Extrapolación de Richardson

Sea $f \in C^{\infty}[a;b]$; $h \simeq 0$

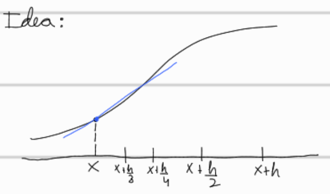
$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) \dots$$

$$f'(x) = \underbrace{\frac{f(x+h)-f(x)}{h}}_{D_0} + \underbrace{h\alpha_1 + h^2\alpha_2 \dots}_{O(h)}$$

$$f'(x) = D_0(h) + O(h) \quad (1)$$

Idea:



$$\frac{h}{2}: f'(x) = D_0\left(\frac{h}{2}\right) + \frac{h}{2}\alpha_1 + \frac{h^2}{4}\alpha_2 + \frac{h^3}{8}\alpha_3 \dots \quad (2)$$

$2 \times (2) - (1)$

$$(-)\begin{cases} 2f'(x) = 2D_0\left(\frac{h}{2}\right) + h\alpha_1 + \frac{h^2}{2}\alpha_2 + \frac{h^3}{4}\alpha_3 \dots \\ f'(x) = D_0(h) + h\alpha_1 + h^2\alpha_2 \dots \end{cases}$$

$$f'(x) = \overbrace{\boxed{2D_0\left(\tfrac{h}{2}\right) - D_0(h)}}^{D_1} + \underbrace{h^2\beta_2 + h^3\beta_3 \dots}_{O(h^2)}$$

$$D_1(h) = 2D_0\left(\frac{h}{2}\right) - D_0(h)$$

Así:

$$f'(x) = D_1(h) + h^2\beta_2 + h^3\beta_3 \dots \quad (3)$$

$$f'(x) = D_1\left(\frac{h}{2}\right) + \frac{h^2}{4}\beta_2 + \frac{h^3}{8}\beta_3 \dots \quad (4)$$

$4\times(4) - (3)$

$$f'(x) = \overbrace{\boxed{\frac{4D_1\left(\frac{h}{2}\right) - D_1(h)}{3}}}^{D_2(h)} + \overbrace{h^3\gamma_3 + h^4\gamma_4 \dots}^{O(h^3)}$$

$$\therefore D_2(h) = \frac{4D_1\left(\frac{h}{2}\right) - D_1(h)}{3}$$

Obs:

$$f'(x) = D_2(h) + \underbrace{h^3\gamma_3 + h^4\gamma_4 \dots}_{O(h^3)}$$

$$D_3(h) = \frac{8D_2\left(\frac{h}{2}\right) - D_2(h)}{7}$$

en general:

$$D_{i+1}(h) = \frac{2^{i+1} D_i\left(\frac{h}{2}\right) - D_i(h)}{2^{i+1} - 1}; \quad i \geq 0$$

$\dots O(h^{i+2})$

Ejm: aproxime el valor de $f'(1)$. si $h=1$, $f(x) = e^{-x^2}$ con iteraciones

sol: por tablas

h	D_0	$D_1(h)$	$D_2(h)$
h	$D_0(h)$	$D_1(h)$	$D_2(h)$
$\frac{h}{2}$	$D_0(h/2)$	$D_1(h/2)$	
$\frac{h}{4}$	$D_0(h/4)$		

$$D_0(h) = \frac{f(\overset{x}{1}+h) - f(\overset{x}{1})}{h}$$

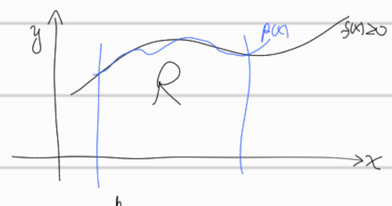
h	$D_0(h)$	$D_1(h)$	$D_2(h)$	$D_3(h)$
1	−0,34956	−0,70036	−0,75478	−0,73742
1/2	−0,52496	−0,74118	−0,73959	
1/4	−0,63307	−0,73999		
1/8	−0,68653			

Será más preciso con $h = 0,5$

$f'(x) = -2xe^{-x^2}$

$f'(1) = \underline{-0,73578\dots}$ $\quad D_3(1) = \underline{-0,7342\dots}$

Integración Numérica



$$\int_a^b f(x)\,dx = A(R)$$

$$f(x) \simeq p(x) \Rightarrow \int_a^b f(x)dx \simeq \int_a^b p(x)dx$$

$$= \sum_{i=0}^{n} A_i y_i$$

A_i = pesos

Suponga $\{x_i\}$ equidistantes

$$f(x) \simeq p(x) = y_0 + \sum_{i=1}^{n} \frac{\Delta^i y_0}{i!} \cdot s^{(i)}$$

Polinomio de diferencias finitas

Por lagrange

$$p(x) = \sum_{i=0}^{n} L_i(x) y_i$$

$$\begin{cases} x = x_0 + hs \\ x_i = x_0 + ih \\ x - x_i = h(s-i) \\ dx = h\,ds \end{cases}$$

Cambio de límites

si $x = x_0 \to s = 0$

$x = x_n \to s = n$

$$\int_a^b f(x)dx = \int_a^b \left(\sum_{i=0}^{n} L_i(x) \cdot y_i\right) dx$$

$$\simeq \sum_{i=0}^{n} \underbrace{\left[\int_a^b L_i(x)dx\right]}_{A_i} y_i \Rightarrow \boxed{A_i = \int_a^b L_i(x)dx}$$

haremos cambio variable

$$A_i = \int_0^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(s-j)\cancel{h}}{(i-j)\cancel{h}} h\,ds$$

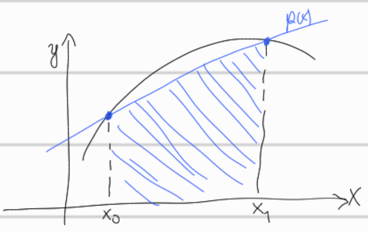
para completar el pol. factorial

$$A_i = h \int_0^n \frac{s(s-1)\dots(s-n)}{i(i-1)(i-2)\dots(i-n)} \cdot \frac{(s-i)}{(s-i)}\,ds$$

$$A_i = h \int_0^n \frac{s^{(n+1)}}{i(i-1)\dots(i-(i-1))\,(i-(i+1))\dots(i-n)} \cdot \frac{1}{s-i}\,ds$$

$$\boxed{\therefore A_i = \frac{(-1)^{n-i} h}{i!\,(n-i)!} \int_0^n \frac{s^{(n+1)}}{s-i}\,ds}$$ F. Newton Cotes

Caso n = 1



$$\int_a^b f(x)dx \simeq A_0 y_0 + A_1 y_1$$

$i=0$

$$A_0 = \frac{-h}{1} \int_0^1 \frac{s^{(2)}}{s}\,ds$$

$$= -h \int_0^1 (s-1)\,ds$$

$$= -h \left(\frac{s^2}{2} - s\right)\Big|_0^1$$

$$\boxed{A_0 = -h\left(\frac{1}{2} - 1\right) = \frac{h}{2}}$$

$$A_1 = \frac{h}{1} \int_0^1 \frac{s^{(2)}}{s-1}\,ds = h \int_0^1 s\,ds = \frac{h}{2}$$

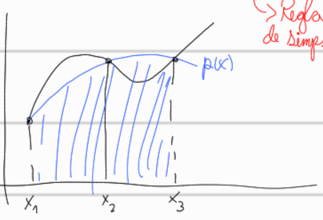
$$\boxed{\int_a^b f(x)dx \simeq \frac{h}{2}(y_0 + y_1)}$$

Formula trapezoidal

Si $n=2$

$$\int_a^b f(x)dx \simeq \boxed{\frac{h}{3}(y_0 + 4y_1 + y_2)}$$

Regla de Simpson



Regla de simpson aproxima mejor que Trapecio

aproxime $\int_0^1 2xe^{x^2}dx$

a) Trapecio

$h=1$; $x_0 = 0$, $x_1 = 1$; $y_0 = 0$, $y_1 = 5,43656$

$$\Rightarrow \int_0^1 2xe^{x^2}dx \simeq \frac{1}{2}(0 + 5,43656) = 2,71828$$

b) Simpson $n=2$, $h = \frac{b-a}{2} = \frac{1}{2}$

$x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$

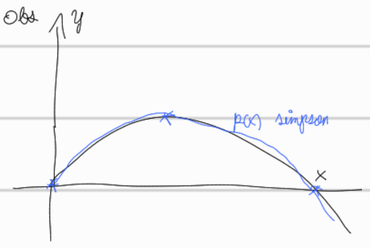
x_i	y_i
$x_0 = 0$	0
$x_1 = \frac{1}{2}$	$y_1 = 1,28403$
$x_2 = 1$	$y_2 = 5,43656$

$$\int_0^1 2xe^{x^2}dx = \frac{1}{3} \cdot \frac{1}{2}\left[0 + 4(1,28403) + 5,43656\right] = 1,76211$$

Obs:

$$\int_0^1 2xe^{x^2}dx = e^{x^2}\Big|_0^1 = e^1 - 1 = 1,71828$$

Obs



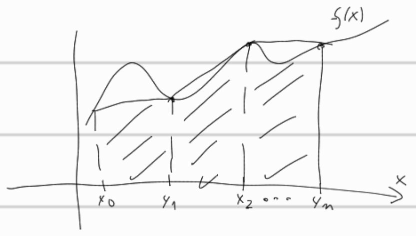
$$I = \int_0^{\pi} \operatorname{sen}x\,dx = 2$$

trapecio $\quad I \simeq \frac{h}{2}(y_0 + y_1) = \pi(0+0) = 0$:(

simpson

Integración compuesta

para $n+1$ nodos:



R.C. Trapecio

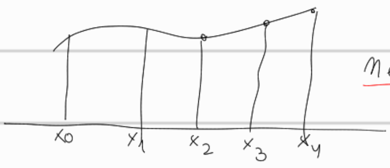
$$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

$$\simeq \sum_{i=0}^{n} \frac{h}{2}(y_i + y_{i+1})$$

$$\simeq \frac{h}{2}(y_1 + y_2 + y_2 + \dots y_{n-1} + y_n)$$

$$\therefore \int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{2}\left(y_0 + 2\sum_{i=1}^{n-1} y_i + y_n\right)$$

R.C. Simpson



n es par

$$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{m} \int_{x_{2i}}^{x_{2i+2}} f(x)dx$$

$$\simeq \sum_{i=0}^{m} \frac{h}{3}(y_{2i} + 4y_{2i+1} + y_{2i+2})$$

$$\simeq \frac{h}{3}\left[y_0 + 4y_1 + y_2 + 4y_3 + y_4 \dots + y_{n-2} + y_{n-1} + y_n\right]$$

$$\therefore \int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{3}\left[y_0 + 4\sigma_1 + 2\sigma_2 + y_n\right]$$

$\sigma_1 = y_1 + y_3 + \dots + y_{n-1}$

$\sigma_2 = y_2 + y_4 + \dots + y_{n-2}$

Ejemplo:

Aproximo $\int_0^1 2xe^{x^2}dx$ con 7 nodos

primero $h = \frac{b-a}{n-1} = \frac{1-0}{6} = \frac{1}{6}$

x_i	y_i	
$x_0 = 0$	$y_0 = 0$	
$x_1 = \frac{1}{6}$	$y_1 = 0{,}34272$	
$x_2 = \frac{2}{6}$	$y_2 = 0{,}74501$ ×	suma y impares $\sigma_1 = 4{,}96441$
$x_3 = \frac{3}{6}$	$y_3 = 1{,}28403$	$\sigma_2 = 2{,}82451$ suma y pares
$x_4 = \frac{4}{6}$	$y_4 = 2{,}07950$ ×	
$x_5 = \frac{5}{6}$	$y_5 = 3{,}33766$	
$x_6 = 1$	$y_6 = 5{,}43656$	

a) trapecio

$$\sum_{i=1}^{5} y_i = 7{,}78892$$

$$\Rightarrow \int_0^1 2xe^{x^2}dx \approx \frac{1}{2}\left(\frac{1}{6}\right)\left[0 + 2(7{,}78892) + 5{,}43656\right]$$

$$\approx 1{,}7512 \, //$$

b) Usando Simpson

$$\int_0^1 2xe^{x^2} \approx \frac{1}{3}\left(\frac{1}{6}\right)\left(0 + 4\sigma_1 + 2\sigma_2 + 5{,}43656\right)$$

$$\approx 1{,}71907$$

Obs:

$$\int_0^1 2xe^{x^2}dx = 1{,}718281\ldots$$