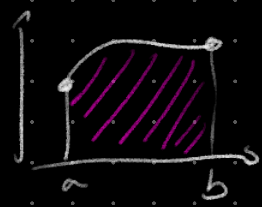
Calculus Lecture 6: Integration techniques

- Recap: definite and indefinite integrals
- Substitution (inverse chain rule)
- Integration by parts (inverse product rule)
- Partial fraction decomposition (rational functions)
- Improper integrals

Adams' Ch. 5.6, 6.1, 6.2, 6.5

Recap

- Definite integral
  - Area below a graph
  - Limit of a Riemann sum of rectangular areas

$$\int_a^b f(x)\,dx =$$ 

NUMBER

- Indefinite integral = anti-derivative

$$\int f(x)\,dx = F(x) + C \quad \text{if} \quad F'(x) = f(x)$$

FUNCTION of x

- The fundamental theorem of Calculus connects definite and indefinite integrals

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad , \text{with} \quad F'(x) = f(x)$$

This lecture: how to calculate integrals

Simple integrals (that are on your formula sheet)

(socrative)

$$\int dx = x + C$$

$r \neq -1$ $$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

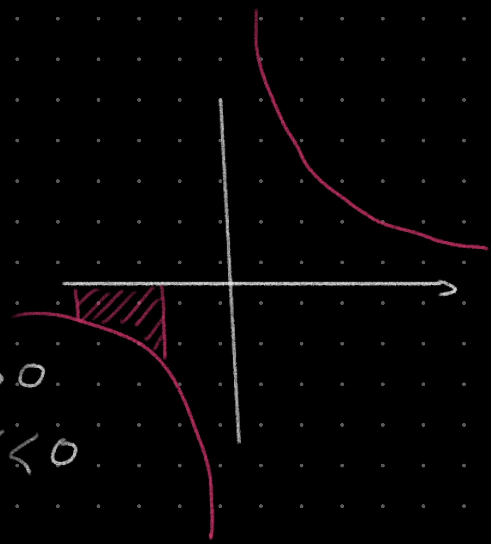
$$\int \cos(x)\,dx$$

$$\int e^x\,dx$$

$$\int e^{-x^2}\,dx$$

$$F(x) = \ln|x|$$

$$F(x) = \ln(x), \quad x > 0$$
$$= \ln(-x), \quad x < 0$$



$$F'(x) = \frac{1}{x} \qquad x > 0$$

$$F'(x) = \frac{d}{dx}\left(\ln(-x)\right) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}$$

$x < 0$

Substitution

chain rule : $\frac{d}{dx}(f(g(x))dx = f'(g(x)) \cdot g'(x)$

$$\int f'(\underbrace{g(x)}) \underbrace{g'(x)\,dx} = \int f'(u)\,du = f(u) + C$$

$u = g(x)$ $\quad du = g'(x)dx$ $\qquad = f(g(x)) + C$

$$\int \sin(3x)\,dx = \int \sin(u) \cdot \frac{du}{3} = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}(-\cos(u)) + C$$
$$= -\frac{1}{3}\cos(3x) + C$$

$u = 3x$

$du = 3 \cdot dx \longrightarrow dx = \frac{du}{3}$

$$\int \frac{dx}{x+1} = \int \frac{du}{u} = \ln|u| + C = \ln|x+1| + C$$

$u = x+1$

$du = 1 \cdot dx = dx$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$

$u = \cos(x)$ $\quad du = -\sin(x) \cdot dx$

$$\int \frac{x\,dx}{x^2+1} = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1) + C$$

$u = x^2+1$ $\quad du = 2x\,dx$

Substitution - definite integrals

$$\int_0^{\pi/4} \tan(x) \cdot dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\, dx = \int_1^{\sqrt{2}/2} \frac{-du}{u} = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u} = \ln(1) - \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$= 0 - \ln\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\ln(2)$$

$u = \cos(x)$
$du = -\sin(x)dx$

$u(0) = 1$
$u\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

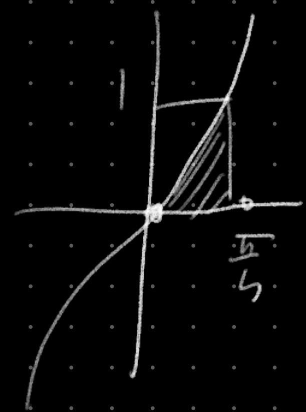
$$-\ln\left(\frac{\sqrt{2}}{2}\right) = \ln\left(\frac{2}{\sqrt{2}}\right)$$

$$= \ln(\sqrt{2}) = \frac{1}{2}\ln(2)$$

In general :

$$\int_a^b f'(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f'(u)\, du$$

$u = g(x)$
$du = g'(x)dx$



Integration by parts (inverse product rule)

$$\frac{d}{dx}(u \cdot v) = u' \cdot v + v' \cdot u \quad \text{product rule}$$

$$\int u(x)\, dv(x) = u \cdot v - \int v(x)\, du(x)$$

$$\int u(x)\, v'(x)\, dx = u \cdot v - \int v(x)\, u'(x)\, dx$$

$$\int x e^x\, dx = x \cdot e^x - \int e^x\, dx = x e^x - e^x + C$$

$$u(x) = x \qquad dv = e^x\, dx$$
$$du = dx \qquad v = e^x$$

$$\int \ln(x)\, dx = \int 1 \cdot \ln(x)\, dx = x \cdot \ln(x) - \int \cancel{x} \cdot \frac{dx}{\cancel{x}} = x\ln(x) - x + C$$

$$u(x) = \ln(x) \qquad du = \frac{dx}{x}$$
$$dv = 1 \cdot dx \qquad v(x) = x$$

$$\int x \sin(x)\, dx = -x \cdot \cos(x) + \int \cos(x)\, dx = -x\cos(x) + \sin(x) + C$$

$$u(x) = x \qquad du = dx$$
$$dv = \sin(x)\, dx \qquad v = -\cos(x)$$

Rational functions - partial fraction decomposition

Idea: we write a rational function $\frac{P(x)}{Q(x)}$ as a sum of a polynomial

and "simple" fractions $\frac{A_k}{x-x_k}$ that we can easily integrate.

poles

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{A_1}{x-x_1} + \dots + \frac{A_n}{x-x_n} \qquad (*)$$

1) factorize $Q(x) = (x-x_1)(x-x_2)(\quad)$

2) If needed, find $P_1(x)$ by long division of polynomials.

3) $A_k = \lim_{x \to x_k} \left((x-x_k) \frac{P(x)}{Q(x)} \right)$

(or solve system of equations) $\longrightarrow$ usually easier.

4) For roots with higher multiplicity: $\frac{P(x)}{(x-x_k)^2} = \frac{A}{x-x_k} + \frac{B}{(x-x_k)^2}$

Example: $\int \frac{dx}{x^2-4} = \int \left(\frac{1}{4} \frac{1}{x-2} - \frac{1}{4} \frac{1}{x+2} \right) dx = \frac{1}{4} \int \frac{dx}{x-2} - \frac{1}{4} \int \frac{dx}{x+2}$

$= \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$

$$\frac{1}{x^2-4} = \frac{A_1}{x-2} + \frac{A_2}{x+2}$$

$$\Rightarrow \quad 1 = A_1(x+2) + A_2(x-2)$$

for $x=-2$: $\quad 1 = A_2(-2-2) = -4A_2 \Rightarrow A_2 = -\frac{1}{4}$

for $x=2$ $\quad 1 = A_1(2+2) = 4A_1 \Rightarrow A_1 = \frac{1}{4}$

$$0 \cdot x + 1 = A_1(x+2) + A_2(x-2) = (A_1+A_2)x + 2(A_1-A_2)$$

$$0 = A_1 + A_2 \Rightarrow A_1 = -A_2$$

$$1 = 2(A_1 - A_2) \Rightarrow A_1 = -A_2 = \frac{1}{4}$$

What if $Q(x) = (x-x_0)^2$? $\quad \frac{P(x)}{Q(x)} = \frac{A_1}{(x-x_0)} + \frac{A_2}{(x-x_0)^2}$

$$\int \frac{x+3}{(x-2)^2}\,dx = \int\left(\frac{1}{x-2} + \frac{5}{(x-2)^2}\right)dx = \int\frac{dx}{x-2} + 5\int\frac{dx}{(x-2)^2} \quad \text{DIY}$$

$$\frac{x+3}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2}$$

$$x+3 = A_1(x-2) + A_2 = A_1 x + (A_2 - 2A_1)$$

$$1 = A_1$$

$$3 = A_2 - 2A_1 \Rightarrow A_2 = 5$$

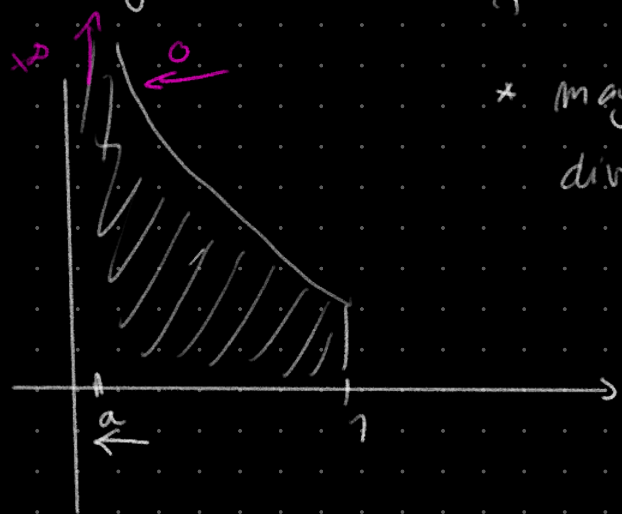
What if $Q(x) = (x-x_0)(\underbrace{ax^2+bx+c}_{\text{no real roots}})$?

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-x_0} + \frac{Bx+C}{ax^2+bx+c}$$

Improper integrals

- Type I: integrating next to a vertical asymptote
- $$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \left(\cancel{\ln(1)} - \ln(a)\right) = +\infty$$

↳ always indeterminate forms!



* may converge : Area is finite

diverge : Area is $\pm\infty$

↳ if $\int_a^b f(x)\,dx = +\infty$, then if $g(x) \geq f(x)$, then $\int_a^b g(x)\,dx = +\infty$

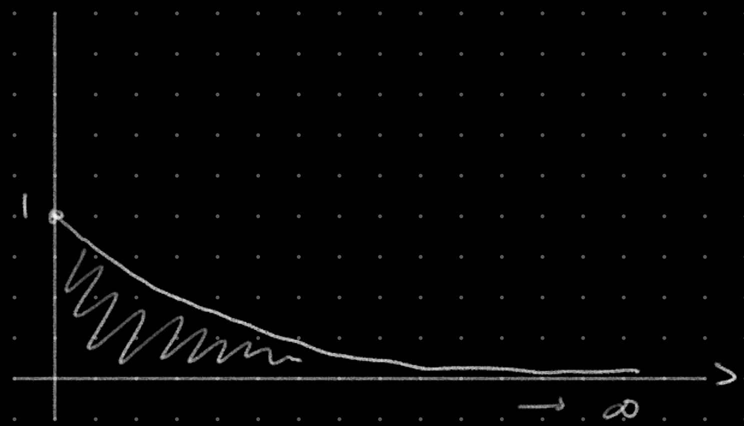
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} 2\left(\sqrt{1} - \sqrt{a}\right) = 2$$

Improper integrals

- Type II: integrating to infinity

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-\overset{\text{to } 0}{e^{-b}} + e^{0}\right)$$

$$= 1$$



- Improper integrals are an INDETERMINATE FORM. They may
  - converge : the area is finite
  - diverge to $\pm\infty$: the area grows arbitrarily large
  - diverge / not exist : it is not possible to calculate the area