Udacity Kinematics Project: Pick & Place

Stephan Hohne

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Abstract

This document contains supporting material for my solution of the *Pick & Place* kinematics project, which is part of the Robotics Nanodegree by Udacity. In the first section I summarize basic trigonometry which is used for describing the kinematics of a robotic arm. Then I describe the Denavit-Hartenberg conventions for assigning reference frames to robotic manipulators. The third sections contains design data for the KUKA KR210 robotic arm.

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1 Geometry

1.1 Euler Rotations

The goal is to find the Euler angles α, β and γ for a given three dimensional rotation which is represented by a 3×3 matrix with nine elements,

$$R_z(\alpha)\ R_y(\beta)\ R_x(\gamma) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} . \tag{1}$$

Then the Euler angles are defined as

$$\alpha = \arctan\left(\frac{r_{21}}{r_{11}}\right) , \tag{2}$$

$$\beta = \arctan\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right) , \tag{3}$$

$$\gamma = \arctan\left(\frac{r_{32}}{r_{33}}\right) . \tag{4}$$

We follow the convention where α is the yaw angle around the z-axis, β is the pitch around the y-axis and γ is the roll around the x-axis.

2 Robot Kinematics

2.1 Denavit-Hartenberg Parameters

In 1955, Jacques Denavit and Richard Hartenberg proposed a systematic method of attaching reference frames to the links of a manipulator that simplified the homogeneous transforms. Their method only requires four parameters to describe the position and orientation of neighboring reference frames.

The convention described here is consistent with (Craig, 2005). We denote the Cartesian base for frame i as $\hat{x}_i$, $\hat{y}_i$, $\hat{z}_i$.

We assign a reference frame to each link so that the frame moves rigidly with the link. We choose $\hat{x}_{i-1}$ so that is normal to the plane spanned by $\hat{z}_{i-1}$ and $\hat{z}_i$. The resulting vector can be determined using the cross product as

$$\hat{x}_{i-1}^{\pm} = \pm \frac{\hat{z}_{i-1} \times \hat{z}_i}{\left\|\hat{z}_{i-1} \times \hat{z}_i\right\|} . \tag{5}$$

The transformation between frame $i-1$ and frame i is then characterized by four parameters.

- The twist angle α_{i-1} is the angle between $\hat{z}_{i-1}$ and $\hat{z}_i$ measured in a right hand sense about $\hat{x}_{i-1}$.
- The link length a_{i-1} is the distance from $\hat{z}_{i-1}$ to $\hat{z}_i$ measured along $\hat{x}_{i-1}$. Beware as this is not always equal to the actual length of the link.
- The link offset d_i is the signed distance from $\hat{x}_{i-1}$ to $\hat{x}_i$ measured along $\hat{z}_i$. This quantity is variable for a prismatic joint.
- The joint angle θ_i is the angle between $\hat{x}_{i-1}$ and $\hat{x}_i$ measured in a right hand sense about $\hat{z}_{i-1}$. This quantity is variable for a revolute joint.

The parameter assignment process for open kinematic chains with n degrees of freedom is summarized as, The assignment algorithm is as follows.

1. Assign labels $\{0, \ldots, n\}$ to links and $\{1, \ldots, n\}$ to joints, starting with the fixed base link.
2. Draw lines through all joints, defining the joint axes.
3. Assign the Z axis of each frame to point along its joint axis.
4. Identify the common normal between the Z axes of subsequent frames according to equation 5.
5. The endpoints of intermediate links (i.e. not the base link or the end effector) are associated with the two joint axes j and $j+1$. For frames $j = 1, \ldots n$ choose the base vectors $\hat{x}_j$ as follows.
   - For skew axes, along the normal between $\hat{z}_j$ and $\hat{z}_{j+1}$. Choose the sign so that $\hat{x}_j$ points from frame j to $j+1$.
   - For intersecting axes, normal to the plane containing $\hat{z}_j$ and $\hat{z}_{j+1}$.
   - For parallel or coincident axes, the assignment is arbitrary. Look for ways to make other DH parameters equal to zero.
6. For the base link, always choose frame 0 to be coincident with frame 1 when the first joint variable θ_1 or d_1 is equal to zero. This will guarantee that $\alpha_0 = a_0 = 0$. If joint 1 is revolute then $d_1 = 0$. If joint 1 is prismatic then $\theta_1 = 0$.
7. For the end effector frame, if joint n is revolute, choose $\hat{x}_n$ to be in the direction of $\hat{x}_{n-1}$ for $\theta_n = 0$. Choose the origin of frame n such that $d_n = 0$.

Special cases involving the Z axes are

- collinear lines: $\alpha = a = 0$
- parallel lines: $\alpha = 0$, $a \neq 0$
- intersecting lines: $\alpha \neq 0$, $a = 0$
- If the common normal intersects the Z axis of frame i at the origin, then $d_i = 0$.

To keep track of the values of these parameters while constructing the reference frames, we write them down in a table. Once the frame assignments are made, the DH parameters are typically presented in tabular form. Each row in the table corresponds to the transform from frame i to frame $i+1$.

2.2 Homogeneous Transformations

The concepts of rotations, translations, and homogeneous transforms are essential to understanding the forward kinematics problem of manipulators: given the joint variables, calculate the location of the end effector. The solution procedure involves attaching a reference frame to each link of the manipulator and writing the homogeneous transforms from the fixed base link to the subsequent links, all the way to the end effector.

The homogeneous transform from frame $i-1$ to frame i is composed of a sequence of four basic transformations, two rotations R and two translations D,

$$ {}^{i-1}_{i}T = R\left(X_{i-1}, \alpha_{i-1}\right) D\left(X_{i-1}, a_{i-1}\right) R\left(Z_i, \theta_i\right) D\left(Z_i, d_i\right) . \tag{6} $$

The corresponding homogeneous transform matrix is of the form

$$ {}^{i-1}_{i}T = \begin{pmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{7} $$

where we introduce the shorthand notations $s\theta$ for $\sin(\theta)$ and $c\theta$ for $\cos(\theta)$.

The overall transformation from the base link to the end-effector is the matrix product of the individual transformations. For a robot with $n+1$ links numbered as $0, \ldots, n$ the overall transformation is given by

$$ {}^0_nT = \prod_{i=1}^{n} {}^{i-1}_{i}T = {}^0_1T\; {}^1_2T\; \ldots {}^{n-1}_{n}T . \tag{8} $$

When we know the Denavit-Hartenberg parameters provided by the robot design and the joint variables determined by the current robot pose, we can insert their values into the overall transformation matrix ${}^0_nT$ and read off the pose of the end effector.

2.3 Inverse Kinematics for a Spherical Wrist

This section follows lesson 2-18. Often the last three joints in a manipulator are revolute joints with intersecting joint axes. Such a design is called a spherical wrist and the common point of intersection is called the wrist center. The advantage this design is that it kinematically decouples the position and orientation of the end effector. Mathematically, this means that instead of solving twelve nonlinear equations simultaneously (one equation for each term in the first three rows of the overall homogeneous transform matrix), it is now possible to independently solve two simpler problems: first, the Cartesian coordinates of the wrist center, and then the composition of rotations to orient the end effector. Physically speaking, a six degree of freedom serial manipulator with a spherical wrist would use the first three joints to control the position of the wrist center while the last three joints would orient the end effector as needed.

We will now formalize the solution procedure for serial manipulators with a spherical wrist. Consider a six degree of freedom manipulator with joints four, five and six comprising the spherical wrist. The location of the wrist center and end effector relative to the base frame are given by $\vec{w}_0$ and $\vec{p}_0$, respectively.

The task is to find joint variable q_1 to q_6 for a manipulator pose given by rotation R_{RPY} and end effector coordinates $\vec{p}_0$. The solution procedure can be split on two parts. First we determine the coordinates of the wrist center with the following steps.

1. Complete the Denavit-Hartenberg parameter table for the manipulator. Place the origin of frames 4, 5, and 6 coincident with the wrist center.

2. Find the location of the wrist center relative to the base frame. The overall homogeneous transform between the base frame and end effector frame 6 has the form

$$ {}^{0}_{6}T = \begin{pmatrix} & {}^{0}_{6}R & & \vec{p}_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \vec{l}_0 & \vec{m}_0 & \vec{n}_0 & \vec{p}_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9} $$

   The column vectors $\vec{l}_0$, $\vec{m}_0$ and $\vec{n}_0$ are the orthonormal basis of frame 6 expressed in terms or the base frame. The wrist center can be found by considering the location vectors. Let d be the distance between wrist center and end effector. Then we have

$$ \vec{w}_0 = \vec{p}_0 - d\,\vec{n}_0 = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} - d \begin{pmatrix} r_{13} \\ r_{23} \\ r_{33} \end{pmatrix} \tag{10} $$

3. Find joint variables q_1, q_2 and q_3 that lead to wrist center coordinates given by 10. This is the hard step. One way to attack the problem is by repeatedly projecting links onto planes and using trigonometry to solve for joint angles. There is no generic recipe that works for all manipulators.

Part two consist of orientation of the end-effector. the given target rotation matrix is denoted by R_{RPY}.

1. Once the first three joint variables are known, calculate ${}^{0}_{3}R$ via application of homogeneous transforms up to the wrist center.

2. Find the last three joint variables q_4, q_5 and q_6. We can determine ${}^3_6R$ from ${}^0_3R$ and the given overall transformation as

$$ {}^3_6R = \left({}^0_3R\right)^T \; R_{RPY} \,. \tag{11} $$

The remaining joint variables are the Euler angles corresponding to this rotation matrix.

3. Choose the correct solution among the set of possible solutions.

joint	parent link	child link	rotation axis	frame origin		
				x	y	z
1	base	1	z	0	0	0.33
2	1	2	y	0.35	0	0.42
3	2	3	y	0	0	1.25
4	3	4	x	0.96	0	-0.054
5	4	5	y	0.54	0	0
6	5	6	x	0.193	0	0
gripper	6	gripper	–	0.11	0	0

Table 1: URDF joint properties for KR210. The rotation axis and the origin are expressed in terms of the joint frame. Lengths are given in meters.

3 Data for KUKA KR210

3.1 Denavit-Hartenberg Parameters

In this section we describe the KUKA KR210 manipulator using the data given in the Udacity kinematics project. We read off the joint properties from the robot description file, see table 1. The resulting Denavit-Hartenberg parameter table for the KR210 robotic arm is presented in table 2.

3.2 URDF vs DH frames

Here we discuss the relation between URDF and DH frame assignments. The parameters in table 2 define a DH transformation matrix ${}^{0}_{G}T$. To demonstrate the properties of this transformation we evaluate it at the default configuration where all joint variables are

i	${}^{i-1}_{i}T$	α_{i-1}	a_{i-1}	d_i	θ_i
1	${}^{0}_{1}T$	0	0	0.75	θ_1
2	${}^{1}_{2}T$	$-\pi/2$	0.35	0	$\theta_2 - \pi/2$
3	${}^{2}_{3}T$	0	1.25	0	θ_3
4	${}^{3}_{4}T$	$-\pi/2$	-0.054	1.5	θ_4
5	${}^{4}_{5}T$	$\pi/2$	0	0	θ_5
6	${}^{5}_{6}T$	$-\pi/2$	0	0	θ_6
G	${}^{6}_{G}T$	0	0	0.303	0

Table 2: Denavit-Hartenberg parameters for KR210. Lengths are given in meters and angles in radians.

zero. The result is

$$
{}^{0}_{G}T\,(0,0,0,0,0,0) = \begin{pmatrix} 0 & 0 & 1 & 2.153 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1.946 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}
$$

We see explicitly now the translation of the end-effector in the fourth column and the orientation of frame O_G relative to O_0 in the first three columns.

We have to make up for the difference between the DH frame definition and the joint frames defined in the URDF file. The solution is to rotate 180 degree around the z-axis and -90 degree around the y-axis. This results in the overall transformation matrix for

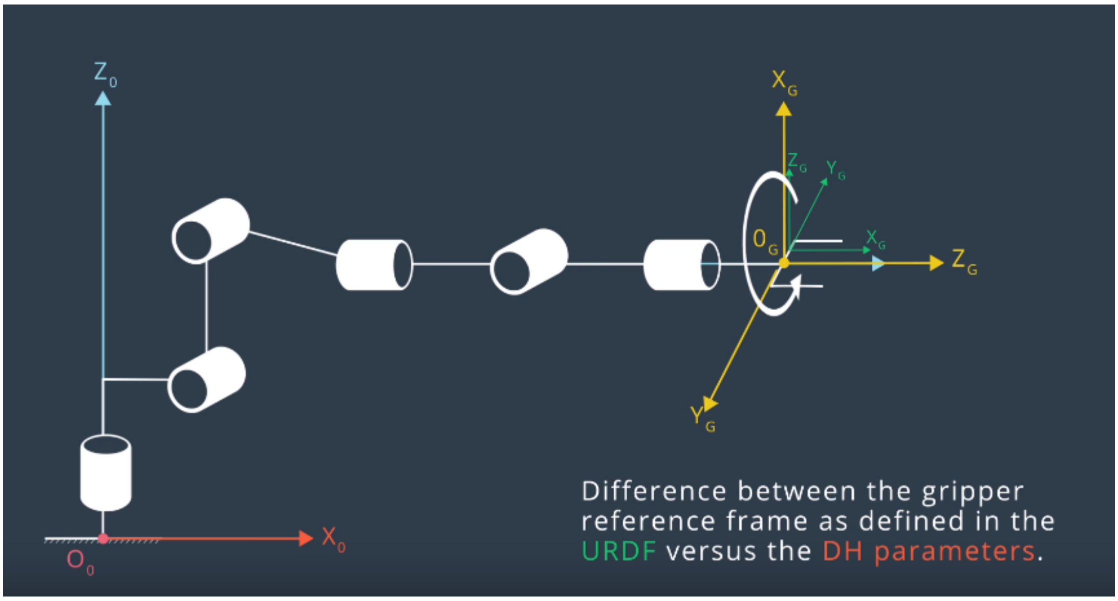

Figure 1: Orientation of gripper frame O_G, URDF frame and base frame O_0. Screenshot from lecture 2-11.

the URDF frame relative to the base frame O_0,

$$ {}^{0}_{G}T_0\,(0,0,0,0,0,0) = \begin{pmatrix} 1 & 0 & 0 & 2.153 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1.946 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13} $$

This matches our expectations, since as we can see in the video in lecture 2-11, the URDF frame has the same orientation as the base frame O_0, and is translated by

$$ \vec{p}_0\,(0,0,0,0,0,0) = \begin{pmatrix} 2.153 \\ 0 \\ 1.946 \end{pmatrix}. \tag{14} $$

See also figure 1.

References

Craig, J. J. (2005). *Introduction to robotics: Mechanics and control.* Pearson Education Inc., NJ.