Escher esque Hyperbolic Tiling Creator - User Guide

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May 13, 2025

1 Introduction

Main goal of the Escher esque Hyperbolic Tiling Creator project is to allow creating tessellations of the hyperbolic space similar to some of the works of Maurits Cornelis Escher.

The purpose of this document is to present the concept on which the program is based. It also discusses how to use the program explains the settings in the tiling configuration file. Finally, it presents sample results generated by the program.

2 Idea

Tessellation in the program involves dividing the entire hyperbolic space using polygons and then filling the space through hyperbolic reflections, about the sides of the polygon, of the image inside a certain selected polygon called the fundamental polygon. The effect is the creation of a some kind of hyperbolic mosaic.

The polygons used to tile space are based on the hyperbolic Schwarz triangles. Which can be optionally connected by omitting one or two of theirs sides and making bigger polygon.

2.1 Schwarz triangle

A Schwarz triangle is a hyperbolic triangle defined by triple of integers $(p,\ q,\ r)$. Each number corresponds to a specific vertex, and the angle in this vertex is related to value of the integer. The Schwarz triangle given by the triple $(p,\ q,\ r)$ has angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$. In general these numbers can be rational numbers but in the program are used integers [8]. The value ∞ can also be used as the vertex value to represent a zero angle.

For a Schwarz triangle to be hyperbolic triangle, the sum of its angles has to be less than π or equivalently the sum of inverses of the integers from defining triple has to be less than one $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$.

The naming convention in the program labels the Schwarz triangle's vertices as p, q, and r, going clockwise starting from p. The sides are labeled as a for the side pr, b for the side rq and c for the side pq. The names of vertices and sides are shown in figure 1.

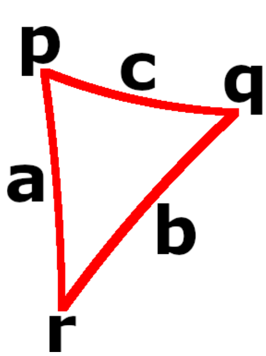

Figure 1. The Schwarz triangle $(3, 4, 5)$ with sides and vertices named according to the convention used in program

2.2 Fundamental polygon

The fundamental triangle is the Schwarz triangle, defined by the settings, that has p-vertex centred at the origin point, being on centre of the screen, the r-vertex directly below the p-vertex and the q-vertex on right from the Y-axis forming the correct angle at the p-vertex. Later will be discussed the setting that moves the fundamental triangle upward to centre it in the foot of the altitude from the vertex r.

The fundamental polygon is created from the fundamental triangle and its neighboring triangles by omitting one or two sides. If no sides are omitted, the triangle itself is the polygon. Some of the fundamental polygons based on the triangle $(3, 4, 5)$ are shown in figure 2.

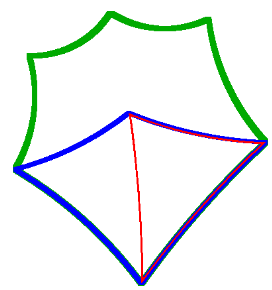

Figure 2. Some of the fundamental polygons based on the red triangle $(3, 4, 5)$, the blue polygon is made by omitting the side a and the green polygon made by omitting sides a and c

2.3 Tessellation with image

Using an image aligned inside the fundamental polygon, whole hyperbolic space is tessellated with this image by reflecting it hyperbolically across the sides of the fundamental polygon. Then continue reflecting it across the sides of all next adjacent polygons, covering whole space.

2.4 Models

The program supports six hyperbolic models for displaying the tessellation and interpreting the input image.

The available models are the Poincaré disk model [6], the Beltrami-Klein disk model [4], the Poincaré half-plane model[7], the Band model [3], the Gans model [5] and the hyperbolic azimuthal equidistant projection model [2].

3 Using the Program

When the user runs the executable file `Escher_esque_Hyperbolic_Tiling_Creator.exe`, two windows will appear. The first is a console window, where the user inputs data and receives prompts. The second window displays the resulting tiling.

At beginning the program shows the sizes of the tiling display window, marked by green underline in figure 3, and the program asks about the path to the txt tiling settings file. If the input path is empty or incorrect, the program uses the default path, marked with red in figure 3. The user enters the file path and presses enter, as shown in red box in figure 3.

After loading the tiling settings the program displays basic information about the fundamental triangle, as marked by green box in figure 3. Meanwhile, on the tiling display window the program renders the tiling. The rendering process may take time, depending on the window and the tiling settings.

At the end the program asks for the path to the png file to save the tiling as image. If the input is empty or lacks the png extension, the program uses the default path, highlighted in blue in figure 3.

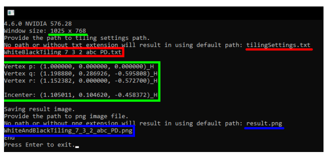

Figure 3. The console program with highlighted the program's output and the user's input

3.1 Tiling settings

The tiling settings file is a plain text file containing configuration values that define the fundamental polygon and tiling behavior. Each setting should be on a separate line, with the setting name ending in a colon. Everything after the hash symbol is treated as a comment and ignored.

3.1.1 Schwarz triangle (p, q, r) values: Followed by a triple (p, q, r), where instead of the letters should be integers or the infinity symbol "oo". Those values define used Schwarz triangle to tile space. With the "vertexCentered" centring option value of the p can't be greater than 50.

3.1.2 separating sides: Indicates which sides a, b, c are separating. If all are listed, the fundamental triangle itself becomes the polygon.

3.1.3 different images for even and odd tiles: Set to "true" to use two different images, one for even tiles and one for odd. Set to "false" to use only the first image.

3.1.4 even tiles image path: and **odd tiles image path:** File paths for the respective images.

3.1.5 source image local rotation angle: Real value determining an angle of a hyperbolic rotation that is applied to the images used to tile.

3.1.6 source image translation hyperboloid vector: The point in the hyperboloid model (w, x, y) indicating where the source images is translated.

3.1.7 source image local rotation angle: Real value determining an angle of a hyperbolic rotation that is applied to the images used to tile after the translation.

3.1.8 tiling image local rotation angle: Real value determining an angle of a hyperbolic rotation that is applied to the final tiling image.

3.1.9 tiling image translation hyperboloid vector: The point in the hyperboloid model (w, x, y) indicating where the final tiling image is translated.

3.1.10 tiling image local rotation angle: Real value determining an angle of a hyperbolic rotation that is applied to the final tiling image after the translation.

3.1.11 source image hyperbolic model: The hyperbolic model of the source images. Possible values are "PoincareDisk" [6], "BeltramiKleinDisk" [4], "PoincareHalfPlane" [7], "Band" [3], "Gans" [5] and "AzimuthalEquidistant" [2].

3.1.12 tiling image hyperbolic model: The hyperbolic model of the final tiling image.

3.1.13 centering type: Choose between "vertexCentered" and "footOfAltitudeCentered". With first option the fundamental triangle has the vertex p centred at the origin point. Using second option "footOfAltitudeCentered" results with the foot of the altitude from the vertex q being at the origin point.

3.2 Window settings

The display window settings are located in the file "res/settings/windowSettings.txt". The file contains two entries: "width:" and "height:". These determine the resolution of the tiling display.

4 Results

At the end are some results of using the program.

4.1 Tiling with the Schwarz triangle $(7, 3, 2)$

The simplest result can be achieved using two solid-color images. Tiles will alternate between these two colors. In cases where an odd number of tiles meet at a vertex, adjacent tiles may share the same color.

Figures 4, 5, 6, 7, 8 and 9 show black and white tiling using the Schwarz triangle $(7, 3, 2)$ across various models.

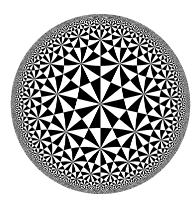

Figure 4. White and black alternating tiling in the Poincaré disk model

4.2 Recreating "Circle limit I"

This example shows that the program can recreate Escher-like tilings. Using a quadrilateral based on the Schwarz triangle $(6, 2, 6)$, and omitting side a, the pattern from "Circle Limit I" [1] is recreated, with the original shown in figure 10. The quadrilateral

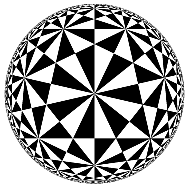

Figure 5. White and black alternating tiling in the Beltrami-Klein disk model

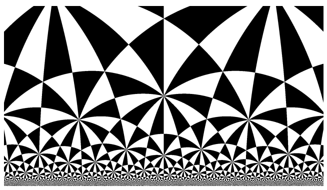

Figure 6. White and black alternating tiling in the Poincaré half-plane model

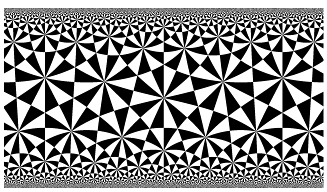

Figure 7. White and black alternating tiling in the Band model

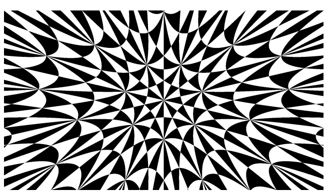

Figure 8. White and black alternating tiling in the Gans model

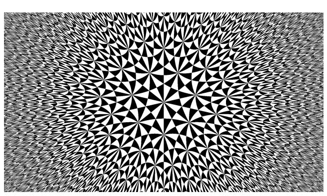

Figure 9. White and black alternating tiling in the azimuthal equidistant model

Figure 10. "Circle Limit I"

is marked in figure 11.

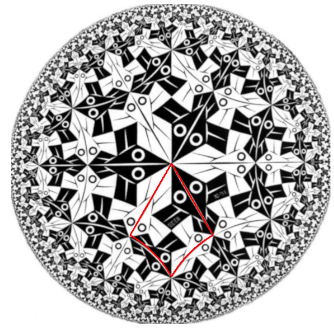

Figure 11. "Circle Limit I" with the fundamental quadrangle marked

Figures 12, 13, 14, 15, 16 and 17 show the recreated tiling in all six supported models.

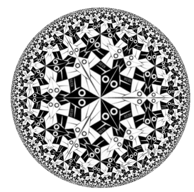

Figure 12. The recreated "Circle Limit I" in the Poincaré disk model

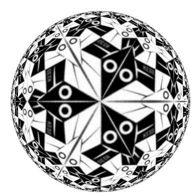

Figure 13. The recreated "Circle Limit I" in the Beltrami-Klein disk model

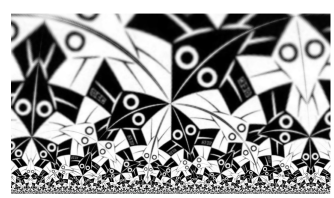

Figure 14. The recreated "Circle Limit I" in the Poincaré half-plane model

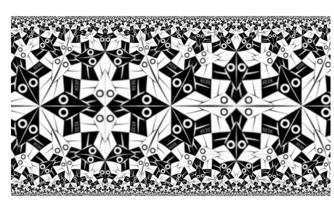

Figure 15. The recreated "Circle Limit I" in the Band model

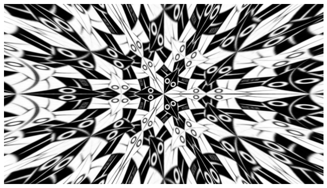

Figure 16. The recreated "Circle Limit I" in the Gans model

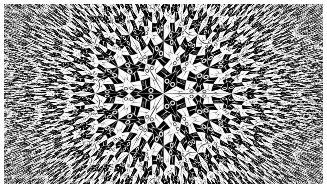

Figure 17. The recreated "Circle Limit I" in the azimuthal equidistant model

References

[1] M.C. Escher. *Circle Limit I. Hosted on WikiArt.* URL: https://www.wikiart.org/en/m-c-escher/circle-limit-i (visited on 09/05/2025).

[2] Inc. Wikimedia Foundation. *Azimuthal equidistant projection.* URL: https://en.wikipedia.org/wiki/Azimuthal_equidistant_projection (visited on 09/05/2025).

[3] Inc. Wikimedia Foundation. *Band model.* URL: https://en.wikipedia.org/wiki/Band_model (visited on 09/05/2025).

[4] Inc. Wikimedia Foundation. *Beltrami–Klein model.* URL: https://en.wikipedia.org/wiki/Beltrami%E2%80%93Klein_model (visited on 09/05/2025).

[5] Inc. Wikimedia Foundation. *Gans model.* URL: https://en.wikipedia.org/wiki/Hyperbolic_geometry#The_Gans_model (visited on 09/05/2025).

[6] Inc. Wikimedia Foundation. *Poincaré disk model.* URL: https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model (visited on 09/05/2025).

[7] Inc. Wikimedia Foundation. *Poincaré half-plane model.* URL: https://en.wikipedia.org/wiki/Poincar%C3%A9_half-plane_model (visited on 09/05/2025).

[8] Inc. Wikimedia Foundation. *Schwarz triangle.* URL: https://en.wikipedia.org/wiki/Schwarz_triangle (visited on 09/05/2025).