TOPOLOGY AND ROBOTICS – DRAFT

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Abstract

The concepts of homotopy and homology are reviewed. The application of these topological methods to swarm robotics and path planning is discussed.

1. Introduction

The goal of the Robot-Swarm-Persistent-Homology project is to apply the methods of algebraic topology to a robotic mapping task. This project aims at generating a connectivity map of a simulated Gazebo environment using encounter information gathered by a swarm of robotic agents. It is based on [3], where a robotic swarm is described which generates a qualitative map of its environment using only encounter information.

This writeup is organized as follows. In section 2, the relevant topological notions are reviewed, following the book [1] and the overview article [2]. In section 3, the implementation of the mapping task is described. The results are discussed in section 4.

2. Background: Algebraic Topology

The goal of algebraic topology is to determine and calculate invariant numbers in order to characterize a topological space. The most important examples are the Betti numbers and the Euler number. The Betti numbers describe the connectivity of a space.

The relevant geometric objects and the relations between them are grouped into categories, for instance manifolds or vector spaces. The mappings between categories are called functors. The advantage of the characteristic numbers is that they are invariant under the action of such a functor. Therefore, the information obtained from the more accessible algebraic category can be used to characterize the objects and relations in the more complex geometric category.

2.1. **Homotopy.** Two continuous functions f and g are called homotopic if they can be continuously deformed into each other, for example by linear interpolation $h_t(f,g) = tf + (1-t)g$ with $t \in (0,1)$. Homotopy is an equivalence relation $f \simeq g$ between those functions. The set of equivalence classes of functions between two topological spaces X and Y is denoted as $[X,Y]$. The elements of this set are called homotopy classes.

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2.2. **Fundamental Group.** A space with base point is a pair (X, x_0) consisting of a topological space X and a fixed point $x_0 \in X$. Let $\Omega(X, x_0)$ be the set of loops beginning and ending at x_0. The concatenation of loops is the mapping

$$\begin{aligned} \circ : \Omega \times \Omega &\rightarrow \Omega \\ (\alpha, \beta) &\mapsto \alpha \circ \beta \end{aligned}$$

defined by

$$(\alpha \circ \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{1}$$

The set of homotopy classes of closed loops at x_0 is defined as

$$\pi_1(X, x_0) := \Omega(X, x_0) / \simeq \ . \tag{2}$$

Loop concatenation is used to define a group operation $[\alpha][\beta] := [\alpha \circ \beta]$ on the set $\pi_1(X, x_0)$, which promotes it to a group called the fundamental group of the space X. The rank of π_1 is equal to the number of homotopy equivalence classes. A topological space is called simply connected if the first fundamental group has rank zero and therefore contains only the identity element. Higher homotopy groups are defined as

$$\pi_n(X, x_0) := [(S^n, N), (X, x_0)] \ . \tag{3}$$

The base point N is a fixed point in S^n, for example the north pole.

2.3. **Contractions.** A topological space is contractible, if it is homotopy equivalent to a point. For instance, the space $(\mathbb{R}^n, x_0)$ can be contracted to the base point x_0 using the linear interpolation formula

$$h_t(x) = (1-t)x + tx_0 \,. \tag{4}$$

A *retraction* is a continuous map $\rho : X \rightarrow A$ with $A \subset X$ and

$$\rho|_A = \mathbf{1}_A \,. \tag{5}$$

For instance, the function 4 is a strong deformation retraction, since

$$h_t(x_0) = x_0 \forall t \in [0, 1] \ . \tag{6}$$

The key topological property of deformation retractions is that they preserve the rank of the first fundamental group,

$$\pi_1(X, x_0) = \pi_1(A, \rho(x_0)) \ . \tag{7}$$

2.4. **Simplices.** Let $\{v_0, \ldots, v_k\}$ be a set of $k+1$ points. A k-simplex in $\mathbb{R}^n$ is the convex hull

$$s(v_0, \ldots, v_k) = \left\{ \sum_{i=0}^{k} \lambda_i v_i \,\middle|\, \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\} . \tag{8}$$

For example, a 1-simplex is a line segment composed of the end points v_0, v_1 and the straight line connecting them. the end points are called vertices when seen as a subset of the simplex. Higher dimensional subsets are edges, faces and so on. A set K of simplices in $\mathbb{R}^n$ is called simplicial complex or polyhedron, if k contains all the subsets for each of its simplices, and the intersection of two simplices is either empty or a common subset. Associated to a simplicial complex is a topological space made of the union of all elements

$$|K| \coloneqq \cup_{\sigma \in K} \sigma \,, \tag{9}$$

where $|K| \subset \mathbb{R}^n$.

2.5. **Homology.** The simplicial homology $H_* = (H_0, H_1, \ldots)$ is constructed as a covariant functor from the simplicial category (polyhedrons and affine transformations of simplices) to the category of graded abelian groups. This construction allows to determine invariant numbers. The homology functor H_* is homotopy invariant. For a topological space X, examples for invariant numbers are the Betti numbers

$$\beta_i (X, \mathbb{F}_2) = \mathrm{rg} H_i (X, \mathbb{F}_2) \tag{10}$$

and the Euler number

$$\chi (X) = \sum_{i=0}^{\infty} (-1)^i \, \mathrm{rg} H_i (X) \ , \tag{11}$$

where $\mathrm{rg} H_i$ is the rank of the homology group.

The mathematical tool of persistent homology allows to infer topological information about an objects based on samples taken from that object.

3. Methods

TODO

A simulated robot swarm in a simulated Gazebo environment collects encounter information. The javaplex package is used to generate the Betti diagrams from the encounter events.

An encounter event between robot I_1 and I_2 at time t_i is described by

$$v_i = [t_i, \alpha_i, \beta_i] \ , \tag{12}$$

where the numbers α_i are taken from the index set for robots. The encounter graph is build from the encounter events as vertices.

4. Results and Discussion

TODO

References

[1] Klaus Janich, *Topologie*, Springer, Berlin, Heidelberg, New York, 6th edition, 1999.
[2] Gunnar Carlsson, *Topology and data*, Bull. Amer. Math. Soc. (N.S.) 46 (2009), no. 2, 255-308.
[3] Alireza Dirafzoon, Edgar Lobaton, *Topological mapping of unknown environments using an unlocalized robotic swarm*, Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems, 5545-5551, 2013, https://ieeexplore.ieee.org/document/6697160