Lab 3 – Written Tasks

1. A pasta company wants to determine whether the average weight of macaroni in their one-pound boxes is equal to 454 grams (1 pound $\approx$ 454 grams). A sample of 14 boxes is selected, and their weights (in grams) are recorded as follows:

(a) State the assumptions that need to be verified before conducting a hypothesis test to answer the company's question.

(b) Assuming the necessary assumptions are met, perform the appropriate hypothesis test at a significance level of 10%. Some summary statistics are provided below:

$$\text{sample mean: } 459.21, \quad \text{sample variance: } 11.57$$

(c) Describe the steps to compute the power of the hypothesis test when the true mean weight is 453.592 grams, assuming the true variance of the box weights is 12. Your description should clearly state the test statistic, its associated parameters (e.g. degrees of freedom for a t-distributed random variable), and the critical values. The exact numerical value of the power is not required.

SOLUTION:

(a) Two assumptions are required for performing the hypothesis test:

- The sample of 14 macaroni boxes should be randomly selected from the population.
- The weight distribution of macaroni boxes follows a normal distribution.

(b) Let μ denote the population mean weight of a macaroni box. The null and alternative hypotheses are:

$$H_0 : \mu = 454 \text{ vs } H_1 : \mu \neq 454$$

As this is a one-sample t-test, the standardised test statistic and its distribution under the assumption that H_0 is true are:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

The value of the standardised test statistic can be calculated as:

$$t_{\text{obs}} = \frac{459.21 - 454}{\sqrt{11.57}/\sqrt{14}} = 5.731$$

As this is a two-sided test, the rejection region is $|t_{\text{obs}}| > t_{1-\alpha/2;n-1} = t_{0.95;13} = 1.7709$. Since t_{obs} falls into the rejection region, we reject the null hypothesis and conclude that there is sufficient evidence to suggest that the average weight of macaroni in the one-pound boxes is different from 454 grams.

(c) The power of the test is the probability of rejecting H_0 when H_1 is true. When H_1 is true, the standardised test statistic will follow a non-central t-distribution with the non-centrality parameter γ calculated as follows:

$$\gamma = \frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{453.592 - 454}{\sqrt{12/14}} = -0.441$$

Following the definition of the power, we get:

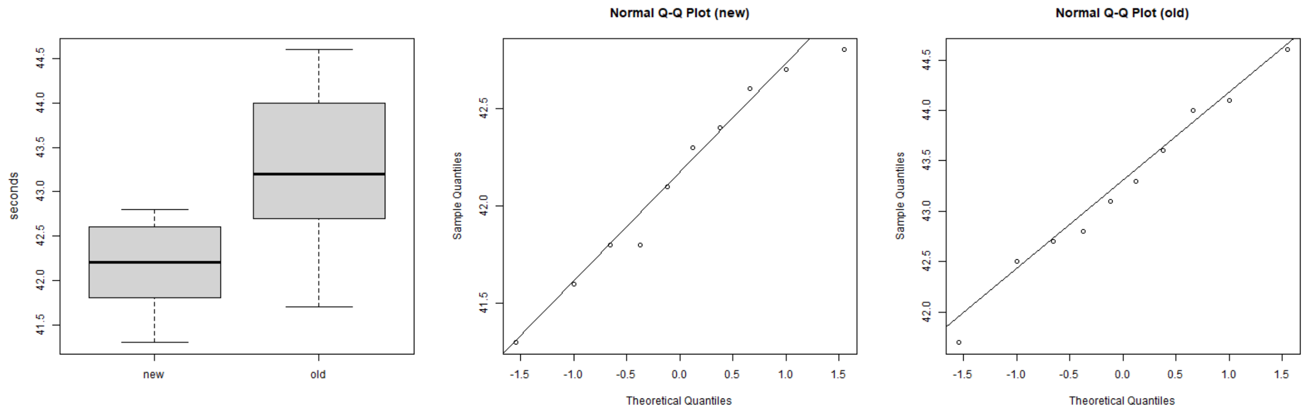
$$\begin{aligned}
\text{Power}(\mu_1 = 453.592) &= P(\text{reject } H_0 | H_1) \\
&= P\big((T < -t_{\alpha/2;n-1}) \cup (T > t_{1-\alpha/2;n-1}) | T \sim t^{\star}_{n-1,\gamma}\big) \\
&= P\big((t^{\star}_{13,-0.441} < -1.7709) \cup (t^{\star}_{13,-0.441} > 1.7709)\big)
\end{aligned}$$

Remark: Calculating the exact value of power requires the use of `R`.

2. In a packing plant, a machine packs cartons with jars. It is supposed that a new machine will pack faster on the average than the machine currently used. To test that hypothesis, the times it takes each machine to pack ten cartons are recorded (in seconds). The results, as well as the sample mean ($\bar{\cdot}$) and sample standard deviation ($s_\cdot$), are shown in the tables below.

New Machine	Old Machine
42.1 41.3 42.4 42.6 41.8	42.7 42.8 42.5 43.1 44.0
41.6 41.8 42.8 42.3 42.7	43.6 43.3 44.6 41.7 44.1
$\bar{x} = 42.14, s_X = 0.504$	$\bar{y} = 43.24, s_Y = 0.864$

The figures below show the boxplots and normal quantile-quantile (Q-Q plot) for the new and old machines.



(a) Comment on the suitability of a normal model for data on the new machine and old machine.

(b) Perform a hypothesis test to determine whether the new machine will pack faster on the average than the machine currently used using a significance level of 5%.

SOLUTION:

(a) A normal model appears to be suitable for both data, as the boxplot shows approximately symmetric data with no obvious skewness or outliers and the data points align with the diagonal line in the QQ plot.

(b) The null and alternative hypotheses are:

$$H_0 : \mu_X - \mu_Y = 0 \text{ vs } H_1 : \mu_X - \mu_Y < 0$$

As the variances of the two machines' packing times are unknown, and they appear to be unequal based on the boxplots, the Welch test should be used. The standardised test statistic and its distribution under the assumption that H_0 is true are:

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}}} \dot{\sim} t_\nu,$$

where

$$\nu = \frac{\left(\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}\right)^2}{\frac{(S_X^2/n_X)^2}{n_X-1} + \frac{(S_Y^2/n_Y)^2}{n_Y-1}} = \frac{\left(\frac{0.504^2}{10} + \frac{0.864^2}{10}\right)^2}{\frac{(0.504^2/10)^2}{9} + \frac{(0.864^2/10)^2}{9}} \approx 14.489.$$

The value of the standardised test statistic can be calculated as:

$$t_{\text{obs}} = \frac{42.14 - 43.24}{\sqrt{\frac{0.504^2}{10} + \frac{0.864^2}{10}}} = -3.477$$

As this is a one-sided test, the rejection region is $t_{\text{obs}} < t_{1-\alpha;\nu} = t_{0.95;14.489} \approx t_{0.95;14} = -1.7613$; here, the degrees of freedom are rounded to the nearest integer to look up the critical value in the statistical table. Since t_{obs} falls into the rejection region, we reject the null hypothesis and conclude that there is sufficient evidence to suggest that the new machine is faster than the old machine.