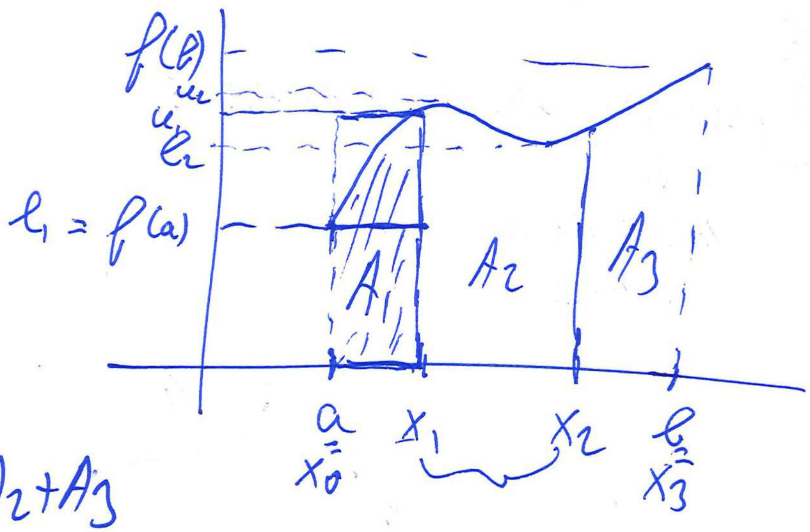
LECTURE 5 : INTEGRATION

$$\mathrm{SGN}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{IF } x > 0 \\ -1 & \text{IF } x < 0 \\ \text{UNDEF.} & \text{IF } x = 0 \end{cases}$$

- DEFINITE INTEGRAL ⟶ AREA UNDER CURVE (AUC)
- IN- " " (ANTIDERIVATIVE) ↳ ROC - CURVE (RECEIVER OPERATING CHARACTERISTIC)
- FUNDAMENTAL THM. OF CALCULUS



$A = A_1 + A_2 + A_3$

- f CONTINUOUS $[a, b]$.
- DIVIDE $[a, b]$ INTO SUBINTERVALS

$a = x_0 < x_1 < x_2 < \dots < x_n = b.$

$[x_{i-1}, x_i]$ PARTITION OF $[a, b]$

f HAS MINIMUM & MAXIMUM ON $[x_{i-1}, x_i]$.

MINIMUM → l_i, MAXIMUM → u_i

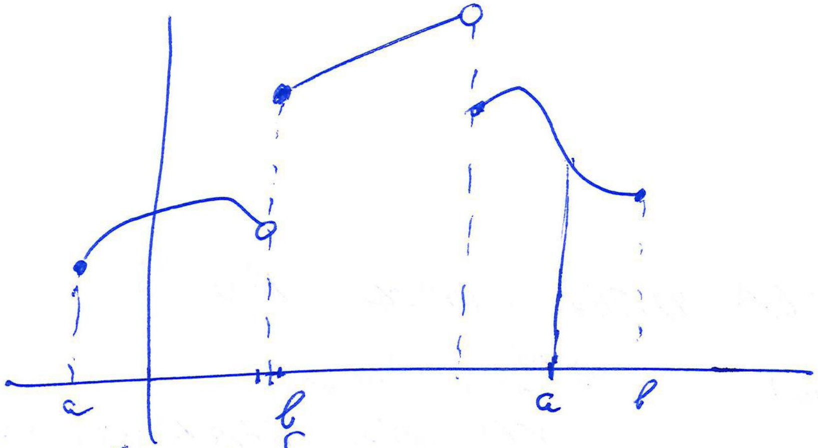
$\Delta x_i = x_i - x_{i-1}$: LENGTH OF i^{TH} SUBINT.

$$l_1 \cdot \Delta x_1 \le A_1 \le u_1 \cdot \Delta x_1$$

LOWER RIEMANN SUM: $L(f, P) \le A$ $\qquad L(f,P) = \sum_{i=1}^{n} \Delta x_i \cdot l_i$

UPPER " " $U(f, P) \ge A.$

$$\|P_n\|_{n\to\infty} = \underset{n\to\infty}{\mathrm{MAX}}\, \Delta x_i = 0 \text{ THEN } L(f, P_n) \,\&\, U(f, P_n) \xrightarrow{n\to\infty} A$$



Piecewise continuous functions are integrable. As long as the number of discontinuities is finite.

Notation: $(A =) \int_a^b f(x)\,dx$: <u>Definite integral</u> : A real number.

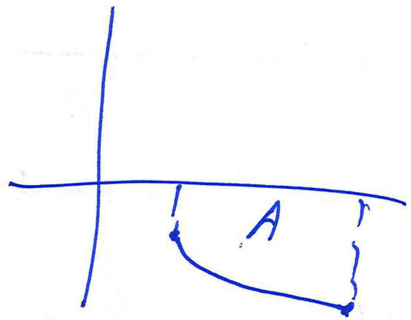
- a = lower limit
- b = upper limit
- x = variable
- dx = differential
- f = integrand
- $\int$ = integration symbol

Properties

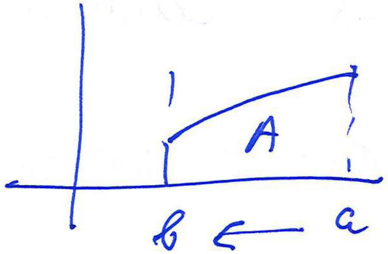
1. $\int_a^b f(x)\,dx$ can be negative

↓

Area below x-axis



2.



$\int_a^b f(x)\,dx < 0$. In fact: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Corollary: $\int_a^a f(x)\,dx = 0$

3. BREAKING UP INTEGRALS INTO SMALLER ONES:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

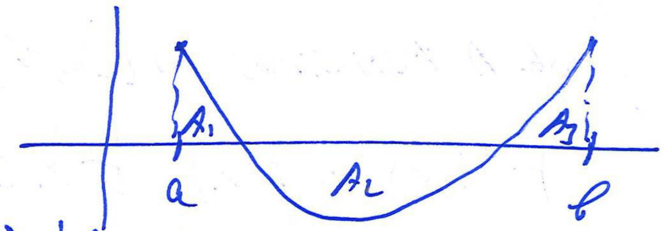
$$\cancel{\int_a^b f(x)\cdot g(x)\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx}$$

4. $$\int_a^b c_1 f(x) + c_2 g(x)\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx$$

$$\int_a^b x^5 + \sin(x)\,dx = \int_a^b x^5\,dx + \int_a^b \sin(x)\,dx$$

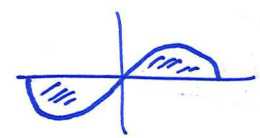
5. TRIANGLE INEQUALITY: $|x|+|y| \geq |x+y| \quad \forall x,y \in \mathbb{R}.$

$$\underbrace{\int_a^b |f(x)|\,dx}_{|A_1|+|A_2|+|A_3|} \geq \left|\int_a^b f(x)\,dx\right| \longrightarrow |A_1 - A_2 + A_3|$$



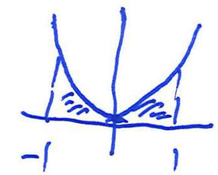
6. f ODD $(f(-x) = -f(x)) \Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

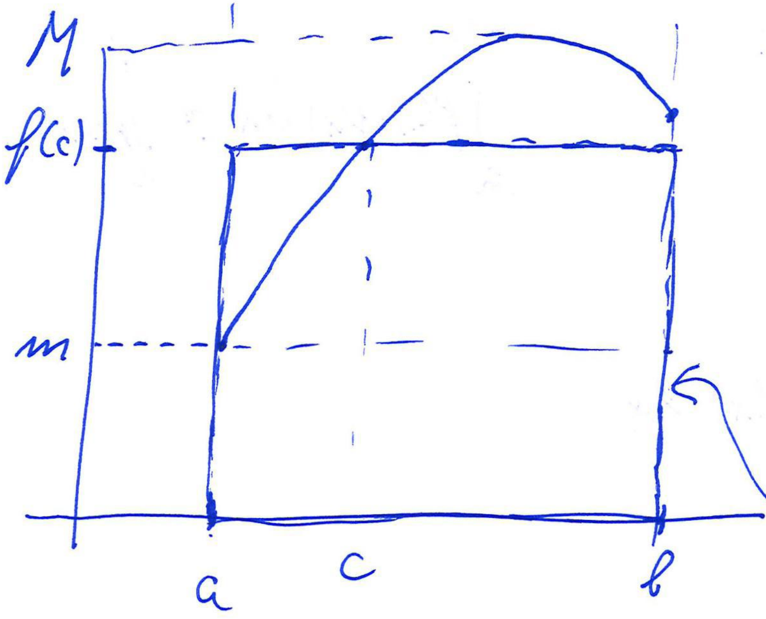
$$\int_{-\pi}^{\pi} \sin(x) = 0$$



f EVEN $(f(-x) = f(x)) \Rightarrow \int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$

$$\int_{-1}^{1} x^2\,dx = 2\cdot\int_0^1 x^2\,dx$$





f CONTINUOUS ON $[a,b]$

m = MINIMUM
M = MAXIMUM

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\exists c \in [a,b] \text{ SUCH THAT } \underbrace{f(c)\cdot(b-a)} = \underbrace{\int_a^b f(x)\,dx}$$

APPL. OF INTERMEDIATE VALUE THEOREM.

LET f BE A FUNCTION ON $[a,b]$. A FUNCTION F ON $[a,b]$ FOR WHICH

$$F'(x) = f(x) \quad \forall x \in [a,b]$$

IS CALLED AN <u>ANTIDERIVATIVE</u> OF f.

<u>THM</u>: 1 $F(x) = \int_a^x f(t)\,dt$ IS AN ANTIDERIVATIVE OF f.

2 IF G IS AN ANTIDERIVATIVE OF f ON $[a,b]$, THEN

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

PROOF: 1 LET $F(x) = \int_a^x f(t)\,dt$

THEN $F'(x) = \frac{d}{dx}F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$

$= \lim_{h \to 0} \frac{1}{h} \cdot \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h} \cdot f(c) \cdot (x+h-x)$ FOR SOME $c \in [x, x+h]$

$= \lim_{h \to 0} f(c)$ FOR SOME $c \in [x, x+h]$

$= f(x)$

2 G BEING AN ANTIDERIVATIVE OF f MEANS $G'(x) = f(x) = F'(x)$ ON $[a,b]$.

SO $G'(x) - F'(x) = (G-F)'(x) = 0$ ON $[a,b]$.

SO $(G-F)(x)$ IS A CONSTANT, C, ON $[a,b]$

SO $G(x) = F(x) + C$

BUT THEN $G(a) = F(a) + C = \cancel{\int_a^a f(t)\,dt} + C = C$

$G(b) = F(b) + C = \int_a^b f(t)\,dt + C$

SO $G(b) - G(a) = \int_a^b f(t)\,dt$.

ELEMENTARY INTEGRALS:

$$\int dx = \int 1\, dx = x + C$$

$$\int x\, dx = \frac{1}{2}x^2 + C$$

$$\int x^2\, dx = \frac{1}{3}x^3 + C$$

$$\int \sin(x)\, dx = -\cos(x) + C$$

$$\int \cos(x)\, dx = \sin(x) + C$$

$$\int e^x\, dx = e^x + C$$

$$\int \frac{1}{\cos^2(x)}\, dx = \tan(x) + C$$

$$\int x^n\, dx = \frac{1}{n+1} \cdot x^{n+1} + C, \quad n \neq -1.$$

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$$