1. $f(x) = \sqrt{1+x^2}$

1) domain : $\mathbb{R}$ $\quad (1+x^2 \geqslant 0$ for all $x \in \mathbb{R})$

2) this function is even $\quad (f(x) = f(-x))$

3) we expect oblique asymptotes, $\left(\text{as } x \to \infty,\ \sqrt{1+x^2} \simeq \sqrt{x^2} = |x|\right)$

- $\lim\limits_{x \to \infty} \dfrac{f(x)}{x} = \lim\limits_{x \to +\infty} \dfrac{\sqrt{1+x^2}}{x} = \lim\limits_{x \to +\infty} \dfrac{\sqrt{x^2(1+\frac{1}{x^2})}}{x}$

$$= \lim_{x \to +\infty} \frac{x\sqrt{1+1/x^2}}{x} = 1 = a$$

- $\lim\limits_{x \to +\infty} (f(x) - ax) = \lim\limits_{x \to \infty} (\sqrt{x^2+1} - x) = \lim\limits_{x \to +\infty} \dfrac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x}$

$$= \lim_{x \to +\infty} \frac{(\cancel{x^2}+1) - \cancel{x^2}}{\sqrt{x^2+1} + x} = \lim_{x \to +\infty} \frac{1}{\sqrt{x^2+1} + x} = 0 = b$$

↳ oblique asymptote $y = x$ (to ∞) (one-sided)

↳ as $f(x)$ is even, we can mirror this asymptote around the y-axis → $(y = ax + b \to y = -ax + b)$

↳ oblique asymptote $y = -x$ (to $-\infty$) (one-sided)

(we can also take $\lim\limits_{x \to -\infty} \dfrac{f(x)}{x}$, but mirroring is easier)

since there are oblique asymptotes, we can exclude horizontal asymptotes. There are also no vertical asymptotes (domain $\mathbb{R}$)

4) $f'(x) = \dfrac{d}{dx}(\sqrt{x^2+1}) = \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}}$ ↳ denominator always +

↳ $f'(x) = 0$ for $x = 0$ critical point

$f'(x) > 0$ for $x > 0$ → $f(x)$ is increasing for $x > 0$

$f'(x) < 0$ for $x < 0$ → $f(x)$ is decreasing $x < 0$

$f(0) = 1$ is a minimum.

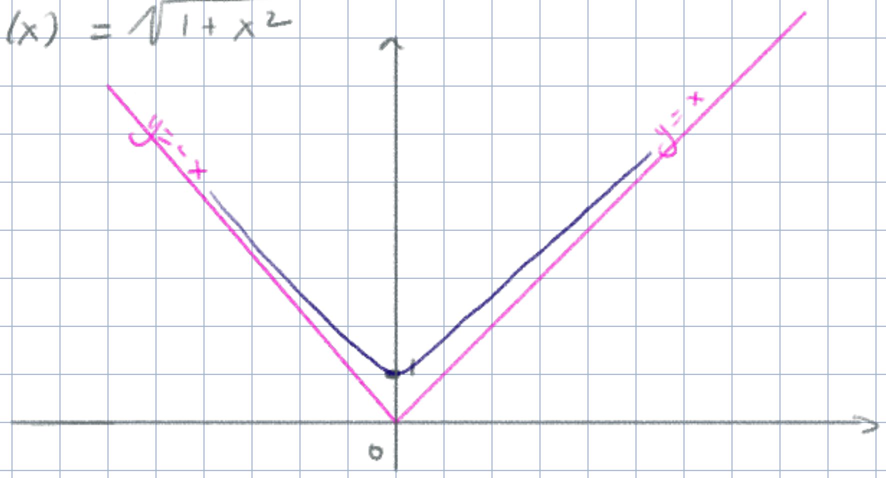
5) $f''(x) = \dfrac{d}{dx}\left(\dfrac{x}{\sqrt{x^2+1}}\right) = \dfrac{\sqrt{x^2+1} - x\dfrac{x}{\sqrt{x^2+1}}}{x^2+1} = \dfrac{(x^2+1) - x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}}$

denominator always +

↳ $f''(x) > 0$ for all $x \in \mathbb{R}$ → the function is always convex

no inflection points

$f(x) = \sqrt{1+x^2}$



$f(x) = \sqrt{x^2+x} - x$

1) domain : $x^2 + x \geqslant 0 \iff x(1+x) \geqslant 0 \iff x \leqslant -1$ or $x \geqslant 0$
domain $(f) = (-\infty, -1] \cup [0, \infty) = \mathbb{R} \setminus (-1, 0)$

2) $f(x) \neq -f(-x)$ not odd
$\neq f(-x)$ not even

3) Asymptotes : no vertical asymptotes (no denominator or ln)

$\rightarrow$ HA? $\lim\limits_{x \to +\infty} \sqrt{x^2+x} - x = \lim\limits_{x \to +\infty} \dfrac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x} =$

$\lim\limits_{x \to +\infty} \dfrac{(\cancel{x^2}+x) - \cancel{x^2}}{\sqrt{x^2+x} + x} = \lim\limits_{x \to +\infty} \dfrac{\cancel{x}}{\cancel{x}(\sqrt{1+\frac{1}{x}}) + 1)} = \lim\limits_{x \to +\infty} \dfrac{1}{\sqrt{1+\frac{1}{x}} + 1} = \dfrac{1}{2}$

$\Rightarrow$ one-sided horizontal asymptote $y = \frac{1}{2}$ (as $x \to +\infty$)

• $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \underbrace{\sqrt{x^2+x}}_{\to +\infty} \underbrace{- x}_{\to +\infty} = +\infty$ so no HA as $x \to -\infty$

$\hookrightarrow$ oblique asymptote?

• $\lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+x} - x}{x} = \lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2}\sqrt{1+\frac{1}{x}} - x}{x}$

if $x < 0$, $\sqrt{x^2} = -x$ $\quad = \lim\limits_{x \to -\infty} \dfrac{-\cancel{x} \cdot \sqrt{1+\frac{1}{x}} - \cancel{x}}{\cancel{x}} = \lim\limits_{x \to -\infty} \dfrac{-\sqrt{1+\frac{1}{x}} - 1}{1} = -2 = a$

• $\lim\limits_{x \to -\infty} (f(x) - ax) = \lim\limits_{x \to -\infty} \left((\sqrt{x^2+x} - x) + 2x\right) =$

$\lim\limits_{x \to -\infty} \sqrt{x^2+x} + x = \lim\limits_{x \to -\infty} \dfrac{(\sqrt{x^2+x} + x)(\sqrt{x^2+x} - x)}{\sqrt{x^2+x} - x} =$

$\lim\limits_{x \to -\infty} \dfrac{(\cancel{x^2} + x) - \cancel{x^2}}{\sqrt{x^2+x} - x} = \lim\limits_{x \to -\infty} \dfrac{x}{\sqrt{x^2}(\sqrt{1+\frac{1}{x}}) - x} =$

if $x < 0$, $\sqrt{x^2} = -x$ $\quad \lim\limits_{x \to -\infty} \dfrac{\cancel{x}}{-\cancel{x}(\sqrt{1+\frac{1}{x}} + 1)} = -\dfrac{1}{2} = b$

$\Rightarrow$ one-sided oblique asymptote (to $-\infty$) $y = -2x - \frac{1}{2}$

4) $f'(x) = \dfrac{d}{dx}(\sqrt{x^2+x} - x) = \dfrac{2x+1}{2\sqrt{x^2+x}} - 1$

critical points? $f'(x) = 0$ for $\frac{2x+1}{2\sqrt{x^2+x}} = 1$

$$\Leftrightarrow \quad 2x+1 = 2\sqrt{x^2+x}$$
$$\Rightarrow \quad (2x+1)^2 = 4(x^2+x)$$
$$\Rightarrow \quad 4x^2+4x+1 = 4x^2+4x \quad \lightning$$

$\hookrightarrow$ no critical points

for $x < -1$ (left part of domain), $f'(x) < 0 \rightarrow$ decreasing function
$\lim\limits_{x \to -1^-} f'(x) = -\infty$

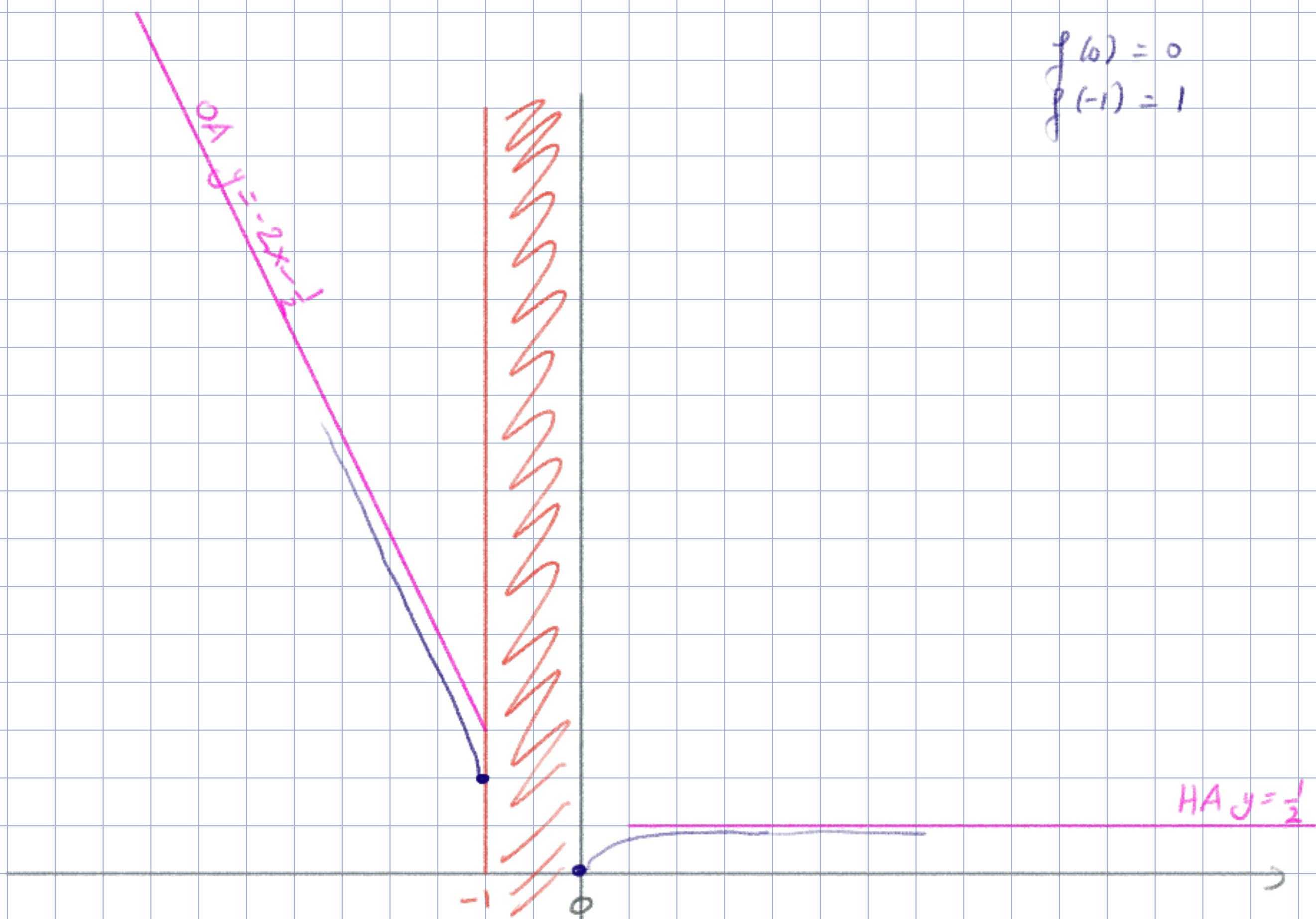
for $x > 0$ (right part of domain), $f'(x) > 0 \rightarrow$ increasing function
$\lim\limits_{x \to 0^+} f'(x) = +\infty$

5) $$f''(x) = \frac{d}{dx}\left(\frac{2x+1}{2\sqrt{x^2+x}} - 1\right) = \frac{2\sqrt{x^2+x}\cdot 2 - (2x+1)^2 \frac{\cancel{2}}{\cancel{2}\sqrt{x^2+x}}}{4(x^2+x)}$$

$$= \frac{4(x^2+x) - (2x+1)^2}{4(x^2+x)^{3/2}} = \frac{-1}{4(x^2+x)^{3/2}}$$ (calculation: see critical pt.)

$\rightarrow f''(x) < 0$ for all $x \in$ domain(f) $\rightarrow$ the function is always concave.

$f(0) = 0$
$f(-1) = 1$



• $f(x) = \dfrac{x^2-5x+6}{x-1} = \dfrac{(x-2)(x-3)}{x-1}$

1) domain: $\mathbb{R} \setminus \{1\}$

2) $f(x) \neq f(-x)$ not even
$f(x) \neq -f(-x)$ not odd

3) Asymptotes

1) VA $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{\overset{-}{(x-2)}\overset{-}{(x-3)}}{\underset{+}{x-1}} = +\infty$ VA : $x = 1$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x\to 1^-} \dfrac{(x-2)(x-3)}{x-1} = -\infty$

2) this is a rational function with the degree of the numerator 1 higher than the degree of the denominator $\rightarrow$ OA

• $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \dfrac{x^2-5x+6}{x(x-1)} = \lim\limits_{x\to\pm\infty} \dfrac{x^2-5x+6}{x^2-x} =$

$= \lim\limits_{x\to\pm\infty} \dfrac{1-\cancel{5/x}+\cancel{6/x^2}}{1-\cancel{\frac{1}{x^2}}} = 1 = a$

• $\lim\limits_{x\to\pm\infty} (f(x)-ax) = \lim\limits_{x\to\pm\infty} \left(\dfrac{x^2-5x+6}{x-1} - x\right) =$

$\lim\limits_{x\to\pm\infty} \dfrac{\cancel{x^2}-5x+6-(\cancel{x^2}-x)}{x-1} = \lim\limits_{x\to\pm\infty} \dfrac{-4x+6}{x-1} = -4 = b$

$\rightarrow f(x)$ has a 2-sided oblique asymptote $y = x-4$

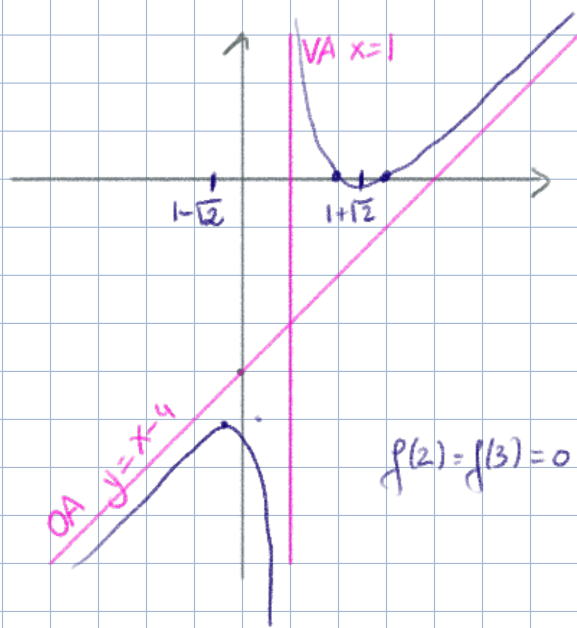
4) $f'(x) = \dfrac{d}{dx}\left(\dfrac{x^2-5x+6}{x-1}\right) = \dfrac{(2x-5)(x-1)-(x^2-5x+6)}{(x-1)^2} = \dfrac{(2x^2-7x+5)-(x^2-5x+6)}{(x-1)^2}$

$= \dfrac{x^2-2x-1}{(x-1)^2} = 1 - \dfrac{2}{(x-1)^2}$

$\hookrightarrow f'(x) = 0 \iff 1 - \dfrac{2}{(x-1)^2} = 0$

$\iff (x-1)^2 = 2 \iff x = 1 \pm \sqrt{2}$

x		$1-\sqrt{2}$		1		$1+\sqrt{2}$	
$f'(x)$	+	0	−		−	0	+
	↗	max	↘	DNE	↘	min	↗



$f(2) = f(3) = 0$

5) $f''(x) = \dfrac{d}{dx}\left(1 - \dfrac{2}{(x-1)^2}\right) = \dfrac{(-2)^2}{(x-1)^3} = \dfrac{4}{(x-1)^3}$

$\rightarrow f''(x) > 0$ for $x > 1$ (convex), $f''(x) < 0$ for $x < -1$ (concave)

$f(x) = \dfrac{x}{x^2+1}$

1) domain : $\mathbb{R}$

2) $f(x) = -f(-x) \Rightarrow$ this function is odd

3) Asymptotes : this is a rational function, with the degree of the denominator higher than the degree of the numerator $\longrightarrow$ HA! No VA, since the denominator has no zeros

HA: $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{x}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{1}{x+\frac{1}{x}} = 0$

2-sided HA $y=0$ ($\hookrightarrow$ therefore, no OA)

4) $f'(x) = \dfrac{d}{dx}\left(\dfrac{x}{x^2+1}\right) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

$\longrightarrow f'(x) = 0$ for $x = \pm 1$

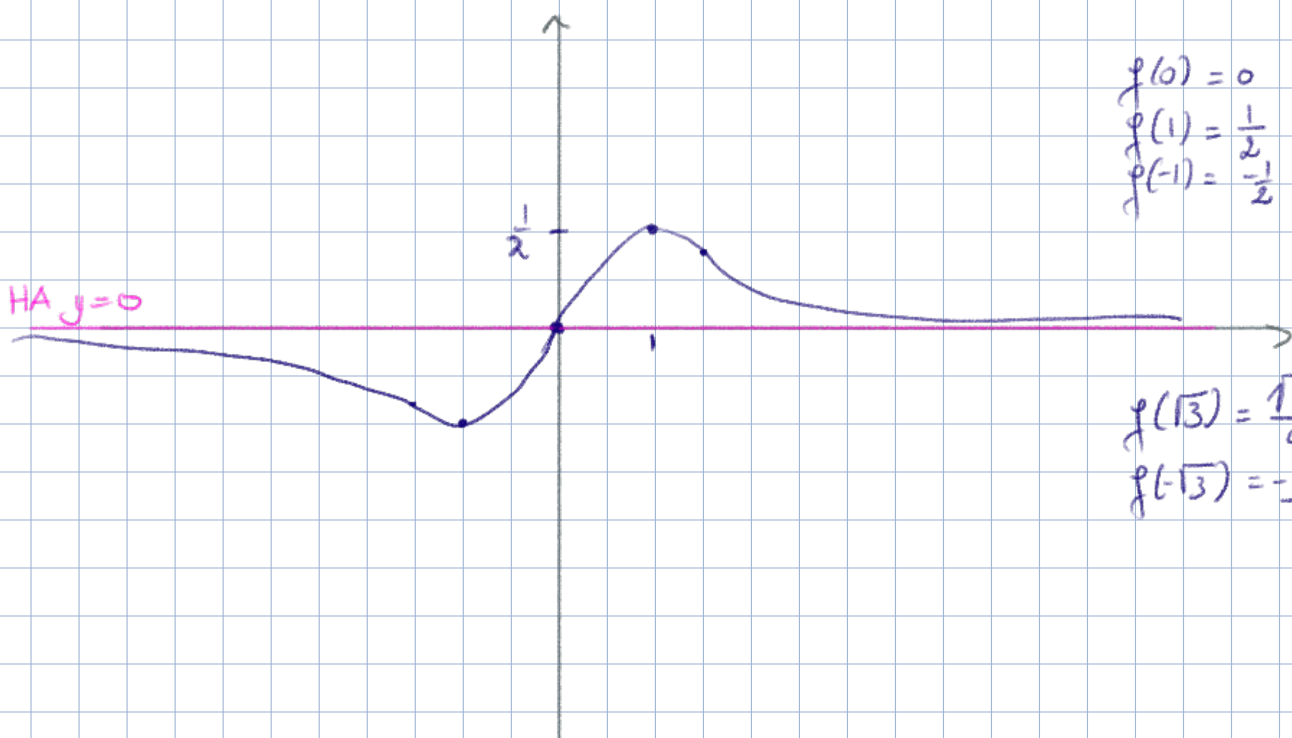
x		-1		1	
$f'(x)$	$-$		$+$		$-$
	$\searrow$	min	$\nearrow$	max	$\searrow$

5) $f''(x) = \dfrac{d}{dx}\left(\dfrac{1-x^2}{(x^2+1)^2}\right) = \dfrac{(x^2+1)^{\cancel{2}}(-2x) - (1-x^2)2\cancel{(x^2+1)}2x}{(x^2+1)^{\cancel{4}\,3}}$

$= \dfrac{-2x}{(x^2+1)^3}\left((x^2+1) + 2(1-x^2)\right) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$ ↖ always +

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
	$\frown$		$\smile$		$\frown$		$\smile$

$f''(x) = 0$ for $x = \pm\sqrt{3}$ and $x = 0$ (inflection pts)



$f(0) = 0$

$f(1) = \frac{1}{2}$

$f(-1) = \frac{-1}{2}$

$f(\sqrt{3}) = \frac{\sqrt{3}}{4}$

$f(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$

$f(x) = x^2 e^{-x^2}$

1) domain $\mathbb{R}$

2) $f(x) = f(-x)$ even function

3) Asymptotes

no VA (domain $\mathbb{R}$)

HA?

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^2 e^{-x^2} = \lim_{x\to\pm\infty} \frac{x^2}{e^{x^2}} \overset{H}{=} \lim_{x\to\pm\infty} \frac{\cancel{2x}^{\,1}}{\cancel{2x}e^{x^2}} = 0$$

↳ 2 sided HA $y=0$ (therefore, no OA)

4) $f'(x) = \frac{d}{dx}(x^2 e^{-x^2}) = 2xe^{-x^2} + x^2 \cdot (-2xe^{-x^2}) = (1-x^2)2xe^{-x^2}$

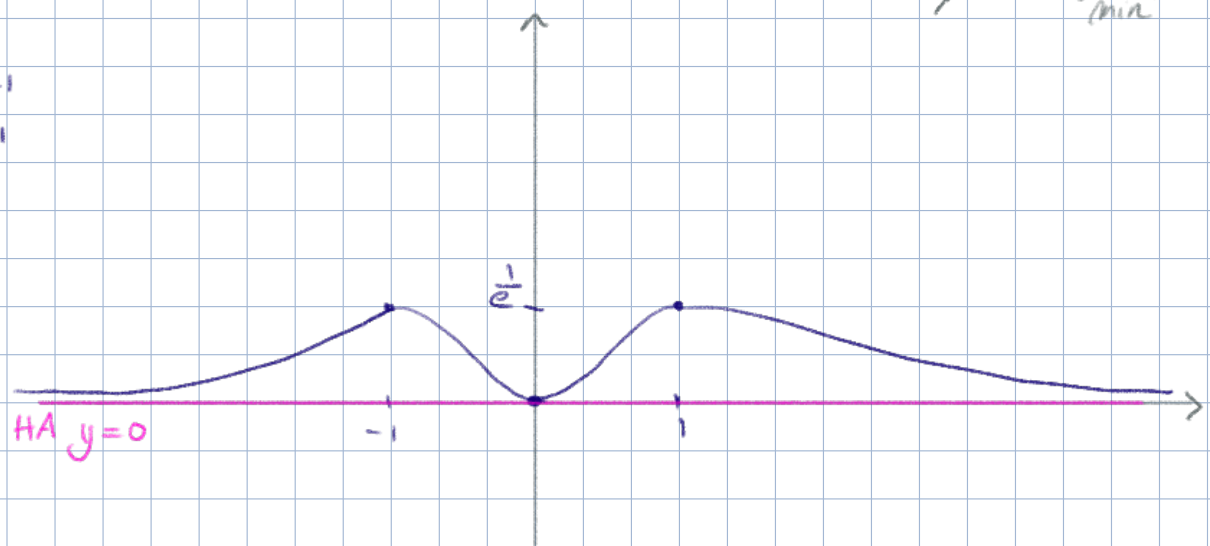
↳ $f'(x) = 0$ for $x = 0$ and $x = \pm 1$

x		-1		0		1	
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
	↗	max	↘	min	↗	max	↘

$f(0) = 0$

$f(1) = e^{-1}$

$f(-1) = e^{-1}$



$f(x) = \sin(x) + 2\sin(2x)$

1) domain $\mathbb{R}$

2) $f(x) = -f(-x)$ odd function

3) No asymptotes

$\longrightarrow$ no OA or HA, this is a periodic function

no VA, domain $\mathbb{R}$.

4) $f'(x) = \cos(x) + 4\cos(2x)$

$f'(x) = 0$ for $\quad \cos(x) + 4\cos(2x) = 0$

$\Leftrightarrow \cos(x) + 4(2\cos^2(x) - 1) = 0$

$\Leftrightarrow 8\cos^2(x) + \cos(x) - 4 = 0$

$\sqrt{129} \approx \sqrt{128} = 8\cdot\sqrt{2}$

$\hookrightarrow \cos(x) = \dfrac{-1 \pm \sqrt{129}}{16} \approx -\dfrac{1}{16} \pm \dfrac{\sqrt{2}}{2}$

$\longrightarrow$ 2 solutions for $\cos(x)$, 4 solutions for $x \in [0, 2\pi]$

$\hookrightarrow$ close to $\pm\dfrac{\sqrt{2}}{2}$ $\qquad$ $\hookrightarrow$ close to $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

5) $f''(x) = -\sin(x) - 8\sin(2x)$

the function is 2π-periodic

we sketch 1 period

$f(0) = 0$

$f(2\pi) = 0$

$f(\pi) = 0$

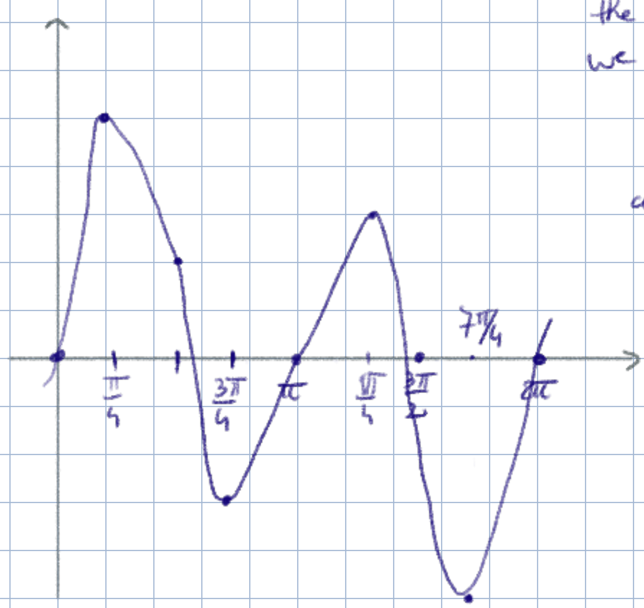
$f\left(\frac{\pi}{2}\right) = 1$

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + 2$

$f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - 2$

$f\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} + 2$

$f\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} - 2$



approx.

(this will not appear on the exam as a function sketch)

$f(x) = \frac{\sin(x)}{x}$

1) domain: $\mathbb{R} \setminus \{0\}$

2) $f(x) = +f(-x)$ even function

3) HA: $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{\sin(x)}{x} = 0$ HA $y = 0$ (no OA)

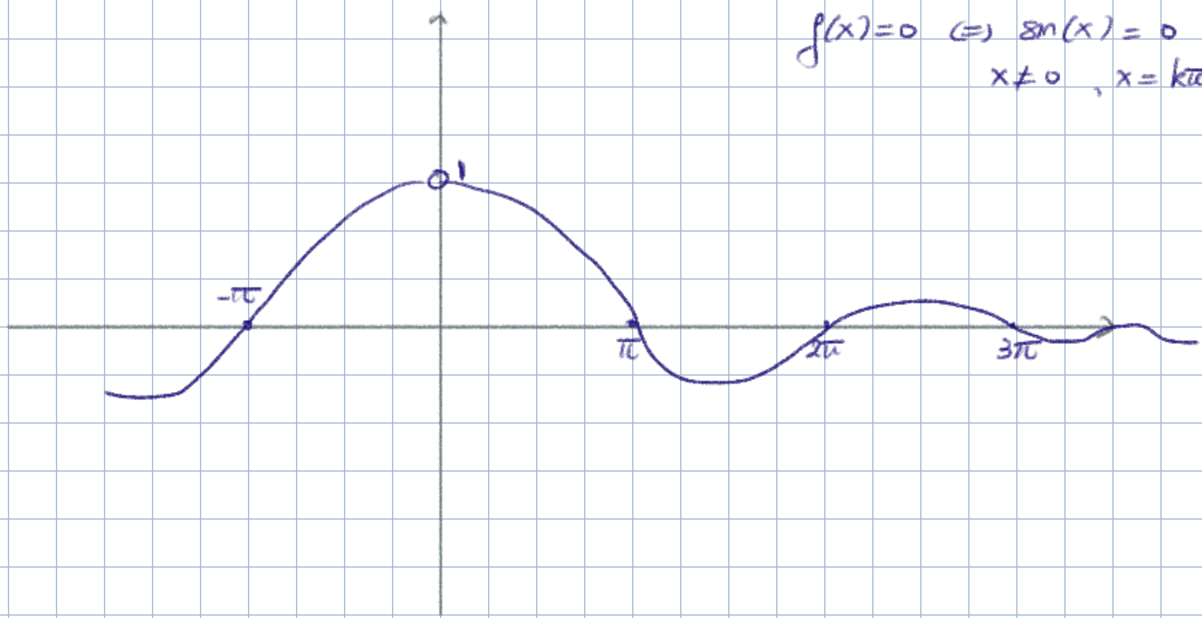
VA? $\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \longrightarrow$ NO VA.

4) $f'(x) = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$

$\hookrightarrow$ as $x \to \pm\infty$, the extrema approximate the zeros of $\cos(x)$

$f(x) = 0 \iff \sin(x) = 0$

$x \neq 0$, $x = k\pi$



$f(x) = x\sqrt{4-x^2}$

1) domain $\quad 4-x^2 \geq 0 \iff x \in [2,2]$

2) $f(-x) = -f(x)$ odd function

3) no asymptotes.

no OA / HA, since the domain is bounded

no VA, since no denominator / ln

4) $f'(x) = \sqrt{4-x^2} + x \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2}} = \dfrac{1}{\sqrt{4-x^2}}\left(4-x^2-x^2\right) = \dfrac{2(2-x^2)}{\sqrt{4-x^2}}$

$\to f'(x) = 0$ for $x = \pm\sqrt{2}$ (both are in the domain)

x	-2		$-\sqrt{2}$		$\sqrt{2}$		2	
$f'(x)$	////	$-$	0	$+$	0	$-$	////	
		$\searrow$	min	$\nearrow$	max	$\searrow$		

$\lim\limits_{x\to -2^+} f'(x) = -\infty$

$\lim\limits_{x\to 2^-} f'(x) = +\infty$

5) $f''(x) = 2\,\dfrac{(-2x)\cdot\sqrt{4-x^2} - (2-x^2)\cdot\dfrac{-2x}{\sqrt{4-x^2}}}{4-x^2} = \dfrac{(-4x)}{(4-x^2)^{3/2}}\left((4-\cancel{x^2})-(2-\cancel{x^2})\right)$

$= \dfrac{-8x}{(4-x^2)^{3/2}} \leftarrow$ always+ $\quad \to \begin{array}{l} f''(x) > 0 \text{ for } x<0 \text{ (convex)} \\ f''(x) < 0 \text{ for } x>0 \text{ (concave)} \end{array}$

$f(2) = f(-2) = f(0) = 0$

$f(-\sqrt{2}) = -\sqrt{2}\left(\sqrt{4-2}\right) = -2$

$f(\sqrt{2}) = 2$