EXERCISES 2.1

In Exercises 1–12, find an equation of the straight line tangent to the given curve at the point indicated.

1. $y = 3x - 1$ at $(1, 2)$
2. $y = x/2$ at $(a, a/2)$
3. $y = 2x^2 - 5$ at $(2, 3)$
4. $y = 6 - x - x^2$ at $x = -2$
5. $y = x^3 + 8$ at $x = -2$
6. $y = \dfrac{1}{x^2 + 1}$ at $(0, 1)$
7. $y = \sqrt{x + 1}$ at $x = 3$
8. $y = \dfrac{1}{\sqrt{x}}$ at $x = 9$
9. $y = \dfrac{2x}{x + 2}$ at $x = 2$
10. $y = \sqrt{5 - x^2}$ at $x = 1$
11. $y = x^2$ at $x = x_0$
12. $y = \dfrac{1}{x}$ at $\left(a, \dfrac{1}{a}\right)$

Do the graphs of the functions f in Exercises 13–17 have tangent lines at the given points? If yes, what is the tangent line?

13. $f(x) = \sqrt{|x|}$ at $x = 0$
14. $f(x) = (x - 1)^{4/3}$ at $x = 1$
15. $f(x) = (x + 2)^{3/5}$ at $x = -2$
16. $f(x) = |x^2 - 1|$ at $x = 1$
17. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$ at $x = 0$
18. Find the slope of the curve $y = x^2 - 1$ at the point $x = x_0$. What is the equation of the tangent line to $y = x^2 - 1$ that has slope -3?
19. (a) Find the slope of $y = x^3$ at the point $x = a$.
    (b) Find the equations of the straight lines having slope 3 that are tangent to $y = x^3$.
20. Find all points on the curve $y = x^3 - 3x$ where the tangent line is parallel to the x-axis.
21. Find all points on the curve $y = x^3 - x + 1$ where the tangent line is parallel to the line $y = 2x + 5$.
22. Find all points on the curve $y = 1/x$ where the tangent line is perpendicular to the line $y = 4x - 3$.
23. For what value of the constant k is the line $x + y = k$ normal to the curve $y = x^2$?
24. For what value of the constant k do the curves $y = kx^2$ and $y = k(x - 2)^2$ intersect at right angles? *Hint:* Where do the curves intersect? What are their slopes there?

Use a graphics utility to plot the following curves. Where does the curve have a horizontal tangent? Does the curve fail to have a tangent line anywhere?

25. $y = x^3(5 - x)^2$
26. $y = 2x^3 - 3x^2 - 12x + 1$
27. $y = |x^2 - 1| - x$
28. $y = |x + 1| - |x - 1|$
29. $y = (x^2 - 1)^{1/3}$
30. $y = ((x^2 - 1)^2)^{1/3}$
31. If line L is tangent to curve C at point P, then the smaller angle between L and the secant line PQ joining P to another point Q on C approaches 0 as Q approaches P along C. Is the converse true: if the angle between PQ and line L (which passes through P) approaches 0, must L be tangent to C?
32. Let $P(x)$ be a polynomial. If a is a real number, then $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n$$

for some $n \ge 0$. If $\ell(x) = m(x - a) + b$, show that the straight line $y = \ell(x)$ is tangent to the graph of $y = P(x)$ at $x = a$ provided $P(x) - \ell(x) = (x - a)^2 Q(x)$, where $Q(x)$ is a polynomial.

EXERCISES 2.2

Make rough sketches of the graphs of the derivatives of the functions in Exercises 1–4.

1. The function f graphed in Figure 2.18(a).
2. The function g graphed in Figure 2.18(b).
3. The function h graphed in Figure 2.18(c).
4. The function k graphed in Figure 2.18(d).
5. Where is the function f graphed in Figure 2.18(a) differentiable?
6. Where is the function g graphed in Figure 2.18(b) differentiable?

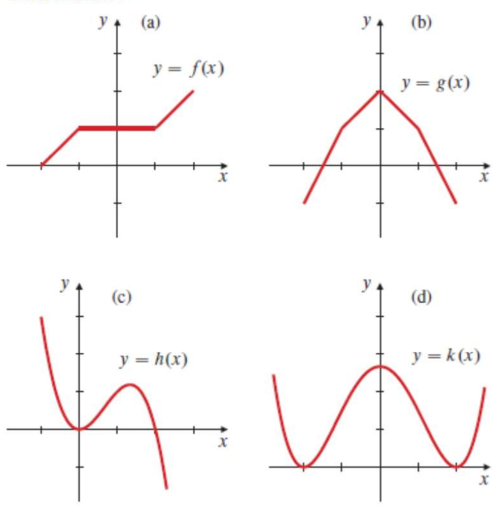

Figure 2.18

Use a graphics utility with differentiation capabilities to plot the graphs of the following functions and their derivatives. Observe the relationships between the graph of y and that of y' in each case. What features of the graph of y can you infer from the graph of y'?

7. $y = 3x - x^2 - 1$
8. $y = x^3 - 3x^2 + 2x + 1$
9. $y = |x^3 - x|$
10. $y = |x^2 - 1| - |x^2 - 4|$

In Exercises 11–24, (a) calculate the derivative of the given function directly from the definition of derivative, and (b) express the result of (a) using differentials.

11. $y = x^2 - 3x$
12. $f(x) = 1 + 4x - 5x^2$
13. $f(x) = x^3$
14. $s = \dfrac{1}{3 + 4t}$
15. $g(x) = \dfrac{2 - x}{2 + x}$
16. $y = \dfrac{1}{3}x^3 - x$
17. $F(t) = \sqrt{2t + 1}$
18. $f(x) = \dfrac{3}{4}\sqrt{2 - x}$
19. $y = x + \dfrac{1}{x}$
20. $z = \dfrac{s}{1 + s}$
21. $F(x) = \dfrac{1}{\sqrt{1 + x^2}}$
22. $y = \dfrac{1}{x^2}$
23. $y = \dfrac{1}{\sqrt{1 + x}}$
24. $f(t) = \dfrac{t^2 - 3}{t^2 + 3}$
25. How should the function $f(x) = x \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?
26. How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?
27. Where does $h(x) = |x^2 + 3x + 2|$ fail to be differentiable?
28. Using a calculator, find the slope of the secant line to $y = x^3 - 2x$ passing through the points corresponding to $x = 1$ and $x = 1 + \Delta x$, for several values of Δx of decreasing size, say $\Delta x = \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$. (Make a table.) Also, calculate $\left.\dfrac{d}{dx}(x^3 - 2x)\right|_{x=1}$ using the definition of derivative.
29. Repeat Exercise 28 for the function $f(x) = \dfrac{1}{x}$ and the points $x = 2$ and $x = 2 + \Delta x$.

Using the definition of derivative, find equations for the tangent lines to the curves in Exercises 30–33 at the points indicated.

EXERCISES 2.3

In Exercises 1–32, calculate the derivatives of the given functions. Simplify your answers whenever possible.

1. $y = 3x^2 - 5x - 7$
2. $y = 4x^{1/2} - \dfrac{5}{x}$
3. $f(x) = Ax^2 + Bx + C$
4. $f(x) = \dfrac{6}{x^3} + \dfrac{2}{x^2} - 2$
5. $z = \dfrac{s^5 - s^3}{15}$
6. $y = x^{45} - x^{-45}$
7. $g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$
8. $y = 3\sqrt[3]{t^2} - \dfrac{2}{\sqrt{t^3}}$
9. $u = \dfrac{3}{5}x^{5/3} - \dfrac{5}{3}x^{-3/5}$
10. $F(x) = (3x - 2)(1 - 5x)$
11. $y = \sqrt{x}\left(5 - x - \dfrac{x^2}{3}\right)$
12. $g(t) = \dfrac{1}{2t - 3}$
13. $y = \dfrac{1}{x^2 + 5x}$
14. $y = \dfrac{4}{3 - x}$
15. $f(t) = \dfrac{\pi}{2 - \pi t}$
16. $g(y) = \dfrac{2}{1 - y^2}$
17. $f(x) = \dfrac{1 - 4x^2}{x^3}$
18. $g(u) = \dfrac{u\sqrt{u} - 3}{u^2}$
19. $y = \dfrac{2 + t + t^2}{\sqrt{t}}$
20. $z = \dfrac{x - 1}{x^{2/3}}$
21. $f(x) = \dfrac{3 - 4x}{3 + 4x}$
22. $z = \dfrac{t^2 + 2t}{t^2 - 1}$
23. $s = \dfrac{1 + \sqrt{t}}{1 - \sqrt{t}}$
24. $f(x) = \dfrac{x^3 - 4}{x + 1}$
25. $f(x) = \dfrac{ax + b}{cx + d}$
26. $F(t) = \dfrac{t^2 + 7t - 8}{t^2 - t + 1}$
27. $f(x) = (1 + x)(1 + 2x)(1 + 3x)(1 + 4x)$
28. $f(r) = (r^{-2} + r^{-3} - 4)(r^2 + r^3 + 1)$
29. $y = (x^2 + 4)(\sqrt{x} + 1)(5x^{2/3} - 2)$
30. $y = \dfrac{(x^2 + 1)(x^3 + 2)}{(x^2 + 2)(x^3 + 1)}$
31. $y = \dfrac{x}{2x + \dfrac{1}{3x + 1}}$
32. $f(x) = \dfrac{(\sqrt{x} - 1)(2 - x)(1 - x^2)}{\sqrt{x}(3 + 2x)}$

Calculate the derivatives in Exercises 33–36, given that $f(2) = 2$ and $f'(2) = 3$.

33. $\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\Bigg|_{x=2}$
34. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2}\right)\Bigg|_{x=2}$
35. $\dfrac{d}{dx}\left(x^2 f(x)\right)\Bigg|_{x=2}$
36. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2 + f(x)}\right)\Bigg|_{x=2}$
37. Find $\dfrac{d}{dx}\left(\dfrac{x^2 - 4}{x^2 + 4}\right)\Bigg|_{x=-2}$.
38. Find $\dfrac{d}{dt}\left(\dfrac{t(1 + \sqrt{t})}{5 - t}\right)\Bigg|_{t=4}$.
39. If $f(x) = \dfrac{\sqrt{x}}{x + 1}$, find $f'(2)$.
40. Find $\dfrac{d}{dt}\Big((1 + t)(1 + 2t)(1 + 3t)(1 + 4t)\Big)\Big|_{t=0}$.
41. Find an equation of the tangent line to $y = \dfrac{2}{3 - 4\sqrt{x}}$ at the point $(1, -2)$.
42. Find equations of the tangent and normal to $y = \dfrac{x + 1}{x - 1}$ at $x = 2$.
43. Find the points on the curve $y = x + 1/x$ where the tangent line is horizontal.
44. Find the equations of all horizontal lines that are tangent to the curve $y = x^2(4 - x^2)$.

2.4

In Exercises 22–29, express the derivative of the given function in terms of the derivative f' of the differentiable function f.

22. $f(2t + 3)$
23. $f(5x - x^2)$
24. $\left[f\left(\dfrac{2}{x}\right)\right]^3$
25. $\sqrt{3 + 2f(x)}$
26. $f\left(\sqrt{3 + 2t}\right)$
27. $f\left(3 + 2\sqrt{x}\right)$
28. $f\Big(2f(3f(x))\Big)$
29. $f\Big(2 - 3f(4 - 5t)\Big)$

EXERCISES 2.5

1. Verify the formula for the derivative of $\csc x = 1/(\sin x)$.
2. Verify the formula for the derivative of $\cot x = (\cos x)/(\sin x)$.

Find the derivatives of the functions in Exercises 3–36. Simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.

3. $y = \cos 3x$
4. $y = \sin \dfrac{x}{5}$
5. $y = \tan \pi x$
6. $y = \sec ax$
7. $y = \cot(4 - 3x)$
8. $y = \sin((\pi - x)/3)$
9. $f(x) = \cos(s - rx)$
10. $y = \sin(Ax + B)$
11. $\sin(\pi x^2)$
12. $\cos(\sqrt{x})$
13. $y = \sqrt{1 + \cos x}$
14. $\sin(2\cos x)$
15. $f(x) = \cos(x + \sin x)$
16. $g(\theta) = \tan(\theta \sin\theta)$
17. $u = \sin^3(\pi x/2)$
18. $y = \sec(1/x)$
19. $F(t) = \sin at \cos at$
20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$
21. $\sin(2x) - \cos(2x)$
22. $\cos^2 x - \sin^2 x$
23. $\tan x + \cot x$
24. $\sec x - \csc x$
25. $\tan x - x$
26. $\tan(3x)\cot(3x)$
27. $t\cos t - \sin t$
28. $t\sin t + \cos t$

57. Use the method of Example 1 to evaluate $\lim_{h\to 0}\dfrac{1 - \cos h}{h^2}$.
58. Find values of a and b that make

$$f(x) = \begin{cases} ax + b, & x < 0 \\ 2\sin x + 3\cos x, & x \ge 0 \end{cases}$$

differentiable at $x = 0$.

EXERCISES 2.6

Find y', y'', and y''' for the functions in Exercises 1–12.

1. $y = (3 - 2x)^7$

2. $y = x^2 - \dfrac{1}{x}$

3. $y = \dfrac{6}{(x-1)^2}$

4. $y = \sqrt{ax + b}$

5. $y = x^{1/3} - x^{-1/3}$

6. $y = x^{10} + 2x^8$

7. $y = (x^2 + 3)\sqrt{x}$

8. $y = \dfrac{x-1}{x+1}$

9. $y = \tan x$

10. $y = \sec x$

11. $y = \cos(x^2)$

12. $y = \dfrac{\sin x}{x}$

In Exercises 13–23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

13. $f(x) = \dfrac{1}{x}$

14. $f(x) = \dfrac{1}{x^2}$

15. $f(x) = \dfrac{1}{2-x}$

16. $f(x) = \sqrt{x}$

17. $f(x) = \dfrac{1}{a + bx}$

18. $f(x) = x^{2/3}$

19. $f(x) = \cos(ax)$

20. $f(x) = x \cos x$

21. $f(x) = x \sin(ax)$

22. $f(x) = \dfrac{1}{|x|}$

23. $f(x) = \sqrt{1 - 3x}$

24. If $y = \tan kx$, show that $y'' = 2k^2 y(1 + y^2)$.

25. If $y = \sec kx$, show that $y'' = k^2 y(2y^2 - 1)$.

26. Use mathematical induction to prove that the nth derivative of $y = \sin(ax + b)$ is given by the formula asserted at the end of Example 5.

27. Use mathematical induction to prove that the nth derivative of $y = \tan x$ is of the form $P_{n+1}(\tan x)$, where P_{n+1} is a polynomial of degree $n + 1$.

28. If f and g are twice-differentiable functions, show that $(fg)'' = f''g + 2f'g' + fg''$.

29. State and prove the results analogous to that of Exercise 28 but for $(fg)^{(3)}$ and $(fg)^{(4)}$. Can you guess the formula for $(fg)^{(n)}$?