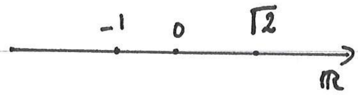
CALCULUS - lecture I (Adams, P1-6)

* P1 : Real numbers, intervals, absolute value (distance) inequalities
* P2 : Cartesian coordinates, equation of a line
* P3-5 Functions, even / odd functions, composite functions
* P6 : polynomials, rational functions

I. Real numbers

- $\mathbb{R}$: "the real line" (no gaps)



$\mathbb{N}$: natural numbers $1, 2, \ldots$

$\mathbb{Z}$: whole numbers (positive, negative, 0): $\ldots -2, -1, 0, 1, \ldots$

$\mathbb{Q}$: rational numbers, $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

cfr : discrete mathematics

↳ real numbers are "complete" (irrational numbers are also real)

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

- Real numbers can be added, subtracted, multiplied, divided (not by 0) and the result is a real number

$\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$

$x - y \in \mathbb{R}$

$xy \in \mathbb{R}$

$\forall x \in \mathbb{R}, \forall y \in \mathbb{R} \setminus \{0\} : \frac{x}{y} \in \mathbb{R}$

- Real numbers are ordered (total order) : $\forall x, y \in \mathbb{R} : x \leq y \vee y \leq x$

↳ calculation rules for inequalities

$\forall x, y, z \in \mathbb{R} : x \leq y \Rightarrow x + z \leq y + z$

$x \leq y,\ z > 0 \Rightarrow xz \leq yz$

$x < y,\ z < 0 \Rightarrow xz > yz$ (sign flips when multiplying with negative number)

$x > 0 \Leftrightarrow \frac{1}{x} > 0$

$0 < x < y \Leftrightarrow 0 < \frac{1}{y} < \frac{1}{x}$ (sign flips when inverting)

- Intervals are subsets of $\mathbb{R}$

[a,b] closed: end points included

(a,b] half open: one end point included

(a,b) open: end points not included

• " $(-\infty, a)$ is an infinite interval, $\mathbb{R} = (-\infty, +\infty)$

↳ inequality solving ! (solution = interval on real line)

Example: (ex 2) $2x-1 > x+3 \iff$

$+1$ ↓ $2x > x+4$ ↓ $+1$ $\iff$

$-x$ ↓ $x > 4$ ↓ $-x$

Solution: $x \in (4, \infty)$

Example (ex 3) $3 \leq 2x+1 \leq 5 \iff (3 \leq 2x+1) \wedge (2x+1 \leq 5)$

$\iff (2 \leq 2x) \wedge (2x \leq 4)$

$\iff 1 \leq x \wedge x \leq 2$

$\iff 1 \leq x \leq 2$

solution

$x \in [1, 2]$

↳ socrative

• Absolute value / magnitude (distance from 0)

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

↳ distance between 2 real numbers x, y: $|x-y|$

↳ $|x| = |-x|$, $|xy| = |x| \cdot |y|$, $|x \pm y| \leq |x| + |y|$

↳ triangle inequality

↳ equation-solving with absolute values : $|x-a| = D \iff x = a \pm D$

$x - a = \pm D$

inequality-solving " : $|x-a| \leq D \iff$

$-D \leq x-a \leq D \iff$

$a-D \leq x \leq a+D$

· $|x-a| > D \iff x-a > D \vee x-a < -D$

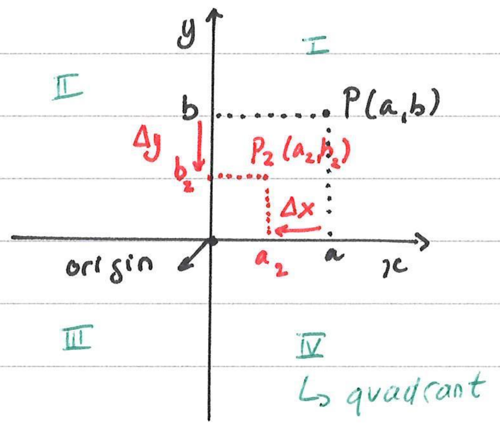
$\iff x > a+D \vee x < a-D$

Example: $|2x+5| < 1 \iff -1 < 2x+5 < 1 \iff -6 < 2x < 4$

$\iff -3 < x < 2$: $x \in (-3, 2)$

↳ socrative 2x

II Cartesian coordinates (the plane $\mathbb{R}^2$ $(\mathbb{R} \times \mathbb{R})$)



increment $\Delta x = a_2 - a$

$\Delta y = b_2 - b$

distance on the real line $\mathbb{R}$: $|x-y|$

↳ distance in the plane $\mathbb{R}^2$: $\sqrt{\Delta x^2 + \Delta y^2}$ (Euclidean distance)

solution of inequality / equation in $\mathbb{R}$ → interval / → point(s)

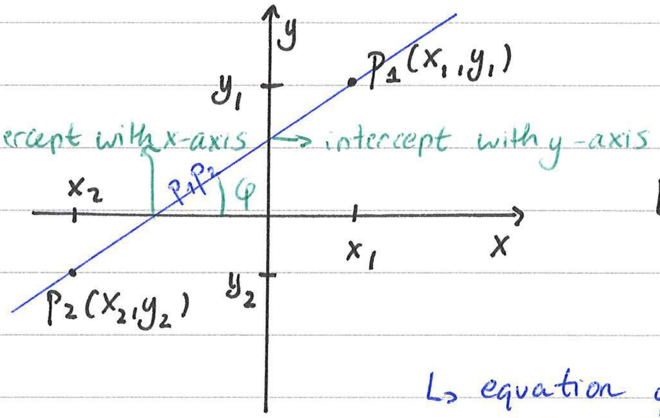
↳ solution of inequality in $\mathbb{R}^2$ → area in the plane / equation → curve "

Example

$x^2 + y^2 = 1$ $\quad$ $x^2+y^2 \leq 1$ $\quad$ $x + y = 1$ $\quad$ $x + y < 1$

* straight lines

P_1P_2 = the line through P_1 and P_2



* slope $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

↳ is the same for every 2 points on the line

↳ equation of a line :

$$y = \underset{\text{slope}}{m}\, x + \underset{y\text{-intercept}}{b}$$

↳ equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

equation of a line = linear equation.

$$y = mx + b = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{\text{slope}} \cdot x + \underbrace{\left(y_1 - \frac{y_2 - y_1}{x_2 - x_1} \cdot x_1\right)}_{y\text{-intercept}}$$

OR $y - y_1 = m(x - x_1) \Rightarrow y = \underbrace{(y_1 - m x_1)}_{b} + mx$

* two lines are perpendicular : $m_2 = -\frac{1}{m_1}$

Example : equation of the line through (1,-1) and (3,5)

$m = \frac{5-(-1)}{3-1} = \frac{6}{2} = 3$ $y+1 = 3(x-1) \Rightarrow y = 3x-4$

↳ soccrative.

III Functions (cfr. Discrete mathematics)

$f: D \to S$ assigns a UNIQUE $f(x) \in S$ to EVERY $x \in D$

domain ↘ co-domain. . Range : $\{y \in S : \exists x \in D : f(x) = y\}$

* example from physics : position x as a function of time t
  - you are always somewhere ($\forall t\ \exists x$)
  - you cannot be at 2 places at the same time.

↳ soccrative.

* in calculus, unless otherwise mentioned, we have a default domain
  ↳ $\{x \in \mathbb{R}$, for which $f(x)$ is defined$\}$
  co-domain : $\mathbb{R}$
  range : $\{f(x) \mid x \in D\}$

Examples : domain of $\sqrt{x^2-1}$: $\mathbb{R} \setminus (-1,1)$

$\frac{1}{x-1}$: $\mathbb{R} \setminus \{1\}$

↳ plot / graph of a function : $y = f(x)$ in $\mathbb{R}^2$

* functions can have symmetry
  - Even functions : $\forall x \in D : f(x) = f(-x)$
  - Odd functions : $\forall x \in D : f(x) = -f(-x)$

Even : x^2, $|x|$, $\cos(x)$ (symmetric around y-axis)

Odd : x, x^3, $\sin(x)$ (symmetric around origin)

* we can add / subtract / multiply / divide ~~and~~ functions
* composite functions $(f \circ g)(x) = f(g(x))$
  ↳ only if Range $(g) \subseteq$ Domain (f)

↳ Soccrative

BRUNNEN

IV Polynomials and rational functions

- a polynomial is a function of the form

  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

  ↳ $a_n \neq 0$, $n \in \mathbb{N}$ or $n=0$ is the ORDER of the polynomial

  $\{a_n\}$ are the coefficients

  domain : $\mathbb{R}$

  a linear function is a first order polynomial $y = mx + b$

  a constant is a zeroth order polynomial $y = c$.

  ↳ fundamental theorem of algebra :

  Every polynomial of degree $n \geq 1$ has at least 1 (possibly complex) root.

  * r is a "root" of a polynomial $P(x)$ if $P(r) = 0$
  * in this case, $P(x) = (x-r)\,Q(x)$, with Q a polynomial of degree $n-1$ (factor theorem)
  * a root that appears m times has multiplicity m.

    In this case, $P(x) = (x-r)^m \, Q(x)$

    ↳ degree $n-m$

- a rational function $R(x)$ is a function of the form

  $R(x) = \dfrac{P(x)}{Q(x)}$ with $P(x)$ and $Q(x)$ polynomials

  ↳ domain : $\mathbb{R} \setminus \{\text{zeros of } Q(x)\}$

  ↳ "poles"