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A review on quantum search algorithms

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Abstract The use of superposition of states in quantum computation, known as quantum parallelism, has significant advantage in terms of speed over the classical computation. It is evident from the early invented quantum algorithms such as Deutsch's algorithm, Deutsch–Jozsa algorithm and its variation as Bernstein–Vazirani algorithm, Simon algorithm, Shor's algorithms, etc. Quantum parallelism also significantly speeds up the database search algorithm, which is important in computer science because it comes as a subroutine in many important algorithms. Quantum database search of Grover achieves the task of finding the target element in an unsorted database in a time quadratically faster than the classical computer. We review Grover's quantum search algorithms for a singe and multiple target elements in a database. The partial search algorithm of Grover and Radhakrishnan and its optimization by Korepin called GRK algorithm are also discussed.

Keywords Quantum algorithms · Grover search algorithm · Partial database search

Contents

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1 Introduction

Quantum computation has the advantage of speed [1–3] over its classical counterpart which makes the quantum computation more favorable. Although building a full-fledged quantum computer [4] is still far from reality, some of the research works such as Shor's algorithm and Grover algorithm have attracted much attention in the theoretical side. In the experimental side, some success with a small number of quantum bits have already been achieved.

Peter Shor showed [5,6] that it is possible for a quantum algorithm to compute factorization in polynomial time. Grover, on the other hand, showed [7–9] that it is possible to search for a single target item in an unsorted database, i.e., the elements of the database are not arranged in any specific order, in a time which is quadratically faster than what a classical computer needs to complete the same task. Here time is measured in terms of the number of queries to the *oracle* one needs to complete a task. Grover algorithm needs $\mathcal{O}(\sqrt{N})$ queries to the *oracle*. Although Grover algorithm cannot perform a task exponentially faster than classical computer, still it is quite popular because of its wide range of applications such as a subroutine of some large algorithms in computer science. It can be shown that the quantum algorithm of Grover is the fastest algorithm, i.e., optimal [10–12] to search in an unsorted database.

Instead of looking for the target element in the whole database at once, it is sometimes natural to divide the database into several blocks and then look for the particular block which contains the target element. This is called quantum partial search algorithm, first studied by Grover and Radhakrishnan [13], which can be optimized [14–17] and further generalized to hierarchical quantum partial search algorithm [2,18,19].

The purpose of this article is to review the basic concepts of quantum search algorithms. In our daily life, we encounter databases which contain many elements. The database may be arranged in a particular order, i.e., sorted or may not have any order at all, i.e., unsorted. For example, consider the telephone directory which has a large number of contact details of individuals. This example is particularly interesting because it serves both as a sorted and an unsorted database. When we look for the names, which

are arranged in lexicographical order, then the telephone directory is an example of sorted database. However, when we look for a telephone number, then the telephone directory becomes an example of an unsorted database. The job of a quantum search algorithm is to find a specific element, usually called the target item or the solution from the vast number of elements in a database. Typically, classical computer takes a time proportional to the size of the database. Quantum search algorithms, which are based on the principle of quantum mechanics, promise to significantly reduce the computation time for the same database search.

This review article is arranged in the following fashion. In Sect. 2 we give an elaborate account of the famous Grover search algorithm. In Sect. 3 we discuss the quantum partial search algorithm and its optimized version known as GRK algorithm [2]. Finally, in Sect. 4 we conclude.

2 Full database search

Let us consider a set $\mathcal{D} = \{a_0, a_1, \ldots, a_{N-1}\}$ containing N number of elements. Assume that one of the N elements is a marked one, which we have to find out. One of the legitimate questions in computing is how fast one can find out the marked element or the solution. If the elements in the set are completely unsorted, then the classical computer can find the marked element in $\mathcal{O}(N)$ queries/time. Grover investigated the same problem quantum mechanically and found that it is possible to devise a quantum algorithm, now known as the Grover algorithm, which can find the marked element in $\mathcal{O}(\sqrt{N})$ queries [7]. This is a quadratic speed up in time over the classical algorithm. Bellow we discuss the famous Grover algorithm which has been extensively investigated in the literature.

2.1 Grover algorithm

A database which we encounter in practice may have a single target item/element or sometimes it may have multiple target elements. Grover search can efficiently search both types of database; however, the database with multiple target elements are faster to search than with single target element as can be understood from the following two sub-subsections.

2.1.1 Single target Grover algorithm

We associate the N elements of the set $\mathcal{D}$ with the basis vectors of a N-dimensional Hilbert space $\mathcal{H}$ spanned by orthonormal basis vectors $\{|a_i\rangle | \langle a_i | a_j \rangle = \delta_{ij}, i = 0, 1, \ldots, N-1\}$. Now consider an initial unit vector $|\Theta\rangle$, which can be written in terms of the basis vectors as

$$|\Theta\rangle = \sum_{i=0}^{N-1} \cos\alpha_i |a_i\rangle, \tag{1}$$

where the direction cosines $\cos(\alpha_i)s$ satisfy $\sum_{i=0}^{N-1} \cos^2 \alpha_i = 1$. To start with an equal probability for all the elements we assume the direction cosines to be same in all directions, i.e., $\alpha_i = \pi/2 - \theta$, which simplifies the initial unit vector (1) as

$$|\Theta\rangle = \sum_{i=0}^{N-1} \sin\theta |a_i\rangle = \sum_{i=0}^{N-1} \sqrt{\frac{1}{N}} |a_i\rangle. \tag{2}$$

One of the basis vectors, let $|a_T\rangle$, which has a probability

$$\mathcal{P}_T = |\langle a_T|\Theta\rangle|^2 = \sin^2\theta = \frac{1}{N}, \tag{3}$$

of obtaining it if measured in the state $|\Theta\rangle$, is assigned to the target element. In order to increase the probability of getting the marked state $|a_T\rangle$ Grover exploited an unitary transformation $\mathcal{G}$, which we call Grover iteration:

$$\mathcal{G} = -\mathcal{I}_\Theta \mathcal{I}_T, \tag{4}$$

where the two reflection operators $\mathcal{I}_T$ and $\mathcal{I}_\Theta$ are given as

$$\mathcal{I}_T = \mathbb{I} - 2|a_T\rangle\langle a_T|, \tag{5}$$
$$\mathcal{I}_\Theta = \mathbb{I} - 2|\Theta\rangle\langle\Theta|. \tag{6}$$

To understand the action of both the reflection operators let us consider a general vector

$$|\psi\rangle = \sum_{i=0}^{N-1} c_i |a_i\rangle, \tag{7}$$

where c_is are the constant coefficients. $\mathcal{I}_T$ only reflects the $|a_T\rangle$ component and keeps the other components unchanged as can be seen from the expression

$$\mathcal{I}_T|\psi\rangle = -c_T|a_T\rangle + \sum_{i=0, i\neq T}^{N-1} c_i |a_i\rangle. \tag{8}$$

For the particular case of the state associated with the marked element $|a_T\rangle$ it simply becomes $\mathcal{I}_T|a_T\rangle = -|a_T\rangle$. On the other hand $-\mathcal{I}_\Theta$ inverts the coefficients c_i of the vector $|\psi\rangle$ about the double average of their coefficients as

$$-\mathcal{I}_\Theta|\psi\rangle = \sum_{i=0}^{N-1} (2\bar{c} - c_i)\, |a_i\rangle, \tag{9}$$

where $\bar{c}$ is the average of all the coefficients given by $\bar{c} = \frac{1}{N}\sum_{i=0}^{N-1} c_i$. One Grover iteration $\mathcal{G}$ acts on a general vector $|\psi\rangle$ as

$$\mathcal{G}|\psi\rangle = -\mathcal{I}_\Theta \mathcal{I}_T |\psi\rangle = (2\tilde{c} + c_T)\,|a_T\rangle + \sum_{i=0, i\neq T}^{N-1} (2\tilde{c} - c_i)\,|a_i\rangle, \tag{10}$$

where now the average being $\tilde{c} = \frac{1}{N}\left(-c_T + \sum_{i=0, i\neq T}^{N-1} c_i\right)$. For our purpose it is helpful to consider the action of the Grover iteration $\mathcal{G}$ on the initial state $|\Theta\rangle$ in Eq. (2), which simply gives

$$\mathcal{G}|\Theta\rangle = \sin(2+1)\,\theta |a_T\rangle + \sum_{i=0, i\neq T}^{N-1} \cos(2+1)\,\theta \tan\theta |a_i\rangle. \tag{11}$$

Applying the same Grover iteration j times on the initial state we obtain

$$\mathcal{G}^j|\Theta\rangle = \sin(2j+1)\,\theta |a_T\rangle + \sum_{i=0, i\neq T}^{N-1} \cos(2j+1)\,\theta \tan\theta |a_i\rangle. \tag{12}$$

Assuming that now the initial state is aligned with the target vector, i.e., $\mathcal{G}^j|\Theta\rangle = |a_T\rangle$ after j successive applications of the Grover iteration we obtain the optimal number of quantum query to the *oracle* necessary for large database

$$j = \lim_{N\to\infty}\left(\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right) = \frac{\pi}{4}\sqrt{N}. \tag{13}$$

This is clearly a quadratic speed up over the classical algorithm to search for a marked element on a set of N unsorted elements. Of course j estimated under the above assumption may make it a non-integer in general. In that case, we have to take the integer closest to the number $\frac{\pi}{4}\sqrt{N}$.

To easily understand the action of $\mathcal{G}^j$ on the initial state vector $|\Theta\rangle$ let us consider the eigenvalue problem [2]

$$\mathcal{G}^j|\phi\rangle = E^j|\phi\rangle. \tag{14}$$

On the plane defined by the vectors $|a_T\rangle$ and $|\Theta\rangle$ Eq. (14) has the following two eigenvectors

$$|\phi\rangle_\pm = \frac{1}{\sqrt{2}}|a_T\rangle \pm \frac{i}{\sqrt{2}}\sum_{i=0, i\neq T}^{N-1} \tan\theta |a_i\rangle, \tag{15}$$

with their corresponding eigenvalues $E_\pm^j = e^{\pm i 2\theta j}$. In terms of these eigenvectors the initial state vector can be expressed as

$$|\Theta\rangle = -\frac{1}{\sqrt{2}} i \left(e^{i\theta} |\phi\rangle_+ - e^{-i\theta} |\phi\rangle_- \right). \tag{16}$$

Acting $\mathcal{G}^j$ on the expression of Eq. (16) we immediately obtain

$$\mathcal{G}^j |\Theta\rangle = -\frac{1}{\sqrt{2}} i \left(e^{i(2j+1)\theta} |\phi\rangle_+ - e^{-i(2j+1)\theta} |\phi\rangle_- \right), \tag{17}$$

which once written in terms of the original basis $|a_i\rangle$ reduces to the expression of Eq. (12).

a. Example with single target Let us consider an example, where there are $N = 4$ elements and one of the element is marked. We need to find out the marked element among the four elements. Naively we may think that classically we can find the marked element in one search, two searches, three searches or in the worst case in four searches. On average we need $\frac{1+2+3+4}{4} = 2\frac{1}{2}$ searches to find the target element. However, since we know there is a marked element it is not necessary to perform a fourth search. Therefore, on average we only need to perform $\frac{1+2+3+3}{4} = 2\frac{1}{4}$ number of classical searches to find the target element. However, quantum mechanically, using Grover algorithm, we can find the marked element in just a single query. In this case $\sin\theta = \sqrt{\frac{1}{N}} = \frac{1}{2}$. So, the angle between the initial state and the state perpendicular to the target state is $\theta = 30°$. One query to the black box will further rotate the initial state $2\theta = 60°$ toward the target element. Now the total angle between the initial state and the state perpendicular to the target state is $2\theta + \theta = 90°$, which means the initial state is now completely aligned with the target state.

We can also exploit Eqs. (8) and (9) to understand the above example in an alternative manner. Note that $\mathcal{I}_T$ just inverts the sign of the amplitude of the target element and $\mathcal{I}_\Theta$ inverts the amplitudes of the basis vectors about the double average. For the database of $N = 4$ elements each basis element in the initial state $|\Theta\rangle$ has an amplitude $c_i = \frac{1}{\sqrt{N}} = \frac{1}{2}$. After the action of $\mathcal{I}_T$ the amplitude of only the target element changes from $c_T = \frac{1}{2}$ to $-c_T = -\frac{1}{2}$. The average of the four amplitudes then reduces from $\bar{c} = \frac{1}{2}$ to $\tilde{c} = \frac{1}{4}$. Then, $\mathcal{I}_\Theta$ inverts the amplitude about the double average, which can be seen from state in Eq. (10). The amplitude of the target element after one Grover iteration is thus amplified to $2\tilde{c} + c_T = 1$, and the amplitudes of all the other basis elements vanish $2\tilde{c} - c_i = 0$.

2.1.2 Multiple targets Grover algorithm

In the above analysis there is just a single marked element in the set. We now consider the case when there are M number of marked elements in the set $\mathcal{D}$ of N number of elements. We discuss this algorithm with the help of a generalized method known as the amplitude amplification, which was studied by Brassard et al. [4]. Let us first divide the Hilbert space $\mathcal{H}$ into two mutually orthogonal subspaces $\mathcal{H}_T$ and $\mathcal{H}_{nT}$. $\mathcal{H}_T$ is the target space of dimensions M, where the basis elements are associated with M target elements, and $\mathcal{H}_{nT}$ is the Hilbert space of non-target elements of dimensions

$N-M$, where the basis vectors are associated with all the $N-M$ non-target elements. An unit vector in the target space can be written in terms of the basis elements of the target space as

$$|A_T\rangle = \sum_{i=1}^{M} \tilde{a}_i |a_i\rangle, \qquad \sum_{i=1}^{M} |\tilde{a}_i|^2 = 1, \tag{18}$$

where we have rearranged the basis vectors such that first M basis vectors correspond to the target space and rest belongs to the non-target space. Similarly, an unit vector in the non-target space can be written as

$$|A_{nT}\rangle = \sum_{i=M+1}^{N} \bar{a}_i |a_i\rangle, \qquad \sum_{i=M+1}^{N} |\bar{a}_i|^2 = 1. \tag{19}$$

We again start with the same initial vector (2) but in terms of the unit basis vectors (18) with $\tilde{a}_i = \sqrt{\frac{1}{M}}$ and (19) with $\bar{a}_i = \sqrt{\frac{1}{N-M}}$

$$|\tilde{\Theta}\rangle = \sqrt{\frac{M}{N}} |A_T\rangle + \sqrt{\frac{N-M}{N}} |A_{nT}\rangle. \tag{20}$$

The initial probability of obtaining state (18) in the initial state (20) is given by

$$\tilde{\mathcal{P}}_T = |\langle A_T|\tilde{\Theta}\rangle|^2 = \sin^2\tilde{\theta} = \frac{M}{N}. \tag{21}$$

Here we remark that we chose specific coefficients in the basis vectors (18) and (19) so that the initial state becomes a state with same direction cosines in all directions. However, we could have kept the coefficients arbitrary.

Similar to the single target case we now need to construct the Grover iteration $\tilde{\mathcal{G}}$, which is defined as

$$\tilde{\mathcal{G}} = -\mathcal{I}_{\tilde{\Theta}} \mathcal{I}_{A_T}, \tag{22}$$

where the two reflection operators $\mathcal{I}_{A_T}$ and $\mathcal{I}_{\tilde{\Theta}}$ are given as

$$\mathcal{I}_{A_T} = \mathbb{I} - 2|A_T\rangle\langle A_T|, \tag{23}$$

$$\mathcal{I}_{\tilde{\Theta}} = \mathbb{I} - 2|\tilde{\Theta}\rangle\langle\tilde{\Theta}|. \tag{24}$$

In terms of the two eigenvectors

$$|\tilde{\phi}\rangle_{\pm} = \frac{1}{\sqrt{2}} |A_T\rangle \pm \frac{i}{\sqrt{2}} |A_{nT}\rangle, \tag{25}$$

of the operator $\tilde{\mathcal{G}}^j$ with their corresponding eigenvalues $\tilde{E}_{\pm}^j = e^{\pm i 2\tilde{\theta} j}$ the initial state vector can be expressed as

$$|\tilde{\Theta}\rangle = -\frac{1}{\sqrt{2}} i \left(e^{i\tilde{\theta}} |\tilde{\phi}\rangle_+ - e^{-i\tilde{\theta}} |\tilde{\phi}\rangle_- \right). \tag{26}$$

Acting $\tilde{\mathcal{G}}^j$ on the expression of Eq. (26) we obtain

$$\tilde{\mathcal{G}}^j |\Theta\rangle = -\frac{1}{\sqrt{2}} i \left(e^{i(2j+1)\tilde{\theta}} |\tilde{\phi}\rangle_+ - e^{-i(2j+1)\tilde{\theta}} |\tilde{\phi}\rangle_- \right), \tag{27}$$

which can be rewritten in terms of the basis vectors $|A_T\rangle$ and $|A_{nT}\rangle$ as

$$\tilde{\mathcal{G}}^j |\Theta\rangle = \sin(2j+1)\,\tilde{\theta} |A_T\rangle + \cos(2j+1)\,\tilde{\theta} |A_{nT}\rangle. \tag{28}$$

For a large database of N elements with M target items the optimal number of quantum queries necessary to find a target state thus becomes

$$j = \lim_{N\to\infty} \left(\frac{\pi}{4} \sqrt{\frac{N}{M}} - \frac{1}{2} \right) = \frac{\pi}{4} \sqrt{\frac{N}{M}}. \tag{29}$$

Example with multiple targets Let us consider an example which is similar to the example of four elements in a database discussed in 2.1.1; however, this time there are multiple target elements instead of just one. For our purpose only the ratio of the number of elements N in the database with the number of target elements M matters. We consider the ratio to be $\frac{N}{M} = 4$. The angle between the orthogonal to unit vector $|A_T\rangle$ in the target state and the initial state $|\Theta\rangle$ can be obtained from Eq. (21) as $\tilde{\theta} = 30^\circ$. One Grover iteration rotates the initial state $|\Theta\rangle$ toward the target state $|A_T\rangle$ by an amount $2\tilde{\theta} = 60^\circ$. After one Grover search the angle between the orthogonal to the target state and the initial state is $2\tilde{\theta} + \tilde{\theta} = 90^\circ$, which means the initial state is now completely aligned with the unit target state.

2.1.3 Generic unitary transformation for Grover search

In the discussion of Grover search algorithm in Sect. 2.1.1 we have implicitly exploited the Walsh–Hadamard (WH) transformation $H^{(n)}$ as an unitary transformation. Note that the initial state in Eq. (2), which is an equal weighted superposition of all basis states, can be obtained from the state $|0\rangle^n$ by the application of WH transformation

$$|\Theta_{H^{(n)}}\rangle = |\Theta\rangle = H^{(n)} |0\rangle^n = \sqrt{\frac{1}{N}} \sum_{i=0}^{N-1} |a_i\rangle. \tag{30}$$

Then, the reflection operator $\mathcal{I}_\Theta$ in Eq. (5) can be obtained as

$$\mathcal{I}_\Theta = H^n \left(\mathbb{I} - 2|0\rangle^{nn}\langle 0| \right) (H^n)^{-1} = \mathbb{I} - 2|\Theta\rangle\langle\Theta|. \tag{31}$$

Instead of using $H^{(n)}$ we can also choose any generic unitary operator U [20] which can act on the Hilbert space $\mathcal{H}$ of N basis states describing $N = 2^n$ elements of the Grover search. The initial state we now consider for our purpose is given by

$$|\Theta_U\rangle = U|0\rangle^n. \tag{32}$$

Then, the reflection operator corresponding to the state in Eq. (32) can be written as

$$\mathcal{I}_{\Theta_U} = U\left(\mathbb{I} - 2|0\rangle^{nn}\langle 0|\right)U^{-1} = \mathbb{I} - 2|\Theta_U\rangle\langle\Theta_U|. \tag{33}$$

As usual $|a_T\rangle$ is the target element which we have to find out from the N elements and $\mathcal{I}_T$ is the corresponding reflection operator. The amplitude of the target element $|a_T\rangle$ in the initial state $|\Theta_U\rangle$ is

$$\mathcal{A}_{T\Theta_U} = \sin\theta_U = \langle a_T|\Theta_U\rangle = \langle a_T|U|0\rangle^n. \tag{34}$$

When the probability of getting the target element in the initial state is low, then Eq. (34) can be approximated as

$$\mathcal{A}_{T\Theta_U} = \lim_{\theta_U\to 0}\sin\theta_U = \theta_U. \tag{35}$$

We can now construct the Grover iteration as

$$G_U = -\mathcal{I}_{\Theta_U}\mathcal{I}_T. \tag{36}$$

One Grover iteration moves the initial state by an angle $2\theta_U$ toward the target element. Assuming that after j_U number of iterations the initial state will align with the target element, then we obtain

$$j_U = \lim_{\mathcal{A}_{T\Theta_U}\to 0}\left(\frac{\pi}{4}\frac{1}{\mathcal{A}_{T\Theta_U}} - \frac{1}{2}\right) = \frac{\pi}{4}\frac{1}{\mathcal{A}_{T\Theta_U}}. \tag{37}$$

When the unitary operator $U = H^{(n)}$ and the amplitude of the target element in the initial state becomes $\mathcal{A}_{T\Theta_U} = \sqrt{\frac{1}{N}}$, then Eq. (37) reduces to the standard result in Eq. (13).

Here we remark that when there is no apparent knowledge of the whereabouts of the target element in a database, then the WH transformation is the most suitable unitary transformation because it produces an initial state which is an equal superposition of all the basis states. For many target elements the average amplitude of the target elements in the initial state is largest and the amplitude of the target elements are known.

However, there can be some problems where we may have more knowledge about the target element/elements or there are some order/structure in the database. The generic unitary transformation then becomes important, because one can choose the unitary operator U accordingly so as to get faster search. The Grover search is then a search of a structured database as opposed to the unstructured search discussed in Sects. 2.1.1 and 2.1.2.

a. Example of a structured Grover search Here we consider an example of a structured Grover search which is discussed in Refs. [21,22]. Let us consider a function $F(a_i, b_i)$ which takes two n-bits (a_i, b_i), $i = 1, 2, \ldots, N$ as inputs and the output is zero for all (a_i, b_i)s except at (a_T, b_T), where $F(a_T, b_T) = 1$. This is an example of a database of N^2 elements with one of the elements (a_T, b_T) being the target element. Classical computer needs $\mathcal{O}(N^2)$ time in the worst case to find the target element. However, Grover algorithm needs $\mathcal{O}(N)$ *oracle* calls to find out the target element with close to one probability.

The number of *oracle* calls can further be reduced if we know there is some structure which can help to minimize the time of search. Let us assume that there is another function $G(a_i)$ which takes one n-bits a_i, $i = 1, 2, \ldots, N$ as input and the output is zero for all a_is except for $M \leq N$ a_is, where $G(a_i) = 1$ and a_T also belongs to those M a_i, i.e., $G(a_T) = 1$.

The case $M = N$ is not interesting because $G(a_i) = 1$ for all the inputs and therefore does not reduce the search time for the target element (a_T, b_T). For the case $M = 1$ we may first use $G(a_i)$ to find a_T in $\frac{\pi}{4}\sqrt{N}$ number of Grover iterations. Then, we can use $F(a_T, b_i)$ to find a_T, b_T in $\frac{\pi}{4}\sqrt{N}$ number of Grover iterations; in total $\frac{\pi}{2}\sqrt{N}$ iterations are needed.

Let us now consider the case $1 < M < N$, and assume that M is known. The result is also valid for $M = 1$ and $M = N$ cases. Now the classical computer can find the target element in $\mathcal{O}(MN)$ repetitions. The quantum algorithm can find the target element in $\mathcal{O}(\sqrt{MN})$ *oracle* calls which is a quadratic speed up in time.

The function $F(a_i, b_i)$ acts on a tensor product space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of dimensions N^2, and basis elements are $|a_i\rangle \otimes |b_i\rangle$, where $|a_i\rangle$ are the basis elements of $\mathcal{H}_1$ and $|b_i\rangle$ are the basis elements of $\mathcal{H}_2$. Both of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ have dimensions N. The initial state we consider is given by

$$|\Theta_{12}\rangle = |\Theta_1\rangle \otimes |\Theta_2\rangle, \tag{38}$$

where the initial state on both the Hilbert spaces is given by

$$|\Theta_1\rangle = \left(\sqrt{\frac{1}{N}} \sum_{i=1}^{N} |a_i\rangle\right) \otimes \mathbb{I}, \tag{39}$$

$$|\Theta_2\rangle = \mathbb{I} \otimes \left(\sqrt{\frac{1}{N}} \sum_{i=1}^{N} |b_i\rangle\right). \tag{40}$$

With all the basis states corresponding to $G(a_i) = 1$ we prepare another state by equal superposition

$$|\Theta_0\rangle = \left(\sqrt{\frac{1}{M}} \sum_{G(a_i)=1} |a_i\rangle\right) \otimes \mathbb{I}. \tag{41}$$

We can now construct the reflection operators corresponding to $|\Theta_1\rangle$, $|\Theta_2\rangle$ and $|\Theta\rangle$ as

$$\mathcal{I}_{\Theta_1} = (\mathbb{I} - 2|\Theta_1\rangle\langle\Theta_1|) \otimes \mathbb{I}, \tag{42}$$
$$\mathcal{I}_{\Theta_2} = \mathbb{I} \otimes (\mathbb{I} - 2|\Theta_2\rangle\langle\Theta_2|), \tag{43}$$
$$\mathcal{I}_{\Theta_0} = (\mathbb{I} - 2|\Theta_0\rangle\langle\Theta_0|) \otimes \mathbb{I}. \tag{44}$$

The other two reflection operators we need are

$$\mathcal{I}_{T_1} = \left(\mathbb{I} - 2 \sum_{G(a_i)=1} |a_i\rangle\langle a_i|\right) \otimes \mathbb{I}, \tag{45}$$
$$\mathcal{I}_{T_{12}} = \mathbb{I} \otimes \mathbb{I} - 2|a_T\rangle\langle a_T| \otimes |b_T\rangle\langle b_T|. \tag{46}$$

Firstly, the Grover iteration

$$\mathcal{G}_1 = -\mathcal{I}_{\Theta_1}\mathcal{I}_{T_1}, \tag{47}$$

is performed $j_1 = \frac{\pi}{4}\sqrt{\frac{N}{M}}$ times on the initial state $|\Theta_{12}\rangle$, which only transforms the initial state vector $|\Theta_1\rangle$ to the state $|\Theta_0\rangle$

$$\mathcal{G}_1{}^{j_1}|\Theta_{12}\rangle = \mathcal{G}_1{}^{j_1}|\Theta_1\rangle \otimes |\Theta_2\rangle \cong |\Theta_0\rangle \otimes |\Theta_2\rangle. \tag{48}$$

We now define a reflection operator $\mathcal{I}_{T_0}$ as

$$\mathcal{I}_{T_0} = \mathcal{G}_{12}^{j_{12}{}^\dagger}\mathcal{I}_{T_{12}}\mathcal{G}_{12}^{j_{12}}, \tag{49}$$

where

$$\mathcal{G}_{12} = -\mathcal{I}_{\Theta_2}\mathcal{I}_{T_{12}}. \tag{50}$$

Note that after $j_{12} = \frac{\pi}{4}\sqrt{N}$ iterations by $\mathcal{G}_{12}$ we can obtain the target state in the following way

$$\mathcal{G}_{12}{}^{j_{12}}|a_i\rangle \otimes |\Theta_2\rangle = |a_i\rangle \otimes |\Theta_2\rangle, \quad \text{for } a_i \neq a_K, \tag{51}$$
$$= |a_T\rangle \otimes |b_T\rangle, \quad \text{for } a_i = a_K. \tag{52}$$

The reflection operator $\mathcal{I}_{T_0}$ defined in Eq. (49) will act on the M dimensional Hilbert space with basis elements a_i for which $G(a_i) = 1$. It reflects the target element a_T about a plane perpendicular to $|a_T\rangle$. In particular its action is given by

$$\mathcal{I}_{T_0}|a_i\rangle \otimes |\Theta_2\rangle = |a_i\rangle \otimes |\Theta_2\rangle, \quad \text{for } a_i \neq a_K, \tag{53}$$
$$= -|a_T\rangle \otimes |\Theta_2\rangle, \quad \text{for } a_i = a_K. \tag{54}$$

We can now define a Grover iteration

$$\mathcal{G}_0 = -\mathcal{I}_{\Theta_0}\mathcal{I}_{T_0}, \tag{55}$$

which will find a target element $|a_T\rangle$ from the database of M elements for which $G(a_i) = 1$. Applying $\mathcal{G}_0$ on the state of Eq. (48) $j = \frac{\pi}{4}\sqrt{M}$ times we obtain

$$\mathcal{G}_0^j \mathcal{G}_1{}^{j_1} |\Theta_{12}\rangle \cong \mathcal{G}_0^j |\Theta_0\rangle \otimes |\Theta_2\rangle \cong |a_T\rangle \otimes |\Theta_2\rangle. \tag{56}$$

Finally, iterating the state in Eq. (56) j_{12} times by $\mathcal{G}_{12}$ we obtain

$$\mathcal{G}_{12}^{j_{12}} \mathcal{G}_0^j \mathcal{G}_1{}^{j_1} |\Theta_{12}\rangle \cong \mathcal{G}_{12}^{j_{12}} |a_T\rangle \otimes |\Theta_2\rangle \cong |a_T\rangle \otimes |b_T\rangle. \tag{57}$$

From the expansion

$$\mathcal{G}_{12}^{j_{12}} \mathcal{G}_0^j \mathcal{G}_1{}^{j_1} = \mathcal{G}_{12}^{j_{12}} \left(-\mathcal{I}_{\Theta_0} \mathcal{G}_{12}^{j_{12}}{}^{\dagger} \mathcal{I}_{T_{12}} \mathcal{G}_{12}^{j_{12}} \right)^j \mathcal{G}_1{}^{j_1}, \tag{58}$$

we obtain the total *oracle* queries j_T in large database N and large M limit

$$j_T = \lim_{N,M\to\infty} (j_{12} + 2 j_{12} j + j_1) = \frac{\pi^2}{8}\sqrt{NM}. \tag{59}$$

This is quadratically faster than the classical time of $\mathcal{O}(NM)$ and even faster than the quantum unstructured Grover search for $M < N$ which takes time of $\mathcal{O}(N)$.

2.1.4 Proof of optimization of Grover algorithm

Grover search is the fastest [23] algorithm for the problem of finding the target element from an unstructured database. No other algorithm can search for the target element shorter than $\mathcal{O}(\sqrt{N})$ *oracle* queries.

Consider an initial state $|\psi_0\rangle$ which evolves to a state $|\psi_J^{a_i}\rangle = U_{a_i}|\psi_0\rangle$ after J *oracle* queries. We assume that after J number of queries the evolved state is very close to the target state $|a_i\rangle$

$$\langle \psi_J^{a_i} | a_i \rangle \approx 1, \quad \text{for } i = 1, 2, \ldots, N. \tag{60}$$

The same initial state $|\psi_0\rangle$ evolves to a state $|\psi_J\rangle = U|\psi_0\rangle$ after J empty *oracle* queries. The distance, how far the state $|\psi_J^{a_i}\rangle$ has drifted from the state $|\psi_J\rangle$, can be qualified in terms of the lower bound as

$$\sum_{i=1}^{N} | \, |\psi_J^{a_i}\rangle - |\psi_J\rangle \, |^2 \geq 2N - 2\sqrt{N}. \tag{61}$$

In Grover's algorithm $|\psi_0\rangle = |\Theta\rangle$ is the state with equal superposition of all the basis elements. The unitary operator U_{a_i} is the Grover iteration applied J times

$$U_{a_i} = (-\mathcal{I}_\Theta \mathcal{I}_{a_i})^J = [-(\mathbb{I} - 2|\Theta\rangle\langle\Theta|)(\mathbb{I} - 2|a_i\rangle\langle a_i|)]^J. \tag{62}$$

Then,

$$|\psi_J^{a_i}\rangle = U_{a_i}|\psi_0\rangle = (-\mathcal{I}_\Theta \mathcal{I}_{a_i})^J |\Theta\rangle \approx |a_i\rangle. \tag{63}$$

The empty *oracle* operator U is given by

$$U = (-\mathcal{I}_\Theta \mathbb{I})^J = [-(\mathbb{I} - 2|\Theta\rangle\langle\Theta|)\mathbb{I}]^J, \tag{64}$$

where the *oracle* operator is just the identity operator. U does not change the initial state at all

$$|\psi_J\rangle = U|\psi_0\rangle = U|\Theta\rangle = |\Theta\rangle. \tag{65}$$

Substituting the results from Eqs. (63) and (65) in the left-hand side of Eq. (61) we obtain $\sum_{i=1}^{N} | \, |\psi_J^{a_i}\rangle - |\psi_J\rangle \, |^2 = 2N - 2\sqrt{N}$, which saturates the inequality.

Given the inequality in Eq. (61) in terms of the number of elements in a database N we now need another inequality which will provide a bound in terms of the number of iterations J. This inequality is given in terms of the lower bound as

$$\sum_{i=1}^{N} | \, |\psi_J^{a_i}\rangle - |\psi_J\rangle \, |^2 \leq 4J^2. \tag{66}$$

From Eqs. (61) and (66) we obtain in large N limit

$$J \geq \sqrt{\frac{N}{2}} = \mathcal{O}(\sqrt{N}). \tag{67}$$

In this proof we have assumed the probability of obtaining a target state to be unity. In general by considering probability close to unity one can refine the lower bound on the number of searches J in Eq. (67). However, up to some small factor the query time is $\mathcal{O}(\sqrt{N})$, which cannot be reduced by any algorithm.

For multiple target elements, the initial probability M/N as given in Eq. (21) of obtaining the target state in the initial state increases which drastically reduces the success probability of obtaining the target state in Grover's original algorithm. However, there are ways to modify [24,25] the Grover's algorithm by introducing phases in the Grover iteration so the success probability is unity. The question of optimal number of searches in modified Grover's algorithm and its comparison with the original Grover's algorithm has been discussed in ref. [25].

2.2 Adiabatic evolution for database search

In recent years there have been several attempts to realize Grover search algorithm by adiabatic evolution [26–28] of a suitably chosen Hamiltonian. In this subsection we state one such work which shows that adiabatic approximation can be utilized to find a target item in $\mathcal{O}(\sqrt{N})$ time which is equivalent to what Grover algorithm needs.

According to the adiabatic theorem if a Hamiltonian changes slowly with time, then the system initially in a ground state will always remain in the instantaneous ground state of the system. We can exploit it by starting from a Hamiltonian whose states are known and then adiabatically evolving the Hamiltonian to a Hamiltonian whose ground state would be the desired state we are looking for, i.e., the target state.

Let us start with the Schrödinger equation of a time-dependent system with Hamiltonian $H(t)$

$$i\hbar \frac{\partial}{\partial t}\psi_A(t) = H(t)\psi_A(t), \tag{68}$$

where $\psi_A(t)$ is a state of the system. The eigenvalue equation for this system is given by

$$H(t)\psi_n(t) = E_n(t)\psi_n(t), \tag{69}$$

where $E_n(t), n = 1, 2, \ldots$ are the time-dependent eigenvalues corresponding to the time-dependent eigenstates $\psi_n(t)$. Note that if the Hamiltonian is time independent, then the eigenvalues are also time independent and the eigenstates only acquire phase factor when it evolves. After a long time of evolution the system initially in $\psi_1(t)$ state will be found in $\psi_2(t)$ state with amplitude ϵ

$$\epsilon \sim | \frac{\langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}t}|\psi_1(t)\rangle}{(E_2(t) - E_1(t))^2} | \ll 1. \tag{70}$$

It is useful to consider even more strict condition to ensure that the system remains in its instantaneous ground state. It is assumed that the maximum of the numerator and the minimum of the denominator in the interval T in Eq. (70) satisfy

$$\frac{\max_{0\le t\le T} \mid \langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}t}|\psi_1(t)\rangle \mid}{\min_{0\le t\le T}(E_2(t) - E_1(t))^2} \le \epsilon. \tag{71}$$

One can exploit the condition (71) to obtain a lower bound on time T to evolve the state from $\psi_1(0)$ to $\psi_1(T)$.

As an explicit example consider the Hamiltonian

$$H_\Theta = \mathbb{I} - |\Theta\rangle\langle\Theta|, \tag{72}$$

whose ground state $|\Theta\rangle$ is the uniform superposition of all the basis elements in the Hilbert space of dimension N defined in Eq. (2). It is assumed that the system is initially in the ground state. Then, to evolve the state $|\Theta\rangle$ to the target state $|a_T\rangle$ we have to consider a Hamiltonian of the form

$$H_T = \mathbb{I} - |a_T\rangle\langle a_T|, \tag{73}$$

whose ground state is the target state $|a_T\rangle$. The Hamiltonian which will evolve the state $|\Theta\rangle$ to the target state $|a_T\rangle$ is given by

$$H(t) = (1 - s(t))H_\Theta + s(t)H_T, \tag{74}$$

where the parameter $s(t)$ depends on time. Consider a simple liner form $s(t) = \frac{t}{T}$, where T is the time over which the system evolves. The difference between the lowest two eigenvalues $E_1(t)$, $E_2(t)$ is given by

$$E_2(t) - E_1(t) = \frac{1}{\sqrt{N}}\sqrt{N - 4(N-1)s(1-s)}. \tag{75}$$

The difference in eigenvalues is minimum i.e., $\min_{0\le t\le T}(E_2(t) - E_1(t))^2 = 1/N$ at $s = 1/2$. The matrix element in the numerator in Eq. (71) can be simplified as

$$\langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}t}|\psi_1(t)\rangle = \frac{\mathrm{d}s}{\mathrm{d}t}\langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}s}|\psi_1(t)\rangle = \frac{1}{T}\langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}s}|\psi_1(t)\rangle \sim \frac{1}{T}. \tag{76}$$

Here we have assumed that the matrix element $\langle\psi_2(t)|\frac{\mathrm{d}H(t)}{\mathrm{d}s}|\psi_1(t)\rangle \sim 1$. Putting the result of Eq. (76) and the minimum eigenvalue difference in Eq. (71) we obtain the time required

$$T \ge \frac{N}{\epsilon}, \tag{77}$$

which is equivalent to what a classical computer would take to find the target element. Since $s = t/T$ does not solve the purpose, we assume that the dependence of s on time t is governed by the adiabatic approximation Eq. (70), which can be rewritten as

$$\frac{\mathrm{d}s}{\mathrm{d}t} \simeq \epsilon(E_2(t) - E_1(t))^2 = \epsilon\frac{1}{N}\left(N - 4(N-1)s(1-s)\right), \tag{78}$$

where again we have assumed $\langle\psi_2(t)|\frac{dH(t)}{ds}|\psi_1(t)\rangle \sim 1$. Integrating Eq. (78) we obtain

$$t = \frac{1}{2\epsilon}\frac{N}{\sqrt{N-1}}\left(\arctan\sqrt{N-1}(2s-1) + \arctan\sqrt{N-1}\right). \tag{79}$$

The evolution time T can be obtained by setting $s = 1$ in Eq. (79)

$$t = \frac{1}{\epsilon}\frac{N}{\sqrt{N-1}}\arctan\sqrt{N-1}. \tag{80}$$

When the number of elements in a database is large $N \gg 1$ we get the time required to find the target element from Eq. (80) as

$$T = \frac{\pi}{2\epsilon}\sqrt{N}. \tag{81}$$

The is a quadratic speed up apart from a factor of inverse of error probability.

This algorithm by adiabatic evolution can be extended to the cases when there are many target elements. This time we consider a Hamiltonian of the form

$$\tilde{H}_T = \mathbb{I} - \sum_{\text{target elements}} |a_i\rangle\langle a_i|. \tag{82}$$

Then, the time-dependent Hamiltonian under which the initial state $|\Theta\rangle$ will be evolved is given by

$$\tilde{H}(t) = (1 - s(t))H_\Theta + s(t)\tilde{H}_T. \tag{83}$$

The difference in energy between the ground state and the first excited state is now given by

$$E_2(t) - E_1(t) = \frac{1}{\sqrt{N}}\sqrt{N - 4(N - M)s(1 - s)}. \tag{84}$$

If we consider $s = t/T$, then we obtain

$$T \geq \frac{N}{M\epsilon}, \tag{85}$$

However, if the adiabatic change is considered to be local in the parameter s, then the required evolution time becomes

$$T = \frac{\pi}{2\epsilon}\sqrt{\frac{N}{M}}, \tag{86}$$

which is in agreement with the Grover algorithm with multiple targets.

3 Partial database search

In reality sometimes we do not need a full search of a database; rather, only a partial search is enough. For example, suppose we want to look for details of contacts of a specific surname in a telephone directory. If there are eight different surnames in the telephone directory, then it can be divided into eight blocks each associated with a surname. In terms of binary the state of an element of the telephone directory with $N = 2^n$ entries can be written as $|a_1, a_2, a_3, \ldots, a_n\rangle$. Since there are only eight blocks, we can assign first three binaries a_1, a_2, a_3 to the surnames. Since all the entries in a block share the same surname, the first three binaries of the states in a block will be same.

Another example which can be related to reality is the following. Suppose there is a target city in the world. Now for some practical reason instead of finding the exact target city we want to find the province to which the target city belongs. Here we have assumed that all the countries are divided into many provinces and each province has many cities. However, the number of elements and number of blocks in this example

are not large. In traveling salesman problem, however, there are many elements and therefore is a good candidate for quantum search. Assume that there are a number of cities and each pair of cities has some nonnegative distance. The problem is to find out the route of minimum distance which connects through all the cities. One can divide the total number of possible routes in many blocks and then perform partial search to find the block which contains the target route. These are examples of more general partial search where each block may not have the same number of elements. However, in the discussion bellow we will discuss more restricted type of partial search where each block has the same number of elements.

a. Some attempts to partial search The purpose of a partial search instead of a full Grover search is to achieve a grater speed than the Grover search. However, not all partial searches are always advantageous. Let us consider a naive partial search in which first the database of N elements is divided into K blocks. Just randomly choose a block and make a full Grover search which requires $\frac{\pi}{4}\sqrt{\frac{N}{K}}$ queries. To obtain the target item and the target block one has to perform full Grover search in $K-1$ blocks separately in the worst case, which requires $(K-1)\frac{\pi}{4}\sqrt{\frac{N}{K}}$ queries. One can see that this is $\frac{K-1}{\sqrt{K}}$ times the full Grover search. Only for $K=2$ the factor $\frac{K-1}{\sqrt{K}}$ is less than one. For more than two blocks therefore this naive partial search is not faster than the full Grover search.

Another example which is also inefficient for database search with more than two blocks is the binary search. In this search the number of blocks should be of the form $K=2^k$ for some positive number k. First divide the whole database in two blocks and perform a standard Grover search in any one of the blocks, which requires $\frac{\pi}{4}\sqrt{\frac{N}{2}}$ iterations. If the target item is not found then take the remaining block and divide that into two sub-blocks and repeat the previous procedure. We keep on repeating this procedure until we are left with the last block. The total number of queries is obtained by taking the sum of all the searches as $\frac{\pi}{4}\sqrt{N}\left(\sum_{i=1}^{k}\sqrt{\frac{1}{2^i}}\right)$. Again the factor $\sum_{i=1}^{k}\sqrt{\frac{1}{2^i}}$ is greater than one for $K\geq 4$, making the binary search inefficient compared to the Grover search for more than two blocks.

b. Grover and Radhakrishnan's simple partial search: The fact that the partial search can be advantageous over the full Grover search can be understood from a simple algorithm discussed by Grover and Radhakrishnan. Let us divide the database into K blocks and perform a full Grover search on elements of $K-1$ randomly chosen blocks, which requires $\frac{\pi}{4}\sqrt{N}\left(\sqrt{\frac{K-1}{K}}\right)$ queries. Note that the factor $\sqrt{\frac{K-1}{K}}$ is always less than one, which suggests that this partial search algorithm is always more efficient than the Grover search algorithm.

3.1 Single target GRK partial search algorithm

Partial search algorithm [2] is a combination of both global search and simultaneous local search in each block. Grover and Radhakrishnan [13] first devised a scheme for a partial database search which was latter optimized by Korepin. The database of N elements which are divided into K blocks are first subjected to a global Grover search $\mathcal{G}$. After j_1 Grover iterations the initial state $|\Theta\rangle$ defined in Eq. (2) becomes

$$\mathcal{G}^{j_1}|\Theta\rangle = \sin(2j_1+1)\,\theta|a_T\rangle + \sum_{i=0, i\neq T}^{N-1} \cos(2j_1+1)\,\theta \tan\theta|a_i\rangle. \tag{87}$$

Then, to perform the local iterations let us consider the initial state of α block as

$$|\Theta_\alpha\rangle = \sum_{\alpha\text{block}}^{N/K \text{ elements}} \sqrt{\frac{K}{N}}|a_i\rangle, \quad \alpha = 1, 2, \ldots, K, \tag{88}$$

which is obtained by equal superposition of all the elements in the block. The target element $|a_T\rangle$ should belong to one block which we call target block. If we measure the probability of obtaining the target element in the initial state of a block, then for all initial states of individual blocks the probability will vanish except for the initial state $|\Theta_T\rangle$ of the target block for which the finite probability is given by

$$\mathcal{P}_T = |\langle a_T|\Theta_T\rangle|^2 = \sin^2\theta_1 = \frac{K}{N}. \tag{89}$$

The local iteration in each block $\mathcal{G}_\alpha$ can be written as

$$\mathcal{G}_\alpha = -\mathcal{I}_{\Theta_\alpha}\mathcal{I}_T, \quad \alpha = 1, 2, \ldots, K, \tag{90}$$

where the local reflections $\mathcal{I}_{\Theta_\alpha}$ are given by

$$\mathcal{I}_{\Theta_\alpha} = \mathbb{I} - 2|\Theta_\alpha\rangle\langle\Theta_\alpha|, \quad \alpha = 1, 2, \ldots, K. \tag{91}$$

Taking a direct sum of all the local iterations we obtain the local Grover iteration $\mathcal{G}^L$

$$\mathcal{G}^L = \oplus_{\alpha=1}^K \mathcal{G}_\alpha = -\left(\oplus_{\alpha=1}^K \mathcal{I}_{\Theta_\alpha}\right)\mathcal{I}_T. \tag{92}$$

Note that except for $\mathcal{G}_T$, which acts on the target block component, all the other local iterations $\mathcal{G}_\alpha$ act trivially on $\mathcal{G}^{j_1}|\Theta\rangle$. The action of $\mathcal{G}_\alpha$ on the respective initial states is given by

$$\mathcal{G}_\alpha|\Theta_\alpha\rangle = -\mathcal{I}_{\Theta_\alpha}\mathcal{I}_T|\Theta_\alpha\rangle = -\mathcal{I}_{\Theta_\alpha}|\Theta_\alpha\rangle = |\Theta_\alpha\rangle, \quad \alpha \neq T, \alpha = 1, 2, \ldots, K. \tag{93}$$

On the other hand $\mathcal{G}_\alpha|\Theta_\beta\rangle$, for $\alpha \neq \beta$, vanishes. Therefore, $\mathcal{G}^L$ acts trivially on $|\Theta_\alpha\rangle$, i.e., $\mathcal{G}^L|\Theta_\alpha\rangle = |\Theta_\alpha\rangle$, $\alpha \neq T, \alpha = 1, 2, \ldots, K$.

To know how $\mathcal{G}_T$ acts on the target block state let us consider the eigenvalue equation

$$\mathcal{G}_T|\phi_1\rangle = E|\phi_1\rangle, \tag{94}$$

which has the following two eigenvectors

$$|\phi_1\rangle_\pm = \frac{1}{\sqrt{2}}|a_T\rangle \pm \frac{i}{\sqrt{2}} \overbrace{\sum_{i\neq T}}^{\text{target block}} \tan\theta_1|a_i\rangle, \tag{95}$$

with their corresponding eigenvalues $E_\pm = e^{\pm i2\theta_1}$. Let us now write the state $\mathcal{G}^{j_1}|\Theta\rangle$ in Eq. (87) in terms of the eigenvectors $|\phi_1\rangle_\pm$ and the initial states of the non-target blocks $|\Theta_\alpha\rangle, \alpha \neq T$ as

$$\begin{aligned}\mathcal{G}^{j_1}|\Theta\rangle = &\frac{1}{\sqrt{2}}\left(\sin(2j_1+1)\theta - i\frac{\cos(2j_1+1)\theta_1\tan\theta}{\tan\theta_1}\right)|\phi_1\rangle_+ \\ &+\frac{1}{\sqrt{2}}\left(\sin(2j_1+1)\theta + i\frac{\cos(2j_1+1)\theta_1\tan\theta}{\tan\theta_1}\right)|\phi_1\rangle_- \\ &+\sum_{\alpha=1,\alpha\neq T}^{K}\frac{\cos(2j_1+1)\theta\tan\theta}{\sin\theta_1}|\Theta_\alpha\rangle.\end{aligned} \tag{96}$$

We now consider j_2 operations with the local Grover operator $\mathcal{G}^L$ on the expression of Eq. (96). Since $\mathcal{G}^L$ acts trivially on $|\Theta_\alpha\rangle$, $\mathcal{G}^{Lj_2}$ will also act trivially on the third expression of the right-hand side of Eq. (96). However, the first and second expressions will pick up phase of $e^{i2\theta_1}$ and $e^{-i2\theta_1}$, respectively, as eigenvalues each time $\mathcal{G}^L$ acts on them. Therefore, $\mathcal{G}^{Lj_2}$ will introduce $e^{i2j_2\theta_1}$ and $e^{-i2j_2\theta_1}$, respectively, in the amplitude of first and second expressions, respectively. Thus, we obtain

$$\begin{aligned}\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle = &\frac{e^{i2j_2\theta_1}}{\sqrt{2}}\left(\sin(2j_1+1)\theta - i\frac{\cos(2j_1+1)\theta_1\tan\theta}{\tan\theta_1}\right)|\phi_1\rangle_+ \\ &+\frac{e^{-i2j_2\theta_1}}{\sqrt{2}}\left(\sin(2j_1+1)\theta + i\frac{\cos(2j_1+1)\theta_1\tan\theta}{\tan\theta_1}\right)|\phi_1\rangle_- \\ &+\sum_{\alpha=1,\alpha\neq T}^{K}\frac{\cos(2j_1+1)\theta\tan\theta}{\sin\theta_1}|\Theta_\alpha\rangle.\end{aligned} \tag{97}$$

It is useful to write the above state $\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle$ in terms of the basis vectors $|a_i\rangle$ as

$$\begin{aligned}\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle = \mathcal{C}_T|a_T\rangle + \mathcal{C}_{TB}&\overbrace{\sum_{i\neq T}}^{\text{target block}}\tan\theta_1|a_i\rangle \\ &+\mathcal{C}_{NTB}\overbrace{\sum}^{\text{non-target blocks}}|a_i\rangle,\end{aligned} \tag{98}$$

where the constant coefficients are given by

$$\mathcal{C}_T = \sin(2j_1+1)\,\theta\cos 2j_2\theta_1 + \frac{\cos(2j_1+1)\,\theta\tan\theta}{\tan\theta_1}\sin 2j_2\theta_1, \tag{99}$$

$$\mathcal{C}_{TB} = -\sin(2j_1+1)\,\theta\sin 2j_2\theta_1 + \frac{\cos(2j_1+1)\,\theta\tan\theta}{\tan\theta_1}\cos 2j_2\theta_1, \tag{100}$$

$$\mathcal{C}_{NTB} = \cos(2j_1+1)\,\theta\tan\theta. \tag{101}$$

To calculate Eq. (98) we have used Eqs. (88) and (95).

To eliminate the components associated with the non-target blocks we make a final global Grover iteration to the vector $\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle$ in Eq. (98). For convenience we perform a transformation with $-\mathcal{I}_T\mathcal{I}_\Theta$ instead of the Grover iteration $\mathcal{G}$; however, in large blocks limit both the results are equivalent. There is also other operator such as $\mathcal{I}_\Theta$ which has been used by Grover and Radhakrishnan to perform the final operation. However, in this case the amplitude of the target element becomes negative. The state after final transformation becomes

$$|\mathcal{F}\rangle = (-\mathcal{I}_T\mathcal{I}_\Theta)\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle = \left(\mathcal{C}_T - 2\bar{\mathcal{C}}\right)|a_T\rangle + \left(2\bar{\mathcal{C}} - \mathcal{C}_{TB}\tan\theta_1\right)\overbrace{\sum_{i\neq T}}^{\text{target block}}|a_i\rangle + \left(2\bar{\mathcal{C}} - \mathcal{C}_{NTB}\right)\overbrace{\sum}^{\text{non-target blocks}}|a_i\rangle, \tag{102}$$

where the average amplitude

$$\bar{\mathcal{C}} = \frac{1}{N}\left(\mathcal{C}_T + \mathcal{C}_{TB}\cot\theta_1 + (K-1)\frac{\mathcal{C}_{NTB}}{\sin^2\theta_1}\right) \tag{103}$$

is obtained by taking the average of the amplitudes of the basis vectors in Eq. (98). To evaluate Eq. (102) we have used the formula of Eq. (9) for the action of $-\mathcal{I}_\Theta$ on a generic state. Since the projection of the state $(-\mathcal{I}_T\mathcal{I}_\Theta)\mathcal{G}^{Lj_2}\mathcal{G}^{j_1}|\Theta\rangle$ on non-target blocks should vanish we obtain from Eq. (102)

$$\mathcal{C}_{NTB} = \frac{2}{N}\left(\mathcal{C}_T + \mathcal{C}_{TB}\cot\theta_1 + (K-1)\frac{\mathcal{C}_{NTB}}{\sin^2\theta_1}\right). \tag{104}$$

Substituting the values of $\mathcal{C}_T$, $\mathcal{C}_{TB}$ and $\mathcal{C}_{NTB}$ in Eq. (104) and simplifying we obtain a condition

$$\begin{aligned}
&-\frac{1}{\sin\theta\cos\theta}\left(\frac{1}{2} - \frac{\sin^2\theta}{\sin^2\theta_1}\right)\cos(2j_1+1)\,\theta \\
&= \sin(2j_1+1)\,\theta\cos 2j_2\theta_1 + \frac{\tan\theta}{\tan\theta_1}\cos(2j_1+1)\,\theta\sin 2j_2\theta_1 \\
&\quad - \cot\theta_1\sin(2j_1+1)\,\theta\sin 2j_2\theta_1 + \frac{\tan\theta}{\tan^2\theta_1}\cos(2j_1+1)\,\theta\cos 2j_2\theta_1,
\end{aligned} \tag{105}$$

which ensures that the non-target elements vanish from the final state. Thus, we obtain the final state $|\mathcal{F}_T\rangle$, which is aligned with the target block

$$|\mathcal{F}_T\rangle = \sin\omega|a_T\rangle + \cos\omega \overbrace{\sum_{i\neq T}}^{\text{target block}} \tan\theta_1|a_i\rangle$$
$$= (\mathcal{C}_T - \mathcal{C}_{NTB})\,|a_T\rangle + (\mathcal{C}_{NTB}\cot\theta_1 - \mathcal{C}_{TB}) \overbrace{\sum_{i\neq T}}^{\text{target block}} \tan\theta_1|a_i\rangle. \quad (106)$$

The block angle ω which only depends on the number of blocks K of a database is given by

$$\tan\omega = \frac{\sin(2j_1+1)\,\theta\cos 2j_2\theta_1 + \cos(2j_1+1)\,\theta\tan\theta\left(\frac{\sin 2j_2\theta_1}{\tan\theta_1} - 1\right)}{\sin(2j_1+1)\,\theta\sin 2j_2\theta_1 + \frac{\cos(2j_1+1)\theta\tan\theta}{\tan\theta_1}(1-\cos 2j_2\theta_1)}. \quad (107)$$

3.1.1 Large database limit

Let us now consider the large database limit $N \to \infty$ [2]. We also consider the blocks of the database to be very large $\frac{N}{K} \to \infty$ so that the number of blocks K in a database remains finite. In these limits the two rotation angles in Eqs. (3) and (89), respectively, reduce to

$$\lim_{\theta\to 0}\sin\theta \to \theta \to \sqrt{\frac{1}{N}}, \quad \lim_{\theta_1\to 0}\sin\theta_1 \to \theta_1 \to \sqrt{\frac{K}{N}}. \quad (108)$$

Following Ref. [2] we write the number of iterations j_1 and j_2 in terms of two new parameters η and β as

$$j_1 = \left(\frac{\pi}{4} - \frac{\eta}{\sqrt{K}}\right)\sqrt{N}, \quad j_2 = \frac{\beta}{\sqrt{K}}\sqrt{N}. \quad (109)$$

Putting the expression for j_1 and j_2 of Eq. (109) and the expression of θ and θ_1 of Eq. (108) in the condition for cancellation of amplitudes Eq. (105) of non-target blocks and taking the large database limit we obtain

$$-\sqrt{N}\left(\frac{1}{2} - \frac{1}{K}\right)\sin\frac{2\eta}{\sqrt{K}}$$
$$= \cos\frac{2\eta}{\sqrt{K}}\cos 2\beta + \frac{1}{\sqrt{K}}\sin\frac{2\eta}{\sqrt{K}}\sin 2\beta$$
$$-\sqrt{\frac{N}{K}}\cos\frac{2\eta}{\sqrt{K}}\sin 2\beta + \frac{\sqrt{N}}{K}\sin\frac{2\eta}{\sqrt{K}}\cos 2\beta. \quad (110)$$

Notice that the left-hand side of the above equation is proportional to $\sqrt{N}$, which is a large number in our case. On the right-hand side the last two terms are proportional to $\sqrt{N}$; however, the first two terms are very small compared to the last two terms. Neglecting these small two terms a simple form for the cancellation of the amplitude corresponding to non-target blocks is obtained as

$$\tan\frac{2\eta}{\sqrt{K}} = \frac{2\sqrt{K}\sin 2\beta}{K - 4\sin^2\beta}. \tag{111}$$

The block angle in Eq. (107) can be simplified using Eq. (111) as

$$\lim_{N\to\infty}\tan\omega = \frac{1}{2}\cot\beta + \left(\frac{2}{K} - \frac{1}{2}\right)\tan\beta. \tag{112}$$

Exploiting the physical constraints we can calculate the bounds of the two parameters η and β. Since the number of queries for the global iteration as well as the number of queries for the local iteration given in Eq. (109) should be nonnegative $j_1, j_2 \geq 0$, we obtain

$$\eta \leq \frac{\pi}{4}\sqrt{K}, \quad \beta \geq 0. \tag{113}$$

The partial search algorithm has to have less number of total iterations $j_1 + j_2 + 1$ compared to the Grover's full search algorithm

$$j_1 + j_2 + 1 = \left(\frac{\pi}{4} + \frac{\beta - \eta}{\sqrt{K}}\right)\sqrt{N} \leq \frac{\pi}{4}\sqrt{N}, \tag{114}$$

which implies

$$\beta \leq \eta. \tag{115}$$

From Eqs. (113) and (115) we obtain

$$0 \leq \beta \leq \eta \leq \frac{\pi}{4}\sqrt{K}. \tag{116}$$

The expression for the parameter η for the global iteration can be readily obtained from Eq. (111) as

$$\eta = \frac{\sqrt{K}}{2}\arctan\left[\frac{2\sqrt{K}\sin 2\beta}{K - 4\sin^2\beta}\right], \tag{117}$$

where the $\arctan(x)$ is restricted to the principal branch only because of the constraint in Eq. (113). The bound for the parameter β then becomes

$$0 \leq \beta \leq \frac{\sqrt{K}}{2} \arctan\left[\frac{2\sqrt{K}\sin 2\beta}{K - 4\sin^2\beta}\right] \leq \frac{\pi}{4}\sqrt{K}. \tag{118}$$

3.1.2 Optimization of partial search

As mentioned in the introduction the partial search of Grover and Radhakrishnan has been optimized by Korepin [2] and the optimized version of the partial search is known as the GRK partial search. In large database limit $N \to \infty$ the total number of queries to the *quantum oracle* by a partial search algorithm is given by

$$J(K) = \lim_{N\to\infty}(j_1 + j_2 + 1) = \left(\frac{\pi}{4} + \frac{\beta - \eta}{\sqrt{K}}\right)\sqrt{N}. \tag{119}$$

To obtain least number of queries $J(K)$ we have to minimize

$$\Lambda(\beta) = \beta - \eta(\beta). \tag{120}$$

Note that the partial search will be more efficient than the full global search if the parameter $\Lambda(\beta)$ defined above is negative. Let us assume that the function $\Lambda(\beta)$ has a minima at some point and the first derivative with respect to β vanishes

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(\beta) = \frac{16(K-1)\sin^4\beta - 4K^2\sin^2\beta + K^2}{16(K-1)\sin^4\beta - 8K\sin^2\beta - K^2} = 0. \tag{121}$$

The two solutions of Eq. (121) are given by

$$\sin^2\beta = \begin{cases} \frac{K}{4(K-1)}, \\ \frac{K}{4}, \quad \text{for} \quad K \leq 4. \end{cases} \tag{122}$$

The second derivative of $\Lambda(\beta)$ is given by

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}\Lambda(\beta) = {} & \frac{16K\sin 2\beta(K-1)(K-2)\cos^2 2\beta}{\left(16(K-1)\sin^4\beta - 8K\sin^2\beta - K^2\right)^2} \\ & + \frac{4K\sin 2\beta\left[16(K-1)\cos 2\beta + (K-2)^2(K+2)\right]}{\left(16(K-1)\sin^4\beta - 8K\sin^2\beta - K^2\right)^2}.\end{aligned} \tag{123}$$

Note that for the number of blocks $K = 2, 3$ and 4 we have to consider the two solutions in Eq. (122), where as for $K \geq 5$ only one solution $\sin^2\beta = \frac{K}{4(K-1)}$ is valid.

For $K = 2$ we notice from Eq. (122) that the two solutions coincide. In this case $\sin^2\beta = \frac{K}{4(K-1)} = \frac{K}{4} = \frac{1}{2} \implies \beta = \frac{\pi}{4}$ and $\eta = \frac{\pi}{2\sqrt{2}}$, which correspond to $j_1 = 0$

and $j_2 = \frac{\pi}{4\sqrt{2}}\sqrt{N}$. For $K = 3$ and 4 the global minimum is at $\sin^2\beta = \frac{K}{4(K-1)}$. Therefore, for $K \geq 2$ the global minimum is achieved for

$$\beta = \arcsin\left(\sqrt{\frac{K}{4(K-1)}}\right), \tag{124}$$

$$\eta = \frac{\sqrt{K}}{2}\arctan\left(\frac{\sqrt{3K-4}}{K-2}\right). \tag{125}$$

3.2 Multiple targets GRK partial search algorithm

In the previous section we considered only one target element and therefore partial search was to find out the single target blocks. However, there may have several target elements and several target blocks. Here we provide a generalization of the partial search to find one of the target blocks. Let us assume that we have a database of N elements with K blocks. Blocks with target elements are called target blocks, and rest of the blocks without target elements are called non-target blocks. There are $B = \frac{N}{M}$ numbers of elements in each block. Assume that there are K_T target blocks and each target block has B_T number of target elements. So in total there are $M = K_T B_T$ target elements.

The initial state $|\tilde{\Theta}\rangle$ of Eq. (20) after iterating j_1 times with the global Grover operator $\tilde{\mathcal{G}}$ becomes

$$\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \sin(2j_1+1)\,\tilde{\theta}|A_T\rangle + \cos(2j_1+1)\,\tilde{\theta}|A_{nT}\rangle, \tag{126}$$

where angle between the initial state $|\tilde{\Theta}\rangle$ and the normal to the unite target state $|A_T\rangle$ is given by

$$\sin^2\tilde{\theta} = \frac{M}{N} = \frac{K_T B_T}{N} \tag{127}$$

Now we have to consider the local Grover iteration in each block for which we define the local iteration in each block $\mathcal{G}_\alpha$ as

$$\tilde{\mathcal{G}}_\alpha = -\mathcal{I}_{\tilde{\Theta}_\alpha}\mathcal{I}_{T\alpha}, \quad \alpha = 1, 2, \ldots, K. \tag{128}$$

The local reflections $\mathcal{I}_{\tilde{\Theta}_\alpha}$ and $\mathcal{I}_{T\alpha}$ are given by

$$\mathcal{I}_{\tilde{\Theta}_\alpha} = \mathbb{I} - 2|\tilde{\Theta}_\alpha\rangle\langle\tilde{\Theta}_\alpha|, \tag{129}$$

$$\mathcal{I}_{T\alpha} = \mathbb{I} - 2|A_{T\alpha}\rangle\langle A_{T\alpha}|, \tag{130}$$

where

$$|\tilde{\Theta}_\alpha\rangle = \sin\tilde{\theta}_1|A_{T\alpha}\rangle + \cos\tilde{\theta}_1|A_{nT\alpha}\rangle, \quad \alpha = 1, 2, \ldots, K, \tag{131}$$

$$|A_{T\alpha}\rangle = \sqrt{\frac{1}{B_T}} \sum_{\substack{\text{target elements} \\ \alpha\text{block}}} |a_i\rangle, \quad \alpha = 1, 2, \ldots, K. \tag{132}$$

We also define

$$|A_{nT\alpha}\rangle = \sqrt{\frac{1}{B - B_T}} \sum_{\substack{\text{non-target elements} \\ \alpha\text{block}}} |a_i\rangle, \quad \alpha = 1, 2, \ldots, K. \tag{133}$$

Note that for blocks which do not have target elements $\mathcal{I}_{T\alpha}$ simply becomes the identity operator. The angle $\tilde{\theta}_1$ which measures the probability of obtaining the target unit state within a target block is given by

$$\sin\tilde{\theta}_1 = \sqrt{\frac{B_T}{B}}. \tag{134}$$

Taking a direct sum of all the local iterations we obtain the local Grover iteration $\tilde{\mathcal{G}}^L$

$$\tilde{\mathcal{G}}^L = \oplus_{\alpha=1}^{K} \tilde{\mathcal{G}}_\alpha. \tag{135}$$

Note that except for those $\tilde{\mathcal{G}}_\alpha$s, which act on the target blocks, all the other local iterations $\tilde{\mathcal{G}}_\alpha$ act trivially on $\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$.

Without loss of generality we assume that first K_T blocks are target blocks and the rest $K - K_T$ is non-target blocks. Then, the action of $\tilde{\mathcal{G}}_\alpha$ on the respective initial states is given by

$$\tilde{\mathcal{G}}_\alpha|\tilde{\Theta}_\alpha\rangle = -\mathcal{I}_{\tilde{\Theta}_\alpha}\mathcal{I}_{T\alpha}|\tilde{\Theta}_\alpha\rangle = -\mathcal{I}_{\tilde{\Theta}_\alpha}|\tilde{\Theta}_\alpha\rangle = |\tilde{\Theta}_\alpha\rangle, \quad \alpha = K_T + 1, K_T + 2, \ldots, K. \tag{136}$$

The action of $\tilde{\mathcal{G}}_\alpha$, $\alpha = 1, 2, \ldots, K_T$, on the target blocks can be understood from the eigenvalue equations $\tilde{\mathcal{G}}_\alpha|\phi_{1\alpha}\rangle = \tilde{E}_\alpha|\phi_{1\alpha}\rangle$, which has the following two eigenvectors

$$|\phi_{1\alpha}\rangle_\pm = \frac{1}{\sqrt{2}}|A_{T\alpha}\rangle \pm \frac{i}{\sqrt{2}}|A_{nT\alpha}\rangle, \tag{137}$$

and their corresponding eigenvalues $E_\pm = e^{\pm i2\tilde{\theta}_1}$.

We now write the state $\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$ in Eq. (126) in terms of the eigenvectors $|\phi_{1\alpha}\rangle_\pm$ and the initial states of the non-target blocks as

$$\begin{aligned}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \sum_{\alpha=1}^{K_T} \Bigg[&\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{K_T}} \sin(2j_1 + 1)\,\tilde{\theta} - i\sqrt{\frac{B - B_T}{N - M}} \cos(2j_1 + 1)\,\tilde{\theta}_1 \right) |\phi_{1\alpha}\rangle_+ \\ &+ \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{K_T}} \sin(2j_1 + 1)\,\tilde{\theta} + i\sqrt{\frac{B - B_T}{N - M}} \cos(2j_1 + 1)\,\tilde{\theta}_1 \right) |\phi_{1\alpha}\rangle_- \Bigg]\end{aligned}$$

$$+\sum_{\alpha=K_T+1}^{K}\sqrt{\frac{B}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}|\tilde{\Theta}_\alpha\rangle. \tag{138}$$

After j_2 operations with the local Grover operator $\tilde{\mathcal{G}}^L$ on the expression of Eq. (138) we immediately obtain

$$\begin{aligned}(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \sum_{\alpha=1}^{K_T}\Bigg[&\frac{e^{i2\tilde{\theta}_1 j_2}}{\sqrt{2}}\left(\frac{1}{\sqrt{K_T}}\sin\left(2j_1+1\right)\tilde{\theta}-i\sqrt{\frac{B-B_T}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}\right)|\phi_{1\alpha}\rangle_+\\&+\frac{e^{-i2\tilde{\theta}_1 j_2}}{\sqrt{2}}\left(\frac{1}{\sqrt{K_T}}\sin\left(2j_1+1\right)\tilde{\theta}+i\sqrt{\frac{B-B_T}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}\right)|\phi_{1\alpha}\rangle_-\Bigg]\\&+\sum_{\alpha=K_T+1}^{K}\sqrt{\frac{B}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}|\tilde{\Theta}_\alpha\rangle.\end{aligned} \tag{139}$$

The above state $(\tilde{\mathcal{G}}^L)^{j_2}\mathcal{G}^{j_1}|\tilde{\Theta}\rangle$ can be written in terms of the basis vectors $|a_i\rangle$ as

$$\begin{aligned}(\tilde{\mathcal{G}}^L)^{j_2}\mathcal{G}^{j_1}|\tilde{\Theta}\rangle = \tilde{\mathcal{C}}_T\sum_{\substack{\text{target elements}\\ \text{target blocks}}}|a_i\rangle+\tilde{\mathcal{C}}_{TB}\sum_{\substack{\text{non-target elements}\\ \text{target blocks}}}|a_i\rangle\\+\tilde{\mathcal{C}}_{NTB}\sum_{\substack{\text{all elements}\\ \text{non-target blocks}}}|a_i\rangle,\end{aligned} \tag{140}$$

where the constant coefficients are given by

$$\begin{aligned}\tilde{\mathcal{C}}_T &= \sqrt{\frac{1}{M}}\sin\left(2j_1+1\right)\tilde{\theta}\cos 2j_2\tilde{\theta}_1\\&\quad+\sqrt{\frac{B-B_T}{B_T(N-M)}}\cos\left(2j_1+1\right)\tilde{\theta}\sin 2j_2\tilde{\theta}_1,\end{aligned} \tag{141}$$

$$\begin{aligned}\tilde{\mathcal{C}}_{TB} &= -\sqrt{\frac{1}{K_T(B-B_T)}}\sin\left(2j_1+1\right)\tilde{\theta}\sin 2j_2\tilde{\theta}_1\\&\quad+\sqrt{\frac{1}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}\cos 2j_2\tilde{\theta}_1,\end{aligned} \tag{142}$$

$$\tilde{\mathcal{C}}_{NTB} = \sqrt{\frac{1}{N-M}}\cos\left(2j_1+1\right)\tilde{\theta}. \tag{143}$$

We make a final global Grover iteration to the vector $(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$ in Eq. (140) with $-\mathcal{I}_T\mathcal{I}_{\tilde{\Theta}}$, which gives

$$|\tilde{\mathcal{F}}\rangle = (-\mathcal{I}_T\mathcal{I}_{\tilde{\Theta}})(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \left(\tilde{\mathcal{C}}_T - 2\bar{\tilde{\mathcal{C}}}\right)\sum_{\text{target blocks}}^{\text{target elements}}|a_i\rangle + \left(2\bar{\tilde{\mathcal{C}}} - \tilde{\mathcal{C}}_{TB}\right)\sum_{\text{target blocks}}^{\text{non-target elements}}|a_i\rangle + \left(2\bar{\tilde{\mathcal{C}}} - \tilde{\mathcal{C}}_{NTB}\right)\sum_{\text{non-target blocks}}^{\text{all elements}}|a_i\rangle, \tag{144}$$

where the average amplitude is given by

$$\bar{\tilde{\mathcal{C}}} = \frac{1}{N}\left(M\tilde{\mathcal{C}}_T + K_T(B - B_T)\tilde{\mathcal{C}}_{TB} + (K - K_T)B\tilde{\mathcal{C}}_{NTB}\right). \tag{145}$$

The amplitude of the non-target blocks in $(-\mathcal{I}_T\mathcal{I}_{\tilde{\Theta}})(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$ should vanish, which gives

$$\tilde{\mathcal{C}}_{NTB} = \frac{2}{N}\left(M\tilde{\mathcal{C}}_T + K_T(B - B_T)\tilde{\mathcal{C}}_{TB} + (K - K_T)B\tilde{\mathcal{C}}_{NTB}\right). \tag{146}$$

Similar to the single target case we substitute the values of $\tilde{\mathcal{C}}_T, \tilde{\mathcal{C}}_{TB}$ and $\tilde{\mathcal{C}}_{NTB}$ in Eq. (146) and obtain

$$\begin{aligned} &-\frac{1}{\sin\tilde{\theta}\cos\tilde{\theta}}\left(\frac{1}{2} - \frac{\sin^2\tilde{\theta}}{\sin^2\tilde{\theta}_1}\right)\cos(2j_1+1)\,\tilde{\theta} \\ &= \sin(2j_1+1)\,\tilde{\theta}\cos 2j_2\tilde{\theta}_1 + \frac{\tan\tilde{\theta}}{\tan\tilde{\theta}_1}\cos(2j_1+1)\,\tilde{\theta}\sin 2j_2\tilde{\theta}_1 \\ &\quad - \cot\tilde{\theta}_1\sin(2j_1+1)\,\tilde{\theta}\sin 2j_2\tilde{\theta}_1 + \frac{\tan\tilde{\theta}}{\tan^2\tilde{\theta}_1}\cos(2j_1+1)\,\tilde{\theta}\cos 2j_2\tilde{\theta}_1. \end{aligned} \tag{147}$$

Thus, the final state $|\tilde{\mathcal{F}}_T\rangle$, aligned with the target block, is given by

$$\begin{aligned} |\tilde{\mathcal{F}}_T\rangle &= \sin\tilde{\omega}|A_T\rangle + \cos\omega|A_{nTT}\rangle \\ &= \sqrt{M}\,(\mathcal{C}_T - \mathcal{C}_{NTB})\,|A_T\rangle + \sqrt{K_T(B - B_T)}\,(\mathcal{C}_{NTB} - \mathcal{C}_{TB})\,|A_{nTT}\rangle, \end{aligned} \tag{148}$$

where

$$|A_{nTT}\rangle = \sqrt{\frac{1}{K_T(B - B_T)}}\sum_{\text{target blocks}}^{\text{non-target elements}}|a_i\rangle. \tag{149}$$

The block angle $\tilde{\omega}$ is given by

$$\tan\tilde{\omega} = \frac{\sin(2j_1+1)\,\tilde{\theta}\cos 2j_2\tilde{\theta}_1 + \cos(2j_1+1)\,\tilde{\theta}\cot\tilde{\theta}}{\sin(2j_1+1)\,\tilde{\theta}\sin 2j_2\tilde{\theta}_1 + \cos(2j_1+1)\,\tilde{\theta}\tan\tilde{\theta}\cot\tilde{\theta}_1(1-\cos 2j_2\tilde{\theta}_1)}. \tag{150}$$

3.2.1 Large database limit

In large database limit $N \to \infty$ and large blocks limit $B = \frac{N}{K} \to \infty$ with the fixed number of blocks K the two rotation angles in Eqs. (127) and (134), respectively, reduce to

$$\lim_{\tilde{\theta}\to 0} \sin\tilde{\theta} \to \tilde{\theta} \to \sqrt{\frac{M}{N}}, \quad \lim_{\tilde{\theta}_1\to 0} \sin\tilde{\theta}_1 \to \tilde{\theta}_1 \to \sqrt{\frac{B_K}{B}}. \tag{151}$$

Following Ref. [2] we write the number of iterations j_1 and j_2 in terms of two new parameters $\tilde{\eta}$ and $\tilde{\beta}$ as

$$j_1 = \left(\frac{\pi}{4} - \frac{\tilde{\eta}\sqrt{M}}{\sqrt{K}}\right)\sqrt{\frac{N}{M}}, \quad j_2 = \frac{\tilde{\beta}\sqrt{M}}{\sqrt{K}}\sqrt{\frac{N}{M}}. \tag{152}$$

Putting the expression for j_1 and j_2 of Eq. (152) in the condition for cancellation of amplitudes Eq. (147) of non-target blocks and taking the large database limit we obtain

$$\begin{aligned} &-\sqrt{N}\left(\frac{1}{2} - \frac{K_T}{K}\right)\sin\frac{2\tilde{\eta}\sqrt{M}}{\sqrt{K}} \\ &= \sqrt{M}\cos\frac{2\tilde{\eta}\sqrt{M}}{\sqrt{K}}\cos 2\tilde{\beta}\sqrt{B_T} + \frac{\sqrt{K_T}}{\sqrt{K}}\sin\frac{2\tilde{\eta}\sqrt{M}}{\sqrt{K}}\sin 2\beta\sqrt{B_T} \\ &\quad -\sqrt{\frac{N}{K}}\sqrt{K_T}\cos\frac{2\tilde{\eta}\sqrt{M}}{\sqrt{K}}\sin 2\beta\sqrt{B_T} + \frac{\sqrt{N}}{K}K_T\sin\frac{2\tilde{\eta}\sqrt{M}}{\sqrt{K}}\cos 2\beta\sqrt{B_T}. \end{aligned} \tag{153}$$

Following the same procedure as was done to obtain Eq. (111) from Eq. (110) and re-scaling by $\bar{K} = \frac{K}{K_T}$, $\bar{\eta} = \tilde{\eta}\sqrt{B_T}$, $\bar{\beta} = \tilde{\beta}\sqrt{B_T}$ a simple form for the cancellation of the amplitude is obtained as

$$\tan\frac{2\bar{\eta}}{\sqrt{\bar{K}}} = \frac{2\sqrt{\bar{K}}\sin 2\bar{\beta}}{\bar{K} - 4\sin^2\bar{\beta}}. \tag{154}$$

3.2.2 Optimization of partial search

Similar to section IV A 2, exploiting the physical constraints, we can calculate the bounds of the two parameters $\bar{\eta}$ and $\bar{\beta}$. Since the number of queries for the global

iteration as well as the number of queries for the local iteration given in Eq. (152) should be nonnegative $j_1, j_2 \geq 0$ we obtain

$$\bar{\eta} \leq \frac{\pi}{4}\sqrt{\bar{K}}, \quad \bar{\beta} \geq 0. \tag{155}$$

The partial search algorithm has to have less number of total iterations $j_1 + j_2 + 1$ compared to the Grover's full search algorithm, i.e.,

$$j_1 + j_2 + 1 = \left(\frac{\pi}{4} + \frac{\bar{\beta} - \bar{\eta}}{\sqrt{\bar{K}}}\right)\sqrt{\frac{N}{M}} \leq \frac{\pi}{4}\sqrt{\frac{N}{M}}, \tag{156}$$

which implies

$$\bar{\beta} \leq \bar{\eta}. \tag{157}$$

From Eqs. (155) and (157) we obtain

$$0 \leq \bar{\beta} \leq \bar{\eta} \leq \frac{\pi}{4}\sqrt{\bar{K}}. \tag{158}$$

The expression for the parameter $\bar{\eta}$ for the global iteration can be readily obtained from Eq. (154) as

$$\bar{\eta} = \frac{\sqrt{\bar{K}}}{2}\arctan\left[\frac{2\sqrt{\bar{K}}\sin 2\bar{\beta}}{\bar{K} - 4\sin^2\bar{\beta}}\right], \tag{159}$$

where the $\arctan(x)$ is restricted to the principal branch only because of the constraint in Eq. (155). The bound for the parameter $\bar{\beta}$ then becomes

$$0 \leq \bar{\beta} \leq \frac{\sqrt{\bar{K}}}{2}\arctan\left[\frac{2\sqrt{\bar{K}}\sin 2\bar{\beta}}{K - 4\sin^2\bar{\beta}}\right] \leq \frac{\pi}{4}\sqrt{\bar{K}}. \tag{160}$$

In large database limit $N \to \infty$ the total number of queries to the *quantum oracle* by a partial search algorithm is given by

$$\tilde{J}(\bar{K}) = \lim_{N/M\to\infty}(j_1 + j_2 + 1) = \left(\frac{\pi}{4} + \frac{\bar{\beta} - \bar{\eta}}{\sqrt{\bar{K}}}\right)\sqrt{\frac{N}{M}}. \tag{161}$$

To obtain least number of queries $\tilde{J}(\bar{K})$ we have to minimize

$$\tilde{\Lambda}(\bar{\beta}) = \bar{\beta} - \bar{\eta}(\bar{\beta}). \tag{162}$$

The global minimum is achieved for $\bar{K} \geq 2$ at

$$\bar{\beta} = \arcsin\left(\sqrt{\frac{\bar{K}}{4(\bar{K}-1)}}\right), \tag{163}$$

$$\bar{\eta} = \frac{\sqrt{\bar{K}}}{2}\arctan\left(\frac{\sqrt{3\bar{K}-4}}{\bar{K}-2}\right). \tag{164}$$

3.3 Success probability in partial search

In GRK's partial search the amplitude of non-target blocks should vanish, and in Grover's search in order to vanish the amplitude of non-target items the argument of the cosine should be $\pi/2$. This gives formal expression for the number of iterations, which can be non-integer. Since the number of iterations is integer, the amplitude of non-target blocks or the non-target items does not exactly vanish. Also it becomes small: 1/N (here N is the number of items in the database). Therefore, for the practical purpose we just take the integral value nearest [1] to the number of queries obtained from full or partial search. It introduces some error in the final state obtained after the iterations are done. This problem can be fixed to obtain the target state or the target block with cent percent success probability. In the case of partial search we will discuss here how to obtain the target block with unit success probability in integer number of iterations in sure success type searches. Since we need the group formulation for this purpose, let us first briefly discuss the group aspect of the search algorithm.

3.3.1 Group formulation of search algorithm

The whole discussion of full Grover search discussed in 2.1.1 and 2.1.2 can be understood by $O(2)$ transformation [2] on the initial state. Let us write the initial state $|\Theta\rangle$ in terms of the unit basis vectors $|A_T\rangle$ and $|A_{nT}\rangle$ of eqs. (18) and (19), respectively, as

$$|\tilde{\Theta}\rangle = \begin{pmatrix} \sin\tilde{\theta} \\ \cos\tilde{\theta} \end{pmatrix}. \tag{165}$$

In the same basis the Grover iteration $\tilde{\mathcal{G}}$ can be represented as a rotation matrix in two dimensions

$$\tilde{\mathcal{G}} = \begin{pmatrix} \cos 2\tilde{\theta} & \sin 2\tilde{\theta} \\ -\sin 2\tilde{\theta} & \cos 2\tilde{\theta} \end{pmatrix}. \tag{166}$$

Action of the Grover iteration j times successively on the initial state becomes

$$\tilde{\mathcal{G}}^j|\tilde{\Theta}\rangle = \begin{pmatrix} \cos 2j\tilde{\theta} & \sin 2j\tilde{\theta} \\ -\sin 2j\tilde{\theta} & \cos 2j\tilde{\theta} \end{pmatrix}\begin{pmatrix} \sin\tilde{\theta} \\ \cos\tilde{\theta} \end{pmatrix} = \begin{pmatrix} \sin(2j+1)\tilde{\theta} \\ \cos(2j+1)\tilde{\theta} \end{pmatrix}. \tag{167}$$

By assuming that the initial state has evolved to the target state, i.e.,

$$\begin{pmatrix} \sin(2j+1)\tilde{\theta} \\ \cos(2j+1)\tilde{\theta} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{168}$$

we can arrive at the same result in Eq. (29) and when there is only one target element, then we arrive at Eq. (13). This formalism can be extended to partial database search problem which has $O(3)$ group representation [2]. Again we will discuss the multiple targets and multiple target blocks case, but the discussion is equally valid for single target partial search also. In partial search there are three mutually orthogonal basis vectors: the unit vector A_T with equal superposition of all the target elements, the unit vector A_{nTT} with equal superposition of all the non-target elements in the target blocks and the unit vector A_N with equal superposition of all the elements in non-target blocks. First two unit vectors A_T and A_{nTT} have already been defined in eqs. (18) and (149), respectively. We now define the unit vector A_N as

$$|A_N\rangle = \sqrt{\frac{1}{B(K-K_T)}} \sum_{\text{non-target blocks}}^{\text{all elements}} |a_i\rangle. \tag{169}$$

These three vectors form a three-dimensional vector space on which the initial state $|\tilde{\Theta}\rangle$ can be expressed as

$$|\tilde{\Theta}\rangle = \begin{pmatrix} \sin\gamma \sin\tilde{\theta} \\ \sin\gamma \cos\tilde{\theta} \\ \cos\gamma \end{pmatrix}, \tag{170}$$

where $\sin\gamma = \sqrt{K_T/K}$, $\sin\tilde{\theta} = \sqrt{M/N}$. The global Grover iteration $\tilde{\mathcal{G}}^{j_1}$ can be represented as

$$\tilde{\mathcal{G}}^{j_1} = T M_{j_1} T, \tag{171}$$

where T and M_{j_1} are given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tilde{\theta}_1 \sin\gamma/\cos\tilde{\theta} & \cos\gamma/\cos\tilde{\theta} \\ 0 & \cos\gamma/\cos\tilde{\theta} & -\cos\tilde{\theta}_1 \sin\gamma/\cos\tilde{\theta} \end{pmatrix}, \tag{172}$$

and

$$M_{j_1} = \begin{pmatrix} \cos 2j_1\tilde{\theta} & \sin 2j_1\tilde{\theta} & 0 \\ -\sin 2j_1\tilde{\theta} & \cos 2j_1\tilde{\theta} & 0 \\ 0 & 0 & (-1)^{j_1} \end{pmatrix}. \tag{173}$$

The global Grover iteration $\tilde{\mathcal{G}}^{j_1}$ reads as

$$\tilde{\mathcal{G}}^{j_1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{174}$$

where $a_{11} = \cos 2j_1\tilde{\theta}$, $a_{12} = \sin 2j_1\tilde{\theta}\sin\gamma$, $a_{13} = \sin 2j_1\tilde{\theta}\cos\gamma$, $a_{21} = -a_{12}$, $a_{22} = (-1)^{j_1}\cos^2\gamma + \cos 2j_1\tilde{\theta}\sin^2\gamma$, $a_{23} = \sin\gamma\cos\gamma\left[(-1)^{j_1+1} + \cos 2j_1\tilde{\theta}\right]$, $a_{31} = -a_{13}$, $a_{32} = a_{23}$ and $a_{33} = (-1)^{j_1}\sin^2\gamma + \cos 2j_1\tilde{\theta}\cos^2\gamma$. Representation (174) is valid for large N and large B limit. The local Grover iteration $(\tilde{\mathcal{G}}^L)^{j_2}$ is represented as

$$(\tilde{\mathcal{G}}^L)^{j_2} = \begin{pmatrix} \cos 2j_2\tilde{\theta}_1 & \sin 2j_2\tilde{\theta}_1 & 0 \\ -\sin 2j_2\tilde{\theta}_1 & \cos 2j_2\tilde{\theta}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{175}$$

The full partial search operation can also be represented in a compact form

$$\tilde{\mathcal{G}}(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1} = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ 0 & \xi_2 & -\xi_1 \\ -1 & 0 & 0 \end{pmatrix}, \tag{176}$$

where $\xi_1 = \frac{1}{2\sqrt{K-1}} - \frac{1}{2}\sqrt{\frac{3K-4}{K}}$ and $\xi_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3K-4}{K(K-1)}}$ satisfying $\xi_1^2 + \xi_2^2 = 1$.

3.3.2 Sure success partial search

It has been shown in Ref. [29] that the partial search of Grover–Radhakrishnan–Korepin can be performed in such a way that the probability of success is unity. In multiple targets partial search we here discuss the method of obtaining the target block with certainty. In this case the process of partial search is followed as it is except in the final Grover iteration in which $\mathcal{I}_T$ and $\mathcal{I}_{\tilde{\Theta}}$ are modified by phase factors, which are suitably adjusted to obtain the target block.

After the first global Grover iteration the initial state $|\tilde{\Theta}\rangle$ becomes

$$\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \frac{1}{\cos^2\tilde{\theta}}\begin{pmatrix} k_g\cos\tilde{\theta} \\ l_g\cos\tilde{\theta}_1\sin\gamma \\ l_g\cos\gamma \end{pmatrix}, \tag{177}$$

where $k_g = \sin 2j_1\tilde{\theta}\left(\cos^2\tilde{\theta}_1\sin^2\gamma + \cos^2\gamma\right) + \cos 2j_1\tilde{\theta}\cos\tilde{\theta}\sin\tilde{\theta}$ and $l_g = \cos 2j_1\tilde{\theta}\left(\cos^2\tilde{\theta}_1\sin^2\gamma + \cos^2\gamma\right) - \sin 2j_1\tilde{\theta}\cos\tilde{\theta}\sin\tilde{\theta}$.

Then, j_2 local Grover iterations on $\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$ gives us [14]

$$(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle = \frac{1}{\cos^2\tilde{\theta}}\begin{pmatrix} k_g\cos\tilde{\theta}\cos 2j_2\tilde{\theta}_1 + l_g\sin\gamma\cos\tilde{\theta}_1\sin 2j_2\tilde{\theta}_1 \\ -k_g\cos\tilde{\theta}\sin 2j_2\tilde{\theta}_1 + l_g\sin\gamma\cos\tilde{\theta}_1\cos 2j_2\tilde{\theta}_1 \\ l_g\cos\gamma \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix}. \tag{178}$$

Two reflection operators in the final Grover iteration are modified as

$$\mathcal{I}_T^{ph} = \mathbb{I} - (\mathbb{I} - e^{2i\phi_1})|A_T\rangle\langle A_T|, \tag{179}$$

$$\mathcal{I}_{\tilde{\Theta}}^{ph} = \mathbb{I} - (\mathbb{I} - e^{i(\phi_1-\phi_2)})|\tilde{\Theta}\rangle\langle\tilde{\Theta}|. \tag{180}$$

Now as stated above, the final modified global Grover iteration is given by

$$\tilde{\mathcal{G}}^{final} = -\mathcal{I}_{\tilde{\Theta}}^{ph}\mathcal{I}_T^{ph} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{181}$$

where $b_{11} = -e^{i(\phi_1-\phi_2)}\left[1 - (1 - e^{2i\phi_1})\sin^2\gamma\sin^2\tilde{\theta}_1\right]$, $b_{12} = (1 - e^{2i\phi_1})\sin^2\gamma\sin\tilde{\theta}_1\cos\tilde{\theta}_1$, $b_{13} = (1 - e^{2i\phi_1})\sin\gamma\cos\gamma\sin\tilde{\theta}_1$, $b_{21} = e^{i(\phi_1-\phi_2)}(1 - e^{2i\phi_1})\sin^2\gamma\sin\tilde{\theta}_1\cos\tilde{\theta}_1$, $b_{22} = (1-e^{2i\phi_1})\sin^2\gamma\cos^2\tilde{\theta}_1 - 1$, $b_{23} = (1-e^{2i\phi_1})\sin\gamma\cos\gamma\cos\tilde{\theta}_1$, $b_{31} = e^{i(\phi_1-\phi_2)}(1 - e^{2i\phi_1})\sin\gamma\cos\gamma\sin\tilde{\theta}_1$, $b_{32} = b_{23}$ and $b_{33} = (1 - e^{2i\phi_1})\cos^2\gamma - 1$.

The projection of the final state $\tilde{\mathcal{G}}^{final}(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle$ in the direction of unit vector $|A_N\rangle$ of non-target blocks should vanish

$$|\langle A_N|\tilde{\mathcal{G}}^{final}(\tilde{\mathcal{G}}^L)^{j_2}\tilde{\mathcal{G}}^{j_1}|\tilde{\Theta}\rangle| = 0. \tag{182}$$

We obtain from Eq. (182) the following condition on the phases

$$\begin{aligned} &c_{11}e^{i(\phi_1-\phi_2)}(1 - e^{2i\phi_1})\sin\gamma\cos\gamma\sin\tilde{\theta} \\ &+ c_{21}(1 - e^{2i\phi_1})\sin\gamma\cos\gamma\cos\tilde{\theta} \\ &+ c_{31}\left[(1 - e^{2i\phi_1})\cos^2\gamma - 1\right] = 0, \end{aligned} \tag{183}$$

where c_{11}, c_{21}, c_{31} are the three components of the state in Eq. (178). For simplicity we rewrite the condition in Eq. (183) in the following fashion

$$e^{i(\phi_1-\phi_2)}(1 - e^{2i\phi_1})x + (1 - e^{2i\phi_1})y + 2z = 0, \tag{184}$$

where $x = c_{11}\sin\gamma\cos\gamma\sin\tilde{\theta}$, $y = c_{21}\sin\gamma\cos\gamma\cos\tilde{\theta} + c_{31}\cos^2\gamma$ and $z = -c/2$. Separating the real and imaginary part from Eq. (184) we obtain

$$\sin\phi_2 = -\frac{y}{x}\sin\phi_1 - \frac{z}{x\sin\phi_1},$$
$$\cos\phi_2 = -\frac{y}{x}\cos\phi_1. \tag{185}$$

Eliminating ϕ_2 from Eq. (185) we get a condition on phase ϕ_1 as

$$\cos^2\phi_1 = \frac{x^2-(y+z)^2}{x^2-y^2-2yz}. \tag{186}$$

Note that in order to have a solution for ϕ_1 from Eq. (186) the following inequality has to be satisfied

$$x^2 \geq (y+z)^2. \tag{187}$$

The solution for ϕ_2 then can be obtained from Eq. (185). Numerical study for sure success partial search has been performed in [29]. It has been shown that it is always possible to find the phases ϕ_1, ϕ_2 if the number of global and local iterations is chosen as

$$\tilde{j_1} = \lfloor j_1 \rfloor, \tag{188}$$
$$\tilde{j_2} = \lfloor j_2 \rfloor + \{0, 1, 2\}, \tag{189}$$

where $\lfloor x \rfloor$ is the integer nearest to x. For the local Grover iteration it may require to perform one or two extra steps as given in Eq. (189). Numerically, it works well for $N \leq 10^6$ except for $K = 2, B = 2$ case.

4 Conclusion

We have provided a detailed discussion on database search algorithms in this review. In particular Grover algorithm for database search is discussed. The database of N items can have a single or multiple target elements in it. The elements in a database can have some order (sorted database) or no order (unsorted database) at all. The unsorted database with single target element can be searched with Grover algorithm in $\mathcal{O}(\sqrt{N})$ steps compared to $\mathcal{O}(N)$ steps by a classical computer. This is an example of quadratic speed up in computation time. Similarly in the unsorted database with M target elements one of the target elements can be obtained in $\mathcal{O}(\sqrt{\frac{N}{M}})$ steps by Grover algorithm. If there is any structure/order in the database, then by exploiting the structure the target element can be searched even in less time by Grover algorithm. It is not possible to devise an algorithm which can search in less time than what Grover algorithm needs, i.e., $\mathcal{O}(\sqrt{N})$ *oracle* calls.

Instead of searching the whole database for the target element sometimes it is reasonable to divide the whole database in several blocks and then look for the block which contains the target element. Grover and Radhakrishnan found an algorithm for this type of partial search, which takes $j = \left(\frac{\pi}{4} + \frac{\beta(K)-\eta(K)}{\sqrt{K}}\right)\sqrt{N}$ steps. Korepin

later improved the partial search algorithm by optimizing the coefficient $\beta(K) - \eta(K)$. This can further be generalized to include several target elements, and the final global iteration can be modified by including phase factors so that the target block is obtained with unit success probability.

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