Overview

- Recap: sequences and series
- Functions of multiple variables
- Continuity and limits in 2 dimensions
- Partial derivatives
- Chain rule in multiple dimensions

Adams' Ch. 12.1–5

Sequences and series

- **Sequence $\{a_n\}$:** ordered list of numbers

  - e.g. $\frac{1}{n}$, $\sqrt{n}$; $a_{n+1} = f(a_n)$ (recursive definition)
  - can converge ($\lim_{n\to\infty} a_n = L$, $a_n \to L$) $\frac{1}{n}$, $\frac{1}{2n}$, ..

    diverge $(-1)^n$

    diverge to $\pm\infty$ n, $n!$

  - can be bounded (above / below) ; increasing, decreasing, alternating
  - every converging sequence is bounded

    a sequence that is bounded and monotonous, converges

  - If there is a real function $f(x)$, such that $a_n = f(n)$,
    - if $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$
    - if $f(x)$ is monotonous, then $\{a_n\}$ as well.

Type of questions:

- calculate $\lim_{n\to\infty} a_n$
- show whether $\{a_n\}$ is bounded / converges / increases or decreases.

• Series $\sum_{n=1}^{\infty} a_n$: sum of infinitely many terms

- sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$
  ↳ if the sequence $\{a_n\}$ does NOT converge to 0,
  the series $\sum_{n=1}^{\infty} a_n$ DIVERGES.

- if $a_n \to 0$, $\sum_{n=1}^{\infty} a_n$ is an indeterminate form
  (sum of infinitely many terms that approach 0)

↳ you need to recognize 2 types of series

1) Geometric series, $a_n = a \cdot r^{n-1}$ $\quad a, ar, ar^2, ar^3, \ldots$
   if $|r| < 1$, $\sum_{n=0}^{\infty} (a \cdot r^n) = \sum_{n=1}^{\infty} a \cdot r^n = \frac{a}{1-r}$ converges
   if $|r| \geqslant 1$, the series diverges (to ∞ for $r \geqslant 1$)

2) p-series, $a_n = \frac{1}{n^p}$
   if $p \leqslant 1$, diverges $\quad$ cfr. $\int_1^{\infty} \frac{dx}{x^p}$ diverges
   if $p > 1$, converges $\quad \int_1^{\infty} \frac{dx}{x^p}$ converges
   (we cannot calculate the sum explicitly)

Types of questions :
- calculate $\sum^{\infty} a_n$ for a geometric series
- does $\sum^{\infty} a_n$ converge or diverge ? Explain.

Functions of multiple variables

$f(x_1, x_2, \dots, x_n)$; in 2D : $z = f(x,y)$

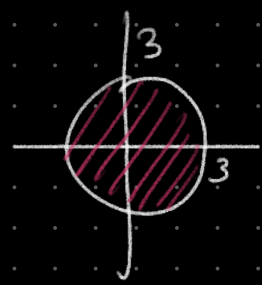
domain : subset of $\mathbb{R}^n$ domain : subset of $\mathbb{R}^2$

domain convention : $\text{domain}(f) = \{(x,y) \in \mathbb{R}^2, f(x,y) \in \mathbb{R}\}$

domain of $\frac{1}{x^2+y^2} = \mathbb{R}^2 \setminus \{(0,0)\}$

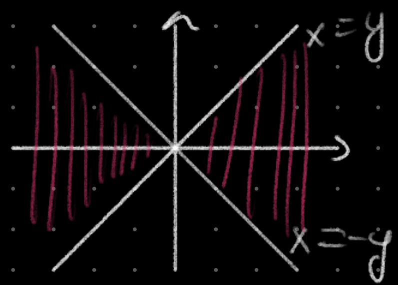
domain of $\sqrt{9-x^2-y^2}$ $x^2+y^2 \leq 9$

→ half sphere

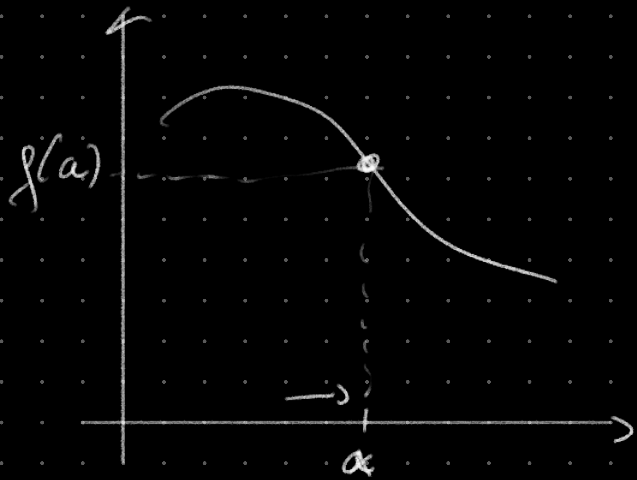


domain of $\sqrt{x^2-y^2}$

$\{(x,y) \in \mathbb{R}^2 : x^2 \geq y^2\}$



Continuity



for 1D (univariate functions)

$f(x)$ is continuous at a $\overset{DEF}{\Longleftrightarrow}$, for $x \in \text{domain}(f)$

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

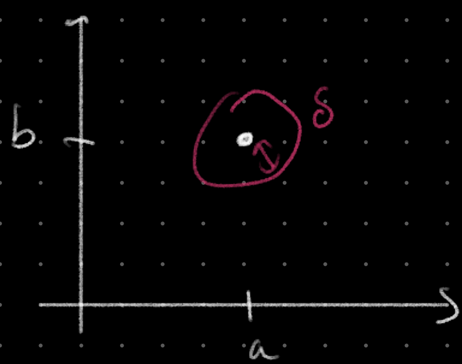
(if x approaches a, then $f(x)$ approaches $f(a)$)

Continuity for multivariate functions)

$f(x,y)$ is continuous at (a,b) iff, for $(x,y) \in$ domain( )

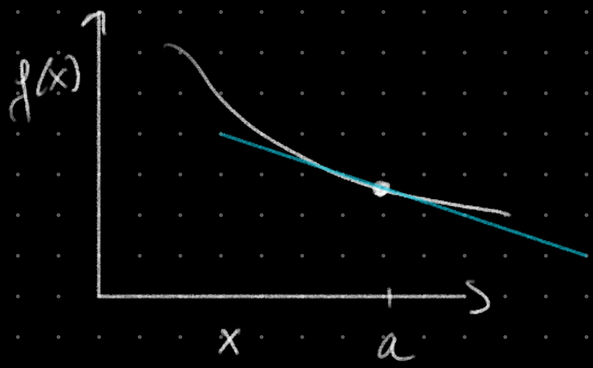
$$\forall \epsilon > 0 \;\; \exists \delta > 0 : \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

$$\Rightarrow |f(x,y) - f(a,b)| < \epsilon$$



Every "regular" function is continuous on its domain.

Partial derivatives



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

the derivative $f'(x)$ indicates how $f(x)$ changes (around $x = a$) if we change x (a little).

PARTIAL DERIVATIVE

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

other notations: $\frac{\partial}{\partial x} f(x,y) = f_1(x,y) = f_x(x,y) = D_x f(x,y)$

$D_1 f(x,y)$

$$\left.\frac{\partial f}{\partial x}\right|_a^b = f_x(a,b)$$

$$\frac{\partial}{\partial x}(xy) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{(x+h)\cdot y - xy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x\cdot y} + \cancel{h}\cdot y - \cancel{xy}}{\cancel{h}} = y$$

$$\frac{\partial}{\partial x}(x+y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{((\cancel{x}+h)+\cancel{y}) - (\cancel{x}+\cancel{y})}{h}$$

$$= 1$$

* how to calculate: like "normal derivatives". If you calculate $\frac{\partial f}{\partial x}$, you treat y like a constant. If you calculate $\frac{\partial f}{\partial y}$, you treat x like a constant.

Examples

$$\frac{\partial}{\partial x}(x \cdot y) = y$$

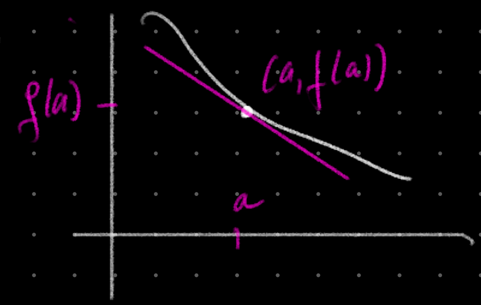
$$\frac{\partial}{\partial y}(x + y) = 1$$

$$\frac{\partial}{\partial x}\left(\ln\left[\frac{x}{y}\right]\right) = \frac{\cancel{y}}{x} \cdot \frac{1}{\cancel{y}} = \frac{1}{x}$$

$$\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \frac{\cancel{2}y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The tangent plane

<u>1D</u>



Equation of tangent line: $y = f(a) + f'(a)(x-a)$

- In 2D, we have a "tangent plane" that touches the surface $f(x,y)$

  Equation of tangent plane at $(a, b, f(a,b))$:

$$z(x,y) = f(a,b) + \left.\frac{\partial f}{\partial x}\right|_a^b (x-a) + \left.\frac{\partial f}{\partial y}\right|_a^b (y-b)$$

Example: $f(x,y) = \sin(xy)$. Tangent plane at $\left(-1, \frac{\pi}{3}, f\left(-1, \frac{\pi}{3}\right)\right)$?

- $f\left(-1, \frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
- $\frac{\partial f}{\partial x} = \cos(xy)\cdot y \rightarrow \frac{\partial f}{\partial x}\left(-1, \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right)\cdot\frac{\pi}{3} = \frac{1}{2}\cdot\frac{\pi}{3}$
- $\frac{\partial f}{\partial y} = \cos(xy)\cdot x \rightarrow \frac{\partial f}{\partial y}\left(-1, \frac{\pi}{3}\right) = -\frac{1}{2}$

Tangent plane: $z = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}(x+1) - \frac{1}{2}\left(y - \frac{\pi}{3}\right)$

Higher order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

↳ there are 4 second order derivatives. $\frac{\partial^2 f}{\partial y^2}$ $\frac{\partial^2 f}{\partial x \partial y}$ $\frac{\partial^2 f}{\partial y \partial x}$

→ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Example : $\frac{\partial^2}{\partial x \partial y}(\sin(x+y)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(x+y)\right) = \frac{\partial}{\partial x}\left(\cos(x+y)\right)$

$= -\sin(x+y)$ ✓

$\frac{\partial^2}{\partial y \partial x}(\sin(x+y)) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\sin(x+y)\right) = \frac{\partial}{\partial y}(\cos(x+y))$

$= -\sin(x+y)$ ✓

$$\frac{\partial^2}{\partial x \partial y}(\sin(xy)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(xy)\right) = \frac{\partial}{\partial x}\left(\cos(xy)\cdot x\right)$$

$$= \cos(xy) - x\cdot\sin(xy)\,y$$

$$= \cos(xy) - xy\cdot\sin(xy)$$

Chain rule in multiple dimensions

(coordinate transformations)

1D: $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$

2D $f(x,y)$, $x(t)$ and $y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$f(x,y)$ $x(r,\theta)$ and $y(r,\theta)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

Example : $z = \frac{1}{(x+y)^2}$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial z}{\partial r} =$$

$$\frac{\partial z}{\partial \theta} =$$

- $z(x,y) = \sqrt{x^2+y^2}$, $x(t) = 2t$, $y(t) = 5-t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\cdot 2 + \frac{y}{\sqrt{x^2+y^2}}(-1) = \frac{2x-y}{\sqrt{x^2+y^2}} = \frac{4t+t-5}{\sqrt{4t^2+(5-t)^2}}$$

$\longrightarrow$ if you write $z = \sqrt{x^2+y^2} = \sqrt{5t^2-10t+25}$, and compute $\frac{dz}{dt}$, you get the same result.