Calculus: overview

1. Real functions and continuity
2. Limits (how does a function behave towards the edges of the domain?)
3. Derivatives (slope of the tangent line - how does a function change?)

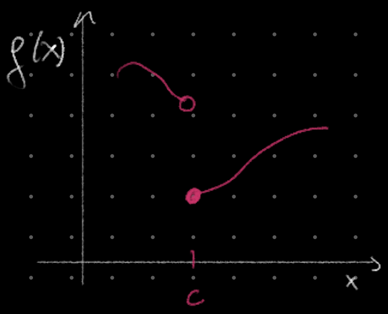
Today: applications of differentiation

- L'Hôpital rules for calculating limits
- Extreme values
    - Increasing and decreasing functions
    - Global and local extrema
    - Min-max theorem
- Concavity and inflections
- Sketching functions

Adams' Ch. 4.3, 4.4, 4.5, 4.6

Recap: continuity and differentiability

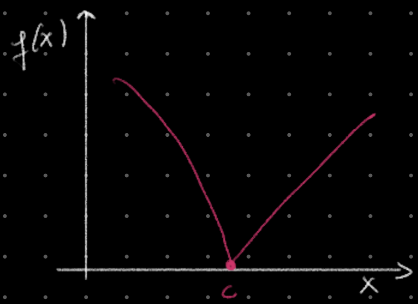
- A function $f(x)$ is continuous at $x=c$ if $f(x)$ approaches $f(c)$ as x approaches c (from both sides) $=$ no gap at $x=c$
- A function $f(x)$ is differentiable at $x=c$ if the tangent line exists at $x=c$ (i.e. the tangent line has a unique (and finite) slope) $= f(x)$ is smooth at $x=c$



not continuous at $x=c$
not differentiable at $x=c$

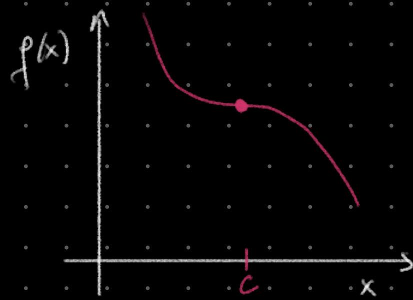
$\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$
left and right limit are different

there is (by default) not a unique tangent line at $x=c$



continuous at $x=c$
not differentiable at $x=c$

$f'_+(c) \neq f'_-(c)$
(there is not a unique tangent line) at $x=c$



continuous at $x=c$
differentiable at $x=c$

Indeterminate forms

- $\lim_{x \to 0^+} x \ln(x)$ $[0 \cdot \infty]$
- $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$ $[1^\infty]$
- $\lim_{x \to +\infty} \left(e^x - e^{(x^2)}\right)$ $[\infty - \infty]$ $\sqrt{x^2+x} - x$
- $\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x^2} - 16}$ $\left[\frac{0}{0}\right]$
- $\lim_{x \to +\infty} x^{\frac{1}{x}}$ $[\infty^0]$
- $\lim_{x \to \infty} \frac{\ln(x)}{x}$ $\left[\frac{\infty}{\infty}\right]$
- $\lim_{x \to +\infty} \left(\frac{1}{x}\right)^{\frac{1}{x}}$ $[0^0]$

1st L'Hôpital rule

(For indeterminate forms $\left[\frac{0}{0}\right]$)

For f(x) and g(x) differentiable functions on (a,b), and g'(x)≠0

↳ can be $(-\infty, +\infty)$

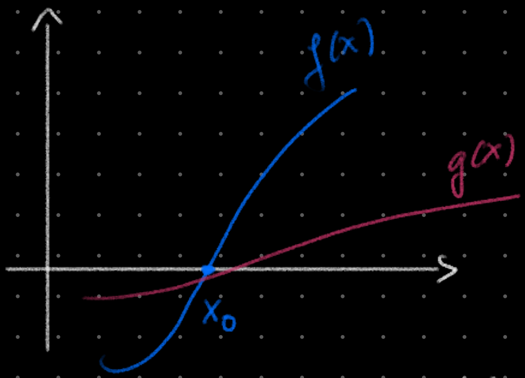
IF $\begin{cases} \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0 \text{ for some } x_0 \in [a,b] \\ \lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)} = L \text{ (can be } \infty, \text{ but exists)} \end{cases}$ $\overset{\Longrightarrow}{\text{then}}$ $\displaystyle \lim_{x\to x_0} \frac{f(x)}{g(x)} = L$

(Also true for left / right limits)

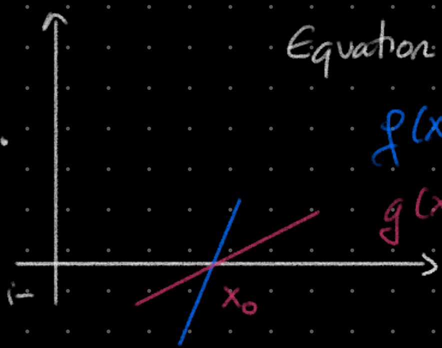
Intuitive explanation: we can replace the functions f(x) and g(x) by their tangent lines

$f(x_0) = 0$

$g(x_0) = 0$



approx. $\approx$



Equation of tangent lines

$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$

$g(x) \simeq g(x_0) + g'(x_0)(x - x_0)$

↑ slope = derivative

$$\frac{f(x)}{g(x)} \approx \frac{\cancel{f(x_0)}^{0} + f'(x_0)(x - x_0)}{\cancel{g(x_0)}_{0} + g'(x_0)(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}$$

around x_0, the approximation is good

2nd L'Hôpital rule

For f(x) and g(x) differentiable functions on (a,b), and g'(x)≠0

↳ can be $(-\infty, +\infty)$

IF $\begin{cases} \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \pm\infty \\ \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L \text{ (can be } \infty, \text{ but exists)} \end{cases}$ $\xRightarrow{\text{then}}$ $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L$

(Also true for $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)}$)

- $$\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x^2} - 16} \overset{H}{=} \lim_{x \to 64} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{3} x^{-1/3}} = \lim_{x \to 64} \frac{3 \sqrt[3]{x}}{4 \sqrt{x}} = \frac{3 \cdot \cancel{4}}{\cancel{4} \cdot 8}$$
  $$= \frac{3}{8}$$

- $$\lim_{x \to \infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

- $$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

- $$\lim_{x \to +\infty} x^{\frac{1}{x}} = \lim_{x \to \infty} \left(e^{\ln\left(x^{\frac{1}{x}}\right)}\right)$$
  $$= e^{\lim_{x \to \infty} \left(\ln\left(x^{\frac{1}{x}}\right)\right)}$$
  $$= e^{0} \quad \text{(see above)}$$
  $$= 1$$

for a continuous function f, if $\lim_{x \to x_0} g(x) = L$ exists, then $\lim_{x \to x_0} f(g(x)) = f(L)$

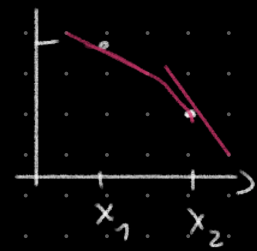
continuous functions and existing limits can be switched.

Increasing and decreasing functions

- increasing function: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

  e^x, x^2 on $(0, \infty)$

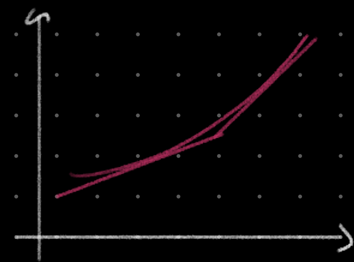
injective



- decreasing function: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

  $-x$, $\cos(x)$ on $(0, \pi)$

- non-increasing: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$



- CONNECTION WITH $f'(x)$:

  $f'(x) < 0 \Rightarrow f(x)$ is decreasing

  $f'(x) > 0 \Rightarrow f(x)$ is increasing

  on an interval

Extreme values

- **Absolute minimum/maximum:**

  $f(x)$ has an absolute $\underset{\text{maximum}}{\text{minimum}}$ at $x = x_0$, if, for all $x \in \text{domain}(f)$,
  $$f(x) \underset{\leq}{\geq} f(x_0)$$

- **Min-max theorem:**

  A continuous function f on a closed and bounded domain $[a,b]$ always has an absolute minimum and maximum.

- **Local minimum/maximum**

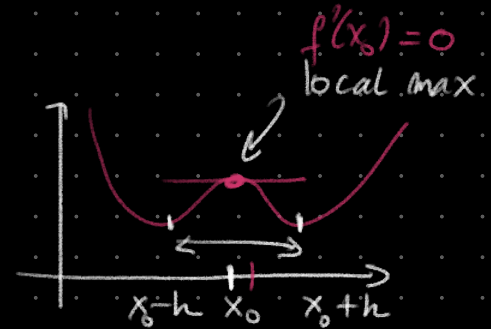
  $f(x)$ has a local $\underset{\text{maximum}}{\text{minimum}}$ at x_0, if there is a neighbourhood $(x-h, x+h)$ $(h>0)$, on wich x_0 is a (absolute) extremum.)

CONNECTION WITH $f'(x)$: if

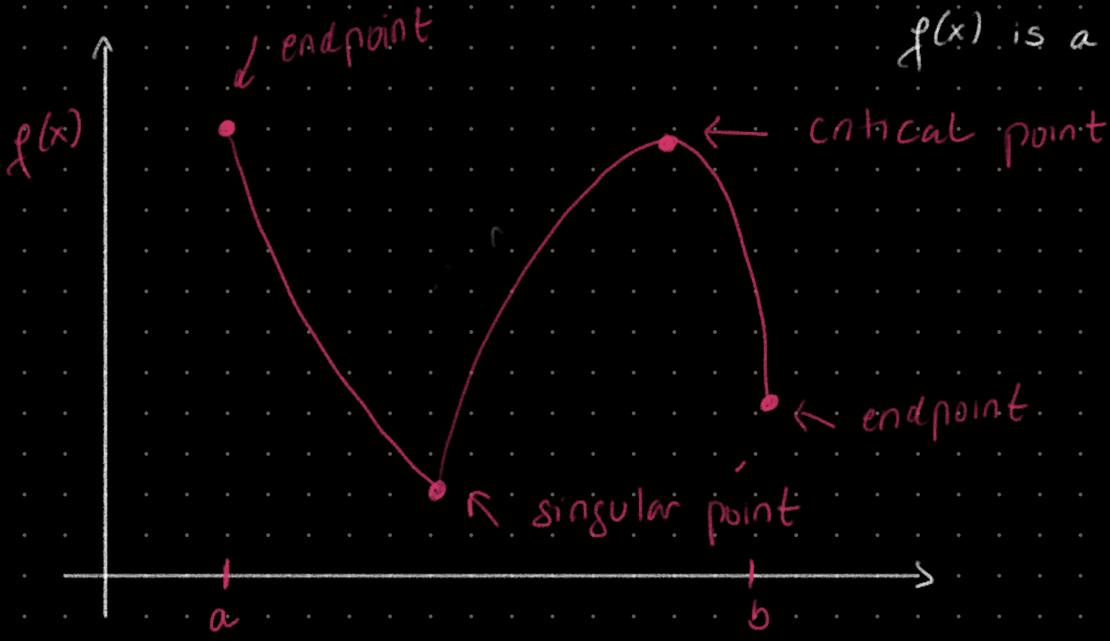
↳ $f'(x_0) = 0$ if

$f'(x_0)$ exists

if $f'(x_0) = 0$, x_0 is CRITICAL POINT



Extreme values



$f(x)$ is a continuous function on $[a,b]$

$f(x) = e^{-x^2}$

$f(0) = 1$

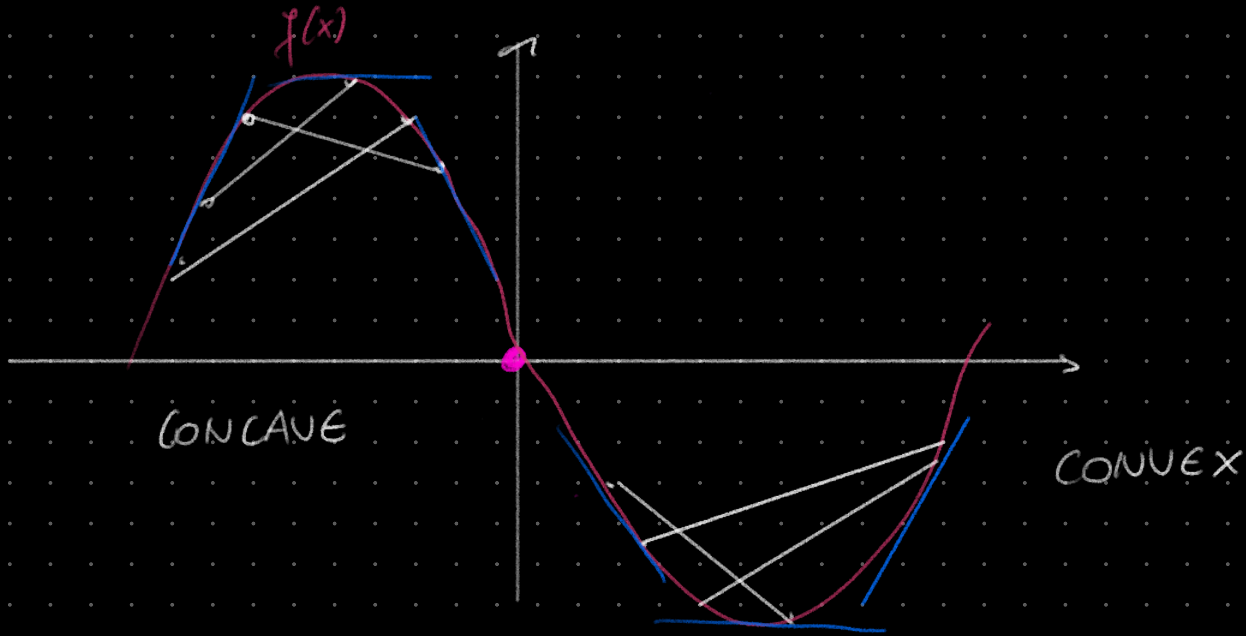
- For a continuous function $f(x)$ on $\begin{cases} \text{an OPEN domain } (a,b) \\ \text{on } \mathbb{R} \end{cases}$

if $\lim_{x \to a^+} f(x) = L$ and there is an x, such that

$\lim_{x \to b^-} f(x) = M$ $\qquad f(x) \underset{\le}{\ge} M$ and $f(x) \underset{\le}{\ge} L$

then there is an absolute $\underset{\text{min}}{\text{max}}$

Concavity and inflections



Convex (concave up) $f''(x) > 0$

all chords above function
all tangent lines below function

Concave (concave down) $f''(x) < 0$

all chords below function
all tangent lines above function

Inflection point: concavity of $f(x)$ changes, if $f''(x)$ exists
$f''(x) = 0$

Sketching functions

- Domain
- Even/odd
- Asymptotes
- First derivative? Increasing and decreasing intervals, extreme values
- Second derivative? Concave and convex intervals, inflection points
- Sketch

Example: $f(x) = x^2 e^{-x}$