Maastricht University
Department of Advanced Computing Science

Algorithmic Design

Week 1:
Introduction, Greedy Algorithms, and More!

David Mestel and **Steven Chaplick**

2024-2025 Period 5

Practicalities

- Steven Chaplick (office: PHS1 C4.006, s.chaplick@maastrichtuniversity.nl)
- David Mestel (office: PHS1 C4.008, david.mestel@maastrichtuniversity.nl)

Lectures:

- Tuesdays & Thursdays, check **timetable** sometimes we have a break!
- Materials posted on Canvas under Modules

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Assessment
- 4 Problem sets (5% each) with programming and mathematical tasks. Submissions via CodeGrade, and Canvas.
- Try using LaTeX – overleaf.com is a great way to get started :)
- Final Exam (80%)

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Background: Data Strctures and Algorithms, Discrete Mathematics

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Complexity Analysis:

- How to measure the “speed” of an algorithm.

Here: deeper look into algorithmic paradigms, and when tasks are inherantly difficult to solve computationally.

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We will assume:

- You know how to build/use data structures
- You can do basic complexity analysis $\rightsquigarrow$ asymptotic runtime, Big-$\mathcal{O}$
- Can apply standard proof techniques $\rightsquigarrow$ cases, contradiction, induction, etc.

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To prove a statement is correct various techniques can be used:

- Proof by cases, proof by contradiction, induction, etc.

Picking the different methods can make the statement easier or more difficult to prove.

- Learning these general techniques gives us tools to find the “easy” proof.

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- Greedy, divide-and-conquer, dynammic programming, etc.

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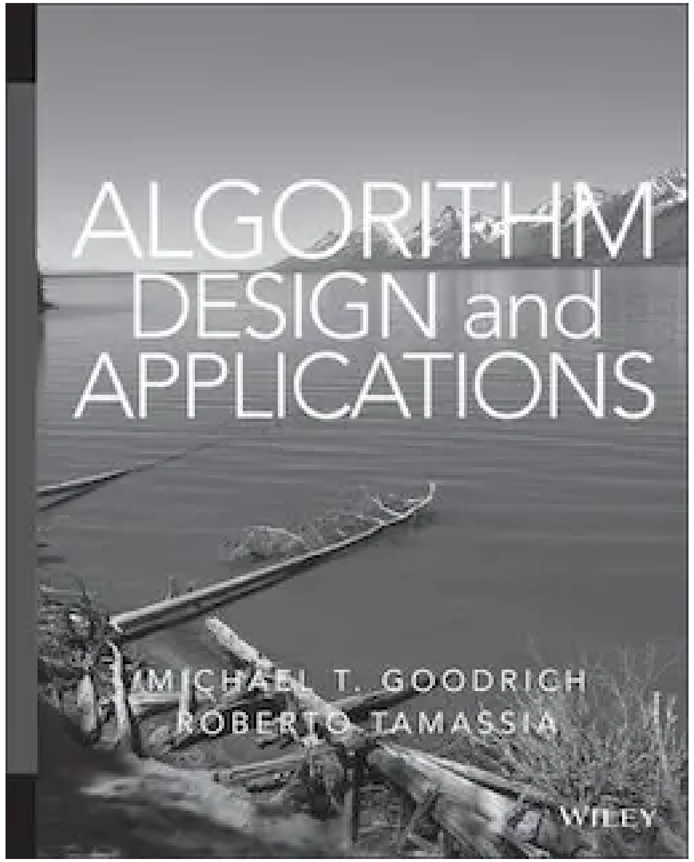
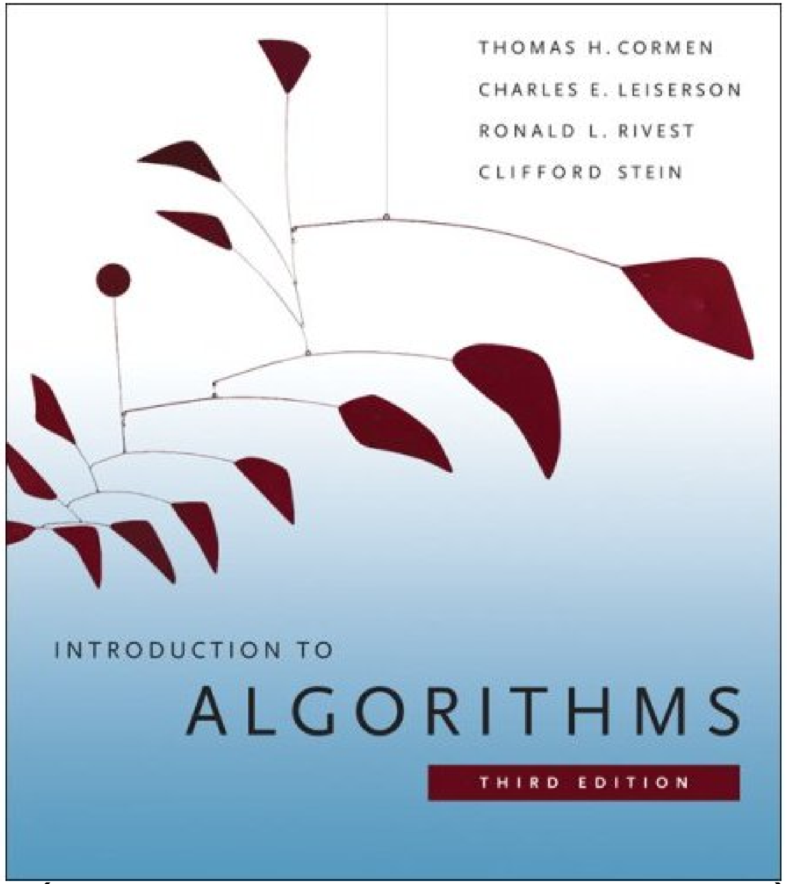
Course Outline:

- Start with these paradigms and corresponding tasks that can be solved with them
- Some tasks lack efficient algorithms to solve them, why? Classification of NP-hard problems.
- But we still need to tackle those problems $\leadsto$ approximations, heuristics, etc.

Books

The ‘official’ textbook for the course is [GT] Goodrich & Tamassia, ‘Algorithm Design and Applications’





(additional standard book) ‘Introduction to Algorithms’ [CLRS]

Week 1:

Greedy Algorithms: [GT §10, CLRS §16]

- Pick what looks the best step-by-step.

Amortised Analysis: [GT §1.4, CLRS §17]

- A different view on measuring runtime of algorithms.

Huffman Codes: [GT §10.3, CLRS §16.3]

- A simple but optimal* compression

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* Beware when optimality is claimed, there can be some strange conditions attached ;)

Greedy algorithms

- **Greedy** means making the choice that seems best now, without thinking about the future
- Algorithmically, “do the what looks the best each step”
- But what does **seems best now** mean? Need to pick the right measure of “best”.

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did you spot the termination bug here?

When greediness goes wrong

Suppose our coins not only have a value, but also a weight.
And, we cannot just take any 10, we are limited by the **weight** of the coins we pick.

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Now it becomes a KnapSack problem:

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Input: Given a weight capacity $W \geq 0$ and n objects with non-negative weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$ respectively, i.e., object i has weight w_i and value v_i.

Goal: Pick a subset S of the given objects of maximum value and total weight at most W.

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Greedy gets the first coin (value$=5$), but it's better to take the other two coins (value$=6$).

When can we use a greedy approach?

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Outline of a greedy algorithm:

- View the problem as a series of choices
- At every point, make the choice that most improves the objective function
- Do this until you have made enough choices to have a solution

Given a particular problem there is not one, single, greedy algorithm for it

Challenges in Greedy Design

- Different algorithms result from varying the objective function, and the starting conditions.
- How to decide which choices to make?
  - It is important that the start position and the rule for choosing a next move allow for *enough* configurations to be reached
- Deciding on an objective function:
  - Technically, the objective function only applies when the configuration is complete
  - One has to create a function to apply to incomplete configurations
  - Usually this is straightforward, but different choices can lead to different algorithms

So when do we use a greedy approach?

Generally we need three properties

1. Optimal substructure:
   - When an optimal solution can be constructed from optimal solutions of its subproblems. ["Grow" a solution without chaging the past]
2. Independent subproblems
   - Two subproblems are independent if they do not share the same resources (Alternatively, two subproblems are independent if the solution to one does not affect the solution to the other)
3. Non-overlapping subproblems
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Let’s try another problem, this time on graphs.

Graphs: (road) networks, relation diagrams, etc.

What is a graph?

- graph G
- vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$
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  where each edge is a pair of vertices.

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- Set notation

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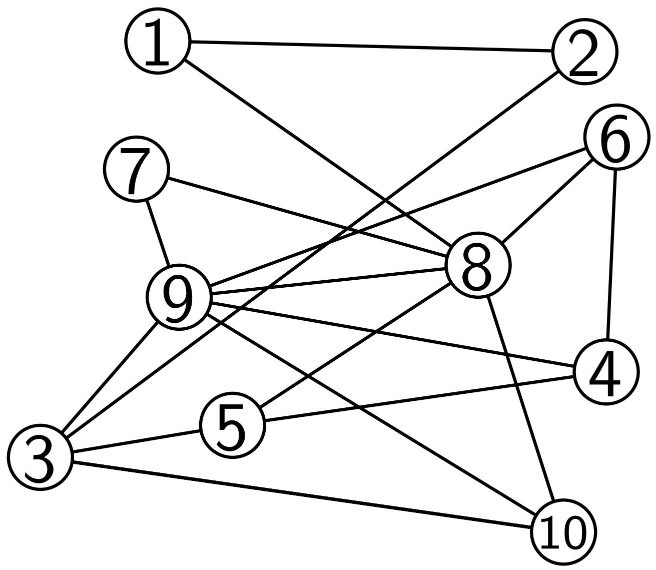
- Drawing



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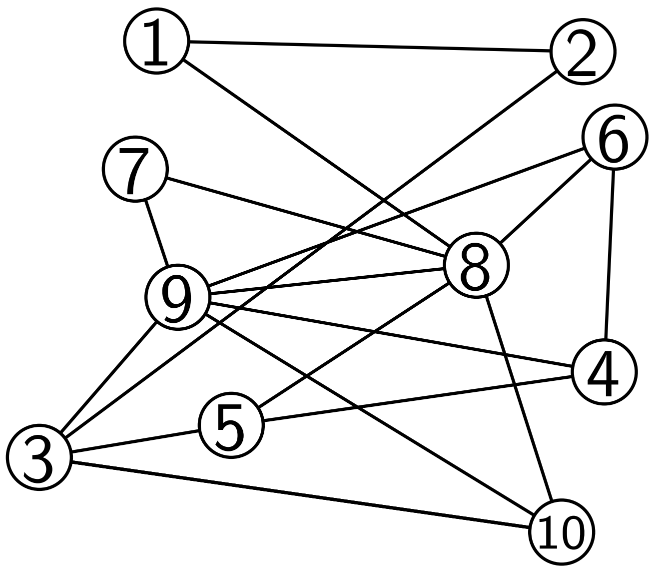

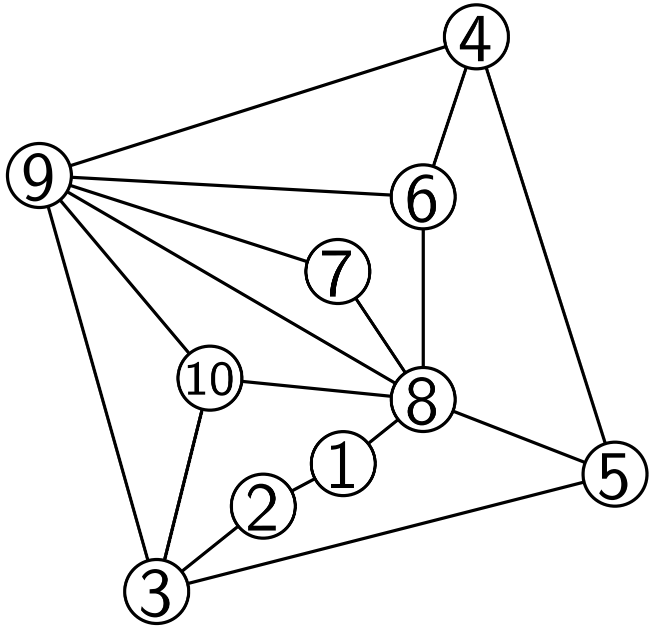

Problem: Efficient Sub-Network Construction

- Need to build a fibre optic network for my kingdom connecting all important locations.
- For ease of planning, cables are built along the road network.
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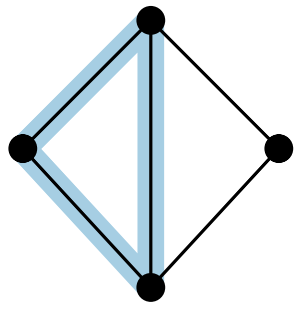
Should our fibre network contain **cycles** ?



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- Need to build a fibre optic network for my kingdom connecting all important locations.
- For ease of planning, cables are built along the road network.
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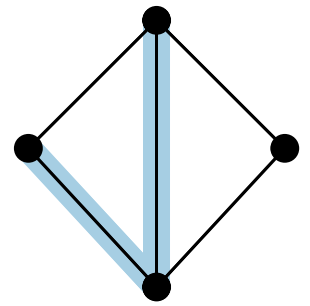
Represent road network as a **graph**: road segments are the edges, locations are the vertices.

Should our fibre network contain **cycles** ?

No, the solution should be a **tree**.



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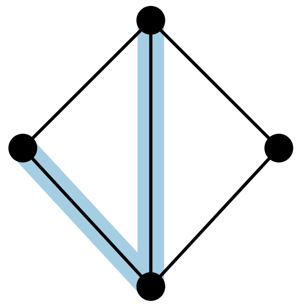

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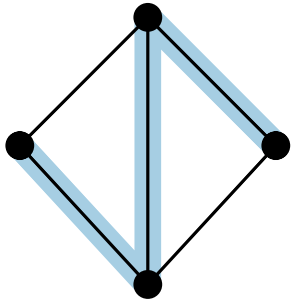

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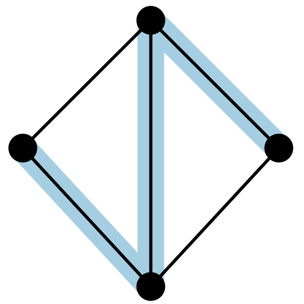

MINIMUM SPANNING TREE (MST)

Input: An n-vertex graph $G = (V, E)$, and edge costs $c : E \to \mathbb{R}$.

Goal: Return a tree T that *spans* G, i.e., $V(T) = V$ and $E(T) \subset E$, where the total edge cost, $\sum_{e \in E(T)} c(e)$, is minimized.

Greedy Algorithm for MST (Prim), [GT §15, CLRS §23]

What are our partial solutions?

How do we "grow" a partial solution greedily?

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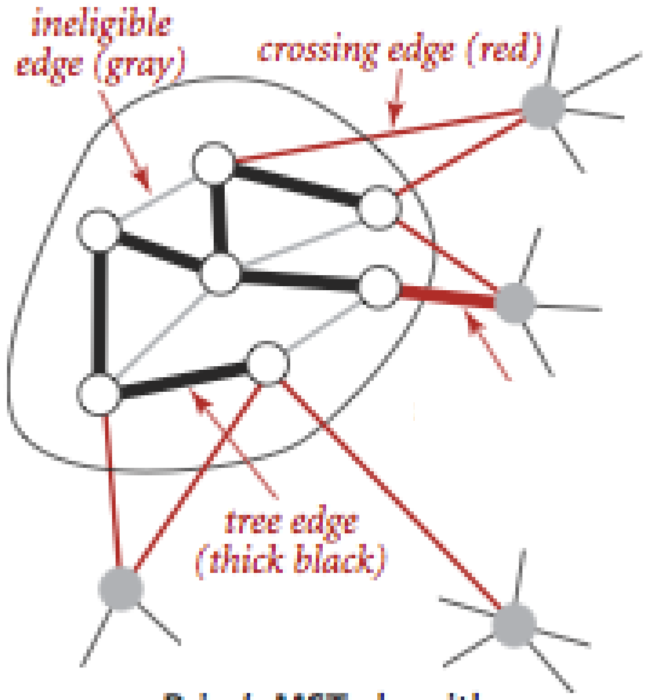

Prim's MST algorithm

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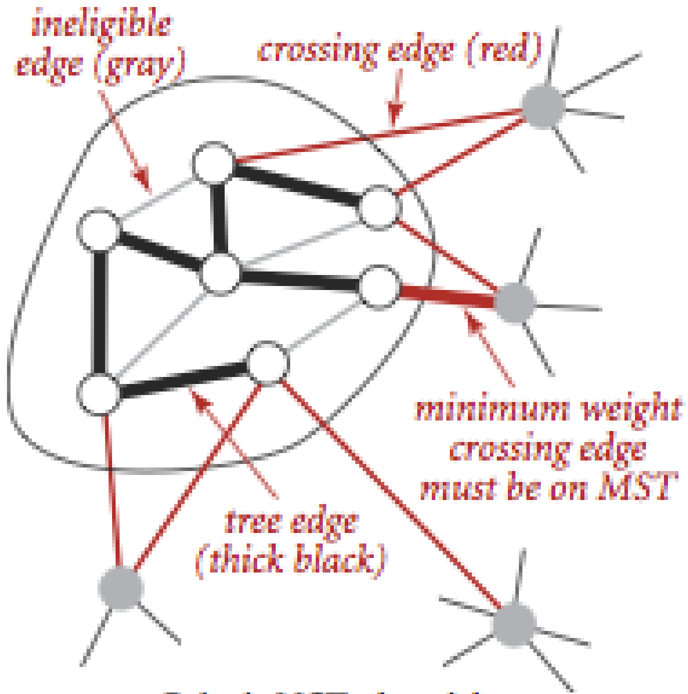

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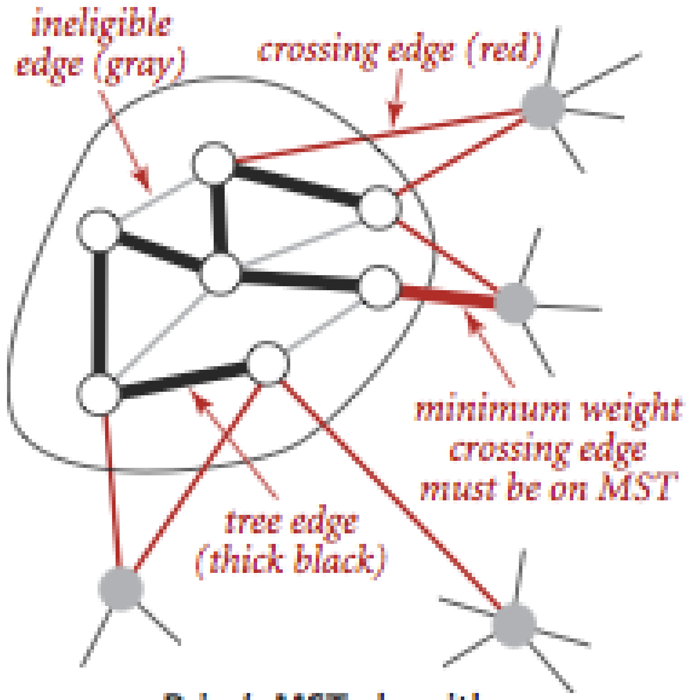

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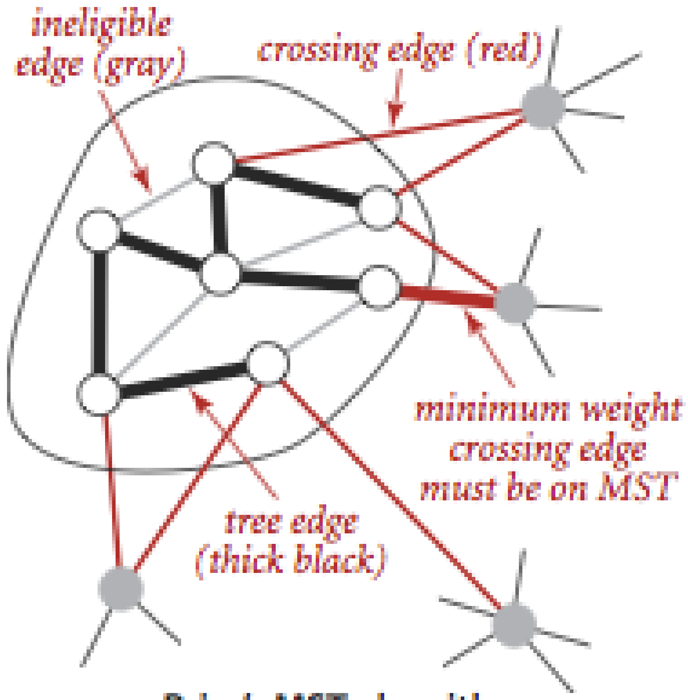

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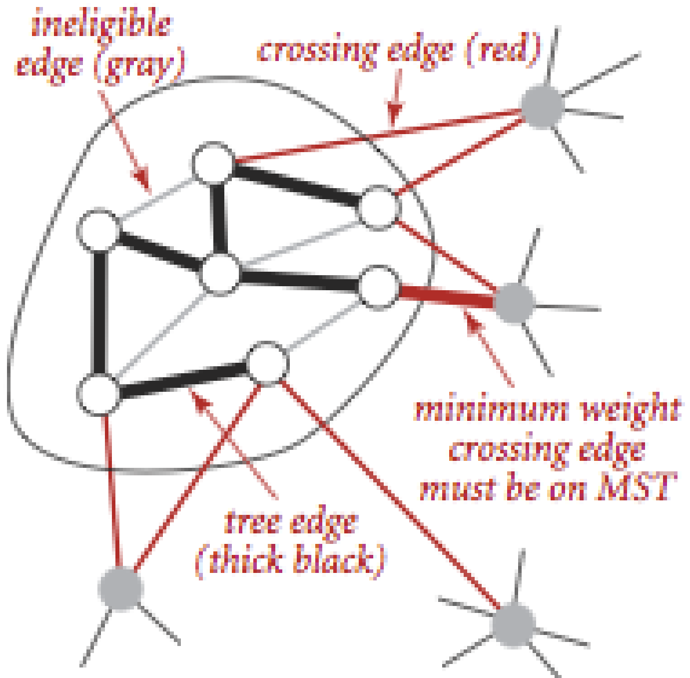

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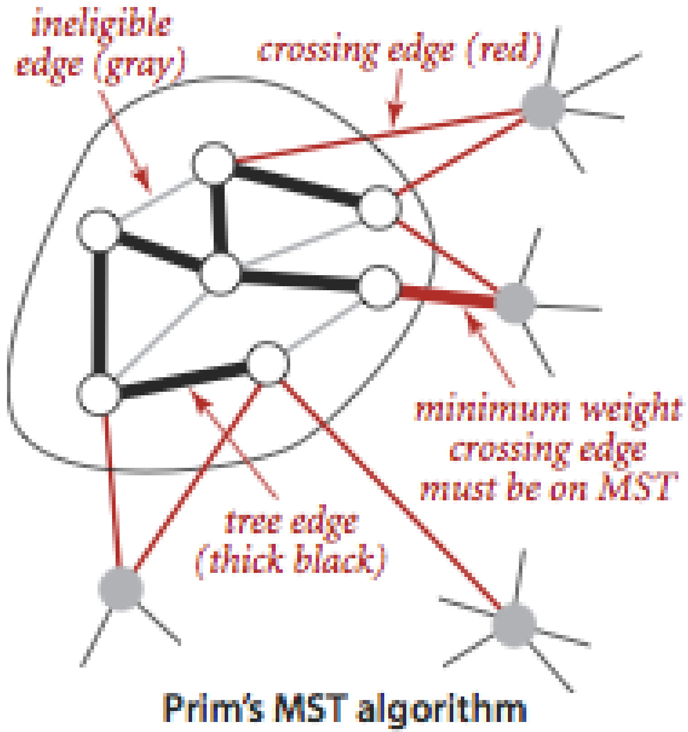

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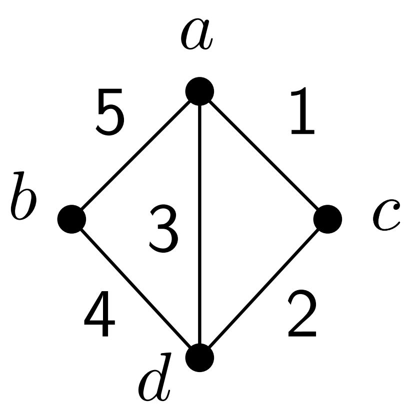
Exercises:

- Formalize this algorithm in pseudocode.
- Analyze the runtime. Think carefully about how you should represent the graph.

But wait, does that really work?

Quick sanity check:



iteration:	
Vertices in our Tree	
Edges (by cost) in the Heap:	

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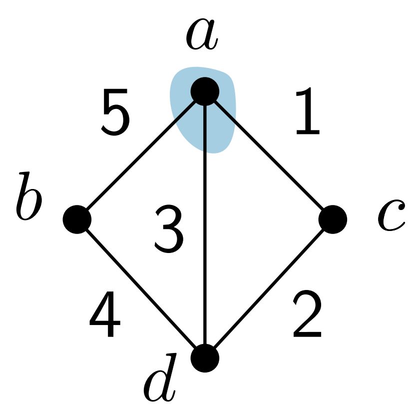

iteration:	1	2	3	4
Vertices in our Tree	$\{a\}$			
Edges (by cost) in the Heap:	$\{1, 3, 5\}$			

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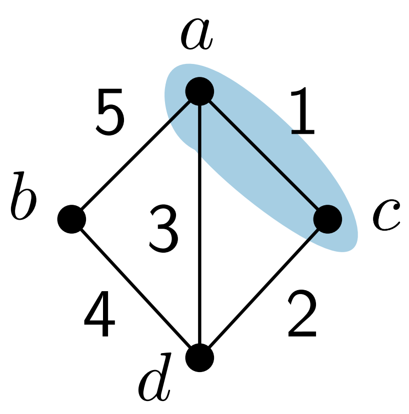

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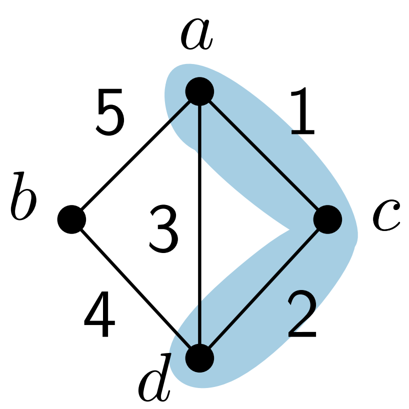

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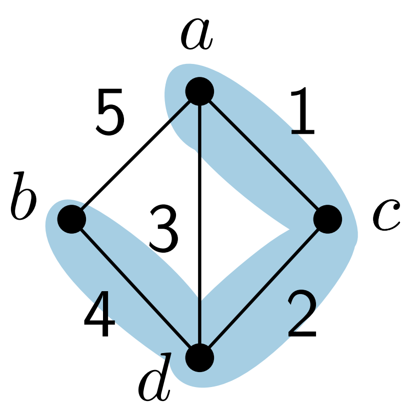

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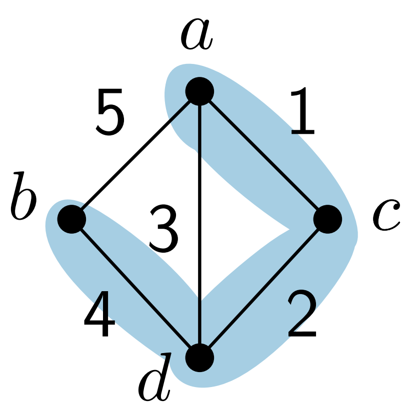

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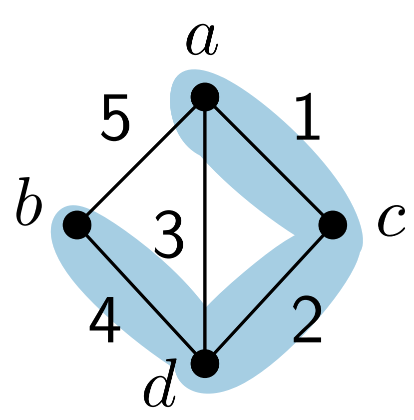

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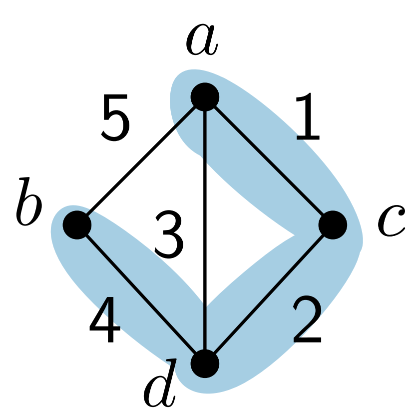

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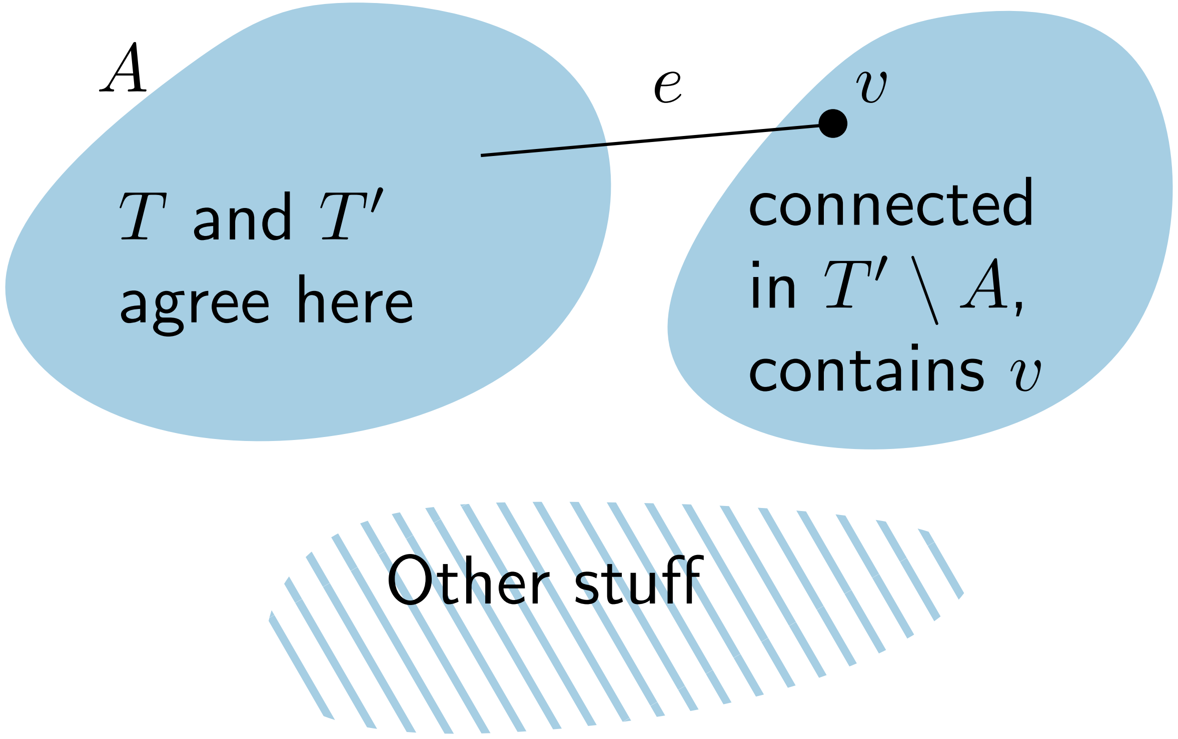

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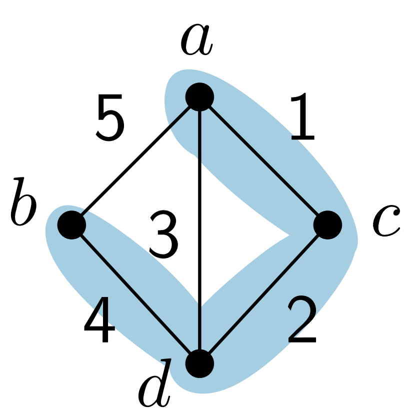

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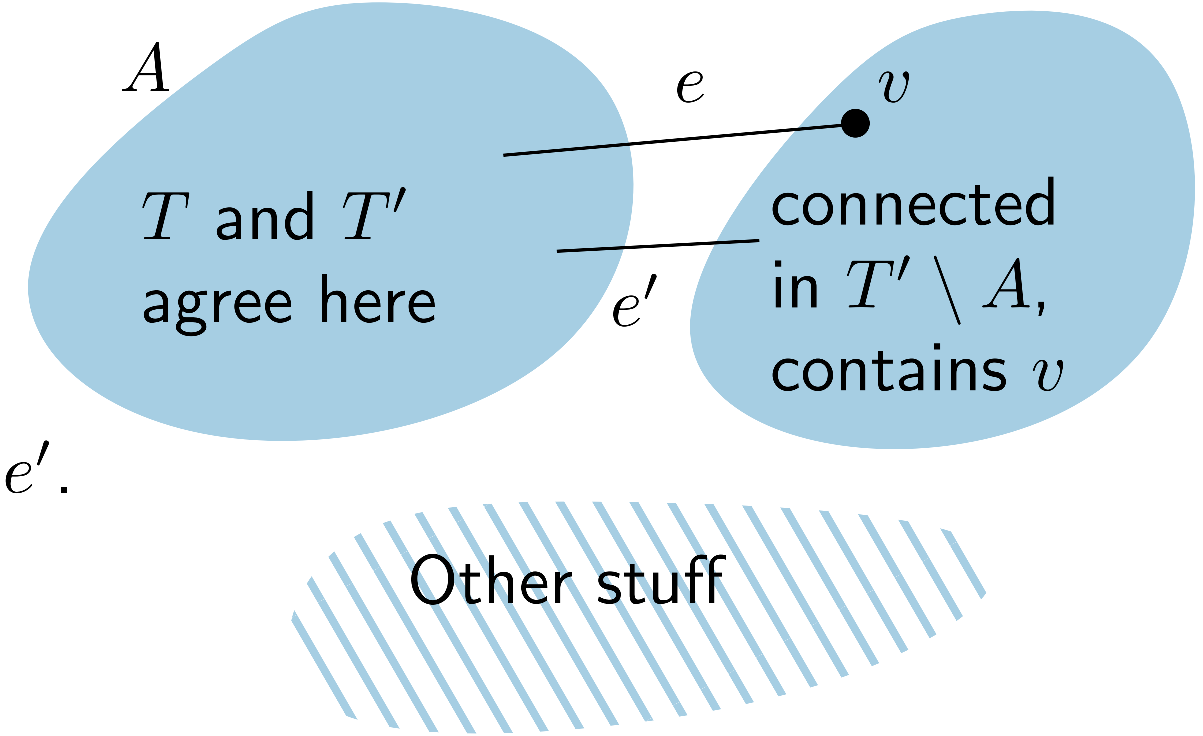

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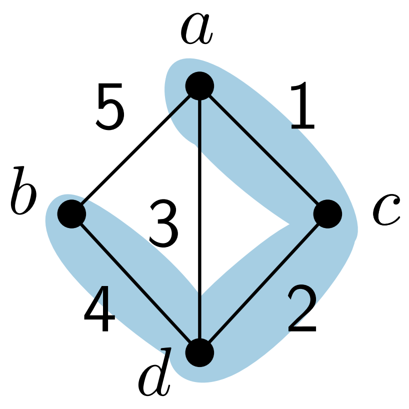

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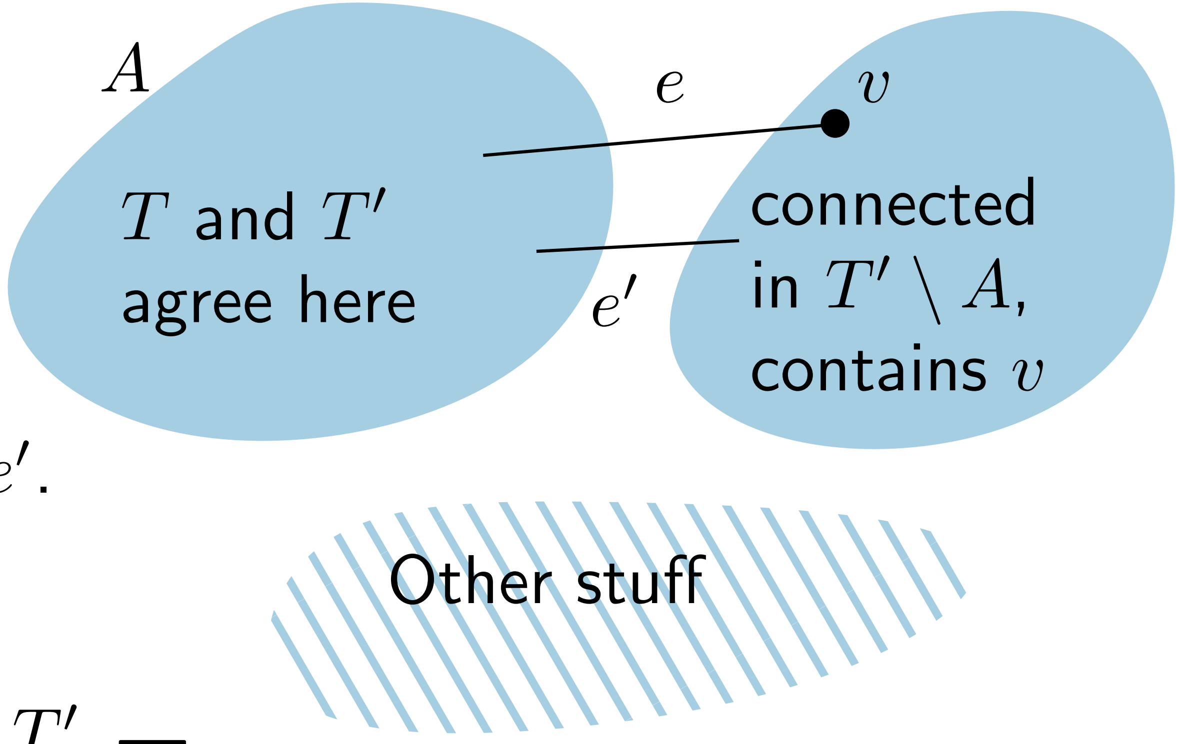

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Let T'' be the result of swapping e' for e in T'.
T'' is a spanning tree contradicting the minimality of T'. ∎

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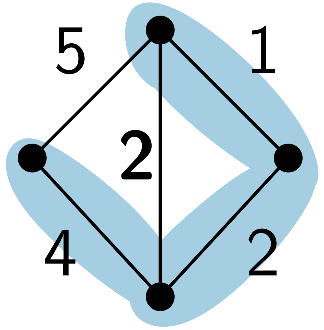
Example from before, now the “middle” edge has cost 2.



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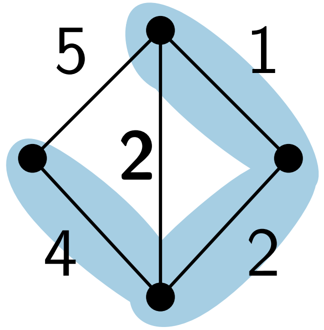

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We now make all the weights unique as follows.

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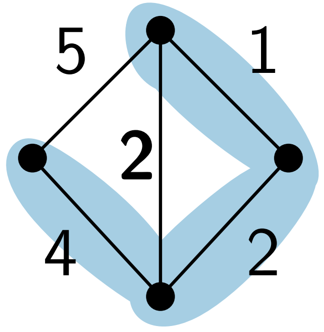

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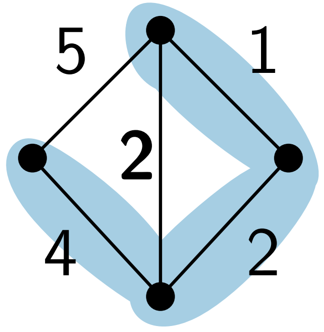

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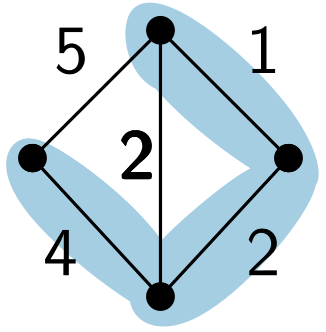

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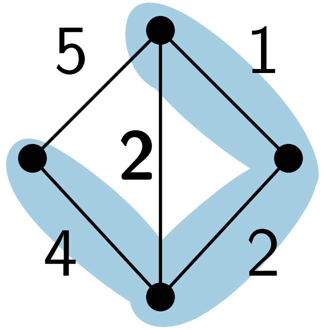

$(1, \quad 2, \quad \mathbf{2}, \quad 4, \quad 5) \rightarrow$

$(1+\frac{1}{4^3}, \; 2+\frac{2}{4^3}, \; \mathbf{2}+\frac{3}{4^3}, \; 4+\frac{4}{4^3}, \; 5+\frac{5}{4^3})$

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Exercise: Try it :) . . . also proven in the books.

Problem: Scheduling

- You are an event coordinator for a convention
- You have a list of talks $T_1, \ldots, T_n$, each T_i has a start and finish time, (s_i, f_i), $s_i < f_i$.
- The venue has an adequate number of rooms to host all talks in any manner
  - They charge an entire day for each room used, even it is only used for one short talk.
- You would like to create a schedule that
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When is a schedule **feasible**?

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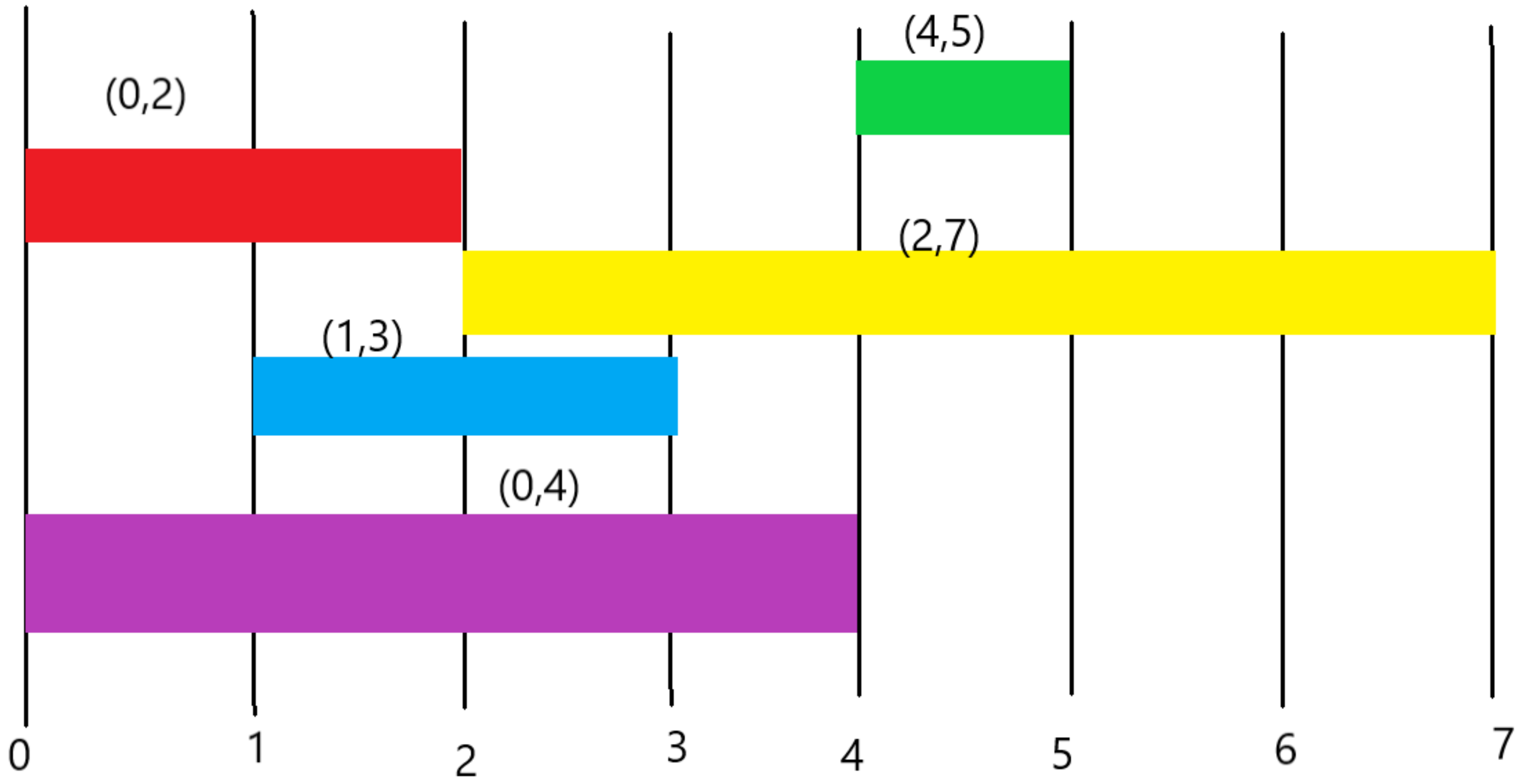
When is a schedule **feasible**? No **conflicting** talks are mapped to the same room.
That is, if $S(T_i) = S(T_j)$ then either $f_i \leq s_j$ or $f_j \leq s_i$.

Scheduling, an example

Here is a graphical example of a set of talks
$\{(4,5),(0,2),(2,7),(1,3),(0,4)\}$



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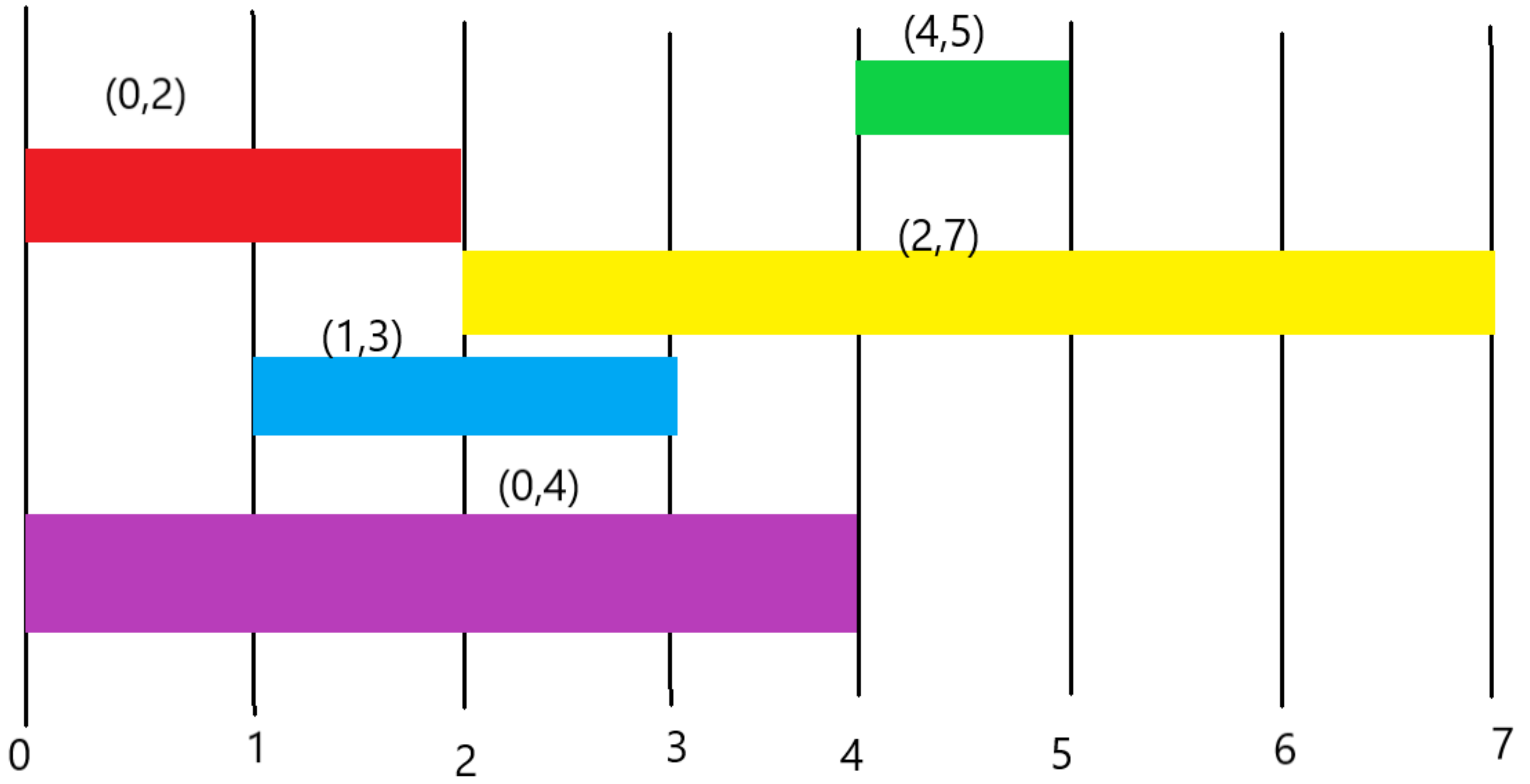

How many rooms do we need?

Scheduling, an example

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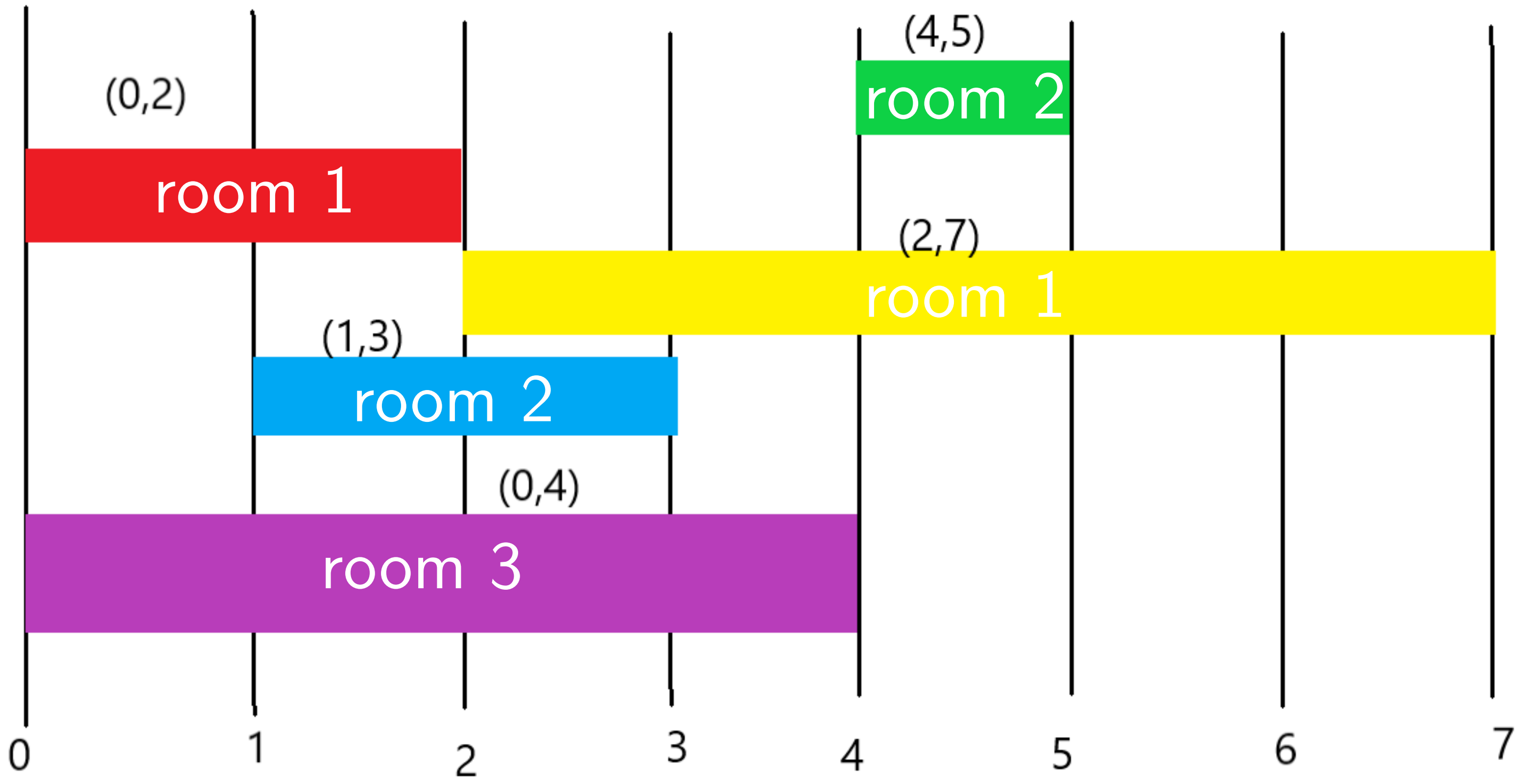

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Greedy Approach

Take the longest talk first, then the next longest, etc. ?

Regardless of our greedy measure, at each point the next talk scheduled either has to

- Be scheduled in room in use, to do so it cannot conflict with another talk in that room.
- Be scheduled in a new room, otherwise

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Taking the longest first could leave space for the shorter talks to “fill in” the unused spots
This sort of reasoning is common when coming up with greedy algorithms

- Pick a likely-sounding rule
- See how it does

Approach #1: longest first

Not optimal :(

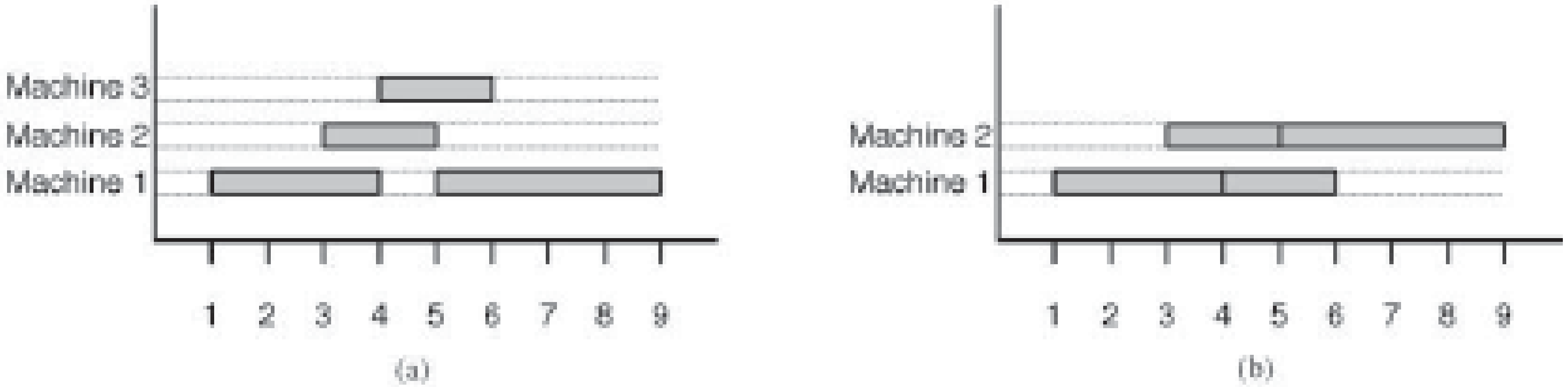

Figure 10.5: Why the longest-first strategy doesn't work, for the pairs of start and finish times in the set $\{(1,4),(5,9),(3,5),(4,6)\}$; (a) the solution chosen by the longest-first strategy; (b) the optimal solution. Note that the longest-first strategy uses three machines, whereas the optimal strategy uses only two.

Next Approach?

Other measures . . . use the start or finish times to greedily fill the schedule?

- Earliest to start first, then next earliest start time, etc.
- Earliest to finish first, then next earliest to finish, etc.
- Latest to finish first, then next latest, etc.
- . . .

Idea: scheduling the earliest talk first fills the rooms from earliest to latest.

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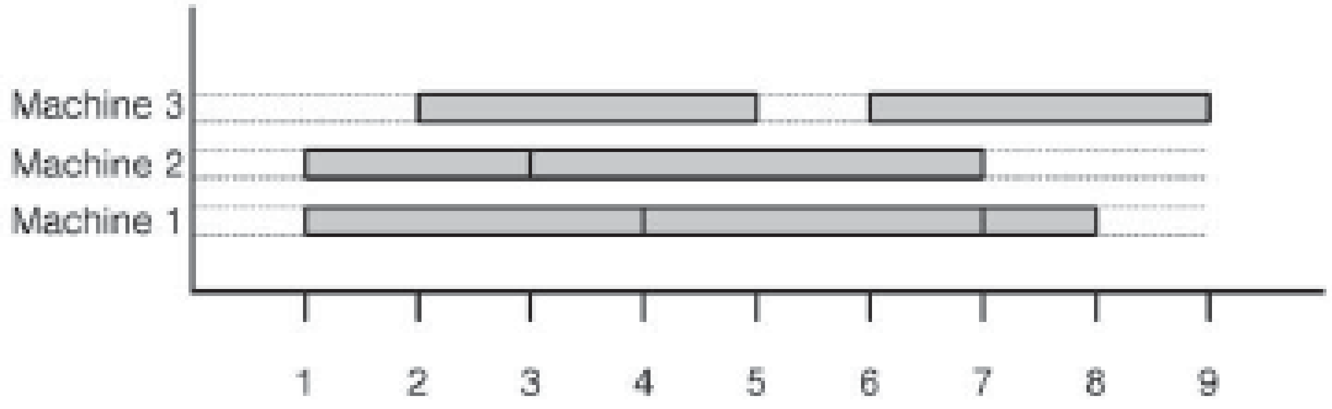

Figure 10.7: An example solution produced by the greedy algorithm based on considering tasks by increasing start times.

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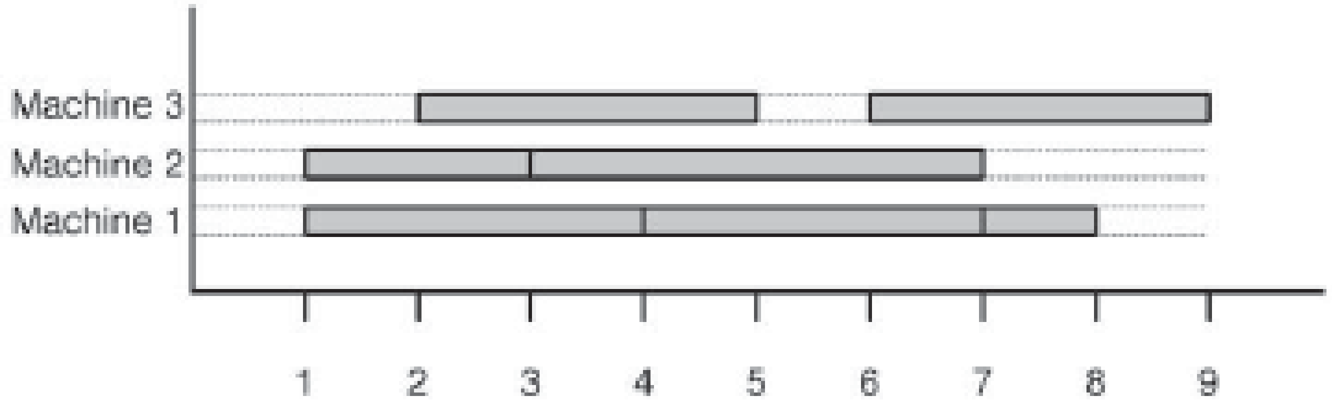

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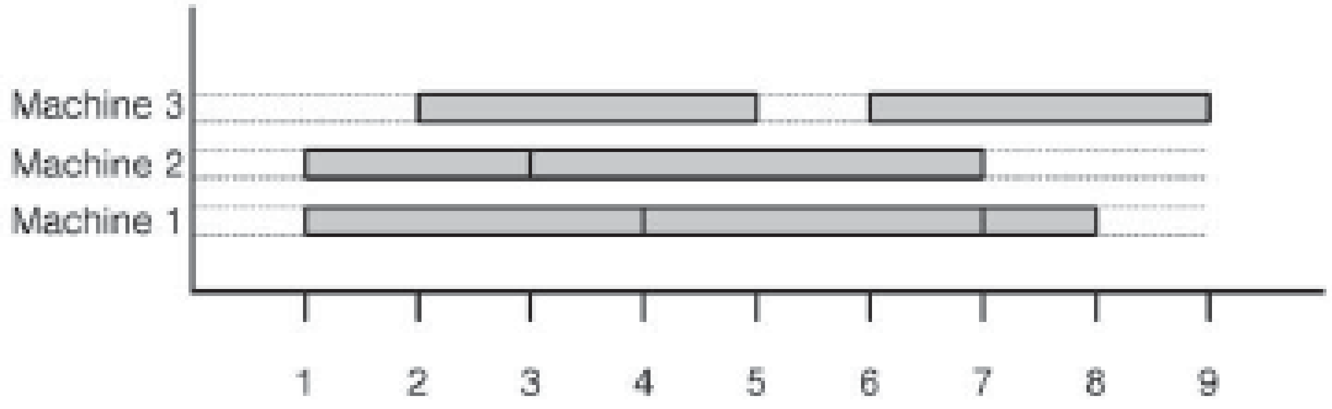

Figure 10.7: An example solution produced by the greedy algorithm based on considering tasks by increasing start times.

Looks promising!

Exercise: write out the pseudocode for this algorithm and analyze the runtime.

But does it really always work?

(The book also provides a proof of this)

Let S be a schedule given by the greedy approach of scheduling talks by earliest start time.

Let S' be an optimal schedule, that is, using the fewest rooms possible.

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Suppose that S uses k rooms and S' uses ℓ rooms. For a contradiction, assume $k > \ell$.

Let T_i be the talk that causes S to start using the room $\ell + 1$. Thus, T_i cannot be added to any of the ℓ active rooms.

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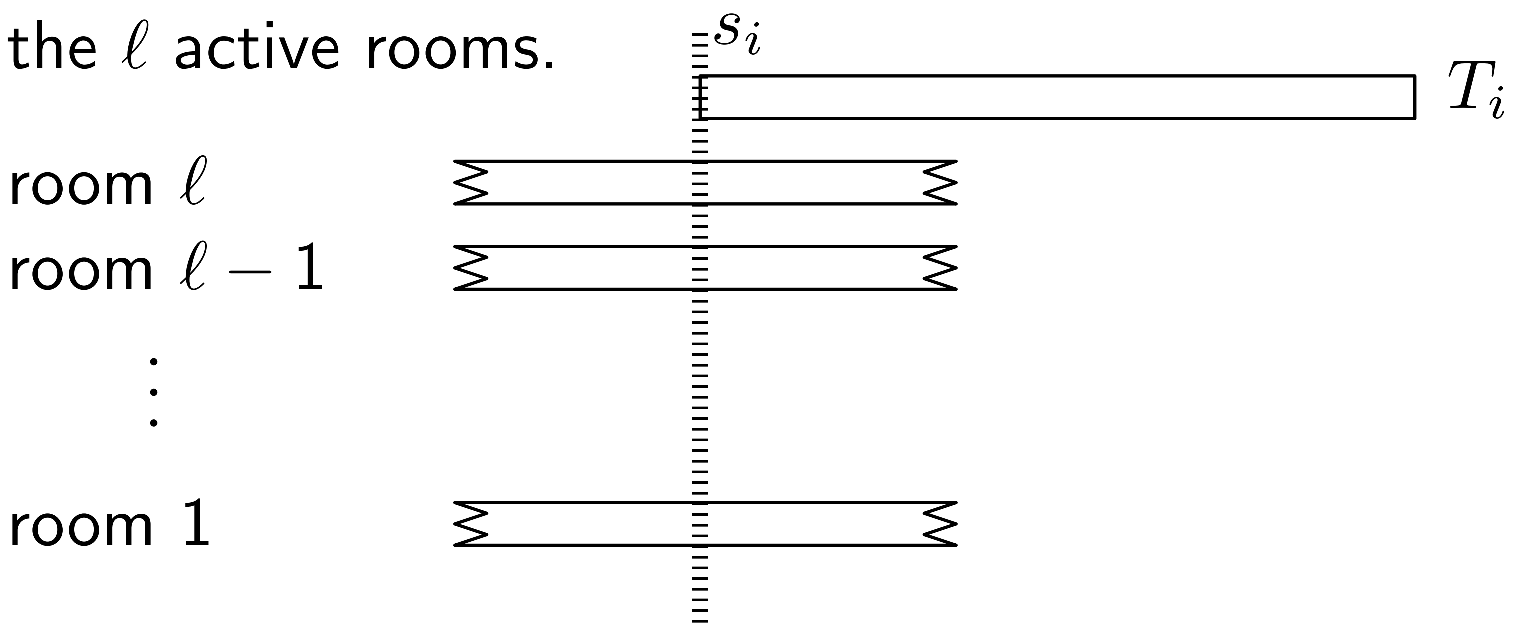

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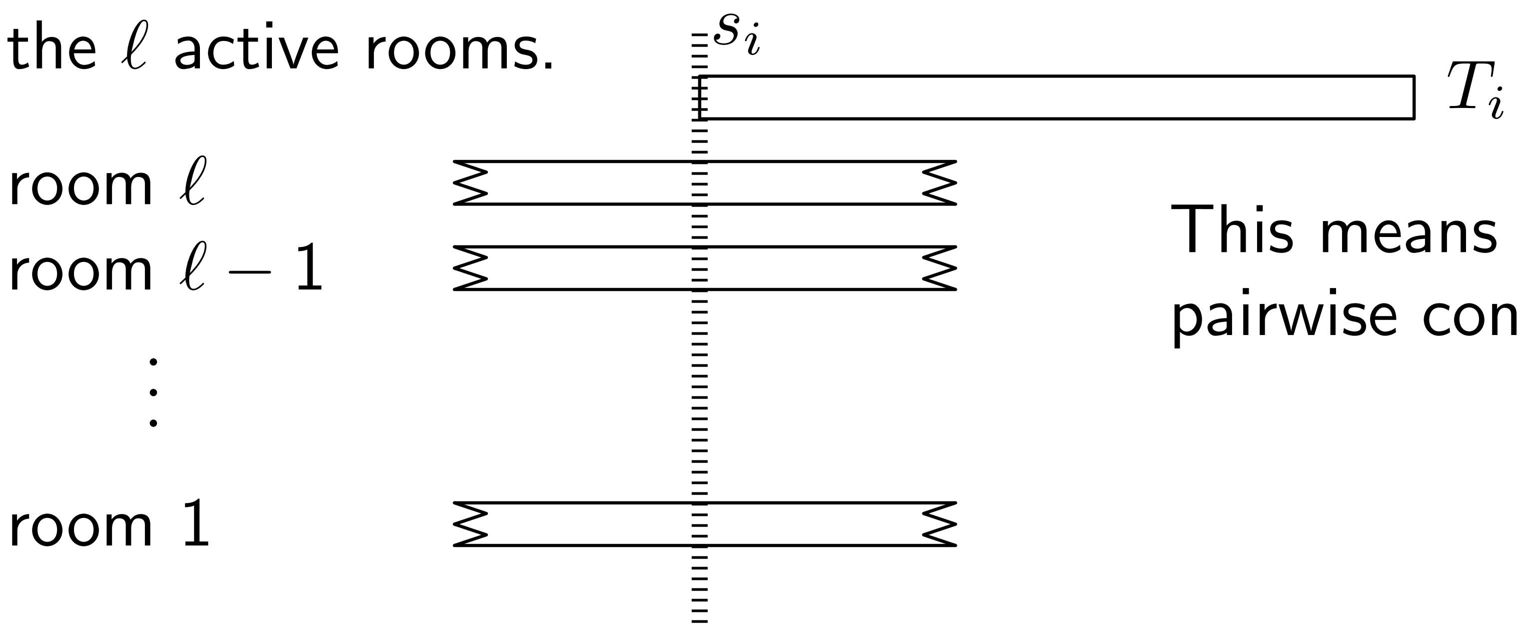

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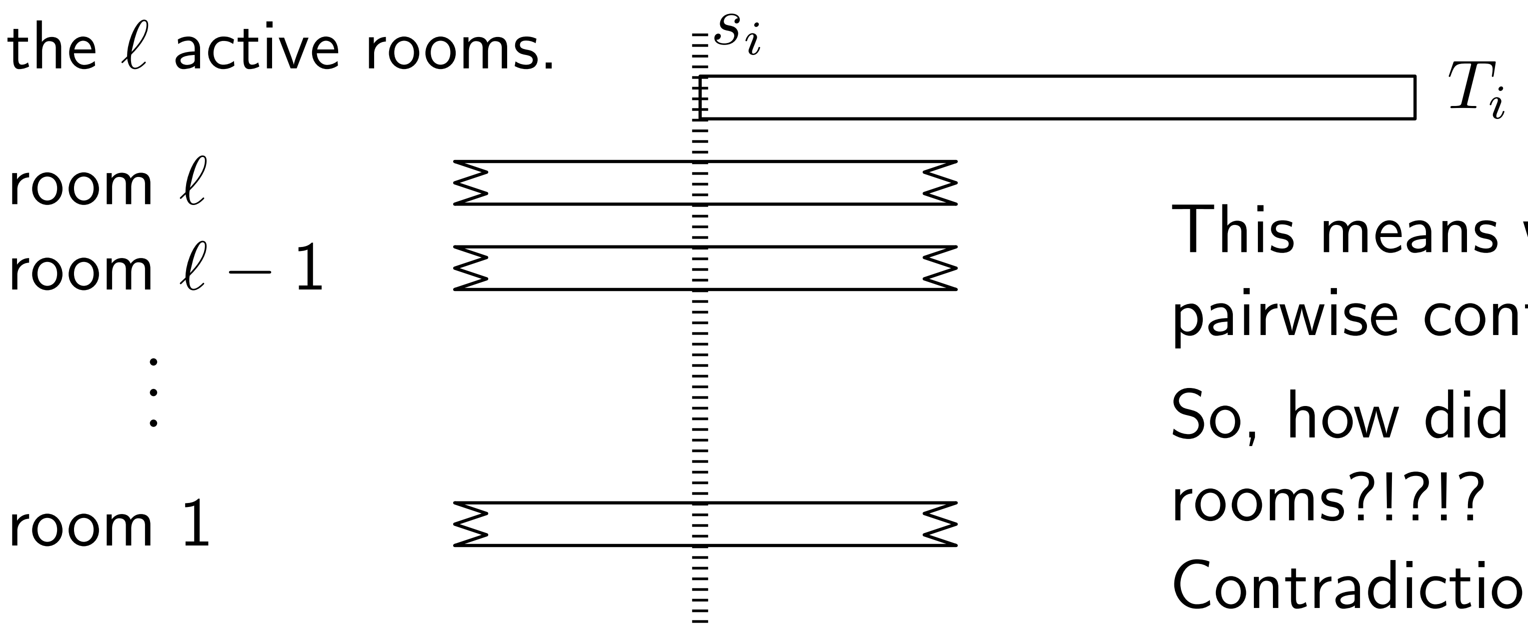

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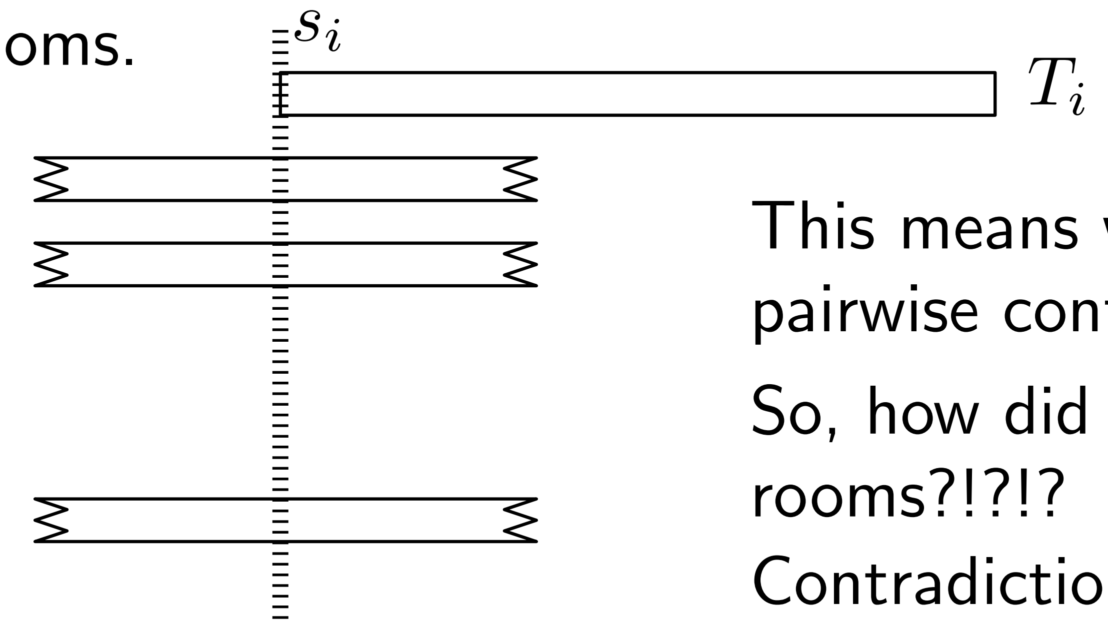
room ℓ

room $\ell - 1$

$\vdots$

room 1



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Thus, our greedy algorithm is optimal!

Exercise What about those other orders by start and finish times, do they also work?

Now for something different [GT §1.4, 10.3, CLRS §17.4, 16.3]

An important part of algorithmic design is understanding how fast (actually, how slow) your algorithms can be.

- Amortized analysis and Accounting. Or:
  Are you sure those worst case iterations happen all the time?

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With data flooding in from all directions, we should probably remember to compress it sometimes. So, let's see a classic compression approach.

GT §10.3, CLRS §16.3

- Huffman Codes. Or:
  Why do we use the same amount of bits for all characters when some occur more often than others?

Amortized Analysis and Accounting [GT §1.4, CLRS §17.4]

The algorithmic analysis tools from the previous course are not enough to analyze some of the algorithms we will be seeing.
For example, consider a **dynamic table**:

- Similar to a regular array, but
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function table_insert(T, newElem)
   if num(T) = size(T)
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      for each elem in T
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You might ask, this looks pretty standard, why do we need new analysis tools?

Dynamic Table: Analysing an Insertion

Assume both `create_table` and `elementary_insert` are constant $O(1)$ operations.

What is the complexity of `table_insert` ?

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Amortization

- Aggregate amortization (or, just, amortization)
- Accounting amortization (or, just, accounting)

(Aggregate) Amortization

Analyze the operation (insertion) over a sequence of calls.

- Can be a small, constant number of steps
- Can be a larger number of steps (to grow the table)

Goal: Find a function that describes the runtime over an **arbitrarily long** sequence of calls (insertions), we will have found our time complexity

$$\frac{\text{total runtime}}{\text{size of the input}}$$

Build Intuition, then Establish the function

Starting with an empty table, the actual costs of a series of inserts would look like:

- Call 1: 0 copy + 1 insert $\rightarrow$ 1 op. (table full)
- Call 2: 1 copy + 1 insert $\rightarrow$ 2 ops. (table full)
- Call 3: 2 copy + 1 insert $\rightarrow$ 3 ops. (table has one free space)
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There is a pattern here

$$1, 2, 3, 1, 5, 1, 1, 1, 9, 1, 1, 1, 1, 1, 1, 1, \ldots$$

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$$= 1 + 2 + 4 + 8 + \ldots + 2^{??}$$

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How many zeros here?

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Uh, but what is that sum?

trick: re-write in binary

How many zeros here?

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So, the amortized runtime is....

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That felt like a lot of work ... let's try the other way, maybe the accountants know better.

Amotization by Accounting (aka Charging/Discharging)

- Consider your personal budget
- There are two (or more) types of days: Regular days and Vacation days
- On regular days you pay for your normal activities, and save some money for vacation
- On vacation days you pay for your normal activities, and spend the money you saved up
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The longer you wait, the more you save.

- a small vacation every month, or
- a bigger one every other month, or an even bigger one every third month, and so on
- saving long enough means affording arbitrarily-expensive vacation

Amotization by Accounting (aka Charging/Discharging)

Applied to algorithms:

- Every operation gets athe same (e.g., constant) **income**
- Simple operations pay for their cost out of this income, and put the rest in **savings**
- Expensive operations we can use **savings** (and income) while ensuring the **savings** are never negative.
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For our dynamic table example, we give `table_insert` an **income of 3**.
When the table doesn't expand, It **pays 1** to do the actual insert and **saves 2** in the bank
Otherwise, it will pay the copying cost plus 1 for the next insert.

Let see how these values progress

In the table we can see how these values change as elements are added.

- **Income** is how much we get paid for each `table_insert`
- **Cost** is the number of operations required for that insertion
- +/- is amount leftover (or deficit) after paying for the operations.
- **Savings** what we have remaining in the bank after covering any deficit.

i	0	1	2	3	4	5	6	7	8	9	10
income	3	3	3	3	3	3	3	3	3	3	3
cost	1	2	3	1	5	1	1	1	9	1	1
+/-	2	1	0	2	-2	2	2	2	-6	2	2
savings	2	3	3	5	3	5	7	9	3	5	7

But how should we prove an income of 3 is enough?

Intuition:

- 1 unit of income gets the element into the table initially
- 1 unit pays for the element to be copied the first time
- 1 unit pays for a previous element to be copied again

With this in mind, we use induction on the expensive calls.

- If i is a power of 2 we need to copy over i elements, for a cost of i.
- By induction, we know that our savings was non-negative at the previous expensive call, $i/2$ calls ago.
- Thus, we have savings of 2 units from each of the last $i/2$ elements, for a total savings of $2(i/2) = i$ units.

Therefore we will never go broke.

Huffman Coding [GT §10.3, CLRS §16.3]

What is the problem to solve?
You have a huge text file (for example xml, json, some other even more horrible format).

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For example, in English the most common letters are E, A, and T
Maybe we can use short bit patterns:

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Fixing the Ambiguity

Use a **prefix code**.

- In a prefix code, no code word is a prefix of any other code word
- The problem before: the code word for "E" (0) is a prefix for the code word for "T" (01)
- The recipient is never sure if a 0 means an "E", or if it just the first bit of a "T"
- New code words:
  - "E" $\rightarrow$ 0
  - "A" $\rightarrow$ 10
  - "T" $\rightarrow$ 11
- Now "EAT" $=$ 01011
- And it can only be interpreted one way

Finally, formalizing the problem

Compute he best way to assign (prefix) code words to letters.
More formally:

- Given a message X
  - X uses characters $C = \{c_1, c_2, ..., c_n\}$
  - For each c_i, let $f(c_i)$ denote how many times c_i occurs in X.
  - Informally, the frequency of each character
- Create a code for C which allows X to be transmitted such that
  - The code allows X to be unambiguously recovered, e.g., it is a prefix code
  - The code ensures that the encoding of X uses the fewest bits possible.

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
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Fewest bits where each character is encoded separately.
Different approaches can do better when encoding more than one character at a time.
But, for single character encodings, there really is a **best** approach.

Huffman Coding

The encoding strategy is formed by the construction of a rooted binary tree where:

- Leaves of the tree map **bijectively** to the characters of the given string.
- For each non-leaf vertex of the tree, one of the two outgoing edges is labelled "0" and the other (if present) is labelled "1".
- The **encoding** of each character C is given by reading the edge labels along the path from the **root** to the **leaf** corresponding to C.
  - "E" $\to$ 0
  - "A" $\to$ 10
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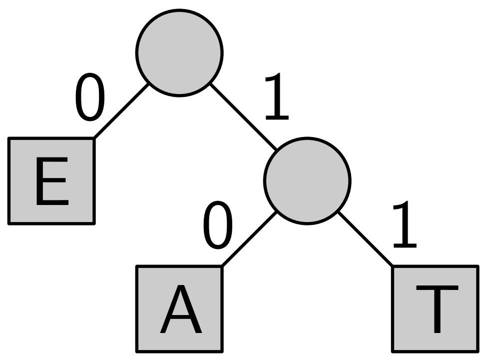

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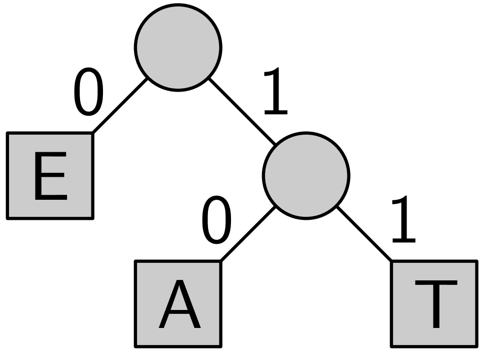

The key insights are:

- The codes arising from such a binary tree are **prefix codes**. Moreover, with such a tree we can unambiguously decode any binary string encoded with the same tree.
- Optimality: Characters whose leaves are close to the root should occur more often in the string than characters whose leaves are further away from the root.

Different trees mean different encoding.

Based on a file with 100,000 characters, but only containing {a,b,c,d,e,f}

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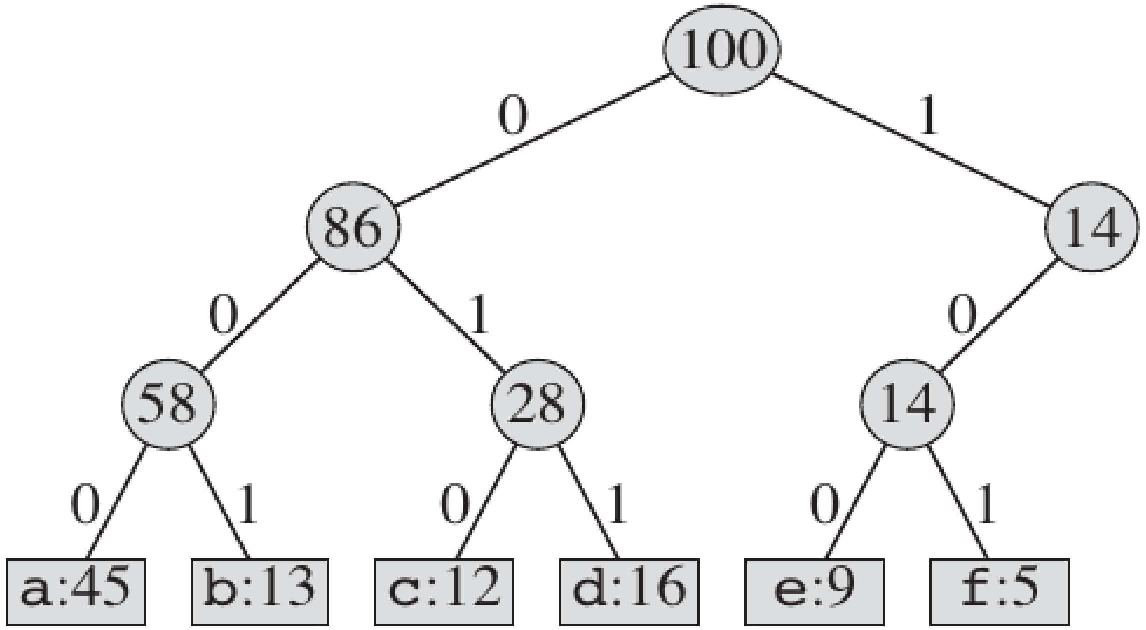

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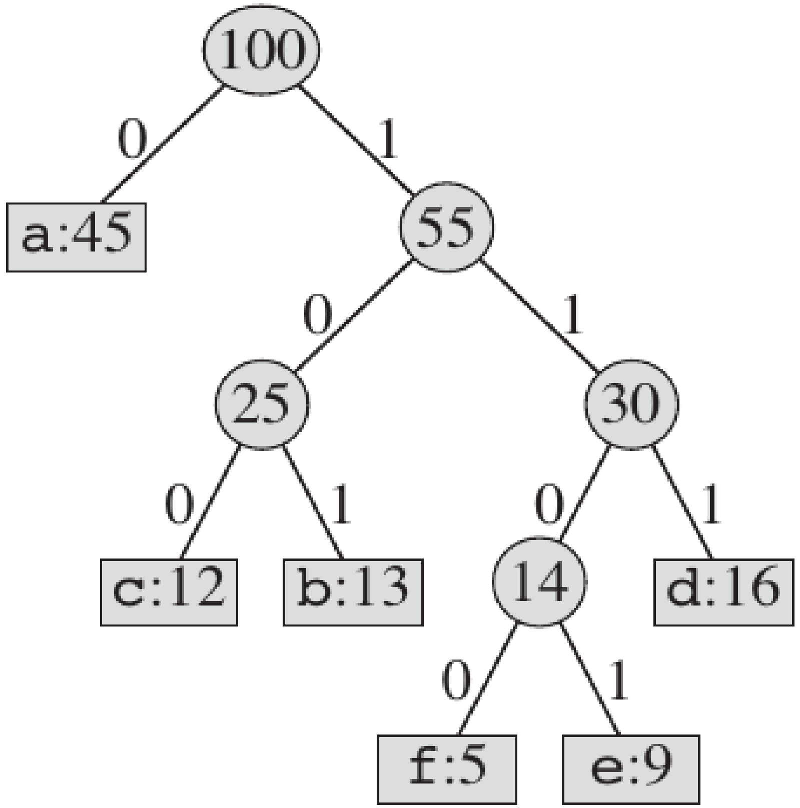

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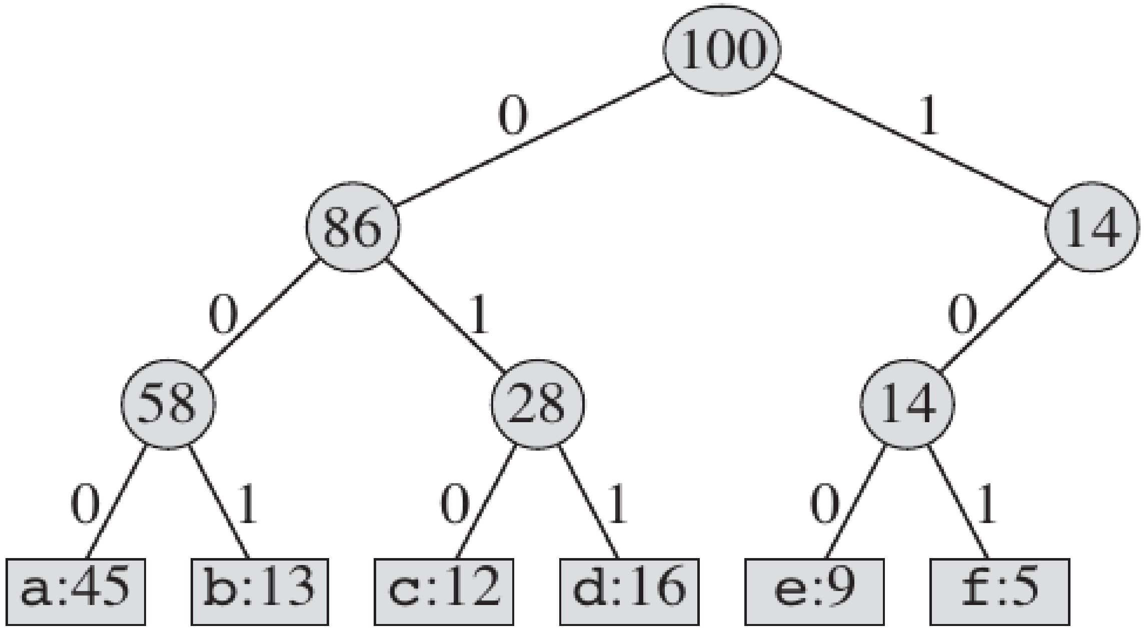

Can this tree be optimal?

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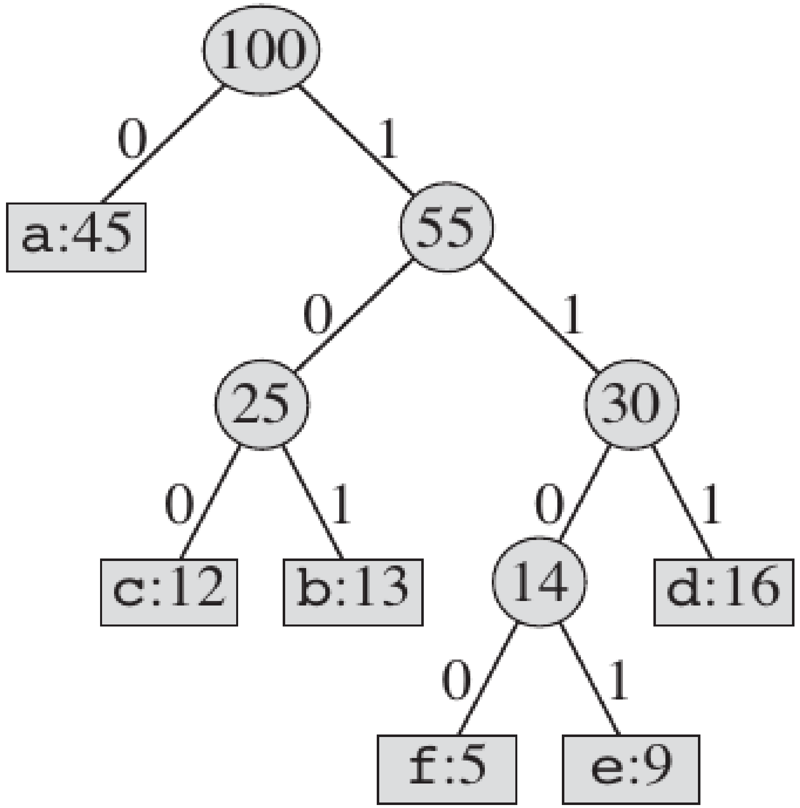

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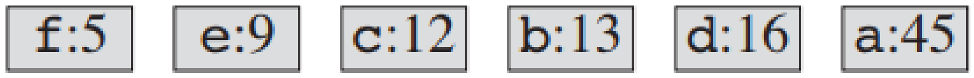

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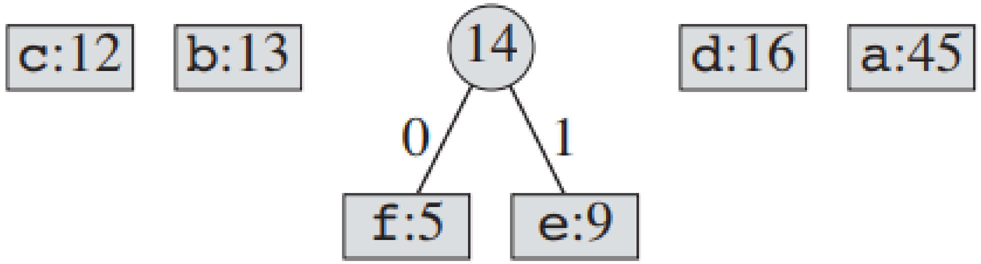

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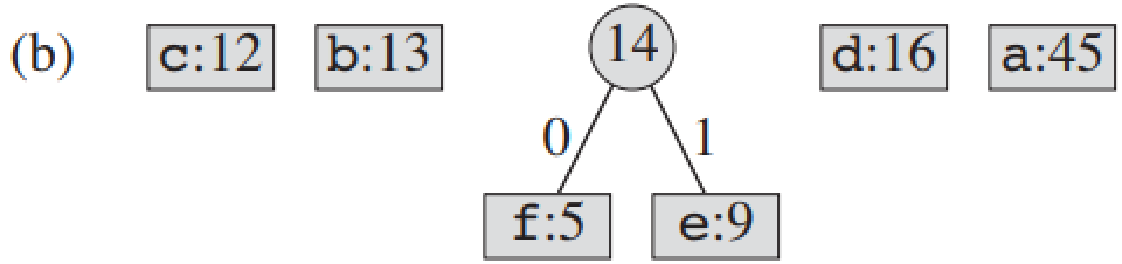

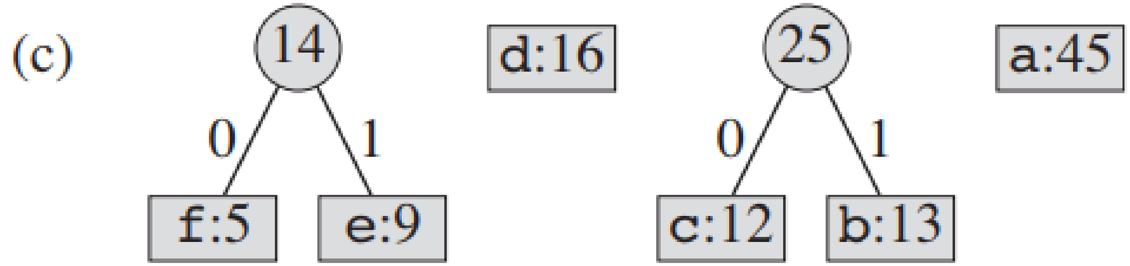

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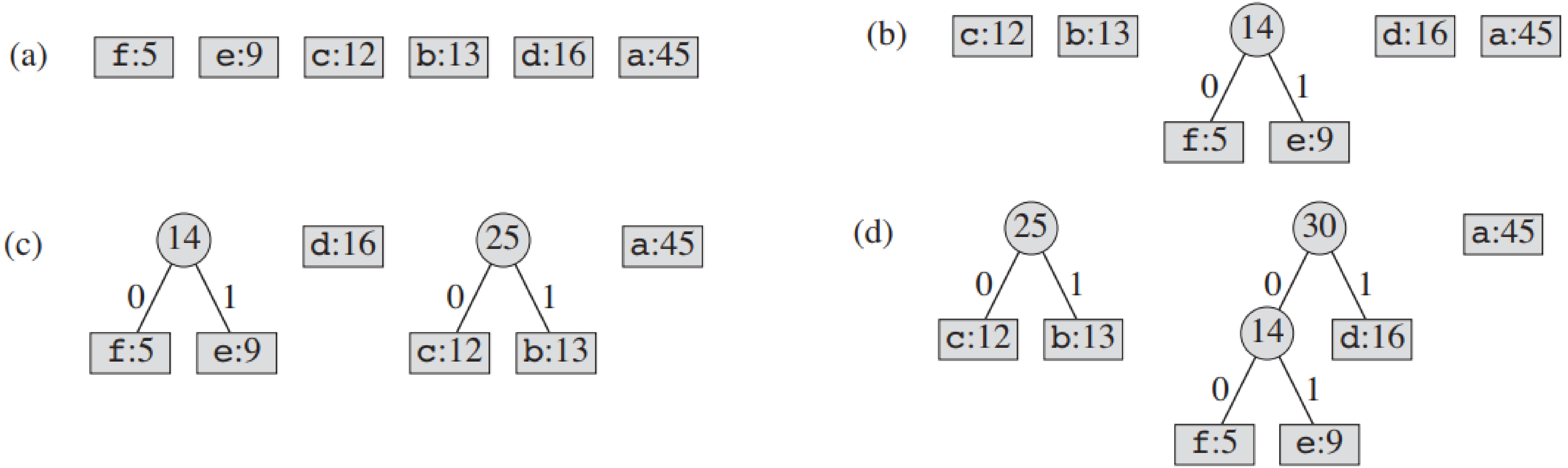

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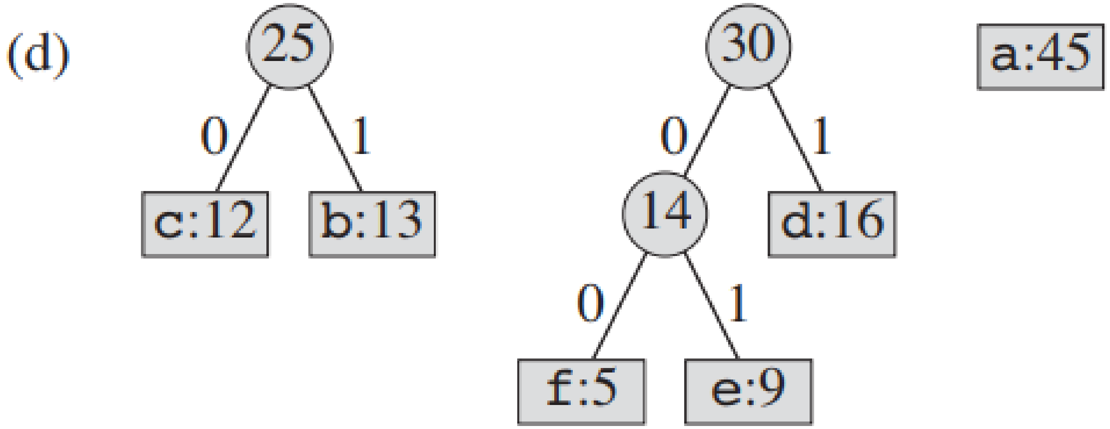

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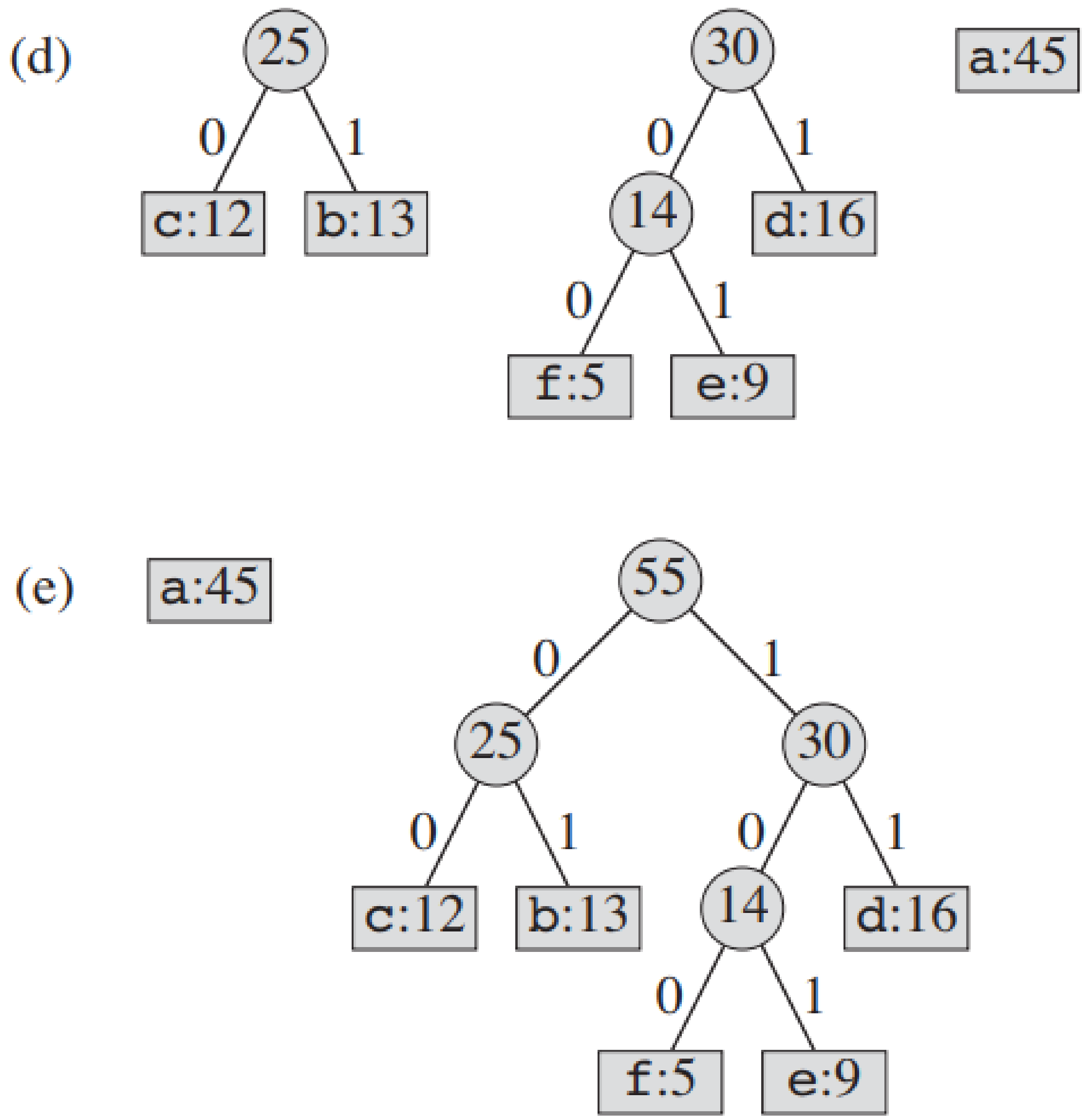

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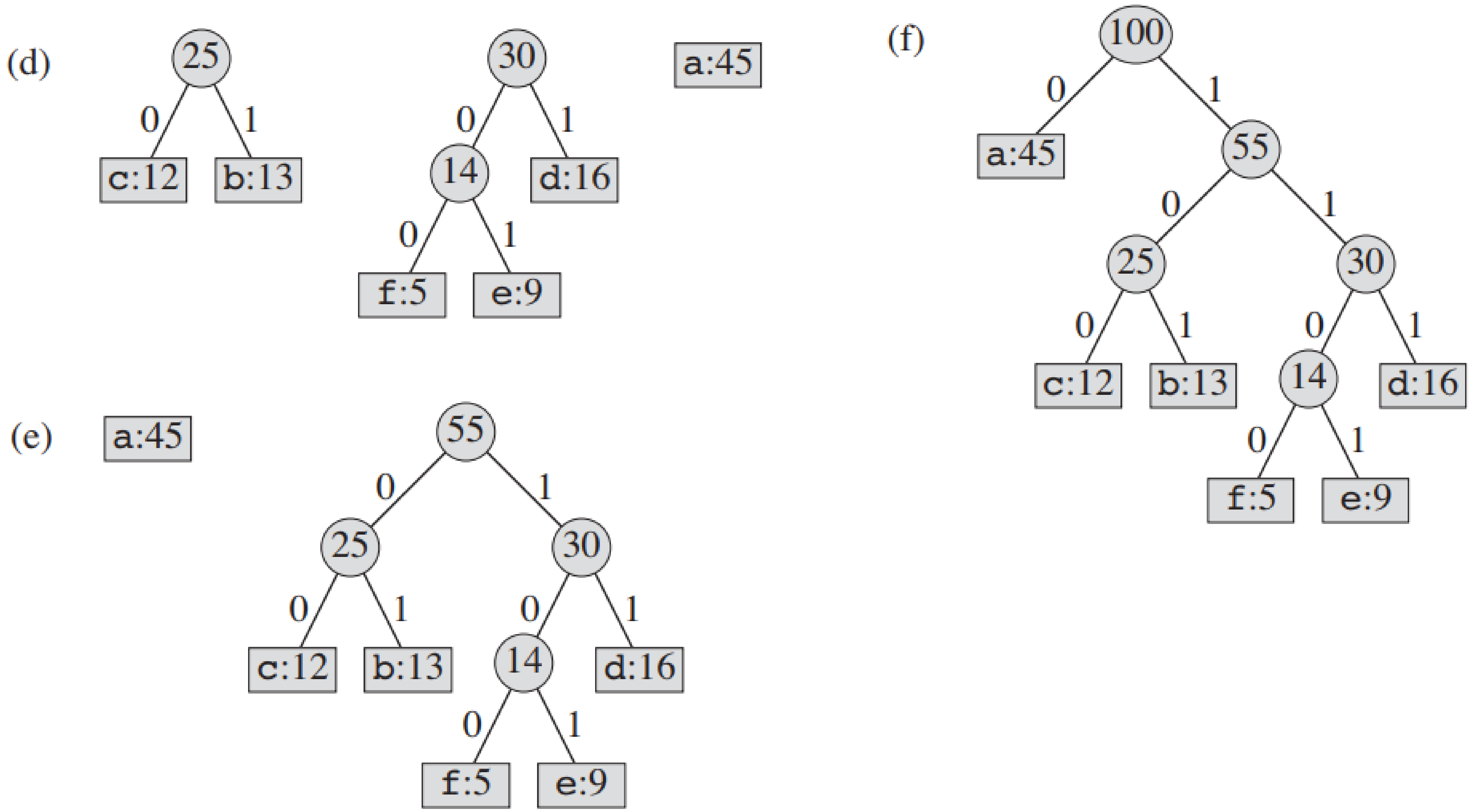

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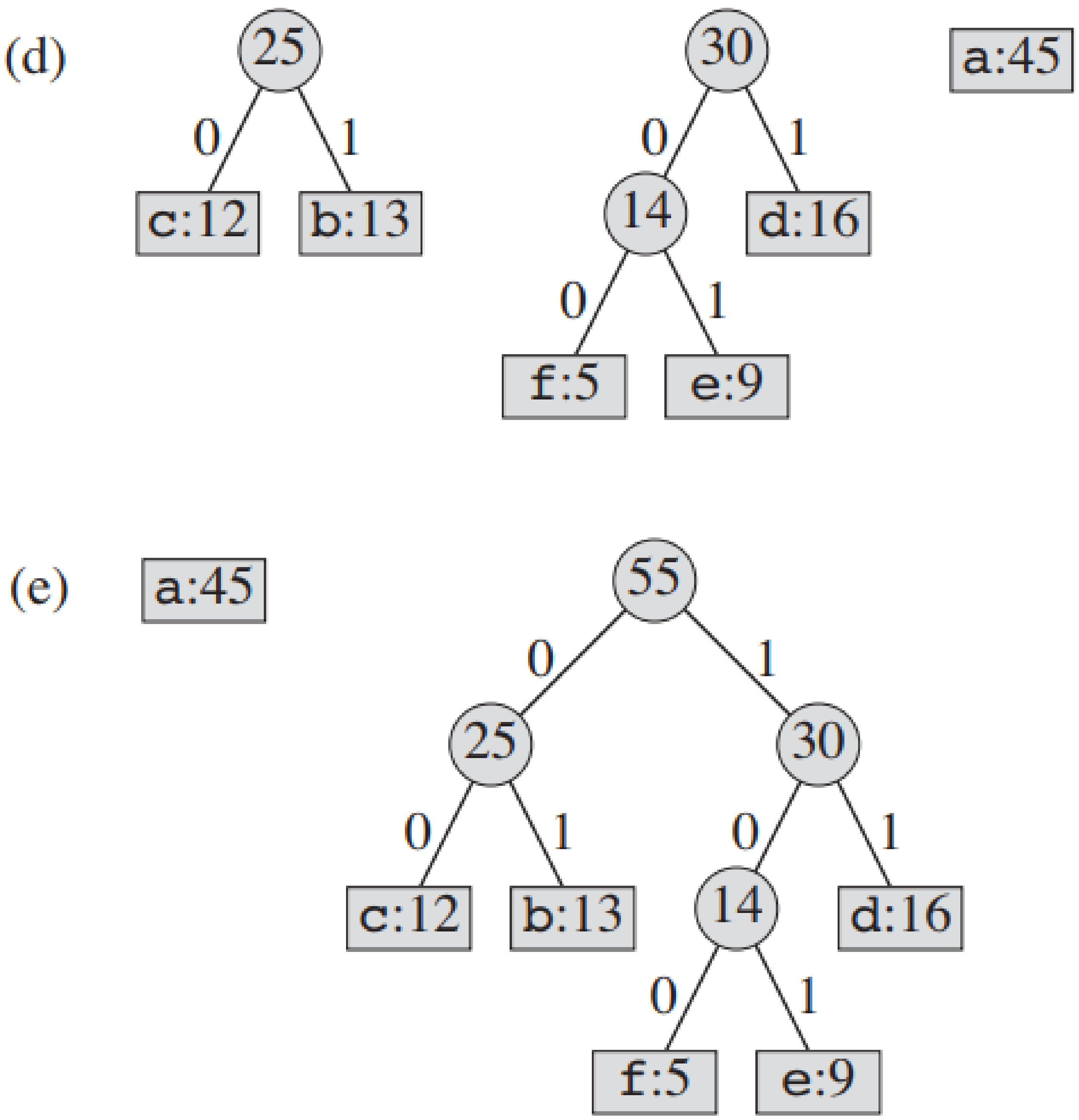

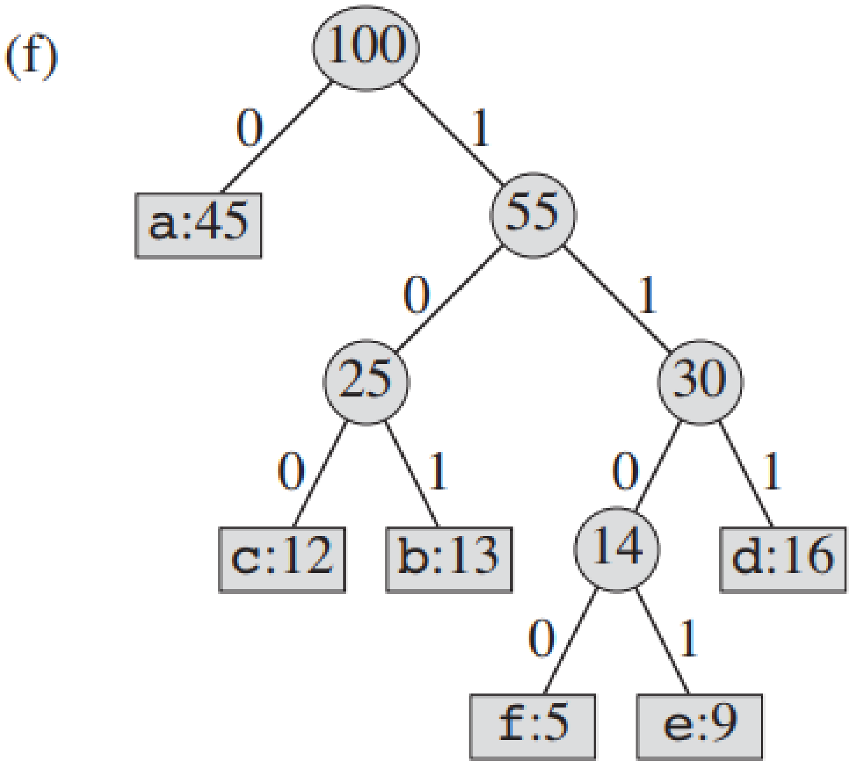

HUFFMAN(C)

$n = |C|$
$Q = C$
for $i = 1$ **to** $n - 1$
    allocate a new node z
    $z.left = x =$ EXTRACT-MIN(Q)
    $z.right = y =$ EXTRACT-MIN(Q)
    $z.freq = x.freq + y.freq$
    INSERT(Q, z)
return EXTRACT-MIN(Q)   // return the root of the tree

But why is this optimal?

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Lemma: Let C, C' be the two characters with the lowest frequency in a string X, and let T be an optimal code for X. The characters C, C' must have the longest codes in T.

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Lemma: Let C, C' be the two characters with the lowest frequency in a string X, and let T be an optimal code for X. The characters C, C' must have the longest codes in T.

Proof (idea) Use contradiction.
Let T be an optimal code, and let C'' be a character distinct from C, C' with the longest code. Swap the code of C (or C') for that of C''.

But why is this optimal?

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HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n - 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)        // return the root of the tree
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Another **exercise:** What is the runtime of our algorithm?

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Another **exercise:** What is the runtime of our algorithm?
How to encode and decode once we have such a code?

Next Week

More algorithmic techniques: Divide and Conquer

e.g., as in Merge Sort.

How to analyze the runtime of recursive methods (e.g., as arising from Divide and Conquer)

The MASTER theorem