UFCFA3-30-1 Exam Formula Booklet

1. Logic

1.1 Truth Tables

Table 1 : **negation** of a proposition	
P	$\neg$P
T	F
F	T

Table 2 : **conjunction** of two propositions		
P	Q	P $\wedge$ Q
T	T	T
T	F	F
F	T	F
F	F	F

Table 3 : **disjunction** of two propositions		
P	Q	P $\vee$ Q
T	T	T
T	F	T
F	T	T
F	F	F

Table 4 : **exclusive-or** of two propositions		
P	Q	P $\oplus$ Q
T	T	F
T	F	T
F	T	T
F	F	F

Table 4 : **implication** P $\rightarrow$ Q		
P	Q	P $\rightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Table 5 : **biconditional** P $\leftrightarrow$ Q		
P	Q	P $\leftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T

1.2 Logical Equivalences (Propositional Logic)

Logical Equivalence	Name of law
$P \wedge True \Leftrightarrow P$ $P \vee False \Leftrightarrow P$	Identity
$P \vee True \Leftrightarrow True$ $P \wedge False \Leftrightarrow False$	Domination
$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$	Idempotent
$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$ $P \leftrightarrow Q \Leftrightarrow Q \leftrightarrow P$	Commutative
$(P \vee Q) \vee R \Leftrightarrow \mathrm{P} \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow \mathrm{P} \wedge (Q \wedge R)$	Associative
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive
$\neg(\neg P) \Leftrightarrow P$	Double negation
$\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$ $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$	de Morgan's
$P \rightarrow Q \Leftrightarrow (\neg P) \vee Q$	Implication
$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	Equivalence
$P \vee (\neg P) \Leftrightarrow \mathrm{True}$	Excluded middle
$P \wedge (\neg P) \Leftrightarrow \mathrm{False}$	Contradiction $(\perp \Leftrightarrow \mathrm{False})$
$P \rightarrow Q \Leftrightarrow (\neg Q) \rightarrow (\neg P)$	Contrapositive
$P \vee (\mathrm{P} \wedge Q) \Leftrightarrow P$ $P \wedge (\mathrm{P} \vee Q) \Leftrightarrow P$	Absorption

1.3 Inference Rules (Propositional Logic)

Inference Rule	Name
$\dfrac{P \quad Q}{P \wedge Q}$	Conjunction
$\dfrac{P \wedge Q}{P}$	Simplification
$\dfrac{P}{P \vee Q}$	Addition
$\dfrac{P \to Q \quad P}{Q}$	Modus Ponens
$\dfrac{P \to Q \quad \neg Q}{\neg P}$	Modus Tollens
$\dfrac{P \vee Q \quad \neg Q}{P}$	Disjunctive Syllogism
$\dfrac{P \to Q \quad Q \to R}{P \to R}$	Hypothetical Syllogism
$\dfrac{P \quad \neg P}{\perp}$	Contradiction ($\perp \Leftrightarrow$ False)

The ‘assumption’ inference laws:

- Conditional Proof:

  Applied to cases where the <u>conclusion</u> is an implication (of the form $\alpha \rightarrow \beta$).

  Assume α is true to get β being true; then you can deduce that $\alpha \rightarrow \beta$ is true.

- Indirect Proof:

  Applies to cases whereby the <u>conclusion</u> is not an implication.

  Assume the negation of the conclusion to get a contradiction; then you have proved the conclusion.

1.4 Quantifiers

Quantifiers of one variable		
Statement	When true?	When false?
$\forall x \cdot P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x \cdot P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Negation of Quantifiers:

Statement	Equivalent Statement
$\neg(\exists x \cdot P(x))$	$\forall x \cdot (\neg P(x))$
$\neg(\forall x \cdot P(x))$	$\exists x \cdot (\neg P(x))$

2. Sets

2.1 Some Standard Sets

$\boldsymbol{N}$ the set of all natural numbers = { 1, 2, 3, …}

$\boldsymbol{N_0}$ the set of all non-negative integers = { 0, 1, 2, 3, …}

$\boldsymbol{Z}$ the set of all integers = { …, –3, –2, –1, 0, 1, 2, 3, …}

$\boldsymbol{Q}$ the set of all rational numbers

$\boldsymbol{R}$ the set of all real numbers

$\boldsymbol{B}$ the set of all bit strings, including the empty string

= {λ, 0, 1, 00, 01, 10, 11, 000, 001, 010, ...}

2.2 Set Notations

- $x \in A$ x is an element of A.
- $x \notin A$ x is not an element of A.
- $A \subseteq B$ A is a subset of B.
- $A \nsubseteq B$ A is not a subset of B.
- $A \subset B$ A is a proper subset of B.

2.3. Set Operations

- The union of *A* and *B*: $A \cup B = \{x | (x \in A) \vee (x \in B)\}$
- The intersection of *A* and *B*: $A \cap B = \{x | (x \in A) \wedge (x \in B)\}$
- The difference of *A* and *B*: $A \backslash B = \{ x \mid (x \in A) \wedge (x \notin B) \}$
- The complement of *A*: $\overline{A} = \{x | x \notin A\}$

- Cartesian Product of *A* and *B*: $A \times B = \{(x, y) \mid (x \in A) \wedge (y \in B)\}$
- Power set of A (the set of all subsets of A): $\boldsymbol{P}(A) = \{ X \mid X \subseteq A \}$

2.4 Cardinality of Sets

Power sets:

Let A be a finite set with $|A| = n$. Then $|\boldsymbol{P}(A)| = 2^n$.

Cartesian Product: $|A \times B| = |A| \times |B|$

Principle of Inclusion Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

Others

- $|A \backslash B| = |A| - |A \cap B|$
- $|\bar{A}| = |U| - |A|$ where U is the universal set.

3 Functions

3.1 Composition of Functions

Let A, B, and C be sets and consider functions

$$f: A \to B$$

and $$g: B \to C$$

Note that the codomain of f equals the domain of g.

The composition of g and f, denoted by $g \circ f$ is the function

$g \circ f: A \to C$ given by $(g \circ f)(x) = g(f(x))$

3.2 Injections, Surjections and Bijections

Let A and B be sets and let $f: A \rightarrow B$ be a function.

Injection: The function f is said to be an injection if and only if:

for all pairs of distinct members of the domain, $x_1, x_2 \in A$ with $x_1 \neq x_2$, then

$f(x_1) \neq f(x_2)$.

Surjection: The function f is said to be a surjection if and only if:

for all members of the codomain, $y \in B$, there exists a member of the domain, $x \in A$, such that $f(x) = y$.

Alternatively, the function f is a surjection if and only if its image set is equal to its codomain.

Bijection: The function f is said to be a bijection if and only if:

it is both an injection and a surjection.

The function f has an inverse if and only if it is a bijection.

4. Relations

4.1 Reflexive, Irreflexive, Symmetric, Transitive

Let A be a set and let $R: A \times A$ be a relation on the set A. Then:

- R is said to be *reflexive* if and only if, for all $x \in A$, we have $(x,x) \in R$.
- R is said to be *irreflexive* if and only if, for all $x \in A$, we have $(x,x) \notin R$.
- R is said to be *symmetric* if and only if, for all $x, y \in A$, we have that:

  if $(x,y) \in R$ then $(y,x) \in R$.
- R is said to be *transitive* if and only if, for all for all $x, y, z \in A$, we have that:

  if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$.

4.3 Composition of Relations

Let R and S be relations. The composition $S \circ R$ exists if and only if the right-set of R is the same as the left-set of $S.$ Let A, B and C be sets and let $R: A\times B$ and $S: B\times C$ be relations, then the relation $S \circ R : A \times C$ exists.

The composition $S \circ R$ is defined as follows:

$$(x, z) \in S \circ R \quad \Leftrightarrow \quad \exists y \cdot \Big(\big((x, y) \in R\big) \wedge \big((y, z) \in S\big)\Big).$$

4.4 Inverse Relation

Let $R: A\times B$ be a relation. The inverse of R, denoted by R^{-1}, is the relation $R^{-1}: B\times A$ given by $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

5 Graph Theory

5.1 Isomorphic Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs.

G_1 and G_2 are isomorphic if and only if there exists a bijection of the form $f: V_1 \to V_2$ with the property that, for all vertices v_1, v_2 in V_1 we have:

$\{v_1, v_2\}$ is an edge of G_1 if and only if $\{f(v_1), f(v_{2)}\}$ is an edge of G_2.

Informally: If it is possible to replace the labelling of the vertices of G_1 with the vertices of G_2 such that the new graph is equal to G_2, then G_1 and G_2 are isomorphic.