Calculus Lecture 3: Differentiation

- Oblique asymptotes
- Tangent lines
- Definition of a derivative
- Calculation of derivatives
  - Product rule
  - Chain rule
  - Trigonometric functions
  - Exponential and logarithmic functions
- Higher order derivatives

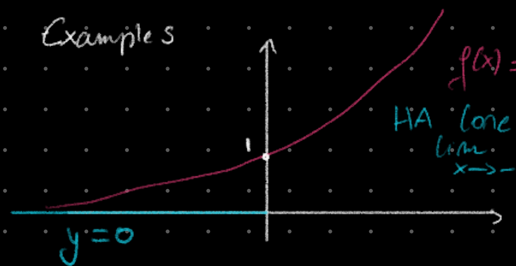
Adams' Ch. 2.1–6, 3.3

Asymptotes

Horizontal asymptote: $\lim_{x \to +\infty} f(x) = a$ OR $\lim_{x \to -\infty} f(x) = b$

the function approaches a horizontal line as $x \to \pm\infty$ ($y = a$, $y = b$)

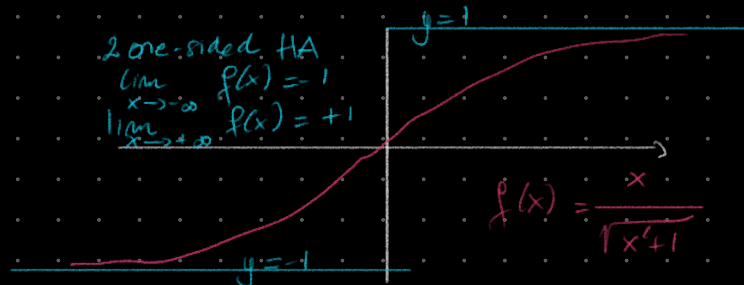
Examples



$f(x) = e^x$

HA (one-sided) $y = 0$

$\lim_{x \to -\infty} f(x) = 0$



2 one-sided HA

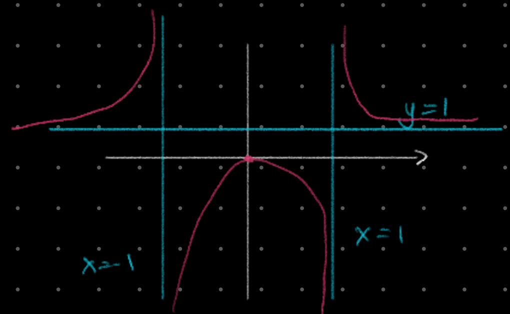
$\lim_{x \to -\infty} f(x) = -1$

$\lim_{x \to +\infty} f(x) = +1$

$f(x) = \dfrac{x}{\sqrt{x^2+1}}$

Vertical asymptote : $\lim_{x \to b^\pm} f(x) = \pm\infty$

the function approaches a vertical line as $x \to b$ (from the right or left side) $x = b$)



$f(x) = \dfrac{x^2}{x^2-1}$

HA : $\lim_{x \to \pm\infty} f(x) = 1$ (two-sided HA)

VA : $\lim_{x \to 1^-} f(x) = -\infty$ $\quad \lim_{x \to -1^+} f(x) = -\infty$

2VA $x = \pm 1$ $\quad \lim_{x \to 1^+} f(x) = +\infty$ $\quad \lim_{x \to -1^-} f(x) = +\infty$

Oblique asymptotes

the function approaches the line $y = ax + b$, $a \neq 0$ as $x \to \pm\infty$

$$\begin{cases} \lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = a \neq 0 \\ \lim\limits_{x \to \pm\infty} (f(x) - ax) = b \end{cases}$$

$\sqrt{x^2}$ is always positive:
$\sqrt{x^2} = x$ for $x \geq 0$
$= -x$ for $x < 0$

Example $f(x) = \sqrt{1 + x^2}$

1) $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = \lim\limits_{x \to +\infty} \frac{\sqrt{1+x^2}}{x} = \lim\limits_{x \to +\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} = \lim\limits_{x \to +\infty} \frac{\cancel{x}\sqrt{1+\cancel{1/x^2}}}{\cancel{x}} = 1 = a$

since $x \to +\infty$, x is positive

2) $\lim\limits_{x \to +\infty} (f(x) - ax) = \lim\limits_{x \to +\infty} (\sqrt{1+x^2} - x)$

$= \lim\limits_{x \to +\infty} \frac{(\sqrt{1+x^2} - x)(\sqrt{1+x^2} + x)}{\sqrt{1+x^2} + x} = \lim\limits_{x \to +\infty} \frac{(1+\cancel{x^2}) - \cancel{x^2}}{\sqrt{1+x^2} + x} = 0 = b$

↳ one-sided oblique asymptote $y = x$
↳ only as $x \to +\infty$

→ since this function is even $(f(x) = f(-x))$, we can mirror the asymptote around the y-axis to find $y = -x$ as $x \to -\infty$

For completeness.

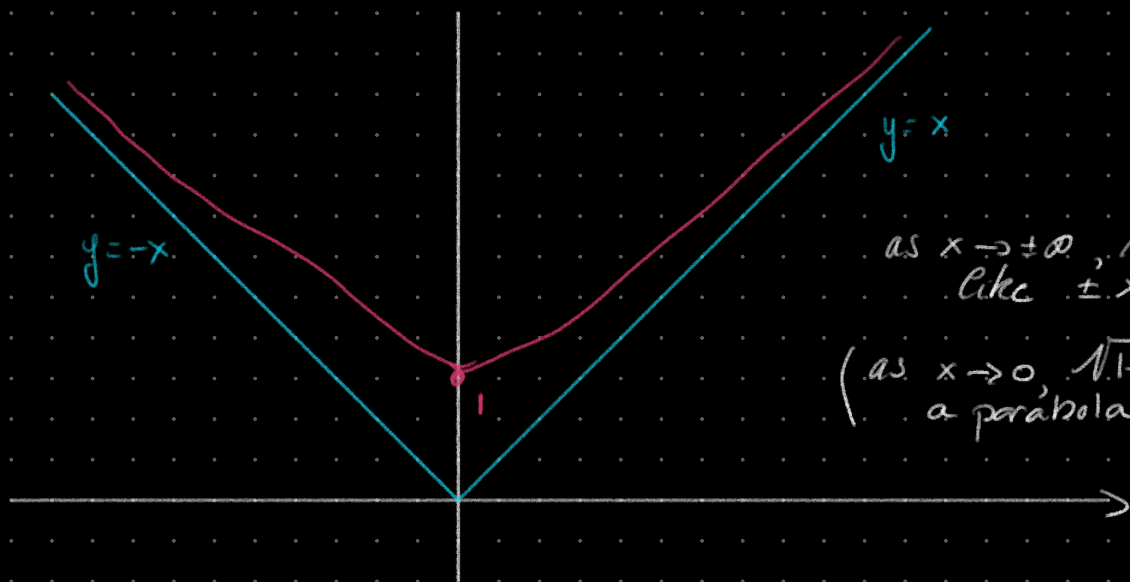
$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\sqrt{1+x^2}}{x} = \lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to-\infty} -\sqrt{1+\frac{1}{x^2}} = -1 = a$$

$\to = -1$, since $x<0$, $\sqrt{x^2}>0$

$$\lim_{x\to-\infty}\left(f(x)-ax\right) = \lim_{x\to-\infty}\left(\sqrt{1+x^2}+x\right) = \lim_{x\to-\infty}\frac{\left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+x^2}-x\right)}{\sqrt{1+x^2}-x}$$

$$= \lim_{x\to-\infty}\frac{(1+\cancel{x^2})-\cancel{x^2}}{\underbrace{\sqrt{1+x^2}-x}_{\text{denominator goes to } +\infty \text{ as } x\to-\infty}} = 0 = b$$

↳ one-sided oblique asymptote $y=-x$



as $x\to\pm\infty$, $\sqrt{1+x^2}$ behaves like $\pm x$.

(as $x\to 0$, $\sqrt{1+x^2}$ resembles a parabola $(1+\frac{x^2}{2}$?)

Asymptotes for rational functions

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$$

- horizontal asymptote $y = 0$ if $m > n$

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + \dots + b_0} = \lim_{x \to \pm\infty} \frac{\overbrace{a_n \frac{1}{x^{m-n}} + \dots + a_0 \frac{1}{x^m}}^{\text{to } 0}}{b_m + \underbrace{\dots + b_0 \frac{1}{x^m}}_{\text{to } 0}} = 0$$

divide all terms by x^m

- horizontal asymptote $y = \frac{a_n}{b_m}$ if $m = n$

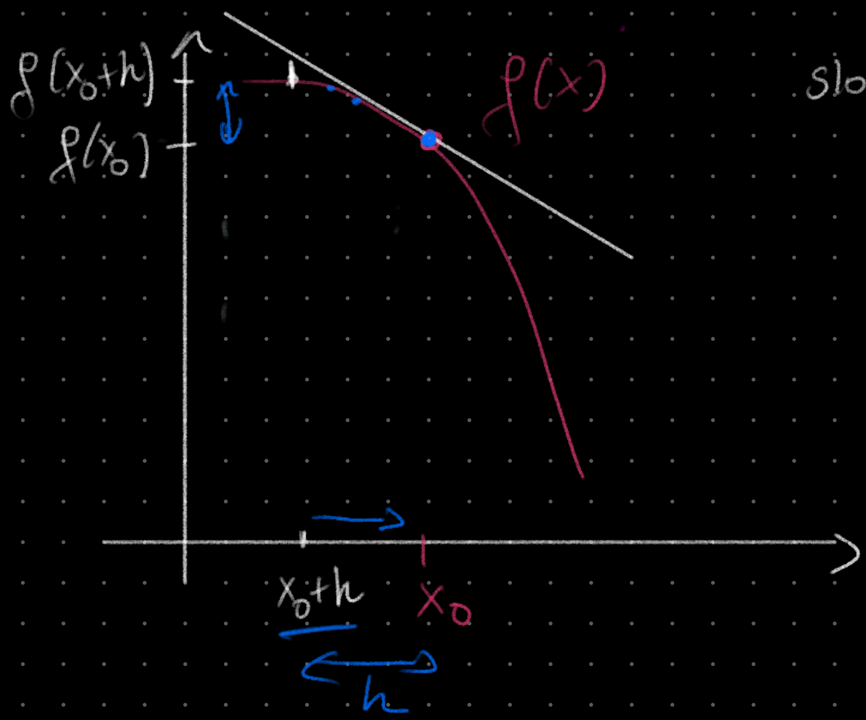
$$\lim_{x \to \pm\infty} \frac{a_n x^n + \dots a_0}{b_m x^m + \dots + b_0} = \lim_{x \to \pm\infty} \frac{a_n + \overbrace{\dots + a_0 \frac{1}{x^m}}^{\text{to } 0}}{b_m + \underbrace{\dots + b_0 \frac{1}{x^m}}_{\text{to } 0}} = \frac{a_n}{b_m}$$

divide all terms by $x^m = x^n$

- oblique asymptote with $a = \frac{a_n}{b_m}$ if $n = m+1$
  (compute $\lim_{x \to \pm\infty} \frac{f(x)}{x}$ to see this)

- vertical asymptotes $x = p$ at the poles, $Q(p) = 0$, $P(p) \neq 0$

Tangent lines and their slopes

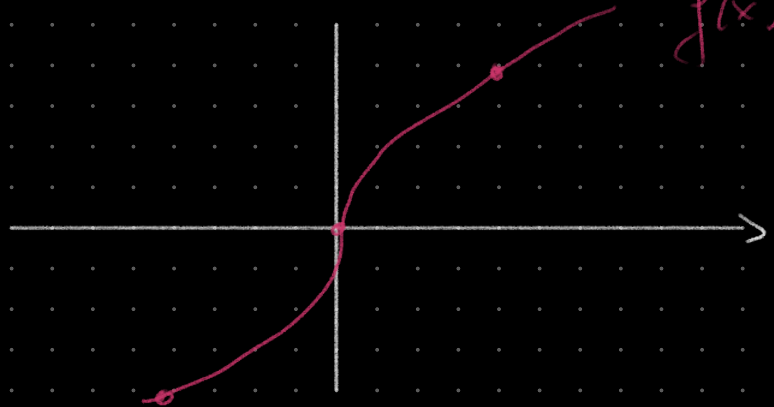


$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\boxed{f(x_0+h) - f(x_0)}}{h}$$

slope of tangent line:

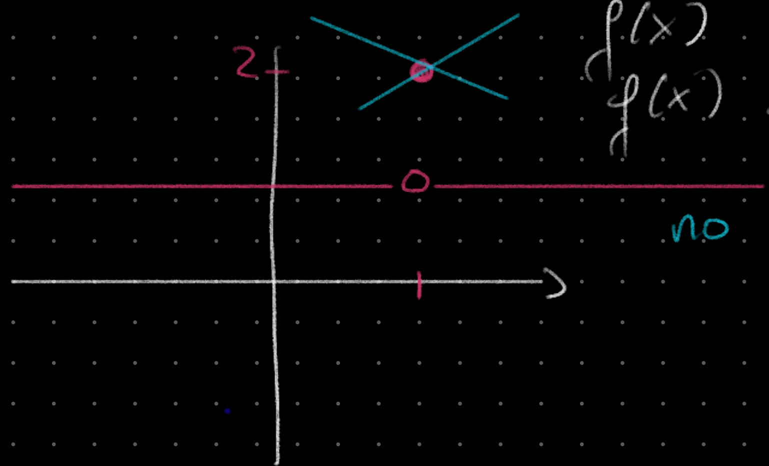
$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

- Can the tangent line be vertical? yes



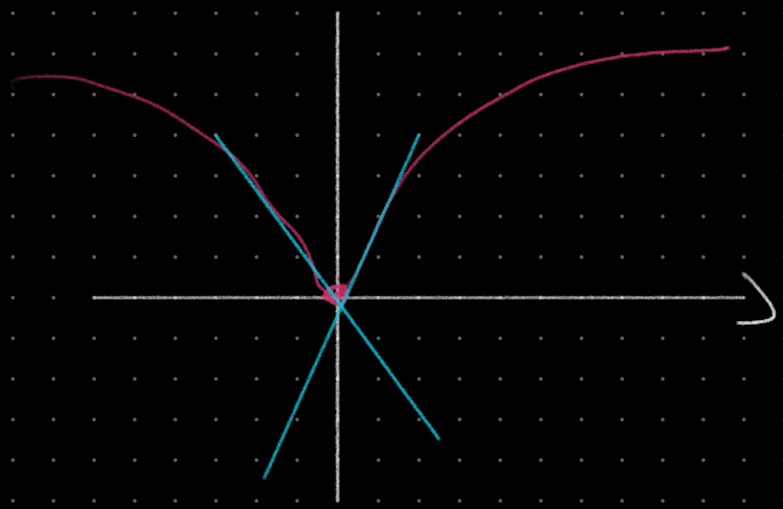
$f(x) = \sqrt[3]{x}$

- Does the tangent line always exist? no



$$\begin{cases} f(x) = 1 & \text{for } x \neq 0 \\ f(x) = 2 & \text{for } x = 1 \end{cases}$$

no tangent line at $x = 1$



the limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

does not exist at $x = 0$, since left and right limit are different.

The tangent line only exists if the limit exists (is finite or ∞)

The derivative

The derivative of a function at a point c of the domain is de defined as the slope of (the tangent of) the function at c.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The derivative only exists if this limit exists and is finite!

f' is also a real function, domain $(f') \subseteq$ domain (f)

- Notations: $f'(x)$, $\frac{d}{dx}$, $y'(x)$, $\frac{dy}{dx}$, $D_x f$, $D_x f_y$
- Differentiable: f is differentiable at x_0 if $f'(x_0)$ exists
- Singular point: x_0 is a singular point if $f'(x_0)$ does not exist
- Left and right derivatives: $f'_{\pm}(x) = \lim_{h \to 0^{\pm}} \frac{f(x+h) - f(x)}{h}$

  analogous to left/right limits

Examples

- $f(x) = ax + b$ (linear function) $\quad f'(x) = \lim_{h \to 0} \frac{(a(\cancel{x}+h) + \cancel{b}) - (a\cancel{x} + \cancel{b})}{h} = a$

the tangent line to a linear function is of corse the linear function itself, so the slope = the derivative.

- $f(x) = c$ (constant function)

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = 0$$

- $f(x) = x^2$ $\quad f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(\cancel{x^2} + 2\cancel{h}x + h^{\cancel{2}}) - \cancel{x^2}}{\cancel{h}}$

$$= \lim_{h \to 0} (2x + h) = 2x$$

$f(x) = \frac{1}{x}$ $\quad f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{x} - (\cancel{x}+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)}$

$$= -\frac{1}{x^2}$$

↳ in general $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ $\quad (n \in \mathbb{R})$ POWER RULE

Differentiation rules

- Differentiable => continuous
- For functions f and g differentiable at x, and k constant,
  - $(f+g)'(x) = f'(x) + g'(x)$
  - $(k \cdot f)' = k \cdot f'(x)$
- (The product rule) For functions f and g differentiable at x
  - $(f(x)g(x))' = f'(x)\, g(x) + f(x) \cdot g'(x)$
  - NOT: $f'(x) \cdot g'(x)$

PROOF: $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x)\, g(x)}{h} =$

$$\lim_{h \to 0} \left[\frac{1}{h} \left(\left(f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) \right) + \left(f(x) \cdot g(x+h) - f(x) \cdot g(x) \right) \right) \right] =$$

$$\underbrace{\lim_{h \to 0} \left(\frac{1}{h} \left(f(x+h) - f(x) \right)\right.}_{f'(x)} \left. \cdot\, g(x+h) \right) + \lim_{h \to 0} \left(\underbrace{\frac{1}{h} \left(g(x+h) - g(x) \right)}_{g'(x)} \, f(x) \right) =$$

$$f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- (The chain rule): If f(u) is differentiable at u=g(x), and g(x) is differentiable at x, then f(g(x)) is differentiable and

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} \quad (\text{intuitive "explanation"})$$

- (reciprocal rule) $g(f(x)) = \frac{1}{f(x)}$, $g(x) = \frac{1}{x}$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{(f(x))^2} \cdot f'(x)$$

- (quotient rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) \overset{\text{product rule}}{=} f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\overbrace{\frac{-1}{(g(x))^2}\, g'(x)}^{\text{reciprocal rule}}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

Examples

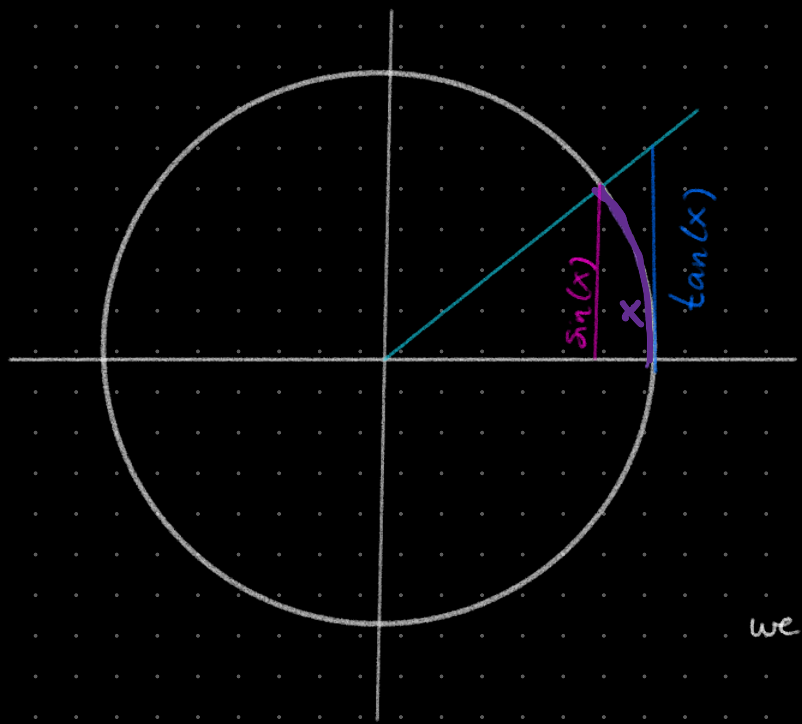
$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-1}{(1+x^2)^2}\cdot\frac{d}{dx}(1+x^2) = \frac{-2x}{(1+x^2)^2}$$

$$\frac{d}{dx}\left(\frac{x^2+1}{\sqrt{x}}\right) = \frac{\sqrt{x}\,\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(\sqrt{x})}{x} = \frac{2x\sqrt{x} - (x^2+1)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{4x^2-(x^2+1)}{2\sqrt{x}\cdot x} = \frac{3x^2-1}{2x\sqrt{x}}$$

Derivatives of trigonometric functions

$$\lim_{x\to 0^+} \frac{\sin(x)}{x}$$



for $x>0$, $\sin(x) \le x \le \tan(x)$

1.) $\sin(x) \le x \iff \frac{\sin(x)}{x} \le 1$

2.) $x \le \tan(x) = \frac{\sin(x)}{\cos(x)}$

$\iff \cos(x) \le \frac{\sin(x)}{x}$

together: $\cos(x) \le \frac{\sin(x)}{x} \le 1$

$\to$ since $\lim_{x\to 0^+} \cos(x) = 1$

and $\lim_{x\to 0^+} 1 = 1$

we have (squeeze theorem) $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$

$\star$ since $f(x) = \frac{\sin(x)}{x}$ is an even function, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$$\rightarrow \frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right) \cdot \cos\left(x+\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{1} \cdot \cos\left(x + \underbrace{\frac{h}{2}}_{\text{to } 0}\right) = \cos(x)$$

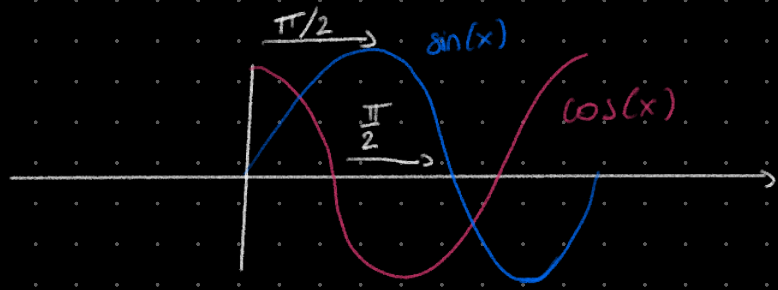
trig identity : $\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

here $a = x+h$
$b = x$

- other trig limits

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(x+\frac{\pi}{2}\right)\right) = \cos\left(x+\frac{\pi}{2}\right) = \sin(x+\pi) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$



if we shift the cosine $\frac{\pi}{2}$ to the right, we get the sine function

$$\sin\left(x+\frac{\pi}{2}\right) = \cos(x)$$

- logarithm and exponential

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

implicit differentiation $y = \ln(x)$

$$\Rightarrow e^y = x$$

$$\Rightarrow e^y \frac{dy}{dx} = 1 \quad \Big\} \frac{d}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Higher order derivatives

$f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right)$ (derivative of the derivative)

notation : $f''(x)$, $\frac{d^2f}{dx^2}$, $D_{xx}f$, $y''(x)$, $\frac{d^2y}{dx^2}$

→ for the n^{th} derivative, you use $f^{(n)}(x)$

example : $f(x) = 3x^3 \longrightarrow f'(x) = 9x^2 \longrightarrow f''(x) = 18x$