac{|x|}{n}\\ &< \frac{|x|^{N-1}}{(N-1)!}\frac{|x|}{N}\frac{|x|}{N}\frac{|x|}{N}\cdots\frac{|x|}{N}\\ &= \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{n-N+1} = K\left(\frac{|x|}{N}\right)^n,\end{aligned}$$

where $K = \dfrac{|x|^{N-1}}{(N-1)!}\left(\dfrac{|x|}{N}\right)^{1-N}$ is a constant that is independent of n. Since $|x|/N < 1$, we have $\lim_{n\to\infty}(|x|/N)^n = 0$ by part (a). Thus, $\lim_{n\to\infty}|x^n/n!| = 0$, so $\lim_{n\to\infty} x^n/n! = 0$.

EXAMPLE 10 Find $\lim_{n\to\infty}\dfrac{3^n + 4^n + 5^n}{5^n}$.

Solution $\displaystyle\lim_{n\to\infty}\frac{3^n+4^n+5^n}{5^n} = \lim_{n\to\infty}\left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1\right] = 0+0+1 = 1$, by Theorem 3(a).

EXERCISES 9.1

In Exercises 1–13, determine whether the given sequence is (a) bounded (above or below), (b) positive or negative (ultimately), (c) increasing, decreasing, or alternating, and (d) convergent, divergent, divergent to ∞ or $-\infty$.

1. $\left\{\dfrac{2n^2}{n^2+1}\right\}$

2. $\left\{\dfrac{2n}{n^2+1}\right\}$

3. $\left\{4 - \dfrac{(-1)^n}{n}\right\}$

4. $\left\{\sin\dfrac{1}{n}\right\}$

5. $\left\{\dfrac{n^2-1}{n}\right\}$

6. $\left\{\dfrac{e^n}{\pi^n}\right\}$

7. $\left\{\dfrac{e^n}{\pi^{n/2}}\right\}$

8. $\left\{\dfrac{(-1)^n n}{e^n}\right\}$

9. $\left\{\dfrac{2^n}{n^n}\right\}$

10. $\left\{\dfrac{(n!)^2}{(2n)!}\right\}$

11. $\left\{n\cos\left(\dfrac{n\pi}{2}\right)\right\}$

12. $\left\{\dfrac{\sin n}{n}\right\}$

13. $\{1, 1, -2, 3, 3, -4, 5, 5, -6, \ldots\}$

In Exercises 14–29, evaluate, wherever possible, the limit of the sequence $\{a_n\}$.

14. $a_n = \dfrac{5-2n}{3n-7}$

15. $a_n = \dfrac{n^2-4}{n+5}$

16. $a_n = \dfrac{n^2}{n^3+1}$

17. $a_n = (-1)^n\dfrac{n}{n^3+1}$

18. $a_n = \dfrac{n^2-2\sqrt{n}+1}{1-n-3n^2}$

19. $a_n = \dfrac{e^n - e^{-n}}{e^n + e^{-n}}$

20. $a_n = n\sin\dfrac{1}{n}$

21. $a_n = \left(\dfrac{n-3}{n}\right)^n$

22. $a_n = \dfrac{n}{\ln(n+1)}$

23. $a_n = \sqrt{n+1} - \sqrt{n}$

24. $a_n = n - \sqrt{n^2-4n}$

25. $a_n = \sqrt{n^2+n} - \sqrt{n^2-1}$

26. $a_n = \left(\dfrac{n-1}{n+1}\right)^n$

27. $a_n = \dfrac{(n!)^2}{(2n)!}$

28. $a_n = \dfrac{n^2 2^n}{n!}$

29. $a_n = \dfrac{\pi^n}{1+2^{2n}}$

30. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ $(n = 1, 2, 3, \ldots)$. Show that $\{a_n\}$ is increasing and bounded above. (*Hint:* Show that 3 is an upper bound.) Hence, conclude that the sequence converges, and find its limit.

31. Repeat Exercise 30 for the sequence defined by $a_1 = 3$, $a_{n+1} = \sqrt{15+2a_n}$, $n = 1, 2, 3, \ldots$. This time you will have to guess an upper bound.

32. Let $a_n = \left(1+\dfrac{1}{n}\right)^n$ so that $\ln a_n = n\ln\left(1+\dfrac{1}{n}\right)$. Use properties of the logarithm function to show that (a) $\{a_n\}$ is increasing and (b) e is an upper bound for $\{a_n\}$.

33. Prove Theorem 2. Also, state an analogous theorem pertaining to ultimately decreasing sequences.

34. If $\{|a_n|\}$ is bounded, prove that $\{a_n\}$ is bounded.

35. If $\lim_{n\to\infty}|a_n| = 0$, prove that $\lim_{n\to\infty} a_n = 0$.

36. Which of the following statements are TRUE and which are FALSE? Justify your answers.

(a) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L > 0$, then $\lim_{n\to\infty} a_n b_n = \infty$.

(b) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty}(a_n + b_n) = 0$.

(c) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n b_n = -\infty$.

(d) If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge.

(e) If $\{|a_n|\}$ converges, then $\{a_n\}$ converges.

9.2 Infinite Series

An **infinite series**, usually just called a **series**, is a formal sum of infinitely many terms; for instance, $a_1 + a_2 + a_3 + a_4 + \cdots$ is a series formed by adding the terms of the sequence $\{a_n\}$. This series is also denoted $\sum_{n=1}^{\infty} a_n$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots .$$

It is sometimes necessary or useful to start the sum from some index other than 1:

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \cdots$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots .$$

Note that the latter series would make no sense if we had started the sum from $n = 1$; the first term would have been undefined.

When necessary, we can change the index of summation to start at a different value. This is accomplished by a substitution, as illustrated in Example 3 of Section 5.1. For instance, using the substitution $n = m - 2$, we can rewrite $\sum_{n=1}^{\infty} a_n$ in the form $\sum_{m=3}^{\infty} a_{m-2}$. Both sums give rise to the same expansion

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = \sum_{m=3}^{\infty} a_{m-2}.$$

Addition is an operation that is carried out on two numbers at a time. If we want to calculate the finite sum $a_1 + a_2 + a_3$, we could proceed by adding $a_1 + a_2$ and then adding a_3 to this sum, or else we might first add $a_2 + a_3$ and then add a_1 to the sum. Of course, the associative law for addition assures us we will get the same answer both ways. This is the reason the symbol $a_1 + a_2 + a_3$ makes sense; we would otherwise have to write $(a_1 + a_2) + a_3$ or $a_1 + (a_2 + a_3)$. This reasoning extends to any sum $a_1 + a_2 + \cdots + a_n$ of finitely many terms, but it is not obvious what should be meant by a sum with infinitely many terms:

$$a_1 + a_2 + a_3 + a_4 + \cdots .$$

Solution The given series is the sum of two geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-(1/3)} = \frac{1}{2} \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-(2/3)} = 4.$$

Thus, its sum is $\frac{1}{2} + 4 = \frac{9}{2}$ by Theorem 7(b).

EXERCISES 9.2

In Exercises 1–18, find the sum of the given series, or show that the series diverges (possibly to infinity or negative infinity). Exercises 11–14 are telescoping series and should be done by partial fractions as suggested in Example 3 in this section.

1. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3^n}$

2. $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots = \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1}$

3. $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$

4. $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$

5. $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$

6. $\sum_{n=0}^{\infty} \frac{1}{e^n}$

7. $\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$

8. $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi)$

9. $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$

10. $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

11. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots$

12. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

13. $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots$

14. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
$= \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$

15. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

16. $\sum_{n=1}^{\infty} \frac{n}{n+2}$

17. $\sum_{n=1}^{\infty} n^{-1/2}$

18. $\sum_{n=1}^{\infty} \frac{2}{n+1}$

19. Obtain a simple expression for the partial sum s_n of the series $\sum_{n=1}^{\infty}(-1)^n$, and use it to show that the series diverges.

20. Find the sum of the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots.$$

21. When dropped, an elastic ball bounces back up to a height three-quarters of that from which it fell. If the ball is dropped from a height of 2 m and allowed to bounce up and down indefinitely, what is the total distance it travels before coming to rest?

22. If a bank account pays 10% simple interest into an account once a year, what is the balance in the account at the end of 8 years if \$1,000 is deposited into the account at the beginning of each of the 8 years? (Assume there was no balance in the account initially.)

23. Prove Theorem 5.

24. Prove Theorem 6.

25. State a theorem analogous to Theorem 6 but for a negative sequence.

In Exercises 26–31, decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counter-example showing the falsehood.

26. If $a_n = 0$ for every n, then $\sum a_n$ converges.

27. If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges to infinity.

28. If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$.

29. If $a_n \ge c > 0$ for every n, then $\sum a_n$ diverges to infinity.

30. If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges.

31. If $a_n > 0$ and $\sum a_n$ converges, then $\sum(a_n)^2$ converges.

EXERCISES 9.3

In Exercises 1–26, determine whether the given series converges or diverges by using any appropriate test. The p-series can be used for comparison, as can geometric series. Be alert for series whose terms do not approach 0.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4-2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \left|\sin\frac{1}{n^2}\right|$

6. $\displaystyle\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n-n^\pi}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

13. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n\sqrt{\ln\ln n}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2e^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n\ln n}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{100}2^n}{\sqrt{n!}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

25. $\displaystyle\sum_{n=4}^{\infty} \frac{2^n}{3^n-n^3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

In Exercises 27–30, use s_n and integral bounds to find the smallest interval that you can be sure contains the sum s of the series. If the midpoint s_n^* of this interval is used to approximate s, how large should n be chosen to ensure that the error is less than 0.001?

27. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$

28. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

29. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

For each positive series in Exercises 31–34, find the best upper bound you can for the error $s-s_n$ encountered if the partial sum s_n is used to approximate the sum s of the series. How many terms of each series do you need to be sure that the approximation has error less than 0.001?

31. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k k!}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

35. Use the integral test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\pi/2$.

36. Show that $\sum_{n=3}^{\infty}(1/(n\ln n(\ln\ln n)^p))$ converges if and only if $p>1$. Generalize this result to series of the form

$$\sum_{n=N}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)\cdots(\ln_j n)(\ln_{j+1} n)^p},$$

where $\ln_j n = \underbrace{\ln\ln\ln\ln\cdots\ln}_{j\ \ln' s} n$.

37. Prove the root test. *Hint:* Mimic the proof of the ratio test.

38. Use the root test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

39. Use the root test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

40. Repeat Exercise 38, but use the ratio test instead of the root test.

41. Try to use the ratio test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happens? Now observe that

$$\begin{aligned}\frac{2^{2n}(n!)^2}{(2n)!} &= \frac{[2n(2n-2)(2n-4)\cdots 6\times 4\times 2]^2}{2n(2n-1)(2n-2)\cdots 4\times 3\times 2\times 1}\\ &= \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\cdots\times\frac{4}{3}\times\frac{2}{1}.\end{aligned}$$

Does the given series converge? Why or why not?

42. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. *Hint:* Proceed as in Exercise 41. Show that $a_n \ge 1/(2n)$.

43. (a) Show that if $k>0$ and n is a positive integer, then $n < \dfrac{1}{k}(1+k)^n$.

(b) Use the estimate in (a) with $0<k<1$ to obtain an upper bound for the sum of the series $\sum_{n=0}^{\infty} n/2^n$. For what value of k is this bound lowest?

(c) If we use the sum s_n of the first n terms to approximate the sum s of the series in (b), obtain an upper bound for the error $s-s_n$ using the inequality from (a). For given n, find k to minimize this upper bound.

44. (Improving the convergence of a series) We know that $\sum_{n=1}^{\infty} 1/\big(n(n+1)\big) = 1$. (See Example 3 of Section 9.2.)

Since $\dfrac{1}{n^2} = \dfrac{1}{n(n+1)} + c_n$, where $c_n = \dfrac{1}{n^2(n+1)}$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} c_n$.

back and forth around 8, converging to that number. Of course, any number other than 8 could also be used in place of 8.

EXERCISES 9.4

Determine whether the series in Exercises 1–12 converge absolutely, converge conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ **2.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+\ln n}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{(n+1)\ln(n+1)}$ **4.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2^n}$

5. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n^2-1)}{n^2+1}$ **6.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi^n}$ **8.** $\displaystyle\sum_{n=0}^{\infty}\frac{-n}{n^2+1}$

9. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{20n^2-n-1}{n^3+n^2+33}$ **10.** $\displaystyle\sum_{n=1}^{\infty}\frac{100\cos(n\pi)}{2n+3}$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(-100)^n}$ **12.** $\displaystyle\sum_{n=10}^{\infty}\frac{\sin(n+1/2)\pi}{\ln\ln n}$

For the series in Exercises 13–16, find the smallest integer n that ensures that the partial sum s_n approximates the sum s of the series with error less than 0.001 in absolute value.

13. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ **14.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2^n}$ **16.** $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$

Determine the values of x for which the series in Exercises 17–24 converge absolutely, converge conditionally, or diverge.

17. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n+1}}$ **18.** $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2 2^{2n}}$

19. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{2n+3}$ **20.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

21. $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{2^n\ln n}$ **22.** $\displaystyle\sum_{n=1}^{\infty}\frac{(4x+1)^n}{n^3}$

23. $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+3)^n}{n^{1/3}4^n}$ **24.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(1+\frac{1}{x}\right)^n$

25. Does the alternating series test apply directly to the series $\sum_{n=1}^{\infty}(1/n)\,\sin(n\pi/2)$? Determine whether the series converges.

26. Show that the series $\sum_{n=1}^{\infty}a_n$ converges absolutely if $a_n=10/n^2$ for even n and $a_n=-1/10n^3$ for odd n.

27. Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth, or give a counter-example to show falsehood.

(a) If $\sum_{n=1}^{\infty}a_n$ converges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(b) If $\sum_{n=1}^{\infty}a_n$ converges and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges absolutely.

(c) If $\sum_{n=1}^{\infty}a_n$ converges absolutely, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

28. (a) Use a Riemann sum argument to show that

$$\ln n! \ge \int_1^n \ln t\,dt = n\ln n - n + 1.$$

(b) For what values of x does the series $\sum_{n=1}^{\infty}\dfrac{n!x^n}{n^n}$ converge absolutely? converge conditionally? diverge? (*Hint:* First use the ratio test. To test the cases where $\rho=1$, you may find the inequality in part (a) useful.)

29. For what values of x does the series $\sum_{n=1}^{\infty}\dfrac{(2n)!x^n}{2^{2n}(n!)^2}$ converge absolutely? converge conditionally? diverge? *Hint:* See Exercise 42 of Section 9.3.

30. Devise procedures for rearranging the terms of the alternating harmonic series so that the rearranged series (a) diverges to ∞, (b) converges to -2.

9.5 Power Series

This section is concerned with a special kind of infinite series called a *power series*, which may be thought of as a polynomial of infinite degree.