Calculus

Lecture 1: Functions and continuity

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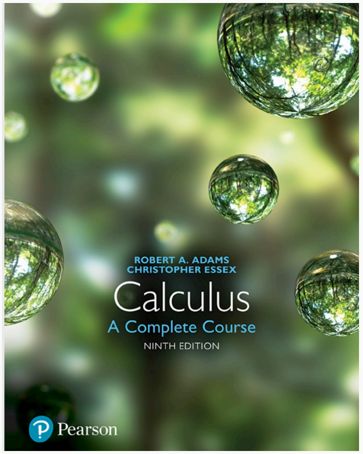
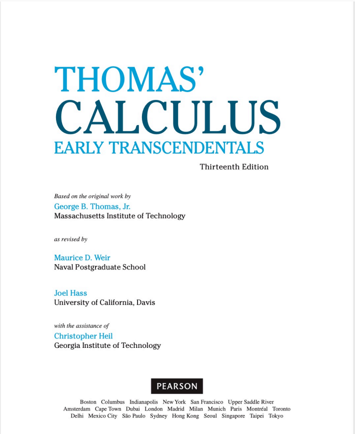
Calculus: Practicalities

- This class has 9 lectures, 7 tutorials, and a Q&A/revision lecture
  - In the tutorials you work on exercises and you have the opportunity to ask questions
- Lecturers: Otti D'Huys (also coordinator), Gijs Schoenmakers
- Teaching Assistants: Juliette Maes, Bregje Derks, Yulin Zhou, Adee Sella, Tommaso Siligardi, Bochen Qiao, Vita Stefanija, Joseph el Khazen, Spyridon Giagtzoglou

Calculus: Practicalities

- Lecture materials (on Canvas):
  - (sometimes) a scan of our handwritten preparation or lecture material
  - the lecture slides
- Tutorial materials (on Canvas):
  - a list of tutorial exercises
  - pdf with their solutions
  - a checklist per course module, with all relevant concepts and an exhaustive list of useful exercises (no need to try them all...)
- Books: mainly Adams (9^{th} ed) and Thomas (13^{th} ed)





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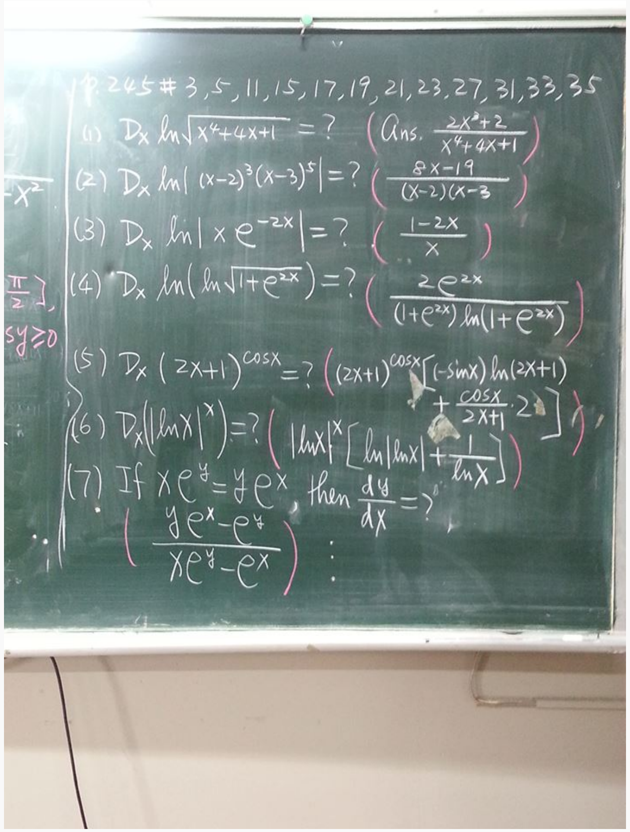
Calculus: Practicalities

- Evaluation = 100% final exam + 10% bonus quizzes
  Exam:
  - Closed book
  - Formula sheet will be provided
  - Calculators are not allowed

  Quizzes
  - 2 Canvas quizzes, worth 5 % each (bonus)
  - Multiple choice and numerical questions
  - The quiz remains open for 5 days, but once you start it, you have limited time to complete it.
  - In weeks 3-4 and 4-5
- How to get help?
  - Discussion Boards on Canvas
  - Ask us during tutorials
  - No emails

Calculus: Course Contents

- Limits and continuity
- Differentiation
- Integration
- Basics of sequences and series
- Basics of multivariate calculus



Why Calculus?

- Calculus was developed to describe motion (mainly by Isaac Newton and Gottfried Leibniz in the 17th century)
- It is a language to describe the world in a numerical way, in terms of functions and their rate of change.
    - Mathematical modelling, control theory, robotics,... all describe systems with differential equations.
    - Probability and statistics: a mathematical description of chance
    - Optimization: finding optimal (extreme) values
    - ...

Functions and continuity - Book chapters

Adams:

- P.1 Real numbers, intervals, absolute value
- P.2 Equation of a line
- P.3 Functions
- P.5 Combining functions
- P.6 Polynomials and Rational functions
- 1.4 Continuity

Real functions

- A function $f : D \to S$ on a set D into a set S is a mapping that assigns a **unique** element $f(x) \in S$ to **each** element $x \in D$.
- Domain D: $\mathbb{R}$ or a subset of $\mathbb{R}$
  - Domain convention:
    domain $= \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$
  - Open interval: (a, b) $\{x \in \mathbb{R} : a < x < b\}$ 

  - Closed interval: $[a, b]$ $\{x \in \mathbb{R} : a \leq x \leq b\}$ 

- Co-domain S: $\mathbb{R}$
- Range: $\{f(x) \mid x \in D\}$
- We can add, subtract, multiply, divide functions
- Composite functions: $f \circ g(x) = f(g(x))$ (only if range$(g) \subseteq$ domain(f))

you need to remember 3 rules to determine the domain of a function

1. no division by 0

   $f(x) = \frac{1}{x}$, domain $(f) = \mathbb{R} \setminus \{0\}$

2. no squareroots of negative numbers

   $f(x) = \sqrt{x}$, domain $(f) = [0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$

3. no logarithms of zero or negative numbers

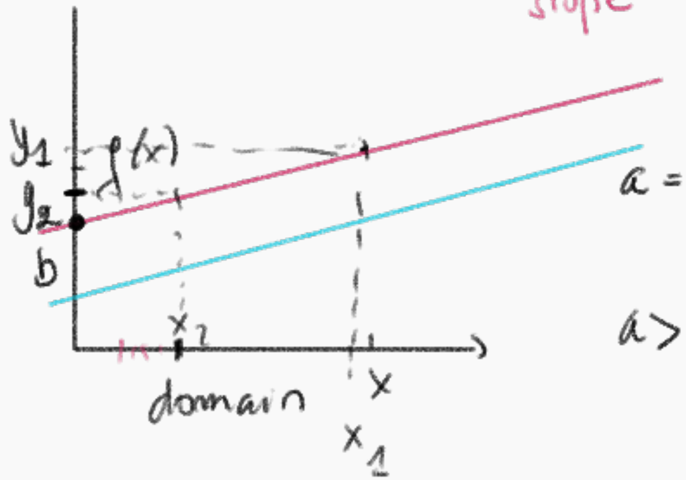
   $f(x) = \ln(x)$, domain $(f) = (0, \infty) = \{x \in \mathbb{R} \mid x > 0\}$

Equation of a line

$f(x) = ax + b$ → y-intercept

slope

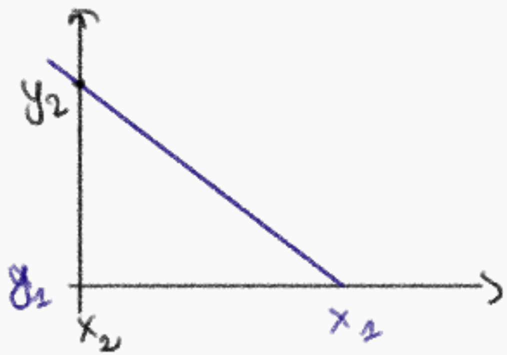
domain: $\mathbb{R}$



$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

$a > 0$: increasing function

• 2 lines are parallel if they have the same slopes, $a_1 = a_2$



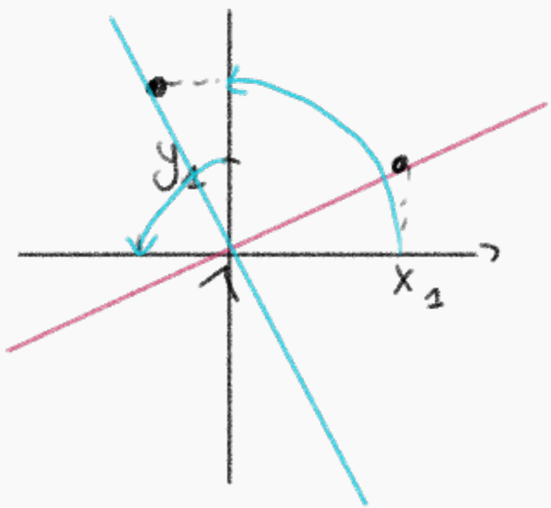
$a < 0$

$$a = \frac{y_2 - y_1}{x_2 - x_1} < 0$$

$a = 0$

↳ $f(x) = b$ constant function

Perpendicular lines



$$a_1 = \frac{y_1}{x_1}$$

$$a_2 = \frac{x_1}{-y_1}$$

$$a_1 \cdot a_2 = \frac{y_1}{x_1} \cdot \frac{x_1}{-y_1} = -1$$

↳ 2 lines are $\left\{\begin{array}{l}\text{perpendicular}\\ \text{orthogonal}\end{array}\right.$ if the product of the slopes $a_1 \cdot a_2 = -1$

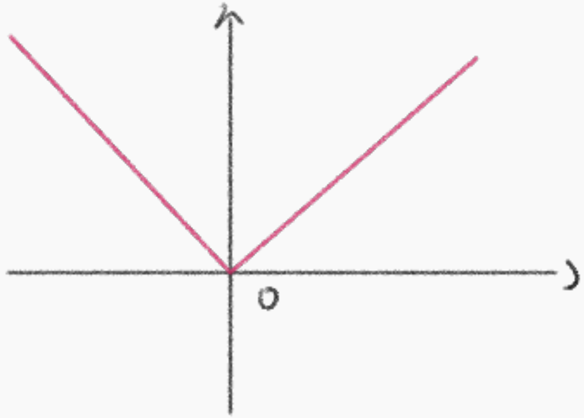
- NOTE : equation of x-axis : $y = 0$ (constant functions $f(x) = b$ are parallel to the x-axis)
  slope $a = 0$

  equation of y-axis : $x = 0$ (this is not a function $y = f(x)$)
  slope $a = \pm\infty$

Absolute value

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

absolute value = distance of a number to



$|a \cdot b| = |a| \cdot |b|$

$|a+b| \leq |a| + |b|$ (triangle inequality)

Polynomials

$f(x) = P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$\hookrightarrow a_n \neq 0$

$\{a_n\}$: coefficients
$\hookrightarrow$ these are real numbers

domain : $\mathbb{R}$

- Degree of a polynomial: n (highest power of x)
- Root of a polynomial: r is a root of $P(x)$ if $P(r) = 0$
  $\hookrightarrow$ if r is a root, $P(x) = (x-r)\underbrace{Q(x)}_{\text{polynomial of degree } n-1}$
- Number of (complex) roots of a polynomial: n
  $\hookrightarrow$ we can have roots with multiplicity m : $P(x) = (x-r)^m Q(x)$
  ex. $x^2 - 2x + 1 = (x-1)^2$, so 1 is a root with multiplicity 2
  $\longrightarrow$ in this case, the sum of the multiplicities of each root add up to n.

- polynomial of degree
  1 : $a_1 x + a_0 \longrightarrow$ linear function
  2 : $a_2 x^2 + a_1 x + a_0 \longrightarrow$ quadratic function (parabola)
  0 : $f(x) = a_0 \longrightarrow$ constant function

Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$ polynomials

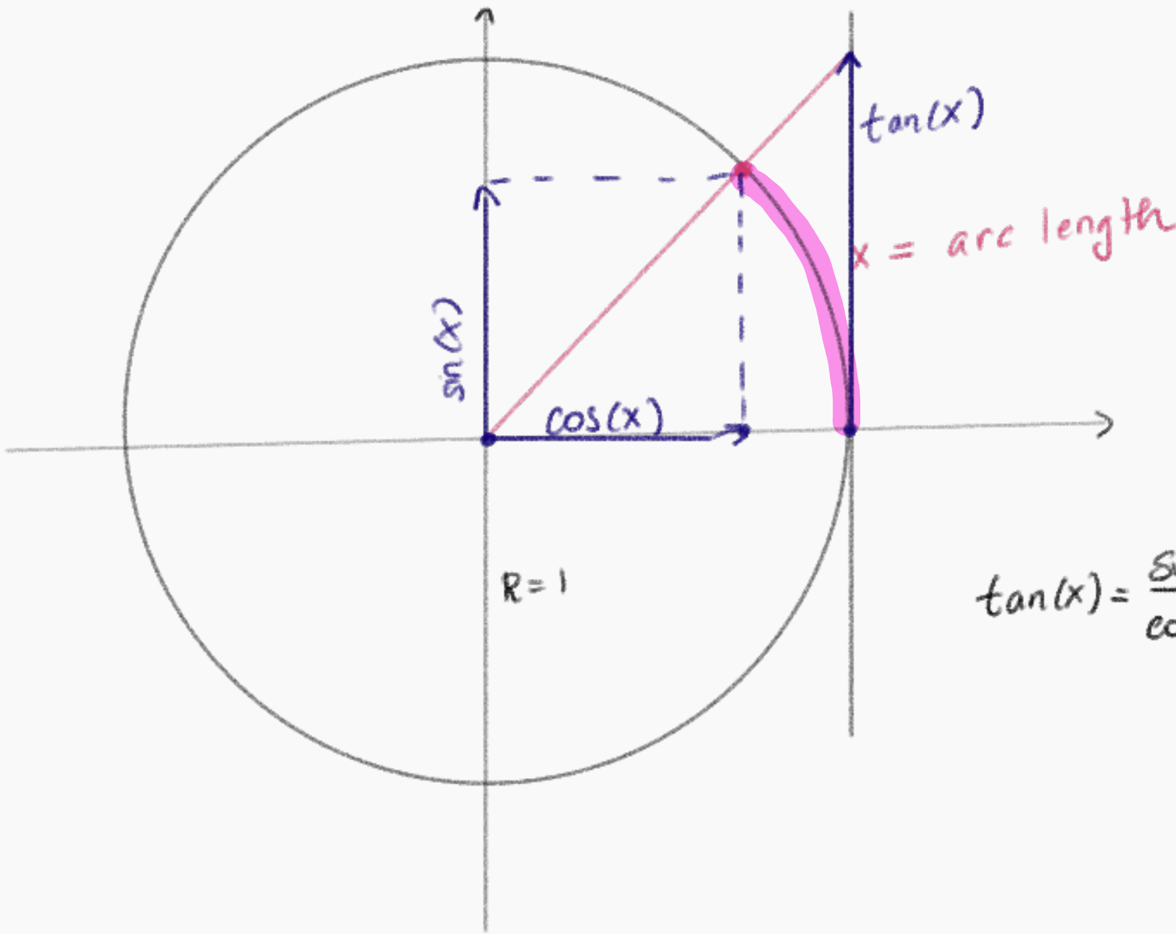
domain : $\mathbb{R} \setminus \underbrace{\{ \text{zeros of } Q(x) \}}_{\text{poles}}$

Example: $\dfrac{2x+3}{x^2-7x+12}$

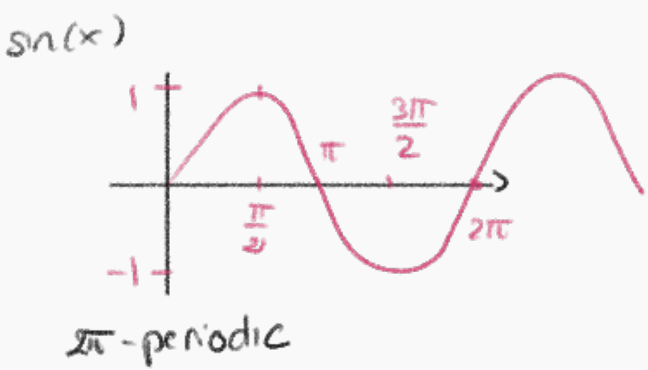
$x^2 - 7x + 12 = (x-3)(x-4)$

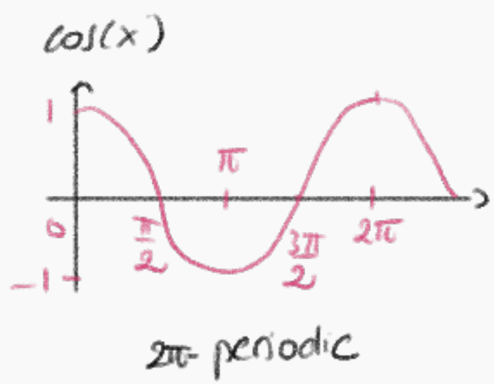
poles : 3, 4

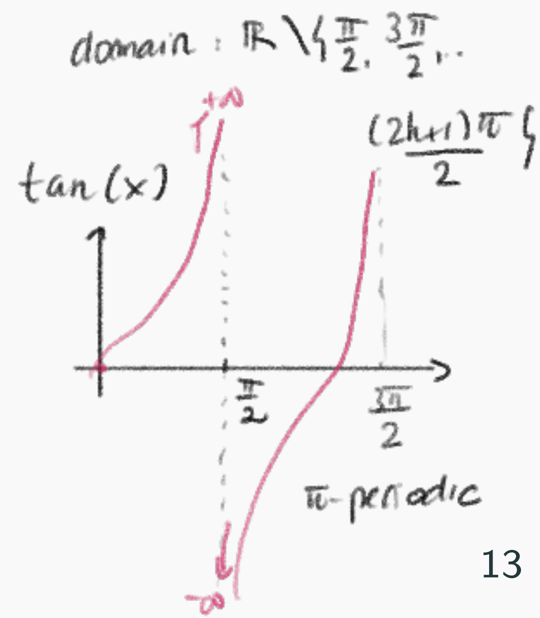
Trigonometric functions



$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$







Exponential and logarithmic functions

$e^x = \exp(x)$

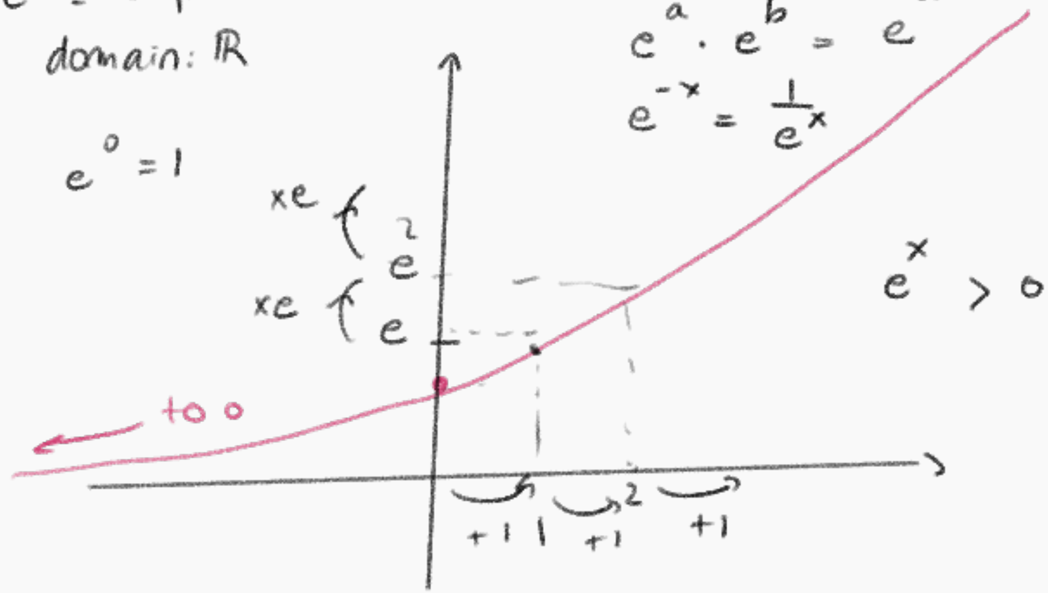
domain: $\mathbb{R}$

$e^0 = 1$

$2^x = e^{\ln(2)\cdot x}$

$e^a \cdot e^b = e^{a+b}$

$e^{-x} = \frac{1}{e^x}$



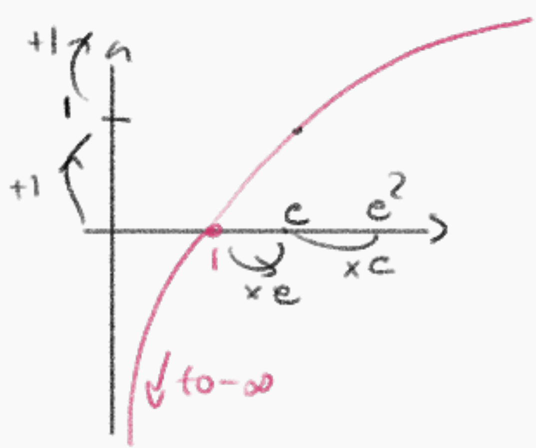
$e^x > 0 \quad \forall x \in \mathbb{R}$

$\ln(x)$ = inverse function of e^x

domain : $(0,\infty)$ (range of e^x)

$\ln(1) = 0$

$\lim_{x \to 0} \ln(x) = -\infty$



Even and odd functions

- Even functions $f(x) = f(-x)$

  $|x|$, $\cos(x)$, x^2, constant functions, $x^4 + x^2$

  mirror around y-axis

- Odd functions $f(x) = -f(-x)$

  x, x^3, $\sin(x)$

$f(x) = \frac{1}{x-1}$

not even, since $f(-x) = \frac{1}{-x-1} \neq \frac{1}{x-1} = f(x)$

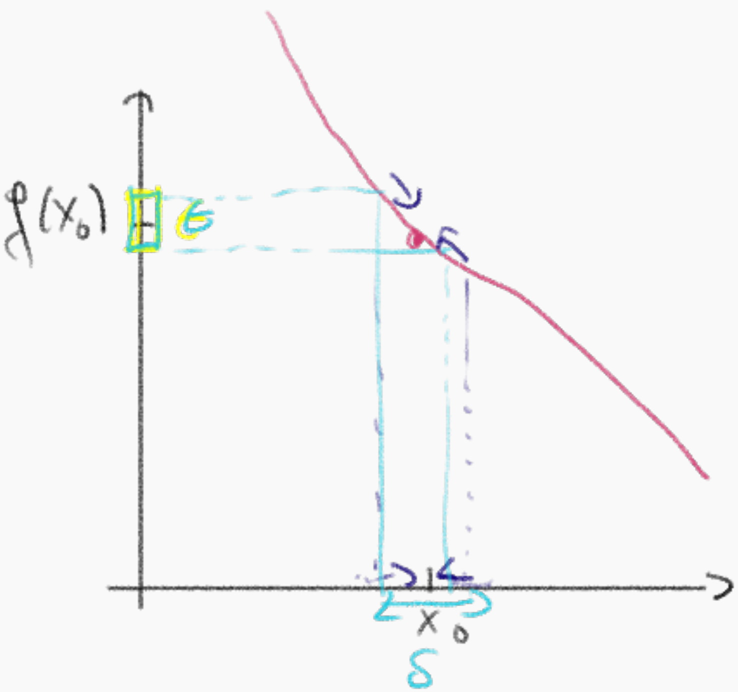
not odd, since $f(-x) = \frac{1}{-x-1} \neq \frac{-1}{x-1} = -f(x)$

Continuity

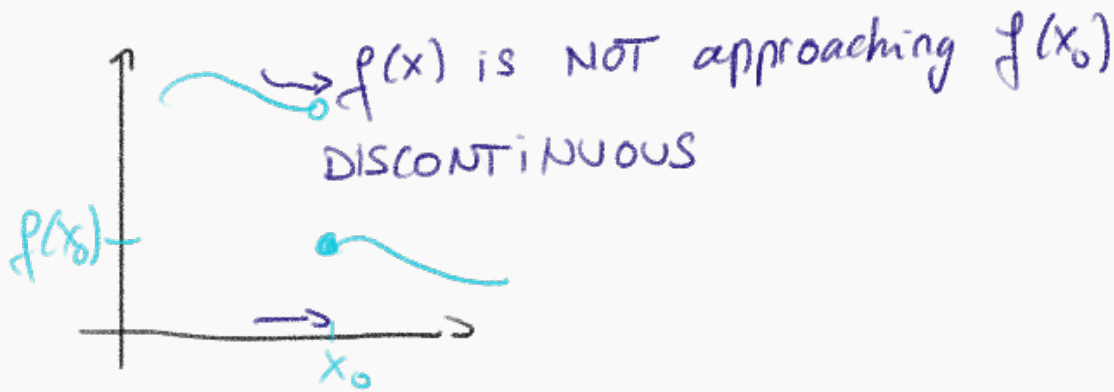
A function $f(x)$ is **continuous** at a point x_0 of its domain if, for all points x in the domain,

$$\forall \epsilon > 0,\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

* no jumps
* you do not need to lift your pen



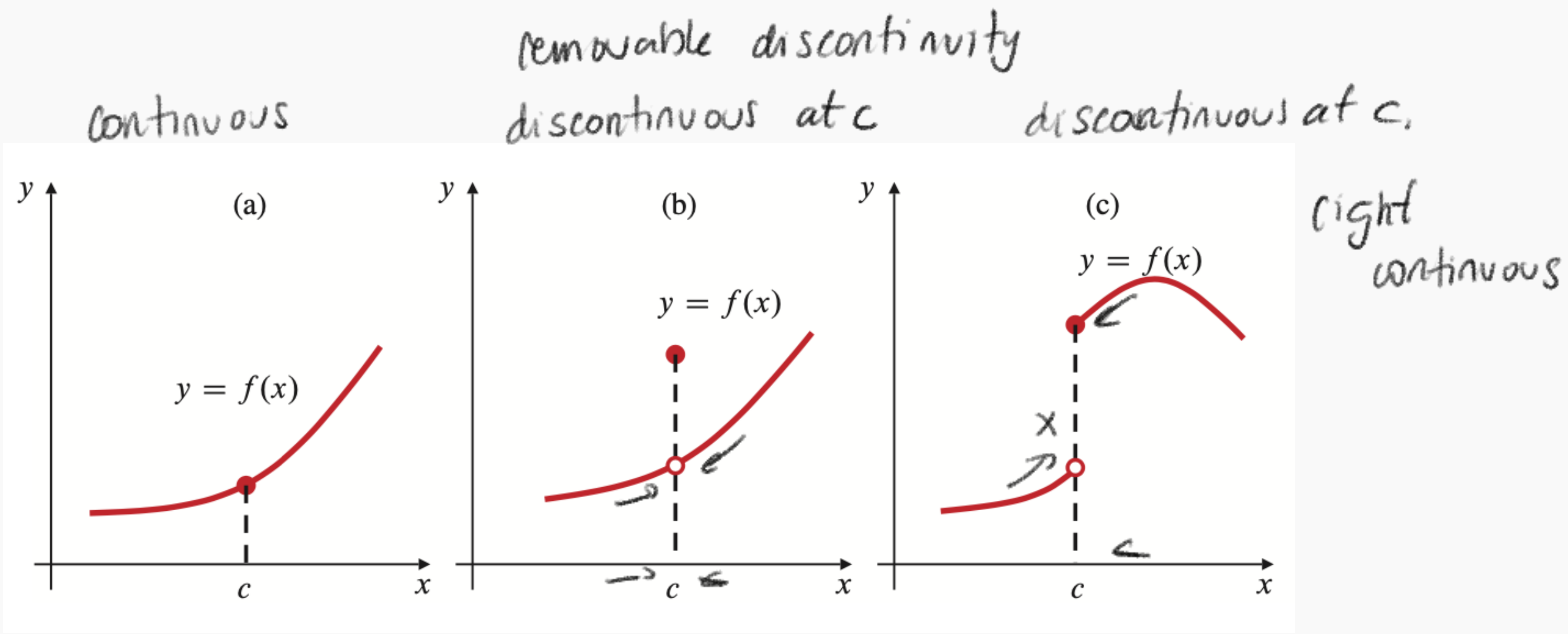
f is continuous at x_0 if, as x approaches x_0, $f(x)$ approaches $f(x_0)$



Continuity

- A function $f(x)$ is **discontinuous** at c if

- Examples:



Note: (in this course) f can only be discontinous at c if c is in the domain of f, i.e. if $f(c)$ exists (in this course). So $f(x) = \frac{1}{x}$ is not discontinous at $x = 0$, it is undefined.

Left and right continuity

A function $f(x)$ is

(from right)

- right continuous at c if $f(x)$ approaches $f(x_0)$ if x approaches x_0, $x > x_0$
- left continuous at c if $f(x)$ approaches $f(x_0)$ if x approaches x_0, $x < x_0$ from left.
- continuous at c if it is both right and left continuous at c.

- right continuous: $\forall \epsilon > 0 \; \exists \delta > 0 : 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$
- left continuous: $\forall \epsilon > 0 \; \exists \delta > 0 : 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Continuous functions on an interval

A function $f(x)$ is

- continuous on an interval $[a, b]$ if
    - it is continuous on all interior points (a,b)
    - it is left continuous a a
    - it is right continuous at b
- piecewise continuous on $[a, b]$ if there are a finite number of discontinuities on [a,b]
  -> i.e. continuous pieces

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L1: functions and continuity

Save and Exit

Align Quiz to Standard

Share

1. Which of the following graphs show a function *y=f(x)*? Multiple answers can be correct.

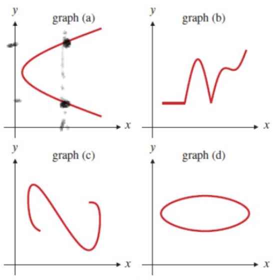


Figure P.56

for a graph to be a (well defined) function, every $x \in$ domain has EXACTLY 1 $y=f(x)$-value.
In graphs (a), (c) and (d), several values of x have 2 y-values associated with it.

A graph (a)

B graph (b)

C graph (c)

D graph (d)

2. For $f(x)=x+5$ and $g(x)=x^2-3$, which combination of $f(x)$ and $g(x)$ is the function $h(x)=x^2+2$?

A $h(x)=g(x)+f(x)$

B $h(x)=f(g(x))$

C $h(x)=g(f(x))$

D $h(x)=f(x)g(x)$

$f(g(x)) = f(x^2-3) = (x^2-3)+5 = x^2-2$

3. What is the equation of the line going through *(2,3)* and *(1,1)*?

A $y=-2x+3$

B $y=x/2+2$

C $y=2x-1$

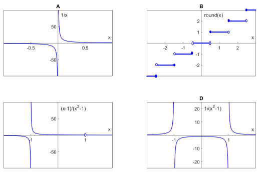
D $y=x/2+1/2$

$y = ax + b$

$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{2-1} = 2$

$b = y_1 - a \cdot x_1 = 1 - 2 \cdot 1 = -1$

4. Which of the following functions is continuous on its domain? Check all that applies



A $f(x) = \frac{1}{x}$

B $f(x) = round(x)$

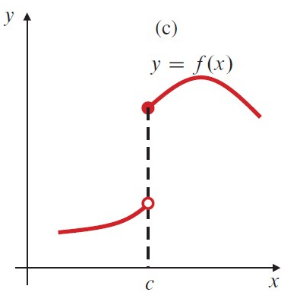
C $f(x) = \frac{x-1}{x^2-1}$

D $f(x) = \frac{1}{x^2-1}$

A, C and D are continuous on their domain, since ON THE DOMAIN, there are no jumps (the "problem zones" are not in the domain)

B is not continuous at $x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$

5. Check all that applies



A $f(x)$ is discontinuous at *x=c*

B $f(x)$ is piecewise continuous (on its domain)

C $f(x)$ is left continuous at *x=c*

D $f(x)$ is right continuous at *x=c*

Add Blank Question