The Greatest Common Divisor and The Least Common Multiple

By Mathew Mouchamel

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The Greatest Common Divisor

The Greatest Common Divisor

What is a divisor?

A divisor is a number that divides another number

However, in the context of integers, it is a number that divides another number completely. Thus, if a number A is the divisor of another number B, it means that when B is divided by A, there is no remainder.

The Greatest Common Divisor

What is the greatest common divisor?

The Greatest Common Divisor (GCD) of two numbers is the largest number that divides both of them without leaving a remainder. Thus, it is the largest divisor that both numbers have in common.

The Greatest Common Divisor

The three common methods of finding the GCD include:

1. The Brute Force Method or Listing Method
2. The Prime Factorization Method
3. The Euclidean Algorithm

Method #1

The Greatest Common Divisor

1. The Brute Force Method

This method involves listing out all the divisors of each number and then finding the largest number that appears on both lists. It's a straightforward approach but can be inefficient for large numbers.

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : 1, 2, 3, 4, 6, 12	18 : 1, 2, 3, 6, 9, 18
12 / 1 = 12	18 / 1 = 18
12 / 2 = 6	18 / 2 = 9
12 / 3 = 4	18 / 3 = 6
12 / 4 = 3	18 / 6 = 3
12 / 6 = 2	18 / 9 = 2
12 / 12 = 1	18 / 18 = 1

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : 1, 2, 3, 4, 6, 12

18 : 1, 2, 3, 6, 9, 18

12 / 2 = 6

18 / 3 = 6

12 / 4 = 3

18 / 6 = 3

12 / 6 = 2

18 / 9 = 2

12 / 12 = 1

18 / 18 = 1

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : 1, 2, 3, 4, 6, 12

18 : 1, 2, 3, 6, 9, 18

$\boxed{12 / 2 = 6}$

$12 / 4 = 3$

$12 / 6 = 2$

$12 / 12 = 1$

$\boxed{18 / 3 = 6}$

$18 / 6 = 3$

$18 / 9 = 2$

$18 / 18 = 1$

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : 1, 2, 3, 4, 6, 12

18 : 1, 2, 3, 6, 9, 18

$\boxed{12 / 2 = 6}$

$\boxed{18 / 3 = 6}$

GCD = 6

The Greatest Common Divisor

Example 2: The GCD of 720 and 840

720: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 30, 36, 40, 45, 48, 60, 72, 80, 90, 120, 144, 180, 240, 360, 720

840 : 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840

The Greatest Common Divisor

Example 2: The GCD of 720 and 840

720: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

840 : 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

The Greatest Common Divisor

Example 2: The GCD of 720 and 840

720: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, $\boxed{120}$

840 : 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, $\boxed{120}$

The Greatest Common Divisor

Example 2: The GCD of 720 and 840

720: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, $\boxed{120}$

840 : 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, $\boxed{120}$

GCD = 120

Method #2

The Greatest Common Divisor

2. The Prime Factorization Method

Here, each number is broken down into its prime factors. The GCD is then found by multiplying the smallest power of all common prime factors present in both numbers. This method provides a clear illustration of the common factors but can be time-consuming if the prime factorization is complicated.

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

The Greatest Common Divisor

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12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

Prime Factors: 2, 3

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

Prime Factors: 2, 3

GCD = (2 ^ min(2, 1)) * (3 ^ min(1, 2))

The Greatest Common Divisor

Example 1:	Greatest Common Divisor of 12 and 18.

12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

Prime Factors: 2, 3

GCD = (2 ^ min(2, 1)) * (3 ^ min(1, 2))

GCD = (2 ^ 1) * (3 ^ 1)

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

Prime Factors: 2, 3

GCD = (2 ^ min(2, 1)) * (3 ^ min(1, 2))

GCD = (2 ^ 1) * (3 ^ 1) = 2 * 3

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

12 : (2 ^ 2) * (3 ^ 1)

18 : (2 ^ 1) * (3 ^ 2)

Prime Factors: 2, 3

GCD = (2 ^ min(2, 1)) * (3 ^ min(1, 2))

GCD = (2 ^ 1) * (3 ^ 1) = 2 * 3

GCD = 6

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

Prime Factors: 2, 3, 5, 7

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

Prime Factors: 2, 3, 5, 7

GCD = (2 ^ min(4, 3)) * (3 ^ min(2, 1)) * (5 ^ min(1, 1)) * (7 ^ min(0, 1))

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

Prime Factors: 2, 3, 5, 7

GCD = (2 ^ min(4, 3)) * (3 ^ min(2, 1)) * (5 ^ min(1, 1)) * (7 ^ min(0, 1))

GCD = (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 0)

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

Prime Factors: 2, 3, 5, 7

GCD = (2 ^ min(4, 3)) * (3 ^ min(2, 1)) * (5 ^ min(1, 1)) * (7 ^ min(0, 1))

GCD = (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 0) = 8 * 3 * 5 * 1

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

720: (2 ^ 4) * (3 ^ 2) * (5 ^ 1)

840: (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 1)

Prime Factors: 2, 3, 5, 7

GCD = (2 ^ min(4, 3)) * (3 ^ min(2, 1)) * (5 ^ min(1, 1)) * (7 ^ min(0, 1))

GCD = (2 ^ 3) * (3 ^ 1) * (5 ^ 1) * (7 ^ 0) = 8 * 3 * 5 * 1

GCD = 120

Method #3

The Greatest Common Divisor

3. The Euclidean algorithm

A more efficient method named after the ancient Greek mathematician Euclid. The Euclidean algorithm is based on the principle that the GCD of two numbers also divides their difference. This method involves a series of divisions where the divisor at each step becomes the dividend for the next step, and the remainder becomes the new divisor. This process is repeated until the remainder is zero, at which point the last non-zero divisor is the GCD.

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

Divide the larger (18) by the smaller (12) to obtain:

$$18 = 12 * 1 + 6$$

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

Divide the larger (18) by the smaller (12) to obtain:

$$18 = \boxed{12} * 1 + \boxed{6}$$

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

Divide the larger (18) by the smaller (12) to obtain:

$$18 = \boxed{12} * 1 + \boxed{6}$$

$$12 = 6 * 2 + 0$$

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

Divide the larger (18) by the smaller (12) to obtain:

$$18 = \boxed{12} * 1 + \boxed{6}$$

$$12 = \boxed{6} * 2 + 0$$

The Greatest Common Divisor

Example 1: Greatest Common Divisor of 12 and 18.

Divide the larger (18) by the smaller (12) to obtain:

$$18 = \boxed{12} * 1 + \boxed{6}$$

$$12 = \boxed{6} * 2 + 0$$

GCD = 6

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

Divide the larger (840) by the smaller (720) to obtain:

$840 = 720 * 1 + 120$

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

Divide the larger (840) by the smaller (720) to obtain:

$$840 = \boxed{720} * 1 + \boxed{120}$$

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

Divide the larger (840) by the smaller (720) to obtain:

$$840 = \boxed{720} * 1 + \boxed{120}$$

$$720 = 120 * 6 + 0$$

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

Divide the larger (840) by the smaller (720) to obtain:

$$840 = \boxed{720} * 1 + \boxed{120}$$

$$720 = \boxed{120} * 6 + 0$$

The Greatest Common Divisor

Example 2: Greatest Common Divisor of 720 and 840

Divide the larger (840) by the smaller (720) to obtain:

$$840 = \boxed{720} * 1 + \boxed{120}$$

$$720 = \boxed{120} * 6 + 0$$

GCD = 120

The Least Common Multiple

The Least Common Multiple

What is a multiple?

A multiple is a number that is obtained by multiplying another number by an integer

Thus, if a number A is the multiple of another number B, it means that when A is divided by B, there is no remainder.

The Least Common Multiple

What is the Least Common Multiple?

The Least Common Multiple (LCM) is the lowest multiple shared by two or more numbers.

The Least Common Multiple

The two common methods of finding the LCM include:

1. The Brute Force Method or Listing Method
2. The Prime Factorization Method

Method #1

The Least Common Multiple

1. The Brute Force Method

This method involves listing out all the multiples of each number and then finding the smallest number that appears on both lists. It's a straightforward approach but can be inefficient for large numbers.

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90

15: 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: 30, 60, 90

15: 30, 60, 90

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: $\boxed{30,}$ 60, 90

15: $\boxed{30,}$ 60, 90

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: $\boxed{30,}$ 60, 90

15: $\boxed{30,}$ 60, 90

LCM = 30

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: 320, 640, 960, 1280, ...

480: 480, 960, 1440, ...

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: 960

480: 960

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: $\boxed{960}$

480: $\boxed{960}$

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: $\boxed{960}$

480: $\boxed{960}$

LCM = 960

Method #2

The Least Common Multiple

2. The Prime Factorization Method

Here, each number is broken down into its prime factors. The LCM is then found by multiplying the largest power of the aggregated prime factors present in both numbers. This method provides a clear illustration of the common factors but can be time-consuming if the prime factorization is complicated.

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(1,0)) * (3 ^ max(1,1)) * (5 ^ max(0,1))

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(1,0)) * (3 ^ max(1,1)) * (5 ^ max(0,1))

LCM = (2 ^ 1) * (3 ^ 1) * (5 ^ 1)

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(1,0)) * (3 ^ max(1,1)) * (5 ^ max(0,1))

LCM = (2 ^ 1) * (3 ^ 1) * (5 ^ 1) = 2 * 3 * 5

The Least Common Multiple

Example 1: The LCM of 6 and 15

6: (2 ^ 1) * (3 ^ 1)

15: (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(1,0)) * (3 ^ max(1,1)) * (5 ^ max(0,1))

LCM = (2 ^ 1) * (3 ^ 1) * (5 ^ 1) = 2 * 3 * 5

LCM = 30

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: (2 ^ 6) * (5 ^ 1)

480: (2 ^ 5) * (3 ^ 1) * (5 ^ 1)

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: (2 ^ 6) * (5 ^ 1)

480: (2 ^ 5) * (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: (2 ^ 6) * (5 ^ 1)

480: (2 ^ 5) * (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(6, 5)) * (3 ^ max(0, 1)) * (5 ^ max(1,1))

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: (2 ^ 6) * (5 ^ 1)

480: (2 ^ 5) * (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(6, 5)) * (3 ^ max(0, 1)) * (5 ^ max(1,1))

LCM = (2 ^ 6) * (3 ^ 1) * (5 ^ 1)

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: $(2 ^\wedge 6) * (5 ^\wedge 1)$

480: $(2 ^\wedge 5) * (3 ^\wedge 1) * (5 ^\wedge 1)$

Prime Factors: 2, 3, 5

LCM = $(2 ^\wedge \max(6, 5)) * (3 ^\wedge \max(0, 1)) * (5 ^\wedge \max(1,1))$

LCM = $(2 ^\wedge 6) * (3 ^\wedge 1) * (5 ^\wedge 1) = 64 * 3 * 5$

The Least Common Multiple

Example 2: The LCM of 320 and 480

320: (2 ^ 6) * (5 ^ 1)

480: (2 ^ 5) * (3 ^ 1) * (5 ^ 1)

Prime Factors: 2, 3, 5

LCM = (2 ^ max(6, 5)) * (3 ^ max(0, 1)) * (5 ^ max(1,1))

LCM = (2 ^ 6) * (3 ^ 1) * (5 ^ 1) = 64 * 3 * 5

LCM = 960

Key Takeaways

- The Euclidean algorithm is the most efficient method for finding the GCD of two or more numbers.
- The Prime Factorization is the most efficient method for finding the LCM of two or more numbers.

Thank you!