Lecture 11 - Calculus

- Limits and continuity
- Differentiation + applications
- Integration
- Sequences and series
- Introduction to multivariate functions
- Double integrals: today last lecture!

Thomas' Ch. 15.1-2 or Adams' Ch. 14.1-2

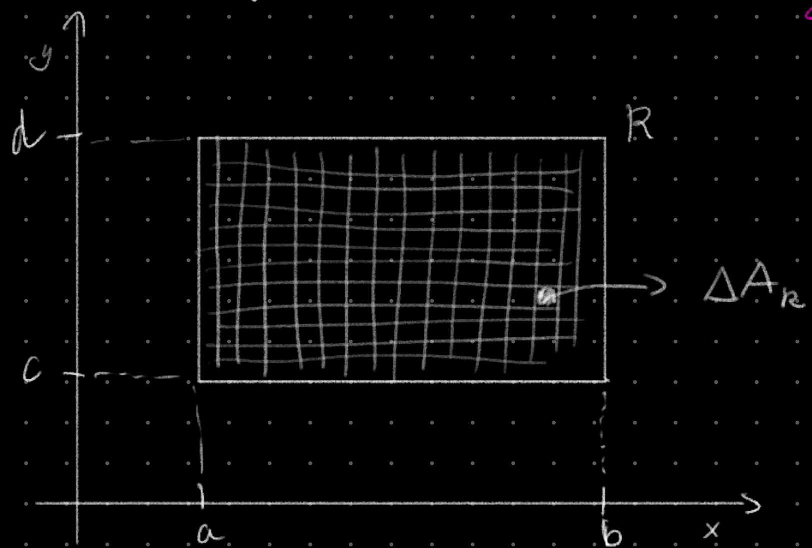
Double integrals

Let $f(x,y)$, continuous on a region R

$R: \; a \le x \le b$
$\quad\;\; c \le y \le d$

How to calculate $\iint_R f(x,y)\, dA$

↳ (signed) volume between R and the surface $f(x,y)$



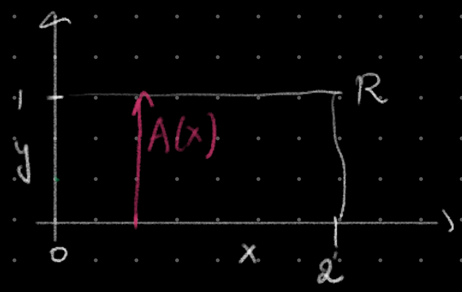
$$\lim_{n\to\infty}\left(\sum_{k}^{n} \Delta A_k \cdot f(x_k,y_k)\right) \quad \text{(Riemann sum)}$$

$$= \iint_R f(x,y)\, dA \quad \text{(if this converges)}$$

Calculation of a double integral

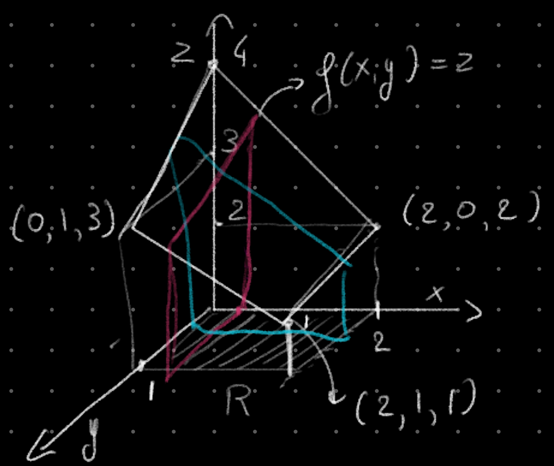
Example : $f(x,y) = 4 - x - y$

$R: \; 0 \le x \le 2$
$\quad\;\; 0 \le y \le 1$



$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^1 f(x,y)\,dy\,dx$$

inner integral $A(x)$



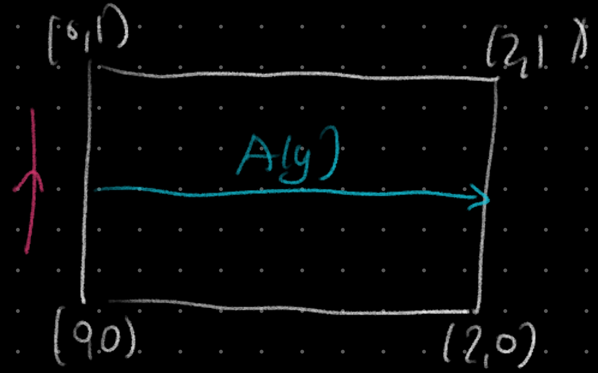
$$A(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 (4 - x - y)\,dy = \left[4y - xy - \frac{y^2}{2}\right]_0^1$$

$$= 4 - x - \frac{1}{2} = \frac{7}{2} - x$$

$$\iint_R f(x,y)\,dA = \int_0^2 A(x)\,dx = \int_0^2 \left(\frac{7}{2} - x\right)dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2$$

$$= 7 - 2 = 5$$



$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^2 f(x,y)\,dx\,dy$$

inner integral $A(y)$

$$A(y) = \int_0^2 (4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_0^2$$

$$= 8 - 2 - 2y = 6 - 2y$$

$$\int_0^1 A(y)\,dy = \int_0^1 (6-2y)\,dy = \left[6y - y^2\right]_0^1 = 6 - 1 = 5$$

Fubini's theorem

If $f(x,y)$ is continuous on the rectangular area R: $a \le x \le b$, $c \le y \le d$,

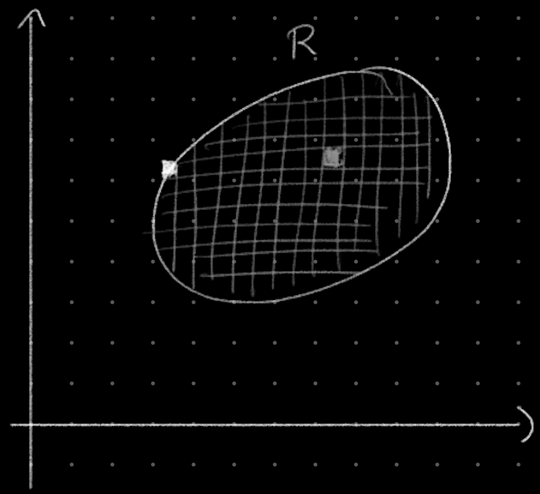
then $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

Example: $\int_{-1}^{1} \int_{0}^{1} (x+y)\,dy\,dx$ (so cative)

$$\int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$$

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)dx = \int_{-1}^{1} \frac{1}{2}\,dx = 1$$

Double integrals over general regions

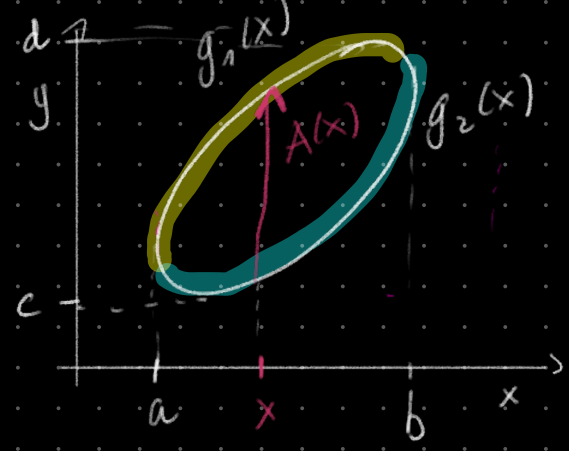


$\iint_R f(x,y)\,dA$ is the volume between $f(x,y)$ and R

The double integral is the limit of the Riemann sums

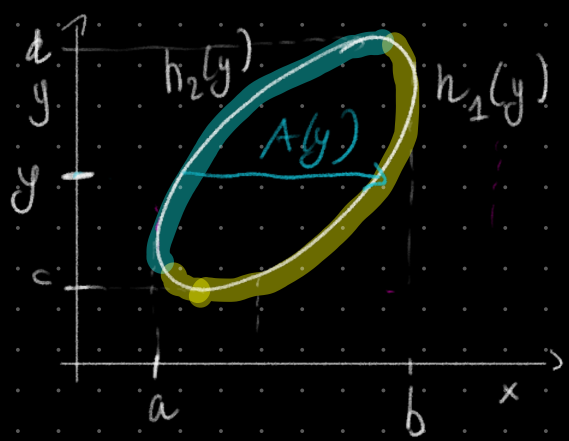
$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

Calculating integrals over general regions



$$\iint_R f(x,y)\,dA = \int_a^b \boxed{\int_{g_2(x)}^{g_1(x)} f(x,y)\,dy}\,dx$$

inner integral A(x)



$$\iint_R f(x,y)\,dA = \int_c^d \boxed{\int_{h_2(y)}^{h_1(y)} f(x,y)\,dx}\,dy$$

inner integral A(y)

Fubini's theorem (stronger form)

If $f(x,y)$ is continuous on a region R

→ if R is defined as $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$
with $g_1(x)$ and $g_2(x)$ continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

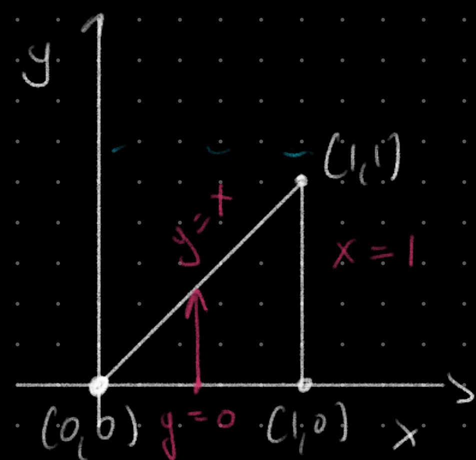
→ if R is defined as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$
with $h_1(y)$ and $h_2(y)$ continuous on $[c,d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example

$f(x,y) = 3 - x - y$ $\qquad$ R = triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

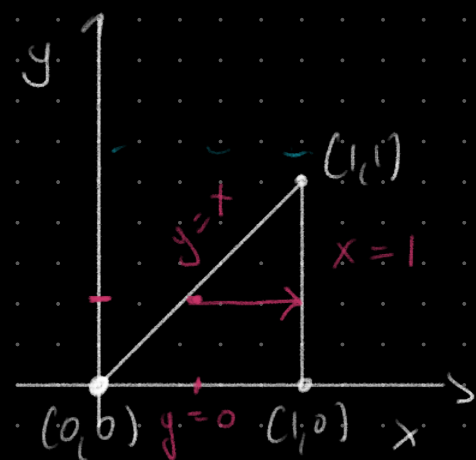


$$A(x) = \int_0^x (3 - x - y)\,dy = \left[3y - xy - \frac{y^2}{2}\right]_0^x$$

$$= 3x - x^2 - \frac{x^2}{2} = 3x - \frac{3}{2}x^2$$

$$\int_0^1 \left(3x - \frac{3}{2}x^2\right) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{2}\right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2} = 1$$



$$\iint_R f(x,y)\,dA = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy$$

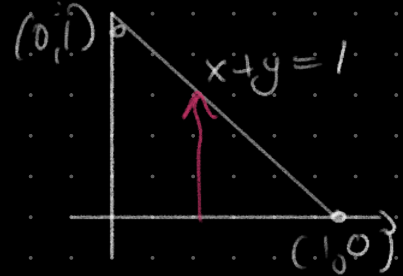
$$A(y) = \int_y^1 (3 - x - y)\,dx = \left[3x - \frac{x^2}{2} - xy\right]_y^1$$

$$= -\left(3y - \frac{y^2}{2} - y^2\right) + \left(3 - \frac{1}{2} - y\right) = \frac{3y^2}{2} - 4y + \frac{5}{2}$$

$$\iint_R (x+y)\, dA = \int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$$

R : triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$

(sorr)



$x : 0 \to 1$

$y : 0 \to 1-x$

$$\int_0^{1-x} (x+y)\, dy = \left[xy + \frac{y^2}{2} \right]_0^{1-x} = x(1-x) + \frac{(1-x)^2}{2}$$

$$= \cancel{x} - x^2 + \frac{1}{2} - \cancel{x} + \frac{x^2}{2}$$

$$= -\frac{x^2}{2} + \frac{1}{2}$$

$$\frac{1}{2}\int_0^1 (1-x^2)\, dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

Properties of double integrals

For $f(x,y)$, $g(x,y)$ continuous on a bounded region R

* $\iint_R c \cdot f(x,y)\,dA = c \cdot \iint_R f(x,y)\,dA$ for $c \in \mathbb{R}$

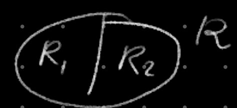
* $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

* if $f(x,y) \geq 0$ on R, then $\iint_R f(x,y)\,dA \geq 0$

  ↳ choose parametrization correctly!

* if $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

* if R is partitioned into R_1 and R_2



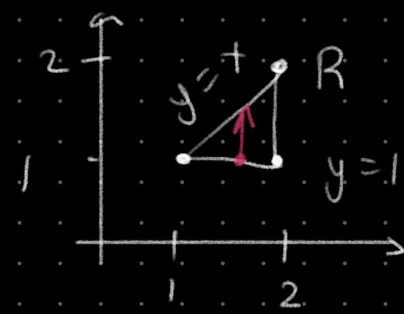
  then $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Examples

(Last year's exam)

$$\iint_T \frac{dA}{(x+y)^3} \quad \text{with } T \text{ the triangle with vertices } (1,1), (2,1), (2,2)$$

$$\int_1^2 \int_1^x \frac{dy\,dx}{(x+y)^3}$$



$$\int_1^x \frac{dy}{(x+y)^3} = \left[\frac{-1}{(x+y)^2}\right]_1^x \cdot \frac{1}{2}$$

$x : 1 \longrightarrow 2$

$y : 1 \longrightarrow x$

$$= \frac{1}{2}\left(\frac{-1}{(2x)^2} - \frac{-1}{(x+1)^2}\right)$$

$$= \frac{1}{2}\frac{1}{(x+1)^2} - \frac{1}{8x^2}$$

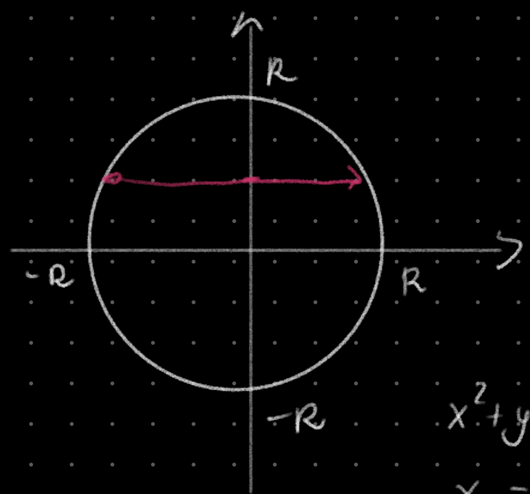
$$\int_1^2 \left(\frac{1}{2}\frac{1}{(x+1)^2} - \frac{1}{8x^2}\right) dx$$

(result: $\frac{1}{48}$)

$$= \frac{1}{2}\int_1^2 \frac{dx}{(x+1)^2} - \frac{1}{8}\int_1^2 \frac{dx}{x^2}$$

$$= \frac{1}{2}\left[\frac{-1}{x+1}\right]_1^2 - \frac{1}{8}\left[\frac{-1}{x}\right]_1^2 = \frac{1}{2}\left(\frac{-1}{3}+\frac{1}{2}\right) - \frac{1}{8}\left(\frac{-1}{2}+1\right)$$

$$= \underbrace{-\frac{1}{6}+\frac{1}{4}}_{\frac{1}{12}} \underbrace{+\frac{1}{16}-\frac{1}{8}}_{\frac{-1}{16}} = \frac{1}{4}\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\frac{1}{12}} = \frac{1}{48}$$

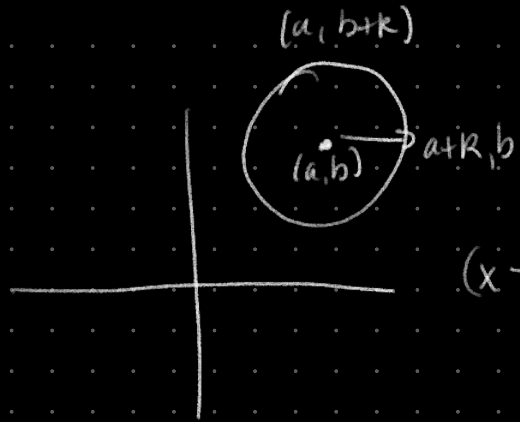


$y : -R \to R$

$x : -\sqrt{R^2-y^2} \to +\sqrt{R^2-y^2}$

$x^2+y^2=R^2$

$x = \pm\sqrt{R^2-y^2}$



$y : b - R \longrightarrow b + R$

$x : a - \sqrt{} \longrightarrow a + \sqrt{}$

$$(x-a)^2 + (y-b)^2 = R^2$$

$$x = a \pm \sqrt{R^2 - (y-b)^2}$$

$$\int_0^1 \frac{2x}{x^2-1}\,dx = \lim_{a \to 1^-} \int_0^a \frac{2x}{x^2-1}\,dx = \lim_{a \to 1^-} \int_{-1}^{a^2-1} \frac{du}{u} = \lim_{a \to 1^-} \ln|u| \Big|_{-1}^{a^2-1}$$

$u = x^2 - 1$

$du = 2x\,dx$

$$= \lim_{a \to 1^-} \ln|a^2-1|$$

$$= -\infty$$