Calculus

Revision

Otti D'Huys, Gijs Schoenmakers

Overview

- Limits and continuity
- Differentiation
- Integration
- Sequences and Series
- Multivariate calculus:
    - partial derivatives
    - double integrals

This overview does only contains the main outlines (many things are omitted). All material in the lecture slides and notes is examinable!

Limits

The **limit** $\lim_{x \to a} f(x) = L$ if $\forall \epsilon > 0 \; \exists \delta > 0 : |x - a| < \delta \Leftrightarrow |f(x) - L| < \epsilon$.

- Left (right) limits: $x < a$ ($x > a$). The limit only exists if left and right limit are equal.
- $\lim_{x \to a} f(x) = +\infty$ if $\forall M > 0 \; \exists \delta > 0 : |x - a| < \delta \Rightarrow f(x) > M$.
- $\lim_{x \to +\infty} f(x) = L$ if $\forall \epsilon > 0 \; \exists N > 0 : x > N \Rightarrow |f(x) - L| < \epsilon$.
- A function is continuous at a if $\lim_{x \to a} f(x) = f(a)$. A function is continuous **on its domain** if it is continuous at every point **of its domain**.
- Asymptotes: a function $f(x)$
  - has a horizontal asymptote $y = a, a \in \mathbb{R}$ if $\lim_{x \to \pm\infty} f(x) = a$.
  - has a vertical asymptote $x = b, b \in \mathbb{R}$ if $\lim_{x \to b^\pm} f(x) = \pm\infty$.
  - has an oblique asymptote $y = ax + b, a, b \in \mathbb{R}$ if $\lim_{x \to \pm\infty}(f(x) - (ax + b)) = 0$.

Derivatives

The **derivative** of $y = f(x)$ with respect to x is given by

$$y'(x) = f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists and is finite.

- The derivative $f'(x_0)$ is the slope of the tangent line to the function at x_0
- Left/right limits lead to left/right derivatives - $f(x)$ is differentiable at a if left and right derivative are equal. $f(x)$ is differentiable on its domain if it is differentiable at every point at its domain.
- 2 important rules to calculate derivatives: chain rule, product rule.
- Sign of $f'(x) \to$ increasing/decreasing intervals/possible extrema.
- Sign of $f''(x) \to$ convex/concave intervals/possible inflection points.
- l'Hopital rules: using derivatives to calculate indeterminate limits of the form $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$:

Integration

- An indefinite integral can be seen as antiderivative:

$$F'(x) = f(x) \Leftrightarrow F(x) + C = \int f(x)dx$$

- The definite integral $\int_a^b f(x)dx = F(b) - F(a)$ is the area under the curve between a and b (Riemann sums)
- Fundamental theorem of Calculus: $\frac{d}{dx} \int_a^x f(t)dt = f(x)$
- Methods to calculate integrals: substitution (inverse chain rule), integration by parts (inverse product rule), partial fraction decompostion,...
- Improper integrals (can converge, diverge, or diverge to $\pm\infty$):
  - $\int_a^\infty f(x)dx) = \lim_{t\to\infty} \int_a^t f(x)dx$
  - if $\lim_{x\to a^+} f(x) = \pm\infty$, $\int_a^b f(x)dx = \lim_{t\to a^+} \int_c^b f(x)dx$

Sequences

- A sequence $\{a_n\}$ is an ordered list of numbers $a_1, a_2, \ldots, a_n, \ldots$
- Main question: does the sequence converge? $\lim_{n\to\infty} a_n = A$?
- How to calculate the limit of a sequence? Calculate the limit of the function!

  If $f(x)$ is defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$, then:

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

  .

- If $a_n \to A$, then $f(a_n) \to f(A)$ for a continuous function f.
- Every converging sequence is bounded, a bounded monotonous sequence converges.

Series

- A series is a formal sum of infinitely many terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- A series is a sequence of **partial sums**:

$$s_n = s_{n-1} + a_n = \sum_{j=1}^{n} a_j$$

- Main question: does a series converge?
- Two important series:
  - The **geometric** series $\sum_{n=0} ar^n$. The geometric series converges absolutely for $|r| < 1$ (by direct calculation of the limit).
  - The **p-series** $\sum_{n=1} n^{-p}$. The p-series converges for $p > 1$ (by integral test).
- If the **sequence** $\{a_n\}$ does not converge to 0, the **series** $\sum_{n=1}^{\infty} a_n$ diverges.

Functions of several variables

- The **limit** $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, $|f(x,y) - L| < \epsilon$
- The **partial derivatives** of $f(x,y)$ with respect to x, y are given by:

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

  if these limits exist.
- The **tangent plane** to $z = f(x,y)$ at $(a,b,f(a,b))$:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Double integrals

- We consider double integrals $\iint_R f(x,y)dA$ for (piecewise) continuous functions $f(x,y)$ on a bounded domain R.
- Just like for one dimension, the integral is a limit of a Riemann sum.
- The double integral is calculated as an iterated (inner + outer) integral

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy\ .$$

- Fubini's theorem: it does not matter how you set up your integral (which variable you choose for your outer integral), as long as your integration limits describe the region R correctly.