LECTURE 4 $\qquad y = f(c) + f'(c)\cdot(x-c)$ $\qquad$ 1

- L'HOPITAL
- EXTREME VALUES
- CONCAVITY / CONVEXITY
- FUNCTION SKETCHING

INDETERMINATE FORMS OF LIMITS: $\underbrace{\frac{0}{0};\ \frac{\infty}{\infty}}_{\text{L'HOPITAL}};\ \underbrace{0\cdot\infty;\ \infty-\infty;\ \infty^0;\ 0^0;\ 1^\infty}_{\text{REWRITE}}$

L'HOPITAL #1 ($\frac{0}{0}$): IF f, g DIFFERENTIABLE ON (a,b) $((\infty,-\infty))$ WITH

1. $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ FOR SOME $c \in [a,b]$

2. $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ (COULD BE ∞ OR $-\infty$)

THEN $\lim_{x\to c} \frac{f(x)}{g(x)} = L$

$$\frac{f}{g} \approx \frac{\cancel{f(c)} + f'(c)\cdot\cancel{(x-c)}}{\cancel{g(c)} + g'(c)\cdot\cancel{(x-c)}}$$

EX: $$\lim_{x\to 3} \frac{x^2-x-6}{x^2-7x+12} \overset{[\frac{0}{0}]}{=} \lim_{x\to 3} \frac{2x-1}{2x-7} = \frac{5}{-1} = -5.$$

$$\lim_{x\to 16} \frac{\sqrt{x}-4}{\sqrt[4]{x}-2} \overset{[\frac{0}{0}]}{=} \lim_{x\to 16} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{4\sqrt[4]{x^3}}} = \frac{\frac{1}{2\cdot 4}}{\frac{1}{4\cdot 8}} = 4.$$

$$\lim_{x\to 1} \frac{x^3-1}{\ln(x)} \overset{[\frac{0}{0}]}{=} \lim_{x\to 1} \frac{3x^2}{\frac{1}{x}} = 3$$

L'HÔPITAL #2: f, g DIFFERENTIABLE ON (a, b) $[(-\infty, \infty)]$ WITH

1. $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \pm\infty$

2. $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ (CAN BE $+\infty$ OR $-\infty$)

THEN $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ (ALSO FOR $x\to b^-$)

EX: $$\lim_{x\to 0^+} x\ln(x) \overset{[0\cdot\infty]}{=} \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

2

$$\lim_{x\to\infty} x^2 \cdot e^{-x} \overset{[0\cdot\infty]}{=} \lim_{x\to\infty} \frac{x^2}{e^x} \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

$$\lim_{x\to 0^+} x^x \overset{[0^0]}{=} \lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^0 = 1 \quad \left[\begin{array}{l} \ln(a^b) = b\cdot\ln(a) \\ e^{\ln(a)} = a \end{array}\right.$$

$$\lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\cdot\ln(x) = 0.$$

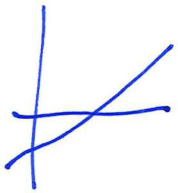
$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \overset{[1^\infty]}{=} \lim_{x\to\infty} e^{\ln\left[\left(1+\frac{1}{x}\right)^x\right]} = e^1 = e.$$

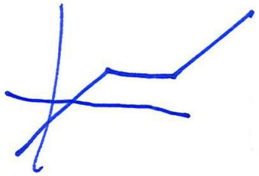
$$\lim_{x\to\infty} \ln\left[\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right) \overset{[\infty\cdot 0]}{=} \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{[\frac{0}{0}]}{=}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

FUNCTION SKETCH:

- FIND ROOTS OF f : WHEN IS f POSITIVE / ZERO / NEGATIVE?
- DOMAIN?
- INCREASING / DECREASING





- f IS INCREASING ON $[a,b]$ IF $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- " " DE " " " $f(x_1) > f(x_2)$
- NONINCREASING IF $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- " DE " $f(x_1) \leq f(x_2)$)

__THM__: f IS CONTINUOUS ON $[a,b]$, DIFFERENTIABLE ON (a,b)

IF $f'(x) > 0$ ON (a,b), THEN f IS INCREASING ON $[a,b]$.
$<$ DE

f INCREASING $\not\Rightarrow f'(x) > 0$: $f(x) = x^3$ ON $[-1,1]$.

$f'(x) = 3x^2$, SO $f'(0) = 0$.

③

EX: $f(x) = x^3 + x^2 - 5x - 5 = (x+1)(x^2-5) = (x+1)(x-\sqrt{5})(x+\sqrt{5})$

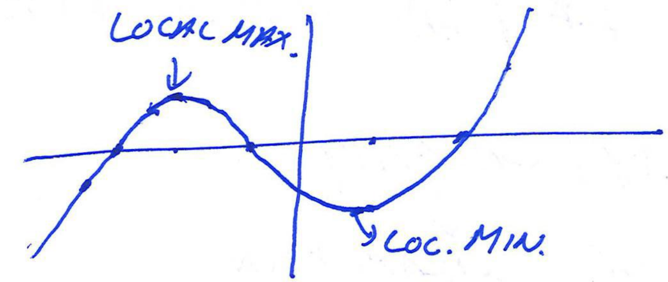
DOMAIN: $\mathbb{R}$, CONTINUOUS.

$f(x) > 0$ IF $x > \sqrt{5}$

AND IF $-\sqrt{5} < x < -1$

$f(x) = 0$ IF $x = -1$

$f'(x) = 3x^2 + 2x - 5 = (x-1)(3x+5) = 0$ IF $x = 1$ OR $x = -\frac{5}{3}$.



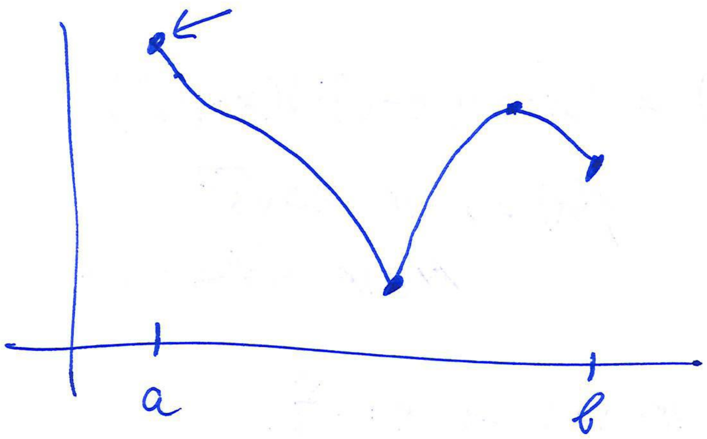
MINIMA & MAXIMA (EXTREME POINTS)

- f HAS GLOBAL/ABSOLUTE MAXIMUM (MINIMUM) AT $c \in D$ IF $f(c) \geq f(x)$ ($\leq$) $\forall x \in D$.

- f HAS LOCAL MAXIMUM (MINIMUM) AT $c \in D$ IF $f(c) \geq f(x)$ ($\leq$) $\forall x \in D$ "CLOSE TO c".

WHERE TO LOOK:
- $f'(x) = 0$: CRITICAL POINTS
- $f'(x)$ D.N.E. : SINGULAR POINTS
- IF $D = [a, b]$, CONSIDER a & b : ENDPOINTS.

THM: IF f CONTINUOUS ON $D = [a, b]$, THEN f HAS A GLOBAL MAX & MIN.



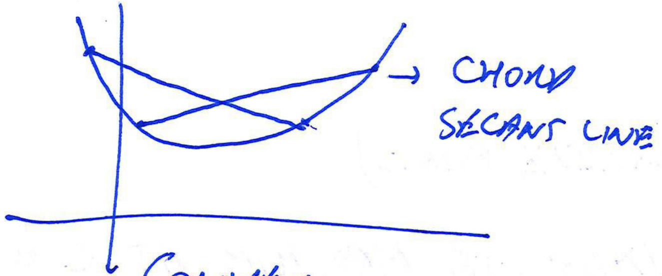
$f(x) = x$ HAS NO MAX/MIN ON (a, b)

THM: IF f HAS LOC/GLOB EXTR. AT c AND $f'(c)$ EXISTS, THEN $f'(c) = 0$

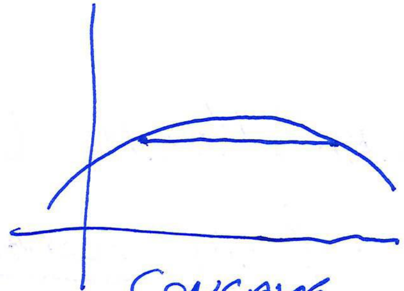
CONVEXITY / CONCAVITY :

↓ CONCAVE UP

↓ CONCAVE DOWN



CONVEX



CONCAVE

INFLECTION POINTS: POINTS WHERE CONCAVITY OF f CHANGES

THM: LET f CONTINUOUS ON $[a, b]$ AND DIFFERENTIABLE ON (a, b).
IF $f''(x) > 0$ ON (a, b), THEN f IS CONVEX ON $[a, b]$
CONCAVE UP

$<$ CONCAVE (DOWN)

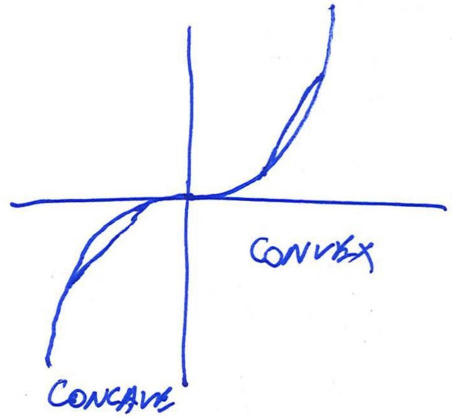
IN AN INFLECTION POINT c WE HAVE $f''(c) = 0$.

$f(x) = x^3$ HAS INFL. POINT AT $x = 0$.

$f'(x) = 3x^2$, $f''(x) = 6x < 0$ IF $x < 0$

$= 0$ IF $x = 0$

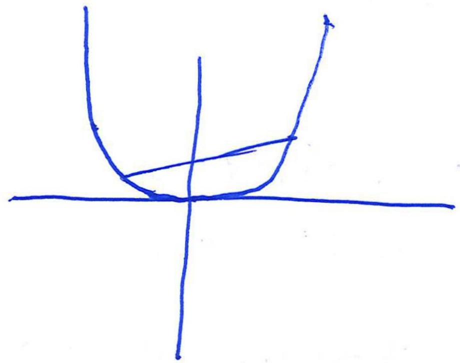
> 0 IF $x > 0$



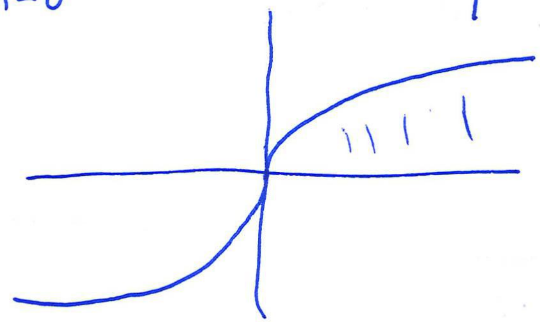
$f(x) = x^4$

$f'(x) = 4x^3$, $f'' = 12x^2 = 0$ IF $x = 0$.

> 0 FOR ALL $x \neq 0$.



$f(x) = \sqrt[3]{x}$ HAS INFLECTION POINT AT $x = 0$



$$f(x) = \frac{x^2+3x-4}{x-2}$$

DOMAIN: $\mathbb{R} \setminus \{2\}$

$$\lim_{x\to 2^-} f(x) = \frac{6}{0^-} = -\infty$$

$$\lim_{x\to 2^+} f(x) = +\infty$$

$$\boxed{V.A. : x=2}$$

$y = ax + b$

O.A. 1) $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2+3x-4}{x^2-2x} = \dots = 1$

2) $\lim_{x\to\pm\infty} f(x) - ax = \lim_{x\to\pm\infty} \frac{5x-4}{x-2} = 5$

O.A. $\boxed{y = x+5}$

$f(x) = 0$ IF $x^2+3x-4=0$

$\Leftrightarrow x=1 \vee x=-4.$

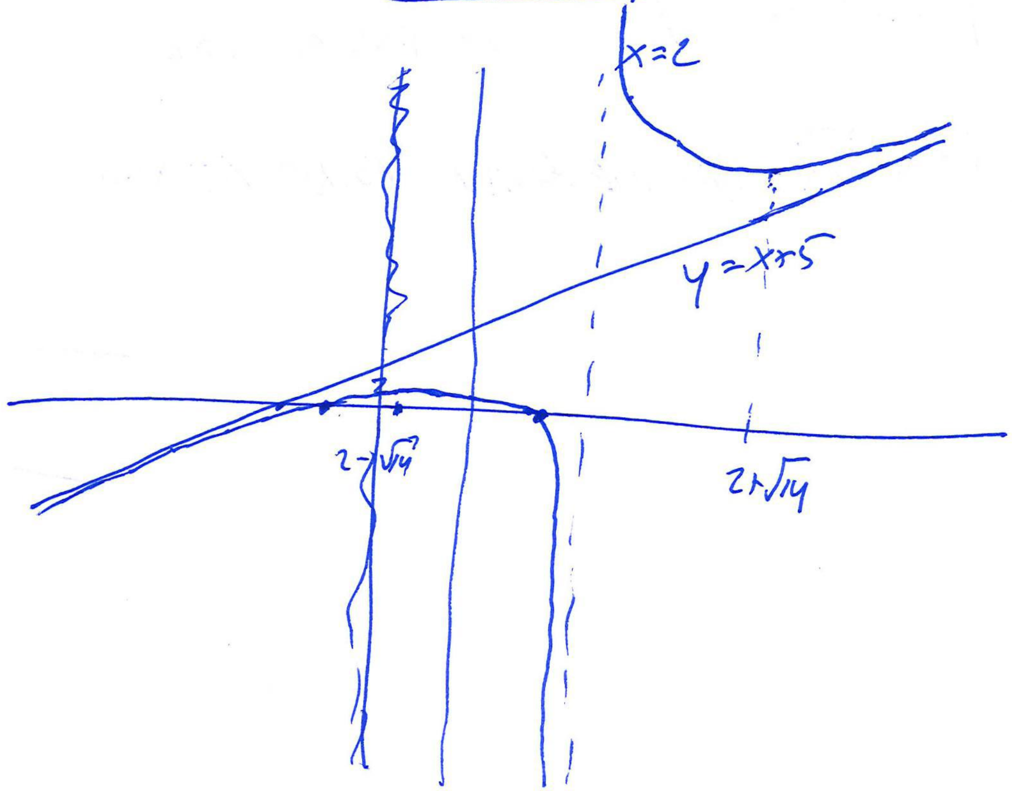
$f(x) > 0$ IF $-4 < x < 1$ OR $x > 2.$

$$f'(x) = \frac{x^2-4x-10}{(x-2)^2}$$

$= 0 \Leftrightarrow x = 2 \pm \sqrt{14}$

$f''(x) = \dots = \frac{20}{(x-2)^3}$ > 0 IF $x > 2$

< 0 IF $x < 2$



$f(x) = x^2 \cdot e^{-x}$

$\boxed{H.A.: y = 0 \text{ FOR } x \to \infty}$

- DOMAIN : $\mathbb{R}$
- $\lim_{x \to \infty} x^2 e^{-x} = 0$
- $\lim_{x \to -\infty} x^2 \cdot e^{-x} = \infty$

$f(x) \geq 0$ AND $f(0) = 0$: GLOBAL MIN. AT $x = 0$.

$f'(x) = x \cdot (2-x) \cdot e^{-x}$

$= 0$ IF $x = 0 \vee x = 2$.

$f'(x) > 0$ IF $0 < x < 2$

$f(2) = \frac{4}{e^2}$

- $f''(x) = (x^2 - 4x + 2) \cdot e^{-x}$

$f''(x) = 0 \Leftrightarrow x = 2 \pm \sqrt{2}$

$f''(x) < 0$ IF $2 - \sqrt{2} < x < 2 + \sqrt{2}$ (CONCAVE)

$f''(x) > 0$ ELSEWHERE. (CONVEX)