LECTURE 8: PARTIAL DERIVATIVES (CH. 12)

$$f(x_1, x_2, x_3, \dots, x_n) : D \to \mathbb{R}$$

$$f(x,y) = \sqrt{4-x^2-y^2} \qquad D = \{(x,y) : x^2+y^2 \le 4\}$$

$$\frac{1}{x^2+y^2} \qquad D = \mathbb{R}^2 \setminus \{(0,0)\}.$$

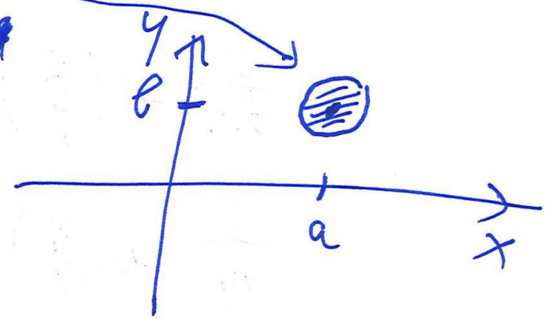
(x,y) : COORDINATES ON MAP

f: HEIGHT, TEMPERATURE, PRESSURE.

$f(x,y) = c$: LEVEL CURVES.

$f(x,y)$ IS CONTINUOUS AT (a,b) IF FOR $(x,y) \in D$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \underbrace{\sqrt{(x-a)^2+(y-b)^2}}_{\text{EUCLIDEAN DISTANCE}} < \delta \Rightarrow |f(x,y) - f(a,b)| < \varepsilon.$$



TYPICAL FUNCTIONS ARE CONTINUOUS:

$\sin(\cdot)$, $e^{\cdot}$, $\sqrt{\cdot}$

$$\frac{\partial f(x,y)}{\partial x}$$: PARTIAL DERIVATIVE.

"DAU"

$$= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial z}{\partial x} = f_1(x,y) = f_x(x,y) = D_1(x,y) = D_x(x,y)$$

$z = f(x,y)$ $\qquad \left.\frac{\partial f(x,y)}{\partial x}\right|_{(a,b)}$

CALCULATION OF $\frac{\partial f(x,y)}{\partial x}$: $\frac{\partial}{\partial x}(x^2+y^2) = 2x$

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{2x}{2\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial x}\ln\left(\frac{x}{y}\right) = \frac{1}{\frac{x}{y}} \cdot \frac{1}{y} = \frac{1}{x}$$

$$\frac{\partial}{\partial y}\ln\left(\frac{x}{y}\right) = \frac{1}{\frac{x}{y}} \cdot -\frac{x}{y^2} = -\frac{1}{y}$$

$$\frac{\partial}{\partial x} \overbrace{e^x \sin(xy)}^{f(x,y)}\Big|_{(1,\pi)} = \left(e^x\cos(xy)\cdot y + e^x \sin(xy)\right)\Big|_{(1,\pi)} = -e\pi$$

$$\frac{\partial}{\partial y} e^x \sin(xy)\Big|_{(1,\pi)} = e^x \cos(xy)\cdot x\Big|_{(1,\pi)} = -e$$

EQ. TANGENT PLANE:

$$z = -e\pi(x-1) - e\cdot(y-\pi)$$

$$(f(1,\pi)=0)$$

TANGENT PLANES OF GRAPH OF $f(x,y)$ AT (a,b)

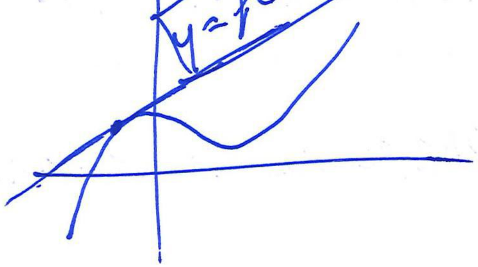
TANGENT LINE y AT $x=c$

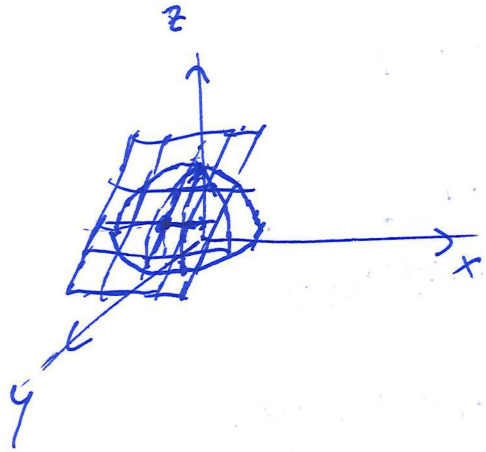
$\rightarrow \cdot\ y(c) = f(c)$

$\rightarrow \cdot\ y'(c) = f'(c)$

$\uparrow$ SLOPE

$$\boxed{y = f(c) + f'(c)\cdot(x-c)}$$





TANGENT PLANE z AT (a,b)

- $z(a,b) = f(a,b)$
- $\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x}\Big|_{(a,b)}$
- $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial y}\Big|_{(a,b)}$

EQUATION TANGENT PLANE:

$$\boxed{z = f(a,b) + \frac{\partial f}{\partial x}\Big|_{(a,b)}\cdot(x-a) + \frac{\partial f}{\partial y}\Big|_{(a,b)}\cdot(y-b)}$$

EX: $f(x,y) = \sin(xy)$, $(a,b) = \left(\frac{\pi}{3}, -1\right)$

- $f\left(\frac{\pi}{3}, -1\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{1}{2}\sqrt{3}$
- $\left.\frac{\partial f}{\partial x}\right|_{\left(\frac{\pi}{3},-1\right)} = \cos(xy)\cdot y\Big|_{\left(\frac{\pi}{3},-1\right)} = -\cos\left(-\frac{\pi}{3}\right) = -\frac{1}{2}$
- $\left.\frac{\partial f}{\partial y}\right|_{\left(\frac{\pi}{3},-1\right)} = \cos(xy)\cdot x\Big|_{\left(\frac{\pi}{3},-1\right)} = \frac{\pi}{3}\cdot\cos\left(-\frac{\pi}{3}\right) = \frac{\pi}{6}.$

EQ. TANGENT ~~SPACE~~ PLANE: $\boxed{z = -\frac{1}{2}\sqrt{3} - \frac{1}{2}\cdot\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}\cdot(y+1)}$

NOTE: $\left.\frac{\partial f}{\partial x}\right|_{(\cdot,\cdot)} = 0$: PARALLEL TO X-AXIS

$f(x,y) = \frac{1}{1+x^2+y}$ AT $(0,0)$; $f(0,0) = \frac{1}{1} = 1$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \frac{-1}{(1+x^2+y)^2}\cdot 2x\Big|_{(0,0)} = 0$; $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \frac{-1}{(1+x^2+y)}\Big|_{(0,0)} = -1$

SO EQ. TANGENT PLANE: $z = 1 + 0x - 1y = 1 - y.$

HIGHER ORDER DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \quad (= f_{11} = f_{xx})$$

$$\frac{\partial^2}{\partial x^2} e^x \sin(xy) = \frac{\partial}{\partial x}\left(e^x \sin(xy) + y\,\cancel{e^x}\cos(x,y)\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{21} = f_{yx}$$

$$= \ldots$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$$

CHAIN RULE IN 2D

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

IN 2D: $z = f(x,y)$ AND x & y ARE FUNCTIONS OF s. $x(s), y(s)$

$$\boxed{\frac{dz}{ds} = \frac{\partial z}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial z}{\partial y}\cdot\frac{dy}{ds}}$$

PROOF: $\dfrac{dz}{ds} = \lim\limits_{h\to 0} \dfrac{z(s+h) - z(s)}{h} = \lim\limits_{h\to 0} \dfrac{f(x(s+h), y(s+h)) - f(x(s), y(s))}{h}$

$$= \lim_{h\to 0} \frac{f(x(s+h), y(s+h)) - f(x(s), y(s+h)) + f(x(s), y(s+h)) - f(x(s), y(s))}{h}$$

$$= \underbrace{\frac{\partial}{\partial x} f(x(s),y(s))}_{\frac{\partial z}{\partial x}} \cdot \frac{dx}{ds} + \underbrace{\frac{\partial}{\partial y} f(x(s),y(s))}_{\frac{\partial z}{\partial y}} \cdot \frac{dy}{ds}$$

$$= \frac{\partial z}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial z}{\partial y} \cdot \frac{dy}{ds}$$

EX: $z = \cos(sxy)$ WHERE $x = s^2$, $y = \frac{1}{s^4+1}$

$$\frac{dz}{ds} = \frac{d}{ds} \cos\left(s \cdot s^2 \cdot \frac{1}{s^4+1}\right) = \frac{d}{ds} \cos\left(\frac{s^3}{s^4+1}\right) = \underline{-\sin\left(\frac{s^3}{s^4+1}\right) \cdot \frac{3s^2 - s^6}{(s^4+1)^2}}$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial z}{\partial y} \cdot \frac{dy}{ds}$$

$$\frac{\partial z}{\partial x} = -\sin(sxy) \cdot sy \, ; \quad \frac{dx}{ds} = 2s$$

$$\frac{\partial z}{\partial y} = -\sin(sxy) \cdot sx \, ; \quad \frac{dy}{ds} = \frac{-4s^3}{(s^4+1)^2}$$

$$= -\sin\left(\frac{s^3}{s^4+1}\right) \frac{s}{s^4+1} 2s + -\sin\left(\frac{s^3}{s^4+1}\right) \cdot s^3 \cdot \frac{-4s^3}{(s^4+1)^2}$$

$$= \ldots$$

X & Y DEPENDING ON 2 VARIABLES S & T.

$x(s,t)\,,\ y(s,t) \qquad z = f(x,y).$

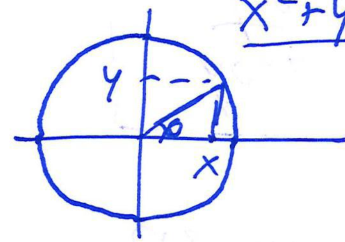
THEN $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial s}$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

POLAR COORDINATES

$z = \dfrac{1}{x^2+y^2}$ WHERE $x = R\cdot\cos(\theta)$

$y = R\cdot\sin(\theta)$



$x^2+y^2=R^2$

FIND $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial r}$

$= \left(\dfrac{-2x}{(x^2+y^2)^2}\cdot\cos(\theta) + \dfrac{-2y}{(x^2+y^2)^2}\cdot\sin(\theta)\right)\Bigg|_{\substack{x=r\cos(\theta)\\ y=r\sin(\theta)}}$

$= \dfrac{-2r\cos^2(\theta) - 2r\sin^2(\theta)}{R^4} = -\dfrac{2}{R^3}$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial}{\partial \theta}\dfrac{1}{R^2} = 0$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial \theta}$

$= \dfrac{2x}{(x^2+y^2)^2}\,r\sin(\theta) - \dfrac{2y}{(x^2+y^2)^2}\cdot r\cos(\theta) = \dfrac{2r^2\cos(\theta)\sin(\theta) - 2r^2\sin(\theta)\cos(\theta)}{r^4} = 0$