EXERCISES 4.3

Evaluate the limits in Exercises 1–32.

1. $\displaystyle\lim_{x\to 0} \frac{3x}{\tan 4x}$

2. $\displaystyle\lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4}$

3. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

4. $\displaystyle\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

5. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

6. $\displaystyle\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$

7. $\displaystyle\lim_{x\to 0} x \cot x$

8. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)}$

9. $\displaystyle\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi}$

10. $\displaystyle\lim_{x\to 0} \frac{10^x-e^x}{x}$

11. $\displaystyle\lim_{x\to \pi/2} \frac{\cos 3x}{\pi-2x}$

12. $\displaystyle\lim_{x\to 1} \frac{\ln(ex)-1}{\sin \pi x}$

13. $\displaystyle\lim_{x\to \infty} x \sin \frac{1}{x}$

14. $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$

15. $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

16. $\displaystyle\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$

17. $\displaystyle\lim_{x\to 0+} \frac{\sin^2 x}{\tan x - x}$

18. $\displaystyle\lim_{r\to \pi/2} \frac{\ln \sin r}{\cos r}$

19. $\displaystyle\lim_{t\to \pi/2} \frac{\sin t}{t}$

20. $\displaystyle\lim_{x\to 1-} \frac{\arccos x}{x-1}$

21. $\displaystyle\lim_{x\to \infty} x(2\tan^{-1} x - \pi)$

22. $\displaystyle\lim_{t\to (\pi/2)-} (\sec t - \tan t)$

23. $\displaystyle\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{te^{at}}\right)$

24. $\displaystyle\lim_{x\to 0+} x^{\sqrt{x}}$

25. $\displaystyle\lim_{x\to 0+} (\csc x)^{\sin^2 x}$

26. $\displaystyle\lim_{x\to 1+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

27. $\displaystyle\lim_{t\to 0} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t}$

28. $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

29. $\displaystyle\lim_{t\to 0} (\cos 2t)^{1/t^2}$

30. $\displaystyle\lim_{x\to 0+} \frac{\csc x}{\ln x}$

31. $\displaystyle\lim_{x\to 1-} \frac{\ln \sin \pi x}{\csc \pi x}$

32. $\displaystyle\lim_{x\to 0} (1+\tan x)^{1/x}$

33. **(A Newton quotient for the second derivative)** Evaluate $\lim_{h\to 0} \dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$ if f is a twice differentiable function.

34. If f has a continuous third derivative, evaluate

$$\lim_{h\to 0} \frac{f(x+3h)-3f(x+h)+3f(x-h)-f(x-3h)}{h^3}.$$

35. **(Proof of the second l'Hôpital Rule)** Fill in the details of the following outline of a proof of the second l'Hôpital Rule (Theorem 4) for the case where a and L are both finite. Let $a < x < t < b$ and show that there exists c in (x, t) such that

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)}.$$

Now juggle the above equation algebraically into the form

$$\frac{f(x)}{g(x)} - L = \frac{f'(c)}{g'(c)} - L + \frac{1}{g(x)}\left(f(t) - g(t)\frac{f'(c)}{g'(c)}\right).$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f'(c)}{g'(c)} - L\right| + \frac{1}{|g(x)|}\left(|f(t)| + |g(t)|\left|\frac{f'(c)}{g'(c)}\right|\right).$$

Now show that the right side of the above inequality can be made as small as you wish (say, less than a positive number ϵ) by choosing first t and then x close enough to a. Remember, you are given that $\lim_{c\to a+}\left(f'(c)/g'(c)\right) = L$ and $\lim_{x\to a+} |g(x)| = \infty$.

EXERCISES 4.4

In Exercises 1–17, determine whether the given function has any local or absolute extreme values, and find those values if possible.

1. $f(x) = x + 2$ on $[-1, 1]$

2. $f(x) = x + 2$ on $(-\infty, 0]$

3. $f(x) = x + 2$ on $[-1, 1)$

4. $f(x) = x^2 - 1$

5. $f(x) = x^2 - 1$ on $[-2, 3]$

6. $f(x) = x^2 - 1$ on $(2, 3)$

7. $f(x) = x^3 + x - 4$ on $[a, b]$

8. $f(x) = x^3 + x - 4$ on (a, b)

9. $f(x) = x^5 + x^3 + 2x$ on $(a, b]$

10. $f(x) = \dfrac{1}{x - 1}$

11. $f(x) = \dfrac{1}{x - 1}$ on $(0, 1)$

12. $f(x) = \dfrac{1}{x - 1}$ on $[2, 3]$

13. $f(x) = |x - 1|$ on $[-2, 2]$

14. $|x^2 - x - 2|$ on $[-3, 3]$

15. $f(x) = \dfrac{1}{x^2 + 1}$

16. $f(x) = (x + 2)^{2/3}$

17. $f(x) = (x - 2)^{1/3}$

In Exercises 18–40, locate and classify all local extreme values of the given function. Determine whether any of these extreme values are absolute. Sketch the graph of the function.

18. $f(x) = x^2 + 2x$

19. $f(x) = x^3 - 3x - 2$

20. $f(x) = (x^2 - 4)^2$

21. $f(x) = x^3(x - 1)^2$

22. $f(x) = x^2(x - 1)^2$

23. $f(x) = x(x^2 - 1)^2$

24. $f(x) = \dfrac{x}{x^2 + 1}$

25. $f(x) = \dfrac{x^2}{x^2 + 1}$

26. $f(x) = \dfrac{x}{\sqrt{x^4 + 1}}$

27. $f(x) = x\sqrt{2 - x^2}$

28. $f(x) = x + \sin x$

29. $f(x) = x - 2\sin x$

30. $f(x) = x - 2\tan^{-1} x$

31. $f(x) = 2x - \sin^{-1} x$

32. $f(x) = e^{-x^2/2}$

33. $f(x) = x\, 2^{-x}$

34. $f(x) = x^2 e^{-x^2}$

35. $f(x) = \dfrac{\ln x}{x}$

36. $f(x) = |x + 1|$

37. $f(x) = |x^2 - 1|$

38. $f(x) = \sin |x|$

39. $f(x) = |\sin x|$

40. $f(x) = (x - 1)^{2/3} - (x + 1)^{2/3}$

In Exercises 41–46, determine whether the given function has absolute maximum or absolute minimum values. Justify your answers. Find the extreme values if you can.

41. $\dfrac{x}{\sqrt{x^2 + 1}}$

42. $\dfrac{x}{\sqrt{x^4 + 1}}$

43. $x\sqrt{4 - x^2}$

44. $\dfrac{x^2}{\sqrt{4 - x^2}}$

45. $\dfrac{1}{x \sin x}$ on $(0, \pi)$

46. $\dfrac{\sin x}{x}$

47. If a function has an absolute maximum value, must it have any local maximum values? If a function has a local maximum value, must it have an absolute maximum value? Give reasons for your answers.

48. If the function f has an absolute maximum value and $g(x) = |f(x)|$, must g have an absolute maximum value? Justify your answer.

49. **(A function with no max or min at an endpoint)** Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ but that it has neither a local maximum nor a local minimum value at the endpoint $x = 0$.

EXERCISES 4.9

In Exercises 1–10, find the linearization of the given function about the given point.

1. x^2 about $x = 3$

2. x^{-3} about $x = 2$

3. $\sqrt{4 - x}$ about $x = 0$

4. $\sqrt{3 + x^2}$ about $x = 1$

5. $1/(1 + x)^2$ about $x = 2$

6. $1/\sqrt{x}$ about $x = 4$

7. $\sin x$ about $x = \pi$

8. $\cos(2x)$ about $x = \pi/3$

9. $\sin^2 x$ about $x = \pi/6$

10. $\tan x$ about $x = \pi/4$

11. By approximately how much does the area of a square increase if its side length increases from 10 cm to 10.4 cm?