Extrapolación de Richardson

Sea $f \in C^\infty[a;b]$; $h \simeq 0$

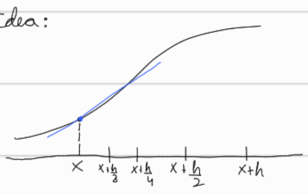
$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots$

$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2!}f''(x) + \frac{h^2}{3!}f'''(x)\dots$

$f'(x) = \underbrace{\frac{f(x+h)-f(x)}{h}}_{D_0} + \underbrace{h\alpha_1 + h^2\alpha_2\dots}_{O(h)}$

$f'(x) = D_0(h) + O(h)$ (1)

Idea:



$\frac{h}{2}: f'(x) = D_0(\frac{h}{2}) + \frac{h}{2}\alpha_1 + \frac{h^2}{4}\alpha_2 + \frac{h^3}{8}\alpha_3 \dots$ (2)

$2\times(2) - (1)$

$(-)\begin{cases} 2f'(x) = 2D_0(\frac h2) + h\alpha_1 + \frac{h^2}{2}\alpha_2 + \frac{h^3}{4}\alpha_3\dots \\ f'(x) = D_0(h) + h\alpha_1 + h^2\alpha_2\dots \end{cases}$

$f'(x) = \underbrace{2D_0(\frac h2) - D_0(h)}_{D_1} + \underbrace{h^2 B_2 + h^3 B_3\dots}_{O(h^2)}$

$D_1(h) = 2D_0(\frac h2) - D_0(h)$

Así:

$f'(x) = D_1(h) + h^2B_2 + h^3B_3\dots$ (3)

$f'(x) = D_1(\frac h2) + \frac{h^2}{4}B_2 + \frac{h^3}{8}B_3\dots$ (4)

$4\times(4) - (3)$

$f'(x) = \underbrace{\frac{4D_1(\frac h2) - D_1(h)}{3}}_{D_2(h)} + \underbrace{h^3\gamma_3 + h^4\gamma_4\dots}_{O(h^3)}$

$\therefore D_2(h) = \frac{4D_1(\frac h2) - D_1(h)}{3}$

Obs:
$f'(x) = D_2(h) + \underbrace{h^3\gamma_3 + h^4\gamma_4\dots}_{O(h^3)}$

$D_3(h) = \frac{8D_2(\frac h2) - D_2(h)}{7}$

en general:

$D_{i+1}(h) = \frac{2^{i+1}D_i(\frac h2) - D_i(h)}{2^{i+1}-1}$; $i\geq 0$

$\dots O(h^{i+2})$

Ejm: aproxime el valor de $f'(1)$. si $h=1$, $f(x) = e^{-x^2}$ con iteraciones

sol: por tablas

h	D_0	$D_1(h)$	$D_2(h)$
h	$D_0(h)$	$D_1(h)$	$D_2(h)$
$\frac h2$	$D_0(h/2)$	$D_1(h/2)$	
$\frac h4$	$D_0(h/4)$		

$\boxed{D_0(h) = \frac{f(1+h) - f(1)}{h}}$

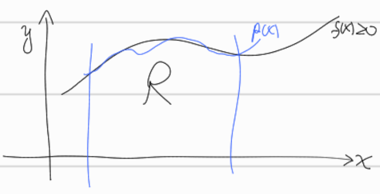
h	$D_0(h)$	$D_1(h)$	$D_2(h)$	$D_3(h)$
1	-0,34956	-0,70036	-0,75478	-0,73742
1/2	-0,52496	-0,74118	-0,73959	
1/4	-0,63307	-0,73999		
1/8	-0,68653			

Será más preciso con $h=0,5$

$f'(x) = -2xe^{-x^2}$

$f'(1) = -0,73578\dots$ $\quad D_3(1) = -0,7342\dots$

Integración Numérica



$\int_a^b f(x)dx = A(R)$

$f(x)\approx p(x) \Rightarrow \int_a^b f(x)dx \approx \int_a^b p(x)dx$
$= \sum_{i=0} A_i y_i$

A_i = pesos

Suponga $\{x_i\}$ equidistantes

$f(x)\approx p(x) = y_0 + \sum_{i=1}^n \frac{\Delta^i y_0}{i!} s^{(i)}$ Polinomio de diferencias finitas

Por lagrange

$p(x) = \sum_{i=0}^n L_i(x) y_i$

$\begin{cases} x = x_0 + hs \\ x_i = x_0 + ih \\ x - x_i = h(s-i) \\ dx = h\,ds \end{cases}$

Cambio de límites
si $x = x_0 \to s = 0$
$x = x_n \to s = n$

$\int_a^b f(x)dx = \int_a^b (\sum_{i=0}^n L_i(x)\cdot y_i)dx$

$\approx \sum_{i=0}^n [\underbrace{\int_a^b L_i(x)dx}_{A_i}] y_i \Rightarrow \boxed{A_i = \int_a^b L_i(x)dx}$

haremos cambio variable

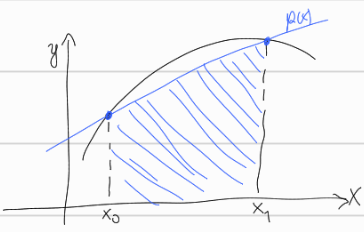
$A_i = \int_0^n \prod_{j=0, j\neq i}^n \frac{(s-j)h}{(i-j)h} h\,ds$

$A_i = h\int_0^n \frac{s(s-1)\dots(s-n)}{i(i-1)(i-2)\dots(i-n)}\cdot\frac{(s-i)}{(s-i)}\, ds$ para completar el pol. factorial

$A_i = h\int_0^n \frac{s^{(n+1)}}{i(i-1)\dots(i-(i-1))\underbrace{(i-(i+1))}_{-1}\dots\underbrace{(i-n)}_{-(n-i)}}\cdot\frac{1}{s-i}\,ds$

$\therefore \boxed{A_i = \frac{(-1)^{n-i}h}{i!(n-i)!}\int_0^n \frac{s^{(n+1)}}{s-i}ds}$ F. Newton cotes

Caso n=1



$\int_a^b f(x)dx \approx A_0y_0 + A_1y_1$

$i=0$ $\quad A_0 = \frac{-h}{1}\int_0^1 \frac{s^{(2)}}{s}ds$

$= -h\int_0^1 (s-1)ds$

$= -h(\frac{s^2}{2} - s)|_0^1$

$\boxed{A_0 = -h(\frac12 - 1) = \frac h2}$

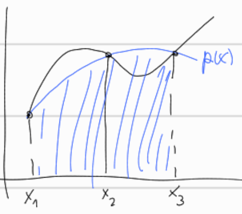
$A_1 = \frac h1 \int_0^1 \frac{s^{(2)}}{s-1}ds = h\int_0^1 s\,ds = \frac h2$

$\boxed{\int_a^b f(x)dx \approx \frac h2(y_0+y_1)}$

Formula trapezoidal

Si $n=2$

$\int_a^b f(x)dx \approx \boxed{\frac h3(y_0+4y_1+y_2)}$ → Regla de Simpson



Regla de simpson aproxima mejor que Trapecio

aproxime $\int_0^1 2xe^{x^2}dx$

a) Trapecio

$h=1$
$x_0=0$, $x_1=1$
$y_0=0$, $y_1=5,43656$

$\Rightarrow \int_0^1 2xe^{x^2}dx \approx \frac12(0+5,43656) = 2,71828$

b) Simpson $n=2$, $h = \frac{b-a}{2} = \frac12$

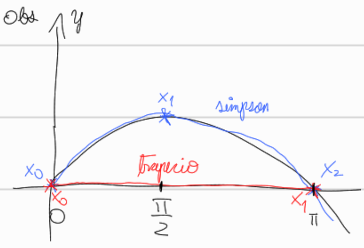
h $\quad h$
$x_0=0$, $x_1=\frac12$, $x_2=1$

x_i	y_i
$x_0=0$	0
$x_1=\frac12$	$y_1=1,28403$
$x_2=1$	$y_2=5,43656$

$\int_0^1 2xe^{x^2}dx = \frac13\cdot\frac12[0+4(1,28403)+5,43656] = 1,76211$

Obs:
$\int_0^1 2xe^{x^2}dx = e^{x^2}|_0^1 = e^1 - 1 = 1,71828$ Simpson lo aproxima mejor

Obs



$I = \int_0^\pi \operatorname{sen}x\,dx = 2$

trapecio $I\approx \frac h2(y_0+y_1) = \pi(0+0) = 0$ ¿Qué pasó? Ö La peculiaridad del método y la función seno nos dan un errollodo nada que ver

simpson $h=\frac\pi2$

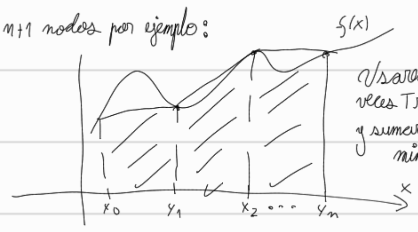
$I\approx\frac h3(y_0+4y_1+y_2) = \frac\pi6(0+4+0) \cong 2,0944$ simpson se salvó...

$\int_0^\pi \operatorname{sen}x\,dx = -\cos x|_0^\pi = -\cos\pi + \cos 0 = 2$

Integración compuesta

Trapecio y Simpson usan 2 y 3 nodos. Esto puede ser limitante. ¿Cómo generalizar con más nodos?

para $n+1$ nodos por ejemplo:



Usaremos muchas veces Trapecio o simpson y sumaremos esas mini-integrales

I.C. Trapecio

$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx \approx$

$\approx \sum_{i=0}^{n-1}\frac h2(y_i+y_{i+1})$ Trapecio

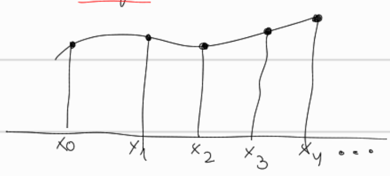
$\approx \frac h2(y_0+y_1+y_1+y_2+y_2+y_3\dots y_{n-1}+y_n)$

$\approx \frac h2(y_0+2y_1+2y_2\dots 2y_{n-1}+y_n)$

$\therefore \int_{x_0}^{x_n} f(x)dx \approx \frac h2(y_0+2\sum_{i=1}^{n-1}y_i+y_n)$

I.C. Simpson

Observación: n es par sí o sí



$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{\frac n2 -1}\int_{x_{2i}}^{x_{2i+2}} f(x)dx$

$\approx \sum_{i=0}^{\frac n2-1}\frac h3(y_{2i}+4y_{2i+1}+y_{2i+2})$ la regla de Simpson

$\approx \frac h3[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+\dots+(y_{n-2}+4y_{n-1}+y_n)]$

$\therefore \int_{x_0}^{x_n} f(x)dx \approx \frac h3[y_0+4\sigma_1+2\sigma_2+y_n]$

$\sigma_1 = y_1+y_3+\dots+y_{n-1}$

$\sigma_2 = y_2+y_4+\dots+y_{n-2}$

Ejemplo:

Aproximo $\int_0^1 2xe^{x^2}dx$ con $N=7$ nodos, $n=6$

primero $h=\frac{b-a}{N-1}=\frac{1-0}{6}=\frac{1}{6}$

x_i	y_i	
$x_0=0$	$y_0=0$	
$x_1=\frac{1}{6}$	$y_1=0{,}34272$	
$x_2=\frac{2}{6}$	$y_2=0{,}74501$ ×	suma y impares $\sigma_1=4{,}96441$
$x_3=\frac{3}{6}$	$y_3=1{,}28403$	$\sigma_2=2{,}82451$ suma y pares
$x_4=\frac{4}{6}$	$y_4=2{,}07950$ ×	
$x_5=\frac{5}{6}$	$y_5=3{,}33766$	
$x_6=1$	$y_6=5{,}43656$	

a) trapecio

$\sum_{i=1}^{5} y_i = 7{,}78892$

$\Rightarrow \int_0^1 2xe^{x^2}dx \approx \frac{1}{2}\left(\frac{1}{6}\right)\left[0+2(7{,}78892)+5{,}43656\right]$

$\approx 1{,}7512$ //

Obs: ahora la aprox es buena

b) Usando Simpson

$\int_0^1 2xe^{x^2} \approx \frac{1}{3}\left(\frac{1}{6}\right)\left(0+4\sigma_1+2\sigma_2+5{,}43656\right)$

$\approx 1{,}71907$

Obs:

$\int_0^1 2xe^{x^2}dx = 1{,}718281\ldots$

Entre más nodos, aproxima mejor