Lecture 7 - Calculus

Overview of the course

- Continuity and limits
- Differentiation
- Integration

END of SECONDARY SCHOOL MATERIAL (for most of you)

- Sequences and series we are here!
- Partial derivatives
- Double integrals

Sequences and series

- Sequences
- Infinite Series
- Geometric series
- P-series

Adams' Ch. 9.1-9.2, Thomas' Ch. 10.1

Sequences

A sequence $\{a_n\}$ is a list of numbers $a_1, a_2, \ldots, a_n, \ldots$ in a given order

* a sequence can be seen as a function $f : \mathbb{N} \to \mathbb{R}$, $n \to a_n = f(n)$

$\{a_n\}$ $\quad$ $a_1, a_2 \to$ index

$\hookrightarrow$ term

Examples : $\sqrt{n}$: $1, \sqrt{2}, \sqrt{3}, \sqrt{4} = 2, \ldots$

$\frac{1}{n}$ $\qquad$ formula for general term.

$(-1)^n$

- $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ $\quad$ FIBONACCI sequence

$1, 1, 2, 3, 5, 8, \ldots$ $\qquad$ recursive formula

$n=0$

- $1, -\frac{x^2}{2}, \frac{x^4}{4!}, -\frac{x^6}{6!}, \ldots$ $\qquad$ (a pattern) $\quad \frac{(-1)^n \cdot x^{2n}}{(2n)!}$

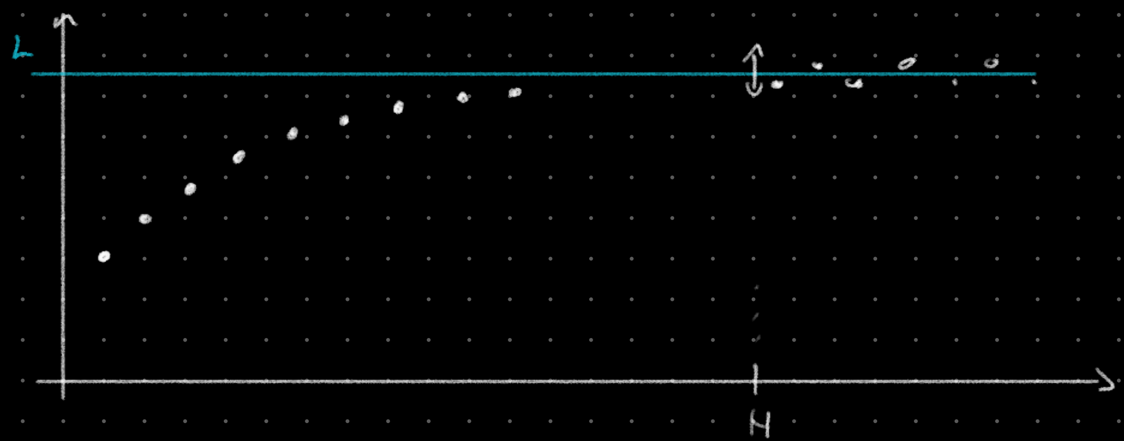
* series and sequences are used to approximate irrational numbers, transcendental functions ($\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, ...) numerically

Convergence of a sequence

A sequence $a_n \to L$ if $\forall \epsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - L| < \epsilon$

$$\lim_{n \to \infty} a_n = L$$

* this means that, after an index N, all terms a_n are within ϵ-distance from the limit L



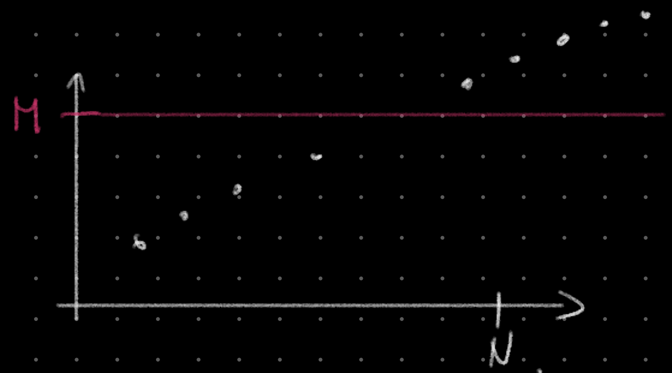
* examples of converging sequences

$\frac{1}{n} \to 0$, $\frac{n}{n+2} \to 1$, $(0.9)^n \to 0$

* NOT all sequences converge

→ a sequence DIVERGES TO INFINITY $(a_n \to +\infty)$ if

$\forall M > 0 \ \exists N$, such that, if $n \geq N$, $a_n > M$



examples: $a_n = n$, $a_n = n!$ $a_n = \sqrt{n}$

→ a sequence DIVERGES if the limit does not exist.

example: $\cos(\pi \cdot n) = (-1)^n = 1, -1, 1, -1$

* if the sequence can be seen as a real function (i.e. if $f(x), x \in \mathbb{R}$ is defined for $x \geq n_0$, and $a_n = f(n)$ for $n \geq n_0$, then

$$\lim_{x \to \infty} f(x) = L \implies a_n \to L \qquad (L \text{ can be } \pm\infty)$$

* this is NOT true for every sequence : $(-1)^n$, $n!$, ...

* if $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to M + L$

  $a_n \to L$, then $k \cdot a_n \to kL$

  if $a_n \to L$, then $f(a_n) \to f(L)$
  f continuous

  only for converging sequences.

→ note: a sequence cannot have vertical asymptotes !

* We can apply a continuous function f on a sequence:

  if $a_n \to L$, then $f(a_n) \to f(L)$

Example $a_n = n^{\frac{1}{n}}$ (socr)

$$\ln(n^{\frac{1}{n}}) = \frac{1}{n}\cdot \ln(n)$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x}, \quad x \in \mathbb{R}^+ \qquad \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$a_n \to e^0 = 1$$

* we can add, subtract, multiply converging sequences

for $\{a_n\}, \{b_n\}$ sequences, $a_n \to A$, $b_n \to B$

then $(a_n \pm b_n) \to A \pm B$; $(a_n \cdot b_n) \to A \cdot B$

$k a_n \to kA$ $(k \in \mathbb{R})$

* squeeze theorem for sequences for $\{a_n\}, \{b_n\}, \{c_n\}$ sequences,

$a_n \le b_n \le c_n \quad \forall n$

if $a_n \to L$, $c_n \to L$, then $b_n \to L$

example: socr

Terminology

A sequence $\{a_n\}$ is

- **Bounded above** if $\exists M$ such that $\forall n \quad a_n \leq M$
- **Bounded below** if $\exists m$ such that $\forall n \quad a_n \geq m$
- **Bounded** if bounded above and below

- **Increasing:** $\forall n \quad a_{n+1} > a_n$
- **Decreasing:** $a_{n+1} < a_n$

$\left.\begin{array}{l} \text{Increasing} \\ \text{Decreasing} \end{array}\right\}$ monotonous

- **Alternating:** $\forall n \quad a_n \cdot a_{n+1} < 0$

- **Positive/negative:** $\forall n \quad a_n \geq 0$ / $a_n \leq 0$

- Every convergent sequence is bounded
- An increasing sequence bounded above converges
  decreasing sequence bounded below converges

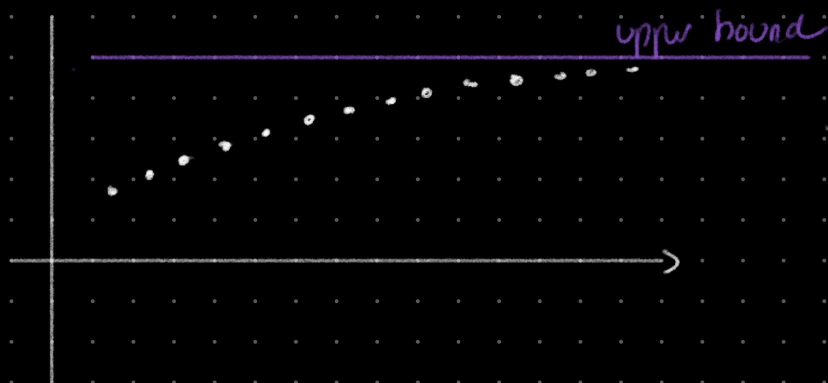
* Intuitive explanation:

- if a sequence converges, $a_n \to L$, then $L+\epsilon$ is an upper bound, $L-\epsilon$ lower bound for the terms $a_N, a_{N+1}, \dots$ (infinitely many)

  → for the first $N-1$ terms there is a minimum term a_{Min}, maximum term a_{Max}

  → As upper bound, take $\max(L+\epsilon, a_{Max})$
  lower bound $\min(L-\epsilon, a_{min})$

- if a monotonous sequence is bounded, than it converges



the sequence cannot increase towards ∞

Example: $a_1 = 1$, $a_{n+1} = \sqrt{6 + a_n}$

* increasing? induction!

Base case: $a_2 > a_1$? $a_2 = \sqrt{6+1} = \sqrt{7} > 1 = a_2$ ✓

Induction: if $a_{n+1} > a_n$, then $a_{n+2} > a_{n+1}$

$$a_{n+2} = \sqrt{6 + a_{n+1}} > \sqrt{6 + a_n} = a_{n+1}$$ ✓

* upper bound? $a_n \leq 3$?

$a_1 = 1 < 3$

if $a_n < 3$, then $a_{n+1} < 3$

$$a_{n+1} = \sqrt{6 + a_n} < \sqrt{6+3} = 3$$

} 3 is an upper bound.

* limit

$$\underset{\downarrow a}{a_{n+1}} = \sqrt{6 + \underset{\downarrow a}{a_n}}$$

$a = \sqrt{6+a}$

$a^2 = 6 + a \iff a^2 - a - 6 = 0$

$(a-3)(a+2) = 0$

$\longrightarrow$ since $a > 0$, $a = 3$!

Infinite series

(Infinite) series = formal sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

* a series can be seen as a sequence of partial sums $\{S_n\}$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

$$S_n = \sum_{k=1}^{n} a_k$$

the series $\sum a_n \to s$ if $S_n \to S$

* SERIES ARE AN INDETERMINATE FORM (usually)
  - ↳ we sum up infinitely many terms (that are infinitely small)
  - ↳ usually, we cannot calculate the sum. We can only conclude whether they converge (the sum exists)

Geometric series

$a_n = a r^{n-1}$, $r = \frac{a_{n+1}}{a_n}$ $(a \neq 0,\ r \neq 1)$

$a,\ ar,\ ar^2,\ ar^3,\ \dots$

* the geometric series is one of the few series where we can calculate the sum

$S_1 = a$

$S_2 = a + a \cdot r = a \cdot (1+r)$

$$\begin{aligned} S_n &= a + a \cdot r + \dots + a \cdot r^{n-1} \\ - \quad r \cdot S_n &= \phantom{a + {}} a \cdot r + a \cdot r^2 + \dots + a \cdot r^n \end{aligned}$$

$S_n (1-r) = a - a r^n \longrightarrow S_n = a \frac{1-r^n}{1-r}$

- if $|r| > 1$, S_n diverges (if $r > 1$, $S_n \to \infty$)
- if $|r| < 1$ $\sum_{n=1}^{\infty} a_n \longrightarrow \frac{a}{1-r}$
- if $r = \pm 1$, $\sum a_n$ diverges

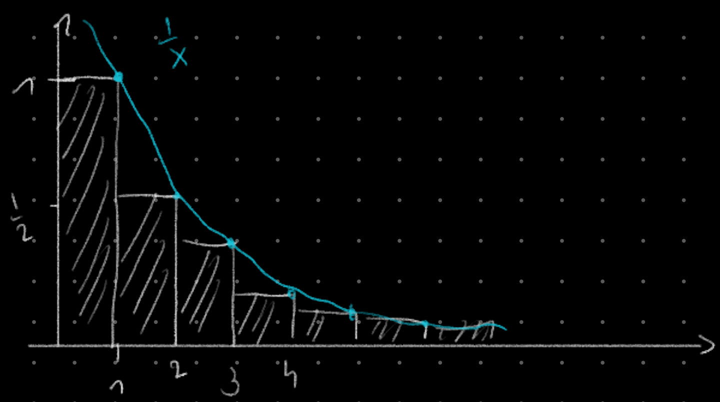
Convergence of the series vs. convergence of the sequence

- If the SEQUENCE does NOT converge to 0, the SERIES diverges.

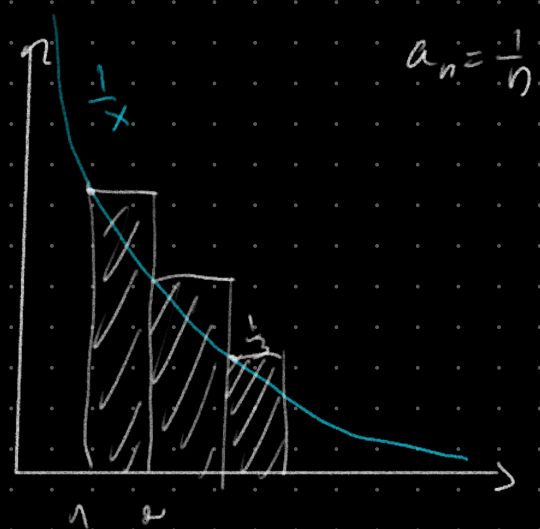
P-series - integral test

- $a_n = \frac{1}{n^p}$

- If the sequence $\{a_n\}$ can be mapped onto a function $f(x)$, the series $\sum_{n=k}^{\infty} a_n$ compares to the improper integral $\int_k^{\infty} f(x)\,dx$



Lower Riemann sum!



Upper Riemann sum.

The series can be seen as both upper and lower Riemann sum of the function $f(x)$

$\rightarrow$ if $\int_n^\infty f(x)\,dx$ converges or diverges, so does the series $\sum_{n=1}^\infty a_n$

- if $a_n \geqslant 0$ and $f(x) \geqslant 0$
- if $a_n = f(n)$

$$\rightarrow \int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \left(\ln(R) - \ln(1)\right) = +\infty$$

$\Rightarrow$ the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ DIVERGES

$$\rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^p} = \lim_{R\to\infty} \left(\frac{1}{1-p}\left(R^{1-p} - 1\right)\right) \begin{cases} = \frac{1}{p-1} & \text{if } p > 1 \\ = +\infty & \text{if } p < 1 \end{cases}$$

$\hookrightarrow \frac{1}{n^p}$ CONVERGES for $p > 1$
DIVERGES for $p \leqslant 1$

* you may know and use the p-series in exercises / the exam without carrying out the integration each time

Series and sequences

- A sequence $\{a_n\}$ is a ordered list of numbers
  - A sequence can converge, diverge or diverge to infinity
  - We can calculate the limit (cfr. Limit to infinity)
  - A sequence can be bounded, monotonous, alternating,...
- A series $\sum_{n=1}^{\infty} a_n$ is an infinite sum of terms - indeterminate form!
- can converge or diverge (but the sum cannot always be calculated)
- Geometric series:
  - Constant ratio between terms $a_n = a \cdot r^{n-1}$; $r = \frac{a_{n+1}}{a_n}$
  - Converges if $|r| < 1$
  - We can calculate the sum $S = \frac{a}{1-r}$
- P-series: $a_n = \frac{1}{n^p}$
  - Converges if $p > 1$
- The converges of other series is typically determined by comparing with p-series or geometric series (not course material)