Lec. 2: Limits

f has limit L at c if $f(x)$ approaches L whenever x appr. c.

Notation: $\lim_{x \to c} f(x) = L$

$$\lim_{x \to 5} \frac{x^2+x-2}{x-1} = \frac{28}{4} = 7.$$

$$\lim_{x \to -2} \frac{x^2-2x-8}{x+2} = \cancel{\frac{0}{0}}$$

$\boxed{\text{CANCEL OUT COMMON FACTOR}}$

$p(x) = x^2-2x-8$ has root -2

So $p(x) = (x+2)(x-4)$

$$= \lim_{x \to -2} \frac{\cancel{(x+2)}(x-4)}{\cancel{x+2}} = \lim_{x \to -2} (x-4) = -2-4 = -6.$$

$x^2-7x-18 = (x-9)(x+2)$

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x^2-7x-18} = \cancel{\frac{0}{0}}$$

$\boxed{\text{MULTIPLY BY CONJUGATE:}}$

$(a-b)(a+b) = a^2-b^2$

$$= \lim_{x \to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(x+2)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{1}{(x+2)(\sqrt{x}+3)} = \frac{1}{66}.$$

FORMAL DEFINITION:

$$\lim_{x\to c} f(x) = L \quad \text{IF} \quad \forall x \in D:$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \underbrace{0 <}_{} |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

LEFT LIMIT & RIGHT LIMIT:

$$\lim_{x\to c^-} f(x) = \lim_{x\uparrow c} f(x) = L$$

$$\lim_{x\to c^+} f(x) = \lim_{x\downarrow c} f(x) = L$$

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

$$\lim_{x\to 1} |x-1| = 0 \ , \ \lim_{x\to 1} x-1 = 0.$$

$$|x| = \sqrt{x^2}$$

$$|f(x)| = \begin{cases} f(x) & \text{IF } f(x) \geq 0 \\ -f(x) & \text{IF } f(x) < 0 \end{cases}$$

$$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \lim_{x\to 1^-} \frac{-\cancel{(x-1)}}{\cancel{x-1}} = -1$$

$$\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

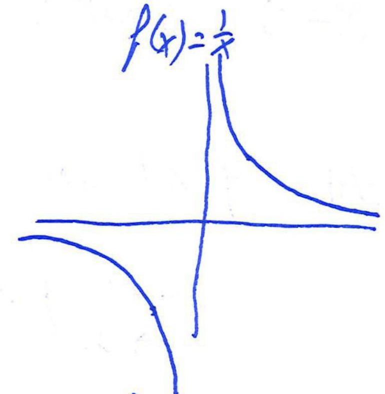
$$\lim_{x\to 1} \frac{|x-1|}{x-1} \ \text{D.N.E.}$$

DIE ROLL EXAMPLE: $F(x) = P(X \leq x)$

$$F(x) = \begin{cases} 0 & \text{IF } x < 1 \\ \frac{1}{6} & \text{IF } 1 \leq x < 2 \\ \vdots \end{cases}$$

$$\lim_{x \to 1^-} F(x) = 0$$

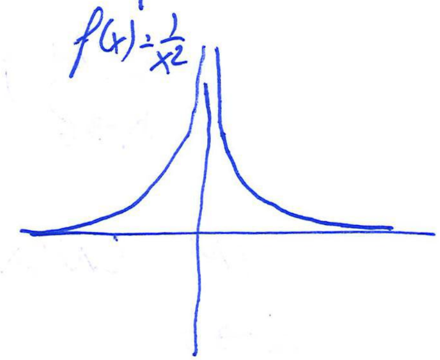
$$\lim_{x \to 1^+} F(x) = \frac{1}{6}$$



INFINITE LIMITS & VERTICAL ASYMPTOTES.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad [\text{D.N.E.}] \qquad \lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{"}$$



IF $\lim_{x \to c^\pm} f(x) = \pm\infty$, WE SAY THAT THE LINE $\underline{\underline{x = c}}$ IS A VERTICAL ASYMPTOTE OF f.

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x^3 - 10x^2 + 33x - 36} = \frac{P(x)}{Q(x)} \qquad P(3) = 0 \qquad Q(3) = 0$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x^2 - 7x + 12)} = \cancel{\frac{5}{0}} \qquad \text{V.A.: } x = 3$$

$f(x)$ ON $[a, b]$.

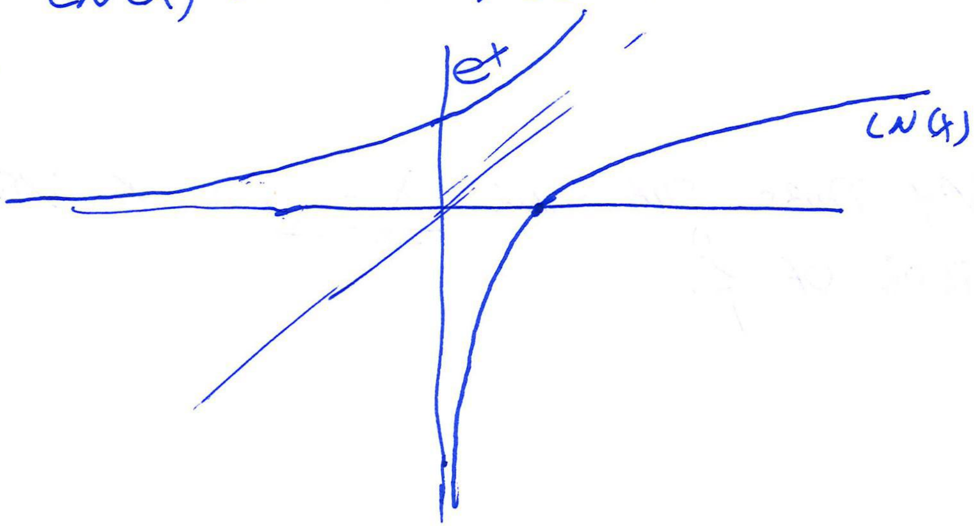
$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$$

$$\lim_{x \to b} f(x) = \lim_{x \to b^-} f(x)$$

$$D = [-1, 1]$$

$$\lim_{x \to 1} \sqrt{1 - x^2} = 0$$

$$\lim_{x \to 0} \ln(x) = -\infty$$, SO $x = 0$ IS A V.A. OF $\ln(x)$

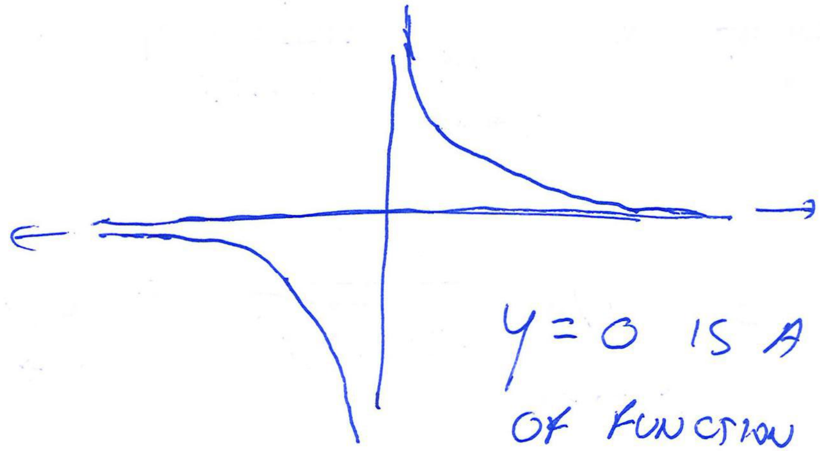


LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES.

IF $f(x)$ APPROACHES $\begin{matrix}L\\M\end{matrix}$ IF X APPROACHES $\begin{matrix}\infty\\-\infty\end{matrix}$, WE SAY THAT

$$\lim_{x\to\infty} f(x) = L$$

$$\lim_{x\to-\infty} f(x) = M$$



$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$$

$y = 0$ IS A H.A.
OF FUNCTION $\frac{1}{x}$.

RATIONAL FUNCTION: $f(x) = \dfrac{P(x)}{Q(x)}$

$\boxed{\text{DEGREE}(P) < \text{DEGREE}(Q)}$

$$\Rightarrow \lim_{x\to\infty} f(x) = 0 \ \& \ \lim_{x\to-\infty} f(x) = 0.$$

$$\lim_{x\to\infty} \frac{5x^2+10x+100}{\frac{1}{10}x^3-7x+1} = \lim_{x\to\infty} \frac{\frac{1}{x^3}(5x^2+10x+100)}{\frac{1}{x^3}(\frac{1}{10}x^3-7x+1)} = \lim_{x\to\infty} \frac{\frac{5}{x}+\frac{10}{x^2}+\frac{100}{x^3}}{\frac{1}{10}-\frac{7}{x^2}+\frac{1}{x^3}}$$

$$= \frac{0+0+0}{\frac{1}{10}-0+0} = 0.$$

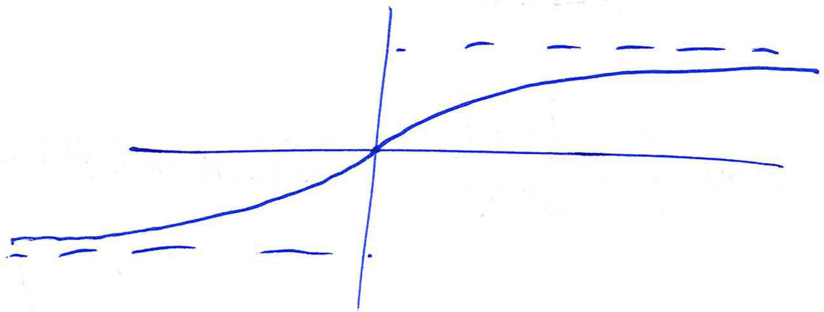
DEG.(P) = DEG.(Q)

$$\lim_{x\to\infty} \frac{5x^2+10x+100}{\frac{1}{10}x^2-7x+1} = \lim_{x\to\infty} \frac{5+\frac{10}{x}+\frac{100}{x^2}}{\frac{1}{10}-\frac{7}{x}+\frac{1}{x^2}} = \frac{5}{\frac{1}{10}} = 50.$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{x}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}$$

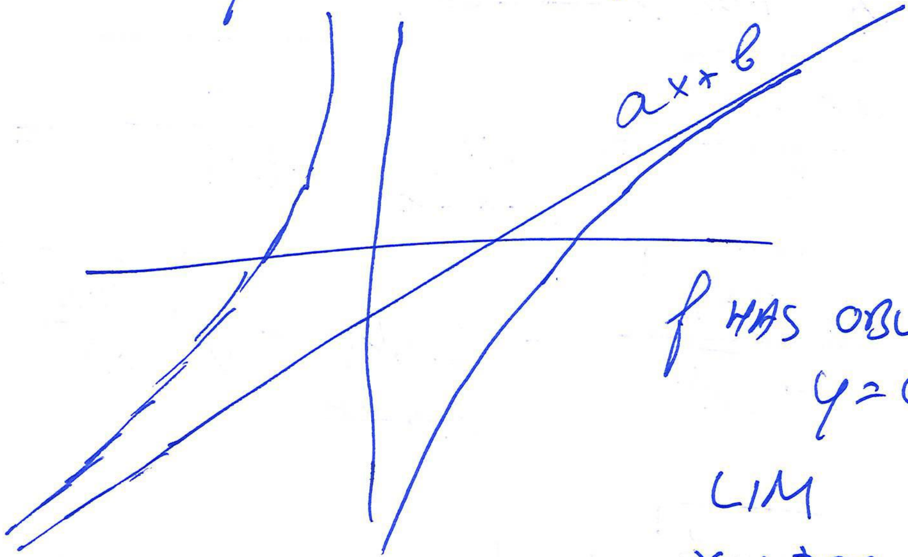
$$= \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\cdot\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1.$$



$$\lim_{x\to-\infty} e^x = 0$$

OBLIQUE ASYMPTOTE: $y = ax + b$, $a \neq 0$.



f HAS OBLIQUE ASYMPTOTE $y = ax + b$ IF

$$\lim_{x \to \pm\infty} \left[f(x) - (ax+b) \right] = 0.$$

RATIONAL FUNCTIONS WITH DEG(P) = DEG(Q) + 1

STEP 1 : $a = \lim\limits_{x \to \pm\infty} \frac{f(x)}{x}$

STEP 2 : $\lim\limits_{x \to \pm\infty} f(x) - ax = b.$

STEP 1 $f(x) = \frac{x^3 + 4x - 5}{x^2 - 3x + 1}$ $\quad \lim\limits_{x \to \infty} \frac{f(x)}{x} = \lim\limits_{x \to \infty} \frac{x^3 + 4x - 5}{x^3 - 3x^2 + x} = 1 = a.$

STEP 2 $\lim\limits_{x \to \infty} f(x) - x = \lim\limits_{x \to \infty} \frac{x^3 + 4x - 5 - x^3 + 3x^2 - x}{x^2 - 3x + 1} = \lim\limits_{x \to \infty} \frac{3x^2 + 3x - 5}{x^2 - 3x + 1} = 3$

$$\boxed{y = x + 3}$$

$\sqrt{x^2+4x}$ HAS AN O.A. FOR $x \to -\infty$.

STEP 1: $a = \lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{-|x|} =$

$= \lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{-\sqrt{x^2}} = \lim_{x\to-\infty} -\sqrt{\frac{x^2+4x}{x^2}}$

$= \lim_{x\to-\infty} -\sqrt{1+\frac{4}{x}} = -1$, SO $\underline{\underline{a=-1}}$.

STEP 2: $b = \lim_{x\to-\infty} f(x) - ax = \lim_{x\to-\infty} \sqrt{x^2+4x} + x \cdot \frac{(\sqrt{x^2+4x} - x)}{(\sqrt{x^2+4x} - x)}$

$= \lim_{x\to-\infty} \frac{\cancel{x^2}+4x-\cancel{x^2}}{\sqrt{x^2+4x} - x} = \lim_{x\to-\infty} \frac{4x}{\sqrt{x^2}\sqrt{1+\frac{4}{x}} - x}$

$= \lim_{x\to-\infty} \frac{4x}{|x|\cdot\sqrt{1+\frac{4}{x}} - x} = \lim_{x\to-\infty} \frac{4x}{-x\sqrt{1+\frac{4}{x}} - x} = \lim_{x\to-\infty} \frac{4}{-\sqrt{1+\frac{4}{x}} - 1} = -2.$

$y = -x - 2.$