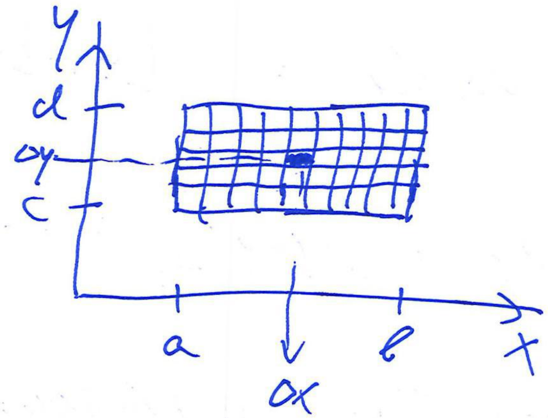
Lecture 9: Double Integration

$f(x,y)$ continuous on D

Rectangle: $a \le x \le b$
$c \le y \le d$



$$\iint_D f(x,y)\,\underbrace{dA}_{\text{"area element"}}$$

$\Delta A = \Delta x \cdot \Delta y$

"Volume under the surface".

Riemann-sum:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\cdot \Delta A_k$$

$n = \#$ "subareas".

$n \to \infty$.

$f(x,y) = 4 - x - y$

$D: \ 0 \le x \le 2, \ 0 \le y \le 1$

$$\iint_D f(x,y)\,dA = \underbrace{\int_0^2 \underbrace{\int_0^1 4-x-y\,dy}_{\text{inner}}\,dx}_{\text{outer}} = \int_0^2 \left[4y - xy - \tfrac{1}{2}y^2\right]_{y=0}^{1} dx$$

$$= \int_0^2 4 - x - \tfrac{1}{2}\,dx = \int_0^2 3\tfrac{1}{2} - x\,dx$$

$$= \left[3\tfrac{1}{2}x - \tfrac{1}{2}x^2\right]_0^2 = 7 - 2 = 5$$

Iterated integration

$$\int_0^1 \int_0^2 4-x-y \, dx \, dy = \int_0^1 \left[4x-\tfrac{1}{2}x^2-xy\right]_{x=0}^{2} dy$$

$$= \int_0^1 6-2y \, dy = \left[6y-y^2\right]_0^1 = 6-1 = 5$$

FUBINI'S THEOREM:

- IF D IS RECTANGULAR & BOUNDED, THEN THE ORDER OF INTEGRATION DOESN'T MATTER
- IF $\iint_D |f(x,y)| \, dA$ IS FINITE, THEN THE ORDER OF INTEGRATION DOESN'T MATTER.

NOTATION BOOK: $\cancel{\int_0^2 dx \int_0^1 4-x-y \, dy}$

JOINT DENSITY FUNCTION: $f(x,y) \geq 0$

$\iint_D f(x,y) \, dA = 1$

$0 \leq x \leq 2$; $0 \leq y \leq 1$

$f(x,y) = c \cdot x^2 y^3$

CHECK THAT $c = \frac{3}{2}$.

PROPERTIES:

$f(x,y)$, $g(x,y)$ CONTINUOUS ON BOUNDED AREA D.

$$\iint_D c\cdot f(x,y)\,dA = c\cdot\iint_D f(x,y)\,dA \qquad \forall c\in\mathbb{R}$$

$$\iint_D f(x,y)\pm g(x,y)\,dA = \iint_D f(x,y)\,dA \pm \iint_D g(x,y)\,dA$$

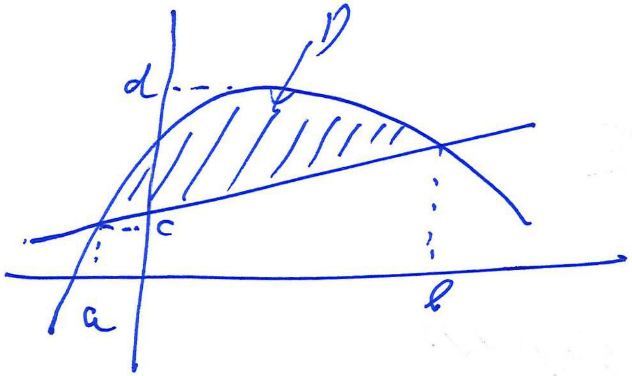
$$g(x,y)\le f(x,y) \text{ ON } D \Rightarrow \iint_D g(x,y)\,dA \le \iint_D f(x,y)\,dA$$

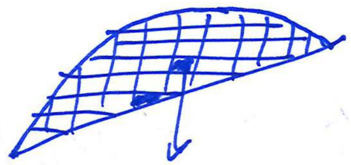
TRIANGLE INEQUALITY: $\left|\iint_D f(x,y)\,dA\right| \le \iint_D |f(x,y)|\,dA$

- IF D HAS AREA 0, THEN $\iint_D f(x,y)\,dA = 0$.
- $\iint_D dA$ = AREA OF D.

WHAT IF D IS NOT RECTANGULAR?

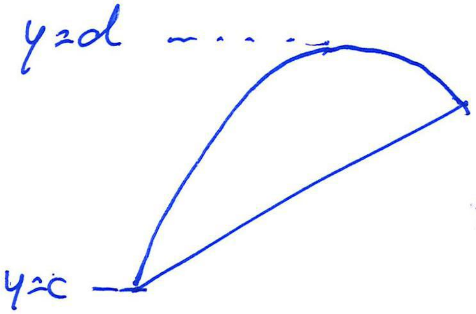
$$S_n = \underbrace{\sum_{k=1}^{n} f(x_k, y_k) \cdot \Delta A_k}$$

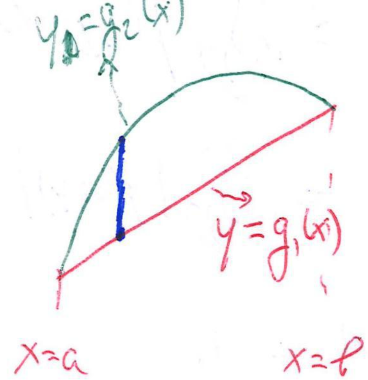


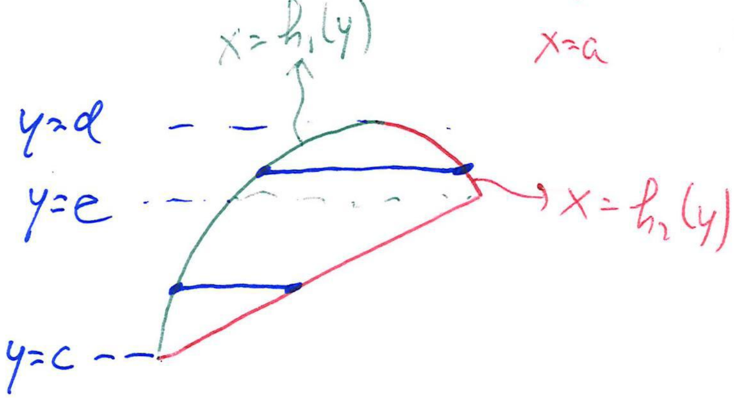


$\Delta A_k = \Delta x \cdot \Delta y$

PARTITION FINE ENOUGH $\Rightarrow$ NON-RECTANGULAR AREAS A_k HAVE ~~VOLUME~~ AREA $\longrightarrow 0$.







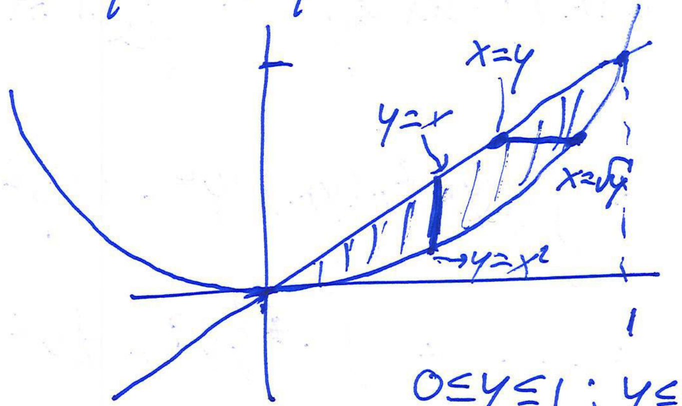
$$\iint_D f(x,y)\, dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

INTEGRATE $f(x,y) = xy$ OVER ~~AREA~~ REGION ENCLOSED BY $y = x$ & $y = x^2$.

$$D: \boxed{0 \le x \le 1 \,;\, x^2 \le y \le x}$$



$0 \le y \le 1 \,;\, y \le x \le \sqrt{y}$

$$\int_0^1 \int_{x^2}^{x} xy \, dy \, dx = \int_0^1 \left[\tfrac{1}{2} x y^2\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \tfrac{1}{2}x^3 - \tfrac{1}{2}x^5 \, dx = \left[\tfrac{1}{8}x^4 - \tfrac{1}{12}x^6\right]_0^1 = \tfrac{1}{24}.$$

$$\int_0^1 \int_y^{\sqrt{y}} xy \, dx \, dy = \ldots = \tfrac{1}{24}.$$

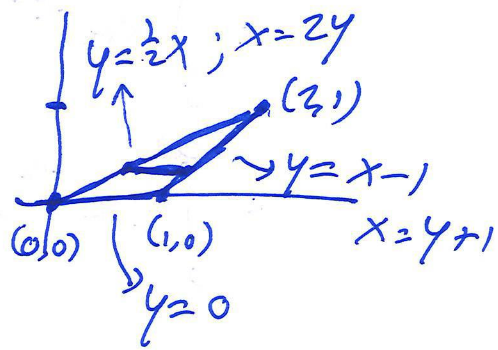
INTEGRATE $f(x,y) = (x+1)\cdot y^2$ OVER TRIANGLE WITH EXTREME POINTS $(0,0)$, $(1,0)$ & $(2,1)$

$$\int_0^1 \int_0^{\frac{1}{2}x} (x+1)y^2 \, dy\, dx + \int_1^2 \int_{x-1}^{\frac{1}{2}x} (x+1)y^2 \, dy \, dx$$

D: $0 \le x \le 2$

If $x \le 1$: $0 \le y \le \frac{1}{2}x$

If $1 \le x \le 2$: $x-1 \le y \le \frac{1}{2}x$



$$\int_0^1 \int_{2y}^{y+1} (x+1)y^2 \, dx \, dy$$

D: $0 \le y \le 1$

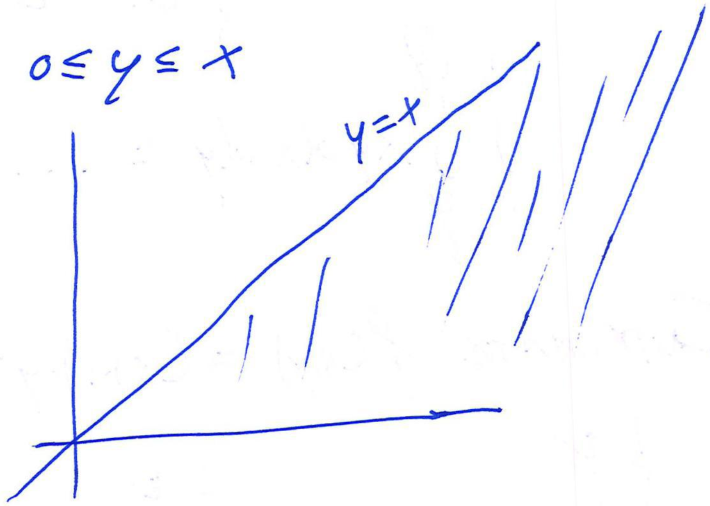
$2y \le x \le y+1$

$$\int_0^1 \int_{2y}^{y+1} (x+1)y^2\,dx\,dy = \int_0^1 \left[\left(\tfrac{1}{2}x^2+x\right)y^2\right]_{x=2y}^{y+1} dy$$

$$= \int_0^1 \left(\tfrac{1}{2}(y+1)^2+(y+1)\right)y^2 - (2y^2+2y)y^2\,dy$$

$$= \dots = \int_0^1 -\tfrac{3}{2}y^4 + \tfrac{3}{2}y^2\,dy = \left[-\tfrac{3}{10}y^5+\tfrac{1}{2}y^3\right]_0^1 = -\tfrac{3}{10}+\tfrac{1}{2} = \tfrac{1}{5}$$

INTEGRATE $f(x,y) = e^{-x^2}$ OVER REGION $0 \le x < \infty$; $0 \le y \le x$



INTEGRATE OVER y FIRST:

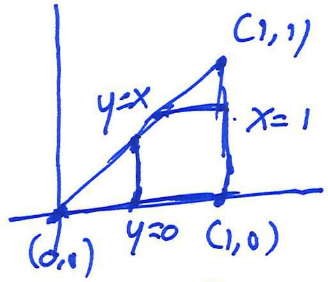
$$\int_0^\infty \int_0^x e^{-x^2}\,dy\,dx = \int_0^\infty \left[y \cdot e^{-x^2}\right]_{y=0}^{x} dx$$

$$= \int_0^\infty x e^{-x^2}\,dx \qquad \text{SUBSTITUTION: } u = x^2, \quad du = 2x\,dx \text{ ; LIMITS: } x=0 \Rightarrow u=0, \quad x=\infty \Rightarrow u=\infty$$

$$= \int_0^\infty \tfrac{1}{2}e^{-u}\,du = \left[-\tfrac{1}{2}e^{-u}\right]_0^\infty = \tfrac{1}{2}.$$

INTEGRATE $f(x,y) = \frac{\sin(x)}{x}$ OVER TRIANGLE WITH EXTREME POINTS $(0,0)$, $(1,0)$ & $(1,1)$.

INTEGRATE OVER y FIRST: D: $0 \leq x \leq 1$, $0 \leq y \leq x$



D: $0 \leq y \leq 1$, $y \leq x \leq 1$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^x \frac{\sin(x)}{x}\,dy\,dx$$

$$= \int_0^1 \left[\frac{\sin(x)}{x} \cdot y\right]_{y=0}^{x} dx = \int_0^1 \sin(x)\,dx = \left[-\cos(x)\right]_0^1 = -\cos(1) + \cos(0)$$

$$= 1 - \cos(1).$$

SOLVE: $$\int_0^1 \int_y^1 \frac{\sin(x)}{x}\,dx\,dy = \int_0^1 \int_0^x \frac{\sin(x)}{x}\,dy\,dx$$