LECTURE 6: INTEGRATION TECHNIQUES

1. METHOD OF SUBSTITUTION (5.6)
2. INTEGRATION BY PARTS (6.1)
3. PARTIAL FRACTION DECOMPOSITION (6.2)

1 SUBSTITUTION

CHAIN RULE: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \underbrace{g'(x)}$

SUBST: $u = g(x)$. THEN $\frac{d}{dx} f(g(x)) = \frac{d}{dx} f(u) = \frac{d}{du} f(u)\Big|_{u=g(x)} \cdot \frac{du}{dx}$

INTEGRATE: $\cancel{\int \frac{d}{dx}} f(g(x)) + C = \int f'(g(x)) \cdot \underbrace{g'(x)\, dx}$

$u = g(x)$. $\frac{du}{dx} = g'(x) \Rightarrow \underbrace{\boxed{du} = g'(x) \cdot dx}$

LINEAR: $\int \cos(6x+1)\, dx \qquad u = 6x+1\ ;\ du = 6 \cdot dx\ ;\ dx = \frac{1}{6} du$

$= \int \cos(u) \cdot \frac{1}{6} du = \frac{1}{6} \sin(u) + C = \frac{1}{6} \sin(6x+1) + C$

$\int \sqrt{3x-5}\, dx \qquad u = 3x-5\ ;\ du = 3dx$

$= \int \sqrt{u}\, \frac{1}{3} du = \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{9} (3x-5)^{\frac{3}{2}} + C$

$$\int x e^{x^2} dx$$

$u = x^2$

$\boxed{du = 2x\,dx}$

$x = 0 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 4$

$$= \int \tfrac{1}{2} e^u \, du = \tfrac{1}{2} e^u + C = \tfrac{1}{2} e^{x^2} + C$$

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)} dx$$

$u = \cos(x)$

$du = -\sin(x)\, dx$

$\cancel{u = \sin(x)}$

$\cancel{du = \cos(x)\, dx}$

$$= \int -\frac{1}{u} du = -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

$$\int_0^2 x e^{x^2} dx = \left[\tfrac{1}{2} e^{x^2}\right]_{x=0}^{2}$$

$$\int_0^2 x e^{x^2} dx = \int_0^4 \tfrac{1}{2} e^u \, du$$

$$\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} dx$$

$u = e^{\sqrt{x}}$

$du = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\, dx$

$$= \int 2 \cos(u)\, du = 2\sin(u) + C = 2 \sin(e^{\sqrt{x}}) + C$$

INTEGRATION BY PARTS

PRODUCT RULE: $\frac{d}{dx} f(x)\cdot g(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$$\underbrace{\int f'(x)g(x)\,dx} = f(x)\cdot g(x) - \underbrace{\int f(x)g'(x)\,dx}$$

TYPICALLY: f TRIGON./EXPON. ; g POLYNOMIAL   LOGARITHMIC

$\int xe^x\,dx$ $f' = e^x,\ f = e^x$
$g = x,\ g' = 1$

$$= xe^x - \int e^x\,dx = xe^x - e^x + C$$

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + \int 2x\cos(x)\,dx = -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$$

$f' = \sin(x) \Rightarrow f = -\cos(x)$ $f' = \cos(x)\quad f = \sin(x)$ $= \dots$
$g = x^2 \Rightarrow g' = 2x$ $g = 2x\quad g' = 2$

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C$$

$f' = 1 \Rightarrow f = x$
$g = \ln(x)\quad g' = \frac{1}{x}$

$$\underbrace{\int e^x \cos(2x)\,dx}_{} = e^x\cos(2x) + 2\int e^x \sin(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x) - 4\underbrace{\int e^x\cos(2x)\,dx}_{}$$

$f' = e^x \quad f = e^x$

$g = \cos(2x) \quad g' = -2\sin(2x)$

$f' = e^x \quad f = e^x$

$g = \sin(2x) \quad g' = 2\cos(2x)$

So $5\int e^x\cos(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x) + C$

PARTIAL FRACTION DECOMPOSITION

RATIONAL FUNCTIONS : $\dfrac{P(x)}{Q(x)} = P_1(x) + \dfrac{P_2(x)}{Q(x)}$ WITH $\deg(P_2) < \deg(Q)$

$\underline{1}$ IF $Q(x) = (x-a_1)\cdot(x-a_2)\cdot(x-a_3)\cdot \ldots \cdot(x-a_n)$

THEN $\dfrac{P_2(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \dfrac{A_2}{x-a_2} + \ldots + \dfrac{A_n}{x-a_n}$ FIND $A_1, A_2, \ldots, A_n$

EXAMPLE: $\displaystyle\int \frac{x+1}{x^2-4}\,dx$

$$\frac{x+1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

So $A(x-2)+B(x+2) = x+1 \quad \forall x$

$\rightsquigarrow A+B = 1 \;\&\; -2A+2B = 1 \Rightarrow A = \frac{1}{4} \;\&\; B = \frac{3}{4}$

$$\int \frac{x+1}{x^2-4}\,dx = \int \frac{\frac{1}{4}}{x+2} + \frac{\frac{3}{4}}{x-2}\,dx = \frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C$$

$$(x-a_j)\cdot\frac{P_2(x)}{Q(x)} = A_1\cdot\frac{x-a_j}{x-a_1} + A_2\cdot\frac{x-a_j}{x-a_2} + \ldots + A_j + \ldots + A_n\cdot\frac{x-a_j}{x-a_n}$$

$$\lim_{x\to a_j} (x-a_j)\cdot\frac{P_2(x)}{Q(x)} = A_j$$

$$A = \lim_{x\to -2} (x+2)\cdot\frac{x+1}{x^2-4} = \lim_{x\to -2}\frac{x+1}{x-2} = \frac{1}{4}.$$

WHAT IF $DEG(P) \geq DEG(Q)$? USE LONG DIVISION

$$\int \frac{2x^3+3x^2-7x-11}{x^2-4}$$

$$\begin{array}{r} 2x+3 \\ x^2-4 \,\big)\, \overline{2x^3+3x^2-7x-11} \\ -\,2x^3 \qquad -8x \\ \hline 3x^2+x-11 \\ -\,3x^2 \qquad -12 \\ \hline x+1 \end{array}$$

$$= \int 2x+3 + \frac{x+1}{x^2-4}\,dx$$

$$= x^2+3x+\frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C$$

$\underline{?}$ WHAT IF $\underline{Q(x) = (x-a)^2}$?

$$\int \frac{x+1}{(x-4)^2}\,dx$$

WRITE: $\dfrac{P_1(x)}{Q(x)} = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} = \dfrac{A(x-a)+B}{(x-a)^2} \longleftrightarrow x+1 \quad \forall x.$

$Ax = 1 \cdot x \rightarrow A = 1$

$-4A + B = 1 \Rightarrow B = 5$

$$= \int \frac{1}{x-4} + \frac{5}{(x-4)^2}\,dx = \ln|x-4| + 5 \cdot \frac{-1}{x-4} + C$$

$\underline{?}$ WHAT IF $Q(x)$ HAS ~~F~~FACTORS THAT HAVE NO REAL ROOTS ?

$Q(x) = x^2 + 1$ $\qquad \int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

$\int \frac{x}{x^2+1}\,dx$ $\qquad u = x^2+1$, $du = 2x\,dx$ $\qquad \cancel{\int \frac{x+1}{x^2+1}\,dx}$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

IMPROPER INTEGRALS

$$\int_a^b f(x)\,dx \quad \text{WHAT IF} \quad \underline{1}\ \ a=-\infty \ \text{AND/OR}\ b=+\infty\,?$$

$$\underline{2}\ \ \lim_{x\to a/b} f(x) = \pm\infty\,?$$

$\underline{1}$

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^2}\,dx$$

$$= \lim_{R\to\infty}\left[-\frac{1}{x}\right]_{x=1}^{R} = \lim_{R\to\infty} -\frac{1}{R}+1 = 1.$$

$\underline{2}$

$$\int_a^b f(x)\,dx \quad \text{WHERE} \quad \lim_{x\to a^+} f(x) = \pm\infty$$

$$= \int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx$$

$$= \lim_{R\to\infty}\left[\ln(x)\right]_{x=1}^{R} = \lim_{R\to\infty}\ln(R) - 0$$

$$= +\infty.$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+}\int_c^b \frac{1}{x^2}\,dx = \lim_{c\to 0^+}\left[-\frac{1}{x}\right]_{x=c}^{1} \overset{\lim_{c\to 0^+}}{=} \left(-1+\frac{1}{c}\right) = -1+\infty = +\infty.$$

$$\int_a^\infty \frac{1}{x^p}\,dx$$: CONVERGES IF $p > 1$ (WE GET A FINITE OUTCOME)

DIVERGES IF $p \leq 1$ (GOES TO ∞)