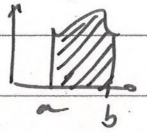
CALCULUS lecture 6 : INTEGRATION TECHNIQUES

Book (Adams & Essex) 5.6, 6.1, 6.2, 6.5

Recap : • definite integral $\int_a^b f(x)dx$ is defined as

- area below a graph



- limit of $\left\{\begin{array}{l}\text{a Riemann sum}\\ \text{upper/lower sums}\end{array}\right.$ of rectangular areas

• indefinite integral $\int f(x)dx = F(x) + C$

$$\Leftrightarrow \frac{d}{dx}(F(x)) = f(x)$$

is an "anti-derivative"

• the fundamental theorem of Calculus connects definite and indefinite integrals

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (\text{for } F'(x) = f(x))$$

This lecture : how to calculate integrals - tips and tricks

note : not every integral ~~be~~ can be analytically calculated, e.g. $\int e^{-x^2}dx$

1) Simple integrals (that are useful to know by heart)

$\int dx = x + C$ $\qquad$ $\int \sin(x)dx = -\cos(x) + C$

$\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$ $\qquad$ $\int \cos(x)dx = \sin(x) + C$

$\int e^x dx = e^x + C$. $\qquad$ $\int \frac{dx}{x} = \ln|x| + C$

(Check by taking the derivative of the result)

2) Substitution

⋆ note : integration is linear $\int(Af(x) + Bg(x))dx = A\int f(x)dx + B\int g(x)dx$

(you can take constant factors out of the integral
– verify with area below curve)

⋆ chain rule : $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

$\int \Downarrow$

$$f(g(x)) + C = f(u) + C = \int f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int f'(u)\,du$$

$u = g(x)$

substitution : $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ for $\left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right.$

for f continuous, g differentiable

↳ substitution comes down to replacing the integration variable x by $u = g(x)$, $du = g'(x)dx$ and rewriting in terms of u.

Examples : $\int \sin(3x)dx = \int \sin(\underbrace{3x}_{u})\frac{1}{3}\underbrace{3dx}_{du} = \frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + C$

$= -\frac{1}{3}\cos(3x) + C$

$u = 3x$
$du = 3dx$

take out of integral

$$\int \frac{dx}{x+1} = \int \frac{du}{u} = \ln|u| + C = \ln|x+1| + C$$

$u = x+1$
$du = dx$

$$\int \frac{x\,dx}{x^2+1} = \int \frac{\frac{1}{2}\cdot \overbrace{2x\,dx}^{du}}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

$u = x^2+1$
$du = 2xdx$

$$\int \tan(x)dx = \int \frac{\sin(x)dx}{\cos(x)} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$

$u = \cos(x)$
$du = -\sin(x)dx$

* substitution is the most useful trick

- substitutions $u = ax + b$ / are often useful
- suitable for integrands of the form $f(\underset{\text{function}}{g(x)})\underset{\text{derivative}}{g'(x)}$
- (substitutions $u = e^x$, $\frac{du}{u} = dx$, $dx = \cos(u)du$, $x = \sin(u)$ are sometimes useful

$x = \ln(u)$, $dx = \frac{du}{u}$

$x = \sin(u)$, $dx = \cos(u)du$

$x = \tan(u)$, $dx = \frac{du}{\cos^2(u)}$

$x = e^u$, $dx = e^u du$

for definite integrals, we can change the integration limits accordingly (or transform back to x before evaluating)

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_A^B f(u)\,du$$

with $A = g(a)$
$B = g(b)$

3) Integration by parts

product rule : $(uv)' = u'v + uv'$

$$\int dx \quad uv = \int v\,du + \int u\,dv$$

$$\Leftrightarrow \int u\,dv = u \cdot v - \int v\,du$$

Example : $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$

$u = x \quad du = dx$

$dv = e^x dx \quad v = e^x$

$\int x \sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx = -x\cos(x) + \sin(x) + C$

$u = x \quad du = dx$

$dv = \sin(x)\,dx \quad v = -\cos(x)$

$\int \ln(x)\,dx = x\ln(x) - \int \cancel{x} \frac{dx}{\cancel{x}} = x\ln(x) - x + C$

$u = \ln(x) \quad du = \frac{dx}{x}$

$dv = dx \quad v = x$

* integration by parts is mainly useful when the integrand contains exponentials, trigonometric functions, logs + polynomials.

4) Rational functions - partial fraction decomposition

* idea : we write a rational function $\frac{P(x)}{Q(x)} = P_1(x) + \frac{A_1}{x-x_1} + \quad + \frac{A_n}{x-x_n}$

as a sum of a polynomial $P_1(x)$ and fractions $\frac{A_k}{x-x_k}$ — each of those is easy to integrate.

How? 1) write $\frac{P(x)}{Q(x)} = P_1(x) + \frac{P_2(x)}{Q(x)}$, with $P_2(x)$ of smaller degree than $Q(x)$

($\int P_1(x)\,dx$ is easy to calculate)

2) ~~P_2~~ Factorize $Q(x) = (x-x_1)(x-x_2)\dots(x-x_n)$

3) $\frac{P_2(x)}{Q(x)} = \frac{A_1}{x-x_1} + \dots + \frac{A_n}{x-x_n}$

$$A_k = \frac{P_2(x_k)}{(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}$$

or set up equation system (more intuitive)

Example $\int \frac{dx}{x^2-4}$

$$\frac{1}{x^2-4} = \frac{A_1}{x-2} + \frac{A_2}{x+2} = \frac{A_1(x+2)+A_2(x-2)}{(x-2)(x+2)}$$

$$\Rightarrow \quad 0 = (A_1 + A_2)x \quad \Rightarrow A_1 = -A_2$$

$$1 = 2A_1 - 2A_2 = 4A_1 \quad \Rightarrow A_1 = \frac{1}{4}, \; A_2 = -\frac{1}{4}$$

↳ $\int \frac{dx}{x^2-4} = \frac{1}{4}\int\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) + C$

★ note : if $Q(x) = (x-x_1)^2$, then $\frac{P_1(x)}{(x-x_1)^2} = \frac{A_1}{x-x_1} + \frac{A_2}{(x-x_1)^2}$

example $\frac{x+3}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} = \frac{A_1(x-2)+A_2}{(x-2)^2}$

$$\Rightarrow \quad x = A_1 x \quad \Rightarrow A_1 = 1$$

$$3 = -2A_1 + A_2 \quad \Rightarrow A_2 = 5$$

$$\frac{x+3}{(x-2)^2} = \frac{1}{x-2} + \frac{5}{(x-2)^2} \quad \Rightarrow \quad \int \frac{x+3}{(x-2)^2}dx = \int \frac{dx}{x-2} + 5\int\frac{dx}{(x-2)^2}$$

$$= \ln|x-2| - \frac{5}{x-2}$$

★ note 2 : if $Q(x) = (x-x_1)(x^2+bx+c)$

↳ no real roots

then $\frac{P_1(x)}{Q(x)} = \frac{A_1}{x-x_1} + \frac{A_2+Bx}{x^2+bx+c}$